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NUCLEAR PHYSICS A

Quantum Chaos

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Are there quantum signatures, for instance in the spectral properties, of the underlying regular or chaotic nature of the corresponding classical motion? Are there universality classes? Within this framework the merging of two at first sight seemingly disconnected fields, namely random matrix theories (RMT) and quantum chaos (QC), is briefly described. Periodic orbit theory (POT) plays a prominent role. Emphasis is given to compound nucleus resonances and binding energies, whose shell effects are examined from this perspective. Several aspects are illustrated with Riemann's ζ-function, which has become a testing ground for RMT, QC, POT, and their relationship.

1. Introduction

Most of the existing physical theories are expressed in a language more or less directly connected to the one of analytical mechanics. New developments in analytical mechanics are therefore expected to influence many branches of physics. One such development has been the discovery and study of chaotic dynamics, which can be considered to have been initiated by Poincaré at the end of the 19th century and has been vigorously developed during the last century, with important contributions from mathematicians and physicists having very different backgrounds. Chaos and dynamical systems is presently a mature, well developed and active branch of physics and mathematics. On the other hand, for most of the phenomena dealt with in the present conference (nuclear physics), the adequate language is based on quantum mechanics. Are the concepts developed in the theory of chaos relevant in the quantum world? If one takes Bohr's correspondence principle as guiding line, one expects that the answer must be in the affirmative. It is the purpose of this contribution to address this type of question. We will not try to define quantum chaos (QC) as such but rather to search and identify specific properties of quantum systems having a classical counterpart which is chaotic. In other words, our aim will be the search for fingerprints in the quantum regime of the chaotic nature of the corresponding classical system. Primarily we will focus on spectral properties of the quantum system.

It may be useful, with that purpose, to examine the different sequences of points on the real line, of very different physical or mathematical origin, which are plotted in Fig.(1). Each of the six 'spectra' contains 50 'levels'. The spectra have been rescaled to the same spectrum span, or stated otherwise, the mean spacing of different sequences is rescaled to unity. In the figure, (a) corresponds to a sequence of uncorrelated points

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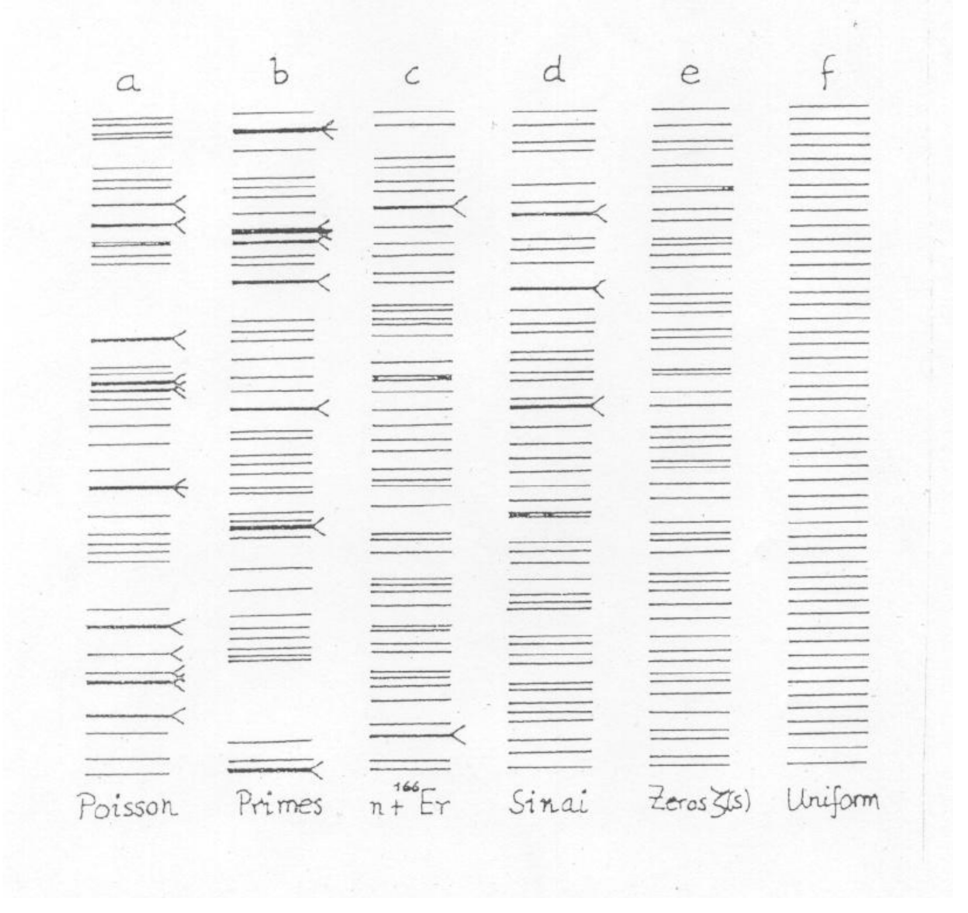

Figure 1. Segments of 'spectra', each containing 50 levels. The arrowheads mark the occurrence of pairs of levels with spacings smaller than 1/4. See text for further explanation (from [1]).

taken at random (Poissonian spectrum); (b) shows an example of a segment containing successive prime numbers in some interval; (c) corresponds to the energy of the compound nucleus resonances observed in the reaction $n +^{166} Er$; (d) displays the eigenvalues of the Laplacian in a plane two-dimensional region with the boundary delimited by a square with an internal disk (Sinai's billiard); (e) shows the position of a stretch of successive zeros on the critical line of the Riemann ζ-function; (f) is a sequence of equally spaced levels (spectrum of the 1-d harmonic oscillator, for instance). Which particular properties are encoded in the statistical behaviour of such sequences? Which are their common or distinct features?

To some extent we will follow the historical sequence of events, thereby underlining that some ideas originated in nuclear physics, specifically random matrix theories (RMT), have subsequently played an important role in the development of QC. Unexpectedly, random matrix ideas and tools have also permeated a particular field of mathematics, namely analytic number theory and the study of the Riemann ζ-function. One goal of the present contribution is to exhibit some of the connections and interactions between RMT, QC and number theory. Some concepts and ideas arising from this perspective will be illustrated in a nuclear physics context. Most of them are by now well established, some are new.

There are books and monographs which cover the different aspects of the material touched here. Concerning RMT theory and their applications let us mention [2], which contains an introduction as well as reprints of important papers prior to 1963, the classical book of Mehta [3], the monographs [4], [5], [7], the nuclear physics oriented review [6]. Concerning QC (as well as connections with RMT) one can consult the books [8], [9], [10], [11], [12], as well as the Proceedings of the Les Houches Lectures [13] and of the Nobel Symposium [14]. Because of the general character of the present contribution, only few original references are given. Extensive bibliography can be found in the books and monographs.

2. Compound Nucleus Resonances, Random Matrix Theories, Zeta-Function

After the end of World War II, military as well as civilian purposes lead to an impressive effort in connection with nuclear fission. It was important, for instance for nuclear reactor purposes, to understand the properties of the compound nucleus resonances. In this general context RMT were introduced in physics by Wigner. The basis of RMT can probably not be stated more succinctly and emphatically than in his own words: '. . . the Hamiltonian which governs the behaviour of a complicated system is a random symmetric matrix with no particular properties except for its symmetric nature'.

The three main ingredients specifying the matrix ensembles are: i) space-time symmetries, ii) isotropy in Hilbert space, iii) statistical independence of matrix elements. i) and ii) are physical requirements, iii) is an (unessential) requirement of mathematical simplicity. The underlying space-time symmetries obeyed by the system put important restrictions on the admissible matrix ensembles. If the Hamiltonian is time-reversal and rotational invariant (the case to which Wigner's quotation corresponds to), the matrices can be chosen real symmetric. If time reversal symmetry does not hold, the matrices are complex hermitean. If the system is time-reversal invariant but not under rotations and with half-integer spin, the matrices are 'quaternion real' (this case corresponds to the presence of Kramer's degeneracies).

The joint probability of eigenvalues $E_1, E_2, \ldots, E_N$ for $N \times N$ matrices is

$$P(E_1, E_2, \ldots, E_N) \sim \exp\left(-\sum_i E_i^2\right) \prod_{i<j} |E_i - E_j|^\beta \tag{1}$$

with β=1, 2 and 4 corresponding to real (Gaussian Orthogonal Ensemble, GOE), hermitean (Gaussian Unitary Ensemble, GUE) and quaternion real (Gaussian Symplectic Ensemble, GSE) matrices respectively. One task accomplished by random matrix theorists, Wigner, Gaudin, Mehta, Dyson,.., was to derive from (1) the n-point correlation functions, spacing distributions, etc. Dyson introduced also ensembles of unitary matrices, the circular ensembles (COE, CUE, CSE). Their eigenvalues are, of course, on the unit circle. However their correlation functions are asymptotically (large N) equivalent to the eigenvalues of the corresponding Gaussian ensembles. As a particularly simple and compact result, as well as for later reference, we give the two-point correlation function R_2 for β=2

$$R_2(s) = 1 - \left(\frac{\sin \pi s}{\pi s}\right)^2 ; \tag{2}$$

in (2) s is the distance between two eigenvalues measured in units of the mean spacing.

