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A quantum central limit theorem for non-equilibrium systems: exact local relaxation of correlated states

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Abstract. We prove that quantum many-body systems on a one-dimensional lattice locally relax to Gaussian states under non-equilibrium dynamics generated by a bosonic quadratic Hamiltonian. This is true for a large class of initial states—pure or mixed—which have to satisfy merely weak conditions concerning the decay of correlations. The considered setting is a proven instance of a situation where dynamically evolving closed quantum systems locally appear as if they had truly relaxed, to maximum entropy states for fixed second moments. This furthers the understanding of relaxation in suddenly quenched quantum many-body systems. The proof features a non-commutative central limit theorem for non-i.i.d. random variables, showing convergence to Gaussian characteristic functions, giving rise to trace-norm closeness. We briefly link our findings to the ideas of typicality and concentration of measure.

Contents

1. Introduction

In what sense can closed local many-body systems in a non-equilibrium situation relax to an apparent equilibrium? Instances of that question have a long tradition in the literature. This apparent contradiction of having entropy preserved in any closed system and at the same time arriving at an equilibrated situation for long times can yet be resolved by acknowledging that merely some observables, and in particular entire subsystems, can well appear as if they had truly relaxed, even in closed systems. In local observations, such a composite system would then look entirely equilibrated.

Early work formulating quite a similar intuition, albeit not in quantum lattice systems, was concerned with fermions freely moving in space [1]. The seminal work in [2, 3] rigorously developed such an intuition for classical harmonic systems on cubic lattices. In [4]–[6], that question is considered in instances of a bosonic or fermionic fully Gaussian setting, where initial states are Gaussian, and the evolution is governed by a quadratic Hamiltonian. The authors of [7] then arrive at a true local relaxation theorem for subsystems where initial states are not taken to be Gaussian: yet, in the course of the non-equilibrium dynamics of the system, locally, the states become Gaussian, and so maximum entropy states for given second moments. This is true without having to invoke a time average. In such lattice models, the intuition is that non-equilibrium generates local excitations at each site that then travel ballistically through the lattice, resulting in a mixing at each site with excitations from further and further separated sites coming in [7, 8].

Some of the revived significant recent interest in equilibration and relaxation in closed quantum many-body systems, needless to say, has been sparked off by novel experiments with cold atoms in optical lattices or on low-dimensional structures. For theory work relating to those experiments, see, e.g. [9]–[14] and references therein. This development is quite intriguing, as questions of an apparent local relaxation in non-equilibrium dynamics can be measured and probed under very precisely tunable laboratory conditions.

In this work, we present a fully rigorous relaxation theorem, generalizing the findings in [7], for one-dimensional quantum lattice systems. Physically speaking, the results presented here can be viewed as a relaxation theorem forming a 'theoretical laboratory', describing an idealized situation of non-equilibrium dynamics in the Bose–Hubbard setting: the system being initially held in some ground or thermal state of the full Bose–Hubbard–Hamiltonian, and then suddenly switched or 'quenched' to a parameter regime of strong hopping.

It is shown that for general pure or mixed initial states satisfying quite weak assumptions concerning the clustering of correlations—properties that are expected to hold in particular for ground states of local many-body models—locally, the system will relax to a Gaussian state under the dynamics generated by a quadratic Hamiltonian. That is to say, we prove the 'local relaxation conjecture' for one-dimensional clustering states under quadratic Hamiltonians on a ring of L sites. Namely, for an initial state $\hat{\varrho}_0$ that is evolved in time under a quadratic Hamiltonian and a given finite subset $\mathcal{S}$ of the chain, we show that for every $\epsilon > 0$ and sufficiently large L there is a relaxation time t_{relax}, and a recurrence time t_{rec} (which grows unboundedly with L), such that the reduction of the time-evolved state to $\mathcal{S}$, $\hat{\varrho}_{\mathcal{S}}(t)$, fulfils

$$\|\hat{\varrho}_{\mathcal{S}}(t) - \hat{\varrho}_G(t)\|_{\text{tr}} \leqslant \epsilon, \quad \text{for all } t_{\text{relax}} \leqslant t \leqslant t_{\text{rec}}, \tag{1}$$

in particular,

$$\lim_{t\to\infty} \lim_{L\to\infty} \|\hat{\varrho}_{\mathcal{S}}(t) - \hat{\varrho}_G(t)\|_{\text{tr}} = 0. \tag{2}$$

The state $\hat{\varrho}_G(t)$ is a Gaussian state with the same second moments as $\hat{\varrho}_{\mathcal{S}}(t)$. The proof features the explicit forms of t_{relax} and t_{rec} in terms of the coupling parameters in the Hamiltonian and the decay behaviour of correlations of the initial state.

The key ingredients of the proof are a quantum Lindeberg central limit theorem, a notion of locality of dynamics as is manifest by a Lieb–Robinson bound, and a Bernstein–Spohn-blocking argument reminiscent of the classical situation. Essentially, we will show that for a given time, excitations arising from far away sites will hardly influence the dynamics at a subset of sites, whereas the neighbouring sites give rise to a mixing such that a Lindeberg central limit theorem can be invoked. In several ways, our quantum treatment reminds us of the classical situation presented in [2, 3], with the blocking argument being essentially identical and with significant differences in many other aspects. In this work, we focus on the important case of a one-dimensional setting and allow for assumptions on the initial state that can be relaxed, in order to render the argument simpler than a fully general argument [15].

After rigorously proving convergence of characteristic functions to Gaussian functions in phase space, we prove closeness in trace-norm of the respective reduced states. In this way, we derive in the trace-norm topology a closeness to maximum entropy states under the second moments. In some ways, the arguments remind us of the situation of discrete time in convergence to stationary states in quantum cellular automata [16]. Finally, we also briefly discuss the findings in the light of recent results concerning concentration of measure and notions of typicality [17]–[19].

2. Preliminaries

In this section, we are collecting preliminaries that are being used in the proof. We also specify the model under consideration here and will specify the assumptions on the initial states in the subsequent section. Note that the assumptions on such initial states are very mild and do by no means include only pure or Gaussian states; merely the model Hamiltonian as such is taken to be a specific nearest-neighbor Hamiltonian.

2.1. System and dynamical setting

We consider a chain $\mathcal{L} = \{1, \ldots, L\}$ equipped with periodic boundary conditions. Each lattice site is associated with a bosonic canonical degree of freedom equipped with the usual symplectic form. The distance between lattice sites $i, j \in \mathcal{L}$ is due to the periodic boundary conditions given by

$$d_{i,j} = \min\{|i-j|, |i-j+L|, |i-j-L|\}\,. \tag{3}$$

For future purposes, we also define distances between subsets of sites. For $\mathcal{A}, \mathcal{B} \subset \mathcal{L}$, we define their distance as

$$d_{\mathcal{A},\mathcal{B}} = \min_{i\in\mathcal{A}, j\in\mathcal{B}} d_{i,j}. \tag{4}$$

Starting from an initial quantum state (restricted to be neither a pure state nor a Gaussian state) $\hat{\varrho}_0$ on $\mathcal{L}$, we are concerned with its time evolution,

$$\hat{\varrho}(t) = \mathrm{e}^{-\mathrm{i}\hat{H}t}\hat{\varrho}_0\mathrm{e}^{\mathrm{i}\hat{H}t} \tag{5}$$

under the Hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{i,j\in\mathcal{L}} \left(\hat{b}_i^\dagger A_{i,j}\hat{b}_j + \hat{b}_i A_{i,j}\hat{b}_j^\dagger\right). \tag{6}$$

$\hat{b}_i$ denotes a bosonic operator at site i. We take the Hamiltonian to be defined by

$$A_{i,j} = -J\delta_{d_{i,j},1}. \tag{7}$$

We will work in units in which $J = 1$ and $\hbar = 1$. This is the Hamiltonian of a harmonic chain, in particular approximating the Hamiltonian of a Bose–Hubbard model in the limit of a hopping that is dominant compared to the interaction. Note that local relaxation may be shown for general quadratic Hamiltonians with sufficiently local couplings (exponentially decaying in the coupling distance) on (not necessarily cubic) lattices of arbitrary spatial dimension [15]. The assumption that $\hat{H}$ is quadratic is essential, however, for the validity of the argument. Yet, the physical intuition that local relaxation is due to a mixing of excitations that essentially travel ballistically through the lattice is expected to be valid more generally. In this sense, the present proof constitutes the mentioned 'theoretical laboratory' in which this mechanism can be studied in great clarity.

In particular, we will consider local properties of $\hat{\varrho}(t)$ and will see that, in a sense, this reduced state will eventually relax in a sense that is yet to be specified. To this end, let $\mathcal{S} \subset \mathcal{L}$ be a subset of sites of the lattice and define the reduced state associated with the degrees of freedom of this subset as

$$\hat{\varrho}_{\mathcal{S}}(t) = \mathrm{tr}_{\mathcal{L}\backslash\mathcal{S}}[\hat{\varrho}(t)]. \tag{8}$$

We write its characteristic function $\chi_{\hat{\varrho}_{\mathcal{S}}(t)}\colon \mathbb{C}^{|\mathcal{S}|} \to \mathbb{C}$ as the expectation value of the Weyl operator, i.e. as

$$\chi_{\hat{\varrho}_{\mathcal{S}}(t)}(\boldsymbol{\beta}) = \mathrm{tr}_{\mathcal{S}}[\hat{\varrho}_{\mathcal{S}}(t)\hat{D}(\boldsymbol{\beta})],\ \hat{D}(\boldsymbol{\beta}) = \prod_{i\in\mathcal{S}} \mathrm{e}^{\beta_i \hat{b}_i^\dagger - \beta_i^* \hat{b}_i}, \tag{9}$$

as a complex-valued phase space function uniquely defining the state. States the characteristic function of which is a Gaussian in phase space are referred to as being quasi-free or Gaussian states.

One finds, after solving Heisenberg's equation of motion and using the cyclic invariance of the trace, for the time evolution of the characteristic function of the reduced state $\hat{\varrho}_{\mathcal{S}}(t)$

$$\chi_{\hat{\varrho}_{\mathcal{S}}(t)}(\boldsymbol{\beta}) = \mathrm{tr}_{\mathcal{L}}\left[\hat{\varrho}_0 \prod_{i\in\mathcal{L}} \mathrm{e}^{\alpha_i(t,\boldsymbol{\beta})\hat{b}_i^\dagger - \alpha_i^*(t,\boldsymbol{\beta})\hat{b}_i}\right] =: \mathrm{tr}_{\mathcal{L}}[\hat{\varrho}_0 \hat{D}(\boldsymbol{\alpha}(t,\boldsymbol{\beta}))], \tag{10}$$

where $\boldsymbol{\alpha} \in \mathbb{C}^{|\mathcal{L}|}$,

$$\alpha_i = \sum_{j\in\mathcal{S}} \beta_j C_{j,i}^*(t), \quad C = \mathrm{e}^{-\mathrm{i}tA}, \tag{11}$$

the latter to be understood as a matrix exponent of the matrix A collecting the coupling coefficients. Note that this is exactly the setting of a sudden quench: one considers a ground or thermal state of some Hamiltonian (or any initial state) and the abrupt switch to the Hamiltonian $\hat{H}$, and follows the free time evolution under this new Hamiltonian. Note that the weak assumptions on the decay of correlations also allow for initial ground states of critical models.

