



# **Exploring equilibration in Fermionic systems: A connection with minimum distance codes**

**Jose Alejandro Montaña Cortes**

Universidad de los Andes  
Faculty of Science, Department of Physics  
Bogotá, Colombia  
2020



# **Exploring equilibration in Fermionic systems: A connection with minimum distance codes**

**Jose Alejandro Montaña Cortes**

**Supervisor:**

Alonso Botero Mejía

A thesis submitted in fulfilment of the requirements for the degree of:

**Master's Degree in Physics**

In the research group of:

Quantum Mechanics and Information Theory

Universidad De Los Andes  
Faculty of Science , Departament of Physics  
Bogotá, Colombia  
2020

*“one can prove that infinitely many more initial states evolve after a long time towards a more uniform distribution of states than to a less uniform one, and that even in that latter case, these states will become uniform after an even longer time”* (L. Boltzmann 1866) [1]

---

## Resumen

La tipicidad canónica ha surgido como una alternativa a los fundamentos de la mecánica estadística, logrando explicar desde una perspectiva de la información cuántica, fenómenos como el de la termalización, el cual emerge como consecuencia del entrelazamiento entre el sistema y su ambiente. Posteriores trabajos han mostrado que esta particular forma de abordar el problema proporciona una idea del mecanismo por el cual se alcanza el equilibrio en sistemas cuánticos. Dentro del marco de estas ideas, promedios temporales juegan un rol principal a la hora de entender como estados reducidos obtenidos a partir de un estado puro, alcanzan el equilibrio. Esto nos conduce a preguntarnos: ¿Es posible encontrar estados reducidos provenientes de un estado puro no estacionario, tales que automáticamente estén equilibrados, es decir, estados reducidos cuyo equilibrio se alcance de forma inmediata?, y en caso de ser así, ¿Es posible determinar el tamaño del conjunto de estados cumpliendo esta propiedad?.

En el presente trabajo, el uso de herramientas de teoría de código, específicamente, códigos aleatorios fermiónicos de distancia mínima, proporcionan una forma diferente de entender los sistemas fermiónicos reducidos. Confirmando así, la existencia de estados reducidos que permanecen en equilibrio para estos sistemas. Se explora así, el espacio de Hilbert generado por el conjunto de estos estados, concretamente, se caracteriza el espacio de Hilbert asociado a estos estados por medio de un exponente de error, el cual proporciona una ley de grandes desviaciones en el sistema. De esta forma, se prueba que el tamaño del espacio de Hilbert asociado a los estados, tales que cumplen la propiedad de equilibrarse de forma inmediata es exponencialmente grande.

**Palabras Clave:** Tipicidad, Sistemas Fermiónicos, códigos aleatorios fermiónicos de distancia mínima .

## Abstract

Canonical typicality has emerged as an alternative to the foundations of statistical mechanics, being able to explain from a viewpoint of quantum information theory, how thermalisation results from entanglement between system and environment. Later results have shown that the ideas used in typicality, provide an explanation form the mechanism of the evolution towards equilibrium for quantum systems. Within this framework of ideas, time averages plays a major rol when it comes to understanding how reduced states obtained from a pure states, reach equilibrium. This led us to ask ourselves, is it possible to find reduced states obtained from a non stationary pure state, such that they equilibrate instantly, this is, reduced states such that immediately reach its equilibrium state?, and if so, is it possible to measure the size of the space fulfilling this property?

In the present work the use of code theory, specifically Fermionic minimum distance codes, provide an alternative to understand fermionic systems. Specifically, we characterised the Hilbert space generated by the states, such that fulfil this property, throughout an error exponent, which provide a characterisation of the large deviation law within the system. Therefore, we prove that the Hilbert space of states fulfilling the property of reaching its equilibrium instantly is indeed exponentially large.

**Key Words:** Typicality, Fermionic systems, Fermionic minimum distance codes.

## Agradecimientos

# Content

<b>Resumen</b>	<b>v</b>
<b>Abstract</b>	<b>v</b>
<b>Agradecimientos</b>	<b>vii</b>
<b>Introduction</b>	<b>1</b>
<b>1 Quantum Thermodynamics And Canonical Typicality</b>	<b>5</b>
1.1 Introduction . . . . .	5
1.2 Postulate of Equal a Priori Probability . . . . .	6
1.2.1 Postulate of Equal a Priori Probability and the Microcanonical Ensemble	8
1.2.2 The Ergodic Hypothesis . . . . .	9
1.2.3 The Quantum Postulate of Equal a Priori Probability . . . . .	10
1.2.4 The canonical Ensemble . . . . .	11
1.3 Entanglement and Canonical Typicality . . . . .	13
1.3.1 Entanglement and the Foundations of Statistical Mechanics . . . . .	13
1.3.2 Canonical Typicality . . . . .	16
1.3.3 Quantitative Arguments behind Typicality . . . . .	17
1.4 Evolution Towards Equilibrium. . . . .	20
<b>2 Fermionic States, its characterisation and the connection with Coding theory.</b>	<b>28</b>
2.1 Overview . . . . .	28
2.2 Majorana Fermions . . . . .	29
2.3 Fermionic Covariance matrix . . . . .	30
2.4 Grassmann Approach . . . . .	32
2.4.1 Fermionic Linear Optics . . . . .	32
2.4.2 Grassmann Calculus . . . . .	33
2.4.3 Gaussian States . . . . .	34
2.5 Error correcting Code Theory . . . . .	36
2.6 Some basic definitions . . . . .	37

2.6.1	Hamming Bound (Sphere packing bound) . . . . .	40
2.6.2	Plotkin Bound . . . . .	40
2.6.3	Gilbert Varshamov Bound . . . . .	42
2.6.4	Error Exponents for Random Minimum Distance Codes . . . . .	44
2.7	Mechanism behind typicality as a correction error method. . . . .	46
2.7.1	“Ultra Orthogonality” on Fermions . . . . .	47
2.7.2	Codes of minimum distance constrained to an energy value . . . . .	49
2.7.3	Case when the numbers of errors is less than $L$ . . . . .	51
<b>3</b>	<b>Getting to understand Ultra Orthogonality in the XY Model.</b>	<b>53</b>
3.1	The XY Model . . . . .	53
3.1.1	The spectrum . . . . .	53
3.1.2	Jordan-Wigner Transformation . . . . .	54
3.1.3	Fourier Transformation . . . . .	54
3.1.4	Bogoliubov -Valantin Transformation . . . . .	55
3.1.5	Fermionic Covariance Matrix for the XY model . . . . .	56
3.2	Exploring “Ultra Orthogonality” . . . . .	60
3.2.1	Error exponent . . . . .	60
3.2.2	Finding Bounds for $\hat{X}_{ij}$ . . . . .	64
<b>Bibliography</b>		<b>71</b>



# Introduction

Alternative considerations about the foundations of statistical mechanics proposed by Popescu et. al. [2], have shown that reliability on subjective randomness, ensemble averaging or time averaging are not required, instead, a quantum information perspective [3] provides an answer, from an uniquely quantum point of view. This way of tackling the problem consider the universe<sup>1</sup> in a quantum pure state. Due to entanglement between system and environment, thermalisation appears as a generic property of the pure states of the universe subjected to a global constraint. This result is mathematically proven and is known as “General Canonical Principle”, specifically, what this principle tell us is that every system will be thermalised for almost all pure states of the universe. In other words, every Hilbert subspace, small enough compared to its environment, will drive to the same state, the canonical state. Even though this principle provides a general viewpoint of thermalisation, this result is kinematic, rather than dynamical. That is, particular unitary evolution of the global state is not considered, and thermalisation is not proven to happen, instead, the key ingredient is Levy’s Lemma [4, 5], which plays a similar role to the law of large numbers and governs the properties of typical states in large- dimensional Hilbert spaces [2], and provide a powerful tool to evaluate functions of randomly chosen quantum states<sup>2</sup>. Additionally, since all states of the universe bring the system to its canonical state, one could anticipate that most evolutions will quickly carry a state in which the system is not thermalised, to the one in which it is, and that the system will remain thermalised for most of its evolution.

The latter problem motivated the research conducted by Linden et. al. [8], in which by using arguments based on typicality’s ideas, reaching equilibrium can be proven to be an universal property of quantum systems. In the frame of these ideas, dynamical aspects are address to explore the evolution which drives systems to equilibrate, and moreover, to study under which circumstances equilibrium is reached on systems. The series of results shown in [8–10], suggest that first, virtually any subsystem of any large enough system will reach equilibrium and fluctuate around it at almost all times, and second, that average speed

---

<sup>1</sup>Here we refer to the Universe as the system together with a sufficiently large environment.

<sup>2</sup>It is important to stress the fact that these ideas were not only proposed by Popescu et. al., contemporarily, Gremmer. et al [6], as well as Goldstein et. al [7], proposed similar ideas, nonetheless, the viewpoint we will discuss here is mostly based on Popescu’s work.

fluctuations are extremely small at almost all times, meaning that the state of the system remains relatively static around its equilibrium.

It turns out that typicality, is an alternative way of studying thermalisation, as well as, equilibration on quantum systems. Even though these techniques are an extremely useful when understanding the foundations of statistical mechanics, there is an implication from typicality which haven not been studied in depth<sup>3</sup>, which is the core idea of this work, and what we consider could be another alternative to understand equilibration in systems. To illustrate what we mean, consider two different pure states living on the same Hilbert subspace ( $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}_R$ ), the one that is obtained by imposing a global constraint over the universe. From typicality we know that the reduce state of  $|\psi_1\rangle$  and  $|\psi_2\rangle$  leads to the same state, the canonical state<sup>4</sup>. Thus, from this fact, we could consider a third state  $|\psi_3\rangle$  which is a generic linear combination of  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , if the condition of typicality is also imposed over the third state, we have therefore that its reduced state will also lead us to the canonical state, meaning that cross terms obtained in the reduced state associated with  $|\psi_3\rangle$  should somehow vanish. Explicitly, when we compute the density matrix associated with the state  $|\psi_3\rangle$  we have

$$|\psi_3\rangle\langle\psi_3| = |c_1|^2|\psi_1\rangle\langle\psi_1| + |c_2|^2|\psi_2\rangle\langle\psi_2| + c_1^*c_2|\psi_2\rangle\langle\psi_1| + c_2^*c_1|\psi_1\rangle\langle\psi_2|, \quad (1)$$

and then its reduced state reads

$$\text{Tr}_{\mathcal{E}}|\psi_3\rangle\langle\psi_3| \equiv \Omega_{\mathcal{S}} = \Omega_{\mathcal{S}} + c_1^*c_2 \text{Tr}_{\mathcal{E}}|\psi_2\rangle\langle\psi_1| + c_2^*c_1 \text{Tr}_{\mathcal{E}}|\psi_1\rangle\langle\psi_2|, \quad (2)$$

where the condition of normalisation was used ( $|c_1|^2 + |c_2|^2 = 1$ ). Notice that the cross terms in equation (2) should vanish in order to satisfy the equality, namely, the condition which has to be satisfied in order to keep the equality is  $\text{Tr}_{\mathcal{E}}|\psi_2\rangle\langle\psi_1| = \text{Tr}_{\mathcal{E}}|\psi_1\rangle\langle\psi_2| = 0$ . We name this condition “Ultra Orthogonality”, since when we compute the partial trace over the exterior product of  $|\psi_1\rangle$  and  $|\psi_2\rangle$  it becomes zero.

The last-mentioned property, may seem a little useless, but indeed this very property is related with the dynamics of the system. To illustrate this, consider an state which is time dependent  $|\Psi(t)\rangle$ , we can expand this state in its energy eigenstates as

$$|\Psi(t)\rangle = \sum_k c_k e^{-iE_k t} |E_k\rangle, \quad (3)$$

---

<sup>3</sup>Of course, there is a possibility these ideas have been pointed out by someone else before, nonetheless, it seems it has not been of great interest for others since there is not a publication nor a public discussion about this subject that we could find.

<sup>4</sup>Explicitly what this means is that

$$\text{Tr}_{\mathcal{E}}|\psi_1\rangle\langle\psi_1| = \text{Tr}_{\mathcal{E}}|\psi_2\rangle\langle\psi_2| = \Omega_{\mathcal{S}},$$

where  $\Omega_{\mathcal{S}}$  corresponds to the canonical state of the system.

where  $\sum_k |c_k|^2 = 1$ , hence

$$\rho(t) = \sum_{k,\ell} c_k c_\ell^* e^{-i(E_k - E_\ell)t} |E_k\rangle \langle E_\ell| = \underbrace{\sum_k |c_k|^2 |E_k\rangle \langle E_k|}_{\omega} + \sum_{k,\ell} c_k c_\ell^* e^{-i(E_k - E_\ell)t} |E_k\rangle \langle E_\ell|, \quad (4)$$

so when we look at the reduced state of  $\rho(t)$  will lead to

$$\text{Tr}_\mathcal{E} \rho(t) \equiv \rho_S(t) = \omega_S + \sum_{k,\ell} c_k c_\ell^* e^{-i(E_k - E_\ell)t} \text{Tr}_\mathcal{E} |E_k\rangle \langle E_\ell|, \quad (5)$$

where  $\omega_S = \text{Tr}_\mathcal{E} \omega$ . Thus, if instead of taking arbitrary states in equation (2), we could take states on the energy eigenbasis we could have that for certain situations the equilibrium could be reached immediately whenever ultra orthogonality holds. Having this in mind, we stress again the motivation of studying ultra orthogonality as an alternative way of reaching equilibrium.

The motivation behind ultra orthogonality is quite related with the equilibration, as well as with thermalisation<sup>5</sup>. Of course, it would be quite interesting to fully study this property to provide a better comprehension of ultra orthogonality as an alternative mechanism for equilibration. Nonetheless, for this work we are not going to talk much about the general problem, instead we will be discussing a particular case from ultra orthogonality. Here we are going to describe the reduced state of a pure and full dynamical states of the universe, such that these reduced states are immediately stationary. Namely, we will be discussing the case when the cross terms in (5) are equal to zero ( $\text{Tr}_\mathcal{E} |E_k\rangle \langle E_\ell| = 0$ ) in the special case of Fermionic systems.

One of the first questions one might wonder is, if there are few or many states that fulfil this distinct property, and for any case, we would like to be able to have a precise measure of the size of the Hilbert subspace associated with these states. We will provide the answers to these questions by using some ideas taken from Fermionic random minimum codes, and we will show that there are exponentially large Hilbert subspaces in which all its states fulfil the condition  $\text{Tr}_\mathcal{E} |E_k\rangle \langle E_\ell| = 0$ .

This document is divided in four chapters, in the first chapter we are going to introduce canonical typicality and its relation to statistical mechanics, starting from introducing the postulate of prior probability and the controversy around this argument, passing through typicality and finishing with a detailed explanation of the problem of ultra Orthogonality. For the second chapter we will state general definitions concerning Fermionic states, in particular, we will discuss the formalism of Grassmann algebras and Grassmann Variables. In

---

<sup>5</sup>Since for most cases we expect thermal typicality to be present in the system, the equilibrium state will coincide with the thermal one.

the third chapter we are going to provide some definitions with regard to correction codes and specifically some of the well known bounds for minimum distance codes, more over, in this chapter we present our main result which we present as follows, first, an explicit connection between the Fermionic systems and random minimum distance codes is formulated in order to understand ultra orthogonality for Fermionic systems, and second, we provide an estimation of the size of the Hilbert subspace associated with the states that fulfil our ultra Orthogonality. Finally in the fourth chapter we present our conclusions as well as our perspective of our work.

# Chapter 1

## Quantum Thermodynamics And Canonical Typicality

### 1.1 Introduction

The foundations of Statistical Mechanics remain a debatable subject. Fundamental questions have emerged from these discussions regarding the role of probabilities, entropy<sup>1</sup> and the relevance of time averages and ensemble averages to individual physical systems. One of the most controversial issue is the validity of the postulate of equal a priori probability, postulate which can not be proven.

In this chapter we are going to discuss some ideas based on typicality addressed by several authors [6, 7, 11], who have abandoned the unprovable aforementioned postulate and replace it with a new viewpoint, which is uniquely quantum, and which does not rely on any ignorance probabilities in the description of the state. Instead, is supported, and can be proved, by means of the entanglement between a physical system and it environment.

This chapter will be divided in three parts, first we are going to introduce the postulate of equal prior probability, discussing its commencement in the foundations of statistical mechanics, the idea of an ensemble as a collection of identical systems will be introduced and the postulate of equal prior probability will be translated to a particular version of ensemble, the microcanonical ensemble. Also a quantum version of the above-mentioned postulate will be exhibited in terms of the random phase postulate [12] and the derivation of the canonical ensemble for a weakly interacting system.

The second part will be dedicated to understand entanglement and therefore we present the phenomenon of canonical typicality, where thermalisation emerges as a consequence of

---

<sup>1</sup>As it is well known, probabilities and entropy are both referred as measures of ignorance.

it. These ideas are taken from *S.Popescu, A. J. short and A. Winter* [2,11] work which we follow very close<sup>2</sup>. Quantitative arguments will be provided and explained with its respective mathematical tools needed to comprehend the ideas in which typicality is built on.