As mentioned, RMT were introduced in order to understand fluctuation properties of the compound nucleus resonances. How far has the comparison with data been pushed? The task was difficult because what was ideally needed were pure and long sequences of consecutive resonances (no missing levels, no spurious levels) having the same exact quantum numbers. By including the most reliable neutron resonance data as well as some additional information coming from high resolution proton resonance data, a global analysis coming from 40 nucleides (Nuclear Data Ensemble, NDE) was performed [15], [16]. Examples of some results are given on Figs. (2) and (3).

On Fig. (2) are displayed the nearest-neighbour spacing distribution and the Dyson-Mehta spectral rigidity $\Delta_3(L)$, which measures the least-square deviation of the counting or staircase function $N(E)$, counting the number of levels up to energy E, from the best straight line fitting it in an interval $[E - L/2, E + L/2]$ of length L. On Fig. (3) is tested the consistency of the data with the following random matrix theorem: Take a sequence of eigenvalues corresponding to GOE; the sequence resulting from picking alternate values from it has the statistical properties of GSE. This is particularly interesting because the form factor (Fourier transform of the two-point correlation function) of GSE has a logarithmic singularity at 1 which, though smoothed because of finite sampling effects, is clearly apparent in the data.

These and similar studies have lead to the conclusion that the fluctuation properties of the compound nucleus resonances are consistent to a high accuracy with the predictions. Remember that GOE is a one-parameter theory (the mean spacing, which provides the scale) with no other system-specific (nuclear physics, in this case) or adjustable parameter. This conveys the notion that one is touching here a general, universal, property of quantum systems. In fact, they have also been observed in atoms, molecules, quantum dots in the adequate regime. Let us also mention a first attempt in this direction concerning the

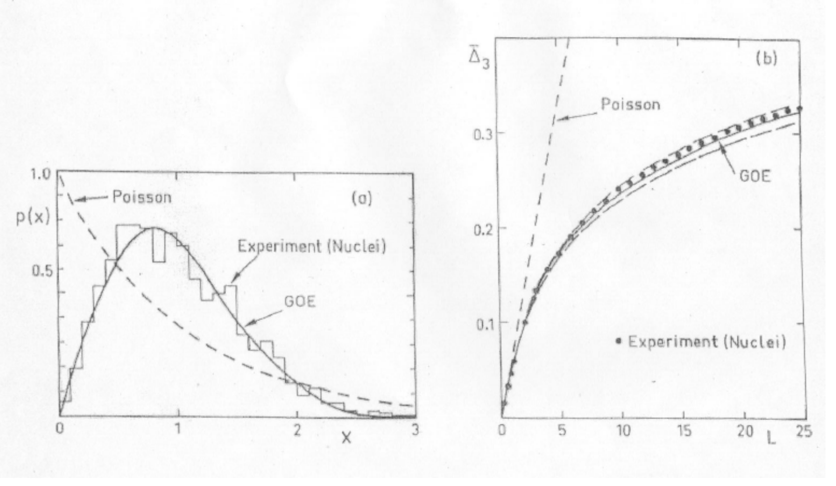

Figure 2. Left: nearest neighbour histogram for the nuclear data ensemble (NDE) [16]. Right: Dyson-Mehta spectral rigidity for the NDE[15]. GOE and Poisson predictions are plotted for the sake of comparison.

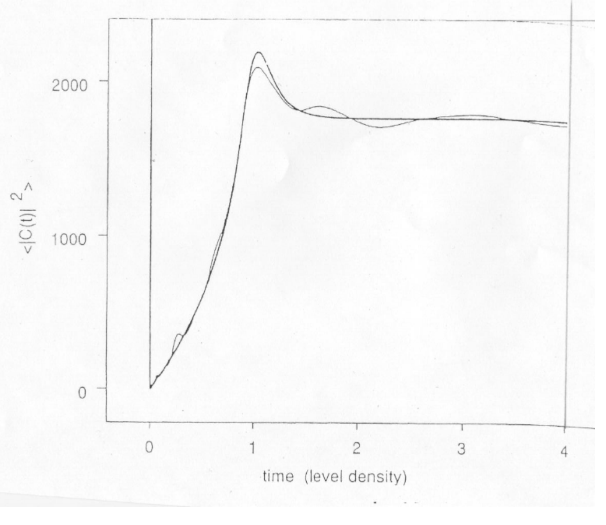

Figure 3. Form factor from the nuclear data ensemble (NDE) omitting each other level compared to the (smoothed) GSE prediction. See text for further explanation (from [17]).

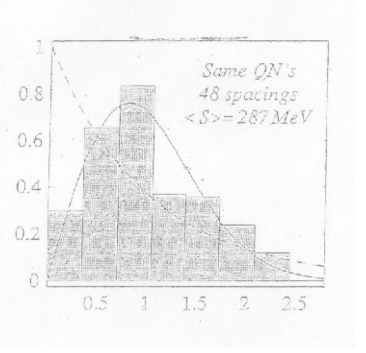

Figure 4. Histograms of the nearest-neighbour mass spacing distribution for hadron (baryon and meson) states with same quantum numbers. Poisson (dashed) and Wigner (solid) distributions, for the sake of comparison (from [18].

spectra of hadrons (see Fig. (4)). The analysis in this case is difficult and not very conclusive, because the available sequences are very short.

Concerning the analysis of data, the very stringent constraint of having a complete sequence may be released, provided that the missing levels are randomly unobserved. It is then possible, and simple, to derive the relevant expressions for analyzing the data as a function of a parameter f $(0 \leq f \leq 1)$ representing the fraction of observed levels ($f = 1$, complete sequence, $f \to 0$, almost all levels unobserved) [19]. This has been recently used to estimate the fraction of missed levels in a high resolution experiment on the fine structure of an isobaric-analog state in ^{93}Tc [20].

We turn now our attention to the announced connection between RMT and number theory. The main role will be played by the Riemann ζ-function. It is a complex function $\zeta(s)$ in the complex plane (see Fig. (5)). In the half-plane $\mathrm{Re}(s) > 1$ it can be defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad n \text{ integer}, \tag{3}$$

and by analytic continuation it can be extended over the whole complex plane. For $\mathrm{Re}(s) > 1$ it admits also the Euler product representation

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \qquad p \text{ prime number}. \tag{4}$$

Comparing (3) and (4) one sees a basic connection between addition (involving integers) and multiplication (involving primes, the building blocks of integers). Gutzwiller, whom we will encounter later again, writes: ‘ $\zeta(s)$ is probably the most challenging and mysterious object of modern mathematics, in spite of its utter simplicity.... The main interest comes from trying to improve the Prime Number Theorem, i.e. getting better estimates

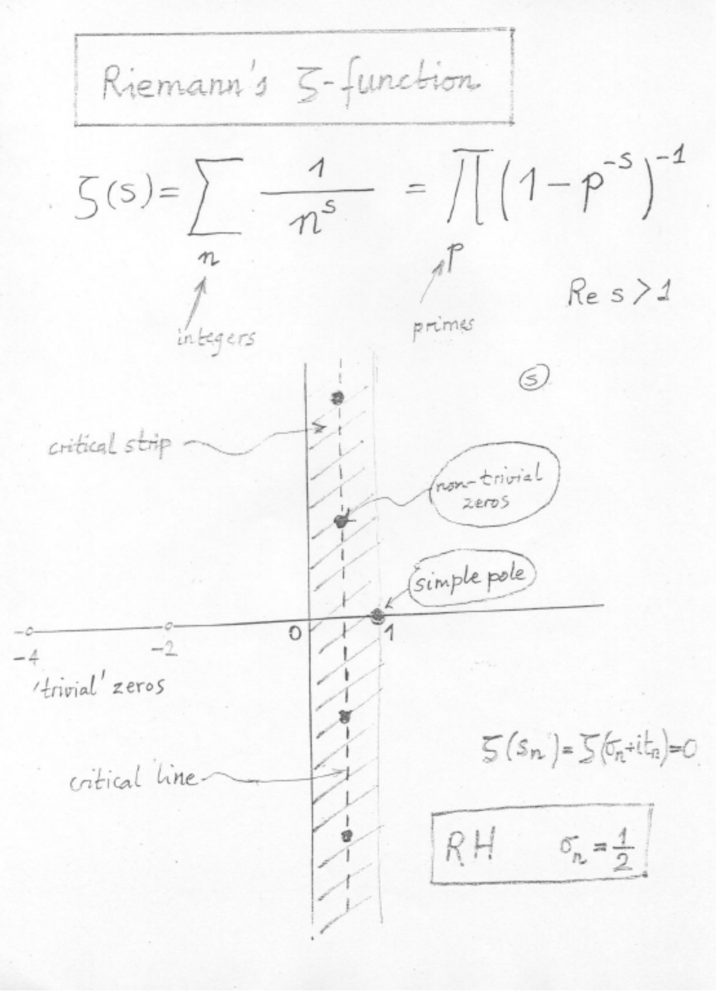

Figure 5. Location of the zeros of Riemann's ζ-function in the complex plane.

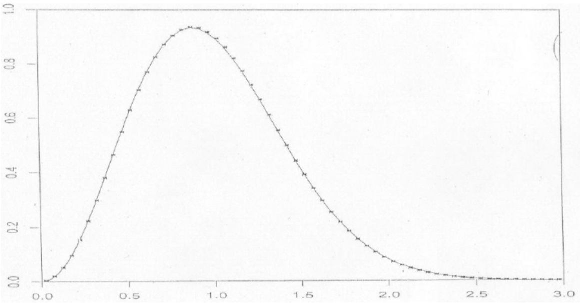

Figure 6. Spacing distribution for Riemann's zeros based on a billion zeros near the $1.3 \cdot 10^{16}$-th zero, compared to the GUE prediction (from [21]).

for the distribution of prime numbers. The secret to the success is assumed to lie in proving a conjecture which Riemann stated in 1859 without much fanfare, and whose proof has since then become the single most desirable achievement for a mathematician'.