2.2. Moments

For operators $\hat{O}$ defined on the Hilbert space associated with the lattice model, we write expectation values as

$$\langle \hat{O} \rangle = \mathrm{tr}[\hat{\varrho}_0 \hat{O}]. \tag{12}$$

For vectors of phase space coordinates $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbb{C}^{|\mathcal{L}|}$, we define the anti-Hermitian operator

$$\hat{b}(\boldsymbol{\alpha}) = \sum_{i\in\mathcal{L}} \left(\alpha_i \hat{b}_i^\dagger - \alpha_i^* \hat{b}_i\right), \tag{13}$$

and instances of second and fourth moments of $\hat{b}(\boldsymbol{\alpha})$ as

$$\sigma(\boldsymbol{\alpha}) = \sigma(\boldsymbol{\alpha}, \boldsymbol{\alpha}), \quad \sigma(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \langle \hat{b}(\boldsymbol{\alpha})\hat{b}(\boldsymbol{\alpha}')\rangle, \quad f(\boldsymbol{\alpha}) = \left\langle \left[\hat{b}(\boldsymbol{\alpha})\right]^4 \right\rangle. \tag{14}$$

For $\mathcal{A} \subset \mathcal{L}$, we define the vector $\boldsymbol{\alpha}_{\mathcal{A}} \in \mathbb{C}^L$ by

$$(\boldsymbol{\alpha}_{\mathcal{A}})_i = \begin{cases} \alpha_i, & \text{if } i \in \mathcal{A}, \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

as the indicator function and write

$$\sigma_{\mathcal{A},\mathcal{B}} = \sigma(\boldsymbol{\alpha}_{\mathcal{A}}, \boldsymbol{\alpha}_{\mathcal{B}}), \quad \sigma_{\mathcal{A}} = \sigma_{\mathcal{A},\mathcal{A}}, \quad f_{\mathcal{A}} = f(\boldsymbol{\alpha}_{\mathcal{A}}), \tag{16}$$

for certain second moments and functions of indicator functions.

3. Assumptions on the initial state and locality of dynamics

In this section, we state the assumptions on the considered initial states and highlight the role of Lieb–Robinson bounds in the argument.

3.1. Assumptions on the initial state

The assumptions on the initial state essentially express some degree of clustering or decay of correlations. This clustering can be assumed to be quite weak, and even a slow algebraic decay of correlations is allowed for. We will also include some natural conditions on the initial state that significantly simplify the argument. In particular, this includes an assumption of the initial state to commute with the total particle number operator. In any bosonic system involving massive particles, this natural requirement will always be satisfied.

More specifically, we assume that the initial state is such that[6]

$$\left[\hat{\varrho}_0, \sum_{i\in\mathcal{L}} \hat{b}_i^\dagger \hat{b}_i\right] = 0. \tag{17}$$

Then, for all $\boldsymbol{\alpha} \in \mathbb{C}^L$ and all $\mathcal{A}, \mathcal{B} \subset \mathcal{L}$, we have $\langle \hat{b}(\boldsymbol{\alpha})\rangle = 0$, and

$$\begin{aligned}
\sigma_{\mathcal{A},\mathcal{B}} &= \sigma(\boldsymbol{\alpha}_{\mathcal{A}}, \boldsymbol{\alpha}_{\mathcal{B}}) = -\sum_{i\in\mathcal{A}}\sum_{j\in\mathcal{B}} \alpha_i \alpha_j^* (\langle \hat{b}_i^\dagger \hat{b}_j\rangle + \langle \hat{b}_j \hat{b}_i^\dagger\rangle),\\
f_{\mathcal{A}} &= f(\boldsymbol{\alpha}_{\mathcal{A}})\\
&= \sum_{i,j,k,l\in\mathcal{A}} \alpha_i\alpha_j\alpha_k^*\alpha_l^* (\langle \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l\rangle + \langle \hat{b}_i^\dagger \hat{b}_k \hat{b}_j^\dagger \hat{b}_l\rangle + \langle \hat{b}_i^\dagger \hat{b}_k \hat{b}_l \hat{b}_j^\dagger\rangle\\
&\quad + \langle \hat{b}_k \hat{b}_j^\dagger \hat{b}_i^\dagger \hat{b}_l\rangle + \langle \hat{b}_k \hat{b}_j^\dagger \hat{b}_l \hat{b}_i^\dagger\rangle + \langle \hat{b}_k \hat{b}_l \hat{b}_i^\dagger \hat{b}_j^\dagger\rangle).
\end{aligned} \tag{18}$$

The first assumption concerns two-point correlations.

Assumption 1 (Two-point correlations). *Let the initial state $\hat{\varrho}_0$ be such that there exist absolute constants $c_1, \mu_1, \epsilon_1 > 0$ such that*

$$|\langle \hat{b}_i^\dagger \hat{b}_j| \leqslant \frac{c_1}{\left[1 + d_{i,j}\right]^{2+\mu_1+\epsilon_1}} \tag{19}$$

for all i, j.

This is a natural assumption on the initial state in a way that does not depend on the system size. This assumption may be relaxed to decay stronger than the spatial dimension of the lattice (i.e. in the setting at hand, the two may be replaced by a one), resulting, however, in a much more involved proof [15]. We note a first immediate consequence of assumption 1 that will be used in the main theorem. We also encounter the recurrence time: the system has to be sufficiently large for the bounds on the second moments to hold.

[6] This simplifies the calculations significantly as expectation values of products involving unequal numbers of annihilation and creation operators vanish; it is, however, not necessary to show local relaxation [15].

Lemma 1 (Variances). *Under assumption 1 and for $L \geqslant |t|^{7/6} \geqslant 1$, one has the following bounds*[7]

$$|\sigma_{\mathcal{A},\mathcal{B}}| \leqslant c_2 \|\boldsymbol{\beta}\|_1^2 \frac{\min\{|\mathcal{A}|,|\mathcal{B}|\}}{|t|^{2/3}\left[1+d_{\mathcal{A},\mathcal{B}}\right]^{1+\mu_1}} \tag{20}$$

for all $\mathcal{A},\mathcal{B} \subset \mathcal{L}$,

$$\left|\sigma_{\mathcal{L}} - \sum_{i=1}^{n} \sigma_{\mathcal{A}_k}\right| \leqslant |\sigma_{\mathcal{L}\setminus\mathcal{A}}| + 2|\sigma_{\mathcal{L}\setminus\mathcal{A},\mathcal{A}}| + c_2\|\boldsymbol{\beta}\|_1^2 \sum_{i=1}^{n} \frac{|\mathcal{A}_i|}{|t|^{2/3}\left[1+d_{\mathcal{A}\setminus\mathcal{A}_i,\mathcal{A}_i}\right]^{1+\mu_1}} \tag{21}$$

for all $\mathcal{A}_i \subset \mathcal{L}$, $i = 1,\ldots,n$, $\mathcal{A} = \cup_i \mathcal{A}_i$, and

$$|\sigma_{\mathcal{A},\mathcal{B}}| \leqslant c_2\|\boldsymbol{\beta}\|_1^2 2^{-d_{\mathcal{A},\mathcal{S}}} \tag{22}$$

for all $\mathcal{A},\mathcal{B} \subset \mathcal{L}$ such that $4\mathrm{e}|t| \leqslant d_{\mathcal{A},\mathcal{S}}$. Here,

$$c_2 = 37^2(4c_1+2)\zeta(1+\epsilon_1) \tag{23}$$

and c_1, ϵ_1, and μ_1 are as in assumption 1.

The second assumption concerns four-point correlations.

Assumption 2 (Four-point correlations). *Let the initial state $\hat{\rho}_0$ be such that there exist absolute constants $c_3, \epsilon_2 > 0$ such that*

$$\left|\left\langle \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l \right\rangle\right| \leqslant \sum_{(r,s,t,u)\in P(i,j,k,l)} \frac{c_3}{([1+d_{r,s}][1+d_{t,u}])^{1+\epsilon_2}} \tag{24}$$

for all i, j, k, l. Here, $P(i,j,k,l)$ contains all permutations of the indices i, j, k and l.

Lemma 2 (Fourth moments). *Under assumptions 1 and 2 and for $L \geqslant |t|^{7/6} \geqslant 1$, one has*

$$\sum_{i=1}^{n} |f_{\mathcal{A}_i}| \leqslant c_4 \|\boldsymbol{\beta}\|_2^2 \|\boldsymbol{\beta}\|_1^2 \frac{\max_i |\mathcal{A}_i|}{|t|^{2/3}} \tag{25}$$

for all $\mathcal{A}_i \subset \mathcal{L}$, $i = 1,\ldots,n$, with $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$. Here,

$$c_4 = 96(6c_3 + 3(4c_1+1))37^2\zeta^2(1+\min\{\epsilon_1,\epsilon_2\}), \tag{26}$$

c_1 and ϵ_1 are as in assumption 1 and c_3 and ϵ_2 as in assumption 2.

We finally put a clustering condition on the initial state.

Assumption 3 (Clustering). *Let the initial state $\hat{\varrho}_0$ be such that there exist absolute constants $c_5, \mu_2 > 0$ such that*

$$|\langle \hat{D}(\boldsymbol{\alpha}_{\mathcal{A}})\hat{D}(\boldsymbol{\alpha}_{\mathcal{B}})\rangle - \langle \hat{D}(\boldsymbol{\alpha}_{\mathcal{A}})\rangle\langle \hat{D}(\boldsymbol{\alpha}_{\mathcal{B}})\rangle| \leqslant \frac{c_5}{[1+d_{\mathcal{A},\mathcal{B}}]^{1/2+\mu_2}} \tag{27}$$

for all $\mathcal{A},\mathcal{B} \subset \mathcal{L}$ and all $\mathcal{L}$.

Note that this condition may again be relaxed [15].

[7] We use the following vector norms for $\boldsymbol{\beta} \in \mathbb{C}^{|\mathcal{S}|}$: $\|\boldsymbol{\beta}\|_1 = \sum_{i\in\mathcal{S}} |\beta_i|$ and $\|\boldsymbol{\beta}\|_2 = \sqrt{\sum_{i\in\mathcal{S}} |\beta_i|^2}$.

3.2. Lieb–Robinson bounds and locality of dynamics

Lieb–Robinson bounds are upper bounds to group velocities of excitations travelling through a quantum lattice system. They define a causal cone of a local excitation, outside of which any influence of this excitation is exponentially suppressed. As such, they provide an upper bound to the speed of any non-negligible information propagation by time evolution of a quantum lattice model [20]–[23]. Originally formulated for spin systems, in the present context, we need to invoke an instance that is suitable for the harmonic, infinite-dimensional individual constituents at hand [24, 25]. Subsequently, we will formulate a useful related bound defining a causal cone for matrix entries of the propagator itself, which will be used in the main theorem.

Lemma 3 (Lieb–Robinson bound). *For all times, all $i, j \in \mathcal{L}$ and all $\mathcal{A} \subset \mathcal{L}$ with $4\mathrm{e}|t| \leqslant d_{\mathcal{A},\mathcal{S}}$, one has*

$$|C_{i,j}| \leqslant \frac{|2t|^{d_{i,j}}}{d_{i,j}!}, \quad \sum_{i\in\mathcal{A}} |\alpha_i| \leqslant 4\|\boldsymbol{\beta}\|_1 2^{-d_{\mathcal{A},\mathcal{S}}}. \tag{28}$$

For all t, L such that $L \geqslant |t|^{7/6} \geqslant 1$, one has for all $i, j \in \mathcal{L}$ that

$$|C_{i,j}| \leqslant \frac{37}{|t|^{1/3}}, \quad |\alpha_i| \leqslant \|\boldsymbol{\beta}\|_1 \frac{37}{|t|^{1/3}}. \tag{29}$$

This bound shows the exponential smallness of the matrix entries of $C = \mathrm{e}^{-\mathrm{i}tA}$ in the distance $d_{i,j}$ away from the main diagonal at a given time. Also, equations (29) signify bounds needed to prove convergence due to mixing within the causal cone for large times. The proof will be given in the appendix.

4. Quantum central limit theorems

Lindeberg central limit theorems are central limit theorems that do not rely on an identically distributed assumption concerning the considered random variables. Here, we formulate a quantum version thereof, which will in the main theorem be extended to a quantum version of a central limit theorem that in addition does not rely on independent random variables. We start by noting a number of useful facts on the quantum characteristic function in order to proceed to formulate the quantum central limit theorem itself.

