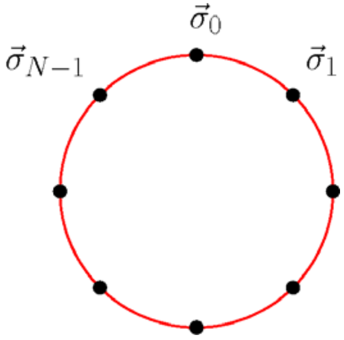
Exact solution of the XY Model on the circle



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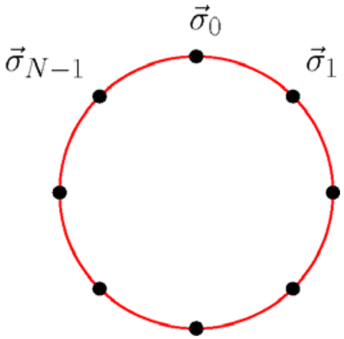
Joint Bilateral Project Italy-Japan (Ministero degli Affari Esteri e Ministero dell'Università)

XY Model on the circle

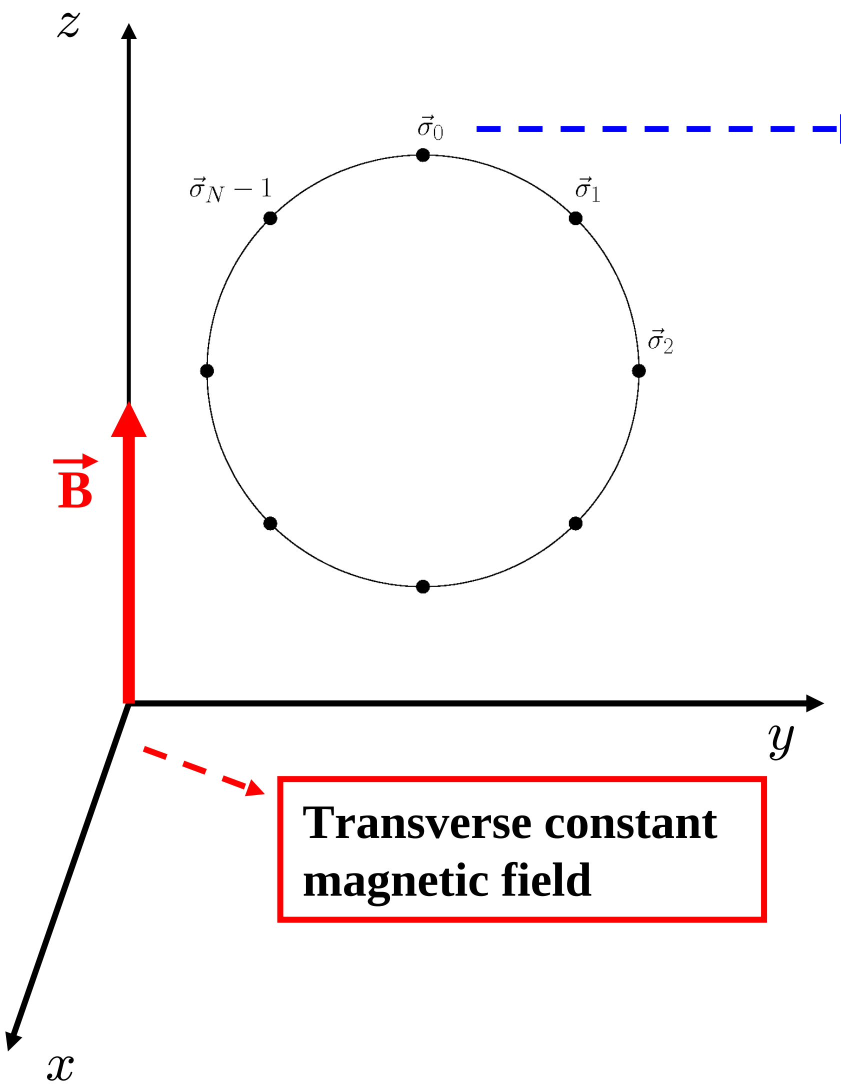


- **XY Model**

- **XX Model: Ground State**

$$\boxed{D = 1}$$

One dimensional spin- $\frac{1}{2}$ chain



Periodic boundary conditions:

$$\vec{\sigma}_0 \equiv \vec{\sigma}_N$$

N-qubits Hilbert space: $(\mathbb{C}^2)^{\otimes N}$

Generic element of the computational basis:

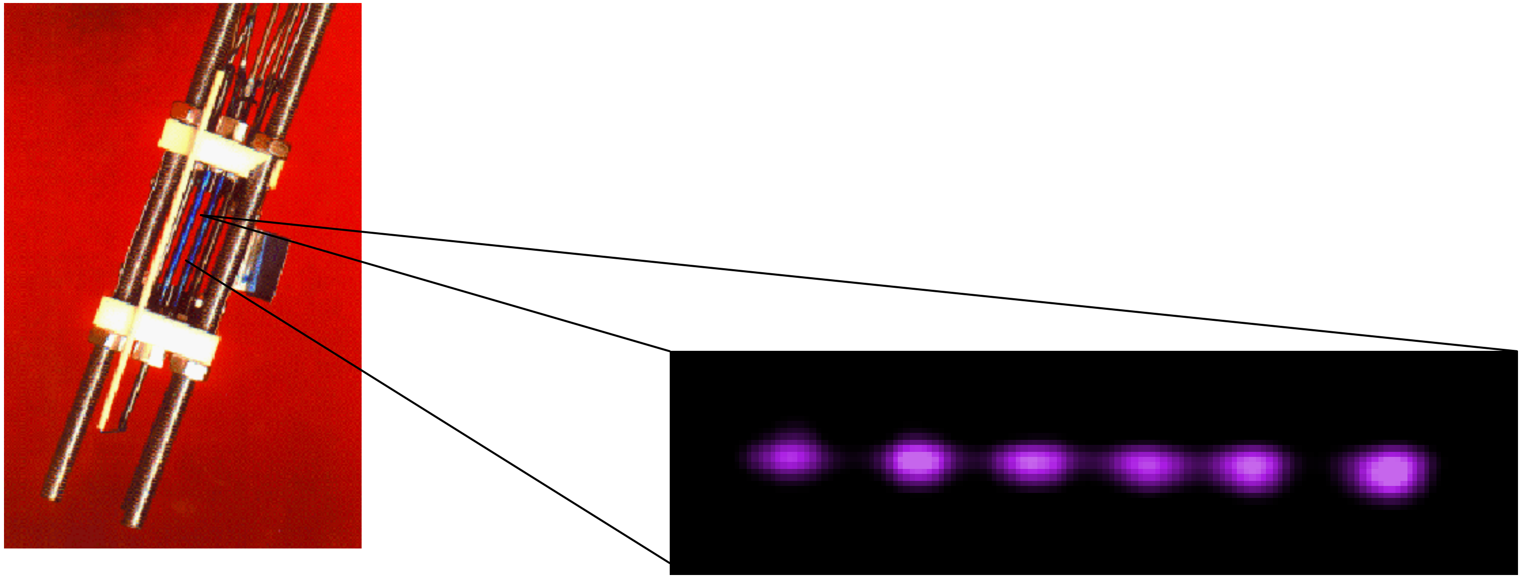
$$|\psi\rangle = |\uparrow\rangle_0 \otimes |\uparrow\rangle_1 \otimes \ldots |\downarrow\rangle_i \otimes \ldots |\downarrow\rangle_{N-1}$$

$|\uparrow\rangle_i =$ **parallel to the z axis**

$|\downarrow\rangle_i =$ **antiparallel to the z axis**

Experimental implementations

- **Ion-traps (N=8) – Innsbruck**



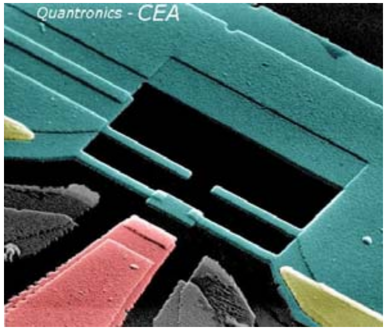
Experimental implementations

- Ion-traps (N=8) – Innsbruck
- Superconducting circuits – Josephson junctions (N=3 , 4)

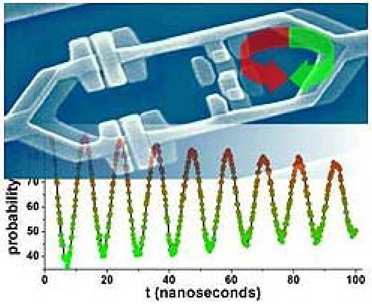
  - Delft and Tokyo

-Charge-Qubit

EUROSQIP



- Flux-Qubit



Experimental implementations

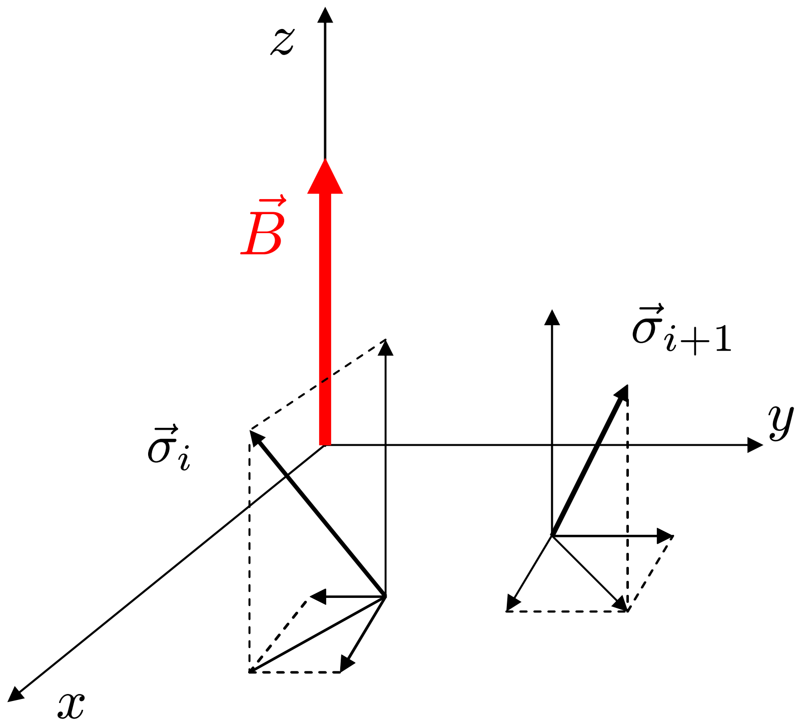
- **Ion-traps (N=8) – Innsbruck**
- **Superconducting circuits – Josephson junctions (N=3 , 4)**

  - Delft and Tokyo

- **Cavity QED (N=5, 6)**

XY Hamiltonian

$$H_\gamma = -J \sum_{i=0}^{N-1} \left[g\sigma_i^z + \left(\frac{1+\gamma}{2}\right)\sigma_i^x\sigma_{i+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_i^y\sigma_{i+1}^y \right]$$



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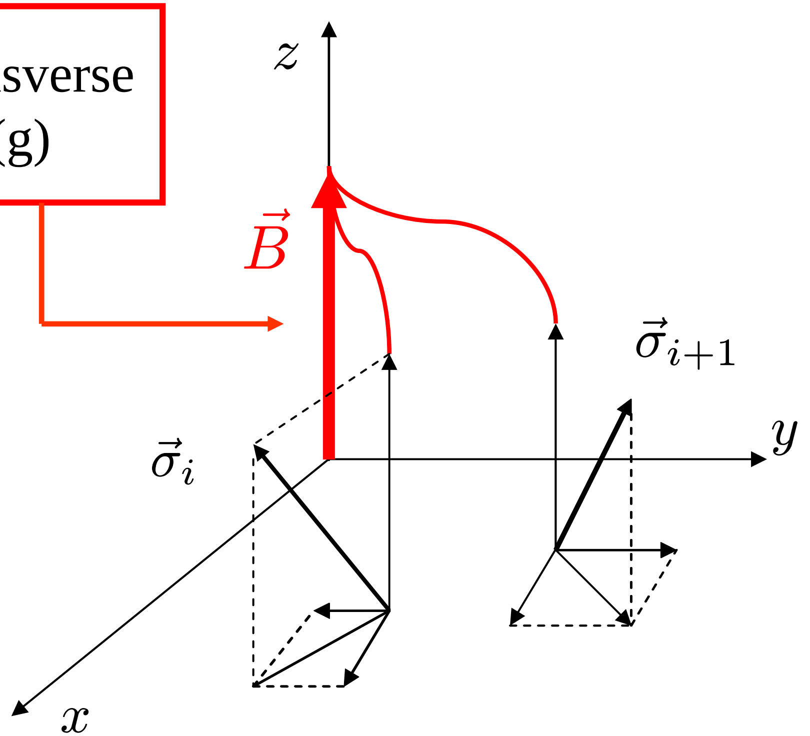
coupling to the transverse magnetic field (g)



XY Hamiltonian

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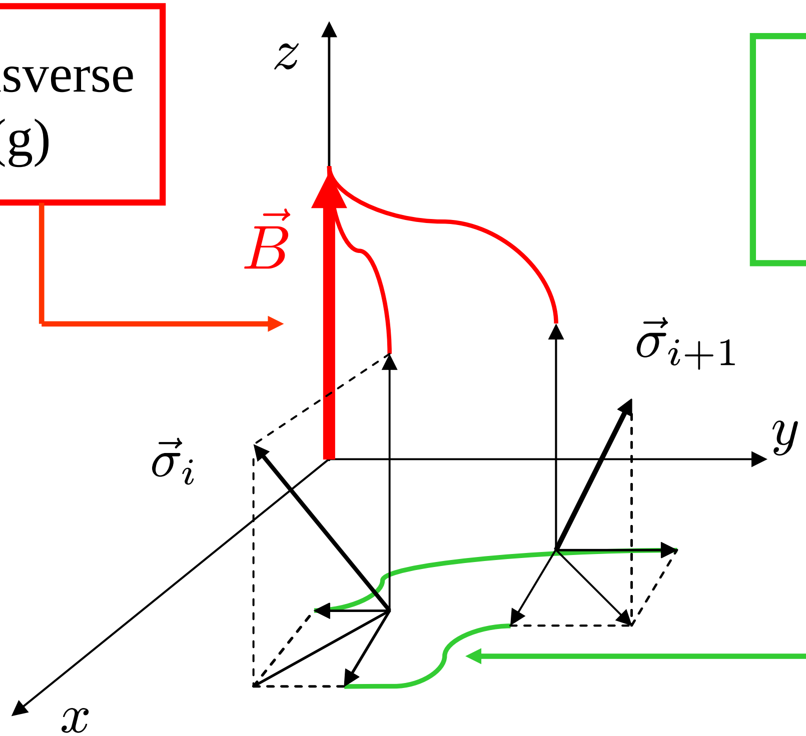
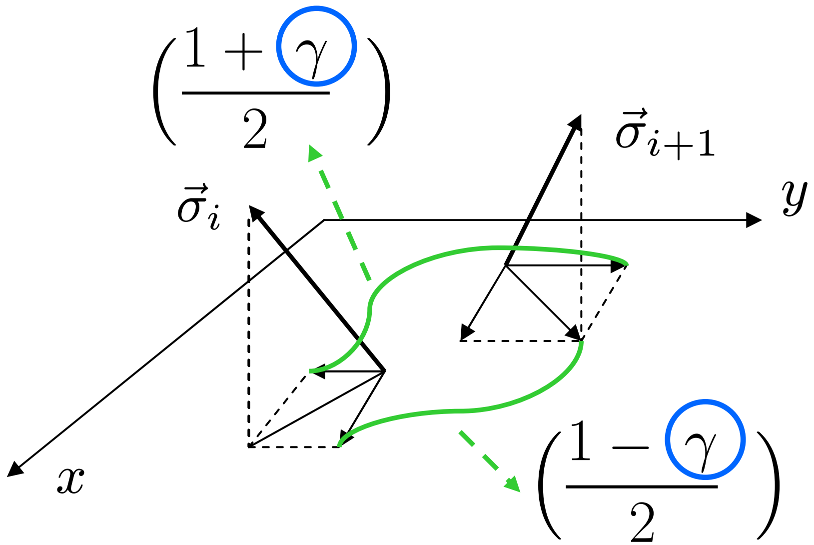
coupling to the transverse magnetic field (g)

nearest neighbours coupling



XY Hamiltonian

$$H_\gamma = -J \sum_{i=0}^{N-1} \left[g\sigma_i^z + \left(\frac{1+\gamma}{2}\right)\sigma_i^x \sigma_{i+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_i^y \sigma_{i+1}^y \right]$$