What is the conjecture Gutzwiller refers to, usually called Riemann Hypothesis (RH)? It is known that $\zeta(s)$, except for $s = 1$ where it has a pole of order one, is an analytic function. Its zeros are: i) the 'trivial' zeros, at $s = -2, -4, -6, \ldots$, ii) an infinity of 'non-trivial' zeros, which lie in the critical strip $0 < \mathrm{Re}(s) < 1$. The non-trivial zeros have a four-fold symmetry (if z is zero, z^*, $1-z$, $1-z^*$ also are). It is known that at least $2/5$ of the non-trivial zeros are on the critical line $\mathrm{Re}(s) = 1/2$ (of course, if $s = 1/2+it$ is a zero, $1/2-it$ also is). The RH assumes that all non-trivial zeros are on the critical line. Edwards refers to the RH in these terms: 'It is now unquestionably the most celebrated problem in mathematics and it continues to attract the attention of the best mathematicians, not only because it has gone unsolved for so long but also because its solution would probably bring to light new techniques of far-reaching importance'. Mathematicians working in this field produce two kind of results, unconditional and conditional, the former without assuming RH, the latter assuming its validity. Both are fruitful in trying to push the subject forward.

In the early 70's, Montgomery, who was unaware of RMT, derived, assuming RH, the pair correlation function of the non-trivial zeros of ζ. He partly obtained the result Eq. (2). Dyson pointed out immediately to Montgomery that it is the same as for eigenvalues of random hermitean matrices. This observation prompted a systematic comparison between statistics of zeros of ζ and RMT results. The 'experimental' data, supplied mainly by Odlyzko, has been a source of extensive work. One example is displayed on Fig. (6), where a stretch of zeros in a window around the 10^{16}-th zero is examined. The 'empirical' nearest-neighbour spacing distribution $p(s)$ is compared to the asymptotic random matrix result ($\beta = 2$). The agreement seems perfect, at least to a physicist eye (see however Section 4). It may look surprising that it is worth to go so high on the critical line. This is partly due to the fact that one wants to reach asymptotic regimes and this is often approached slowly when dealing with ζ. For instance, the average density of zeros $\bar{\rho}(T)$ at a height T on the critical line increases slowly (logarithmically)

$$\bar{\rho}(T) = \frac{1}{2\pi}\log\frac{T}{2\pi} + O(T^{-2}). \tag{5}$$

One important outcome of this type of study has been the following conjecture (Montgomery-Odlyzko): Asymptotically, Riemann's zeros behave locally as GUE (or CUE) eigenvalues. This is an example of a conjecture based on a conjecture, a conditional conjecture.

3. Searching a Basis for RMT Fluctuations: Chaotic Dynamics

Once the ability of RMT to predict fluctuation properties exhibited by the compound nucleus resonances as well as by other physical systems (and by Riemann's zeros) was established, it still remained to understand in physical terms their origin, their domain of validity and their limitations. Wigner qualified a system to which RMT could be applied as a 'complicated system'. Does this mean a many-particle system with many degrees of freedom, like the atomic nucleus, or could it also mean something else? It will be useful at this point to recall two developments, one related to classical mechanics, the other to quantum mechanics.

Let us start by making a short historical interlude [22], [23]. It contains an important Swedish component, welcomed on the present occasion. Once upon a time there was a King in Sweden, Oscar II, who graduated from Uppsala and who kept strong interest in mathematics. There was also a young mathematician, Gösta Mittag-Leffler, who also graduated from Uppsala, in 1872. After his thesis, he spent some time with Hermite, in Paris, and mostly with Weierstrass, in Berlin. Eventually he came back to Sweden and became professor in Stockholm. Mittag-Leffler convinced the King to give financial support for publishing a mathematical journal of high standards, *Acta Mathematica*, and later to create a prize in mathematics. The work obtaining the award would then be published by the journal. Candidates should submit on important subjects selected by Hermite and Weierstrass, who joined by Mittag-Leffler, would constitute the jury. An announcement was published on July 1885 in *Nature* and four different subjects, or rather mathematical subfields, were proposed. The deadline for the (in principle anonymous) submission was June 1888 and it should be awarded on 21st January 1889, Oscar II's 60-th birthday.

One of the proposed subjects concerned solutions of the n-body problem of point masses interacting via gravitational forces, a central problem since Newton's times and to which many famous mathematicians contributed. The 'difficulties' start already when trying to go from the two-body (Kepler) problem to the three-body problem (sun-earth-moon problem, for instance). Among the important achievements in this field the at that time recent work of Hill on the restricted three-body problem, published in 1877 and reprinted in *Acta Mathematica* in 1886, deserves special mention. It concerns the plane motion of two massive particles describing circular motion and a small third particle. This restricted problem can be reduced to the study of a system of four equations, with the four variables representing the small body's position and momentum in the plane. Probably that Hill's results had some influence in the choice of one of the subjects for the price. Twelve memoirs were submitted and the jury selected the one entitled 'Sur le problème des trois corps et les equations de la dynamique'. Its author was Poincaré, already famous at that time. Hermite and Weierstrass understood that the memoir contained very important and original developments of far reaching consequences, though they could not understand every point. So much so because neither Poincaré really could. Indeed, in the process of preparing the final version to be published by *Acta Mathematica*, Edward Phragmén, a young mathematician from Stockholm who was helping Mittag-Leffler, noticed, during July 1889, a number of obscure points in the original manuscript. This information was transmitted to Poincaré who, in the process of improving it, found that on a particular but crucial point there was an important error. It concerned the behaviour of stable and unstable trajectories, it concerned, in modern language, generic properties of trajectories of chaotic systems. After intense efforts, Poincaré succeeded in putting on a rather firm ground a correct picture, quite different, almost at variance in some essential aspects, from the one given in the original manuscript. And only by the end of 1890 did the final ('official') error-free version appear. The honor of Mittag-Leffler, of *Acta Mathematica* and even of Oscar II was finally preserved!

Poincaré worked further very intensively on the subject. The results have given rise to the classical treatise 'Méthodes Nouvelles de la Mécanique Céleste' (1892-99), which constitutes the basis on which the general field of chaotic dynamics has developed. Let us notice one particular outcome of Poincaré's work which is important in some developments of modern physics, namely the relevance of periodic orbits. In his own words: '... what make the periodic orbits so valuable is that they offer, so to speak, the only opening through which we may try to penetrate into the fortress which has the reputation of being impregnable'. The path covered since Poincaré's time has let to put some problems in celestial mechanics upside-down. Contemporary questions consist in establishing time scales before which some planets undergo chaotic motion, not to prove the stability of the solar system.

For what follows a few results concerning properties of dynamical systems are needed. We will only refer to Hamiltonian conservative systems. Due to the presence of instabilities, long time predictions of individual trajectories, despite the fact that the system is deterministic, are impossible (deterministic chaos). When parameters of the Hamiltonian are varied, the system undergoes changes reflected in the geometrical properties of phase space. For instance, the system may undergo a transition from regular to chaotic motion, the regular situation being characterized by the presence of invariant tori in phase space, the fully chaotic one by their absence. For regular systems the periodic orbits appear in continuous families and their number increases with energy as a power law. In contrast, for chaotic systems, the periodic orbits are isolated and their number proliferates with energy exponentially. The effect of a small perturbation on an integrable system is the content of the Kolmogorov-Arnold-Moser (KAM) theorem, a landmark in the field. Generically, systems show mixed phase spaces: part of the phase space is occupied by invariant tori which are surrounded by chaotic seas. The motion may be very complicated, in fact fully chaotic, even if only few degrees of freedom are present (recall the restricted three-body problem). Specifically, two degrees of freedom are sufficient for conservative Hamiltonian systems to exhibit chaos. On Fig. (7) are displayed two examples, one of regular motion, the other of chaotic motion: a free particle inside a two-dimensional box (billiard) moves

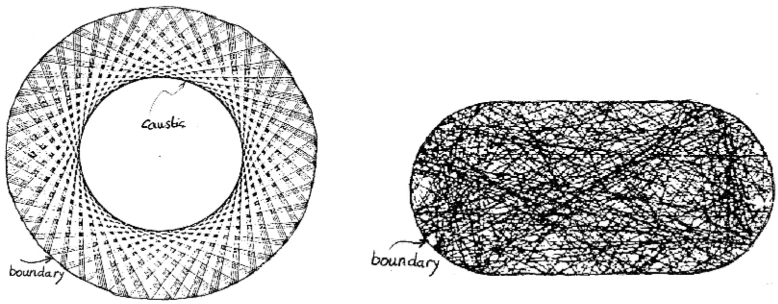

Figure 7. A typical trajectory in a billiard. Left: circular billiard (regular). Right: Bunimovich's stadium (chaotic).

on straight line segments and when hitting the boundary it bounces elastically undergoing a specular reflection.

We shall now revisit the 'complicated' character of a system to which Wigner was referring to, but this time adopting the point of view that complicated means chaotic, a notion that probably he did not have in mind. With this perspective, some 'simple' systems have been investigated, specifically their spectral properties. The example of billiards is particularly suited because they show a large variety of behaviours, going from the most regular (circular billiard, for instance) to the most irregular (hard chaos, for stadium billiard, see Fig.(11) , or Sinai's billiard, see Fig. (1) column (d), for instance).