4.1. Some facts about characteristic functions

In this section, we will relate quantum characteristic functions to classical characteristic functions, by invoking Bochner's theorem. For a similar correspondence of quantum to classical characteristic functions, see, e.g., [26]. For a fixed vector of phase space coordinates $\boldsymbol{\alpha} \in \mathbb{C}^L$, consider the function $\phi: \mathbb{R} \to \mathbb{C}$,

$$\phi(r) = \mathrm{tr}[\hat{\varrho}_0 \hat{D}(r\boldsymbol{\alpha})] = \mathrm{tr}[\hat{\varrho}_0 \mathrm{e}^{r\hat{b}(\boldsymbol{\alpha})}] = \langle \mathrm{e}^{r\hat{b}(\boldsymbol{\alpha})} \rangle. \tag{30}$$

One finds that $\phi^{(n)}(0) = 0$ for odd $n \in \mathbb{N}$ and

$$\phi^{(2)}(0) = \sigma(\boldsymbol{\alpha}), \quad \phi^{(4)}(0) = f(\boldsymbol{\alpha}), \tag{31}$$

where $\phi^{(n)}$ denotes the nth derivative of ϕ. Using the Cauchy–Schwarz inequality, one finds for even $m \in \mathbb{N}$ the following chain of inequalities. We suppress the argument $\boldsymbol{\alpha}$ of $\hat{b}$ defined as in equation (13), use the fact that $\hat{b}^\dagger = -\hat{b}$ and write $\hat{\varrho}_0 = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$ for the spectral decomposition of the state $\hat{\varrho}_0$. Hence we find for the mth derivative,

$$\begin{aligned}|\phi^{(m)}(r)| &\leqslant \sum_n \varrho_n |\langle\psi_n|\hat{b}^{m/2}\mathrm{e}^{r\hat{b}}\hat{b}^{m/2}|\psi_n\rangle| \\ &\leqslant \sum_n \varrho_n |\langle\psi_n|\hat{b}^m|\psi_n\rangle| = \left|\sum_n \varrho_n \langle\psi_n|\hat{b}^m|\psi_n\rangle\right| = |\phi^{(m)}(0)| \\ &\leqslant \sum_n \varrho_n |\langle\psi_n|(\hat{b}^\dagger)^m\hat{b}^m|\psi_n\rangle|^{1/2} \leqslant \sqrt{\sum_n \varrho_n \langle\psi_n|\hat{b}^{2m}|\psi_n\rangle} = \sqrt{\phi^{(2m)}(0)}.\end{aligned} \tag{32}$$

Furthermore, $\phi(0) = 1$, and ϕ is positive semi-definite and continuous at the origin [27]. It follows from Bochner's theorem that ϕ is a classical characteristic function of a classical random variable. Hence, if $\phi^{(2)}(0)$ exists, ϕ is twice continuously differentiable on the entire real line, i.e. as $\phi^{(1)}(0) = 0$, one has from Taylor's theorem

$$|\phi(1) - 1| \leqslant \int_0^1 \mathrm{d}x\, |\phi^{(2)}(x)|(1-x) \leqslant \frac{|\phi^{(2)}(0)|}{2}. \tag{33}$$

Hence,

$$|\langle\hat{D}(\boldsymbol{\alpha})\rangle - 1| \leqslant \frac{|\sigma(\boldsymbol{\alpha})|}{2} \tag{34}$$

for all $\boldsymbol{\alpha} \in \mathbb{C}^L$ as $\boldsymbol{\alpha}$ was arbitrary.

If $\phi^{(4)}(0)$ exists, ϕ is four times continuously differentiable on the entire real line, i.e. one has from Taylor's theorem

$$\left|\phi(1) - 1 - \frac{\phi^{(2)}(0)}{2}\right| \leqslant \int_0^1 \mathrm{d}x\, |\phi^{(4)}(x)|\frac{(1-x)^3}{6} \leqslant \frac{|\phi^{(4)}(0)|}{24}. \tag{35}$$

4.2. A Lindeberg-type quantum central limit theorem

We now turn to the actual quantum instance of a Lindeberg-type central limit theorem. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathcal{L}$ be mutually disjoint sets. These take the role of independent but not necessarily identically distributed random variables. Consider

$$z_i = \phi_i(1) = \langle\exp[\hat{b}(\boldsymbol{\alpha}_{\mathcal{A}_i})]\rangle, \quad i = 1, \ldots, n. \tag{36}$$

If $|z_i - 1| \leqslant 1/2$ for all i, a complex logarithm is defined by the Mercator series. We have

$$\begin{aligned}|\log(z_i) - \phi_i^{(2)}(0)/2| &\leqslant |\log(z_i) - (z_i - 1)| + |z_i - 1 - \phi_i^{(2)}(0)/2| \\ &\leqslant |\log(z_i) - (z_i - 1)| + \frac{|\phi_i^{(4)}(0)|}{24},\end{aligned} \tag{37}$$

where

$$\begin{aligned}|\log(z_i)-(z_i-1)| &= |\log(1+(z_i-1))-(z_i-1)| \leqslant \sum_{n=2}^{\infty}\frac{|z_i-1|^n}{n}\\ &= |z_i-1|^2\sum_{n=0}^{\infty}\frac{|z_i-1|^n}{n+2} \leqslant |z_i-1|^2\sum_{n=0}^{\infty}\frac{|1/2|^n}{n+2}\\ &\leqslant |z_i-1|^2 \leqslant \frac{|\phi_i^{(2)}(0)|^2}{4} \leqslant \frac{|\phi_i^{(4)}(0)|}{4}.\end{aligned}\tag{38}$$

Hence,

$$\left|\sum_i \log(z_i)-\sum_i\frac{\sigma_{\mathcal{A}_i}}{2}\right| \leqslant \frac{7}{24}\sum_i|\phi_i^{(4)}(0)|.\tag{39}$$

Now, let $x, y, z \in \mathbb{R}$. We set out to show that $|\mathrm{e}^{x+\mathrm{i}y}-\mathrm{e}^z| \leqslant |x-z+\mathrm{i}y|\mathrm{e}^{\max\{x,z\}}$. This holds for $x=z$, i.e. we may let $x\neq z$ w.l.o.g. Then

$$\mathrm{e}^{x+\mathrm{i}y}-\mathrm{e}^z=\mathrm{e}^z(\mathrm{e}^{x-z+\mathrm{i}y}-1)=\mathrm{e}^z(x-z+\mathrm{i}y)\int_0^1 \mathrm{d}t\,\mathrm{e}^{t(x-z+\mathrm{i}y)},\tag{40}$$

i.e. using the mean value theorem, there is a c between x and z such that

$$\begin{aligned}|\mathrm{e}^{x+\mathrm{i}y}-\mathrm{e}^z| &\leqslant \mathrm{e}^z|x-z+\mathrm{i}y|\int_0^1\mathrm{d}t\,\mathrm{e}^{t(x-z)} = |x-z+\mathrm{i}y|\frac{\mathrm{e}^x-\mathrm{e}^z}{x-z}\\ &= |x-z+\mathrm{i}y|\mathrm{e}^c \leqslant |x-z+\mathrm{i}y|\mathrm{e}^{\max\{x,z\}}.\end{aligned}\tag{41}$$

Now, we recall that $|z_i-1|\leqslant 1/2$ and $|z_i|\leqslant 1$, i.e.

$$\log|z_i| = -|\log|z_i||.\tag{42}$$

Hence, using the fact that $\sigma_{\mathcal{A}_i}=-|\sigma_{\mathcal{A}_i}|$, we find that

$$\begin{aligned}\left|\prod_i z_i-\mathrm{e}^{\sum_i\sigma_{\mathcal{A}_i}/2}\right| &= \left|\prod_i \mathrm{e}^{\log z_i}-\mathrm{e}^{-\sum_i|\sigma_{\mathcal{A}_i}|/2}\right|\\ &= \left|\mathrm{e}^{-\sum_i|\log|z_i||+\mathrm{i}\sum_i\arg z_i}-\mathrm{e}^{-\sum_i|\sigma_{\mathcal{A}_i}|/2}\right|\\ &\leqslant \left|\sum_i\log z_i-\sum_i\frac{\sigma_{\mathcal{A}_i}}{2}\right|\mathrm{e}^{\max\{-\sum_i|\log|z_i||,-\sum_i|\sigma_{\mathcal{A}_i}|/2\}}\\ &\leqslant \left|\sum_i\log z_i-\sum_i\frac{\sigma_{\mathcal{A}_i}}{2}\right|,\end{aligned}\tag{43}$$

which yields the following theorem.

Theorem 1 (Lindeberg quantum central limit theorem). *Let $\mathcal{A}_i\subset\mathcal{L}$, $i=1,\ldots,n$, be mutually disjoint sets, $|\sigma_{\mathcal{A}_i}|\leqslant 1$, and let $f_{\mathcal{A}_i}<\infty$. Then*

$$\left|\prod_i\langle\hat{D}(\boldsymbol{\alpha}_{\mathcal{A}_i})\rangle-\mathrm{e}^{\sum_i\sigma_{\mathcal{A}_i}/2}\right| \leqslant \frac{7}{24}\sum_{i=1}^n|f_{\mathcal{A}_i}|.\tag{44}$$

A dynamical quantum central limit for initially uncorrelated states $\hat{\varrho}_0 = \otimes_{i=1}^{L} \hat{\varrho}_i$ follows directly from this theorem. The subsequent technicalities are due to the fact that we allow for correlations in the initial state.

5. Main theorem

We now turn to stating the main theorem. We arrive at a statement showing local convergence to a Gaussian state in trace-norm, under the weak assumptions on the initial state specified above. We first introduce a variant of a Bernstein–Spohn blocking argument, and then make use of the above quantum Lindeberg central limit theorem, to show convergence to a Gaussian characteristic function in phase space. We then show that this implies closeness of $\hat{\varrho}_{\mathcal{S}}(t)$ to a Gaussian state—so a maximum entropy state for given second moments.

5.1. The Bernstein–Spohn blocking argument

We divide—for a given time—the lattice into two parts: sites inside the causal cone, where mixing occurs, and sites outside the causal cone, the influence of which is exponentially suppressed. Inside the cone, in turn, we split the region into several subsets, the role of which will soon become obvious. This choice of a blocking is derived from the 'room and corridor' argument on classical lattice systems presented in [2, 3]. To this end, let

$$a \geqslant b \geqslant 1 \text{ such that } n := \left\lfloor \frac{8\mathrm{e}|t|}{a+b} \right\rfloor > 1. \tag{45}$$

Then

$$8\mathrm{e}|t| \geqslant n(a+b) \geqslant 4\mathrm{e}|t|. \tag{46}$$

Now define the set

$$\mathcal{T} = \left\{ j \in L\colon\ d_{j,\mathcal{S}} \geqslant n(a+b) \right\} \tag{47}$$

as the sites that are significantly outside the causal cone, compare lemma 3. Define also the sets $\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$ and $\mathcal{B} = \bigcup_{i=1}^{n} \mathcal{B}_i$, where

$$\begin{aligned} \mathcal{A}_i &= \left\{ j \in \mathcal{L}\colon\ (i-1)(a+b) \leqslant d_{j,\mathcal{S}} < i(a+b) - b \right\}, \\ \mathcal{B}_i &= \left\{ j \in L\colon\ i(a+b) - b \leqslant d_{j,\mathcal{S}} < i(a+b) \right\}. \end{aligned} \tag{48}$$

Then $\mathcal{L} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{T}$—the entire lattice is covered by these sets—and $\mathcal{S} \subset \mathcal{A}_1$. What is more, $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cap \mathcal{T} = \mathcal{T} \cap \mathcal{B} = \mathcal{A}_i \cap \mathcal{A}_j = \mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for $i \neq j$. The sets $\mathcal{A}$ and $\mathcal{B}$ cover hence the complement of $\mathcal{T}$ in an alternating fashion, the first set $\mathcal{A}_1$ containing the subset $\mathcal{S}$ of interest for which we aim at showing relaxation. The goal is now to choose b sufficiently small (such that the influence of sites in $\mathcal{B}$ is small due to the large separation of the $\mathcal{B}_i$), but large enough such that the $\mathcal{A}_i$ are sufficiently far apart (and therefore approximately independent) and the Lindeberg central limit theorem yields closeness of $\chi_{\hat{\varrho}_{\mathcal{S}}(t)}$ to a Gaussian function in phase space. We will see that for sufficiently large $|t|$, this is achieved by the choice

$$a = \frac{|t|^{2/3}}{\log|t|}, \quad b = |t|^{(2-\mu_1/(1+\mu_1))/6}, \tag{49}$$

for which one can prove the following lemma.