$\gamma \in [0, 1]$

anisotropy coefficient

XY Hamiltonian $H_\gamma = -J \sum_{i=0}^{N-1} \left[g\sigma_i^z + \left(\frac{1+\gamma}{2}\right)\sigma_i^x\sigma_{i+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_i^y\sigma_{i+1}^y \right]$

The Hamiltonian of the system can diagonalized by means of the **JORDAN-WIGNER (JW) TRANSFORMATION:**

Hilbert space of N spin- $\frac{1}{2}$	$\Longleftrightarrow$	**Fock space of spinless fermions**
$(\mathbb{C}^2)^{\otimes N}$		$\mathcal{F}_-(\mathbb{C}^N) = \bigoplus_{n=0}^{N}(h^{\otimes n})_- \quad , \; h = \mathbb{C}^N$
$dim \; = \; 2^N$		$dim = \sum_{n=0}^{N} \binom{N}{n} = 2^N$

$$\left.\begin{matrix} |\uparrow\rangle_i \\ |\downarrow\rangle_i \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} |\,\bullet\,\rangle_i \\ |\,\circ\,\rangle_i \end{matrix}\right.$$

$$\sigma_i^- = \frac{\sigma_i^x - \mathrm{i}\sigma_i^y}{2} \qquad \longrightarrow \qquad c_i = e^{\mathrm{i}\pi \mathbf{n}_{i\downarrow}} \sigma_i^-$$

(ANNIHILATION OPERATOR)

$$\sigma_i^+ = \frac{\sigma_i^x + \mathrm{i}\sigma_i^y}{2} \qquad \qquad c_i^\dagger = e^{\mathrm{i}\pi \mathbf{n}_{i\downarrow}} \sigma_i^+$$

(CREATION OPERATOR)

$$\boxed{\textbf{Number operator counting the holes between 0 and i-1}}$$

$$\left.\begin{array}{l} |\uparrow\rangle_i \\ |\downarrow\rangle_i \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} |\bullet\rangle_i \\ |\circ\rangle_i \end{array}\right.$$

$$\sigma_i^- = \frac{\sigma_i^x - \mathrm{i}\sigma_i^y}{2}$$

$$\sigma_i^+ = \frac{\sigma_i^x + \mathrm{i}\sigma_i^y}{2}$$

$$c_i = e^{\mathrm{i}\pi \mathbf{n}_{i\downarrow}} \sigma_i^-$$ (ANNIHILATION OPERATOR)

$$c_i^\dagger = e^{\mathrm{i}\pi \mathbf{n}_{i\downarrow}} \sigma_i^+$$ (CREATION OPERATOR)

Number operator counting the holes between 0 and i-1

$$i \neq j \quad \left[\sigma_i^\pm, \sigma_j^\pm\right] = 0$$

$$i = j \quad \left\{\sigma_i^-, \sigma_j^-\right\} = 0$$

$$i = j \quad \left\{\sigma_i^+, \sigma_j^-\right\} = 1$$

$$\forall\ i, j \quad \{c_i, c_j\} = 0$$

$$\{c_i^\dagger, c_j\} = \delta_{ij}$$

$$\sigma_i^- = \frac{\sigma_i^x - \mathrm{i}\sigma_i^y}{2}$$

$$\sigma_i^+ = \frac{\sigma_i^x + \mathrm{i}\sigma_i^y}{2}$$

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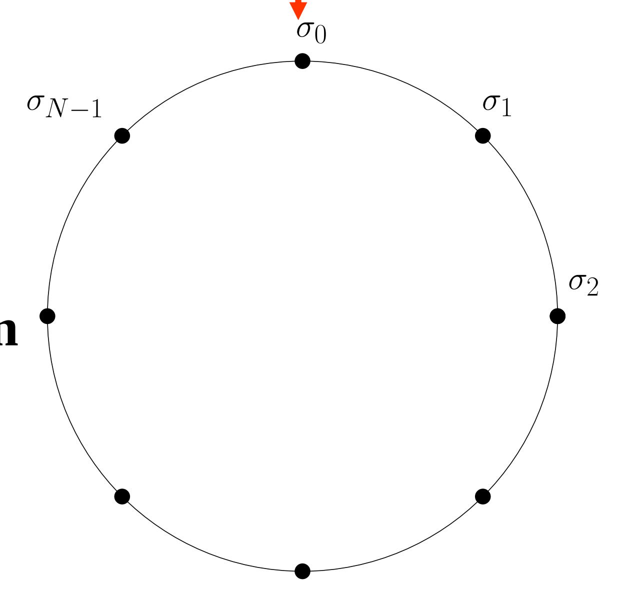
$$c_i^\dagger = e^{\mathrm{i}\pi \mathbf{n}_{i\downarrow}} \sigma_i^+$$ (CREATION OPERATOR)

Number operator counting the holes between 0 and i-1

no phase



The JW transformation introduces an arbitrary dependence of the phase $e^{\mathrm{i}\pi \mathbf{n}_{i\downarrow}}$ on the ordering of the spin on the circle. The phase depends on the state the Hamiltonian is applied to

Periodicity of the
Jordan –Wigner operators

Pauli operators

JW operators

$$\vec{\sigma}_0 \equiv \vec{\sigma}_N \quad \longrightarrow \quad c_0 = \begin{cases} c_N & \text{if } n_\downarrow \text{ odd} \\ -c_N & \text{if } n_\downarrow \text{ even} \end{cases}$$

$$H_\gamma = -J\Bigg\{ \sum_{j=0}^{N-1} g(1-2c_j c_j^\dagger) + \sum_{j=0}^{N-2} (c_j c_{j+1}^\dagger + c_{j+1} c_j^\dagger) + \gamma(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)$$

$$+ \boxed{\left(e^{\mathrm{i}\pi(n_\downarrow+1)} - 1\right)\left[(c_{N-1}c_0^\dagger + c_0 c_{N-1}^\dagger) + \gamma(c_{N-1}c_0 + c_0^\dagger c_{N-1}^\dagger)\right]} \Bigg\}$$

Periodicity of the **Jordan –Wigner** operators

Pauli operators

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BOUNDARY TERM

$$\left. + \cancel{\boxed{\left(e^{i\pi(n_\downarrow+1)} - 1\right)\left[(c_{N-1}c_0^\dagger + c_0 c_{N-1}^\dagger) + \gamma(c_{N-1}c_0 + c_0^\dagger c_{N-1}^\dagger)\right]}} \right\}$$

1961: **Lieb, Schultz , Mattis**

In the termodynamic limit ($N \to \infty$) the boundary term can be neglected since its contrbution scales like 1 / N $\Rightarrow$ **“c-cyclic Hamiltonian”**

Periodicity of the **Jordan –Wigner** operators

Pauli operators

JW operators

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BOUNDARY TERM

$$+ \left(e^{\mathrm{i}\pi(n_\downarrow+1)} - 1\right)\left[(c_{N-1}c_0^\dagger + c_0 c_{N-1}^\dagger) + \gamma(c_{N-1}c_0 + c_0^\dagger c_{N-1}^\dagger)\right]\Big\}$$

For finite size systems (*and applications*) the boundary term CANNOT be neglected

Boundary term

Defining: $c_N \equiv c_0$ the problem introduced by the presence of the boundary becomes:

$$\left(e^{\mathrm{i}\pi(\mathbf{n}_\downarrow+1)} - 1\right)\left[c_{N-1}c_N^\dagger + c_N c_{N-1}^\dagger + \gamma(c_{N-1}c_N + c_N^\dagger c_{N-1}^\dagger)\right]$$

$$\neq \left[c_j c_{j+1}^\dagger + c_{j+1}c_j^\dagger + \gamma(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)\right]\Big|_{j=N-1}$$

$\downarrow$

$\boxed{\textbf{OPERATOR}}$ $\longrightarrow$ **Quadratic Hamiltonian** $\Rightarrow \left[e^{\mathrm{i}\pi\mathbf{n}_\downarrow}, H_\gamma\right] = 0$

The parity operator

- **Consider the parity operator** $\boxed{\mathcal{P} = e^{\mathrm{i}\pi(\mathbf{n}_\downarrow + 1)}}$ **; its spectral decomposition in the basis of the number operator** $\mathbf{n}_\downarrow$ **is:**

$$\mathcal{P} = e^{\mathrm{i}\pi(\mathbf{n}_\downarrow + 1)} \sum_{n_\downarrow = 0}^{N} |n_\downarrow\rangle\langle n_\downarrow| = P_+ - P_- \longrightarrow \begin{cases} P_+ = \sum_{n_\downarrow \text{ odd}} |n_\downarrow\rangle\langle n_\downarrow| \\ P_- = \sum_{n_\downarrow \text{ even}} |n_\downarrow\rangle\langle n_\downarrow| \end{cases}$$

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- $\left[\mathrm{e}^{\mathrm{i}\pi n_\downarrow}, H_\gamma\right] = 0 \Rightarrow$ **The Hamiltonian preserves the parity sectors and can be decomposed as:**

$$\boxed{H = P_+ H P_+ + P_- H P_- = H^{(+)} + H^{(-)}}$$

- **The analysis can then be separately performed in the two parity sectors, where the parity operator** $\mathcal{P}$ **acts as a C-NUMBER**

$$\begin{aligned} H_\gamma \; = \; & -J\left\{ \sum_{j=0}^{N-1} g(1-2c_jc_j^\dagger) + \sum_{j=0}^{N-2} (c_jc_{j+1}^\dagger + c_{j+1}c_j^\dagger) + \gamma(c_jc_{j+1} + c_{j+1}^\dagger c_j^\dagger) \right. \\ & \left. + \boxed{\left(e^{\mathrm{i}\pi(n_\downarrow+1)}-1\right)\left[(c_{N-1}c_0^\dagger + c_0 c_{N-1}^\dagger) + \gamma(c_{N-1}c_0 + c_0^\dagger c_{N-1}^\dagger)\right]} \right\} \end{aligned}$$

$$
\begin{aligned}
H_\gamma \;=\; -J\Bigg\{ & \sum_{j=0}^{N-1} g(1-2c_j c_j^\dagger) + \sum_{j=0}^{N-2} (c_j c_{j+1}^\dagger + c_{j+1} c_j^\dagger) + \gamma(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) \\
& + \boxed{\cancel{\left(e^{\mathrm{i}\pi(n_\downarrow+1)} - 1\right)\Big[(c_{N-1}c_0^\dagger + c_0 c_{N-1}^\dagger) + \gamma(c_{N-1}c_0 + c_0^\dagger c_{N-1}^\dagger)\Big]}}\Bigg\}
\end{aligned}
$$

Lieb, Schultz, Mattis: **discrete**

Pfeuty **Fourier transform on the circle**

$$
c_j = \frac{1}{\sqrt{N}} \sum e^{\frac{2\pi \mathrm{i} j k}{N}} \hat{c}_k
$$

$\downarrow$

PERIODIC

$\downarrow$

$\boxed{\textbf{INCOMPATIBLE WITH THE BOUNDARY TERM}}$

DEFORMED Fourier transform

The arbitrary dependence of the phase $e^{\mathrm{i}\pi \mathbf{n}_{i\downarrow}}$ on the ordering of the spins on the circle can be compensated by deforming the discrete Fourier transform

$$c_j = \frac{1}{\sqrt{N}} \boxed{e^{\frac{2\pi \mathrm{i} \alpha_j}{N}}} \sum_{k=0}^{N-1} e^{\frac{2\pi \mathrm{i} k j}{N}} \hat{c}_k$$

LOCAL GAUGE $\longrightarrow$ $j \in \mathbb{Z}_N$

DEFORMED Fourier transform

The arbitrary dependence of the phase $e^{i\pi \mathbf{n}_{i\downarrow}}$ on the ordering of the spins on the circle can be compensated by deforming the discrete Fourier transform

$$c_j = \frac{1}{\sqrt{N}} \boxed{e^{\frac{2\pi i \alpha_j}{N}}} \sum_{k=0}^{N-1} e^{\frac{2\pi i k j}{N}} \hat{c}_k$$

Imposing that the boundary term after the Fourier transform has the same form of the other N-1 terms, one gets:

$$\alpha_j = \alpha j + \alpha_0 \qquad \boxed{e^{-\frac{2\pi i \alpha}{N}} = e^{\frac{i\pi(n_\downarrow + 1)}{N}}}$$

sector dependent costant

free parameter

DEFORMED Fourier transform

GLOBAL gauge **LOCAL gauge**

$$c_j = \frac{1}{\sqrt{N}} e^{\frac{2\pi i \alpha_0}{N}} \; e^{\frac{2\pi i}{N}\alpha j} \sum_{k=0}^{N-1} e^{\frac{2\pi i k j}{N}} \hat{c}_k$$

$$e^{-\frac{2\pi i \alpha}{N}} = e^{\frac{i\pi(n_\downarrow + 1)}{N}} \longrightarrow \begin{cases} \alpha = \frac{1}{2} \bmod N \,, & n_\downarrow \ \textbf{even} \\ \alpha = 0 \bmod N \,, & n_\downarrow \ \textbf{odd} \end{cases}$$

$n_\downarrow$ odd $\longrightarrow$ the Hamiltonian is "c-cyclic" and one finds the STANDARD FOURIER TRANSFORM

XY Hamiltonian diagonalized

In the Fourier space the Hamiltonian has no more the boundary term:

$$\begin{aligned} H_\gamma \;&=\; -J\Bigg\{\sum_{k=0}^{N-1} g + 2\hat{c}_k\hat{c}_k^\dagger\left(\cos\left(2\pi\frac{\alpha+k}{N}\right) - g\right) \\ &\quad + \mathrm{i}\,\gamma\;\sin\left(2\pi\frac{\alpha+k}{N}\right)\left(e^{\frac{2\pi\mathrm{i}(2\alpha_0)}{N}}\hat{c}_{\bar{k}}\hat{c}_k + e^{-\frac{2\pi\mathrm{i}(2\alpha_0)}{N}}\hat{c}_{\bar{k}}^\dagger\hat{c}_k^\dagger\right)\Bigg\} \end{aligned}$$

$$\bar{k} = (-2\alpha - k) \bmod N$$

The Hamiltonian diagonalized by means of the Bogoliubov transformation is given by:

$$H_\gamma = \mp 2J\sum_{k=0}^{N-1}\left(\hat{b}_k^\dagger\hat{b}_k - \frac{1}{2}\right)\varepsilon_k$$

$$\varepsilon_k = \sqrt{\cos^2\left(2\pi\frac{\alpha+k}{N}\right) + \gamma^2\sin^2\left(2\pi\frac{\alpha+k}{N}\right) + g^2 - 2g\cos\left(2\pi\frac{\alpha+k}{N}\right)}$$