The spectrum of the Laplacian in an enclosure (Schrödinger operator for free motion with energy $E = k^2$)

$$(\Delta + k^2)\Psi = 0, \tag{6}$$

with Dirichlet boundary conditions, say, is discrete and infinite. It can be obtained analytically for the regular case, numerically for the chaotic one. For the average density of levels one has

$$\bar{\rho}(E) = \frac{S}{4\pi} - \frac{L}{8\pi\sqrt{E}} + \ldots, \tag{7}$$

where S is the surface of the boundary enclosed, L its perimeter, etc. $\bar{\rho}(E)$ is determined by the global geometric properties of the enclosure, irrespective of the regular or chaotic character.

What about fluctuations around the average? Results for regular systems (the circular billiard, for instance) were derived by Berry and Tabor [24]. Over a restricted range, their spectral fluctuations are those of an uncorrelated Poissonian sequence (see Fig. (1), column (a)). Corresponding results for chaotic systems are qualitatively different.

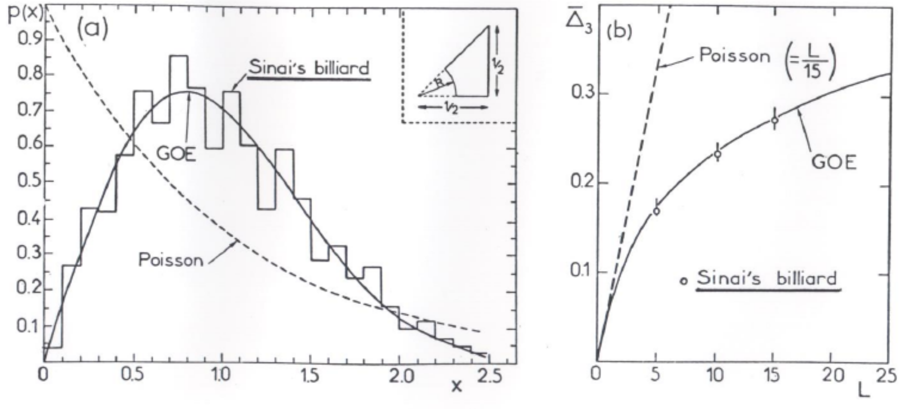

Figure 8. Spectral fluctuations of Sinai's billiard. (a) nearest- neighbour spacing distribution; (b) Dyson-Mehta spectral rigidity. Poisson and GOE predictions for the sake of comparison (from [25]).

Some illustrative examples are displayed on Fig.(8) for a chaotic (Sinai's) billiard. As can be seen, the spectral fluctuations seem to be consistent with GOE predictions, at least in the restricted range investigated. These studies led to conjecture that locally, in the semiclassical limit (large energies), the spectral fluctuations of chaotic systems are described by RMT [25]. This conjecture, or elaborate versions of it, is often referred to in the literature as the random-matrix or BGS conjecture. It has been reinforced by many other numerical studies as well as with increasing theoretical understanding (see next Section).

4. Towards a Dynamical Theory of Quantum Spectral Fluctuations

Gutzwiller, starting in the late 60's, devised a method to deal with the quantization of chaotic systems. It is derived from Feynman's path integral formulation of quantum mechanics and uses the stationary phase approximation. The quantization scheme requires only the knowledge of the properties of the periodic orbits of the corresponding classical system. This approach is often referred to as the periodic orbit theory (POT) [9]. Work on closely related lines was performed independently and almost simultaneously by Balian and Bloch [26]. The physical motivation and the emphasis was however different: Gutzwiller was mainly interested in the quantization of chaotic systems, Balian and Bloch on shell effects, namely strong departures from uniformity of the quantum spectrum, bunching of levels, effects which are mainly connected to regular motion.

The quantum level density can be separated in a smooth part $\bar{\rho}(E)$ and a fluctuating, or more appropriately denoted oscillating part $\rho_{osc}(E)$

$$\rho(E) = \bar{\rho}(E) + \rho_{osc}(E). \tag{8}$$

The oscillating part is given by

$$\rho_{osc}(E) = 2\sum_{p}\sum_{r=1}^{\infty} A_{p,r}(E)\cos[rS_p(E)/\hbar + \pi\mu_{p,r}/2]. \tag{9}$$

The summation is over primitive periodic orbits, labeled by p, r are the repetitions, $\hbar$ is Planck's constant, S_p is the action of the corresponding periodic orbit and $\mu_{p,r}$ its Maslov index related to the number of conjugate points. The amplitudes $A_{p,r}$ have different properties depending on the system being considered, regular or chaotic. Gutzwiller showed that the amplitude in (9) for chaotic systems is given by

$$A_{p,r}(E) = \frac{\tau_p}{2\pi\hbar|\det(M_p^r - I)|^{1/2}}; \tag{10}$$

in (10) τ_p is the period of the primitive orbit and M_p its monodromy matrix, which contains information about its instability. For billiards, for instance, (9) takes the following form

$$\rho_{osc}(E) = \frac{1}{2\pi}\sum_{p}\sum_{r=1}^{\infty}\frac{l_p}{\sinh(r\alpha_p/2)}\cos[r(kl_p + \pi\mu_{p,r}/2)]; \tag{11}$$

in (11) $k(= E^{1/2})$ is the wave number, l_p is the length of the primitive periodic orbit and α_p its instability exponent.

A theoretical connection between RMT fluctuations and those exhibited by (quantum) chaotic systems was found by Berry in the mid 80's [27]. It is based on the Gutzwiller periodic orbit sum. To study the two-point correlation function, for instance, one has to work out the average of the product $\rho(E)\cdot\rho(E+\epsilon)$ or its Fourier transform with respect to ϵ (the spectral form factor $K(\tau)$). It contains a double sum over periodic orbits p and p'. Using the diagonal approximation (only terms $p = p'$ are retained) and the (classical) Hannay-Ozorio sum-rule, which is based on the fact that in chaotic systems the phase-space distribution of very long orbits is uniform on the energy surface irrespective of the details of the system considered, one can not only recover in the semiclassical limit some of the basic results of RMT (linear behaviour at the origin of $K(\tau)$, for instance), but also obtain departures from it, in particular due to the presence of short periodic orbits. The universality of RMT spectral fluctuations reflect the universality properties of long periodic orbits. However, short periodic orbits, which differ from system to system, give rise to long range oscillations in the fluctuations and have characteristic effects in several of the statistical measures considered previously, spectral rigidity for instance. These effects are not contained in RMT, a one parameter theory. For instance, by extending the range of L in Fig. (8), departures from GOE behaviour are predicted and found.

Let us also mention that motivated by these theoretical findings, experimental programs to study 'wave chaos' in microwave cavities have been developed (Stöckmann at Marburg, Sridhar at Boston, Richter at Darmstadt; see for instance their contributions in [14] or [12]). Many results have been analyzed on the light of RMT and POT.

Much effort has been devoted since the seminal work of Berry to go beyond the diagonal approximation. The work of Bogomolny and Keating [28], of Sieber and Richter (see their contribution in [14]) and extensions of it [29], represent recent major advances.

It is a remarkable fact that the properties of Riemann's ζ discussed before also fit in this general scheme. Indeed, if one considers the density ρ of the imaginary part of the non-trivial zeros ($s_\mu = 1/2 + it_\mu$), it is known that

$$\rho(T) = \frac{1}{2\pi}\log\frac{T}{2\pi} + O(T^{-2}) - \frac{1}{\pi}\sum_p\sum_{r=1}^{\infty}\frac{\log p}{r^{r/2}}\cos(Tr\log p), \tag{12}$$

in close analogy with (8), (9) and (10). Notice however that (12) corresponds to the case $\beta = 2$ and that to derive it, in contrast to (8)–(10), no stationary phase approximation is needed. The 'quantum spectrum' of an (unknown) system (the sequence of t_μ's) is related to its 'classical dynamics' specified by quantities related to the prime numbers. A dictionary translating properties of ζ in dynamical terms can therefore be established [30]. This correspondence is illustrated below

Label of periodic orbits	prime number p
Planck constant	$\hbar \to 1$
Symmetry class	$\beta \to 2$
Asymptotic limit	$T \to \infty$
(Asymptotic) density	$\bar\rho \to \log(T/2\pi)/(2\pi)$
Heisenberg time	$\tau_H = h\bar\rho \to \log(T/2\pi)$
Action	$S_p \to T\log(p)$
Period	$\tau_p \to \log(p)$
Lyapounov exponent	$\lambda_p \to 1$
Stability factor	$\lvert\det(M_p^r - 1)\rvert \to p^r$

Some of the methods already used in RMT, QC and POT have been developed by Berry, Keating, Bogomolny, ... for the ζ case. A wealth of results concerning statistical properties of Riemann's zeros, some of them completely unexpected, have followed [31]. Let us give a few examples.

Some statistical properties of prime numbers are well known, others are conjectured. Though prime numbers may appear at first sight as uncorrelated, like a Poissonian sequence (see columns (a) and (b) of Fig. (1)), they are in fact weakly correlated. There is a long standing conjecture (Hardy-Littlewood) on their correlation function. In order to have RMT spectral fluctuations, subtle correlations among classical actions of periodic orbits must be present (see (9)). For the Riemann case, extending some of Montgomery's results, Bogomolny and Keating have shown that if the Hardy-Littlewood conjecture holds ('classical action correlations'), the Montgomery-Odlyzko conjecture referred to above follows ('quantum spectral correlations') [32]. All these are asymptotic results.