To conclude, we use all the previous results in the second part to present in the third part, an approach studied by *Linden et. al.* [8] in which rather than a kinematic insight, they address the dynamical aspects of thermalisation, explaining a mechanism in which systems far from equilibrium would reach a generic state<sup>3</sup> [8]. Furthermore, we will talk about the main subject of this work, an idea which intuition is built on the view point of typicality, and that could provide us an alternative way of understanding equilibration. Expressly, we provide a thorough explanation about what we call ultra orthogonality and under which circumstances is expected to happen. Furthermore, we state the problem we are going to tackle.

## 1.2 Postulate of Equal a Priori Probability

Statistical physics have been historically inspired in phenomenological thermodynamics. In fact quantities such as temperature, pressure, heat and entropy emerged as concepts to describe empirical and macroscopical principles. the fact that we could describe a system by understanding how its microscopic constituents behave via some effective laws, known as *the law of thermodynamics* [13]<sup>4</sup>. A well known result from these studies was *the second law of thermodynamics* which let us know what kind of natural phenomena could occur.

It was not until Boltzmann, Gibbs and Maxwell that statistical physics became known as a fundamental theory, providing a bridge between experimental thermodynamics and the theoretical notions of the atomic world. However, this approach has been a subject of strong debates. Statistical physics differ from phenomenological thermodynamics by two main reasons. The first is the consideration that the behaviour of microscopic constituents could be described with classical dynamics, and the second is the introduction of probability and statistics to deduce the laws of thermodynamics. We are going to introduce the main theoretical concepts proposed at the end of the 19th century, related to the description of the microscopic world<sup>5</sup>.

---

<sup>2</sup>However, it is important to stress that an independent work done by *Goldstein et. al.* [7] discuss a similar issues addressed by *S.Popescu et. al.* in [2,11].

<sup>3</sup>Here a generic state has to be understood as a typical state, meaning the one that corresponds to its canonical state.

<sup>4</sup>For this very section we follow the ideas provided in [13] which turned up to be a very interesting historical review of statistical mechanics. This text is one of the most completed documented review in which not only the work of others is exposed but the controversy at the time related with these subjects. Even though is longer than i expected, it worth the time and i sincerely recommend to take a look at it.

<sup>5</sup>An interesting fact is that, it was Maxwell who really marks the rebirth of kinetic theory. His ideas about

Consider a mechanical system of  $N$  ( $N \gg 1$ ) particles subject to a time-independent potential  $V$ . Supposing every particle in the system has mass  $m$  and coordinates in its phase space  $(\vec{q}_i, \vec{p}_i)$  we easily see that the state of the system is represented by a point in the phase space  $\Gamma$   $(\vec{q}, \vec{p}) = (\vec{q}_1, \dots, \vec{q}_N, \vec{p}_1, \dots, \vec{p}_N)$ , and its time evolution follows the dynamics induced by its Hamiltonian. Moreover, imposing the restriction that the energy  $E$  is conserved, the evolution is confined to a set of Hamiltonians such that fulfil this restriction ( $\Sigma_E = \{(\vec{q}, \vec{p}) \in \Gamma | H(\vec{q}, \vec{p}) = E\}$ ). At this moment is when probability comes to play an important role. In [1] Boltzmann proposed to interpret the probability associated to a particular state as the relative time spent by the system in that state<sup>6</sup>. This assumption relies on the hypothesis that the total time of a measurement is extremely long if compared with the intrinsic time scales of the evolution.

By Liouville's theorem,  $f$  is a constant function on all the admissible trajectories on  $\Sigma_E$ . Moreover, if we assume that the trajectory of a single point in phase space fills densely  $\Sigma_E$  (Ergodic hypothesis)

$$\rho(\vec{q}, \vec{p}) \propto \delta(H(\vec{q}, \vec{p}) - E). \quad (1.1)$$

Based on this argument, Boltzmann was able to prove that thermal equilibrium could be described in terms of Maxwell's distribution of velocities.

In fact in both of his works [14], and [15], Boltzmann mentioned his dissatisfaction about the Ergodic hypothesis and slowly abandoned it. It seems that he only considered it an useful assumption for his general result

For any system for which the hypothesis is true, its equilibrium state is characterized by<sup>7</sup>,

$$\rho_{mc}(p|\vec{q}_1, \dots, \vec{q}_N) dp = \frac{1}{\sqrt{2m\pi}} \frac{\Gamma(\frac{3N}{2})}{\Gamma(\frac{3N-1}{2})} \frac{\left(E - U - \frac{p^2}{2m}\right)^{(3N-2)/2}}{(E - U)^{(3N-3)/2}} dp, \quad (1.2)$$

from which an analogy to the Maxwell distribution may be recovered in the limit  $N \rightarrow \infty$ , regardless of any details of the inter-particle interactions, or indeed whether the system represented is a gas, fluid, solid or any other thermal body.

---

the how in a system with a large number of particles could be described taking a frequency interpretation of the probability was the base for Boltzmann's work.

<sup>6</sup>Concretely the probability of the phase point lies in an infinitesimal region of  $\Sigma_E$  is given by

$$\rho(q, p)dqdp = f(q, p)\delta(H(q, p) - E)dqdp,$$

where  $dqdp$  is the Lebesgue measure of  $\Gamma$  and  $f$  is a suitable function.

<sup>7</sup>This equation was first derived by Boltzmann in [1], and this is the first occasion where probability considerations are applied to the state of the mechanical system as whole, instead of its individual particles.

### 1.2.1 Postulate of Equal a Priori Probability and the Microcanonical Ensemble

An alternative path was taken by Gibbs who made use of an alternative idea of probability in what he named as “ensembles”. In his book [16] Gibbs introduce the probability as a distribution function on a collection of identical systems instead of introducing the probability as an ingredient associated to the state of a single system, as Boltzmann proposed. Gibbs considered “ensembles” in his pioneering treatment of statistical mechanics. His ensembles were infinite sets of macroscopically identical systems, each represented by a point in his phase space (also called *microstates*) being compatible with a single macroscopic configuration (*macrostate*). It is worth stressing that this approach is not interested in following the temporal evolution of a single microscopic configuration, but rather is concerned about the distribution of all the available microscopic configurations, even if the *microstate* associated, is entirely fictitious.

More precisely an ensemble is introduced as a probability density function on the phase space  $\Gamma$ , such that the average number of *microstates* in a region  $R$  of  $\Gamma$  is given by  $\int_R \rho(\vec{q}, \vec{p}) d\vec{q}d\vec{p}$ . Furthermore, the expectation value of an observable  $f : \Gamma \rightarrow \mathbb{R}$  is the average of  $f$  over  $\Gamma$

$$\langle f \rangle = \int_{\Gamma} f((\vec{q}, \vec{p})) \rho(\vec{q}, \vec{p}) d\vec{q}d\vec{p}. \quad (1.3)$$

In this approach we take the condition that the ensembles must be stationary<sup>8</sup>, to derive the microcanonical ensemble, which is the only compatible with the conservation of energy is such that

$$\rho_{\mu c}(\vec{q}, \vec{p}) = \frac{1}{|\Sigma_E|} \delta(H(\vec{q}, \vec{p}) - E), \quad (1.4)$$

with  $|\Sigma_E|$  the measure of the set  $\Sigma_E$ .

As mentioned before Classical statistical mechanics is built on the postulate of Equal Probability. This result can be interpreted as follows:

*All microstates accessible to an isolated system are equally probable, because there is no evidence that certain microstate should be more probable than others.*

Namely, when a macroscopic system is at equilibrium, every state compatible with the constrain of the system, is equally available (likely). Which formally is translated translates into the choice of a constant density function, called the microcanonical ensemble.

---

<sup>8</sup>This condition reads as

$$\frac{\partial \rho_t}{\partial t} = \{H, \rho_t\} = 0$$

### 1.2.2 The Ergodic Hypothesis

We dedicate this special section to discuss a little more about the ergodic hypothesis and how Boltzmann could have understood this to give a formal argument of why it is needed that independent of the initial distribution of the system, this has to evolve towards a uniform distribution (equal a prior probability). First we will show some previous concepts to fully understand this ideas.

The ergodicity provides us an equality between ensemble and time averages of observables. Each measurement of an observable  $f$  at time  $t_0$  takes a certain time to accomplish. In this period of time the observable  $f$  samples different values so that the effectively quantity is the time average

$$\frac{1}{t} \int_0^t f(T_s(\vec{q}_0, \vec{p}_0)) ds, \quad (1.5)$$

with  $(\vec{q}_0, \vec{p}_0)$  is the initial microstate (at  $t = 0$ ), and  $\{T_s\}_{s \in \mathbb{R}}$  is the Hamiltonian flow generated by  $H$ <sup>9</sup>. Thus we are sampling  $f$  trajectories whose initial points are  $(\vec{q}_0, \vec{p}_0)$ . Furthermore, due to the difference in scales of time between the measurement and the intrinsic time of the microscopical constituents, it is possible to consider the limit

$$f^*(\vec{q}_0, \vec{p}_0) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(T_s(q_0, p_0)) ds. \quad (1.6)$$

The problem about the ergodic hypothesis is to determine when the average ensemble average equals the time averages. Of course the general answer to this question is that this equality does not always holds, since  $f^*(\vec{q}_0, \vec{p}_0)$  depends on the initial conditions whereas  $\langle f \rangle$  does not. nevertheless these differences, the cases when the equality between time and ensemble averages holds, we say that the ergodic hypothesis is valid.

Perhaps the most well-known, interpretation to this problem is the one provided by the Ehrenfests [17]. In essence, they suggest that Boltzmann somehow relied on the ergodic hypothesis in his argument, even though he never explicitly mentioned it.

It is indeed evident that if the ergodic hypothesis holds, a state will spend time in the various regions of the energy hypersurface in phase space in proportion to their volume. In other words, during the evolution of the system along its trajectory, regions with small volume, corresponding to highly non-uniform distributions of state, are visited only sporadically, and

---

<sup>9</sup>Here  $H$  is the Hamiltonian function describing the system of  $N$  particles subject to a time-independent potential  $V$

$$H(\vec{q}, \vec{p}) = \frac{1}{2m} \sum_{i=1}^N \vec{p}_i^2 + V(\vec{q}_1, \dots, \vec{q}_N).$$

regions with larger volume, corresponding to more uniform distributions of state, more often.

Therefore, the latter could also make plausible the idea that if a system starts out from a very small region (a very improbable state), it will display a tendency to evolve towards the overwhelmingly larger equilibrium state.

Thus, as is mentioned in [17], it is suggested that Boltzmann relied on the ergodic hypothesis in order to equate time averages and phase averages, or in other words, to equate two meanings of probability (relative time and relative volume in phase space.) There is however no evidence that Boltzmann ever followed this line of reasoning neither in the 1870s, nor later. He simply never gave any justification for equivocating time and particle averages, or phase averages, at all. Presumably, he thought nothing much depended on this issue and that it was a matter of taste<sup>10</sup>.

### 1.2.3 The Quantum Postulate of Equal a Priori Probability

The latter discussion is completely classical. Since our world is quantum, a quantum formulation of statistical mechanics is needed. Instead of probability distributions that live in phase space, one should consider density matrices, which encode the whole physical content of the system. Therefore, the way we write the postulate of equal a priori probability has to be adapted to the formalism of quantum mechanics.

In quantum mechanics a system is described in a Hilbert space  $\mathcal{H}$  and its evolution is generated by a Hamiltonian operator  $\hat{H}$ . By fixing the energy to belong to an energy shell around the value  $E$  ( $[E, E + \delta]$ ), with  $\delta \ll E$ , but  $\delta$  large enough so that the shell contains many energy eigenvalues of  $\hat{H}$ . Rather than considering a phase space region  $\Gamma_E$ , as we did in the classical case, we consider a subspace spanned by all eigenvectors with energy eigenvalues belonging to the energy shell ( $\mathcal{H}_R = \mathcal{H}_{[E, E + \delta]}$ ).

The postulate of equal prior probability affirms that all the states compatible with the energy  $E$  are equiprobable<sup>11</sup>. However, in the quantum case this is not enough. In fact, these states must be in an incoherent superposition. This what is known as *the random phase postulate* and it's a contribution to the foundations of statistical mechanics which is purely quantum.

The idea here is to assume that the coupling between the system and its environment is sufficiently weak such that the energy of the system is found with in a macroscopically narrow range  $[E, E + \delta]$  containing many possible states of the system. Transitions between

---

<sup>10</sup>For further readings on ergodic hypothesis i sincerely recommend [13, 18–21]

<sup>11</sup>In the quantum mechanical frame this reads as every eigenstate belonging to  $\mathcal{H}_R$

this energy are mediated by  $\Delta H$ . All states within energy range which can be connected by range which can be connected by  $\Delta H$  are considered accessible. We assume that the environment is sufficiently complex, its states so numerous, and its transitions so rapid that phase relationships between different states of the system can not be maintained over microscopically long time intervals. Formally the idea is then that the Fourier coefficient of a vector state in  $\mathcal{H}_R$  should have equal probability and completely random phases, due to the unavoidable interactions between the environment and the system [22].

Using these two postulates we conclude that the state of the universe is described in terms of a projector of the Hamiltonian  $\hat{H}$  in the constrained Hilbert space  $\mathcal{H}_R$ . We define  $\mathcal{E}_R$ , the equiprobable state of the universe corresponding to the restriction  $R$  by

$$\mathcal{E}_R = \frac{\mathbb{I}_R}{d_R}, \quad (1.7)$$

where  $\mathbb{I}_R$  is the identity (projection) operator on  $\mathcal{H}_R$ , and  $d_R$  is the dimension of  $\mathcal{H}_R$ .  $\mathcal{E}_R$  is the maximally mixed state in  $\mathcal{H}_R$ , in which each pure state has equal probability. This corresponds to the standard intuition of assigning equal a priori probabilities to all states of the universe consistent with the constraints.

#### 1.2.4 The canonical Ensemble

In this part we will show that under some quite general assumptions it is possible to show that if the universe is in the equiprobable state, every small part of it (system) will be in a canonical state  $\Omega_S$  characterised by a Boltzmann distribution among its eigenstates at a given temperate  $\beta^{-1}$ , this is

$$\Omega_S \propto e^{-\beta H_S}. \quad (1.8)$$

To prove this we make a division of the universe in two parts. The system  $S$  and its environment  $E$ . So Hilbert space will then be described by  $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ . Then the Hamiltonian of the universe will be written as

$$\hat{H} = \hat{H}_S \otimes \hat{\mathbb{I}}_E + \hat{\mathbb{I}}_S \otimes \hat{H}_E + \hat{H}_{\text{int}}, \quad (1.9)$$

where  $\hat{H}_S$  and  $\hat{H}_E$  act separately on the system and its environment respectively,  $\hat{\mathbb{I}}_S$  and  $\hat{\mathbb{I}}_E$  are the identity operators on  $\mathcal{H}_S$  and  $\mathcal{H}_E$ , and  $\hat{H}_{\text{int}}$  describes the interaction between the system and its environment. Supposing the interaction is weak<sup>12</sup> and assuming the dimension of  $\mathcal{H}_E$  is much larger than the dimension of  $\mathcal{H}_S$  ( $d_E = \dim \mathcal{H}_E \gg d_S = \dim \mathcal{H}_S$ ).

---

<sup>12</sup>This can be understood in terms of the Hamiltonians as

$$\|\hat{H}_{\text{int}}\| \ll \| \hat{H}_S \|, \| H_E \|.$$

We suppose the macroscopic energy  $E$  belongs to a small energy interval  $([E, E + \delta])^{13}$ . Let  $\mathbb{I}_R$  be the identity (projector) operator of  $\hat{H}$  on the energy shell

$$\mathcal{H}_R = \mathcal{H}_{[E, E + \delta]}. \quad (1.10)$$

Assuming the universe is in the equiprobable state  $\mathcal{E}_R$  given by (1.7). The state of the system  $S$  can be obtained by partial trace the state of the universe over the environment, that is,

$$\Omega_S = \text{Tr}_E \mathcal{R}. \quad (1.11)$$

From now on the main idea will be to show that  $\Omega_S$  corresponds to the thermal state at a given temperature<sup>14</sup>.