Lemma 4 (Correlations). *Let assumptions 1 and 2 hold and let t and L be such that*

$$L^{6/7} \geqslant |t| \geqslant 2, \quad \log|t| \leqslant |t|^{1/3+\mu}, \tag{50}$$

where $\mu = \mu_1/(6(\mu_1+1))$. Then,

$$\frac{|t|^{2/3}}{\log|t|} = a \geqslant b = |t|^{1/3-\mu} \geqslant 1, \quad \left\lfloor \frac{8\mathrm{e}|t|}{a+b} \right\rfloor = n > 1, \tag{51}$$

and for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{T}$ as above, one has

$$\begin{aligned}
&|\sigma_{\mathcal{T}}|, |\sigma_{\mathcal{B},\mathcal{T}}|, |\sigma_{\mathcal{A},\mathcal{T}}| \leqslant c_2 \|\boldsymbol{\beta}\|_1^2 2^{-4\mathrm{e}|t|},\\
&|\sigma_{\mathcal{B}}|, |\sigma_{\mathcal{A},\mathcal{B}}| \leqslant 16\mathrm{e}c_2|\partial\mathcal{S}|\|\boldsymbol{\beta}\|_1^2 \frac{\log|t|}{|t|^{\mu}},\\
&\left|\sigma_{\mathcal{L}} - \sum_{i=1}^{n} \sigma_{\mathcal{A}_i}\right| \leqslant c_2\|\boldsymbol{\beta}\|_1^2 \left(\frac{2|\mathcal{S}|}{|t|^{2/3}} + 80\mathrm{e}|\partial\mathcal{S}|\frac{\log|t|}{|t|^{\mu}}\right),\\
&\sum_{i=1}^{n} |f_{\mathcal{A}_i}| \leqslant c_4 \|\boldsymbol{\beta}\|_2^2 \|\boldsymbol{\beta}\|_1^2 \left(\frac{|\mathcal{S}|}{|t|^{2/3}} + \frac{2|\partial\mathcal{S}|}{\log|t|}\right).
\end{aligned} \tag{52}$$

The proof is again presented in the appendix. Here, the constant μ_1 is defined as specified in the assumptions on the initial state, c_2 and c_4 as in the lemma following the assumptions, and

$$\partial\mathcal{S} = \left\{ i \in \mathcal{S} \colon d_{i,\mathcal{L}\setminus\mathcal{S}} = 1 \right\} \tag{53}$$

is the boundary of $\mathcal{S}$.

5.2. The main theorem

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{T}$ be as above and write $\hat{D}_{\mathcal{A}} = \hat{D}(\boldsymbol{\alpha}_{\mathcal{A}})$ and similarly for the other sets. We have (we employ the triangle inequality three times)

$$\begin{aligned}
\left|\chi_{\hat{\varrho}_{\mathcal{S}}(t)}(\boldsymbol{\beta}) - \mathrm{e}^{\sigma_{\mathcal{L}}/2}\right| \leqslant\ & |\langle\hat{D}_{\mathcal{L}}\rangle - \langle\hat{D}_{\mathcal{A}}\rangle| + \left|\langle\hat{D}_{\mathcal{A}}\rangle - \prod_{i=1}^{n}\langle\hat{D}_{\mathcal{A}_i}\rangle\right| + |\mathrm{e}^{\sum_{i=1}^{n}\sigma_{\mathcal{A}_i}/2} - \mathrm{e}^{\sigma_{\mathcal{L}}/2}|\\
&+ \left|\prod_{i=1}^{n}\langle\hat{D}_{\mathcal{A}_i}\rangle - \prod_{i=1}^{n}\mathrm{e}^{\sigma_{\mathcal{A}_i}/2}\right|.
\end{aligned} \tag{54}$$

The first term is bounded by (we write again $\hat{\varrho}_0 = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$ for the spectral decomposition of $\hat{\varrho}_0$ and employ the Cauchy–Schwarz inequality)

$$\begin{aligned}
\left|\langle\hat{D}_{\mathcal{A}}(\hat{D}_{\mathcal{B}\cup\mathcal{T}} - \mathbb{1})\rangle\right| &\leqslant \sum_n \varrho_n |\langle\psi_n|\hat{D}_{\mathcal{A}}(\hat{D}_{\mathcal{B}\cup\mathcal{T}} - \mathbb{1})|\psi_n\rangle|\\
&\leqslant \sum_n \sqrt{\varrho_n}(\varrho_n\langle\psi_n|(\hat{D}_{\mathcal{B}\cup\mathcal{T}} - \mathbb{1})^\dagger(\hat{D}_{\mathcal{B}\cup\mathcal{T}} - \mathbb{1})|\psi_n\rangle)^{1/2}\\
&\leqslant (\langle(\hat{D}_{\mathcal{B}\cup\mathcal{T}} - \mathbb{1})^\dagger(\hat{D}_{\mathcal{B}\cup\mathcal{T}} - \mathbb{1})\rangle)^{1/2}\\
&= (2\Re[\langle(\mathbb{1} - \hat{D}_{\mathcal{B}\cup\mathcal{T}})\rangle])^{1/2} \leqslant |\sigma_{\mathcal{B}\cup\mathcal{T}}|^{1/2}\\
&\leqslant (|\sigma_{\mathcal{B}}| + |\sigma_{\mathcal{T}}| + 2|\sigma_{\mathcal{B},\mathcal{T}}|)^{1/2}.
\end{aligned} \tag{55}$$

To bound the second term in equation (54), we let $I = \{m \in \mathbb{N} \colon m \text{ even}, 1 \leqslant m \leqslant n\}$, $J = \{1, \dots, n\} \backslash I$. Then, using the fact that $d_{\mathcal{A}_i, \mathcal{A}_j} \geqslant b$ for $i \neq j$, we find under assumption 3

$$\begin{aligned}
\left| \left\langle \prod_{i=1}^{n} \hat{D}_{\mathcal{A}_i} \right\rangle - \prod_{i=1}^{n} \left\langle \hat{D}_{\mathcal{A}_i} \right\rangle \right| &\leqslant \left| \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \prod_{j\in J} \hat{D}_{\mathcal{A}_j} \right\rangle - \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle \left\langle \prod_{j\in J} \hat{D}_{\mathcal{A}_j} \right\rangle \right| \\
&\quad + \left| \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle \left\langle \prod_{j\in J} \hat{D}_{\mathcal{A}_j} \right\rangle - \prod_{i=1}^{n} \langle \hat{D}_{\mathcal{A}_i} \rangle \right| \\
&\leqslant \frac{c_5}{b^{1/2+\mu_2}} + \left| \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle \left\langle \prod_{j\in J} \hat{D}_{\mathcal{A}_j} \right\rangle - \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle \prod_{j\in J} \langle \hat{D}_{\mathcal{A}_i} \rangle \right| \\
&\quad + \left| \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle \prod_{j\in J} \langle \hat{D}_{\mathcal{A}_i} \rangle - \prod_{i=1}^{n} \langle \hat{D}_{\mathcal{A}_i} \rangle \right| \\
&\leqslant \frac{c_5}{b^{1/2+\mu_2}} + \left| \left\langle \prod_{j\in J} \hat{D}_{\mathcal{A}_j} \right\rangle - \prod_{j\in J} \langle \hat{D}_{\mathcal{A}_j} \rangle \right| + \left| \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle - \prod_{i\in I} \langle \hat{D}_{\mathcal{A}_i} \rangle \right|
\end{aligned} \tag{56}$$

and, using the fact that $d_{\mathcal{A}_i, \mathcal{A}_j} \geqslant a$ for $i, j \in I$, $i \neq j$,

$$\begin{aligned}
\left| \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle - \prod_{i\in I} \langle \hat{D}_{\mathcal{A}_i} \rangle \right| &\leqslant \left| \left\langle \prod_{i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle - \langle \hat{D}_{\mathcal{A}_2} \rangle \left\langle \prod_{2\neq i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle \right| + \left| \left\langle \prod_{2\neq i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle - \prod_{2\neq i\in I} \langle \hat{D}_{\mathcal{A}_i} \rangle \right| \\
&\leqslant \frac{c_5}{a^{1/2+\mu_2}} + \left| \left\langle \prod_{2\neq i\in I} \hat{D}_{\mathcal{A}_i} \right\rangle - \prod_{2\neq i\in I} \left\langle \hat{D}_{\mathcal{A}_i} \right\rangle \right| \\
&\leqslant \cdots \leqslant |I| \frac{c_5}{a^{1/2+\mu_2}}.
\end{aligned} \tag{57}$$

This means that

$$\begin{aligned}
\left| \left\langle \prod_{i=1}^{n} \hat{D}_{\mathcal{A}_i} \right\rangle - \prod_{i=1}^{n} \langle \hat{D}_{\mathcal{A}_i} \rangle \right| &\leqslant \frac{c_5}{b^{1/2+\mu_2}} + \frac{c_5 n}{a^{1/2+\mu_2}} \\
&\leqslant \frac{c_5}{|t|^{(1/12+\mu_2/6)}} + 8c_5 \mathrm{e} \frac{(\log|t|)^{3/2+\mu_2}}{|t|^{2\mu_2/3}},
\end{aligned} \tag{58}$$

which bounds the second term in equation (54).

Now, for $x, y \in \mathbb{R}$, there is a x_0 between x and y such that

$$\mathrm{e}^x - \mathrm{e}^y = (x-y)\mathrm{e}^{x_0}, \text{ i.e., } |\mathrm{e}^x - \mathrm{e}^y| = |x-y|\mathrm{e}^{x_0} \leqslant |x-y|\mathrm{e}^{\max\{x,y\}}. \tag{59}$$

Hence, for the third term in equation (54), we find (we recall that $\sigma_{\mathcal{L}}, \sigma_{\mathcal{A}_i} \leqslant 0$)

$$|\mathrm{e}^{\sum_{i=1}^{n}\sigma_{\mathcal{A}_i}/2} - \mathrm{e}^{\sigma_{\mathcal{L}}/2}| \leqslant \left|\sigma_{\mathcal{L}} - \sum_{i=1}^{n}\sigma_{\mathcal{A}_i}\right| \Big/ 2. \tag{60}$$

If $|\sigma_{\mathcal{A}_i}| \leqslant 1$, the last term in equation (54) may be bounded using the Lindeberg-type central limit theorem and we finally have

$$\begin{aligned}\left|\chi_{\hat{\varrho}_{\mathcal{S}(t)}}(\boldsymbol{\beta}) - \mathrm{e}^{\sigma_{\mathcal{L}}/2}\right| &\leqslant \left(|\sigma_{\mathcal{B}}| + |\sigma_{\mathcal{T}}| + 2|\sigma_{\mathcal{B},\mathcal{T}}|\right)^{1/2} + \left|\sigma_{\mathcal{L}} - \sum_{i=1}^{n}\sigma_{\mathcal{A}_i}\right| \Big/ 2 + \frac{7}{24}\sum_{i=1}^{n}|f_{\mathcal{A}_i}| \\ &\quad + \frac{c_5}{|t|^{(1/12+\mu_2/6)}} + 8c_5\mathrm{e}\frac{(\log|t|)^{3/2+\mu_2}}{|t|^{2\mu_2/3}},\end{aligned} \tag{61}$$

where bounds on the moments may be read off from lemma 4. This yields the main theorem.

Theorem 2 (Convergence of characteristic functions). *For all $\mathcal{L} = \{1, \ldots, L\} \subset \mathbb{N}$, all $\mathcal{S} \subset \mathcal{L}$, and every $\hat{\varrho}_0$ fulfilling assumptions 1–3 there exists a $F_{\mathcal{S}}(\boldsymbol{\beta}, t, L) > 0$ such that*

$$\left|\chi_{\hat{\varrho}_{\mathcal{S}(t)}}(\boldsymbol{\beta}) - \mathrm{e}^{\sigma_{\mathcal{L}}/2}\right| \leqslant F_{\mathcal{S}}(\boldsymbol{\beta}, t, L). \tag{62}$$

The function $F_{\mathcal{S}}\colon \mathbb{C}^{|\mathcal{S}|} \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ has the following properties. There is a recurrence time $t_{\mathrm{rec}} > 0$, which increases polynomially with L, and a relaxation time t_{relax} such that $F_{\mathcal{S}}$ is decreasing in time for all $|t| \in [t_{\mathrm{relax}}, t_{\mathrm{rec}}]$, and all $\mathcal{S} \subset \{1, \ldots, L\}$ with

$$\frac{|\partial\mathcal{S}|}{\log|t|}, \frac{|\mathcal{S}|}{|t|^{2/3}} \underset{|t|\to\infty}{\longrightarrow} 0. \tag{63}$$

If the latter holds and $|\mathcal{S}|$ does not depend on L, one has in particular

$$\lim_{|t|\to\infty}\lim_{L\to\infty} F_{\mathcal{S}}(\boldsymbol{\beta}, t, L) = 0. \tag{64}$$

This theorem indeed shows convergence of characteristic functions of reduced states associated with some sublattice $\mathcal{S}$ to Gaussian characteristic functions in phase space. Note that the size of this sublattice may even slowly grow in time; one would still end up with a Gaussian characteristic function. Surely, in any finite system, there will be recurrences, but with increasing L, these recurrence times t_{rec} can be made arbitrarily large.