(exact) DISPERSION RELATION

Anisotropy coefficient

$$\gamma \in [0,1]$$

XY HAMITONIAN: $$H_\gamma = -J \sum_{i=0}^{N-1} \left[g\sigma_i^z + \left(\frac{1+\gamma}{2}\right)\sigma_i^x\sigma_{i+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_i^y\sigma_{i+1}^y \right]$$

coupling with the transverse magnetic field (g)

nearest neighbours coupling

Anisotropy coefficient

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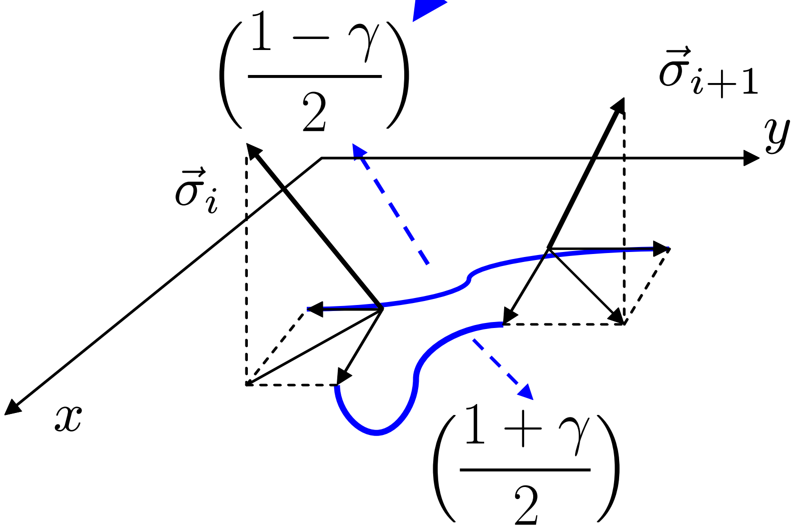
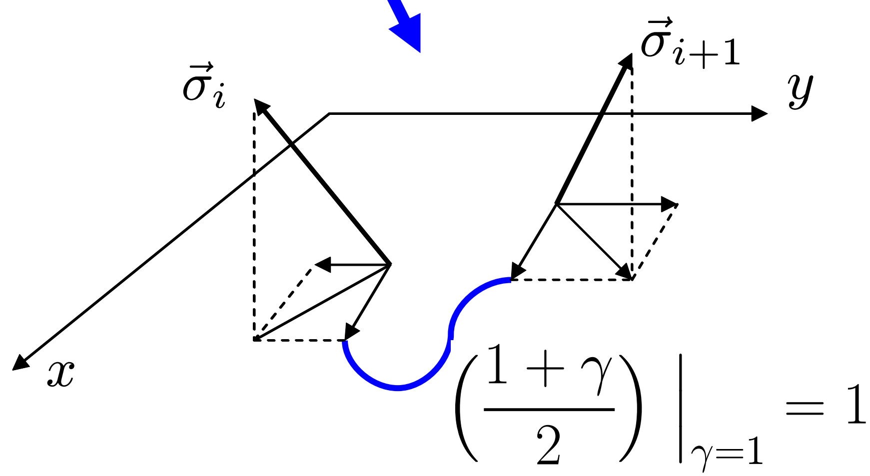
XY HAMITONIAN: $$H_\gamma = -J \sum_{i=0}^{N-1} \left[g\sigma_i^z + \left(\frac{1+\gamma}{2}\right)\sigma_i^x \sigma_{i+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_i^y \sigma_{i+1}^y \right]$$

$$H_{\text{Ising}} = -J \left[\sum_{i=0}^{N-1} g\sigma_i^z + \sum_{i=0}^{N-1} \sigma_i^x \sigma_{i+1}^x \right] = H_\gamma \Big|_{\gamma=1}$$

$\boxed{\gamma \in [0,1]}$



$\boxed{\gamma = 1}$



XX Model (γ=0)

$$H_0 = -J \sum_{k=0}^{N-1} \left[g\sigma_i^z + \frac{1}{2}\sigma_i^x \sigma_{i+1}^x + \frac{1}{2}\sigma_i^y \sigma_{i+1}^y \right]$$

$$\boxed{\left(\frac{1+\gamma}{2}\right)\Big|_{\gamma=0}} \qquad \boxed{\left(\frac{1-\gamma}{2}\right)\Big|_{\gamma=0}}$$

ISOTROPIC INTERACTION IN THE XY PLANE

XX Model (γ=0)

$$H_0 = -J \sum_{k=0}^{N-1} \left[g\sigma_i^z + \frac{1}{2}\sigma_i^x\sigma_{i+1}^x + \frac{1}{2}\sigma_i^y\sigma_{i+1}^y \right]$$

The diagonalized Hamiltonian is given by:

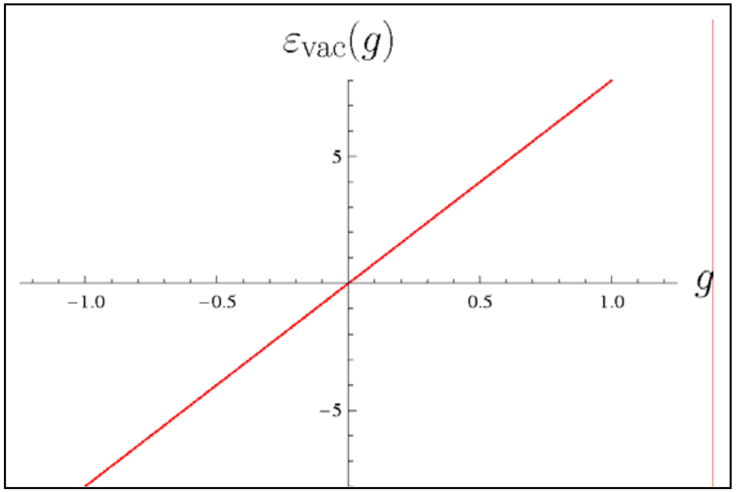
$$H_0 = -2J \sum_{k=0}^{N-1} \left(\hat{b}_k^\dagger \hat{b}_k - \frac{1}{2} \right) \varepsilon_k \quad \Longrightarrow \quad \boxed{\varepsilon_k = g - \cos\left(2\pi \frac{\alpha + k}{N} \right)}$$

XX DISPERSION RELATION

XX Spectrum

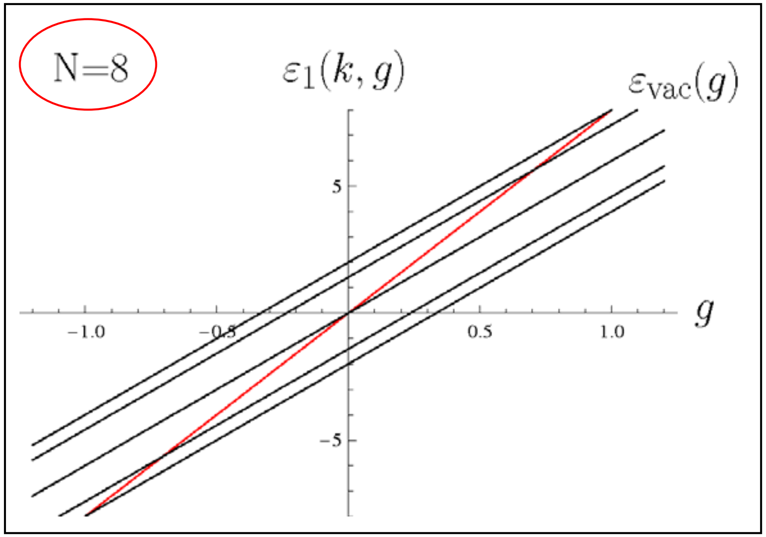
- **Vacuum state (zero fermions):**

$$\frac{E_{\text{vac}}}{N} = \varepsilon_{\text{vac}}(g) = g$$



- **State with 1 fermion with impulse k = 0,1,…N-1:**

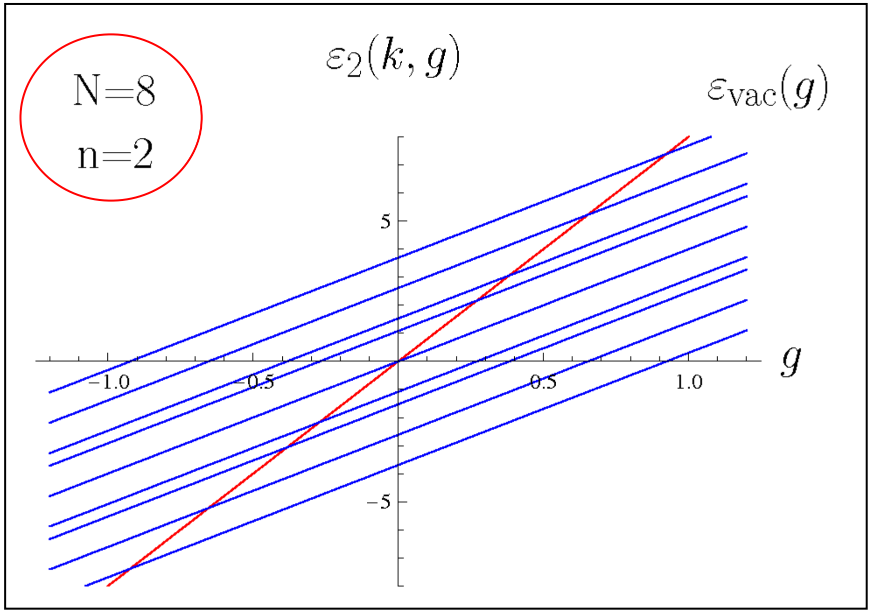
$$\begin{aligned}\frac{E_1}{N} &= \varepsilon_1(k,\alpha,g) \\ &= \varepsilon_{\text{vac}} + 2\left(-\frac{g}{N} + \frac{1}{N}\cos\left(2\pi\frac{\alpha+k}{N}\right)\right)\end{aligned}$$



XX Spectrum

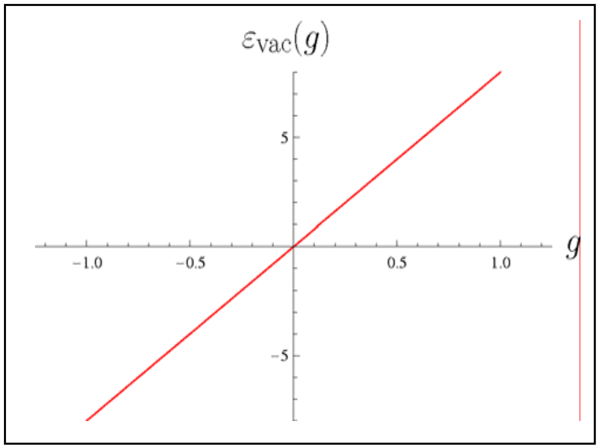
- **State with n fermions of momenta $k_1 \neq k_2 \neq \ldots k_n$, $n \leq N$**

$$\varepsilon_n(k_1, k_2, \ldots, k_n, \alpha, g) = \varepsilon_{\text{vac}} + 2\left(-\frac{ng}{N} + \sum_{1\leq i\leq n} \frac{1}{N}\cos\left(2\pi\frac{\alpha + k_i}{N}\right)\right)$$