Let  $\{|E_k\rangle\}_{k=1}^{d_E} \subset \mathcal{H}_E$  and  $\{|\varepsilon_\alpha\rangle\}_{k=1}^{d_S} \subset \mathcal{H}_S$  be the energy basis of  $\hat{H}_E$  and  $\hat{H}_S$  respectively<sup>15</sup>. Since we use the hypothesis that the interaction is weak, we have that the equiprobable state  $\mathcal{E}_R$  is written as<sup>16</sup>.

$$\mathcal{E}_R = \frac{\mathbb{I}_R}{d_R} \approx \frac{1}{d_R} \sum_{\alpha, k} \chi_{[E, E + \delta]}(\varepsilon_\alpha + E_k) |\varepsilon_\alpha\rangle \langle \varepsilon_\alpha| \otimes |E_k\rangle \langle E_k|. \quad (1.12)$$

By tracing over the environment we get

$$\Omega_S = \text{Tr}_E \mathcal{E}_R = \frac{1}{d_R} \sum_{\alpha, k} \chi_{[E, E + \delta]}(\varepsilon_\alpha + E_k) |\varepsilon_\alpha\rangle \langle \varepsilon_\alpha| = \frac{1}{d_R} \sum_{\alpha} d_\alpha^{(E)} |\varepsilon_\alpha\rangle \langle \varepsilon_\alpha|, \quad (1.13)$$

where

$$d_\alpha^{(E)} = \sum_k \chi_{[E, E + \delta]}(\varepsilon_\alpha + E_k) = \sum_k \chi_{[E - \varepsilon_\alpha, E - \varepsilon_\alpha + \delta]}(E_k). \quad (1.14)$$

Since  $\hat{H}_E = \sum_k E_k |E_k\rangle \langle E_k|$ , we get that

$$d_\alpha^{(E)} = \text{Tr} \chi_{[E - \varepsilon_\alpha, E - \varepsilon_\alpha + \delta]}(H_B) = \dim \mathcal{H}_{[E - \varepsilon_\alpha, E - \varepsilon_\alpha + \delta]}^{(E)}, \quad (1.15)$$

where  $\mathcal{H}_{[E_1, E_2]}^{(E)} \subset \mathcal{H}_E$  is the subspace generated by all eigenstates with energy in  $[E_1, E_2]$ . Thus  $d_\alpha^{(E)}$  is a non-negative integer.

---

<sup>13</sup>As mentioned before here we suppose  $\delta \ll E$  on a macroscopic scale but large enough to contain many eigenvalues on  $H_E$

<sup>14</sup>The following prove was taken from [22]

<sup>15</sup>In other words this mean that the Hilbert  $\mathcal{H}$  space associated with the universe has basis given by

$$\{|\varepsilon_\alpha\rangle \otimes |E_k\rangle : 1 \leq \alpha \leq d_S, 1 \leq k \leq d_E\}$$

<sup>16</sup>Here the sums is over indexes  $k$  and  $\alpha$  such that  $\varepsilon_\alpha + E_k \in [E, E + \delta]$ . Which is equivalent to say where the characteristic function does not vanish

Defining the bath entropy at energy  $E$  as the logarithm of the number on energy levels in the bath,

$$S_E(E) = \ln \left( \dim \mathcal{H}_{[E, E+\delta]}^{(E)} \right), \quad (1.16)$$

we deduce,

$$d_\alpha^{(E)} = \dim \mathcal{H}_{[E-\varepsilon_\alpha, E-\varepsilon_\alpha+\delta]}^{(E)} = e^{S_E(E-\varepsilon_\alpha)}. \quad (1.17)$$

As the dimension of  $\mathcal{H}_E$  is very large,  $d_R \ggg 1$ , we can therefore assume that the spectrum of  $\hat{H}_E$  is quasi-continuous, so that  $S_E(E)$  can be considered a continuous differentiable function of  $E$ . If we assume that the microscopic energy is much smaller than the macroscopic energy  $E$  ( $\varepsilon \lll E$ ), we can write

$$S_E(E - \varepsilon_\alpha) \approx S_E(E) - \frac{dS_E(E)}{dE} \varepsilon_\alpha \quad (1.18)$$

Thus, we get

$$\Omega_S = \frac{1}{d_R} \sum_{\alpha} d_\alpha^{(E)} |\varepsilon_\alpha\rangle \langle \varepsilon_\alpha| \approx \frac{1}{Z} \sum_{\alpha} e^{-\beta \varepsilon_\alpha} |\varepsilon_\alpha\rangle \langle \varepsilon_\alpha| = \frac{1}{Z} e^{-\beta \hat{H}_S}, \quad (1.19)$$

with  $Z = \text{Tr} \left( e^{-\beta \hat{H}_S} \right)$ , and

$$\beta = \frac{dS_E(E)}{dE}. \quad (1.20)$$

is the thermodynamical expression of the inverse temperature of the bath.

## 1.3 Entanglement and Canonical Typicality

As mentioned before we dedicate this section to show how the postulate of equal a priori probability can be proved rather than postulated as we quoted before. The most interesting thing is that this postulate emerges from the very structure of quantum mechanics and it is a simply consequence of entanglement.

### 1.3.1 Entanglement and the Foundations of Statistical Mechanics

When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own (E. Schrödinger 1995 [23]).

To understand what Schrödinger meant by this, we can illustrate the situation by considering a composite system of two spins on the Hilbert space  $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ , where  $\mathcal{H}_S = \mathcal{H}_E = \mathbb{C}^2$ .

Denoting  $\{|\uparrow\rangle, |\downarrow\rangle\}$ , our computational basis of  $\mathbb{C}^2$  [24–26]. If the bipartite system is described in terms of a factorized state, for example,

$$|\phi\rangle = |\uparrow\rangle_S \otimes |\uparrow\rangle_E, \quad (1.21)$$

then the system and its environment can be described independently. On the contrary, when the global state is not factorized (Bell state),

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_S \otimes |\uparrow\rangle_E + |\downarrow\rangle_S \otimes |\downarrow\rangle_E) \quad (1.22)$$

then the state is entangled. In fact the density matrix associated with this system is

$$\rho_S = \text{Tr}_E (|\Phi^+\rangle \langle \Phi^+|) = \frac{1}{2} \hat{\mathbb{I}}_2 \quad (1.23)$$

where  $\hat{\mathbb{I}}$  is the identity operator on  $\mathcal{H}_S = \mathbb{C}^2$ . the latter equation tells us that the system  $S$  is in a totally mixed state, that is, it is uniformly random distributed in  $\mathcal{H}$ .

Now consider the general parametrization of the latter example,

$$|\Phi_\alpha\rangle = \sqrt{\alpha} |\uparrow\rangle_S \otimes |\uparrow\rangle_E + \sqrt{1-\alpha} |\downarrow\rangle_S \otimes |\downarrow\rangle_E, \quad \alpha \in [0, 1], \quad (1.24)$$

which condensate the separable state  $|\phi\rangle$  ( $\alpha = 0$ ), and the Bell state  $|\Phi^+\rangle = |\Phi_{1/2}\rangle$  ( $\alpha = 1/2$ ). The reduced density matrix is then given by

$$\rho_S^\alpha = \alpha |\uparrow\rangle \langle \uparrow| + (1 - \alpha) |\downarrow\rangle \langle \downarrow|, \quad (1.25)$$

which is a state whose mixture depends on  $\alpha$ .

Summarising, if the state of the composite system is factorised, the information on the whole state and on every subsystem is completely accessible. On the other hand, if the global state is entangled, even though we have complete knowledge of the state in the universe, a prior only a partial knowledge of the subsystem can be obtained. Particularly, if the global state is maximally entangled ( $\alpha = 1/2$ ) one has no information at all on the subsystem. Mathematically speaking, the information content of a state  $\rho$  is described by the von Neumann entropy [27].

$$S(\rho) = -\text{tr}(\rho \ln \rho) = -\sum_k p_k \ln p_k, \quad (1.26)$$

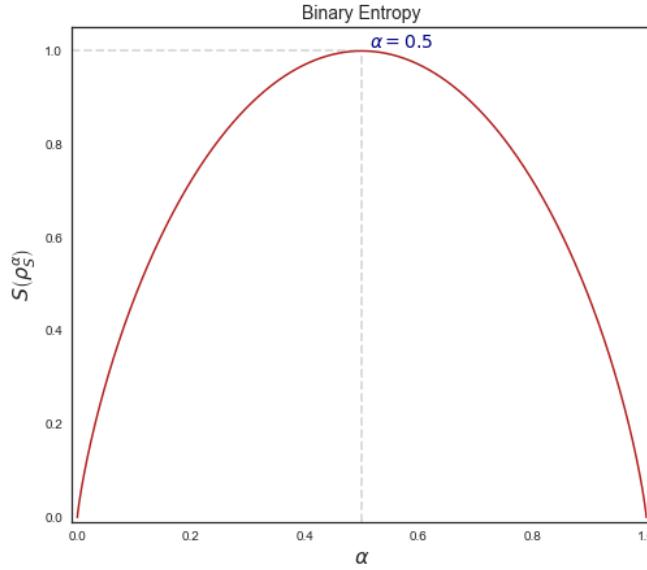
where  $p_k$  are the eigenvalues of  $\rho$ , subject to the conditions

- $0 \leq p_k \leq 1$ ,
- $\sum_k p_k = 1$ .

It is straightforward to realise that every pure state has 0 entropy, which means that the information encoded in the state is completely available. In our case, for example we see that  $S(|\Phi_\alpha\rangle\langle\Phi_\alpha|) = 0$  for every  $\alpha$ . And contrarily, for the general parametrised reduced state we get

$$S(\rho_S^\alpha) = -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha), \quad (1.27)$$

that is a positive metric function of  $\alpha \in [0, 1]$ , which is 0 for separable global states ( $\alpha = 0, 1$ )



**Figure 1.1:** Binary Entropy as a function of binary outcome probability  $\alpha$ .

and reaches its maximum for the maximally entangled Bell state ( $\alpha = 1/2$ ). In this latter example the entropy is maximal and it corresponds to a complete ignorance on the subsystem  $S$ . Observe that the expression given in (1.27) is nothing but the binary entropy function [28] which is equivalent to the Shannon entropy [29] of the probability vector  $(\alpha, 1 - \alpha)$ .

From the latter example it is easy to generalise it to a given pure state  $|\Psi\rangle$  of a composite system  $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$  with generic dimensions  $d_S = \dim \mathcal{H}_S \leq d_E = \dim \mathcal{H}_E$ , one gets that

$$0 \leq S(\rho_S) \leq \ln d_S. \quad (1.28)$$

Here  $S(\rho_S) = 0$  for separable state,  $|\Psi\rangle = |u\rangle \otimes |v\rangle$ , while  $S(\rho_S) = \ln d_S$  for maximally entangled states,

$$|\Psi\rangle = \frac{1}{\sqrt{d_S}} \sum_{k=1}^{d_S} |u_k\rangle \otimes |v_k\rangle, \quad (1.29)$$

with  $\{u_k\}, \{v_k\}$  are the orthogonal basis for each of system and environment. The von Neumann entropy is a measure of entanglement which leads to an objective lack of knowledge<sup>17</sup>. In fact, this objective “lack of knowledge” is related to the state of the system, because even if we knew everything about the state of the universe, every subsystem (small portion of the universe) could be mixed [31, 32]. This is something astonishing since this lack of knowledge came from nothing but the true nature of quantum mechanics, and no randomness is introduced<sup>18</sup>.

In the following part we will show that thermalisation appear as a generic property of pure states of the universe, meaning that for the overwhelming majority of them, the reduced state of the system is the canonical mixed state. As a conclusion we can state that the postulate of equal a prior probability, which refers to ensembles or time averages of states of the universe, and as such relies on a subjective lack of information, can be disregarded and one can refer only to pure state of the universe. The lack of information which will give a canonical density matrix for the system is just a physical consequence of entanglement between the system and its environment.

### 1.3.2 Canonical Typicality

In this part we are going to show that the principle of equal prior probability, which can not be proved, should be replaced with the principle of *Canonical typicality*, which is based on individual states rather than ensembles or time averages, and most importantly, can be proved.

Consider a large quantum mechanical system, “The Universe”, which we decompose in two parts, the system  $S$  and the environment  $E$ <sup>19</sup>. Now, suppose the universe has to obey some global constraint  $R$ , which translates into the choice of a subspace of the total Hilbert space, say

$$\mathcal{H}_R \subset \mathcal{H}_S \otimes \mathcal{H}_E \quad (1.30)$$

As we mentioned before, the dimensions of  $\mathcal{H}_S$ ,  $\mathcal{H}_E$  and  $\mathcal{H}_R$  will be denoted by  $d_S$ ,  $d_E$  and  $d_R$  respectively. In the standard approach to statistical mechanics, as we aforementioned, the restriction is imposed on the total energy. However, as Popescu *et. al.* mention in [2, 11],

---

<sup>17</sup>In general entropy provides a tool that can be used to quantify entanglement, although other entanglement measures exist [30] If the overall system is pure, the entropy of one subsystem can be used to measure its degree of entanglement with the other subsystems. In particular for bipartite pure states, the von Neumann entropy of reduced states is the unique measure of entanglement in the sense that it is the only function on the family of states that satisfies certain axioms required of an entanglement measure. Being this the reason why we only introduce this entropy.

<sup>18</sup>Classically the complete knowledge of the state of the universe implies a complete knowledge of the state of any subsystem. Understanding why in the case of classical mechanics, randomness was artificially added to the description.

<sup>19</sup>Here it is implicit that the dimension of the environment is much larger than the one of the system.

this restriction can be let to be completely arbitrary.

Using the definition of the equiprobable state in (1.7), we know that  $\mathcal{E}_R$  is the maximally mixed state in  $\mathcal{H}_R$ , in which pure state has equal probability<sup>20</sup>.

The *canonical state* of the system  $S$  is defined as the trace over the environment,

$$\Omega_S = \text{Tr}_E \mathcal{E}_R. \quad (1.31)$$

Instead of considering the universe in the equiprobable state  $\mathcal{E}_R$ , which describes subjective ignorance, we consider the universe to be in a random pure state  $|\phi\rangle \in \mathcal{H}_R$ . In such a case the system will be described by the its reduced density matrix

$$\rho_S = \text{tr}_B(|\phi\rangle\langle\phi|) \quad (1.32)$$

Here we ask ourselves, how different is  $\rho_S$  form the canonical state  $\Omega_S$ . The answer to this is provided by Popescu *et. al.* in [2,11], which states that  $\rho_S$  is very close to  $\Omega_S$  for every pure state compatible with the constraint  $R$ . That is, for almost every pure state of the universe, the system behaves as if the universe were actually in the equiprobable mixed state  $\mathcal{E}_R$ .

Nevertheless, it is important to stress that  $\Omega_S$  is not necessarily the thermal canonical state (1.19), but rather a generalised canonical state with respect to the arbitrary restriction  $R$  chosen<sup>21</sup>.

As an illustrative example, we can interpret this result by representing our universe obeying the global constraint  $R$  as a map chart in which we see nothing but a vast ocean representing the equiprobable states in  $\mathcal{H}_R$  and some islands representing those states who differ drastically from the canonical state. Particularly we represent with a boat a random picked state in the universe, as one can see, for the vast majority of random choices we have, this boat would land in a portion of this ocean.

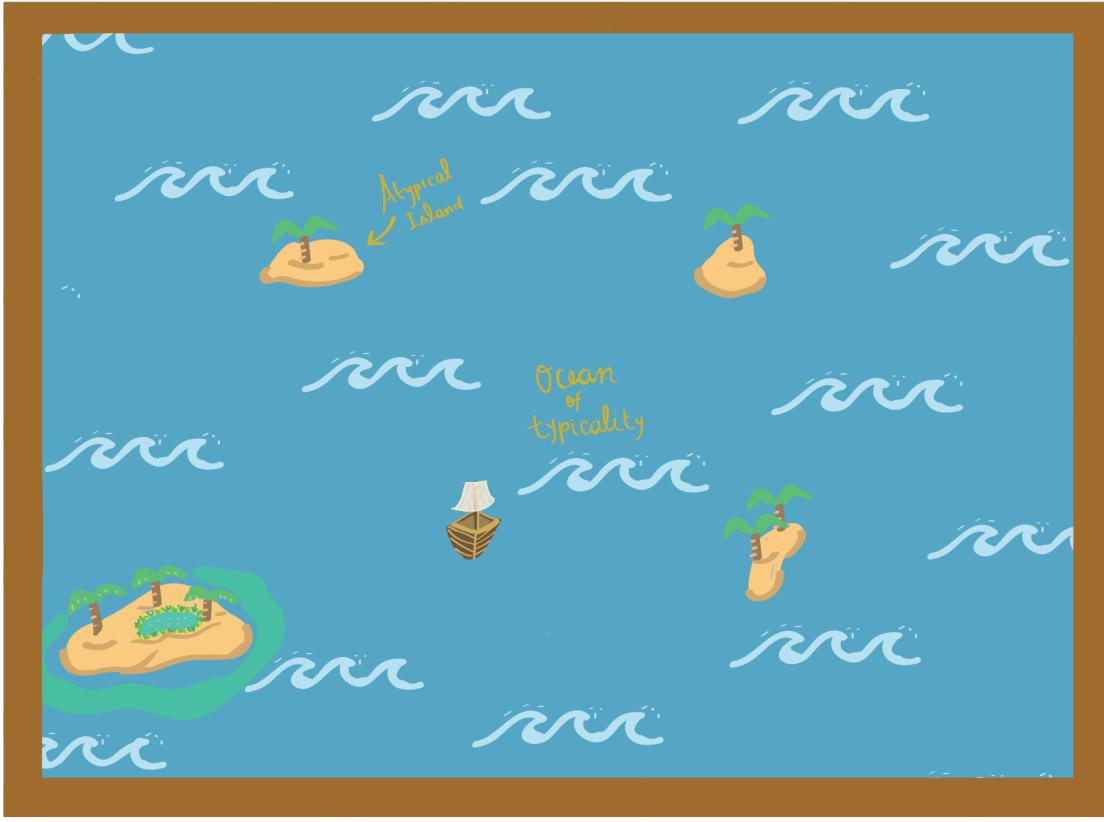
### 1.3.3 Quantitative Arguments behind Typicality

In order to formally express these ideas, it is important provide more details. To start, it is necessary to first define a notion of distance between states  $\rho_S$  and  $\Omega_S$ , as well as a measure over which pure states  $|\phi\rangle$  are defined.