Let us now examine how some asymptotic results are approached. On Fig. (6) the empirical nearest-neighbour spacing distribution $p(s)$ was compared to the ($\beta = 2$) RMT asymptotic result. By magnifying the scale and plotting their difference, a well defined structure becomes apparent. Is it also reproduced by RMT? The appropriate ensemble to consider is the CUE. It is known that in this case

$$p_{CUE}(s) = p^0(s) + \frac{1}{N^2}p^1(s) + O(N^{-4}), \tag{13}$$

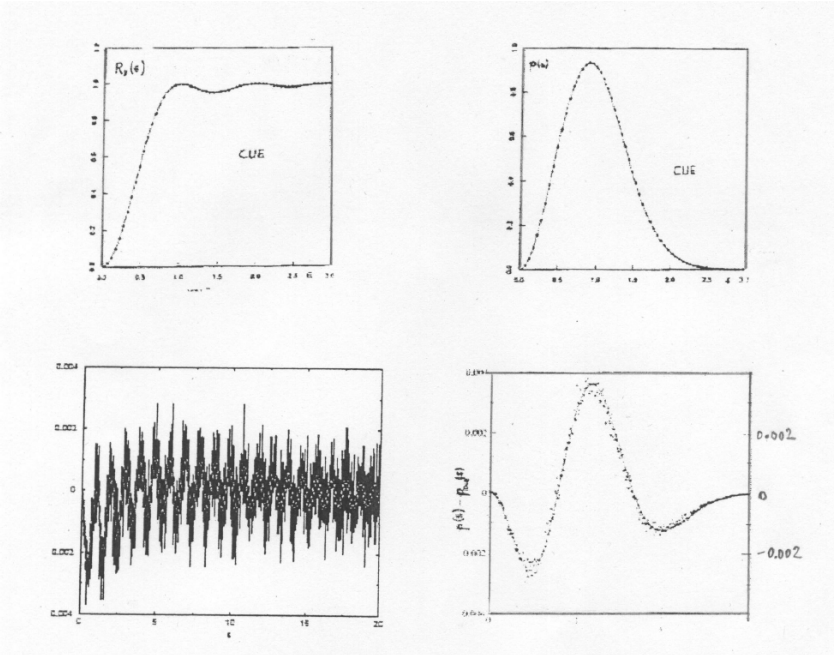

Figure 9. Statistical behaviour of Riemann's zeros. Data from Odlyzko, around the 10^{23}th zero. Upper part: left, two-point correlation function $R_2(\epsilon)$; right, spacing distribution $p(s)$; continuous curves, asymptotic CUE distributions. Lower part: difference between Riemann's zeros values and asymptotic CUE distributions; continuous curves, theory (from [34],[33]).

where N is the matrix dimensionality, with p^0 (the asymptotic distribution) and p^1 known functions. The simplest guess when analyzing ζ-data at a height T is to equate the mean spacings, namely to take $N = N_0$ where

$$\frac{N_0}{2\pi} = \frac{1}{2\pi} \log \frac{T}{2\pi} \tag{14}$$

in (13). It turns out however that this is incorrect. By following POT it can be seen [33] that $p(s)$ is very well approximated by (13) as illustrated on Fig. (9), but with N replaced by $N_0/\sqrt{(12K)}$ where $K = \gamma_0^2 + 2\gamma_1 + c_0 = 1.57314\ldots$, γ_0, γ_1 are Stieltjes constants and

$$c_0 = \sum_p \frac{\log^2 p}{(p-1)^2} = 1.38559\ldots \tag{15}$$

Therefore, though the asymptotics is given by RMT, the approach to asymptotics contains number theoretic information, absent, of course, in RMT. The previous results are derived taking the ones of [28], which go beyond the diagonal approximation, as starting point. The corresponding result (departures from asymptotics for finite T) for the two-point correlation function (Eq. (2)) is also illustrated on Fig. (9). The agreement with theory is excellent.

Detailed, microscopic, features in statistical properties of zeros of ζ can also be exhibited. Consider the autocovariances $C(n)$ of two spacings between consecutive zeros located n levels apart ($n = 1, 2, \ldots$). For uncorrelated spacings, $C(n) = 0$. For eigenvalues of random matrices

$$C(n) = \frac{1}{\beta\pi^2} \log(1 - \frac{1}{n^2}) \qquad \beta = 1, 2, 4. \tag{16}$$

Results are displayed on Fig. (10). Superimposed to the random matrix prediction one observes small amplitude resonant structures. A resonance-type formula can be derived, in which the role of the pole, the trivial and non-trivial zeros of ζ appear. The RMT result, Eq. (16), is due to the pole. Successive zeros give rise to resonances (harmonics) and subharmonics. We therefore see that , when properly examined, even at a height around the 10^{12}-th zero on the critical line, the first, second,.. zeros at 14.1347.., 21.0220.., .. are lurking (resurgences).

There are also asymptotic properties for which RMT don't give the leading order result. This is because they are not local properties. Take, for instance, the imaginary part of the non-trivial zeros as the single particle energies of a Fermi gas. The ground state energy Ω is obtained by filling the single-particle states from the lowest Riemann zero up to a 'Fermi energy' T. Following nuclear physics terminology, we may refer to this 'element' as the 'Riemannium' [36]. The total energy can, as usual, be decomposed in a smooth part $\bar{\Omega}$

$$\bar{\Omega}(T) = -\frac{T^2}{4\pi} \log \frac{T}{2\pi} + \frac{3}{8\pi} T^2 + \ldots \tag{17}$$

and an oscillating part $\tilde{\Omega}$. It can be shown: i) that there is a well defined distribution $P(\tilde{\Omega})$ as $T \to \infty$ (see Fig.(11)), ii) $P(\tilde{\Omega})$ is asymmetric, with a strange-looking shape, iii) the moments of $P(\tilde{\Omega})$ may be computed with very good accuracy from POT, their value being dominated by the contribution of the lowest prime numbers (shortest periodic orbits). For instance

$$< \tilde{\Omega}^2 >= \frac{1}{2\pi^2} \sum_p \sum_{r=1}^{\infty} \frac{1}{r^4 p^r \log^2 p} \approx 7.9 \times 10^{-2}. \tag{18}$$

In contrast, a RMT estimate of $P(\tilde{\Omega})$ would give a gaussian distribution, qualitatively different from the actual result.

As a last example, let us mention the work of Keating and Snaith [37] concerning moments $M(\lambda)$ of the absolute value of ζ on the critical line

$$M(\lambda) = \lim_{T\to\infty} \frac{1}{(\log T)^{\lambda^2}} \frac{1}{T} \int_0^T |\zeta(\frac{1}{2} + it)|^{2\lambda} dt, \tag{19}$$

a quantity of current interest in this field since almost hundred years. Number theoretic arguments suggest a decomposition of $M(\lambda)$ in two factors $f(\lambda) \cdot a(\lambda)$. The random matrix analogy suggest that $f(\lambda)$ could be estimated from

$$f_{CUE}(\lambda) \equiv \lim_{N\to\infty} \langle |Z(\theta)|^{2\lambda} \rangle_{U(N)} \tag{20}$$

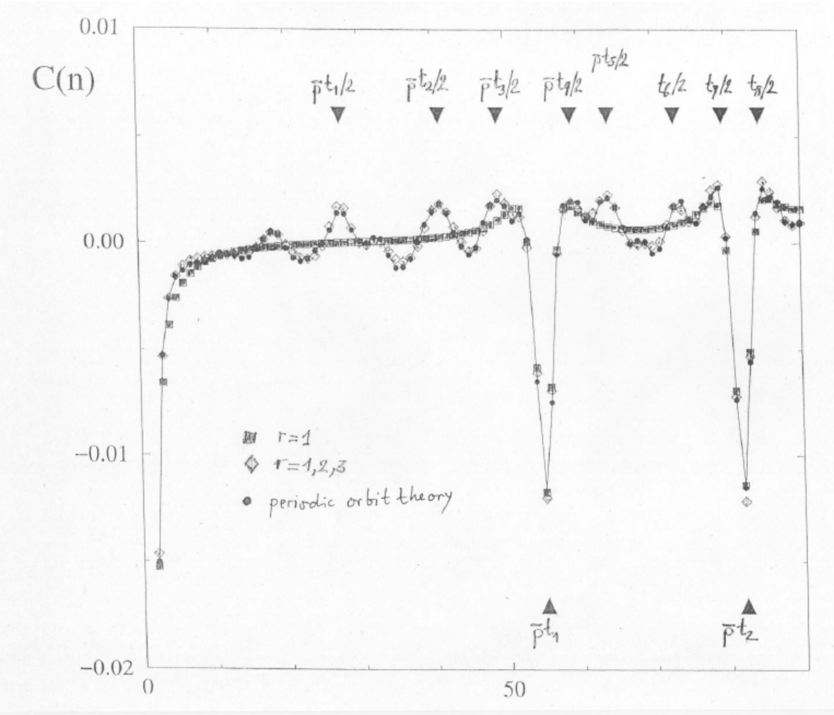

Figure 10. Spacing autocovariances $C(n)$ for the Riemann zeros around the 10^{12}th zero ($\bar{\rho}$=3.8953..). Results obtained from the resonance formula with one (squares) and three (diamonds) terms are compared to the full calculation (open circles). Arrows in the lower (upper) part indicate the rescaled position $\rho \cdot \tau_\mu$ of the first two ($\rho \cdot \tau_\mu/2$ of the first eight) resonances (subresonances), respectively (from [35]).