The characteristic function

$$\chi_{\hat{\varrho}_{G(t)}}(\boldsymbol{\beta}) = \mathrm{e}^{\sigma_{\mathcal{L}}/2} = \mathrm{e}^{-\boldsymbol{\beta}^{\dagger}\gamma(t)\boldsymbol{\beta}/2} \tag{65}$$

corresponds to the Gaussian state $\hat{\varrho}_G(t)$ with second moments

$$[\gamma(t)]_{l,k} = \langle\hat{b}_k^{\dagger}(t)\hat{b}_l(t)\rangle + \langle\hat{b}_l(t)\hat{b}_k^{\dagger}(t)\rangle = [\mathrm{e}^{-\mathrm{i}tA}\gamma(0)\mathrm{e}^{\mathrm{i}tA}]_{l,k} \tag{66}$$

and one has[8]

$$\begin{aligned}\mathrm{tr}[\hat{\varrho}_G(t)\hat{b}_i^{\dagger}\hat{b}_i] &= \mathrm{tr}[\hat{\varrho}_{\mathcal{S}}(t)\hat{b}_i^{\dagger}\hat{b}_i] \\ &= \frac{[\gamma(t)]_{i,i} - 1}{2} \leqslant \|\mathrm{e}^{-\mathrm{i}tA}\gamma(0)\mathrm{e}^{\mathrm{i}tA}\| = \|\gamma(0)\|.\end{aligned} \tag{67}$$

[8] Here, $\|\cdot\|$ denotes the spectral norm.

Note that if the initial correlations are translationally invariant, $\gamma(0)$ and A commute, i.e. the second moments are conserved, $\gamma(t) = \gamma(0)$. By virtue of lemma 5, which we prove in the next section, trace-norm convergence of states is also inherited from this main theorem. We state this corollary for a sublattice $\mathcal{S}$ that is constant in time.

Corollary 1 (Convergence of reduced states to Gaussian states). *Let $|\mathcal{S}|$ and $\|\gamma(0)\|$ be bounded by an absolute constant. Under the assumptions of theorem 2 and for any $\varepsilon > 0$ there exists a recurrence time $t_{\rm rec} > 0$, which increases polynomially with L, and a relaxation time $t_{\rm relax}$ such that*

$$\|\hat{\varrho}_{\mathcal{S}}(t) - \hat{\varrho}_G(t)\|_{\rm tr} \leqslant \varepsilon, \tag{68}$$

for all $|t| \in [t_{\rm relax}, t_{\rm rec}]$, where $\hat{\varrho}_G(t)$ is the Gaussian state with characteristic function $\mathrm{e}^{\sigma_{\mathcal{L}}/2}$.

This means that in the sense of trace-norm closeness, local reduced states become Gaussian—and hence maximum entropy—states.

6. Convergence of quantum states

In this section, we will show that pointwise convergence of characteristic functions is inherited by trace-norm convergence on the level of quantum states. This argument generalizes the one in [27]—showing weak convergence for states from pointwise convergence of characteristic functions—and the one in [28]—relating trace-norm and weak convergence. The central idea is to consider two sets: on the one hand, this is the subset of phase space that supports, in a sense yet to be made precise, 'most' of the characteristic function. On the other hand, this is the subspace of Hilbert space with the property that the projection of the state onto it is a positive operator that eventually has almost unit trace. One can then bound the error made when neglecting the complement of these sets and, in turn, relate the relevant sets in phase and state space. Inside these relevant sets, one can relate weak convergence to pointwise convergence of characteristic functions and use the equivalence of norms. Here, we present an argument going beyond the results in [27, 28] and use the gentle measurement lemma of Winter [29].

6.1. Preliminaries

To prepare the argument, consider, for a single mode, the upper bound to matrix elements of Weyl operators

$$\begin{aligned} |\langle n|\hat{D}(\alpha)|m\rangle| &\leqslant \mathrm{e}^{-|\alpha|^2/2}|\alpha|^{|n-m|}\left(\frac{\min\{m,n\}!}{\max\{m,n\}!}\right)^{1/2}|L^{|n-m|}_{\min\{m,n\}}(|\alpha|^2)| \\ &=: \mathrm{e}^{-|\alpha|^2/2}p_{n,m}(|\alpha|^2), \end{aligned} \tag{69}$$

where $\alpha \in \mathbb{C}$ and L_n^k are the associated Laguerre polynomials, i.e. $p_{n,m}(|\alpha|^2)$ is a polynomial of degree $\min\{m,n\} + |n-m|/2 = (n+m)/2$ in $|\alpha|^2$. Here $\{|n\rangle\}_{n=0,1,\ldots}$ is the standard number basis of the Hilbert space associated with a single mode. We can make use of this in the situation of having N modes: define for some $T > 0$

$$\bar{\mathcal{M}} := \left\{\boldsymbol{\alpha} \in \mathbb{C}^N : \|\boldsymbol{\alpha}\|_2^2 > T\right\} \tag{70}$$

as the region far away in 2-norm from the origin of phase space and $\mathcal{M} := \mathbb{C}^N \backslash \bar{\mathcal{M}}$ as its complement. For a continuous function $f : \mathbb{C}^N \to \mathbb{C}$ with $|f(\boldsymbol{\alpha})| \leqslant 2$, consider the bound (we write $p_{\boldsymbol{n},\boldsymbol{m}}(\boldsymbol{\alpha}) = \prod_i p_{n_i,m_i}(|\alpha_i|^2)$ and $|\boldsymbol{n}\rangle = |n_1 \cdots n_N\rangle$)

$$\begin{aligned}
\left| \int_{\mathbb{C}^N} \mathrm{d}\boldsymbol{\alpha}\, f(\boldsymbol{\alpha}) \langle \boldsymbol{n}| \hat{D}(\boldsymbol{\alpha}) |\boldsymbol{m}\rangle \right| &\leqslant \int_{\bar{\mathcal{M}}} \mathrm{d}\boldsymbol{\alpha}\, |f(\boldsymbol{\alpha})| \left| \langle \boldsymbol{n}| \hat{D}(\boldsymbol{\alpha}) |\boldsymbol{m}\rangle \right| \\
&\quad + \int_{\mathcal{M}} \mathrm{d}\boldsymbol{\alpha}\, |f(\boldsymbol{\alpha})| \left| \langle \boldsymbol{n}| \hat{D}(\boldsymbol{\alpha}) |\boldsymbol{m}\rangle \right| \\
&\leqslant 2 \int_{\bar{\mathcal{M}}} \mathrm{d}\boldsymbol{\alpha} \left| \langle \boldsymbol{n}| \hat{D}(\boldsymbol{\alpha}) |\boldsymbol{m}\rangle \right| + |\mathcal{M}| \max_{\boldsymbol{\alpha}\in\mathcal{M}} |f(\boldsymbol{\alpha})| \\
&\leqslant 2 \int_{\bar{\mathcal{M}}} \mathrm{d}\boldsymbol{\alpha}\, \mathrm{e}^{-\|\boldsymbol{\alpha}\|_2^2/2} p_{\boldsymbol{n},\boldsymbol{m}}(\boldsymbol{\alpha}) + |\mathcal{M}| \max_{\boldsymbol{\alpha}\in\mathcal{M}} |f(\boldsymbol{\alpha})| \\
&\leqslant 2\mathrm{e}^{-T/4} \int_{\mathbb{C}^N} \mathrm{d}\boldsymbol{\alpha}\, \mathrm{e}^{-\|\boldsymbol{\alpha}\|_2^2/4} p_{\boldsymbol{n},\boldsymbol{m}}(\boldsymbol{\alpha}) + |\mathcal{M}| \max_{\boldsymbol{\alpha}\in\mathcal{M}} |f(\boldsymbol{\alpha})| \\
&=: c_{\boldsymbol{n},\boldsymbol{m}} \mathrm{e}^{-T/4} + |\mathcal{M}| \max_{\boldsymbol{\alpha}\in\mathcal{M}} |f(\boldsymbol{\alpha})|.
\end{aligned} \tag{71}$$

These findings may be summarized as follows. For all $N, n_i, m_i \in \mathbb{N}$ and every $\epsilon > 0$ there is a $c_0 > 0$ and a compact set $\mathcal{C} \subset \mathbb{C}^N$ such that

$$\left| \int_{\mathbb{C}^N} \mathrm{d}\boldsymbol{\alpha}\, f(\boldsymbol{\alpha}) \langle \boldsymbol{n}| \hat{D}(\boldsymbol{\alpha}) |\boldsymbol{m}\rangle \right| \leqslant \epsilon + c_0 \max_{\boldsymbol{\alpha}\in\mathcal{C}} |f(\boldsymbol{\alpha})| \tag{72}$$

for all continuous $f : \mathbb{C}^N \to \mathbb{C}$ with $|f(\boldsymbol{\alpha})| \leqslant 2$.

Now, for $M \subset \mathbb{N}^N$, let

$$\hat{P}_M = \sum_{\boldsymbol{n}\in M} |\boldsymbol{n}\rangle\langle \boldsymbol{n}|. \tag{73}$$

This will take the role of a projection onto the relevant part of state space. For states $\hat{\varrho}_1$ and $\hat{\varrho}_2$, one finds (see the appendix for a proof)

$$\|\hat{\varrho}_1 - \hat{\varrho}_2\|_{\mathrm{tr}} \leqslant \|\hat{\varrho}_1 - \hat{P}_M \hat{\varrho}_1 \hat{P}_M + \hat{P}_M \hat{\varrho}_2 \hat{P}_M - \hat{\varrho}_2\|_{\mathrm{tr}} + |M|^{3/2} \max_{\boldsymbol{n},\boldsymbol{m}\in M} \left| \langle \boldsymbol{n}| (\hat{\varrho}_1 - \hat{\varrho}_2) |\boldsymbol{m}\rangle \right|. \tag{74}$$

What is more, using the Heisenberg–Weyl correspondence, one has

$$\langle \boldsymbol{n}| (\hat{\varrho}_1 - \hat{\varrho}_2) |\boldsymbol{m}\rangle \propto \int_{\mathbb{C}^N} \mathrm{d}\boldsymbol{\alpha}\, [\chi_{\hat{\varrho}_1}(\boldsymbol{\alpha}) - \chi_{\hat{\varrho}_2}(\boldsymbol{\alpha})] \langle \boldsymbol{n}| \hat{D}(\boldsymbol{\alpha}) |\boldsymbol{m}\rangle. \tag{75}$$

That is, combining equations (72) and (74), one has that for every $N \in \mathbb{N}$, $\epsilon > 0$ and $M \subset \mathbb{N}^N$ there is a constant $c_0 > 0$ and a compact $\mathcal{C} \subset \mathbb{C}^N$ such that

$$\|\hat{\varrho}_1 - \hat{\varrho}_2\|_{\mathrm{tr}} \leqslant \epsilon + c_0 \max_{\boldsymbol{\alpha}\in\mathcal{C}} \left| \chi_{\hat{\varrho}_1}(\boldsymbol{\alpha}) - \chi_{\hat{\varrho}_2}(\boldsymbol{\alpha}) \right| + \|\hat{\varrho}_1 - \hat{P}_M \hat{\varrho}_1 \hat{P}_M + \hat{P}_M \hat{\varrho}_2 \hat{P}_M - \hat{\varrho}_2\|_{\mathrm{tr}} \tag{76}$$

for all N-mode states $\hat{\varrho}_1$ and $\hat{\varrho}_2$.