---

<sup>20</sup>As Popescu mention in his paper, this corresponds to the standard intuition in classical mechanics of assigning equal a priori probabilities to all states of the universe consistent with the constraints [2,11].

<sup>21</sup>The thermodynamic interpretation is recover is the restriction imposed coincide with the total energy. In that case we can state that almost every pure state  $|\phi\rangle$  of the universe is such that the system  $S$  is approximately in the canonical thermal state  $e^{-\beta\hat{H}_S}/Z$ .



**Figure 1.2:** Illustration of the typicality in a map chart. The boat here refers to a random picked state in our universe. Here the islands refer to those states who radically differ from the Canonical state.

Defining the distance of two operators is not a trivial task. As we will see depending on the distance we choose connections to a physical quantity could emerge. We start by defining the trace distance between  $\rho_S$  and the canonical state  $\Omega_S$ , by  $\|\rho_S - \Omega\|_1$ , which is defined via the trace norm

$$\|\rho\|_1 = \text{Tr} |\rho| = \text{Tr} \left( \sqrt{\rho^\dagger \rho} \right). \quad (1.33)$$

An interesting fact about this distance is that it represents the maximal difference in probability of obtaining any outcome for any measurement performed on the states  $\rho_S$  and  $\Omega_S$ . This can be clearly seen if we write explicitly the definition of an induced norm

$$\|\rho\| = \sup_{\|M\|=1} |\text{Tr}(\rho M)|. \quad (1.34)$$

Using this, we can say that the trace distance measures how difficult is to know  $\rho_S$  and  $\Omega_S$  via a measurement  $M$ , explicitly this is

$$|\text{Tr} (\rho_S M) - \text{Tr} (\Omega_S M)| \leq \|\rho_S - \Omega_S\|_1 \|M\|, \quad (1.35)$$

An important fact to stress is the choice of distance, one could wonder, why do not we use other operator distance such like Hilbert-Schmidt norm, which is defined by taking the root square of the trace, instead of the trace of the root square, as the trace norm is defined. The reason to do so is simply because the Hilbert-Schmidt distance in higher dimensions can tend to be very small even when the states have a disjoint support [33]. To clarify this, consider in  $\mathbb{C}^{2d}$  the two states  $\rho_1 = \mathbb{I}_1/d$  and  $\rho_2 = (1 - \mathbb{I}_1)/d$ , where  $\mathbb{I}_1$  is a projector over  $d$  dimensions. An important fact over these states is that they have disjoint supports just as we mentioned before. So computing the distance between these two states give us

$$\|\rho_1 - \rho_2\|_1 = 2, \quad \|\rho_1 - \rho_2\|_2 = \sqrt{\frac{2}{d}}, \quad (1.36)$$

and then the trace norm give us a constant while the Hilbert-Schmidt distance decreases as  $d$  increases  $1/\sqrt{d}$ .

Consider  $|\phi\rangle$  to be a pure state in  $\mathcal{H}_R$ , with respective dimension  $d_R$ . As the state is normalized ( $\langle\phi|\phi\rangle = 1$ ) we know that the pure state  $|\phi\rangle$  lives in the real sphere  $(2d_R - 1)$ -dimensional. Therefore, the states we are interested on, live over the surface of a sphere of  $d_R$  dimensions, thus by random sample one of this states we will have to sample with the measure  $\sigma(\mathbb{S}^{2d_R-1})$ , which is known as the Haar measure. From this definition, it is clear to see if the selected random state over the Haar measure, the average state of the universe in  $\mathcal{H}_R$  is nothing but the equiprobable state, and then we get that average state of the system is nothing but the canonical state  $\Omega_S = \langle\rho_S\rangle$ .

Now with the notion of the distance we chose to work with, and the space where these pure states live in, we are ready to announce the general result in typicality.

**Theorem 1.3.1** (Theorem of Canonical Typicality [2, 11]). *For a random chosen state, sampled with the Haar measure,  $|\phi\rangle \in \mathcal{H}_R \subset \mathcal{H}_S \otimes \mathcal{H}_B$  and arbitrary  $\varepsilon > 0$  the distance between the reduced density matrix  $\rho_S = \text{Tr}_E(|\phi\rangle\langle\phi|)$  and the canonical state  $\Omega_S = \text{Tr}_E \mathcal{E}_R$  is given probabilistically by:*

$$\text{Prob} (\|\rho_S - \Omega_S\|_1 \geq \eta) \leq \eta', \quad (1.37)$$

where

$$\eta = \varepsilon + \sqrt{\frac{d_S}{d_E^{\text{eff}}}}, \quad \eta' = 2 \exp(-Cd_R\varepsilon^2), \quad (1.38)$$

with

$$C = \frac{1}{18\pi^3}, \quad d_E^{\text{eff}} = \frac{1}{\text{Tr} \Omega_E^2} \geq \frac{d_R}{d_S}, \quad \Omega_E = \text{Tr}_S \mathcal{E}_R \quad (1.39)$$

From the latter result we notice that  $\eta$  and  $\eta'$  are small quantities, and then, the state will be close to the canonical state with high probability, whenever  $d_E^{\text{eff}} \gg d_S$  and  $d_R\varepsilon^2 \gg$

$1 \gg \varepsilon$ . The latter condition can be ensured when  $d_R \gg 1$ .

So what our latter results holds is that probabilistically speaking if the dimension of the accessible space ( $d_R$ ) is large enough, we will have that for the overwhelming majority of choices of random pure states, we will have always (almost) that every system, with small enough dimension, will be indistinguishable from the canonical state. Even more, *Popescu et. al.* get an expression to show how the fluctuations around the average behave. For this they get the next result

$$\langle \|\rho_S - \Omega_S\|_1 \rangle \leq \sqrt{\frac{d_S}{d_E^{\text{eff}}}} \leq \sqrt{\frac{d_S^2}{d_R}}, \quad (1.40)$$

which tells us, if the dimension of our accessible space  $d_R$  is large enough, compared to the dimension of the system, the fluctuations on the system are also very small. Thus these two results give us the quantitative arguments of typicality.

Up to this moment we have provided the qualitative as well quantitative arguments to show why typicality gives us an approach different than the equal a priori probability, that can not be proved, and replace it with a key quantum property, entanglement, which provides us a subjective lack of knowledge and more important that can be proved, to show that thermalisation turns up to be a generic property of pure states of the universe. However, despite this result explains very well the reason why by choosing randomly a state  $|\phi\rangle$  over the Haar measure, it coincides with the canonical state in almost all cases, it does not explain the way a state out of equilibrium (atypical state) under certain unitary evolution can reach its thermalisation. This means that the latter result is Kinematic rather than dynamical and therefore can not be used to explain how a thermalisation in a system can occur. The goal of the next section is to introduce a work done by *Linden et. al.* [8] in which a dynamical approach is done in order to explain how thermalisation can occur in a system reliant on a given unitary dynamic.

## 1.4 Evolution Towards Equilibrium.

As we above-mentioned, the latter result is only valid for a given time and state, meaning that this can not be used to study how thermalisation occurs. Now we are interested in states that are atypical, in the sense that are going to be those states such that drastically differ from the canonical state. Even though, we might think that if typicality holds, most evolutions will quickly drive us from a state in which the system is not thermalised into one that is, and that system will remain in this state of “thermalisation” for most of its evolution. Nevertheless, there are some problems which are worth to mention about why this is much harder problem to solve. In the following we will show what could be named as the roadmap of what is needed to be proved in order to show that a thermal state has

thermalised.

- **Equilibration:** It is possible to affirm that a system will equilibrate if its states evolve towards a particular state, which can be in the more general case a mixed state, and remains in that state, or at least quite close to it, for every time<sup>22</sup>
- **Environment state independence:** The equilibrium state that the system reaches has to be independent of the initial state of the environment, this is, when the system reaches its state of equilibrium, this state should depend only on macroscopic parameters of the environment, like temperature or similar macroscopic parameters.
- **System state independence:** If the system is much smaller than the environment, the state of equilibrium should not depend of its initial state.

Having this in mind, it is possible to tackle these problems one by one. An interesting result is provided by *Linden et. al.* [8], where they manage to prove that with relatively full generality, equilibrium is an universal property of Quantum systems and even more that equilibrium state does not depend on the state of the environment.

To completely understand *Linden et. al.* work is necessary to first understand couple of definitions they provide in their paper [8].

- **Universe:** Here we will refer always to a large quantum universe living in a Hilbert space  $\mathcal{H}$ . As previously, we are considering an universe that can be decomposed in two, in this decomposition we refer the system  $S$  to a small part of the Hilbert space and the rest we will call it the environment. Explicitly we decompose the Hilbert space of the universe as a tensor product of the Hilbert space of the system and the environment,  $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ , where  $d_S$  and  $d_E$  the respective dimension of the system and the environment. Notice that here the environment nor the system have provided with any special property. Here the system could be a single particle or even a section of a lattice.
- **Hamiltonian:** The evolution of the universe will be governed by a Hamiltonian given by

$$\hat{H} = \sum_k E_k |E_k\rangle \langle E_k|. \quad (1.41)$$

with  $|E_k\rangle$  the eigenstate in the energy basis with energy  $E_k$ . The main assumption here is related with the possible values of energies this Hamiltonian can have. The only requirement needed the Hamiltonian to have non-degenerate energy gaps.

---

<sup>22</sup>Even though this definition does not specify the sort of equilibrium state of the system, the state of equilibrium will strongly depend on the initial condition of the system as well as the initial conditions over its environment.

It is said that a Hamiltonian has no-degenerate energy gaps if any non-zero difference of eigenvalues of energy determine the two energy values involved. That is, for any four eigenstates with energy  $E_k, E_\ell, E_m, E_n$ , satisfy that if  $E_k - E_\ell = E_m - E_n$ , then  $m = n$  and  $k = \ell$ , or  $k = m$  and  $\ell = n$ . Which turns out to be the same condition of imposing that the energy levels have to be non-degenerate.

Notice that the restriction imposed to the Hamiltonian is a extremely natural constraint, this is because all Hamiltonians that lack of symmetries have non-degenerate energies, so we refer to a set of Hamiltonians with measure 1.

**Notation:** We will work here with pure states for the universe represented by  $|\Psi(t)\rangle$  with a density matrix state given by  $\rho(r) = |\Psi(t)\rangle \langle \Psi(t)|$ . As above-mentioned, the state of the system is obtained by tracing out the environment at a time  $t$ , that is,  $\rho_S(t) = \text{Tr}_E \rho(t)$ . Similarly we define the state of the environment as  $\rho_E(t) = \text{Tr}_S \rho(t)$ .

We define a convenient quantity which is the transient state, or the time averaged state  $\omega$

$$\omega = \langle \rho(t) \rangle_t = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \rho(t) dt, \quad (1.42)$$

and similarly we define  $\omega_s$  and  $\omega_E$  as the time averaged state of the system and the environment respectively.

It is also convenient to re introduce a concept we have already used, the effective dimension of a mixed state  $\rho$ :

$$d^{\text{eff}}(\rho) = \frac{1}{\text{Tr}(\rho^2)}, \quad (1.43)$$

which is generally a better measurement of the effective dimension than the dimension of the support of  $\rho$ . This measure roughly tells us how many states contribute to the mixture<sup>23</sup>, carrying the probabilistic weight of different states in the mixture, and is a continuous measure.

With the concepts aforementioned Linden et. al. [8] are able to mathematically prove

*Every pure state of a quantum universe, composed by a large number of eigenstates of energy<sup>24</sup> such that evolves under an arbitrary Hamiltonian, is such that every system small enough will equilibrate.*

The reason why the latter statement requires the universe to have many changes in its time evolution, is because for equilibration to take place it is needed that part of the information

---

<sup>23</sup>Particularly, a mixture of  $n$  orthogonal states with equal probability has effective dimension of  $n$

<sup>24</sup>The reason why we need the global state to have many eigenstates of energy is because by imposing this we can assure that there will be a large quantity of changes throughout the evolution of the system.

of the initial state of the system leaves the system and enters in the environment. This notion of evolving through many states can be mathematically encapsulated via the effective dimension of the time average state  $\omega = \langle \rho(t) \rangle_t$ , and the connection between this and the number of eigenstates is with ease seen by expanding  $|Psi(t)\rangle$  as

$$|\Psi(t)\rangle = \sum_k c_k e^{-iE_k t} |E_k\rangle \quad (1.44)$$

where  $\sum_k |c_k|^2 = 1$  and hence

$$\rho(t) = \sum_{k,l} c_k c_l^* e^{-i(E_k - E_l)t} |E_k\rangle \langle E_l|, \quad (1.45)$$

which can be expanded and written as

$$\begin{aligned} \rho(t) &= \underbrace{\sum_n \|c_n\|^2 |E_n\rangle \langle E_n|}_{\omega} + \underbrace{\sum_{m \neq n} c_n c_m^* |E_n\rangle \langle E_m| e^{-it(E_n - E_m)}}_{\lambda(t)} \\ &= \omega + \lambda(t), \end{aligned} \quad (1.46)$$

Now in the case of non-degeneracy of the energy levels we have

$$\omega = \langle \rho(t) \rangle_t = \sum_k |c_k|^2 |E_k\rangle \langle E_k|, \quad (1.47)$$

leading us to

$$d^{\text{eff}}(\omega) = \frac{1}{\text{Tr}(\omega^2)} = \frac{1}{\sum_k |c_k|^4}. \quad (1.48)$$

Thus formally the statement above-mentioned can be mathematically written in terms of central quantity  $D[\rho_S(t), \omega_S]$ <sup>25</sup>, the distance between  $\rho_S(t)$ , the state of the system at a time  $t$ , and its time average,  $\omega_S = \langle \rho_S(t) \rangle_t$ . The difference between  $\rho_S(t)$  and  $\omega_S$  in the energy eigenstates can be written as

$$\rho_S(t) - \omega_S = \sum_{m \neq n} c_m c_n^* e^{-i(E_m - E_n)t} \text{Tr}_E |E_m\rangle \langle E_n|. \quad (1.49)$$

Since in general we know that  $\rho_S(t)$  fluctuates around the state  $\omega_S$ , it is evident that the distance between them will change over time. Therefore, in order to characterise these fluctuations we will be interested in the time average of distance  $\langle D[\rho_S(t), \omega_S] \rangle_t$ , so the value this average takes will tell us about where the system is spending most of its time. In other words  $\langle D[\rho_S(t), \omega_S] \rangle_t$  will be small when the system equilibrates to  $\omega_S$ .

---

<sup>25</sup>Here has in the case of typicality we use the trace distance.

To be able to prove what is announced as the *Theorem 1* in [8] it is useful to relate the trace distance to the square of the Hilbert-Schmidt distance using a standard bound provided in [34]

$$D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr}_S \sqrt{(\rho_1 - \rho_2)^2} \leq \frac{1}{2} \sqrt{d_S \text{Tr}_S (\rho_1 - \rho_2)^2}. \quad (1.50)$$

By using the concavity of the square-root function, we therefore have

$$\langle D[\rho_S(t), \omega_S] \rangle_t \leq \sqrt{d_S \langle \text{Tr}_S [\rho_S(t) - \omega_S]^2 \rangle_t}, \quad (1.51)$$

which will provide us the bound we need to proof the theorem. Now using (1.49) we write

$$\langle \text{Tr}_S [\rho_S(t) - \omega_S]^2 \rangle_t = \sum_{m \neq n} \sum_{k \neq l} \mathcal{T}_{klmn} \text{Tr}_S (\text{Tr}_E |E_k\rangle \langle E_l| \text{Tr}_E |E_m\rangle \langle E_n|), \quad (1.52)$$

where  $\mathcal{T}_{klmn} = c_k c_l^* c_m c_n^* e^{-i(E_k - E_l + E_m - E_n)t}$ . Computing this time average taking into account that the Hamiltonian has non-degenerate energy gaps<sup>26</sup> we find that

$$\begin{aligned} \langle \text{Tr}_S [\rho_S(t) - \omega_S]^2 \rangle_t &= \sum_{k \neq l} |c_k|^2 |c_l|^2 \text{Tr}_S (\text{Tr}_E |E_k\rangle \langle E_l| \text{Tr}_E |E_l\rangle \langle E_k|) \\ &= \sum_{k \neq l} |c_k|^2 |c_l|^2 \sum_{ss'bb'} \langle sb|E_k\rangle \langle E_l|s'b\rangle \langle s'b'|E_l\rangle \langle E_k|sb'\rangle \\ &= \sum_{k \neq l} |c_k|^2 |c_l|^2 \sum_{ss'bb'} \langle sb|E_k\rangle \langle E_k|sb'\rangle \langle s'b'|E_l\rangle \langle E_l|s'b\rangle \\ &= \sum_{k \neq l} |c_k|^2 |c_l|^2 \text{Tr}_E (\text{Tr}_S |E_k\rangle \langle E_k| \text{Tr}_S |E_l\rangle \langle E_l|) \\ &= \sum_{k \neq l} \text{Tr}_E [\text{Tr}_S (|c_k|^2 |E_k\rangle \langle E_k|) \text{Tr}_S (|c_l|^2 |E_l\rangle \langle E_l|)] \\ &= \text{Tr}_E \omega_E^2 - \sum_k |c_k|^4 \text{Tr}_S [(\text{Tr}_E |E_k\rangle \langle E_k|)^2] \\ &\leq \text{Tr}_E \omega_E^2, \end{aligned} \quad (1.53)$$

where  $\omega_E = \text{Tr}_S \omega$ . To obtain a further bound, we invoke weak sub-additivity of the Rényi entropy [35]

$$\text{Tr}(\omega^2) \geq \frac{\text{Tr}_E(\omega_E^2)}{\text{rank}(\rho_S)} \geq \frac{\text{Tr}_E(\omega_E^2)}{d_S}, \quad (1.54)$$

and therefore combining (1.51), (1.53) and (1.54) we get

$$\langle D[\rho_S(t), \omega_S] \rangle_t \leq \frac{1}{2} \sqrt{d_S \text{Tr}_E(\omega_E^2)} \leq \frac{1}{2} \sqrt{d_S^2 \text{Tr}(\omega^2)}, \quad (1.55)$$

---

<sup>26</sup>This condition is reflected in the evaluation by considering the terms where  $k \neq \ell$  and  $m \neq n$ , leading to  $m = \ell$  and  $k = n$  are the only terms that contribute.

which by taking the definition of effective dimension, we get the main result shown in [8]

$$\langle D [\rho_S(t), \omega_S] \rangle_t \leq \frac{1}{2} \sqrt{\frac{d_S}{d^{\text{eff}}(\omega_E)}} \leq \frac{1}{2} \sqrt{\frac{d_S^2}{d^{\text{eff}}(\omega)}}. \quad (1.56)$$

As we can see, the result obtained by Linden et. al. tell us that the vast majority of quantum systems, in which the dynamic of the universe is governed by a Hamiltonian with no gaps, will spend most of its time close to its equilibrium state independently of its initial state<sup>27</sup>.