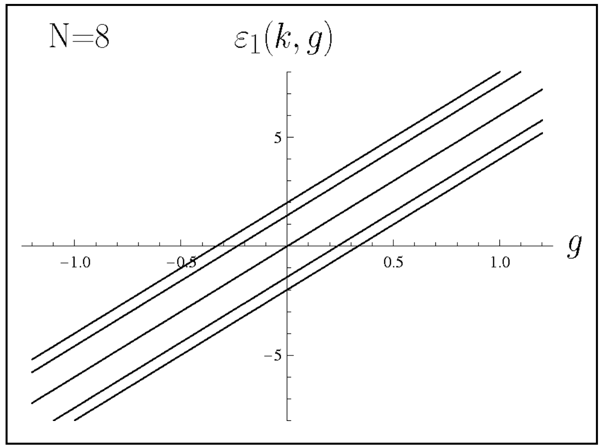


XX Spectrum

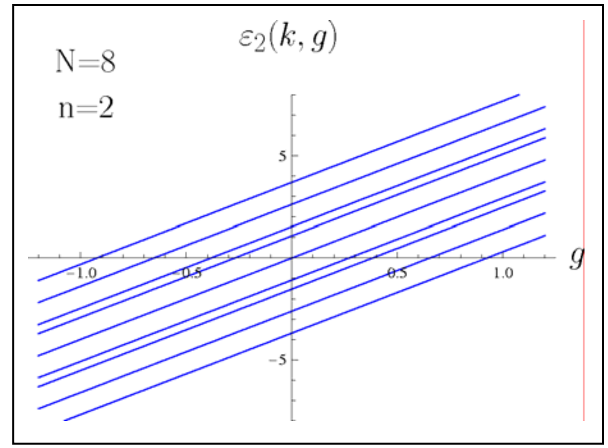
n = 0



n = 1



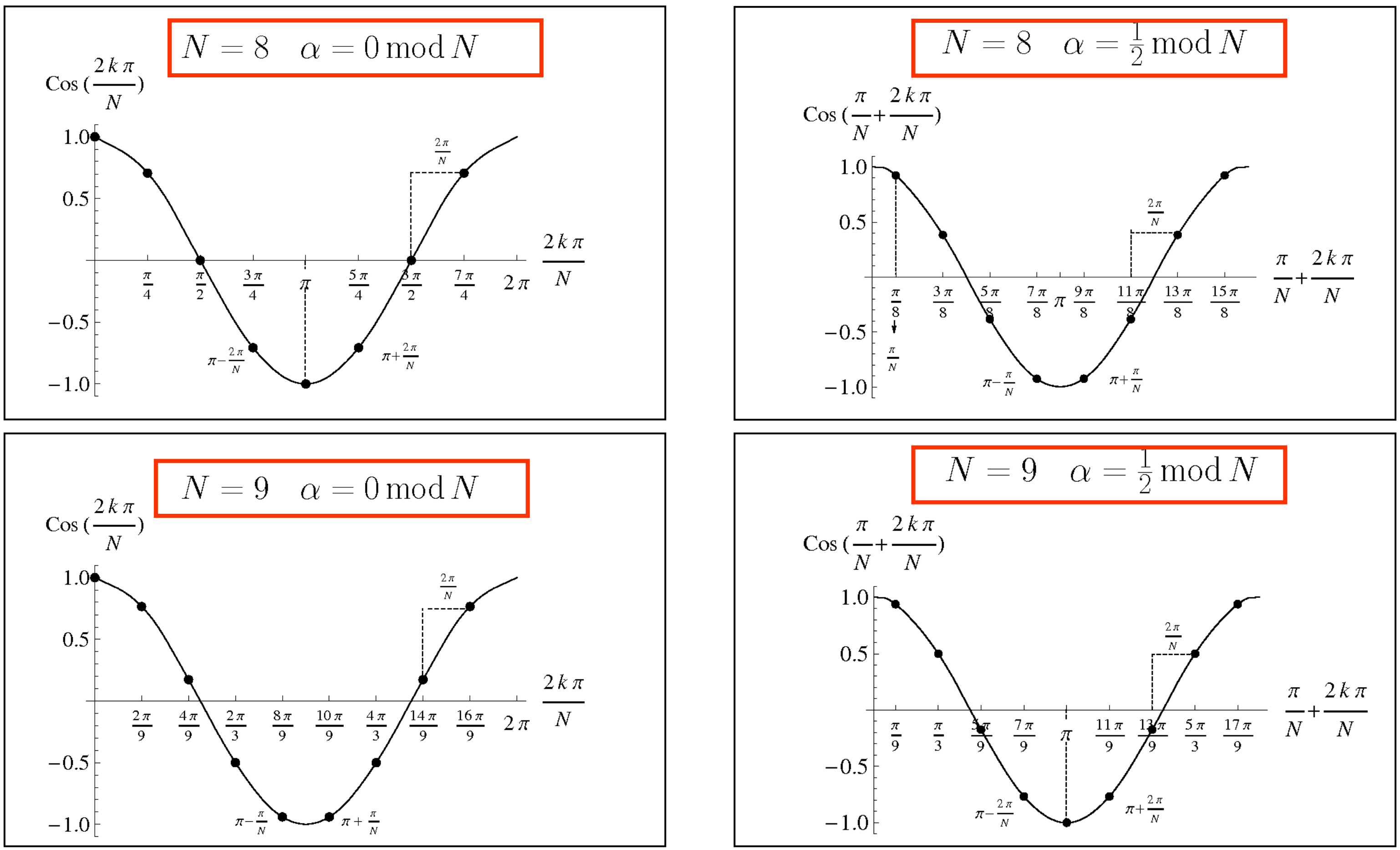
n = 2



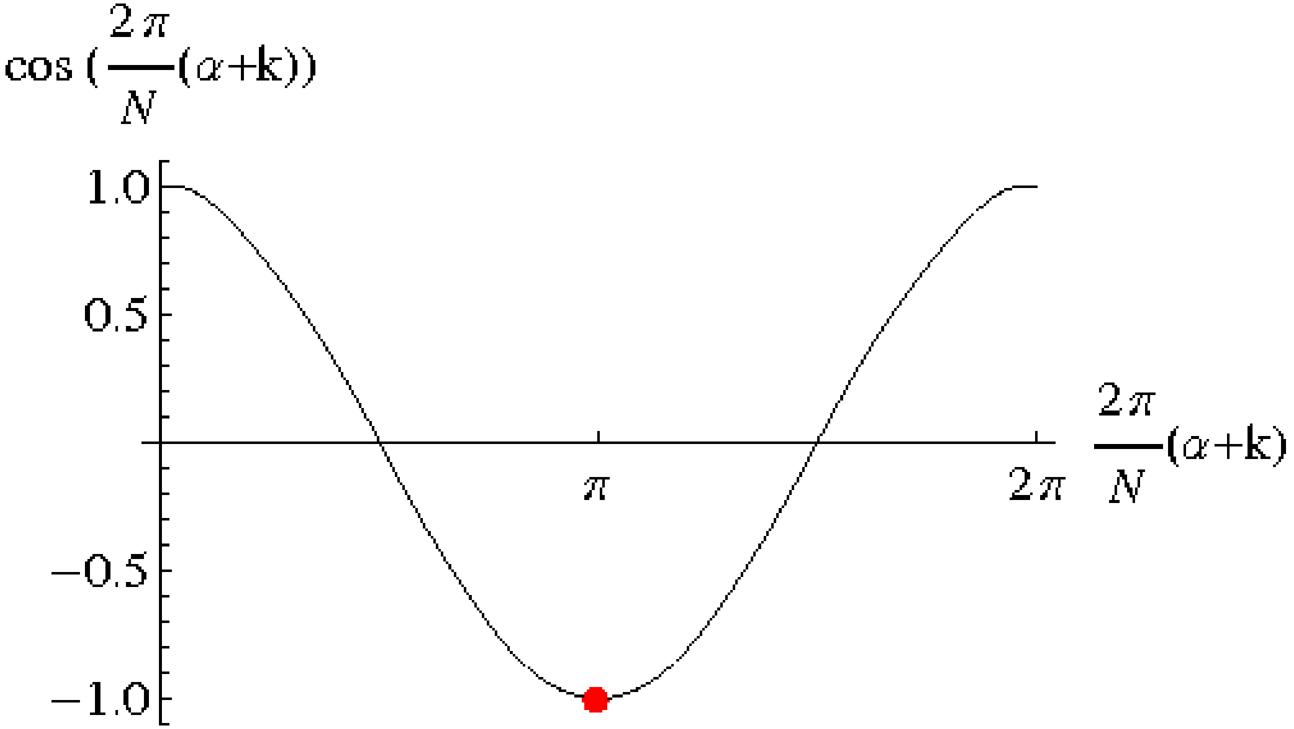
- **Observe that as the number of fermion increases the angolar coefficient decreases**

Lowest energy levels

We focus on the lowest energy levels and consider the values taken by the function $\boxed{\cos\left(2\pi\frac{\alpha+k}{N}\right)}$ **in the four possible cases [2 parities of number of sites N and 2 values of α]**



Filling the lowest energy levels



N=8

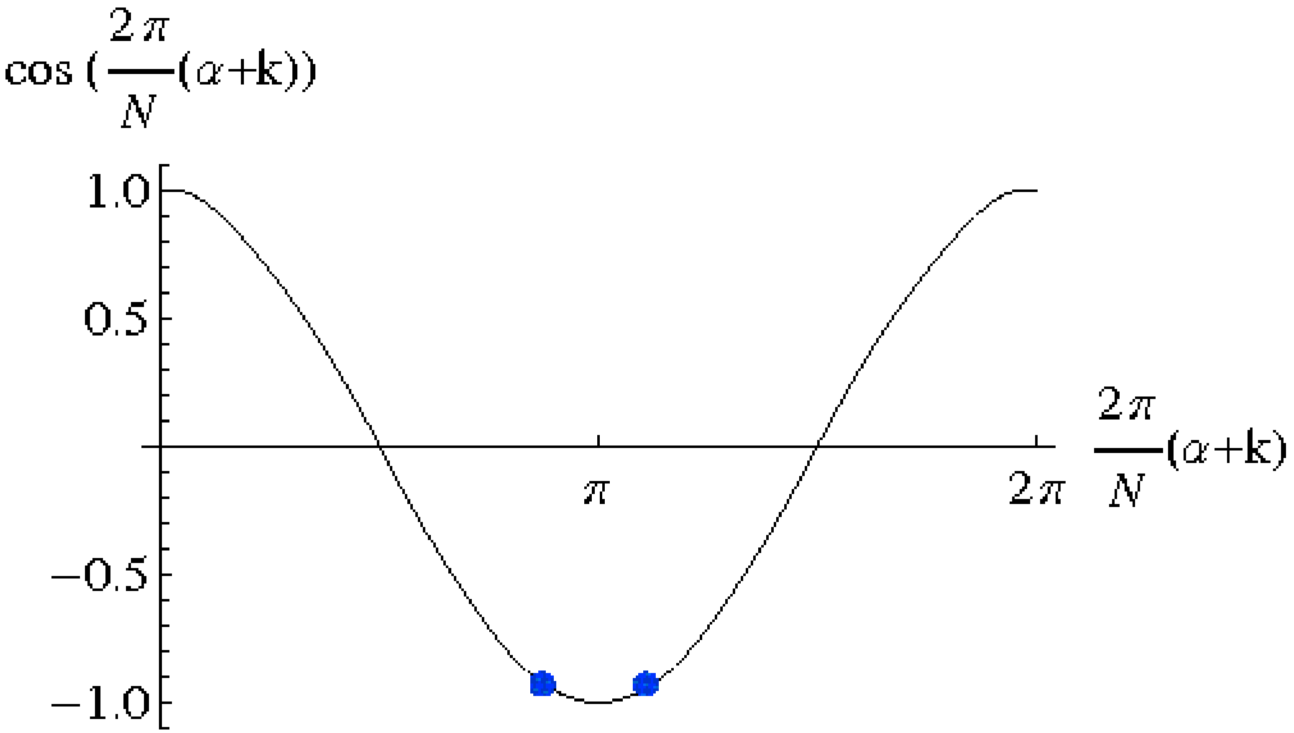
$\alpha = 0 \bmod N$

holes number = N-n odd

$\alpha = \frac{1}{2} \bmod N$

holes number = N-n even

Filling the lowest energy levels



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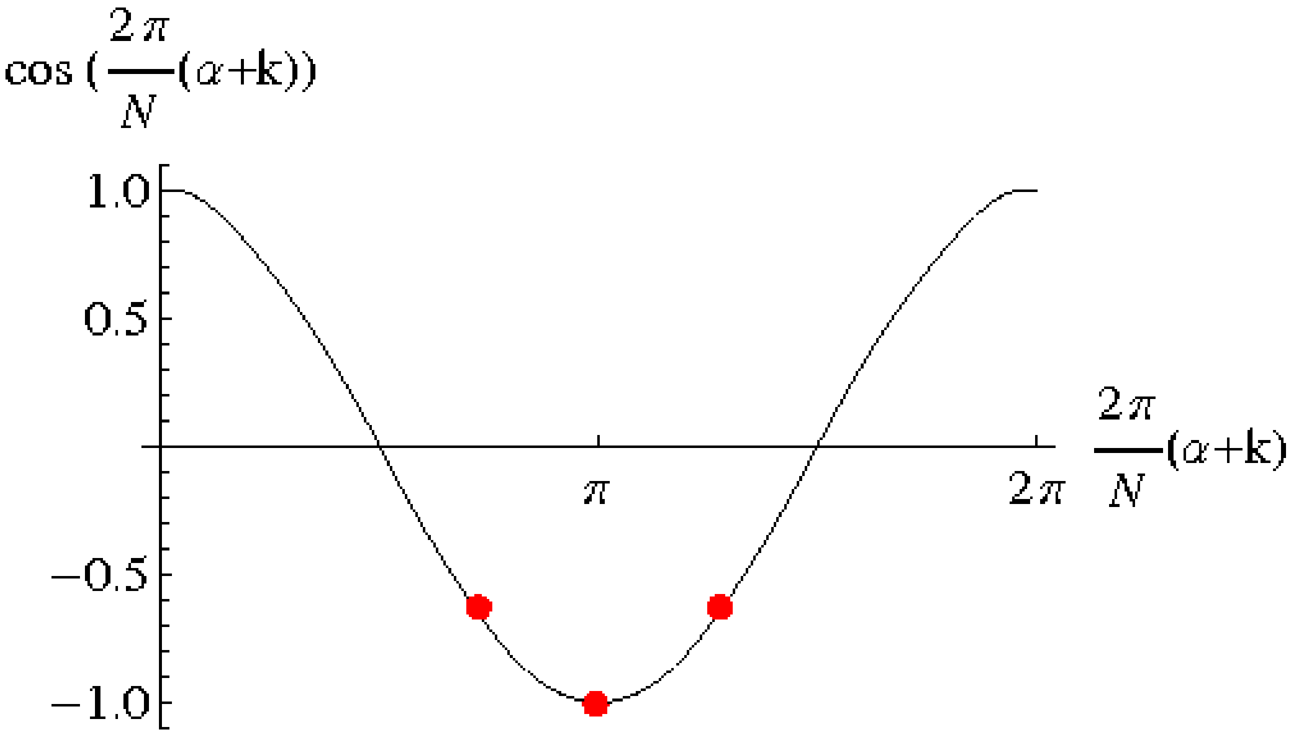

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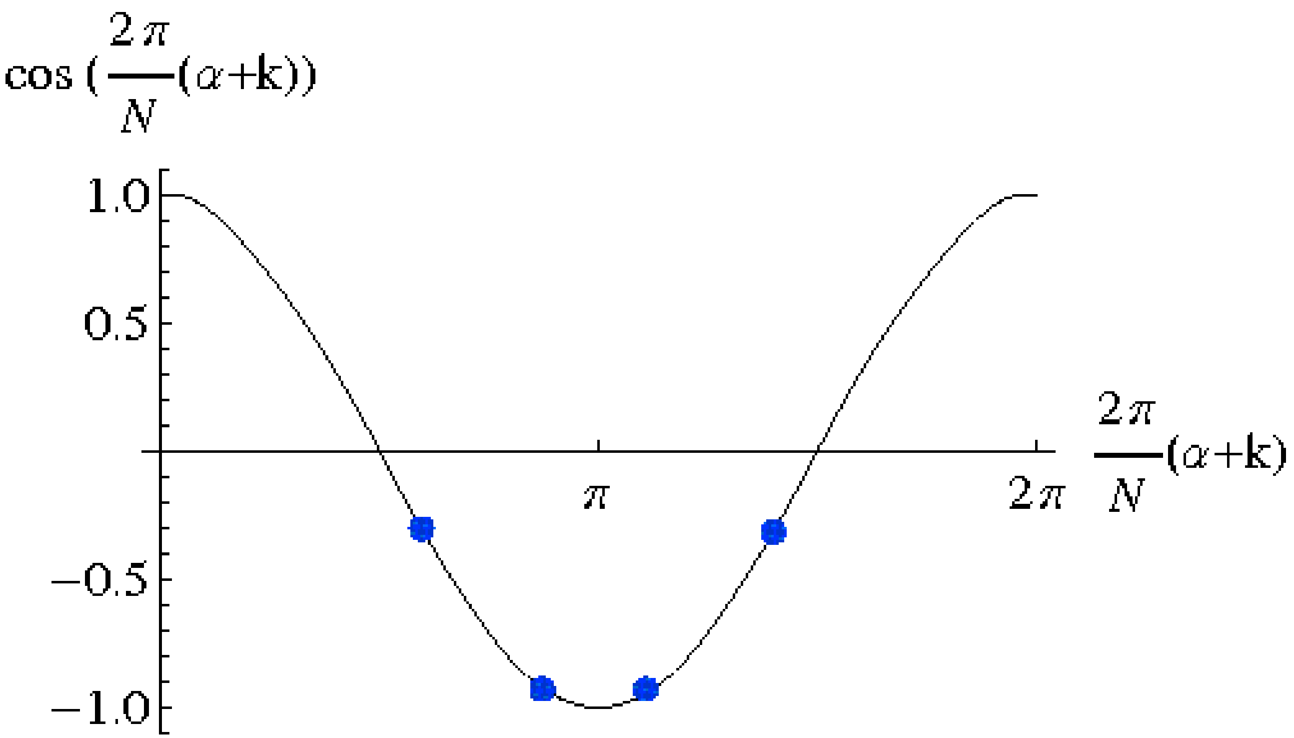