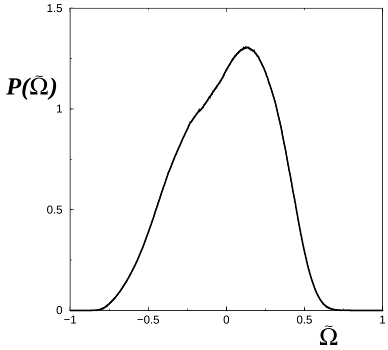

Figure 11. Statistics of Riemann's zeros. Distribution $P(\tilde{\Omega})$ computed numerically at a height $T \approx 1.44 \cdot 10^{20}$ on the critical line, from data supplied by A.Odlyzko (from [36]).

where $Z(\theta)$ is the characteristic polynomial of a random CUE matrix. The calculation of (20) is straightforward and gives

$$f(\lambda) = \prod_{n=0}^{\lambda-1} \frac{n!}{(\lambda+n)!}. \tag{21}$$

Eq. (21) gives: for $\lambda = 1$ and 2, 1 and 1/2, known as theorems since 1919 and 1926 respectively; for $\lambda = 3$ and 4 it gives 42/9! and 24024/16!, obtained as conjectures, via complicated and subtle derivations, since 1992 and 1998 respectively. A conclusion seems inescapable: when properly used RMT are able, in some cases, to guess (presumably) exact results previously unknown from mathematicians.

This excursion in the ζ-function realm had two purposes: i) to illustrate that ζ has become a testing ground on which ideas and approximations from QC, RMT and POT can be investigated in detail (there is no problem of scarcity of data nor of enumeration of periodic orbits), ii) to illustrate that tools developed in RMT, QC and POT may bring new ideas and may provide advances in some number theoretic problems. In particular, these developments reinforce the indications on which many researchers base their belief that the Hilbert-Polya conjecture (the RH is true because the imaginary part of the zeros of ζ correspond to the eigenvalues of a hermitean operator) is the most promising approach to prove the RH.

5. Binding Energies: Regular (and Chaotic) Dynamics

In this Section, which is largely based on the work of Leboeuf and collaborators on thermodynamics of small Fermi systems and its fluctuations [38], [39], [40], we shall discuss binding energies of many-fermion systems at the light of the preceding discussion (see also [41]). Till now, we have discussed properties of highly excited states (compound nucleus resonances, for instance) whereas now we will focus on ground state properties. Again, the emphasis will be on the interplay between regular and chaotic motion, between universal and system-specific properties, and POT and RMT will be the main theoretical tool.

That POT can give deep physical insight can be illustrated, almost as a textbook example, by considering properties of small metallic particles. The simplest model is the one of non-interacting electrons in a spherical cavity. One expresses the total energy Ω in terms of the single-particle density $\rho(E)$ (see Eqs.(8),(9)), and a sum over periodic orbits results. Superimposed to a smooth variation of $\bar{\Omega}$ with respect to the number of particles, there are quantum oscillations $\tilde{\Omega}$ (shell effects). As we know, to a spherical cavity corresponds regular motion, which, in turn, gives rise to the occurrence of families of orbits. The importance of their contribution decreases rapidly with their length. The shortest (contributing) periodic orbits for a spherical cavity are the ones corresponding to the triangular and square families. They have a similar length. One can then easily see that a beating pattern (supershells) should result (predicted by Balian and Bloch [26], worked out in detail in [42] and found experimentally), on which the more familiar shell oscillations are superimposed (see Fig. 12). The simplest model is, in this case, almost quantitatively in agreement with observation (see [41]).

Let us also mention that shell effects have recently been observed in a different context, namely in metallic nanowires. Gold and alkali nanowires, for instance, have a remarkable property of being able to support extremely high current densities, which would vaporize a macroscopic wire, and magic conductance values are observed [43].

Shell effects in nuclei, of course, is a very old subject intensively studied for more than fifty years. Here we are interested in shell effects associated to binding energies. From the experimental point of view (see contribution of C.Scheidenberger to this conference), the determination of nuclear binding energies has been making impressive uninterrupted progress since its origin, with an improvement of roughly one order of magnitude in precision every ten years. The last breakthrough is due to the advent of ion traps. Comparatively, the rate of progress on the theoretical side has been deceptively slow and in the foreseeable future theory will remain very far from the precision attained by experiments. However this does not mean that physical insight cannot be gained from theoretical studies.

Let us start with the simplest computation of shell effects [40]. Again, the total energy Ω of a nucleus is decomposed in a smooth part $\bar{\Omega}$ and an oscillating part $\tilde{\Omega}$:

$$\Omega(A,x) = \bar{\Omega}(A,x) + \tilde{\Omega}(A,x), \tag{22}$$

where x denotes, for instance, shape parameters. $\bar{\Omega}$ can be described by the liquid drop model (Bethe-Weizsäcker). In one of its simplest versions, it reads

$$\bar{\Omega} = a_v A - a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_A \frac{(N-Z)^2}{A} - a_p \frac{t_1}{A^{1/2}}, \tag{23}$$

with a_v, a_s, a_c, a_A, a_p associated to volume , surface, Coulomb, symmetry, pairing energies, respectively. Shell effects, namely $\tilde{\Omega}$, are computed using POT. Consider the nucleons as moving freely in a spheroidal cavity, whose shape is determined, as usual, by minimizing the energy. Results are displayed on Fig.13. Despite the rusticity of the model (in

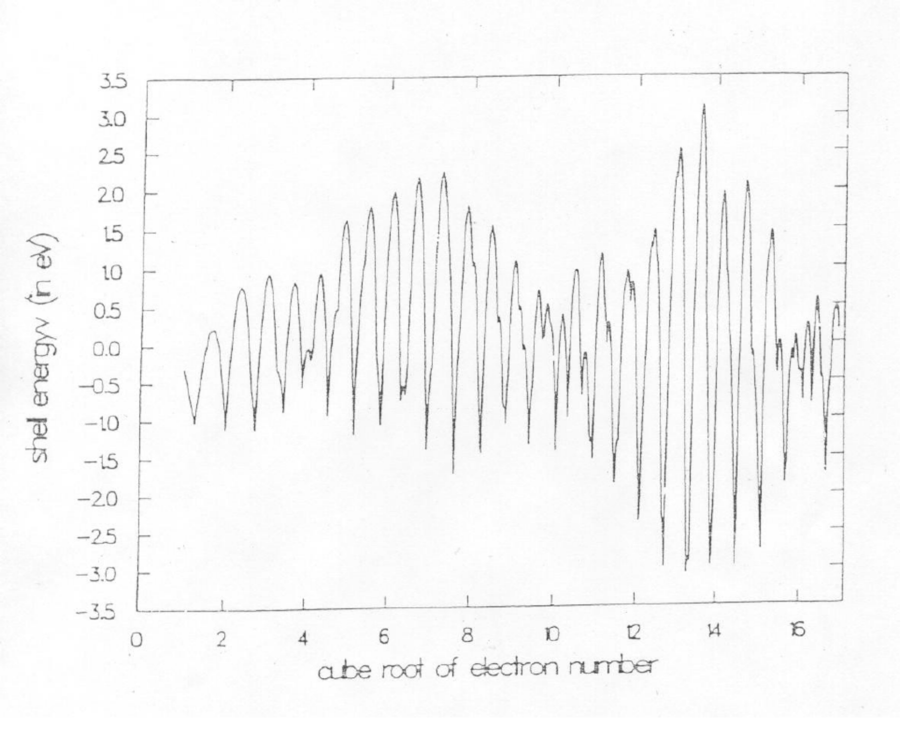

Figure 12. Metallic clusters. Shell energy $\tilde{\Omega}$ (in eV) as a function of $N^{1/3}$. Shells and supershells are clearly visible (from [42]).

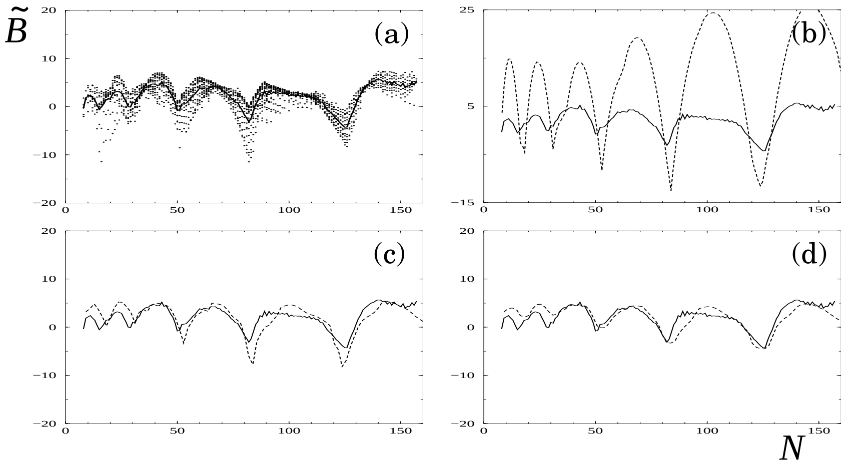

Figure 13. Shell energy $\tilde{\Omega}$ (in MeV) in nuclei as a function of neutron number. (a): from experiment (dots), with its average (full line). (b): theory, spherical cavity, (dashed line), compared to experiment (full line). (c): as (b), but allowing for deformations. (d): as (c) but including a finite coherent length (from [40]).

particular spin-orbit effects are ignored) , many qualitative, even semi-quantitative, effects agree with the data: a spherical cavity gives magic numbers which are not far from their correct value and the period of the oscillations of $\tilde{\Omega}$ is essentially correct. Allowing for deformations largely improves the description and the amplitude of these oscillations are fairly well described. This is obtained by using six parameters, all of them having a clear physical meaning: five in the liquid drop formula plus another one which fixes the size ($\sim A^{1/3}$). Results are further improved by accounting for a finite-coherence length of the nucleon motion due to inelastic effects (quasi-particle lifetime), thereby introducing an additional parameter. Let us introduce an overall figure of merit of a model through the root mean square $< \delta^2 >^{1/2}$ of the deviation δ between the experimental Ω_{exp} and the calculated Ω_{cal} values of the binding energies Ω. The closer to zero, the better the figure of merit is.

$$\delta = \Omega_{exp} - \Omega_{cal}. \tag{24}$$

The one corresponding to the model discussed (see Fig. 13 (d)) is 1.3 MeV.