To illustrate this result we will picture an universe as the one described before in which the islands refer to the systems regions of the universe such that are atypical and the ocean the region which represent the region where the system equilibrates. As shown in the figure 1.3 we can start from a region in the island (out of the equilibrium) and going out to the ocean (region of equilibrium), as is shown we jump from the island to the ocean so that in average we spend most of the time out of the island. Even though this results tell us that apparently the quantum property of systems is to be close to its equilibrium state, it does not take into consideration how fast it is getting to this state. Since the time averages are taken, we can say that the term of  $\lambda$  goes to zero in trace norm, that is  $\langle \|\lambda\|_1 \rangle_t \rightarrow 0$ , this can be assured since the *Random phase postulate* described above, holds. However, it is easy to see that if we take a state which its energy eigenvalues are very close to each others, the time it will take to equilibrate is exponentially large. It is not of our purpose to measure how fast a system presenting but instead we present an alternative to the *Random phase postulate*, for the special case in which the energy eigenvalues are very close to each other.

To better explain our idea, consider an universe  $|\Psi\rangle$  which is decomposed on two states that are very close to each other (meaning that live on the shell of a defined energy), for example a state of the universe belonging to an shell of a defined energy, the energy eigenstates  $|E_n\rangle$  and  $|E_m\rangle$ , the state of the universe can be written as

$$\begin{aligned} \rho &= |\Psi\rangle\langle\Psi| \\ &= \|c_n\|^2 |E_n\rangle\langle E_n| + \|c_m\|^2 |E_m\rangle\langle E_m| \\ &\quad + c_n c_m^* |E_n\rangle\langle E_m| + c_m c_n^* |E_m\rangle\langle E_n|, \end{aligned} \quad (1.57)$$

if we take the partial trace of the equation (1.57) this will lead us to the state of the system,

$$\begin{aligned} \rho_S &= \text{Tr}_E \rho = \text{Tr}_E |\Psi\rangle\langle\Psi| \\ &= \|c_n\|^2 \text{Tr}_E |E_n\rangle\langle E_n| + \|c_m\|^2 \text{Tr}_E |E_m\rangle\langle E_m| \\ &\quad + c_n c_m^* \text{Tr}_E |E_n\rangle\langle E_m| + c_m c_n^* \text{Tr}_E |E_m\rangle\langle E_n|. \end{aligned} \quad (1.58)$$

---

<sup>27</sup>Notice that the last result is not necessarily considering that the state of equilibration will coincide with the canonical state, this result is very general. Exponential bounds can be achieve if we consider the energy eigenvalues of the Hamiltonian to have no rational dependencies, this restriction leads to exponential bound in  $d^{\text{eff}}(\omega)$  (check Appendix C in [8]).



**Figure 1.3:** Illustration of the result from Linden et. al. drawn in a map chart. The ship here refers to the state sailing in our universe. Whereas the islands refer to the region in which our ship is out of equilibrium.

Something we notice from (1.58) is that since we consider the state to have its eigenvalues of energy to be quite close to each other, by typicality we know that  $\text{Tr}_E |E_n\rangle \langle E_n| = \text{Tr}_E |E_m\rangle \langle E_m| = \Omega(E)$ , with  $\Omega(E)$  the canonical state. Therefore the equation (1.58) can be written as

$$\rho_S = \Omega(E) + c_n c_m^* \text{Tr}_E |E_n\rangle \langle E_m| + c_m c_n^* \text{Tr}_E |E_m\rangle \langle E_n|. \quad (1.59)$$

So in order to get the fulfil the equality, we need that the last terms of (1.59) to vanish, and happens if and only if ultra orthogonality holds. This name refers to an apparent orthogonality in the states when applying partial traces, that is

$$\text{Tr}_E |E_n\rangle \langle E_m| = \text{Tr}_E |E_m\rangle \langle E_n| \approx O\left(\sqrt{\frac{d_S}{d^{\text{eff}}(\omega_E)}}\right), \quad (1.60)$$

which tell us that these crossed terms should be of the order of the fluctuations in the system, because dimension generally grows exponentially with the number of particles, we would expect any fixed power of  $d_S$  to be much smaller than than  $d^{\text{eff}}(\omega_E)$ . Considering this, we state that an alternative mechanism which makes the systems to equilibrate emerge naturally as

a consequence of typicality.

Inspired on this, we decided to explore Ultra Orthogonality for the special case in which the terms in (1.60) of the left hand side are equal to zero, strictly speaking. What this implies is that whenever we look at the reduced state of this universe, it will be immediately stationary, that is

$$\mathrm{Tr}_{\mathcal{E}} \rho(t) \equiv \rho_{\mathcal{S}} = \omega_{\mathcal{S}}, \quad (1.61)$$

meaning that the correspondent reduced state of a fully interacting universe remains constant<sup>28</sup>. Up to this point, one might think that this condition is quite restrictive and that just few states will satisfy it. We are going to provide arguments to show that it is not the case in Fermions, and that indeed there is an exponentially large Hilbert subspace of states fulfilling this condition. More over we will show how this is possible to prove by using tools taken from Fermionic random minimum codes.

---

<sup>28</sup>Note that we are not expecting to stumble upon the thermal state, this result only tell us about the correspondent state of equilibrium for the reduced state. Nonetheless, since thermal typicality is expected on the system, one might think that this equilibrium state correspond to the canonical one.

# Chapter 2

## Fermionic States, its characterisation and the connection with Coding theory.

Up to this point we provided a detailed explanation of how in quantum mechanical description, reaching equilibrium emerges as a consequence of the structure of quantum mechanics. Even more, we stated that in the case we deal with states which its eigenenergies are close to each other, typicality will also provide us an answer of how this kind of states will reach equilibrium. As we would like to illustrate this phenomena occur, we focus our study in the fermionic case. In this Chapter we are Going to provide a background to understand how fermionic states are usually treated and why we choose to work with them. More specifically, we provide the overview of solvable fermionic systems, its connection to Majorana fermions and Gaussian states, the formalism of Grassmann for anticommuting variables, and the link between all the formalism for fermions and coding theory.

### 2.1 Overview

In many areas of physics one has to deal with solving quantum many body problems, which is often a computationally difficult if not impossible task. However, the cases which can be analitically solved are very well known, and some assumptions have to be taken into account. In spite of this considerations it has been found that a wide class of complicated Hamiltonians with many-body interactions can be often be mapped onto Hamiltonians that are quadratic in annihilation and creation operators and have the generic form [36]

$$\hat{H} = \sum_{ij} C_{ij} \hat{a}_i^\dagger \hat{a}_j + \sum_{ij} \left( A_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger + \text{h.c.} \right), \quad (2.1)$$

where  $i, j$  run from 1 to the number of modes in the system ( $N$ ) and  $\hat{a}_i, \hat{a}_i^\dagger$  are Fermionic annihilation and creation operators which satisfy the canonical anti-commutation relations (CAR) [37]

$$\{\hat{a}_k, \hat{a}_l\} = \left\{ \hat{a}_k^\dagger, \hat{a}_l^\dagger \right\} = 0, \quad \left\{ \hat{a}_k, \hat{a}_l^\dagger \right\} = \delta_{kl}. \quad (2.2)$$

A convenience when working with these kind of Hamiltonians is that can be diagonalized via a Bogoliubov- Valantin transformations transformation (i.e., canonical transformations), which maps Fermionic creation and annihilation operators on the creation and annihilation operators of non-interacting quasi-particles [38, 39]. Explicitly the transformation looks like

$$\begin{aligned} \hat{a}_i &\mapsto \gamma_i \hat{q}_i + \kappa_i \hat{q}_i^\dagger, \\ \hat{a}_i^\dagger &\mapsto \bar{\gamma}_i \hat{q}_i^\dagger + \bar{\kappa}_i \hat{q}_i. \end{aligned} \quad (2.3)$$

where  $\gamma_i, \kappa_i$  are complex numbers such that preserves the canonical anti-commutation relations given by (2.2) for  $\hat{q}, \hat{q}^\dagger$ <sup>1</sup>.

Many relevant physics models are diagonalizable via a Bogoliubov-Valantin transformations, some examples are the Hubbard model, BCS theory of superconductivity in the mean field or Hartree-Fock approximation, and certain solvable spin-chain models (After a Jordan-Wigner transformation) [37]. As we will later explain, an important feature about the class of Hamiltonians described by (2.1), if not the most important for the purpose of this project, is that not only the ground state (quasi-particle vacuum) but all eigenstates describing a excitation in a set of quasi-particles, belong to the class of so-called Fermionic Gaussian states [36]. What is important about this class of states is that are fully characterized by second order correlations, because all the higher moments factorize. This result is very well known and as Wick theorem [40, 41].

## 2.2 Majorana Fermions

Majorana fermions are fermions such that they are their own antiparticle. In the frame of condensed matter Majorana Fermions (quasi-particles) can be interpreted as a superposition of a electron state and a hole [42].

The formalism of Majorana fermions or Majorana modes of the system (with  $N$  modes) can be introduce with the operators

$$\hat{c}_{2j-1} = \hat{a}_j^\dagger + \hat{a}_j, \quad \hat{c}_{2j} = (-i) \left( \hat{a}_j^\dagger - \hat{a}_j \right). \quad (2.4)$$

---

<sup>1</sup>This relation can also be expresses as a condition over  $\gamma_i, \kappa_i$ ,

$$\gamma_i^2 + \kappa_i^2 = 1,$$

and

$$\{\hat{q}_k, \hat{q}_l\} = \left\{ \hat{q}_k^\dagger, \hat{q}_l^\dagger \right\} = 0, \quad \left\{ \hat{q}_k, \hat{q}_l^\dagger \right\} = \delta_{kl}.$$

In which case its canonical anti-commutation relations (CAR) take the form

$$\{\hat{c}_k, \hat{c}_l\} = 2\delta_{kl}. \quad (2.5)$$

The anti-commutation relations is seen to be a consequence of an  $\mathbb{R}^{2N}$  Clifford algebra<sup>2</sup>. Transformation in between Fermionic and Majorana operators is achieved by matrix of the block form

$$\Omega = \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ i\mathbb{I} & -i\mathbb{I} \end{pmatrix}, \quad (2.6)$$

and then the map from Fermionic ( $\vec{\hat{a}}^T = (\hat{a}_1, \dots, \hat{a}_1^\dagger, \dots)$ ) and Majorana ( $\vec{\hat{c}}^T = (\hat{c}_1, \dots, \hat{c}_1^\dagger, \dots)$ ) operators is written as  $\Omega \vec{\hat{a}} = \vec{\hat{c}}$ .

By changing from Fermionic operators to Majorana operators is possible, and convenient, to define de Fermionic covariance matrix which as we mentioned before, fully characterise Gaussians states<sup>3</sup>.

## 2.3 Fermionic Covariance matrix

A system of  $N$  fermion modes, described by a set of creation and annihilation operators  $\hat{a}^\dagger, \hat{a}$  and satisfying the canonical anti-commutations relations in (2.2), is Gaussian if for such system any state  $\rho$  can be written as [43]

$$\rho = \bigotimes_{k=1}^N \tilde{\rho}_k, \quad \tilde{\rho}_k = \frac{1}{2} \left( 1 - \lambda_i [\tilde{a}_i^\dagger, \tilde{a}_i] \right), \quad (2.7)$$

for a certain choice of mode basis  $\tilde{a} = u_i^j \hat{a}_j + v_i^j \hat{a}_j^\dagger$ , and with  $|\lambda_i| \leq 1$ , where the equality holds for pure states. Equivalently, as mentioned before, Gaussian States are fully characterized by their second moments, so an equivalent form of writing  $\rho$  is

$$\rho = \frac{1}{Z} \cdot \exp \left[ -\frac{i}{4} \hat{c}^T G \hat{c} \right], \quad (2.8)$$

with  $\hat{c} = (\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{2N})$ , the vector of Majorana operators (2.4),  $Z$  a normalization constant and  $G$  real anti-symmetric  $2N \times 2N$  matrix.

Since  $G$  is a skew-symmetric matrix, it can always be brought to the block diagonal form

$$OGO^T = \bigoplus_{i=1}^N \begin{pmatrix} 0 & -\beta_j \\ \beta_j & 0 \end{pmatrix} \quad \text{with} \quad O \in SO(2N), \quad (2.9)$$

---

<sup>2</sup>By inspection of (2.5) we see that any linear transformation of the form  $\tilde{\gamma}_\alpha = O_{\alpha\beta}\gamma_\beta$  where  $O \in SO(2N)$ , the special orthogonal group in  $2N$  dimensions

<sup>3</sup>In comparison to its boson counterpart the fermion Gaussian states have the property that correlation functions for the creation/annihilation operators are completely determined by the two-point functions according to Wick's theorem [40], and moreover, since this property is extensible to correlation function pertaining to a reduced subset of the modes, it follows that any partial (reduced) density matrix obtained from  $\rho$  remains Gaussian.

by a special orthogonal matrix  $O \in SO(2N)$  where the  $\beta_j$  are called the Williamson eigenvalues of the matrix  $G$ . From equation (2.8) it is clear that Gaussian states have an interpretation as thermal (Gibbs) states corresponding to a Hamiltonian of the form

$$\hat{H} = \frac{i}{4} \hat{c}^T G \hat{c} = \frac{i}{4} \sum_{k>l} G_{kl} [\hat{c}_k, \hat{c}_l] \quad (2.10)$$

and (2.9) shows that every Gaussian state has a normal mode decomposition in terms of  $N$  single mode ‘‘thermal states’’ of the form (2.7) ( $\sim \exp(-\beta \hat{a}^\dagger \hat{a})$ ) [44]. From this is clear, that the state can be fully determined by the expectation values of quadratic operators ( $\hat{a}_i^{(\dagger)} \hat{a}_j^{(\dagger)}$  and  $\hat{a}_i^\dagger \hat{a}_j$ ). So collecting these expectation values in a real and skew-symmetric covariance matrix  $\Gamma$  which is defined via

$$\Gamma_{kl} = \frac{i}{2} \text{tr} (\rho [c_k, c_l]). \quad (2.11)$$

We will be able to bring this anti-symmetric matrix to its block diagonal form, via a canonical transformation.

$$\tilde{\Gamma} = O \Gamma O^T = \bigoplus_{i=1}^M \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}, \quad (2.12)$$

with  $\lambda_i \geq 0$  the Williamson eigenvalues.

In terms of the creation/annihilation operators obtained from the transformed  $\tilde{c} = O \hat{c}$ , the Gaussian state  $\rho$  takes the form (2.8) with  $\tilde{\Gamma}$  as its Fermionic covariance matrix. It is easy to see that the relation between  $G$  and  $\Gamma$  is given by  $\lambda_i = \tanh(\beta_i/2)$ , for  $i = 1, 2, \dots, N$  [44]. The equivalence between the special orthogonal group in  $2N$  dimensions ( $SO(2N)$ ) and the Fermionic Gaussian states, drives to an interesting property about states describing multi-particles excitations. If  $|vacuum\rangle$  is the ground state of some Hamiltonian, with annihilation operators  $\hat{a}_i$  in a given quasi-particle basis, then  $\hat{a}_i^\dagger |vacuum\rangle = \hat{c}_{2i} |vacuum\rangle$ . Meaning that any multi-particle state of this kind is obtained from some transformation (that preserves the canonical anti-commutation relations) of the ground state  $|vacuum\rangle$ , remains Gaussian. In other words, Gaussian states are preserved under any unitary transformation that preserves anti-commutation relations.