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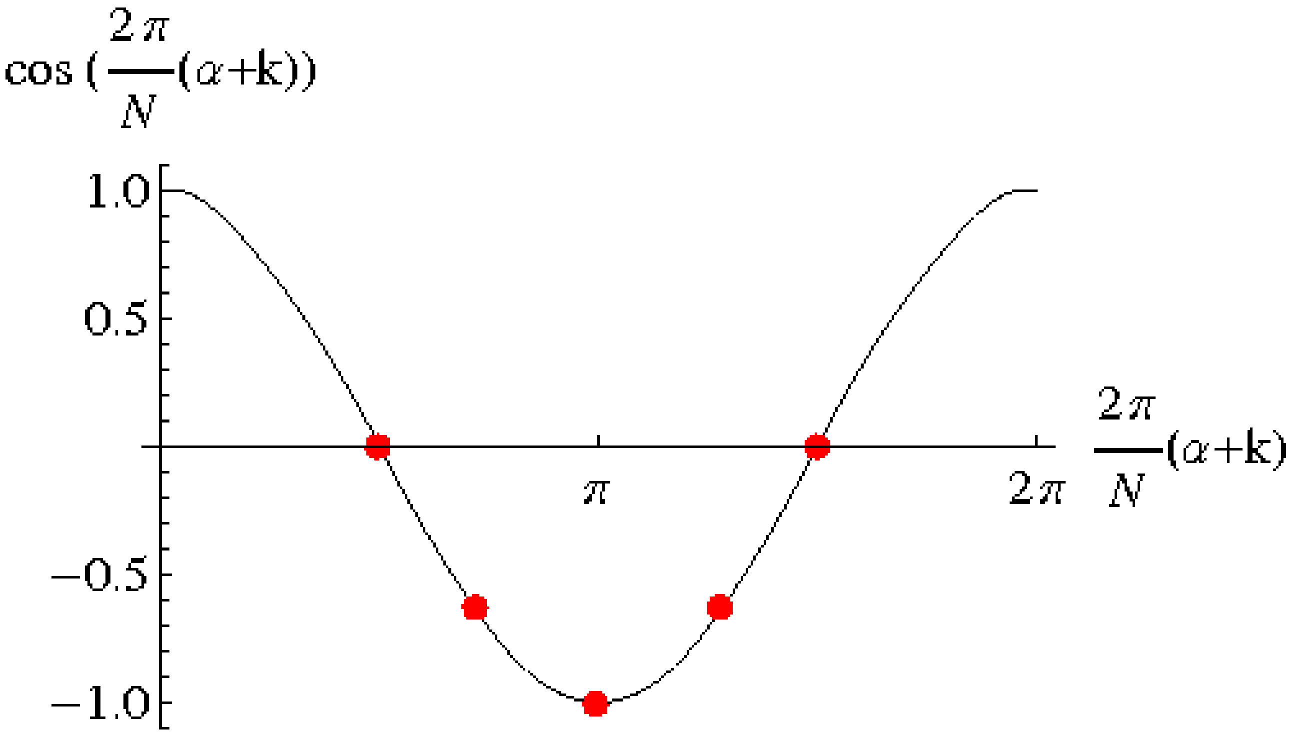

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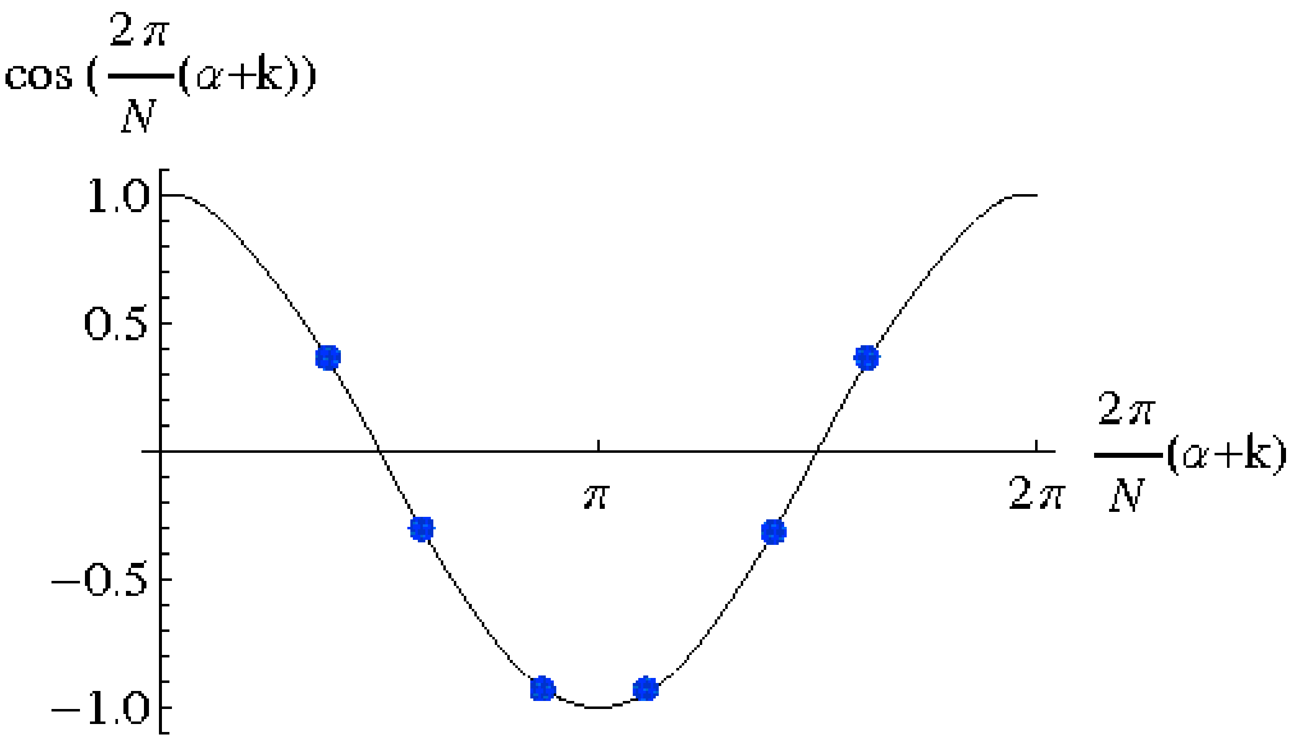

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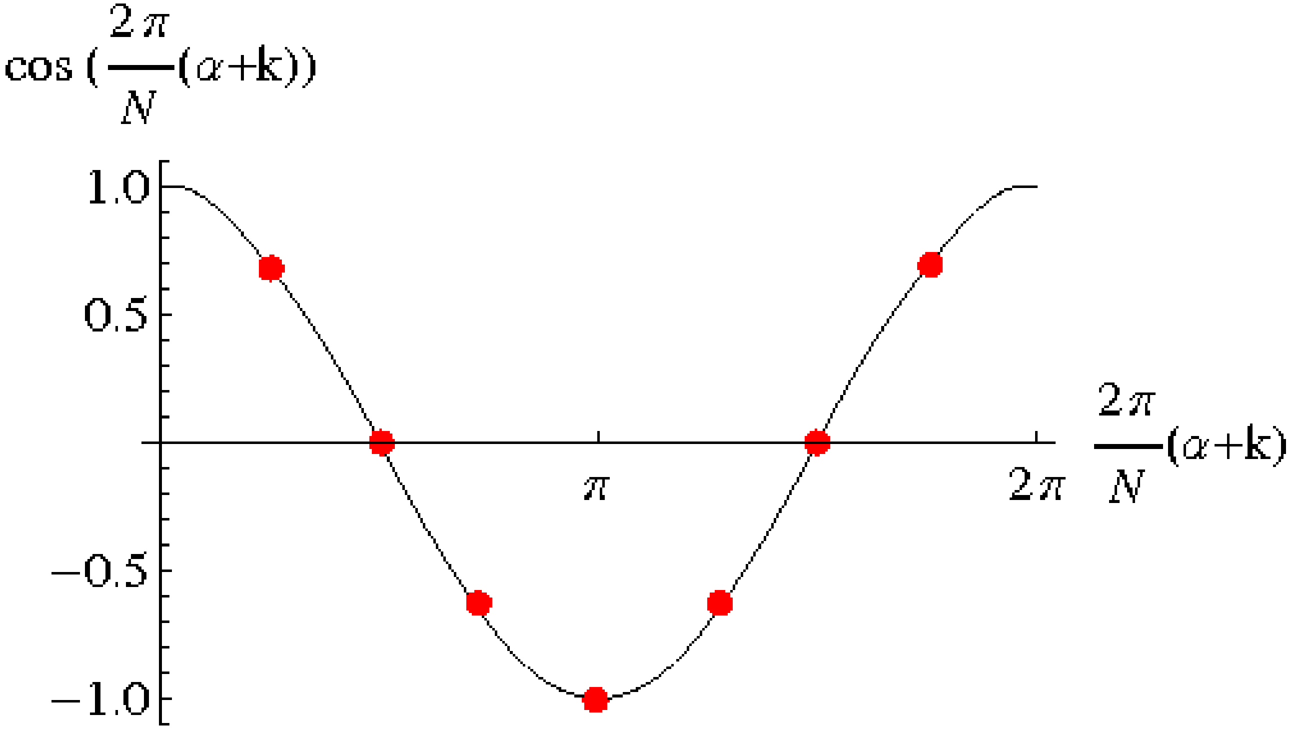
Filling the lowest energy levels



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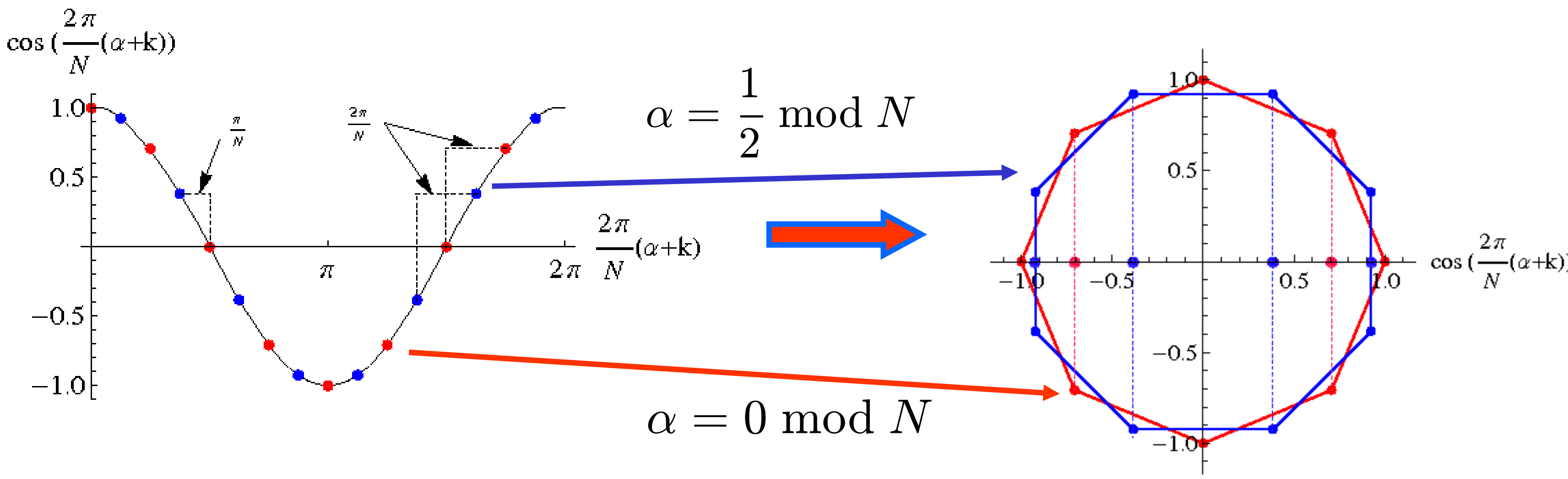
Lowest energy levels

For fixed N (e.g. N=8) one gets:



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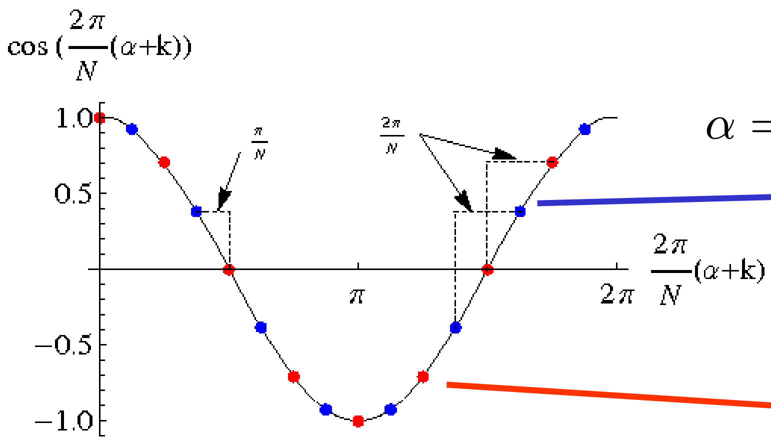

$$\alpha = \frac{1}{2} \bmod N$$

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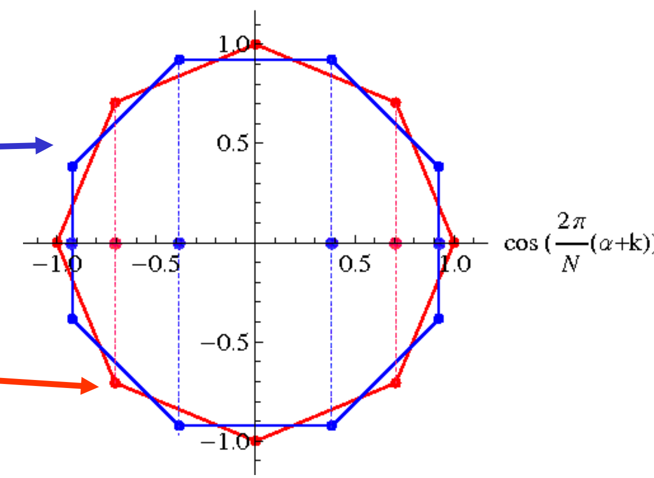


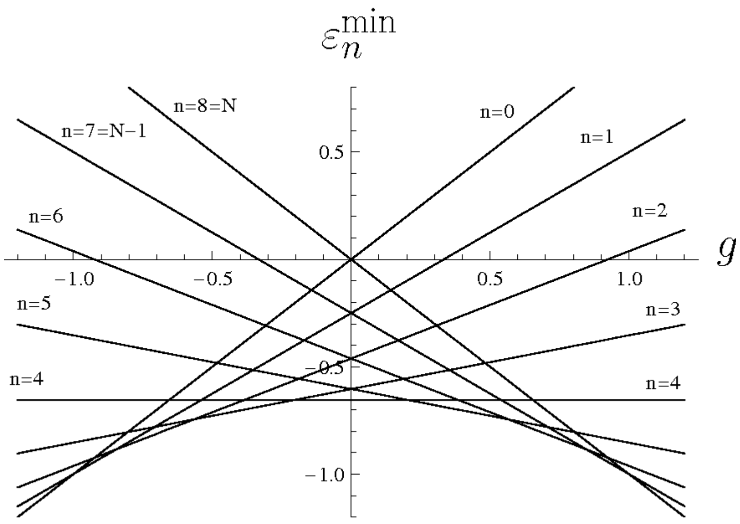
It can be shown that the general expression of the lowest energy levels in the different n-particle sectors (0 $\leq$n$\leq$N) does not depend on the parity of N:

$$\varepsilon_n^{\min}(g) = g\left(1 - 2\frac{n}{N}\right) - \frac{2}{\pi}\frac{\sin\left(\frac{\pi n}{N}\right)}{\chi_N} \quad \textbf{with} \quad \boxed{\chi_N = \frac{\sin\left(\frac{\pi}{N}\right)}{\frac{\pi}{N}}}$$

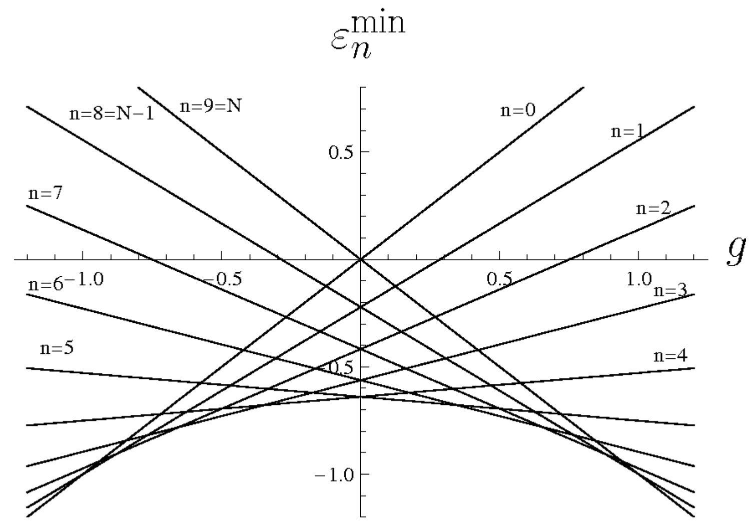
"detects finiteness" ⟵ **FINITE SIZE PARAMETER**