Much work has been devoted in computing binding energies of nuclei covering a large range of the periodic table. As far as shell effects are concerned, the seminal work of Strutinsky, largely based on POT, is worth mentioning [44], [45]. The models used range from macroscopic-microspocic models to more or less fully microscopic and selfconsistent ones [46] (for a review, see [47]). They usually contain a certain number of adjustable parameters. A typical figure of merit of the most successful models is in the range 0.5-1

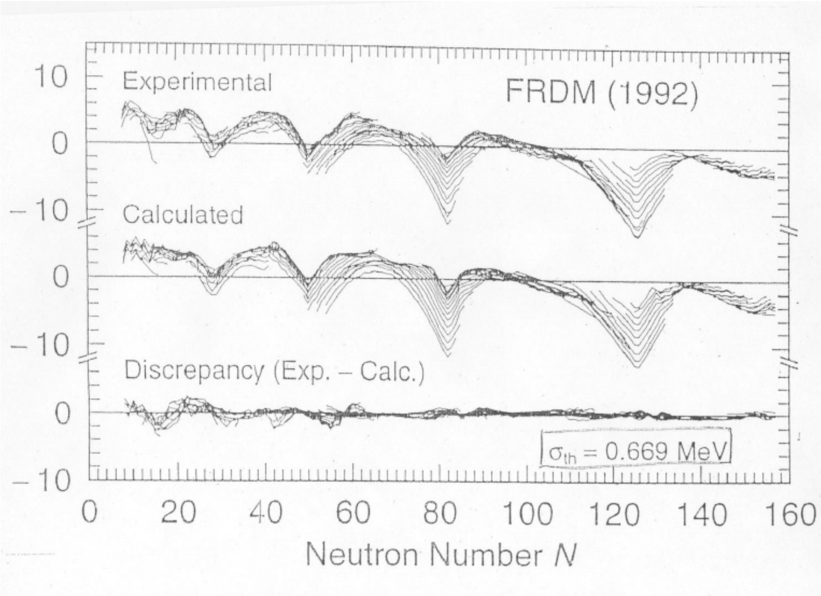

Figure 14. Shell energy $\tilde{\Omega}$ (in MeV) in nuclei as a function of neutron number. Upper part: from experiment. Middle part: theory, from the finite-range droplet model (FRDM). Bottom part: deviation δ (difference between the previous two quantities) (from [48]).

MeV, quite independent of the details of the model used and which is only by a factor of the order of 2 better than the 'rustic' model of nucleons in a cavity discussed above. An example of one of the most thorough analysis is given in Fig. 14. The upper part is the same as in Fig. 13 (a), except that the liquid drop formula used in [48] is much more sophisticated than (23). Consider now the bottom part of the figure, where the deviation δ as a function of neutron number is plotted. No obvious structure is apparent, in contrast to similar older analyses, suggesting that systematic effects are taken care of correctly. One can also guess a slight decrease of δ with neutron number. One question at issue will now be : can one, on the light of the discussion of the previous sections, make a theoretical estimate of an expected figure of merit?

We have seen that generically systems show mixed dynamics, where regular and chaotic dynamics coexist. Though many ground state properties are dominated by the close to regular motion with pronounced symmetries of the mean field, it is extremely unlikely that the motion of many interacting particles is completely regular. What is then expected from the, even if small, chaotic components? In the framework of a Fermi gas description, the separation of the single-particle level density into a smooth and an oscillating part results in a corresponding separation of the ground state energy Ω

$$\Omega = \bar{\Omega} + \tilde{\Omega}. \tag{25}$$

One wants to estimate typical sizes of $\tilde{\Omega}$, namely to compute $< \tilde{\Omega}^2 >$ by using POT but including, this time, the contribution of both regular and chaotic orbits

$$\tilde{\Omega} = \tilde{\Omega}_{reg} + \tilde{\Omega}_{ch}. \tag{26}$$

It can be shown [38], [49] that

$$< \tilde{\Omega}^2 > \approx \frac{\hbar^2}{2\pi^2} int_0^\infty \frac{d\tau}{\tau^4} K(\tau), \tag{27}$$

where $K(\tau)$ is the Fourier transform of the two-point correlation function (form factor), an object already met several times. It is reasonable to assume that $< \tilde{\Omega}_{reg}\tilde{\Omega}_{ch} > \approx 0$ because the dominant contributions come from short orbits (fourth power in the denominator in (27)) and then the regular and chaotic trajectories lie in different regions of phase space (see [50] for an estimate). One is therefore finally lead to estimate $< \tilde{\Omega}^2_{reg} >$ and $< \tilde{\Omega}^2_{ch} >$ by using (27).

The general characteristics of $K(\tau)$ are known. They are very different for regular and chaotic systems (remember the discussion on spectral fluctuations and the connection to RMT). Two time (or energy) scales are relevant, one, the Heisenberg time $\tau_H = h/D$, the long time scale, related to the single-particle mean spacing D at the Fermi energy (small energy scale), the other τ_{min}, the short time scale, given by the period of the shortest periodic orbit, to which a large energy scale $E_c = h/\tau_{min}$ is associated. The dimensionless quantity

$$g = \frac{\tau_H}{\tau_{min}} = \frac{E_c}{D} \ (>> 1) \tag{28}$$

represents the number of single-particle levels contained in a shell. The final expressions are simple

$$< \tilde{\Omega}^2_{reg} > = \frac{1}{24\pi^4} g E_c^2 \tag{29}$$

and

$$< \tilde{\Omega}^2_{ch} > = \frac{1}{8\pi^4} E_c^2. \tag{30}$$

By comparing (29) and (30) one can see an enhancement given by g of the contribution of the regular oscillations with respect to the chaotic ones. Notice that $< \tilde{\Omega}^2_{ch} >$ is independent of the Heisenberg time τ_H, and therefore of the relative phase space volume occupied by chaotic layers. (30) is also valid in arbitrary dimensions and the period of the shortest chaotic orbit in the multidimensional space, which may include residual interaction effects, is probably not very different from the one of a free particle in the 3-dimensional space.

Taking into account the variation of the size of with particle number one is lead to

$$\sigma_{reg} = < \tilde{\Omega}^2_{reg} >^{1/2} \approx 2.9 MeV \tag{31}$$

in good agreement with experimental results, and

$$\sigma_{ch} =< \tilde{\Omega}^2_{ch} >^{1/2} \approx \frac{2.8}{A^{1/3}} MeV. \tag{32}$$

Because most of the arguments used in deriving (29) and (30) are genericity arguments and because results are given in terms of well identified scales, the estimates (31) and (32) are reliable. They should be taken, however, for what they are, namely estimates. The numerical value in (32), for instance, can be changed by, say, a factor of two if a different estimate of τ_{min} is used and/or if spin and isospin effects are properly included.

The calculations mentioned before of binding energies Ω_{cal} take accurately into account mean field effects related to regular motion but are probably much less efficient in dealing with residual interaction effects. Remember also that the spectral fluctuations exhibited by the compound nucleus resonances, well described by RMT, typical of chaotic systems and present in shell model calculations, should be attributed to strong configuration mixing due to residual interactions [4]. It is then tempting and plausible that one should identify the calculated binding energies Ω_{cal} with $\bar{\Omega} + \tilde{\Omega}_{reg}$ and their deviations δ (bottom part of Fig. 14) with $\tilde{\Omega}_{ch}$. Results are displayed on Fig. 15. The different analyses indicate a slow decrease of δ with mass number, consistent with (32). The size predicted by (32) is in the correct range. Data corresponding to [53] (figure of merit 375 keV) are systematically lower than those corresponding to [48] and [52] (figure of merit twice larger). This is probably due to the fact that the analysis of [53] is i) largely based on shell-model calculations, which include residual interaction effects, ii) contains 28 parameters, not all of them under physical control. These results support the (generic) picture we are advocating , namely coexistence of order and chaos in the ground state and, as a corollary, identification of $\tilde{\Omega}_{ch}$ with δ, unless an almost complete and exact treatment of residual interaction effects is performed.

Further evidence supporting this picture can be given. Indeed, a whole chapter of RMT concerns extensions to systems depending on external parameters. In particular, it has been shown that, in the chaotic regime and when properly scaled, the parametric correlations in the level density are also universal (see for instance [54]). This correlations, in turn, reflect in the autocorrelation $< \tilde{\Omega}_{ch}(A+\delta A) \cdot \tilde{\Omega}(A) >$ of $\tilde{\Omega}_{ch}$ [38]. Preliminary results are displayed on Fig.16. Again, the data seem to be consistent with the identification of $\tilde{\Omega}_{ch}$ (A) with δ (A) . Work to improve and extent this analysis is in progress [55].