Now, from the gentle measurement lemma of Winter [29], it follows that

$$\|\hat{\varrho} - \hat{P}_M \hat{\varrho} \hat{P}_M\|_{\mathrm{tr}} \leqslant 2(\mathrm{tr}[\hat{\varrho}(\mathbb{1} - \hat{P}_M)])^{1/2}, \tag{77}$$

and for $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$, one finds that (we write $\bar{M} = \mathbb{N}^N \backslash M$)

$$N \max_i \mathrm{tr}[\hat{\varrho}\hat{n}_i] \geqslant \sum_{i=1}^N \mathrm{tr}[\hat{\varrho}\hat{n}_i] = \sum_{i=1}^N \sum_{\boldsymbol{n}\in M} n_i \langle \boldsymbol{n}|\hat{\varrho}|\boldsymbol{n}\rangle + \sum_{i=1}^N \sum_{\boldsymbol{n}\in \bar{M}} n_i \langle \boldsymbol{n}|\hat{\varrho}|\boldsymbol{n}\rangle$$

$$\geqslant \mathrm{tr}[\hat{\varrho}(\mathbb{1} - \hat{P}_M)] \min_{\boldsymbol{n}\in\bar{M}} \sum_{i=1}^N n_i > \mathrm{tr}[\hat{\varrho}(\mathbb{1} - \hat{P}_M)]m, \tag{78}$$

where the last inequality holds for the particular choice of

$$M = \left\{ \boldsymbol{n} \in \mathbb{N}^N \colon \sum_i n_i \leqslant m \right\}. \tag{79}$$

To summarize, for every $m, N \in \mathbb{N}$ and $\epsilon > 0$, there is a constant $c_0 > 0$ and a compact $\mathcal{C} \subset \mathbb{C}^N$ such that

$$\|\hat{\varrho}_1 - \hat{\varrho}_2\|_{\mathrm{tr}} \leqslant \epsilon + c_0 \max_{\boldsymbol{\alpha}\in\mathcal{C}} \left|\chi_{\hat{\varrho}_1}(\boldsymbol{\alpha}) - \chi_{\hat{\varrho}_2}(\boldsymbol{\alpha})\right| + 2\left(\frac{N}{m}\right)^{1/2} (\max_i \sqrt{\mathrm{tr}[\hat{\varrho}_1\hat{n}_i]} + \max_i \sqrt{\mathrm{tr}[\hat{\varrho}_2\hat{n}_i]}) \tag{80}$$

for all N-mode states $\hat{\varrho}_1$ and $\hat{\varrho}_2$.

6.2. *Trace-norm bounds*

We summarize the above findings in the following lemma. The lemma shows that closeness in trace-norm is inherited by pointwise closeness of characteristic functions. The last line corresponds essentially to a bound to the local energy per mode, which can, in all relevant settings, be bounded from above. Note that all the constants on the right-hand side of the inequality can be made explicit.

Lemma 5 (Trace-norm convergence from pointwise phase space convergence). *For every $\epsilon, c_1 > 0$ and $N \in \mathbb{N}$, there is a constant $c_0 > 0$ and a compact $\mathcal{C} \subset \mathbb{C}^N$ such that*

$$\|\hat{\varrho}_1 - \hat{\varrho}_2\|_{\mathrm{tr}} \leqslant \epsilon + c_0 \max_{\boldsymbol{\alpha}\in\mathcal{C}} \left|\chi_{\hat{\varrho}_1}(\boldsymbol{\alpha}) - \chi_{\hat{\varrho}_2}(\boldsymbol{\alpha})\right| \tag{81}$$

for all states $\hat{\varrho}_1$ and $\hat{\varrho}_2$ on a collection of N sites with, for all i,

$$\mathrm{tr}[\hat{\varrho}_1\hat{n}_i],\ \mathrm{tr}[\hat{\varrho}_2\hat{n}_i] < c_1. \tag{82}$$

Using theorem 2, the pointwise distance between characteristic functions and ϵ may, hence, be subsumed in ε to yield the trace norm estimate in corollary 1.

7. Summary, discussion and the link of the results to notions of typicality

The above theorem shows that weakly clustering states locally relax to maximum entropy states. This holds true in the strong sense of trace-norm convergence, to states that have maximum entropy for given second moments. Physically speaking, this means that, locally, it appears as if the system had entirely relaxed, without the need for an external environment. Every part of the system forms the environment of the other, and, when merely mediated by local interactions, the system relaxes. This is a proven instance of a situation in which, under non-equilibrium quenched dynamics, closed systems do locally relax. As mentioned earlier, it can be viewed as a proof of the 'local relaxation conjecture', generalizing the findings in [7] and constituting, to the knowledge of the authors, a first proof of local relaxation in a situation not requiring product or Gaussian, meaning quasi-free, initial states. That is to say, the statement shows that for a large class of initial states, locally, the system quickly becomes relaxed and stays like that for long periods.

We finally mention a potential link of the above results to the concept of 'typicality'. The term typicality refers—roughly speaking—to the idea that when quantum states are randomly chosen from a suitable ensemble of large-dimensional pure states, almost all realizations will share properties like reduced states of subsystems having large entropy, or being close to a Gibbs state. This effect is a manifestation of the phenomenon of concentration of measure. The simplest such situation is realized when drawing a random pure state according to the Haar measure on $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^D$. Then the von-Neumann entropy of the d-dimensional subsystem will, in the case of large D, be, quite certainly, nearly maximum, with any deviations being exponentially suppressed in probability. Similarly, if states are drawn from an ensemble of pure states that are eigenvectors of a Hamiltonian with eigenvalues close to a limiting point in the spectrum, defining an energy, then one can specify the technical conditions under which one can prove that, quite certainly, subsystems will locally look like the reduction of a Gibbs state to that subsystem [17, 18]. That is to say, almost all states appear to have high local entropy. This is strictly proven only in the case of finite-dimensional constituents, although it is to be expected that the same intuition carries over also to the situation of infinite-dimensional constituents. In the light of this observation, it appears plausible that starting from an 'untypical initial situation', systems will eventually be driven to a 'typical one' by virtue of unitary time evolution.

Needless to say, time evolution defines merely a one-dimensional manifold, parameterized by time, and from the above argument alone there is no guarantee that the system will indeed arrive at a typical situation (although steps can be taken that show that for most times, albeit unknown ones, the system appears locally typical [19] in a non-equilibrium situation). Here—for a class of models—a significantly stronger statement is proven, in that after a short time, the system will certainly come arbitrarily close to a maximum entropy state under the constraints of second moments and will stay there. It would be interesting to take steps to combine such kinematical and dynamical approaches towards understanding equilibration in closed quantum systems.

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Appendix. Proofs

A.1. The Lieb–Robinson bound (lemma 3)

Under periodic boundary conditions, the entries of C take the form $C_{i,j} = C_{i-j} = C_{d_{i,j}}$,

$$C_l = \frac{1}{L}\sum_{k=1}^{L} \mathrm{e}^{2\pi \mathrm{i} k l/L}\mathrm{e}^{-\mathrm{i}t\lambda_k}, \qquad \lambda_k = -2\cos(2\pi k/L). \tag{A.1}$$

As $A_{i,j} = 0$ for $d_{i,j} > 1$, we find from Taylor's theorem the estimate

$$\begin{aligned}|C_{d_{i,j}}| &= \left|\frac{1}{L}\sum_{k=1}^{L}\mathrm{e}^{2\pi\mathrm{i}kd_{i,j}/L}\sum_{n=d_{i,j}}^{\infty}\frac{(-\mathrm{i}t\lambda_k)^n}{n!}\right| \\ &\leqslant \max_{k\in\mathcal{L}}\left|\sum_{n=d_{i,j}}^{\infty}\frac{(-\mathrm{i}t\lambda_k)^n}{n!}\right| \\ &\leqslant \max_{k\in\mathcal{L}}\frac{|t\lambda_k|^{d_{i,j}}}{d_{i,j}!} \leqslant \frac{|2t|^{d_{i,j}}}{d_{i,j}!} \leqslant \left(\frac{2\mathrm{e}|t|}{d_{i,j}}\right)^{d_{i,j}},\end{aligned} \tag{A.2}$$

i.e. for $\mathcal{A}\subset\mathcal{L}$ we find from equation (11) that

$$\begin{aligned}\sum_{i\in\mathcal{A}}|\alpha_i| &\leqslant \sum_{i\in\mathcal{A}}\sum_{j\in\mathcal{S}}|\beta_j||C_{j,i}| \leqslant \|\boldsymbol{\beta}\|_1\max_{j\in\mathcal{S}}\sum_{i\in\mathcal{A}}\left(\frac{2\mathrm{e}|t|}{d_{i,j}}\right)^{d_{i,j}} \\ &\leqslant \|\boldsymbol{\beta}\|_1\max_{j\in\mathcal{S}}\sum_{i\in\mathcal{A}}\left(\frac{2\mathrm{e}|t|}{d_{\mathcal{A},\mathcal{S}}}\right)^{d_{i,j}} \\ &= \|\boldsymbol{\beta}\|_1\max_{j\in\mathcal{S}}\sum_{d=d_{\mathcal{A},\mathcal{S}}}^{\infty}\left(\frac{2\mathrm{e}|t|}{d_{\mathcal{A},\mathcal{S}}}\right)^{d}\sum_{i\in\mathcal{A}}\delta_{d_{i,j},d} \\ &\leqslant 2\|\boldsymbol{\beta}\|_1\left(\frac{2\mathrm{e}|t|}{d_{\mathcal{A},\mathcal{S}}}\right)^{d_{\mathcal{A},\mathcal{S}}}\sum_{d=0}^{\infty}\left(\frac{2\mathrm{e}|t|}{d_{\mathcal{A},\mathcal{S}}}\right)^{d} \leqslant 4\|\boldsymbol{\beta}\|_1 2^{-d_{\mathcal{A},\mathcal{S}}},\end{aligned} \tag{A.3}$$

where the last inequality holds for $4\mathrm{e}|t| \leqslant d_{\mathcal{A},\mathcal{S}}$.

Furthermore,

$$\begin{aligned}C_l &= \frac{1}{L}\sum_{k=1}^{L}\mathrm{e}^{2\mathrm{i}t\cos(2\pi k/L)}\mathrm{e}^{2\pi\mathrm{i}kl/L} \\ &=: \frac{1}{L}\sum_{k=1}^{L} g(2\pi k/L)\mathrm{e}^{2\pi\mathrm{i}kl/L},\end{aligned} \tag{A.4}$$

where $g(\phi)=\mathrm{e}^{2\mathrm{i}t\cos(\phi)}$ can be written as

$$
\begin{aligned}
g(\phi) &= \sum_{n=-\infty}^{\infty} g_n \mathrm{e}^{\mathrm{i}n\phi}, \\
g_n &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, g(\phi)\mathrm{e}^{-\mathrm{i}n\phi} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{2\mathrm{i}t\cos(\phi)}\mathrm{e}^{\mathrm{i}n\phi} = \mathrm{i}^n J_n(2t) \\
&= \frac{1}{2\pi(\mathrm{i}n)^2}\int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}n\phi}\partial_\phi^2 \mathrm{e}^{2\mathrm{i}t\cos(\phi)},
\end{aligned}
\tag{A.5}
$$

where J_n is a Bessel function of the first kind. Hence, for $|t|\geqslant 1$,

$$
|g_n| \leqslant \frac{1}{2\pi n^2}\int_0^{2\pi} \mathrm{d}\phi\, |\partial_\phi^2 \mathrm{e}^{2\mathrm{i}t\cos(\phi)}| \leqslant \frac{2|t|+4|t|^2}{n^2} \leqslant 6\frac{|t|^2}{n^2} \tag{A.6}
$$

and

$$
\begin{aligned}
C_l &= \frac{1}{L}\sum_{n=-\infty}^{\infty} g_n \sum_{k=1}^{L} \mathrm{e}^{2\pi\mathrm{i}k(n+l)/L} \\
&= \sum_{n=-\infty}^{\infty} g_n \delta_{n+l\in L\mathbb{Z}} \\
&= \sum_{z=-\infty}^{\infty} g_{Lz-l} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, g(\phi)\mathrm{e}^{\mathrm{i}l\phi} + \sum_{z=1}^{\infty}\left(g_{Lz-l}+g_{Lz+l}\right),
\end{aligned}
\tag{A.7}
$$

where, for $|t|\geqslant 1$ and $0\leqslant l\leqslant L/2$, which we can assume to be $C_{i,j}=C_{d_{i,j}}$,

$$
\sum_{z=1}^{\infty} |g_{Lz-l}+g_{Lz+l}| \leqslant \frac{6|t|^2}{L^2}\sum_{z=1}^{\infty}\left(\frac{1}{(z-l/L)^2}+\frac{1}{(z+l/L)^2}\right) \leqslant \frac{6(\pi^2-4)|t|^2}{L^2}, \tag{A.8}
$$

i.e. for $L\geqslant |t|^{7/6}\geqslant 1$ (we use a bound obtained in [30]),

$$
\begin{aligned}
\left|C_{d_{i,j}}\right| &\leqslant \left|C_{d_{i,j}} - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\, g(\phi)\mathrm{e}^{\mathrm{i}d_{i,j}\phi}\right| + |J_{d_{i,j}}(2t)| \\
&\leqslant \frac{6(\pi^2-4)|t|^2}{L^2} + \frac{1}{|2t|^{1/3}} \leqslant \frac{37}{|t|^{1/3}}.
\end{aligned}
\tag{A.9}
$$