The fact that all eigenstates of the Hamiltonian in (2.1) are Gaussian as well as the extension of this property to a reduced subset of the modes, is quite important since a big part of this work is focused on the study of the Fermionic covariance matrix of the  $XY$  model.

Up to this point we have talked about some generalities about the quadratic Hamiltonians and how these can be brought to its diagonal form to be analytically solved. However, one may wonder how is this connected to some observables, if the Fermionic covariance matrix has something to do in the observables. The answer to these questions will be boarded in the next section in which we will provide the formalism needed to compute observables over anti-commuting variables as well as the differences we will have between bosons and fermions.

## 2.4 Grassmann Approach

Fermions are one kind of fundamental particles which can be used to represent quantum information, being one possibility to do quantum computation. From Knill, Laflamme and Milburn [45] work, we know that it is possible to provide an universal set of operations for quantum computation by just using passive linear objects. Right after this discovery, Terhal, DiVincenzo [46] and knill [47] provided a description of the computational capability for Fermionic linear optics (FLO)<sup>4</sup>. These results show that FLO is not a promising way to do quantum computation, since it can be efficiently simulated by classical means, on the other hand, this provides a very interesting tool to study general properties quantum channels for quantum communication.

When working with fermions one could think, weather or not is possible to use the same formalisms than the ones used when working with bosons<sup>5</sup>, and as expected many features between them are shared. More specifically, in the case of FLO, it shares most of the features with a limited version of photon linear optics, where mostly Gaussian states appears [45]<sup>6</sup>. It is worth mentioning that some of the results in here are quite general and can be used even beyond the scope of our work.

### 2.4.1 Fermionic Linear Optics

In order provide a more specific description of FLO, we start by defining  $N$  abstract Fermionic modes which can be defined by creation  $\hat{a}_j$  and annihilation  $\hat{a}_j^\dagger$ ,  $j = 1, \dots, N$ , satisfying (2.2). As mentioned above, any state of the system can be generated from the vacuum state  $|vacuum\rangle$  of the Fock basis, which we define as

$$|N_1, \dots, N_N\rangle = \left(\hat{a}_1^\dagger\right)^{N_1} \cdots \left(\hat{a}_n^\dagger\right)^{N_n} |vacuum\rangle, \quad (2.13)$$

with  $N_i \in \{0, 1\}$ . We have then that any sequence  $\{N_i\}$  can be understood as a Fermionic state which comes from a superposition of the Fock basis.

When working with FLO, quadratic Hamiltonians govern the dynamics, in which terms associated with individual energy modes, tunneling and bulk dynamics are included. The structure of these Hamiltonians is what allow us conveniently work with operators which generate the *Clifford* algebra ( $\mathcal{C}_{2N}$ ), described in (2.5). So we will have that any arbitrary

<sup>4</sup>Whenever we are referring to this term, it basically has in consideration a system consisting of non-interacting electrons in a controllable external potential and a detector that measures projectively occupation numbers of single-electron modes.

<sup>5</sup>By this we mean that it could be also written in its correspond operators of annihilation and creation  $\hat{a}^\dagger, \hat{a}$  [48]

<sup>6</sup>the main reason for this is that a set of states that can be achieved by FLO operations starting from the Fock vacuum is a set of Fermionic Gaussian states, just as the case of its bosonic counterpart.

operator  $X \in \mathcal{C}_{2N}$  will be written as a polynomial in the operators  $\{\hat{c}_a\}$  as

$$X = \alpha \hat{I} + \sum_{p=1}^{2n} \sum_{1 \leq a_1 < \dots < a_p \leq 2n} \alpha_{a_1, \dots, a_p} \hat{c}_{a_1} \cdots \hat{c}_{a_p}, \quad (2.14)$$

with  $\alpha = 2^{-n} \text{Tr}(X)$ . Particularly for the Hamiltonian operator we have that

$$H = \frac{i}{4} \sum_{a,b=1}^{2N} H_{ab} \hat{c}_a \hat{c}_b, \quad (2.15)$$

with  $\{H_{ab}\}$  an anti-symmetric  $2N \times 2N$  matrix.

We may think that when working with fermions could be the same as working with bosons by changing commutators with anti-commutators and changing to a fine algebra. However, this is not the case, as we will show in the next section observables have to be computed by using the corresponding calculus for fermionic systems.

### 2.4.2 Grassmann Calculus

There are a lot of results from the earlier seventies which describe techniques to study systems with infinite number of fermionic modes [49], nonetheless, the formalism we are going to present here is a specific case where a finite number of modes is considered, and consist in a customization of the Lagrangian representation for infinite number of Fermionic modes, presented in [50]. Consider  $\theta_1, \dots, \theta_N$  span a N-dimensional complex linear space  $\mathbb{C}^N$ . From the anti-commutation relation we get that for  $\theta_1, \dots, \theta_N$ ,

$$\theta_a^2 = 0 \quad \text{and} \quad \theta_a \theta_b + \theta_b \theta_a = 0, \quad (2.16)$$

so it is clear that the most general function we can built over the Grassmann algebra with complex coefficients  $\mathcal{G}_N$  is a polynomial of  $\theta$ 's

$$f(\theta) = \alpha + \sum_{p=1}^n \sum_{1 \leq a_1 < \dots < a_p \leq N} \alpha_{a_1, \dots, a_p} \theta_{a_1} \cdots \theta_{a_p}, \quad (2.17)$$

where the coefficients  $\alpha_*$  are complex numbers. A polynomial  $f(\theta)$  will be called *even* if it involves only even powers of  $\theta$ , nonetheless something to stress is that even elements constitute the center of the Grassmann algebra.

It is possible to differentiate functions of Grassmann variables. A partial derivative over  $\theta_a$  is a linear operator from  $\mathcal{G}_N$  to  $\mathcal{G}_N$ , which is defined by

$$\frac{\partial}{\partial \theta_a} 1 = 0, \quad \frac{\partial}{\partial \theta_a} \theta_b = \delta_{ab}, \quad (2.18)$$

and Liebniz's rule

$$\frac{\partial}{\partial \theta_a} (\theta_b f(\theta)) = \delta_{ab} f(\theta) - \theta_b \frac{\partial}{\partial \theta_a} f(\theta). \quad (2.19)$$

The equality (2.19) implies that

$$\left\{ \frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b} \right\} = 0. \quad (2.20)$$

Since a derivative  $\frac{\partial}{\partial \theta_a} f(\theta)$  does not depend upon variable  $\theta_a$ , it is sometimes convenient to think about differentiation as a linear operator mapping  $\mathcal{G}_n$  into  $\mathcal{G}_{n-1}$ . Such operator is called integration and is denoted as [51]<sup>7</sup>

$$\int d\theta_a \equiv \frac{\partial}{\partial \theta_a} : \mathcal{G}_n \rightarrow \mathcal{G}_{n-1}. \quad (2.21)$$

From equation (2.20) we have that

$$\int D\theta \frac{\partial}{\partial \theta_a} f(\theta) = 0, \quad (2.22)$$

this equation combined with (2.19) give us the anti-commuting version of integration by parts.

With these properties we show the main equation that will help us calculate what is need for the purpose of this work.

*For a vectors of Grassmann variables  $\vec{\theta}$ ,  $\vec{\eta}$  and complex anti-symmetric matrix  $M$  we have*

$$\int D\theta \exp\left(\frac{i}{2}\theta^T M \theta\right) = i^n \text{Pf}(M), \quad (2.23)$$

and

$$\int D\theta \exp\left(\eta^T \theta + \frac{i}{2}\theta^T M \theta\right) = i^n \text{Pf}(M) \cdot \exp\left(-\frac{i}{2}\eta^T M^{-1} \eta\right). \quad (2.24)$$

In these formulas  $\text{Pf}(N)$  is the Pfaffian of a complex antisymmetric matrix  $N$  defined as anti-symmetrized product  $\mathcal{A}(N_{1,2}N_{3,4}\cdots N_{2n-1,2n})$  that is

$$\text{Pf}(N) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) N_{\sigma_1, \sigma_2} \cdots N_{\sigma_{2n-1}, \sigma_{2n}} \quad (2.25)$$

### 2.4.3 Gaussian States

For this part we are going to introduce in a formal way the states we have been talking about in at the beginning of this chapter. Informally we can say that any Gaussian operator can be

---

<sup>7</sup>Here we will use a compact notation

$$\int D\theta \equiv \int d\theta_n \cdots \int d\theta_2 \int d\theta_1,$$

and the order is chosen such that

$$\int D\theta \theta_1 \cdots \theta_n = 1$$

represented as an exponent of another operator which is quadratic in creation/annihilation operators. When considering an operator  $X \in \mathcal{C}_{2N}$  It is natural to assign a polynomial  $\omega(X, \theta) \in \mathcal{G}_{2N}$  of  $2N$  Grassmann variables defined by

$$\omega(\hat{c}_p \hat{c}_q \cdots \hat{c}_r, \theta) = \theta_p \theta_q \cdots \theta_r, \quad \omega(\hat{\mathbb{I}}, \theta) = 1, \quad (2.26)$$

where this definition comes from the general functions in the Grassmann algebra described in (2.17).

To illustrate what how this map can be done, consider as an example the projector  $\hat{a}_1 \hat{a}_1^\dagger$ , which its action is nothing but to project into a state where the first mode is empty. For this precise case we have that the map to Grassmann variables will look as

$$\omega(\hat{a}_1 \hat{a}_1^\dagger, \theta) = \frac{1}{2} (\mathbb{I} + i\theta_1 \theta_2) = \frac{1}{2} \exp(i\theta_1 \theta_2), \quad (2.27)$$

where all exponents were defined by a Taylor series.

For any two operators  $X, Y \in \mathcal{C}_{2N}$  one can compute the trace by just using a simple formula,

$$\text{Tr}(XY) = (-2)^N \int D\theta D\mu e^{\theta^T \mu} \omega(X, \theta) \omega(Y, \mu), \quad (2.28)$$

which can be directly verified. It is also easy to see that canonical transformations of an operator  $X \in \mathcal{C}_{2N}$  are equivalent to an orthogonal change of basis in the space of the Grassmann variables, this is

$$\omega(VXV^\dagger, \theta) = \omega(X, \eta), \quad \eta_a = \sum_{b=1}^{2n} R_{ab} \theta_b. \quad (2.29)$$

So we will define a Gaussian states of  $N$  Fermionic modes as via its density operator  $\rho \in \mathcal{C}_{2N}$ . We say that a state is Gaussian in  $\mathcal{C}_{2N}$  iff its Grassmann representation is as well Gaussian

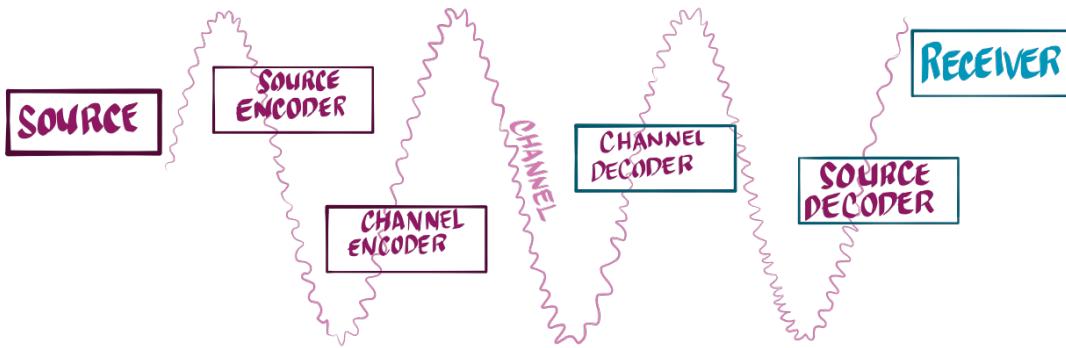
$$\omega(\rho, \theta) = \frac{1}{2^n} \exp\left(\frac{i}{2} \theta^T M \theta\right), \quad (2.30)$$

for some  $2N \times 2N$  antisymmetric matrix  $M$ . The matrix  $M$  is defined as

$$M_{ab} = \frac{i}{2} \text{Tr}(\rho [\hat{c}_a, \hat{c}_b]) = \begin{cases} \text{Tr}(\rho i \hat{c}_a \hat{c}_b) & \text{for } a \neq b \\ 0 & \text{for } a = b \end{cases}, \quad (2.31)$$

which is nothing but the covariance Fermionic matrix defined in (2.11), meaning that it can be brought to its Williamson form to get its eigenvalues.

Therefore the connection between the Grassmann formalism for Gaussian states and the Fermionic covariance matrix described at the beginning of the chapter, is easily understood



**Figure 2.1:** Representation of the scheme of communication. In the image the noise in the channel is represented by the noise wavy connection between the parts in the communication.

by simply assigning the correspondent covariant matrix to the states we work with. Nonetheless, one could be wondering how is this formalism connect to the error correcting code theory. In the next section we will provide a little historical background of how this theory was done and how one can think states of fermions as binary codes. More specifically we will provide the necessary background to link the theory of correcting errors with Fermionic states which are constrained over a shell of energy in the Hilbert Space.

## 2.5 Error correcting Code Theory

In this section we will provide some concepts and definitions in order to understand how the Coding theory can be linked to Fermionic states.

In 1948, Claude Shannon presents his extraordinary work named “A Mathematical Theory of Communication” [29] in which he provided a precise measure of the information content of a random variable in terms of its entropy. His work is divided in two parts the noiseless coding theorem and the noisy channel theorem. For the purpose of our needs we are going to focus only in the second part of his work, which states that a reliable communication is possible if we use schemes such that its rate is less than the capacity of the channel. Even though he never provided an idea of how this schemes could be found, his work is considered one of the most relevant discovery of the century.

Here we will consider codes in communication scenario, as the one showed in figure 2.1,

meaning that there will be a sender who wants to send  $k$  message symbols over a noisy channel and there will be a receiver who has to correct possible errors over the sent code to fully interpret it. The sender will first encode the  $k$  message symbols into  $n$  symbols. The receiver then tries to recover the original  $k$  message symbols. thus, encoding is the process of adding redundancy and decoding is the process of removing errors and the communication can only be done over the channel [28]. The most fundamental question one can ask is what will be the relation between the amount of redundancy and the errors that can be corrected, and in order to answer this question we will provide some useful definitions.

## 2.6 Some basic definitions

**Definition 2.6.1** (Code). *A code of block  $C$  length  $n$  over an alphabet  $\Sigma$  is a subset of  $\Sigma^n$ . If  $|\Sigma| = q$ , we say that  $C$  is a  $q$ -ary code.*

it is worth mention that, associated with a code there is also an encoding map  $E$  which maps the message set  $\mathcal{M}$ , identified in some canonical way with  $\{1, 2, \dots, |C|\}$  say, to code words belonging to  $\Sigma^N$ , and thus we have to understand the code as the image of the encoding map [52].

**Definition 2.6.2** (Dimension of a code). *Given a code  $C \subset \Sigma^n$ , its dimension is given by*

$$k \stackrel{\text{def}}{=} \log_q |C|, \quad (2.32)$$

An interesting fact about defining the dimension of the code in this way is that implicitly it is telling us that when working with codes exponential growth will be always taken into account.

We have to provide here a way to measure the amount of redundancy in a given message.

**Definition 2.6.3** (Rate of a code). *The rate of a code with dimension  $k$  and block length  $n$  is given by*

$$R \stackrel{\text{def}}{=} \frac{k}{n}, \quad (2.33)$$

this definition is nothing but the average amount of non redundant information each of the  $n$  symbols transmitted over the channel.

However, an alternative, and more general way of defining this is via the size of the code and the alphabet as

$$R(C) = \frac{\log |C|}{n \log |\Sigma|}. \quad (2.34)$$

**Definition 2.6.4** (Hamming distance). *The Hamming distance between two strings  $x$  and  $y$  of the same length over a finite alphabet  $\Sigma$ , denoted  $\Delta(x, y)$ , is defined as the number of positions at which the two strings differ, i.e.,  $\Delta(x, y) = |\{i | x_i \neq y_i\}|$ . The fractional Hamming distance or relative distance between  $x, y \in \Sigma^n$  is given by  $\delta(x, y) = \frac{\Delta(x, y)}{n}$ .*

It is trivial to check that the Hamming distance defines a metric on  $\Sigma^n$ .

**Definition 2.6.5. Hamming weight:** *The Hamming weight of a string  $x$  over alphabet  $\Sigma$  is defined as the number of non-zero symbols in the string. More formally, the Hamming weight of a string  $\mathcal{W}(x) = |\{i|x_i \neq 0\}|$ . Note that  $\mathcal{W}(x - y) = \Delta(x, y)$ .*

Given a string  $x \in \Sigma^N$ , the Hamming ball of radius  $r$  around  $x$  is the set  $\{y \in \Delta^n | \Delta(x, y) \leq r\}$ .