6. Summary and Conclusions

The field quantum chaos (QC) is a meeting area for physicists as well as for some mathematicians. One is interested in finding fingerprints of the chaotic nature of the classical dynamics of the corresponding system in the quantum regime, in establishing time and energy scales, in identifying universal as well as system-specific properties. The origin of the studies of classical chaos can be traced back to Poincaré, who already pointed out that periodic orbits provide the skeleton on which the understanding of the motion can be built. Among many others, three important notions emerged: i) for a chaotic system, even if governed by deterministic laws, long time predictions of individual unstable trajectories are impossible (deterministic chaos), ii) it is not necessary to have many degrees of freedom to exhibit chaos (for conservative Hamiltonian systems, for instance,

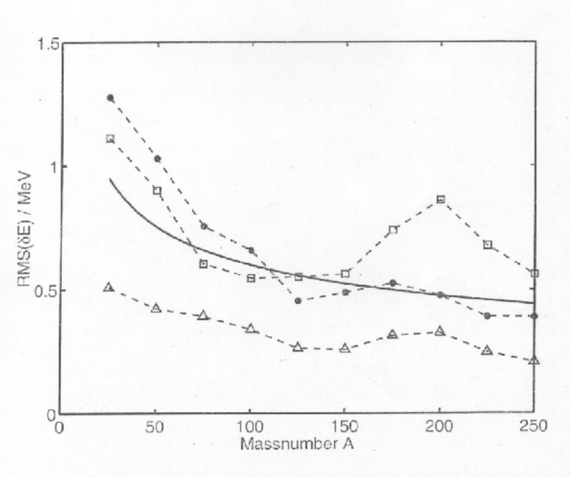

Figure 15. Root-mean-square (RMS) of the deviation δ (in Mev), as a function of mass number. Full circles: from [48]; squares: from [52]; triangles: from [53] (from [51]). Continuous curve: theoretical estimate of $< \tilde{\Omega}^2_{ch} >$ using POT and RMT (from [49]).

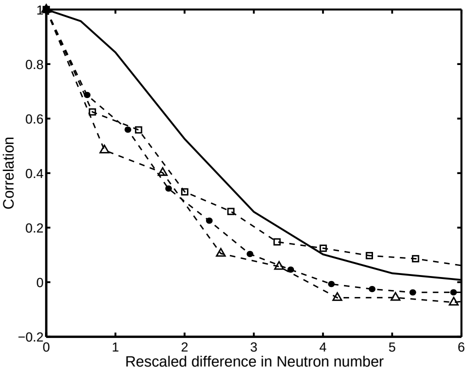

Figure 16. Autocorrrelation of $\delta(A)$ over the mass table (see caption of Fig. (15) . Continuous curve: autocorrelation of $\tilde{\Omega}_{ch}$ using POT and RMT (from [51]).

two degrees of freedom suffice), iii) generically systems show mixed dynamics (regular and chaotic dynamics coexist).

Independently of all these developments, statistical nuclear physics was standing as a well established subfield. One particular topic was the study of the properties of the compound nucleus resonances. In the mid 50's, it gave rise to the birth and development of random matrix theories (RMT) which, subsequently, have known a large number of applications in different fields of physics and mathematics.

It was in the mid 80's that the merging of RMT and QC, two at first sight seemingly disconnected fields, took place. It was the use of periodic orbit theory (POT) which provided the clues for a theoretical understanding. Since then, RMT, QC and POT have known many mutual and often fruitful influences. Progress in one of them has often given rise to a counterpart in one of the other 'partners'. The universality of RMT spectral fluctuations reflect the universality properties of long periodic orbits of chaotic systems. Chaos being ubiquituous, RMT spectral fluctuations have been exhibited not only in compound nucleus resonances, but also in atomic, molecular and (to a lesser extent) hadron spectra, in the adequate regime.

Riemann's ζ-function has played a particular role. Unexpectedly, many of its properties can be shaped in the language of QC and POT. In contrast to physical systems, there is for ζ practically no limitation concerning the amount of 'data' available. Besides its unquestionable intrinsic interest, ζ has become a privileged territory were many ideas and approximations developed in QC can be tested. Several examples are discussed here, covering universal as well as system-specific (number-theoretic) aspects.

In the last Section are addressed questions related to binding energies from the above general perspective, Focus is on shell effects. POT have been thoroughly used in the past to compute them , since the work of Strutinsky. However only effects related to regular motion (in turn related to symmetries) have been discussed. Here we consider the natural extension of including the contribution of chaotic motion as well. It gives rise, superimposed to the large oscillations of the familiar shell effects, to additional oscillations whose size decrease with the number of particles in the system. We argue that the deviations between experimental and calculated binding energies resulting from systematic analyses covering the periodic table can be attributed to the contribution of chaotic motion.

To conclude, some general remarks may be in order. It is a fact that long time predictions of unstable individual trajectories of classically chaotic systems is impossible (think on weather forecast; progress is reflected on more accurate short term predictions, but detailed long time predictions remain out of sight). Is there a corresponding result for (closed) quantum mechanical systems which are classically chaotic?

Let us answer by discussing an example, the hydrogen atom in a homogeneous magnetic field (see, for instance, Delande's contribution in [13]). It is simple, it corresponds to a real physical system and the important actors, chaos and symmetry, are at work. Without magnetic field the classical system (Kepler problem) is regular and integrable (ellipses). Quantum-mechanically the problem can be solved exactly (SO4 symmetry). A weak magnetic field produces the Zeeman effect (perturbative regime). When the magnetic field is extremely strong (Coulomb term can be neglected) the system becomes again regular (spiraling orbits of the electron) and the quantum problem can also be solved

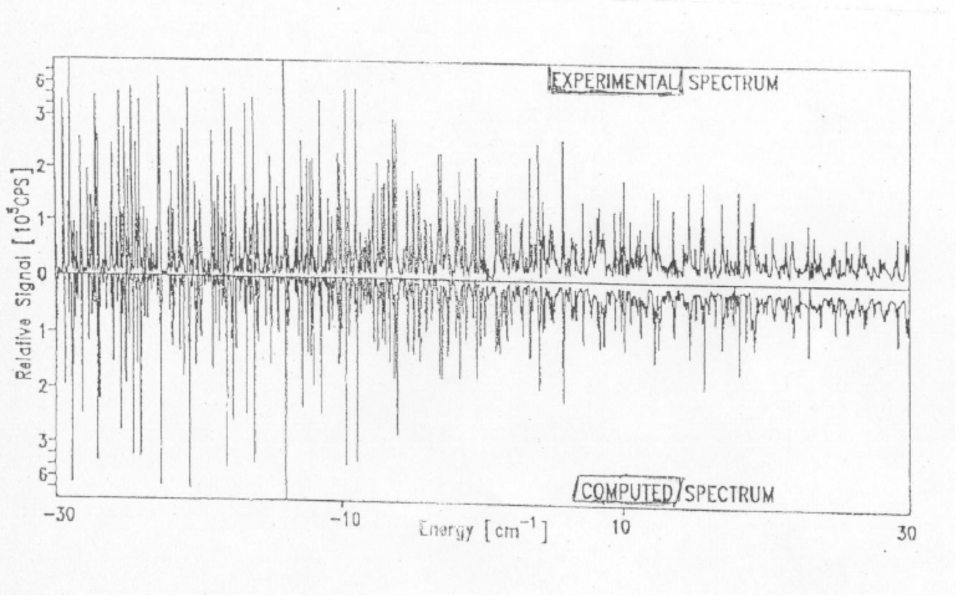

Figure 17. Comparison of the experimental spectrum of the Lithium Rydberg atom in a strong magnetic field with the theoretical spectrum (from [56]).

(Landau oscillator-like spectrum). In the regime where the contribution of the Coulomb and Lorenz forces are similar, the classical motion, as a result of two different conflicting symmetries, becomes chaotic (diamagnetic Kepler problem). It is then impossible to make predictions of individual long unstable orbits. However, if the severe numerical difficulties are carefully treated, the quantum spectrum can be accurately computed (see Fig.17). The calculation contains no adjustable parameter and the experiment is performed, in order to attain the adequate regime, with Rydberg atoms. The agreement between theory and experiment is close to perfect [56]. Notice that the spectral fluctuations have been also analyzed and compared successfully to RMT predictions.

The previous example illustrates the fact that the extreme wilderness of classical chaotic trajectories, with an ever increasing complication preventing their long time prediction, does not reflect in the quantum regime. This is due to the fact that in quantum mechanics the uncertainty Heisenberg relations apply and there is a built-in scale, namely Planck's constant. Planck's constant blurres and smoothes the ever more complicated and sharp patterns of classical dynamics. There is no chaos in its genuine classical sense of instability and unpredictability for isolated quantum system. At the price of a substantial increase of numerical effort, quantum predictions are accurate.

What should be then concluded, concerning calculations of nuclear binding energies, if one accepts that the precision presently reached make the inclusion of chaotic effects (full residual interaction) necessary? Present binding energy calculations are at the edge of what we may call the 'chaotic barrier'. The penetration of this barrier may be challenging. However, without a significant improvement in the determination of the nuclear Hamiltonian and without a virtually exact solution of the nuclear many-body problem, it seems unlikely that this barrier can be deeply penetrated.

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