Lowest energy levels

$\boxed{N = 8}$



$\boxed{N = 9}$



It can be shown that the general expression of the lowest energy levels in the different n-particle sectors (0 ≤n ≤N) does not depend on the parity of N:

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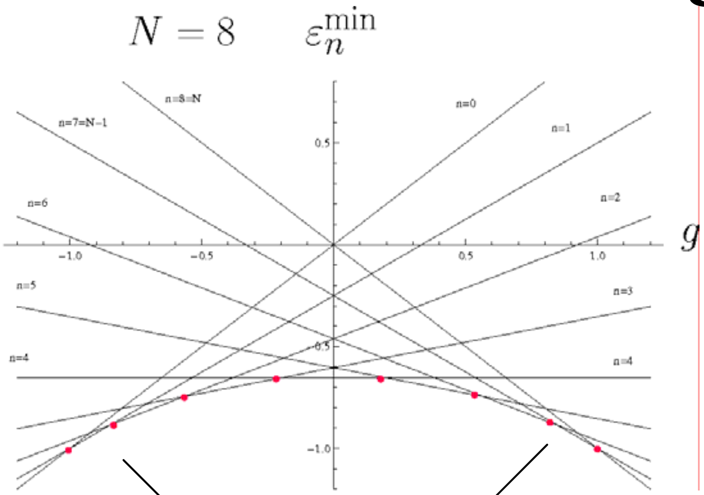
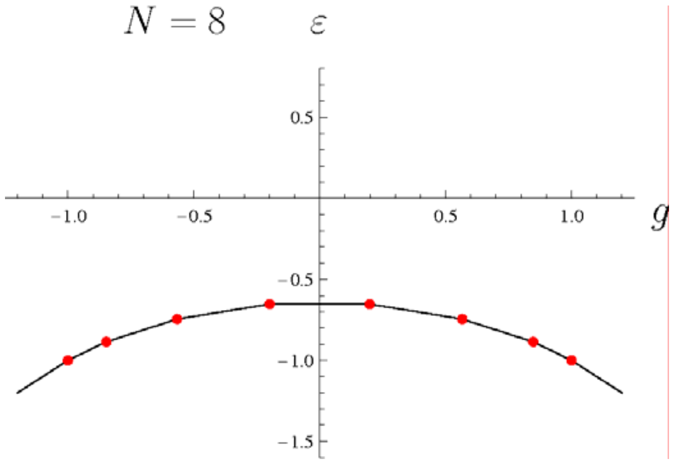
"detects finiteness" ⟵ **FINITE SIZE PARAMETER**

The ground state



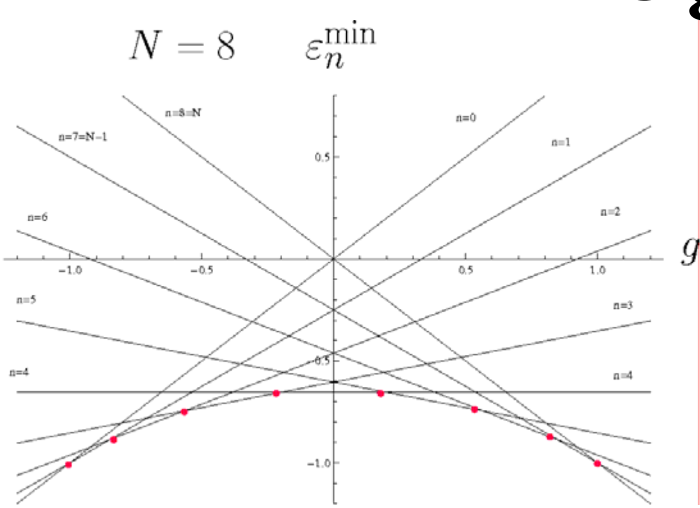


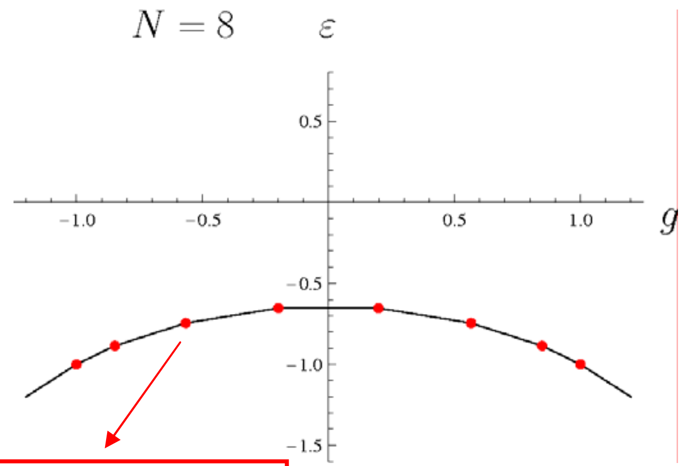
Level Crossing points

"forerunners" of the quantum phase transition (QPT) points in the thermodynamic limit

Ground state

The ground state



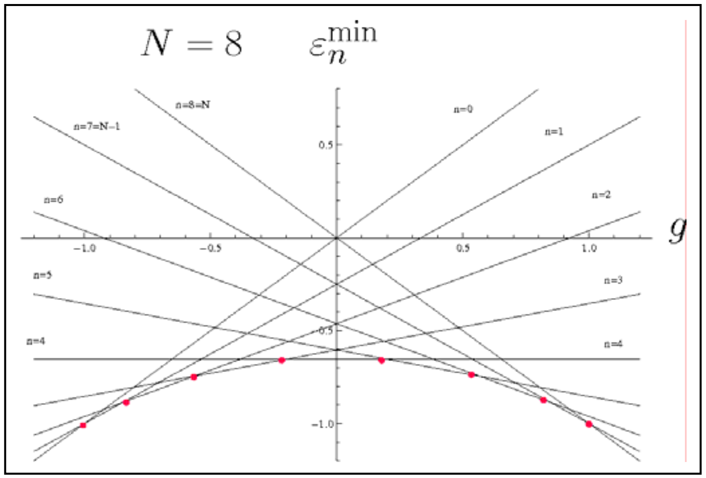


$$\boxed{g_c(n) = \frac{\sin\left(\frac{\pi n}{N}\right) - \sin\left(\frac{\pi(n+1)}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}}$$

$$\varepsilon_{\text{ground state}}(g) = \begin{cases} g & g \leq g_c(0) = -1 \\ g(1-\frac{2}{N}) - \frac{2}{N} & g_c(0) \leq g \leq g_c(1) \\ g(1-\frac{4}{N}) - \frac{4}{N}\cos\left(\frac{\pi}{N}\right) & g_c(1) \leq g \leq g_c(2) \\ \vdots & \vdots \\ g(1-2\frac{n}{N}) - \frac{2}{N}\frac{\sin\left(\frac{\pi n}{N}\right)}{\sin\left(\frac{\pi}{N}\right)} & g_c(n-1) \leq g \leq g_c(n) \\ \vdots & \vdots \\ -g(1-\frac{2}{N}) - \frac{2}{N} & g_c(N-2) \leq g \leq g_c(N-1) \\ -g & 1 = g_c(N-1) \leq g \end{cases}$$

Ground state envelope

The number of LEVEL CROSSING points grows with N.



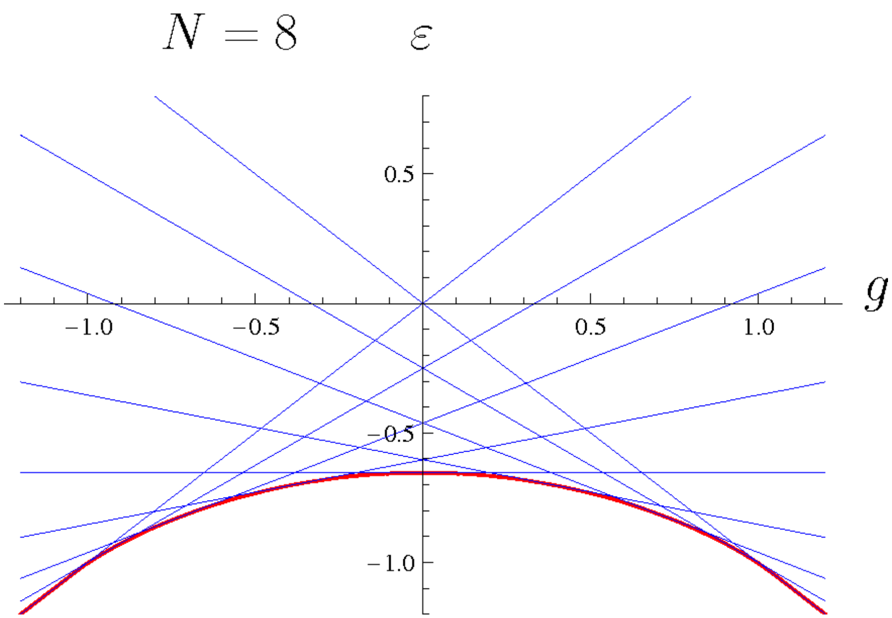
Envelope

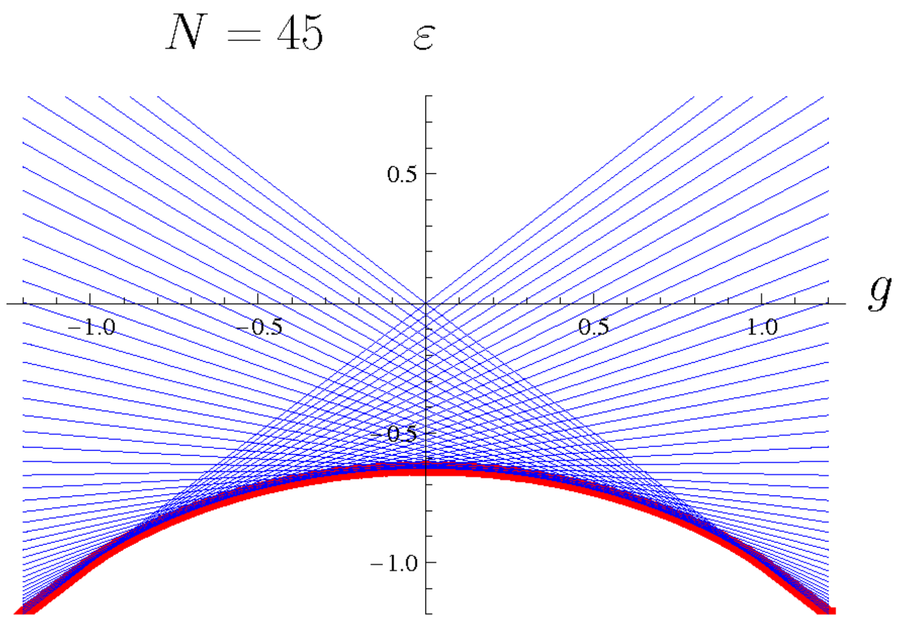
$$\frac{\partial\, \varepsilon_n^{\min}}{\partial\, n} = 0$$

$$\varepsilon_{\text{env}}(g) = \begin{cases} g\left(1 - \dfrac{2}{\pi}\arccos(-g\chi_N)\right) - \dfrac{2}{\pi}\dfrac{\sqrt{1-g^2\chi_N^2}}{\chi_N} & , \quad |g| < \dfrac{1}{\chi_N} \\ -|g| & , \quad |g| > \dfrac{1}{\chi_N} \end{cases}$$

$\chi_N =$ Finite size parameter

Ground state envelope





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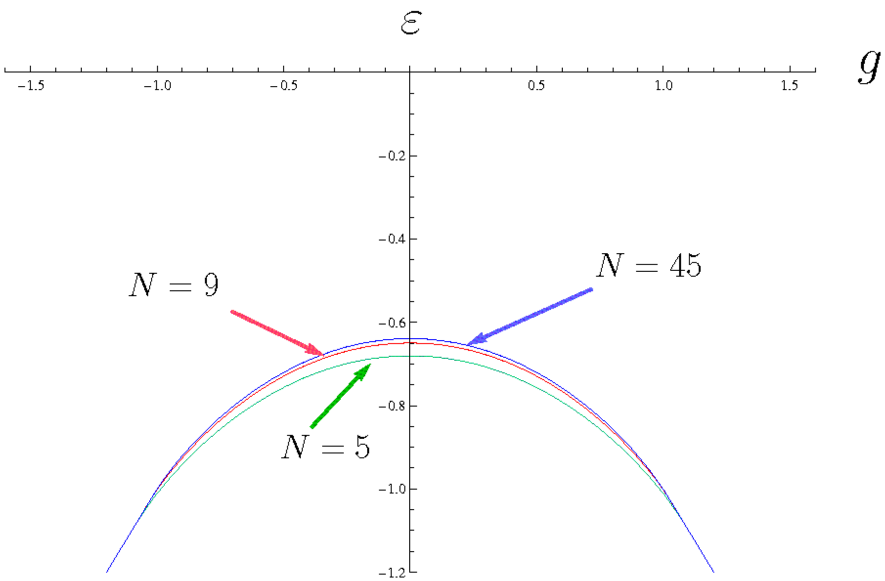
$\boxed{\chi_N = \text{Finite size parameter}}$

Thermodynamical limit

Consider the ground state envelope of systems with different number of sites N



$$\lim_{N\to\infty} \varepsilon_{\text{env}}(g) = \begin{cases} g\left(1 - \dfrac{2}{\pi}\arccos(-g)\right) - \dfrac{2}{\pi}\sqrt{1-g^2} & , \ |g| \le 1 \\ -|g| & , \ |g| \ge 1 \end{cases}$$