A.2. Moments

We will need the following bound. Let ϵ, $\mu > 0$ and $\mathcal{A} \subset \mathcal{L}$. Then

$$\begin{aligned}\sum_{i\in\mathcal{A}} \frac{1}{\left[1+d_{i,j}\right]^{2+\mu+\epsilon}} &= \sum_{r=d_{\mathcal{A},j}}^{\infty} \frac{1}{[1+r]^{1+\epsilon}\,[1+r]^{1+\mu}} \sum_{i\in\mathcal{A}} \delta_{d_{i,j},r} \\ &\leqslant \frac{2}{\left[1+d_{\mathcal{A},j}\right]^{1+\mu}} \sum_{r=0}^{\infty} \frac{1}{[1+r]^{1+\epsilon}} \\ &= \frac{2\zeta(1+\epsilon)}{\left[1+d_{\mathcal{A},j}\right]^{1+\mu}}. \end{aligned} \tag{A.10}$$

A.2.1. Second moments (lemma 1). In the following, we assume that t and L are such that $L \geqslant |t|^{7/6} \geqslant 1$. Under assumption 1, we find from equations (18) and (A.10) and lemma 3 that

$$\begin{aligned}|\sigma_{\mathcal{A},\mathcal{B}}| &\leqslant \sum_{\substack{i\in\mathcal{A}\\ j\in\mathcal{B}}} |\alpha_i\alpha_j|(|\langle \hat{b}_i^{\dagger}\hat{b}_j\rangle| + |\langle \hat{b}_j\hat{b}_i^{\dagger}\rangle|) \\ &\leqslant \frac{37^2}{|t|^{2/3}} \|\boldsymbol{\beta}\|_1^2 \sum_{\substack{i\in\mathcal{A}\\ j\in\mathcal{B}}} (2|\langle \hat{b}_i^{\dagger}\hat{b}_j\rangle| + \delta_{i,j}) \\ &\leqslant \frac{37^2(2c_1+1)}{|t|^{2/3}} \|\boldsymbol{\beta}\|_1^2 \sum_{\substack{i\in\mathcal{A}\\ j\in\mathcal{B}}} \frac{1}{[1+d_{i,j}]^{2+\mu_1+\epsilon_1}} \\ &\leqslant 37^2(4c_1+2)\zeta(1+\epsilon_1)\|\boldsymbol{\beta}\|_1^2 \frac{\min\{|\mathcal{A}|, |\mathcal{B}|\}}{|t|^{2/3}[1+d_{\mathcal{A},\mathcal{B}}]^{1+\mu_1}} \end{aligned} \tag{A.11}$$

for all $\mathcal{A}, \mathcal{B} \subset \mathcal{L}$. Similarly, for all $\mathcal{A}_i \subset \mathcal{L}$, $i = 1, \ldots, n$, with $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$, we find ($\mathcal{A} := \bigcup_i \mathcal{A}_i$)

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^{n} |\sigma_{\mathcal{A}_i,\mathcal{A}_j}| &\leqslant \frac{37^2(2c_1+1)}{|t|^{2/3}} \|\boldsymbol{\beta}\|_1^2 \sum_{i=1}^{n} \sum_{k\in\mathcal{A}_i} \sum_{l\in\mathcal{A}\setminus\mathcal{A}_i} \frac{1}{[1+d_{k,l}]^{2+\mu_1+\epsilon_1}} \\ &\leqslant c_2\|\boldsymbol{\beta}\|_1^2 \sum_{i=1}^{n} \frac{|\mathcal{A}_i|}{|t|^{2/3}\left[1+d_{\mathcal{A}_i,\mathcal{A}\setminus\mathcal{A}_i}\right]^{1+\mu_1}}. \end{aligned} \tag{A.12}$$

Now, for all $\mathcal{A}, \mathcal{B} \subset \mathcal{L}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$, one has $\sigma_{\mathcal{A}\cup\mathcal{B}} = \sigma_{\mathcal{A}} + \sigma_{\mathcal{B}} + 2\sigma_{\mathcal{A},\mathcal{B}}$, i.e.

$$\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}\setminus\mathcal{A}} + 2\sigma_{\mathcal{L}\setminus\mathcal{A},\mathcal{A}} + \sigma_{\mathcal{A}} = \sigma_{\mathcal{L}\setminus\mathcal{A}} + 2\sigma_{\mathcal{L}\setminus\mathcal{A},\mathcal{A}} + \sum_{i=1}^{n} \sigma_{\mathcal{A}_i} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \sigma_{\mathcal{A}_i,\mathcal{A}_j}, \tag{A.13}$$

i.e.

$$\left|\sigma_{\mathcal{L}} - \sum_{i=1}^{n} \sigma_{\mathcal{A}_i}\right| \leqslant |\sigma_{\mathcal{L}\setminus\mathcal{A}}| + 2|\sigma_{\mathcal{L}\setminus\mathcal{A},\mathcal{A}}| + c_2\|\boldsymbol{\beta}\|_1^2 \sum_{i=1}^{n} \frac{|\mathcal{A}_i|}{|t|^{2/3}\left[1+d_{\mathcal{A}\setminus\mathcal{A}_i,\mathcal{A}_i}\right]^{1+\mu_1}}. \tag{A.14}$$

Finally, under assumption 1 and for t, $\mathcal{A}$ such that $4e \leqslant 4e|t| \leqslant d_{\mathcal{A},\mathcal{S}}$, we find from equations (18) and (A.10) and lemma 3 that

$$\begin{aligned}
|\sigma_{\mathcal{A},\mathcal{B}}| &\leqslant \sum_{\substack{i\in\mathcal{A}\\ j\in\mathcal{B}}} \frac{(2c_1+1)|\alpha_i\alpha_j|}{\left[1+d_{i,j}\right]^{1+\mu_1+\epsilon_1}} \\
&\leqslant \sum_{i\in\mathcal{A}} |\alpha_i| \max_{i\in\mathcal{A}} \sum_{j\in\mathcal{B}} \frac{37(2c_1+1)\|\boldsymbol{\beta}\|_1}{\left[1+d_{i,j}\right]^{1+\mu_1+\epsilon_1}} \\
&\leqslant 37(4c_1+2)\|\boldsymbol{\beta}\|_1 \zeta(1+\epsilon_1) \sum_{i\in\mathcal{A}} |\alpha_i| \\
&\leqslant c_2 \|\boldsymbol{\beta}\|_1^2 2^{-d_{\mathcal{A},\mathcal{S}}}.
\end{aligned} \tag{A.15}$$

A.2.2. Fourth moments (lemma 2). From equation (18), we find

$$\begin{aligned}
|f_{\mathcal{A}}| \leqslant & \sum_{i,j,k,l\in\mathcal{A}} |\alpha_i\alpha_j\alpha_k\alpha_l| (|\langle \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l\rangle| + |\langle \hat{b}_i^\dagger \hat{b}_k \hat{b}_j^\dagger \hat{b}_l\rangle| + |\langle \hat{b}_i^\dagger \hat{b}_k \hat{b}_l \hat{b}_j^\dagger\rangle| \\
& + |\langle \hat{b}_k \hat{b}_j^\dagger \hat{b}_i^\dagger \hat{b}_l\rangle| + |\langle \hat{b}_k \hat{b}_j^\dagger \hat{b}_l \hat{b}_i^\dagger\rangle| + |\langle \hat{b}_k \hat{b}_l \hat{b}_i^\dagger \hat{b}_j^\dagger\rangle|) \\
\leqslant & \, 3 \sum_{i,j,k,l\in\mathcal{A}} |\alpha_i\alpha_j\alpha_k\alpha_l| (2|\langle \hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l\rangle| + 4\delta_{k,j} |\langle \hat{b}_i^\dagger \hat{b}_l\rangle| + \delta_{k,j}\delta_{i,l}),
\end{aligned} \tag{A.16}$$

where, under assumption 1, we have

$$\begin{aligned}
\left(\delta_{i,j} + 4|\langle \hat{b}_i^\dagger \hat{b}_j\rangle|\right)\delta_{k,l} &\leqslant \frac{(4c_1+1)\delta_{k,l}}{[1+d_{i,j}]^{1+\epsilon_1}} \\
&\leqslant \frac{(4c_1+1)}{\left([1+d_{i,j}][1+d_{k,l}]\right)^{1+\epsilon_1}} \\
&\leqslant \sum_{(r,s,t,u)\in P(i,j,k,l)} \frac{(4c_1+1)}{\left([1+d_{r,s}][1+d_{t,u}]\right)^{1+\epsilon_1}},
\end{aligned} \tag{A.17}$$

i.e. under assumptions 1 and 2, we have

$$|f_{\mathcal{A}}| \leqslant \sum_{i,j,k,l\in\mathcal{A}} |\alpha_i\alpha_j\alpha_k\alpha_l| \sum_{(r,s,t,u)\in P(i,j,k,l)} \frac{6c_3+3(4c_1+1)}{\left([1+d_{r,s}][1+d_{t,u}]\right)^{1+\min\{\epsilon_1,\epsilon_2\}}}. \tag{A.18}$$

Now let (r,s,t,u) be a given permutation of (i,j,k,l). Then, using the geometric mean inequality and equation (A.10),

$$\begin{aligned}
\sum_{i,j,k,l\in\mathcal{A}} \frac{|\alpha_i\alpha_j||\alpha_k\alpha_l|}{\left([1+d_{r,s}][1+d_{t,u}]\right)^{1+\min\{\epsilon_1,\epsilon_2\}}} &= \sum_{r,s,t,u\in\mathcal{A}} \frac{|\alpha_r||\alpha_s||\alpha_t||\alpha_u|}{\left([1+d_{r,s}][1+d_{t,u}]\right)^{1+\min\{\epsilon_1,\epsilon_2\}}} \\
&= \left[\sum_{r,s\in\mathcal{A}} \frac{|\alpha_r||\alpha_s|}{[1+d_{r,s}]^{1+\min\{\epsilon_1,\epsilon_2\}}}\right]^2
\end{aligned}$$

$$\leqslant \left[\sum_{r,s\in\mathcal{A}} \frac{|\alpha_r|^2}{[1+d_{r,s}]^{1+\min\{\epsilon_1,\epsilon_2\}}}\right]^2$$
$$\leqslant 4\zeta^2(1+\min\{\epsilon_1,\epsilon_2\})\left[\sum_{i\in\mathcal{A}}|\alpha_i|^2\right]^2, \tag{A.19}$$

i.e. using lemma 3 and the fact that $\sum_{i\in\mathcal{L}}|\alpha_i|^2 = \|\boldsymbol{\beta}\|_2^2$ and $|P| = 24$, we find, for $L \geqslant |t|^{7/6} \geqslant 1$,

$$\begin{aligned}\sum_{i=1}^{n}|f_{\mathcal{A}_i}| &\leqslant 96(6c_3+3(4c_1+1))\zeta^2(1+\min\{\epsilon_1,\epsilon_2\})\sum_{i=1}^{n}\left[\sum_{j\in\mathcal{A}_i}|\alpha_j|^2\right]^2\\ &\leqslant 96(6c_3+3(4c_1+1))\zeta^2(1+\min\{\epsilon_1,\epsilon_2\})\|\boldsymbol{\beta}\|_2^2\max_i\sum_{j\in\mathcal{A}_i}|\alpha_j|^2\\ &\leqslant 96(6c_3+3(4c_1+1))37^2\zeta^2(1+\min\{\epsilon_1,\epsilon_2\})\|\boldsymbol{\beta}\|_2^2\|\boldsymbol{\beta}\|_1^2\frac{\max_i|\mathcal{A}_i|}{|t|^{2/3}}.\end{aligned} \tag{A.20}$$

A.3. Blocking argument (lemma 4)

In the following, we write $\tau = 8\mathrm{e}|t|$.