The minimum distance, or simply distance, of a code  $C$ , denoted  $\Delta(C)$ , is defined to be the minimum Hamming distance between two distinct code words of  $C$ . That is

**Definition 2.6.6** (Minimum distance). *The minimum distance, or simply distance, of a code  $C$ , denoted  $\Delta(C)$ , is defined to be the minimum Hamming distance between two distinct code words of  $C$ . That is*

$$\Delta(C) = \min_{\substack{c_1, c_2 \in C \\ c_1 \neq c_2}} \Delta(c_1, c_2). \quad (2.35)$$

In particular, for every pair of distinct code words in  $C$  the Hamming Distance between them is at least  $\Delta(C)$

The relative distance of  $C$ , denoted  $\delta(C)$ , is the quantity  $\frac{\Delta(C)}{N}$ , where  $N$  is the block length of  $C$ . Thus any two code words of  $C$  differ in at least a fraction  $\Delta(C)$ .

**Definition 2.6.7** (Notation). *A  $q$ -ary code of block length  $N$  and dimension  $k$  will be referred to as an  $[N, k]_q$  code. Further, if the code has minimum distance  $d$ , it will be referred to as an  $[N, k, d]_q$  code. When the alphabet size  $q$  is clear from the context, or not very relevant to the discussion, we omit the subscript.*

Up to this point we have only described specific codes, codes with fixed block length and dimension. However, since we are interested in the asymptotic behaviour, it turns out to be more useful the study of families of codes instead of an specific code.

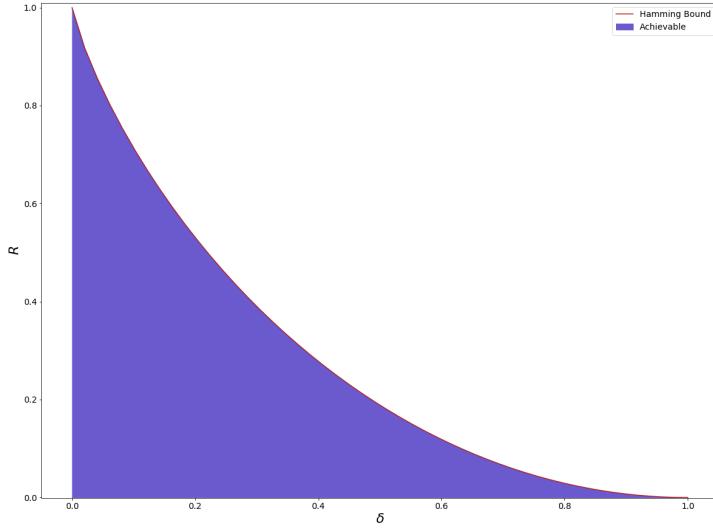
**Definition 2.6.8** (Family of Codes). *Let  $q \geq 2$ . let  $\{n_i\}_{i \geq 1}$  be and increasing sequence of block lengths and suppose there exists sequences  $\{k_i\}_{i \geq 1}$  and  $\{d_i\}_{i \geq 1}$  such that for all  $i \geq 1$  there exist an  $[n_i, k_i, d_i]_q$  code  $C_i$ . then the sequence  $C = \{C_i\}_{i \geq 1}$  is a family of codes. The rate of  $C$  is defined as*

$$R(C) = \lim_{i \rightarrow \infty} \left\{ \frac{k_i}{n_i} \right\}, \quad (2.36)$$

and the relative distance of  $C$  is defined as

$$\delta(C) = \lim_{i \rightarrow \infty} \left\{ \frac{d_i}{n_i} \right\}, \quad (2.37)$$

from now on whenever we talk about a code we are implicitly referring to a family of codes.



**Figure 2.2:** An illustration of the Hamming bound for the case of  $q = 2$ . Note any code above this bound could exist. The shaded region shows the codes that could exist.

For the purpose of this work, we are going to focus on a particular class of codes, name minimum distance code. This special kind of codes came to our interest since they seem to appear naturally when study typical Fermionic states. We will then show some of the main results on this particular kind of codes to after show how this results could be extended to our particular case.

When we talk about codes of minimum distance  $d$  we refer to codes which have the property that for every pair  $x, y$  of codewords we have that

$$\Delta(x, y) = d, \quad (2.38)$$

and therefore, the its relative distance  $\delta$  is given by  $N\delta = d$ .

when working with this kind of codes one may wonder what is the best rate we can achieve. Particularly we are going to show 3 results, 1 positive and 2 negative results<sup>8</sup>. Even though there are other known bounds, we will not talk about others but Gilbert-Varshamov, Hamming and Plotkin bounds, the reason for this is due to the fact that the other bound apply for large enough alphabets, so for binary codes we are not interested at all in these kind of bounds [28].

---

<sup>8</sup>Note that a negative result refer to an upper bound on the rate, meaning that the maximum achievable rate we could get can not exceed some value. Whereas a positive result refer to lower bound on the rates we can achieve.

### 2.6.1 Hamming Bound (Sphere packing bound)

**Definition 2.6.9** (Volume of Hamming ball). *Let  $q \geq 2$  and  $n \geq r \geq 1$  be integers. Then the volume of a Hamming ball of radius  $r$  is given by*

$$\text{Vol}_q(r, n) = |B_q(\mathbf{0}, r)| = \sum_{i=0}^r \binom{n}{i} (q-1)^i, \quad (2.39)$$

where the choice of  $\mathbf{0}$  as the center of the Hamming ball is chosen arbitrary, since the volume of the Hamming ball is independent of its center.

It is simple to show that

$$\frac{k}{n} \leq 1 - \frac{\log_q \text{Vol}_q\left(\left\lfloor \frac{d-1}{2} \right\rfloor, n\right)}{n}, \quad (2.40)$$

where the volume in (2.40) correspond to the definition in (2.39). With some algebra and using the Stirling asymptotic approximation one can show that

$$\text{Vol}_q\left(\left\lfloor \frac{d-1}{2} \right\rfloor, n\right) \geq q^{H_q\left(\frac{\delta}{2}\right)n-o(n)}, \quad (2.41)$$

where the latter inequality immediately provide us an upper bound on the rate

$$R \leq 1 - H_q\left(\frac{\delta}{2}\right) + o(1). \quad (2.42)$$

The inequality in (2.42) is known as the Hamming Bound.

### 2.6.2 Plotkin Bound

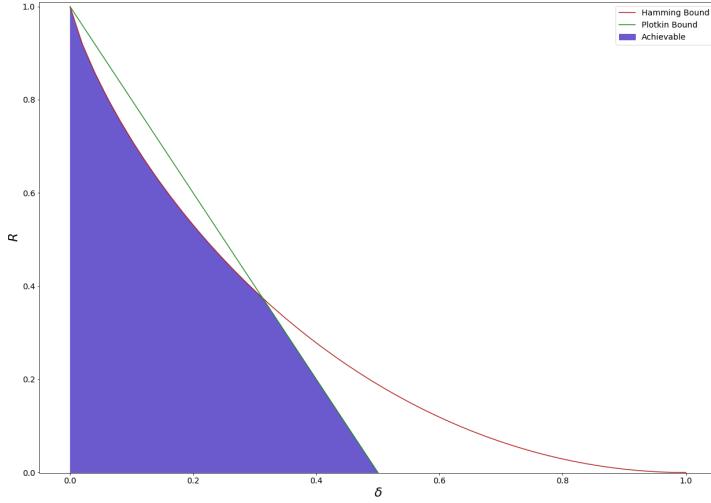
**Definition 2.6.10** (Plotkin Bound). *The following holds for any code  $\mathcal{C} \subset [q]^n$*

- If  $d = \left(1 - \frac{1}{q}\right)n$ ,  $|\mathcal{C}| \leq 2qn$ .
- If  $d > \left(1 - \frac{1}{q}\right)n$ ,  $|\mathcal{C}| \leq \frac{qd}{qd-(q-1)n}$ .

Note that the Plotkin Bound implies that a code with relative distance  $\delta \geq 1 - 1/q$ , must necessarily have  $R = 0$ .

**Definition 2.6.11.** *For any  $q$ -ary code with relative distance  $0 \leq \delta \leq 1 - \frac{1}{q}$ ,*

$$R \leq 1 - \left(\frac{q}{q-1}\right)\delta + o(1). \quad (2.43)$$



**Figure 2.3:** An illustration of Plotkin and Hamming bounds for the case of  $q = 2$ . For this case the shaded region changes indicating us that there are not codes with  $R > 0$  when  $\delta = 1 - 1/q$ .

To illustrate the proof of this bound we can consider the distance  $d = n\delta$ . So we can shorten the codewords and group them in a way such that they agree on the first  $n - n'$  symbols, with  $n' = \left\lfloor \frac{qd}{q-1} \right\rfloor - 1$ . Then in particular for any  $x \in [q]^{n-n'}$ , define the prefix code

$$\mathcal{C}_x = \{(c_{n-n'+1}, \dots, c_n) \mid (c_1 \dots c_N) \in \mathcal{C}, (c_1 \dots c_{n-n'}) = x\}. \quad (2.44)$$

for all  $x$ ,  $\mathcal{C}_x$ , has distance  $d$  as  $\mathcal{C}$  has distance  $d$ . Additionally, it has block length  $n' < \left(\frac{q}{q-1}\right)d$ , and thus  $d > \left(1 - \frac{1}{q}\right)n'$ . From the Plotkin bound, this implies that

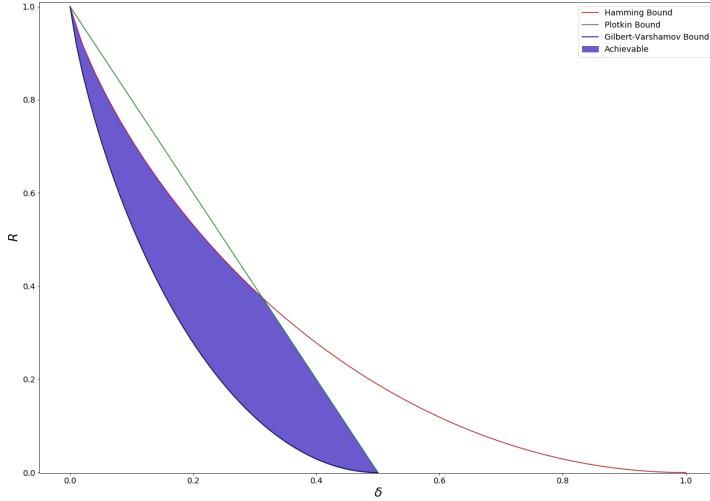
$$|\mathcal{C}_x| \leq \frac{qd}{qd - (q-1)n'} \leq qd, \quad (2.45)$$

where the second inequality follows from the fact that  $qd - (q-1)n'$  is an integer. Note that from the definition of  $\mathcal{C}_x$

$$|\mathcal{C}| = \sum_{x \in [q]^{n-n'}} |\mathcal{C}_x|, \quad (2.46)$$

which tell us that

$$|\mathcal{C}| \leq \sum_{x \in [q]^{n-n'}} qd = q^{n-n'} \cdot qd \leq q^{n-\frac{q}{q-1}d+o(n)} = q^{n(1-\delta \cdot \frac{q}{q-1} + o(1))}, \quad (2.47)$$



**Figure 2.4:** An illustration of 3 bounds Hamming, Plotkin and Gilbert-Varshamov for the case of  $q = 2$ . The lower bound correspond to the Gilvert-Varshamov Bound whereas the other 2 are the upper bound for the rate.

in other words this provides another upper bound to the rate given by

$$R \leq 1 - \left( \frac{q}{q-1} \right) \delta + o(1) \quad (2.48)$$

To close this part we will show the latter but positive result which provide us lower bound on the code rates, Gilbert Varshamov bound.

### 2.6.3 Gilbert Varshamov Bound

We now switch gears to provide a positive result. We will only provide the main ideas for the proof of this result an we will discuss why this result turn out to be one of the most important results.

**Definition 2.6.12** (Gilbert-Varshamov Bound). *Let  $q \geq 2$ . For every  $0 \leq \delta < 1 - \frac{1}{q}$  and  $0 < \varepsilon \leq 1 - H_q(\delta)$ . There exist a code with rate  $R \geq 1 - H_q(\delta) - \varepsilon$  and relative distance  $\delta$ .*

To provide a main idea about the proof we can consider a greedy approach. First we start with an empty code  $\mathcal{C}$  and we keep adding vectors that are not in  $\mathcal{C}$  and that have Hamming distance at least  $d$  from all the existing codewords in  $\mathcal{C}$ . Notice that by doing so we can assure that we will never add a vector  $c$  that will make that will make the distance of  $\mathcal{C}$  fall below  $d$ . Indeed it is easy to see that after doing so, we have

$$\bigcup_{\mathbf{c} \in \mathcal{C}} B(\mathbf{c}, d-1) = [q]^n, \quad (2.49)$$

this is easily checked, because if it was not true, then there would exist a vector  $\mathbf{v} \in [q]^n \setminus C$ , such that  $\Delta(\mathbf{v}, \mathbf{c}) \geq d$  and therefore  $\mathbf{v}$  can be added. How ever, this would contradict the fact that we have finished the procedure. So

$$\left| \bigcup_{\mathbf{c} \in C} B(\mathbf{c}, d - 1) \right| = q^n. \quad (2.50)$$

It isn't hard to see that

$$\sum_{\mathbf{c} \in C} |B(\mathbf{c}, d - 1)| \geq \left| \bigcup_{\mathbf{c} \in C} B(\mathbf{c}, d - 1) \right|, \quad (2.51)$$

which implies that

$$\sum_{\mathbf{c} \in C} |B(\mathbf{c}, d - 1)| \geq q^n, \quad (2.52)$$

but as mentioned before, the volume of the Hamming ball is translation invariant,

$$\sum_{\mathbf{c} \in C} \text{Vol}_q(d - 1, n) \geq q^n. \quad (2.53)$$

Since  $\sum_{\mathbf{c} \in C} \text{Vol}_q(d - 1, n) = \text{Vol}_q(d - 1, n) \cdot |C|$

$$\begin{aligned} |C| &\geq \frac{q^n}{\text{Vol}_q(d - 1, n)} \\ &\geq \frac{q^n}{q^{nH_q(\delta)}} \\ &= q^{n(1 - H_q(\delta))}. \end{aligned} \quad (2.54)$$

And therefore concluding the proof. It is worth mention that this way of proceeding the code have not any special structure but as one might think this algorithm will take exponentially long time to finish. How ever, one may wonder if there is a special kind of code that also achieve this rate. Indeed is possible to show that random linear codes lies, with high probability, on the Gilbert-Varshamov Bound. To pick a linear random code we only need pick a random  $k \times n$  matrix, in which each entry is chosen uniformly and independently at random according to its alphabet [28, 52]. Aside from these result providing some bounds on the rate of the minimum distance codes. We are interested in the performance of a special kind of random codes, more specifically binary codes over a binary-symmetric channel (BSC). In here we derive the minimum distance, distance distribution and error exponent of a typical random code (TRC) from a random code ensemble (RCE), as well as the one correspondent to a typical linear code (TLC) from a linear code ensemble (LCE) [53]. As mentioned by *A Barg, and G. D. Forney, Jr* most of the important of these results are expressed in terms of the Gilbert-Varshamov distance  $\delta_{GV}(R)$ .

### 2.6.4 Error Exponents for Random Minimum Distance Codes

It is very well known that on a BSC with crossover probability  $p$ , the channel capacity is  $C = 1 - H_2(p)$ <sup>9</sup>. The error coding exponent  $E_r(R)$  is positive for  $0 \leq R < C$  and given by [54, 55].

$$E_r(R) = \begin{cases} R_0 - R, & 0 \leq R \leq R_{\text{crit}} \\ E_{\text{sp}}(R), & R_{\text{crit}} \leq R \leq C \end{cases}, \quad (2.55)$$

where  $R_0$ ,  $R_{\text{crit}}$  and  $E_{\text{sp}}(R)$ , are known as cutoff rate, critical rate and the sphere-packing exponent respectively. In [56] Gallager has shown that the random coding exponent is the true error exponent for the RCE on any discrete memoryless channel. Here we will show the main results provided in [53] and we will provide the ideas to derive these results, as we will show later this ideas of error exponents will be quite helpful to understand how it is possible to make a connection between minimum distance codes and typical Fermionic states.