Thermodynamical limit

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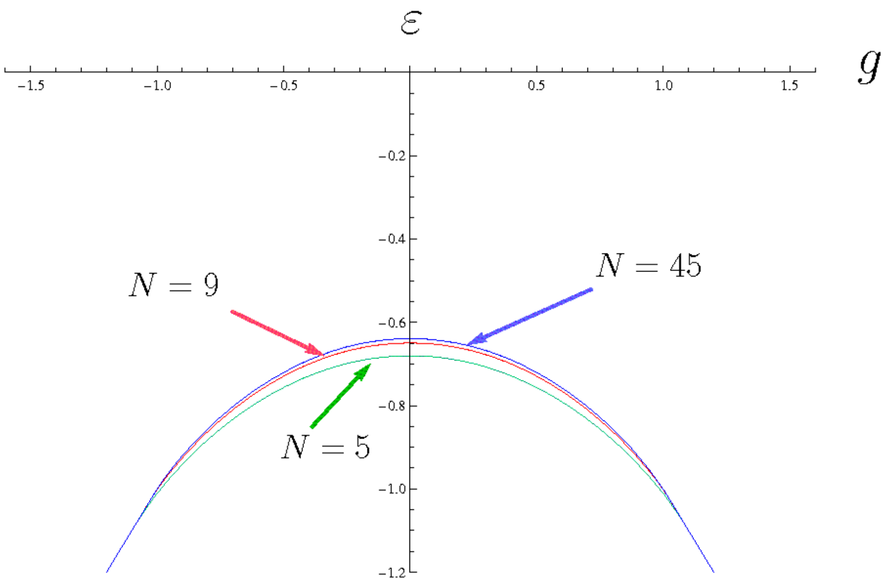
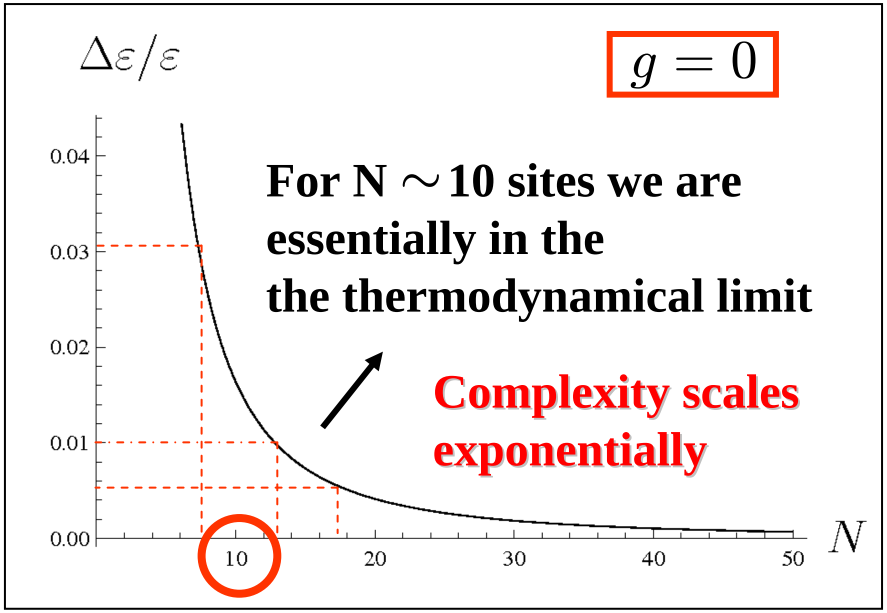




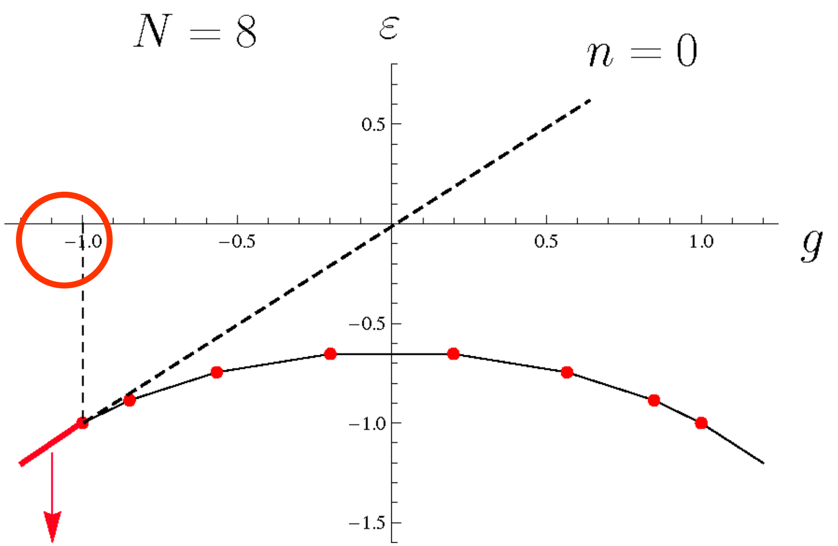
$$\lim_{N\to\infty} \varepsilon_{\text{env}}(g) = \begin{cases} g\left(1 - \frac{2}{\pi}\arccos(-g)\right) - \frac{2}{\pi}\sqrt{1-g^2} & , \ |g| \leq 1 \\ -|g| & , \ |g| \geq 1 \end{cases}$$

Ground state interpreted in terms of spins

$g \leq -1 \Leftrightarrow$ **Strong and negative magnetic field**

ALL SPINS DOWN

$// \vec{B}$



$$|\psi_{\rm vac}\rangle = |\psi\rangle_0 = |\downarrow\rangle_0 |\downarrow\rangle_1 |\downarrow\rangle \ldots |\downarrow\rangle |\downarrow\rangle_N$$

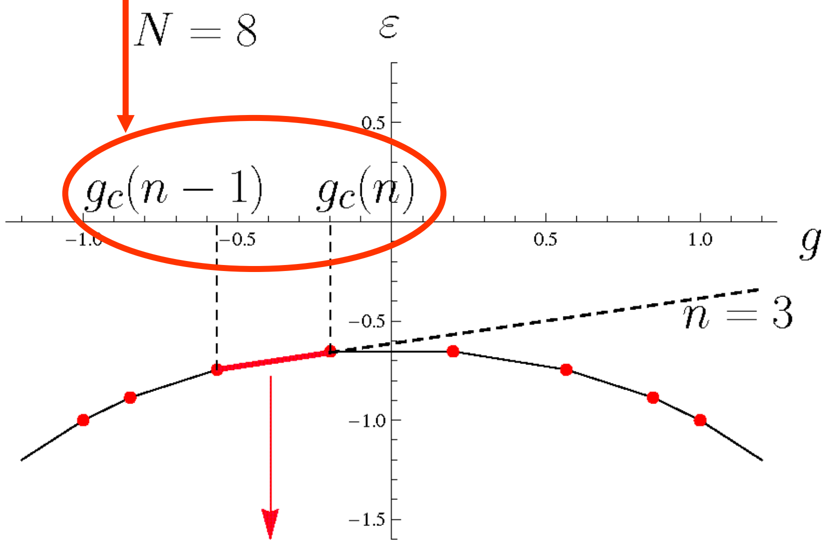
Ground state interpreted in terms of spins

$$g \in [g_c(n-1), g_c(n)]$$

Weaker magnetic fields

SUPERPOSITION OF STATES WITH:

$$\begin{cases} \mathbf{n} \longrightarrow |\downarrow\rangle \\ \mathbf{N\text{-}n} \longrightarrow |\uparrow\rangle \end{cases}$$



$$|\psi\rangle_n = \frac{1}{N} \sum_{j_1<j_2<\ldots<j_n} \Big[\lambda_{j_1,j_2,\ldots,j_n} (-1)^{nj_1}(-1)^{(n-1)(j_2-j_1)}(-1)^{(n-2)(j_3-j_2)} \ldots$$

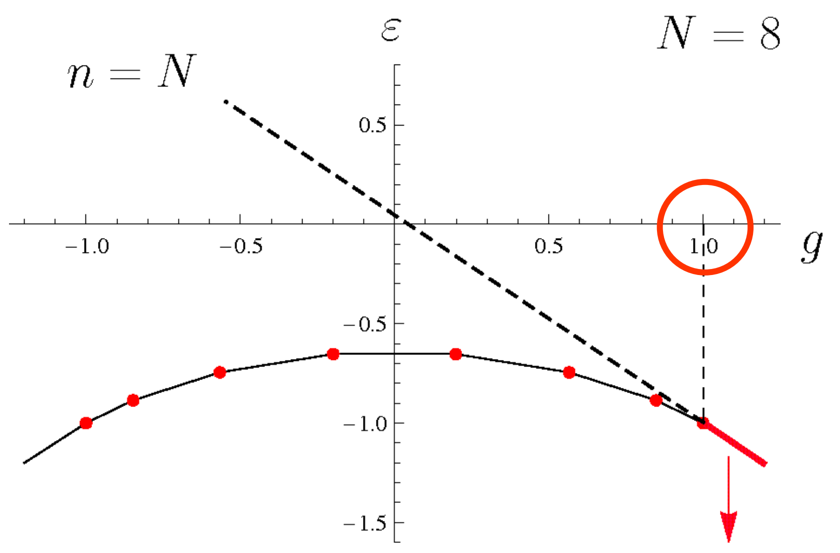
$$(-1)^{j_n-j_{n-1}}\Big] |\downarrow\rangle_0 \ldots |\uparrow\rangle_{j_1} \ldots |\uparrow\rangle_{j_2} \ldots |\uparrow\rangle_{j_n} |\downarrow\rangle_{j_n+1} \ldots |\downarrow\rangle_{N-1}$$

Ground state interpreted in terms of spins

$g \geq 1 \Leftrightarrow$ **Strong and positive magnetic field**

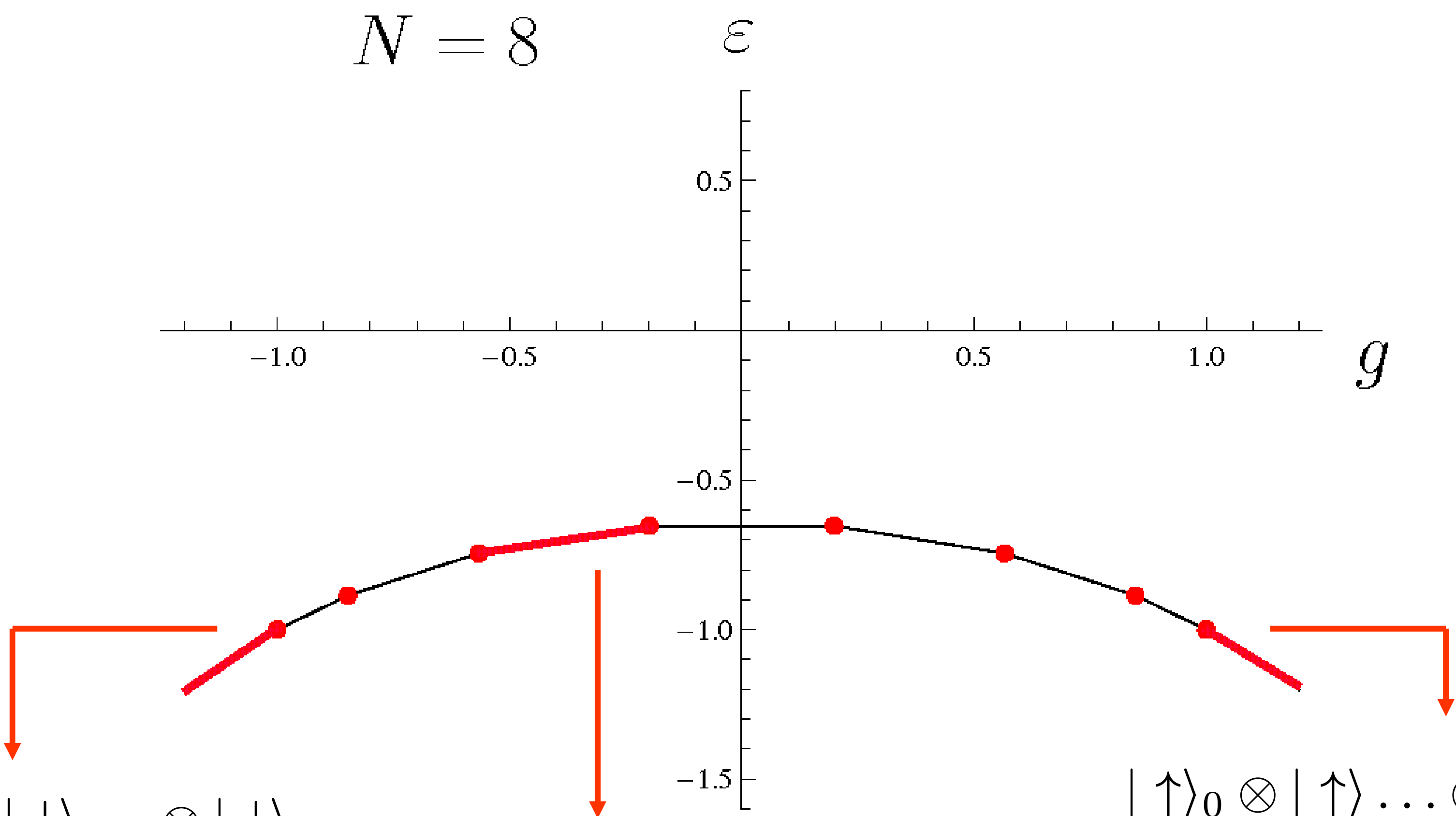
ALL SPINS UP

$// \vec{B}$



$$|\psi\rangle_N = |\uparrow\rangle|\uparrow\rangle \ldots |\uparrow\rangle|\uparrow\rangle$$

Ground state interpreted in terms of spins



$|\downarrow\rangle_0 \otimes |\downarrow\rangle \ldots \otimes |\downarrow\rangle_N$

SEPARABLE state

ENTANGLED state

$|\uparrow\rangle_0 \otimes |\uparrow\rangle \ldots \otimes |\uparrow\rangle_N$

SEPARABLE state

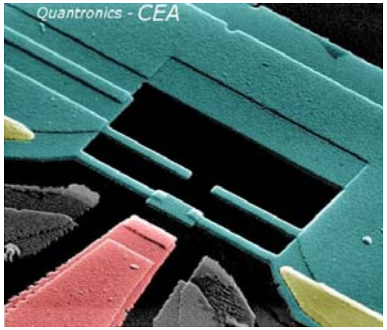
Experimental implementations

- Ion-traps (N=8) – Innsbruck
- Cavity QED (N=5, 6)
- Superconducting circuits – Josephson junctions (N=3 , 4)

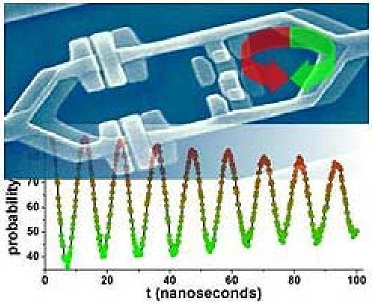
- Delft and Tokyo

-Charge-Qubit

EUROSQIP



- Flux-Qubit



Multipartite entanglement of the ground state

- **Our approach to multipartite entanglement is to analyze the statistical properties of bipartite entanglement over all**

balanced bipartitions

$$\underset{\text{total spins}}{n} = \underset{\text{subsystem A}}{n_A} + \underset{\text{subsystem B}}{n_B} \longrightarrow \begin{cases} n_A = [n/2] \\ n_B = [(n+1)/2] \end{cases}$$

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$$n = n_A + n_B \longrightarrow \begin{array}{l} n_A = [n/2] \\ n_B = [(n+1)/2] \end{array}$$

total spins (n) — **subsystem A** (n_A) — **subsystem B** (n_B)

- **The ground state ($|\psi_{gs}\rangle$) entanglement can be evaluated introducing the purity of the subsystem :**

$$\pi_{AB}(|\psi_{gs}\rangle) = \mathrm{Tr}_A \rho_A^2 = \mathrm{Tr}_B \rho_B^2$$

where:

$$\rho = |\psi_{\mathrm{gs}}\rangle\langle\psi_{\mathrm{gs}}|$$

$$\rho_A = \mathrm{Tr}_B \rho$$

Multipartite entanglement of the ground state

$$\frac{1}{N_A} = 2^{-n_A} \leq \pi_{AB} = \mathrm{Tr}_A \rho_A^2 \leq 1$$

maximally entangled state

pure state

$$\rho_A = \begin{pmatrix} \frac{1}{N_A} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{N_A} & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & \frac{1}{N_A} \end{pmatrix}$$

$$\rho_A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Multipartite entanglement of the ground state

$$\frac{1}{N_A} = 2^{-n_A} \le \underbrace{\pi_{AB}}_{\text{depends on the bipartition}} = \mathrm{Tr}_A \rho_A^2 \le 1$$

depends on the bipartition

the key idea is:

$$\begin{cases} \mu = \langle \pi_{AB} \rangle \\ \sigma^2 = \langle \pi_{AB}^2 \rangle - \langle \pi_{AB} \rangle^2 \end{cases}$$