A.3.1. Distances. For all $k \in A_i$, $l \in A_j$, we have (we pick $s \in \mathcal{S}$ such that $d_{l,s} = d_{l,\mathcal{S}}$)

$$\begin{aligned}(i-1)(a+b) \leqslant d_{k,\mathcal{S}} &\leqslant d_{k,s}\\ &\leqslant d_{k,l}+d_{l,s}\\ &= d_{k,l}+d_{l,\mathcal{S}}\\ &< d_{k,l}+j(a+b)-b\end{aligned} \tag{A.21}$$

and (we pick $s \in \mathcal{S}$ such that $d_{k,s} = d_{k,\mathcal{S}}$)

$$\begin{aligned}(j-1)(a+b) \leqslant d_{l,\mathcal{S}} &\leqslant d_{l,s}\\ &\leqslant d_{l,k}+d_{k,s}\\ &= d_{l,k}+d_{k,\mathcal{S}}\\ &< d_{k,l}+i(a+b)-b,\end{aligned} \tag{A.22}$$

i.e.

$$d_{\mathcal{A}_i,\mathcal{A}_j} = \min_{\substack{k\in\mathcal{A}_i\\ l\in\mathcal{A}_j}} d_{k,l} > |i-j|(a+b)-a. \tag{A.23}$$

Furthermore

$$d_{\mathcal{S},\mathcal{T}} = \min_{l\in\mathcal{T}} d_{\mathcal{S},l} \geqslant n(a+b) \geqslant \frac{\tau}{2}. \tag{A.24}$$

A.3.2. Cardinalities. For $i \in \mathcal{L}\backslash\mathcal{S}$ pick $s_i \in \partial\mathcal{S}$ such that $d_{i,\mathcal{S}} = d_{i,s_i} = d_{i,\partial\mathcal{S}}$. Then

$$
\begin{aligned}
|\mathcal{A}_1| &= \sum_{0\leqslant l<a} \sum_{i\in\mathcal{L}} \delta_{d_{i,\mathcal{S}},l} \\
&= |\mathcal{S}| + \sum_{1\leqslant l<a} \sum_{i\in\mathcal{L}\backslash\mathcal{S}} \delta_{d_{i,\mathcal{S}},l} \\
&= |\mathcal{S}| + \sum_{1\leqslant l<a} \sum_{i\in\mathcal{L}\backslash\mathcal{S}} \delta_{d_{i,s_i},l} \\
&\leqslant |\mathcal{S}| + \sum_{1\leqslant l<a} \sum_{s\in\partial\mathcal{S}} \sum_{i\in\mathcal{L}\backslash\mathcal{S}} \delta_{d_{i,s},l} \\
&\leqslant |\mathcal{S}| + 2|\partial\mathcal{S}|a.
\end{aligned} \tag{A.25}
$$

Similarly, for $i \neq 1$

$$|\mathcal{A}_i| \leqslant 2|\partial\mathcal{S}| \sum_{(i-1)(a+b)\leqslant l<i(a+b)-b} 1 \leqslant 2|\partial\mathcal{S}|a, \tag{A.26}$$

i.e.

$$|\mathcal{A}| = \sum_{i=1}^{n} |\mathcal{A}_i| \leqslant |\mathcal{S}| + 2|\partial\mathcal{S}|na \leqslant |\mathcal{S}| + 2|\partial\mathcal{S}|\tau. \tag{A.27}$$

Furthermore,

$$|\mathcal{B}| \leqslant 2|\partial\mathcal{S}| \sum_{i=1}^{n} \sum_{i(a+b)-b\leqslant l<i(a+b)} 1 \leqslant 2|\partial\mathcal{S}|nb \leqslant 2|\partial\mathcal{S}|\frac{\tau b}{a}. \tag{A.28}$$

A.3.3. Moments. In the following, let t and L be such that

$$L^{6/7} \geqslant |t| \geqslant 2, \quad \log|t| \leqslant |t|^{1/3+\mu}, \tag{A.29}$$

where $\mu = \mu_1/(6(\mu_1+1))$. Then

$$\frac{|t|^{2/3}}{\log|t|} = a \geqslant b = |t|^{1/3-\mu} \geqslant 1, \quad \left\lfloor \frac{8\mathrm{e}|t|}{a+b} \right\rfloor = n > 1. \tag{A.30}$$

Combining the above bounds on distances and cardinalities with lemmas 1 and 2, we find under assumptions 1 and 2 that

$$|\sigma_{\mathcal{T}}|,\ |\sigma_{\mathcal{B},\mathcal{T}}|,\ |\sigma_{\mathcal{A},\mathcal{T}}| \leqslant c_2\|\boldsymbol{\beta}\|_1^2 2^{-4\mathrm{e}|t|} \tag{A.31}$$

and

$$
\begin{aligned}
|\sigma_{\mathcal{B}}|,\ |\sigma_{\mathcal{A},\mathcal{B}}| &\leqslant 2c_2|\partial\mathcal{S}|\|\boldsymbol{\beta}\|_1^2 \frac{\tau b}{a|t|^{2/3}} \\
&= 16\mathrm{e}c_2|\partial\mathcal{S}|\|\boldsymbol{\beta}\|_1^2 \frac{|t|^{1/3}b}{a} \\
&= 16\mathrm{e}c_2|\partial\mathcal{S}|\|\boldsymbol{\beta}\|_1^2 |t|^{-\mu} \log|t|,
\end{aligned}
$$

$$
\begin{aligned}
\left|\sigma_{\mathcal{L}} - \sum_{i=1}^{n} \sigma_{\mathcal{A}_i}\right| &\leqslant |\sigma_{\mathcal{B}\cup\mathcal{T}}| + 2|\sigma_{\mathcal{B}\cup\mathcal{T},\mathcal{A}}| + c_2\|\boldsymbol{\beta}\|_1^2 \frac{|\mathcal{A}|}{|t|^{2/3} b^{1+\mu_1}} \\
&\leqslant |\sigma_{\mathcal{B}}| + |\sigma_{\mathcal{T}}| + 2|\sigma_{\mathcal{B},\mathcal{T}}| + 2|\sigma_{\mathcal{B},\mathcal{A}}| + 2|\sigma_{\mathcal{A},\mathcal{T}}| + c_2\|\boldsymbol{\beta}\|_1^2 \left(\frac{|\mathcal{S}|}{|t|^{2/3}} + 16\mathrm{e}|\partial\mathcal{S}|\frac{|t|^{1/3}}{b^{1+\mu_1}}\right) \\
&\leqslant 5c_2\|\boldsymbol{\beta}\|_1^2 2^{-4\mathrm{e}|t|} + c_2\|\boldsymbol{\beta}\|_1^2 \left(\frac{|\mathcal{S}|}{|t|^{2/3}} + 16\mathrm{e}|\partial\mathcal{S}|\left[3|t|^{-\mu}\log|t| + \frac{|t|^{1/3}}{b^{1+\mu_1}}\right]\right) \\
&\leqslant c_2\|\boldsymbol{\beta}\|_1^2 \left(\frac{2|\mathcal{S}|}{|t|^{2/3}} + 80\mathrm{e}|\partial\mathcal{S}||t|^{-\mu}\log|t|\right), \\
\sum_{i=1}^{n} |f_{\mathcal{A}_i}| &\leqslant c_4\|\boldsymbol{\beta}\|_2^2\|\boldsymbol{\beta}\|_1^2 \frac{|\mathcal{A}_1|}{|t|^{2/3}} \\
&\leqslant c_4\|\boldsymbol{\beta}\|_2^2\|\boldsymbol{\beta}\|_1^2 \frac{|\mathcal{S}| + 2|\partial\mathcal{S}|a}{|t|^{2/3}} \\
&= c_4\|\boldsymbol{\beta}\|_2^2\|\boldsymbol{\beta}\|_1^2 \left(\frac{|\mathcal{S}|}{|t|^{2/3}} + \frac{2|\partial\mathcal{S}|}{\log|t|}\right).
\end{aligned}
\tag{A.32}
$$

A.4. Closeness of quantum states

We have for any states $\hat{\varrho}_1, \hat{\varrho}_2$,

$$
\|\hat{\varrho}_1 - \hat{\varrho}_2\|_{\mathrm{tr}} \leqslant \|\hat{\varrho}_1 - \hat{P}_M\hat{\varrho}_1\hat{P}_M + \hat{P}_M\hat{\varrho}_2\hat{P}_M - \hat{\varrho}_2\|_{\mathrm{tr}} + \|\hat{P}_M\hat{\varrho}_1\hat{P}_M - \hat{P}_M\hat{\varrho}_2\hat{P}_M\|_{\mathrm{tr}}, \tag{A.33}
$$

treating the relevant part of the state space. Within that relevant part, we can make use of

$$
\begin{aligned}
\|\hat{P}_M\hat{\varrho}_1\hat{P}_M - \hat{P}_M\hat{\varrho}_2\hat{P}_M\|_{\mathrm{tr}} &= \|\hat{P}_M(\hat{\varrho}_1 - \hat{\varrho}_2)\hat{P}_M\|_{\mathrm{tr}} \\
&= \mathrm{tr}[(\hat{P}_M(\hat{\varrho}_1 - \hat{\varrho}_2)\hat{P}_M(\hat{\varrho}_1 - \hat{\varrho}_2)\hat{P}_M)^{1/2}] \\
&= \sum_{\boldsymbol{n}} \langle \boldsymbol{n}|(\hat{P}_M(\hat{\varrho}_1 - \hat{\varrho}_2)\hat{P}_M(\hat{\varrho}_1 - \hat{\varrho}_2)\hat{P}_M)^{1/2}|\boldsymbol{n}\rangle,
\end{aligned}
\tag{A.34}
$$

giving rise to the bound

$$
\begin{aligned}
\|\hat{P}_M\hat{\varrho}_1\hat{P}_M - \hat{P}_M\hat{\varrho}_2\hat{P}_M\|_{\mathrm{tr}} &\leqslant \sum_{\boldsymbol{n}} \left(\langle \boldsymbol{n}|\hat{P}_M(\hat{\varrho}_1 - \hat{\varrho}_2)\hat{P}_M(\hat{\varrho}_1 - \hat{\varrho}_2)\hat{P}_N|\boldsymbol{n}\rangle\right)^{1/2} \\
&= \sum_{\boldsymbol{n}} \left(\sum_{\boldsymbol{m}} |\langle \boldsymbol{n}|\hat{P}_M(\hat{\varrho}_1 - \hat{\varrho}_2)\hat{P}_M|\boldsymbol{m}\rangle|^2\right)^{1/2} \\
&= \sum_{\boldsymbol{n}\in M} \left(\sum_{\boldsymbol{m}\in M} |\langle \boldsymbol{n}|(\hat{\varrho}_1 - \hat{\varrho}_2)|\boldsymbol{m}|^2\right)^{1/2} \\
&\leqslant |M|^{3/2} \max_{\boldsymbol{n},\boldsymbol{m}\in M} |\langle \boldsymbol{n}|(\hat{\varrho}_1 - \hat{\varrho}_2)|\boldsymbol{m}\rangle|.
\end{aligned}
\tag{A.35}
$$

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