$\mu = \langle \pi_{AB} \rangle \longrightarrow$ **“ 1/(amount of entanglement)’’**

$\sigma^2 = \langle \pi_{AB}^2 \rangle - \langle \pi_{AB} \rangle^2 \longrightarrow$ **entanglement distribution**

$$\langle \ldots \rangle = \binom{n}{[n/2]}^{-1} \sum_{\text{bipartitions}} \ldots$$

Multipartite entanglement of the ground state

$$\frac{1}{N_A} = 2^{-n_A} \leq \underbrace{\pi_{AB}}_{\text{depends on the bipartition}} = \mathrm{Tr}_A \rho_A^2 \leq 1$$

the key idea is:

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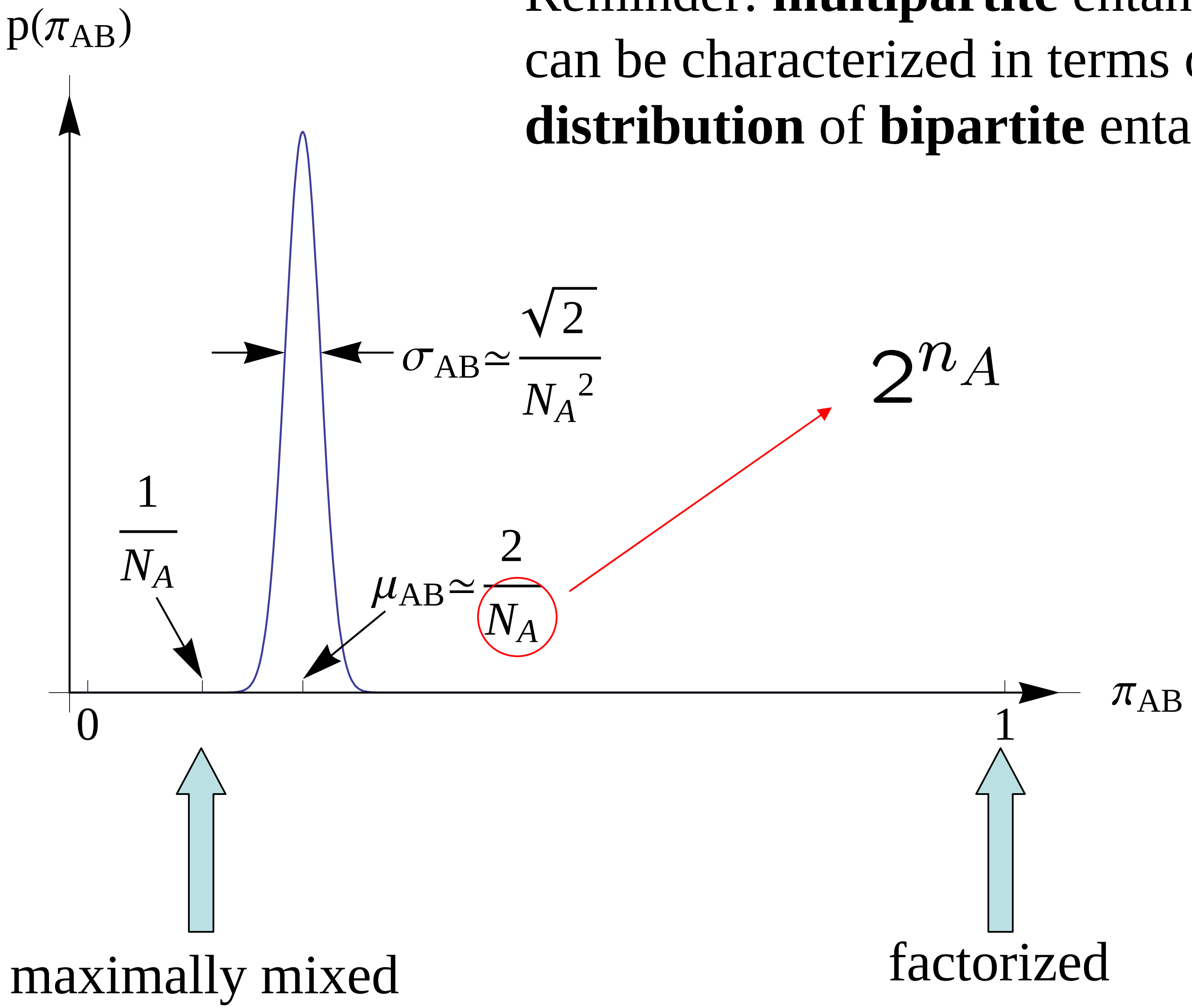
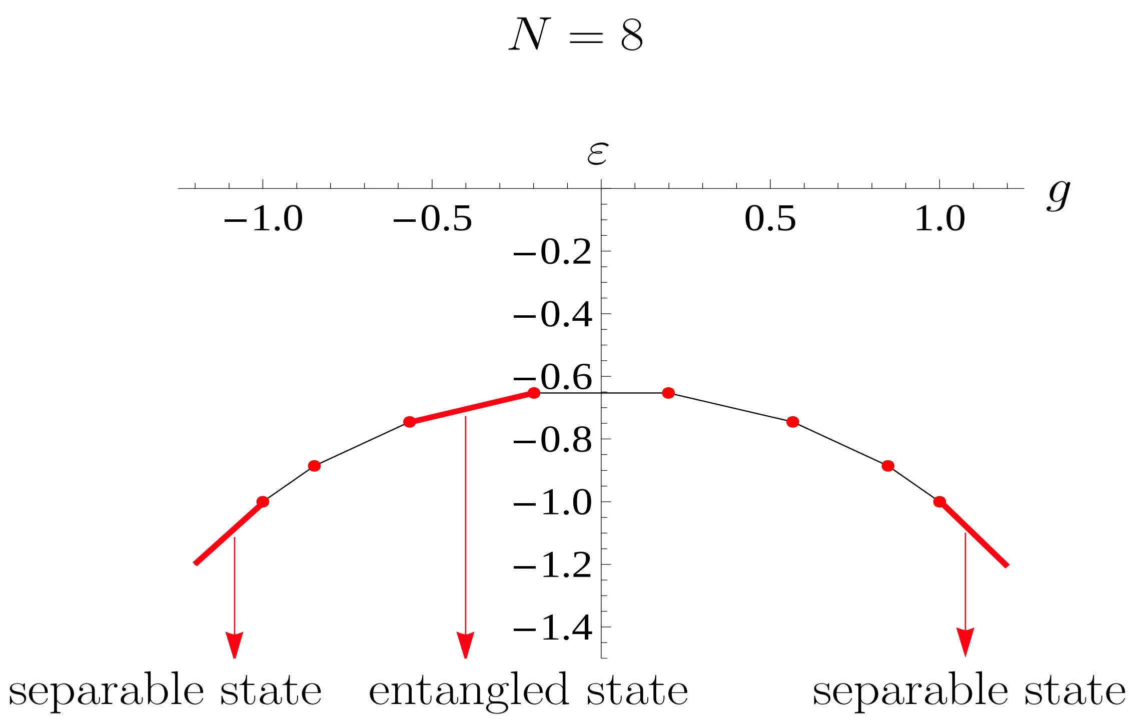
MULTIPARTITE ENTANGLEMENT

A *smaller variance* will correspond to a larger insensitivity to the choice of the bipartition and will witness if *entanglement is really multipartite*

Reminder: **multipartite** entanglement can be characterized in terms of the **distribution** of **bipartite** entanglement

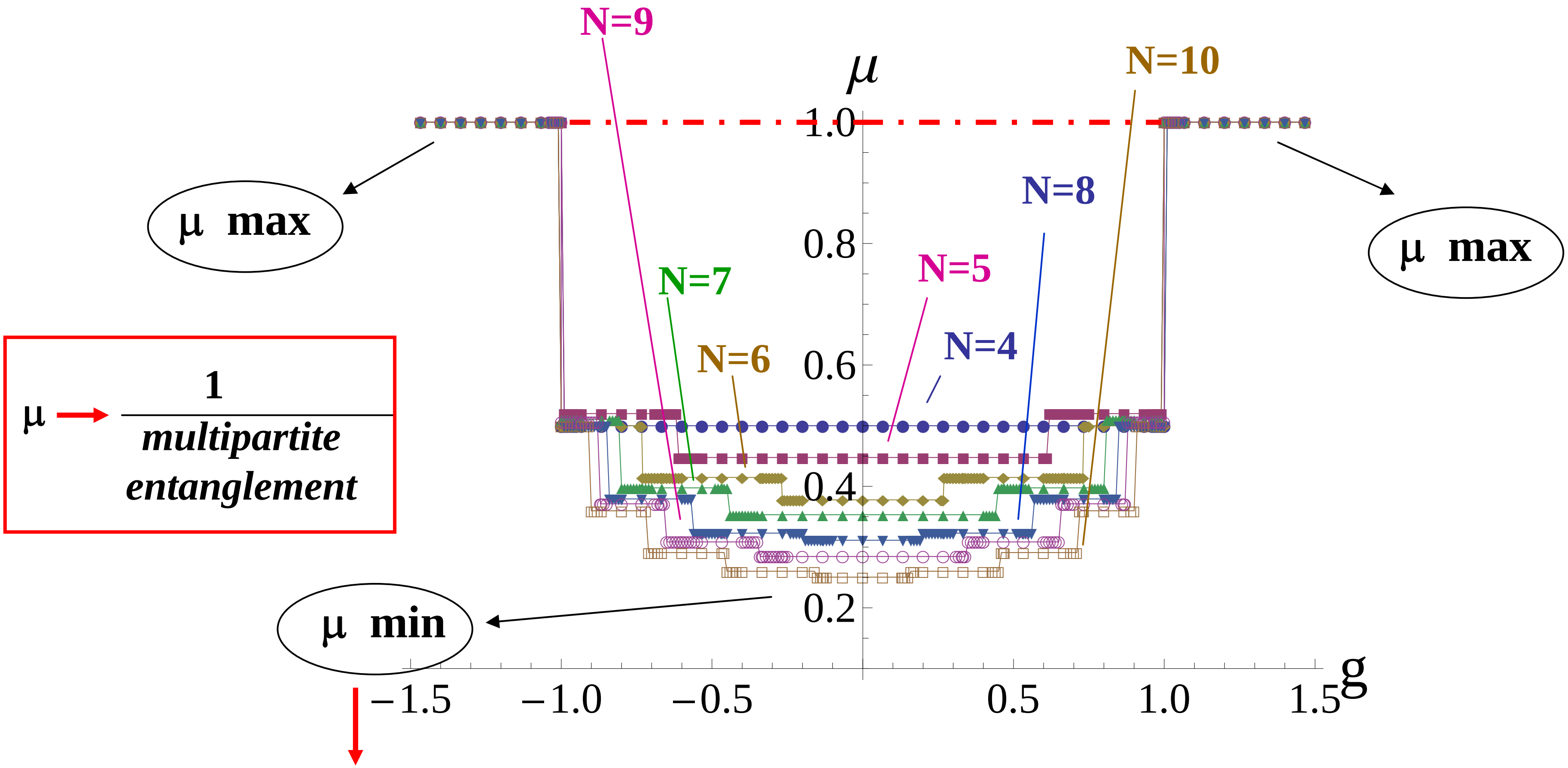


Multipartite entanglement of the ground state



We expect the maximum entanglement is reached at g=0

Multipartite entanglement of the ground state



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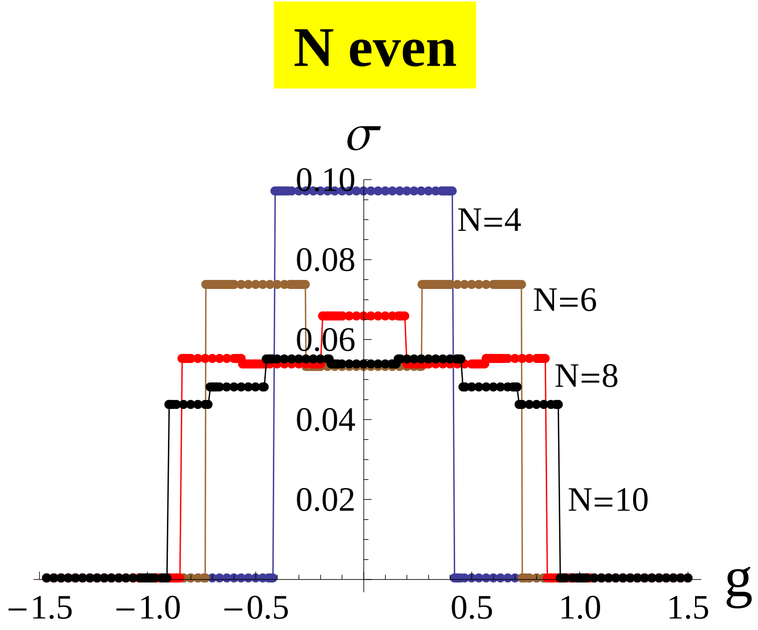
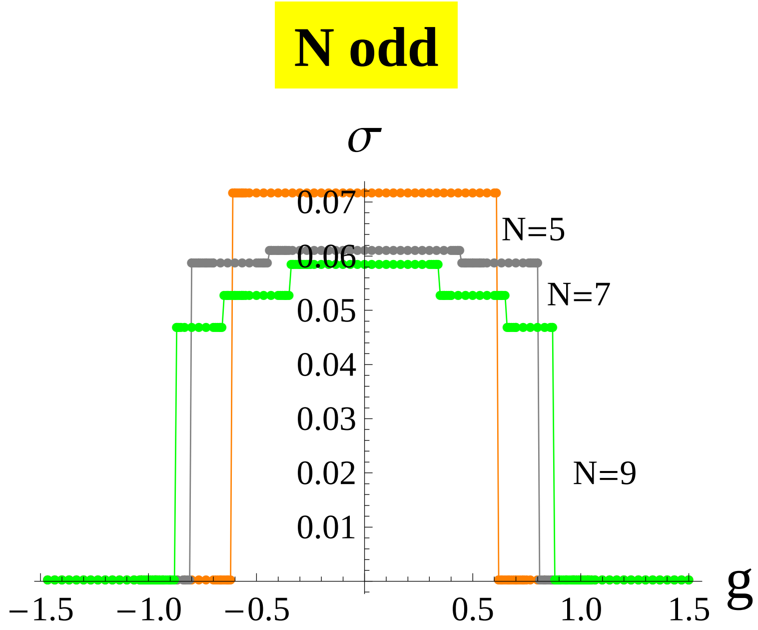
Multipartite entanglement of the ground state





For N even the maximum of σ is not a decreasing function of N. However, as a *general trend* for both even and odd N, σ tends to decrease with N.

Multipartite entanglement and thermodynamical limit

If one defines the relative width at maximum entanglement:

$$\sigma_{\mathrm{rel}} = \frac{\sigma(\mu_{\mathrm{max}})}{\mu_{\mathrm{max}}}$$

there are two possible scenarios in the thermodynamical limit *:

$\sigma_{\mathrm{rel}} \longrightarrow c > 0 \ (\mathrm{eventually} \infty)$**: distribution of entanglement is not optimal**

$\sigma_{\mathrm{rel}} \longrightarrow 0$ **: the entanglement of the ground state is macroscopically insensitive to the choice of the bipartition**

$\downarrow$

possible link between Quantum Phase Transition and Multipartite entanglement

* [*G. Costantini et al. J. Phys. A: Math. Theor.* **40** *(2007)* 8009]