#### Random Binary Codes

Consider a binary code  $C$  of length  $N$  and rate  $R$  bits per symbol is a set of  $M = 2^{NR}$ . For the case of RCE one compute the probability that by taking a random codeword  $\mathbf{x}_i$  of length  $N$  it would have Hamming distance  $d = N\delta$  from an arbitrary binary  $N$ -tuple  $\mathbf{b}$  and see that it will be independent of  $\mathbf{b}$  and equals to

$$\Pr\{d_H(\mathbf{x}_i, \mathbf{b}) = d\} = \binom{N}{d} \tilde{p}^d (1 - \tilde{p})^{N-d}, \quad (2.56)$$

where  $\tilde{p}$  corresponds to the probability of having a one. Under this RCE, two distances  $d_H(\mathbf{x}_i, \mathbf{x}_j)$  and  $d_H(\mathbf{x}_{i'}, \mathbf{x}_{j'})$  are independent random variables. So if we consider the number of unordered pairs of codewords  $(\mathbf{x}_i, \mathbf{x}_j)$  with  $i \neq j$  in  $C$  at a distance  $d$  apart

$$S_C(d) = \sum_{i=0}^{M-1} \sum_{j=0}^{i-1} \Phi\{d_H(\mathbf{x}_i, \mathbf{x}_j) = d\}, \quad (2.57)$$

where  $\Phi\{d_H(\mathbf{x}_i, \mathbf{x}_j) = d\}$  is equal to 1 if the condition  $d_H(\mathbf{x}_i, \mathbf{x}_j)$  is satisfied and 0 otherwise. For the case of RCE on a BSC  $S_C(d)$  is a sum of  $\binom{M}{2}$  pairwise independent, identically distributed random variables, so we have

$$ES_C(d) = \binom{M}{2} E\Phi \doteq 2^{N(2R-1+\mathcal{H}(\delta))}. \quad (2.58)$$

Therefore we are ready to state the following theorem

**Theorem 2.6.1** (Minimum distance in RCE). *For  $0 \leq R < 1/2$  and any  $\varepsilon > 0$ , the probability that a code length  $N$  and rate  $R$  from the RCE has relative minimum distance*

---

<sup>9</sup>We will be using the notation  $\mathcal{H}$  to refer to the binary entropy  $\mathcal{H} \equiv H_2$

less than  $\delta_{GV}(2R) - \varepsilon$  goes to zero exponentially as  $N \rightarrow \infty$ . For  $0 \leq R < 1$ , if  $d = N\delta$  is such that

$$\delta_{GV}(2R) + \varepsilon \leq \delta \leq 1 - \delta_{GV}(2R) - \varepsilon, \quad (2.59)$$

then the probability that the number of codeword pairs at a distance  $d$  satisfies  $S_C(d) \doteq 2^{N(2R-1+\mathcal{H}(\delta))}$  goes to one as  $N \rightarrow \infty$ .

*Proof.* For a given value of the code value of the code rate  $R$  we can choose  $d$  such that  $d/N \rightarrow \delta \leq \delta_{GV}(2R) - \varepsilon$ . Then

$$\Pr \{S_C(d) \geq 1\} \leq \mathbb{E}S_C(d) \doteq 2^{-N(1-\mathcal{H}(\delta)-2R)} \rightarrow 0, \quad (2.60)$$

which in other words it tell us that with probability differing from 1 by an exponentially falling quantity, there will be no pairs at distance  $d$ . Notwithstanding this result, if  $\delta_{GV}(2R) + \varepsilon < \delta < 1 - \delta_{GV}(2R) - \varepsilon$ , then  $1 - \mathcal{H}(\delta) < 2R$  and the average of number of pairs  $\mathbb{E}S_C(d)$  at a distance  $d$  is exponentially large. To see this, we can use the Chebyshev inequality, so for any  $\delta > 0$ , we have

$$\Pr \left\{ |S_C(d) - \mathbb{E}S_C(d)| \geq \binom{M}{2} \alpha \right\} \leq \frac{\mathbb{E}\Phi}{\binom{M}{2} \alpha^2}, \quad (2.61)$$

by choosing  $\alpha \doteq 2^{-N(1-\mathcal{H}(\delta)+\Delta)} < \mathbb{E}\Phi$  for any  $\Delta > 0$ , we have

$$\Pr \left\{ |S_C(d) - \mathbb{E}S_C(d)| > \binom{M}{2} \alpha \right\} \leq \frac{2\mathbb{E}\Phi}{M(M-1)\alpha^2} \doteq 2^{-N(2R-1+\mathcal{H}(\delta)-2\Delta)}. \quad (2.62)$$

The exponent on the right-hand side can be made positive by choosing  $\Delta$  small enough. This establishes the fact that  $S_C(d) \doteq 2^{N(2R-1+\mathcal{H}(\delta))}$  for the chosen value of  $d$  with probability tending to one as  $N \rightarrow \infty$ .  $\square$

### Random Linear Codes

A binary linear code  $C$  of length  $N$  and rate  $K/N$  is a set of  $M = 2^K$  binary  $N$ -tuples that is generated by  $K$   $N$ -tuples  $\mathbf{g}_j$ ,  $1 \leq j \leq K$ . This is

$$\mathbf{x}(u) = \sum_k \mathbf{u}_k \mathbf{g}_j, \quad (2.63)$$

where  $\mathbf{u}$  is an arbitrary binary  $K$ -tuple. Here we will be considering the case in which each of the  $2^{NK}$  matrices are chosen with equal probability. For the case of linear codes, the distribution of distance  $\{d_H(\mathbf{x}_i, \mathbf{x}_j), i \neq j\}$  from any given codeword  $\mathbf{x}_i$  is independent of  $i$ . The average distance distribution of a linear code  $C$  therefore reduces to

$$\mathcal{N}_C(d) = \sum_{j \neq i} \Phi \{d_H(\mathbf{x}_i, \mathbf{x}_j) = d\} \quad (d = 1, 2, \dots, N), \quad (2.64)$$

where  $\mathbf{x}_i$  is an arbitrary codeword. Typically,  $\mathbf{x}_i$  is taken as the all zero codeword  $\mathbf{0} = \mathbf{x}(\mathbf{0})$ . If  $(\mathbf{u}_j, \mathbf{u}_k)$  is any pair distinct non-zero  $K$ -tuples, then the corresponding codewords  $(\mathbf{x}(\mathbf{u}_j), \mathbf{x}(\mathbf{u}_k))$  are a pair of independent random binary  $N$ -tuples. It follows that two distinct distances  $d_H(\mathbf{x}_i, \mathbf{x}_j)$  and  $d_H(\mathbf{x}_i, \mathbf{x}_k)$  from a given codeword  $\mathbf{x}_i$  are pairwise-independent and distributed as in the RCE. In particular

$$\Pr \{d_H(\mathbf{x}_i, \mathbf{x}_j) = N\delta\} \doteq 2^{-N(1-\mathcal{H}(\delta))}, \quad (2.65)$$

for any  $d = N\delta$ , the quantity  $\mathcal{N}_C(d)$  in (2.64) is a sum of  $M - 1 \doteq 2^{NR}$  pairwise-independent, identically distributed random variables with mean  $E\Phi \doteq 2^{-N(1-\mathcal{H}(\delta))}$ . Its mean value is therefore equal to

$$\mathcal{N}_{\text{LCE}}(d) \doteq 2^{N(R-1+\mathcal{H}(\delta))}. \quad (2.66)$$

Therefore we say that the relative minimum distance of a code chosen at random from the LCE will be, with probability  $1 - 2^{-\Omega(N)}$ , approximately equal to the GV relative distance  $\delta_{\text{GV}}(R)$ .

In summary, the typical minimum distance in the LCE is better than that in the RCE because the minimum of only  $M - 1$  pairwise-independent distances, whereas in the RCE it is the minimum of  $\binom{M}{2}$  pairwise independent distances.

Up to this point one may be wonder how all this theory of correcting errors and minimum distance codes are connected to our problem. In following section we are going to show how this connection emerge as a natural consequence of the structure in the Clifford algebra and therefore provide an explanation of what we named as “Super-Orthogonality”.

## 2.7 Mechanism behind typicality as a correction error method.

As we mentioned in the first chapter, we are interested in study the behaviour of states which are close in energy. First we will show why it is possible to study “Ultra Orthogonality” for Fermionic states in two possibilities. The first one is in terms of minimum distance codes and we will see that “Ultra Orthogonality” turns out to be an exact result. For the second part, we deduce its correspondent exponent error. For the last part, we will tackle the problem when we can not work over a minimum distance and we will show how “Ultra orthogonality” can be acting over this specific case. Even though for this case we were not able to generalised this to every Fermionic system, we will discuss why this result should holds in general.

### 2.7.1 “Ultra Orthogonality” on Fermions

To start this part we want to recapitulate a couple of things. First, as we showed in equation (1.49), when we take the partial trace the remaining cross terms will be of our interest. When we discuss the property of Ultra Orthogonality, we stated that the cross terms of  $\text{Tr}_E |E_n\rangle \langle E_m|$  should be the ones that had to be near to zero, in order for thermalisation to occur on the specific case we discussed in chapter 1. Here we are going to formalise all these ideas.

We can start by choosing an arbitrary state such that it can be decomposed in its vectors of the Fock basis as<sup>10</sup>

$$|\psi\rangle = \sum_{\vec{n}} \psi(\vec{n}) |\vec{n}\rangle. \quad (2.67)$$

Therefore we are interested in studying the terms of the form<sup>11</sup>

$$\hat{X}_{ij} \doteq \text{Tr}_E (|\vec{n}_i\rangle \langle \vec{n}_j|) \equiv \text{Tr}_{N/L} (|\vec{n}_i\rangle \langle \vec{n}_j|). \quad (2.68)$$

What Ultra orthogonality tell us is that this term has to zero or in the worst scenario something of the order of fluctuations. Since we are working on the very special case of Quadratic Hamiltonian describing Fermionic systems, we recall the fact that the operators are generated by the Majorana operators, and form the so called Clifford algebra, described in section 2.2. We also saw that these operators can be mapped to the Grassman variables, which allow us to compute things like observables. Taking into account that we started with an space in which we had to deal with  $N$  Pauli operator and we changed to a new space in which we work with  $2N$  spinless operators and the fact that the most general function we can built over the Grassman algebra is polynomial of the Grassmann variables. Thus we construct a function of the Grassman variables which takes two binary sequences  $(\vec{x}, \vec{y})$ ,  $x_i, y_i \in \{0, 1\}$ , lets call it  $\gamma(\vec{x}, \vec{y})$ . the reason for defining this function is that we are going to describe the system in the space of size  $L$  with its correspondent operators, meaning that these sequences can not be arbitrary, they must have to be sequences such that after its first  $L$  elements, they must have only zeros. To illustrate this consider the following example, let  $\vec{x} = (0100 \dots 0)$ ,  $\vec{y} = (1100 \dots 0)$ , here  $L = 3$ , so our function will be described by<sup>12</sup>

$$\gamma(\vec{x}, \vec{y}) = \gamma_1^{x_1=0} \gamma_4^{y_1=1} \gamma_2^{x_2=1} \gamma_5^{y_2=1} \gamma_3^{x_3=0} \gamma_6^{y_3=1} = \gamma_4 \gamma_2 \gamma_5 \gamma_6, \quad (2.70)$$

<sup>10</sup>Note that we can use the Fock basis since it is naturally the basis in which we diagonalise our Hamiltonian, therefore it is an energy basis as well.

<sup>11</sup>Here we denote the partial trace by  $\text{Tr}_{N/L}$  meaning that for the corresponding sequence of length  $N$  in the Fock Space we take the respective  $L$  elements, meaning that  $N - L$  elements will be regarded as our environment.

<sup>12</sup>Here we have changed the original notation given in (2.4) and change it by

$$\hat{c}_{2j-1} \rightarrow \gamma_j, \quad \hat{c}_{2j} \rightarrow \gamma_{N+j}, \quad (2.69)$$

for the case of  $N$  operators.

Note that in this way we are able to write any product of Grassmann operators. in general, for two sequences, and a fixed size  $L$  this function will be given by

$$\gamma(\vec{x}, \vec{y}) = \gamma_1^{x_1} \gamma_{L+1}^{y_1} \gamma_2^{x_2} \gamma_{L+2}^{y_2} \cdots \gamma_L^{x_L} \gamma_{2L}^{y_L}, \quad (2.71)$$

where this function has the property that

$$\gamma(\vec{x}, \vec{y})\gamma(\vec{x}', \vec{y}') = e^{i\phi(\vec{x}, \vec{y}, \vec{x}', \vec{y}')} \gamma(\vec{x} + \vec{x}', \vec{y} + \vec{y}'). \quad (2.72)$$

The phase appear as a consequence of the anti-commutation relation and  $\phi(\vec{x}, \vec{y}, \vec{x}', \vec{y}')$  is a function that would depend on the weight of the sequences  $\vec{x}, \vec{y}, \vec{x}'$  and  $\vec{y}'$ .

This provide a set of operators that live in the space of size  $L$ , and that will allow us to expand our operator  $\hat{X}_{ij}$ ,<sup>13</sup>

$$\hat{X}_{ij} = \sum_{\vec{x}, \vec{y}} f(\vec{x}, \vec{y}) \gamma(\vec{x}, \vec{y}). \quad (2.73)$$

Our task will then be to find the coefficient  $f(\vec{x}, \vec{y})$ , this can be achieve by multiplying on both sides by  $\gamma(\vec{x}', \vec{y}')$  and taking the trace over  $L$

$$\text{Tr}_L \left( \hat{X}_{ij} \gamma^\dagger(\vec{x}', \vec{y}') \right) = \sum_{\vec{x}, \vec{y}} f(\vec{x}, \vec{y}) \text{Tr}_L \left( \gamma(\vec{x}, \vec{y}) \gamma^\dagger(\vec{x}', \vec{y}') \right). \quad (2.74)$$

The right hand side of this equation can be combined with equation (2.72) and deduce that it will give us a delta ( $\delta_{\vec{x}+\vec{x}'} \delta_{\vec{y}+\vec{y}'}$ ). Thus the coefficients  $f(\vec{x}, \vec{y})$  are given by

$$f(\vec{x}', \vec{y}') = \frac{1}{2^L} \text{Tr}_L \left( \hat{X}_{ij} \gamma^\dagger(\vec{x}', \vec{y}') \right) = \frac{1}{2^L} \langle \vec{n}_j | \gamma^\dagger(\vec{x}', \vec{y}') | \vec{n}_i \rangle. \quad (2.75)$$

Thus we first have to know how the operators  $\gamma$  acts over the states  $|\vec{n}_i\rangle$ , for this we recall the fact that the Hamiltonians are diagonalised via an orthogonal transformation which links what we call the spacial modes and the normal modes

$$\overbrace{\gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_k}}^{\text{Spacial modes}} = O_{i_1 \alpha_1} O_{i_2 \alpha_2} \cdots O_{i_k \alpha_k} \underbrace{\gamma_{\alpha_1} \gamma_{\alpha_2} \cdots \gamma_{\alpha_k}}_{\text{Normal modes}}. \quad (2.76)$$

Note that this operators are diagonal over our  $\vec{x}$ 's and  $\vec{y}$ 's, which will allow us to operate over the states  $|\vec{n}_i\rangle$ .<sup>14</sup> The equation (2.75) will be simplified to

$$\langle \vec{n}_j | \gamma(\vec{x}, \vec{y}) | \vec{n}_i \rangle = \delta_{\vec{n}_i + \vec{x} + \vec{y}, \vec{n}_j} e^{i\phi(\vec{n}_i, \vec{n}_j, \vec{x}, \vec{y})}, \quad (2.78)$$

---

<sup>13</sup>For illustrate these ideas we first suppose we have the operators in order. However, we will generalise it later.

<sup>14</sup>By using the function  $\gamma(\vec{x}, \vec{y})$  there are two ways of getting an specific state  $|\vec{n}_i\rangle$

$$|n_i\rangle = \gamma(\vec{n}_i, 0) |0\rangle, \quad |n_i\rangle = \gamma(0, \vec{n}_i) |0\rangle e^{i\phi(\vec{n}_i)}, \quad (2.77)$$

therefore we can say that the  $\vec{x}$ 's takes the 0 and turn them into a 1, and the  $\vec{y}$ 's take 0 and transform it into a one multiplied by a phase.

whereas the coefficients  $f(\vec{x}, \vec{y})$  will be given by

$$f(\vec{x}, \vec{y}) = \frac{1}{2^L} \sum_{\vec{x}', \vec{y}'} \mathcal{U}_{\vec{x}\vec{x}'} \mathcal{V}_{\vec{y}\vec{y}'} \underbrace{\langle \vec{n}_j | \gamma(\vec{x}, \vec{y}) | \vec{n}_i \rangle}_{\propto \delta_{\vec{n}_i + \vec{n}_j, \vec{x} + \vec{y}}} \quad (2.79)$$

In equation (2.78) the term with the delta can be change by  $\delta_{\vec{n}_i + \vec{n}_j, \vec{x} + \vec{y}}$ , an the reason to do so is because we are working with arithmetic mod 2 and we can see that the term  $\vec{n}_i + \vec{n}_j$  is nothing but the vector of differences, which means that the only values different than zero in this vector are when  $n_{i_k} \neq n_{j_k}$ . This result is extremely important because it tell us that when ever we work with states like  $\hat{X}_{ij}$ , if the vector of differences have more ones than the vector of differences  $\vec{x} + \vec{y}$ . Note that the vector of difference given by  $\vec{x} + \vec{y}$  can have at most  $L$  errors, which means that whenever the number of ones in the difference vector defined by the states  $\vec{n}_i + \vec{n}_j$  exceeds  $L$  the state  $\hat{X}_{ij}$  will be immediately zero. This turns out to be an astonishing result and bring even more questions, like how likely is to have more than  $L$  errors when  $N \gg L$ ? what happen when we have less errors is this quantity still small enough as we expected?. These questions will be tackled in a moment but something to stress is the fact that it is the branch point of our study. We will be first addressing the first question and afterwards we will talk about the second one.

### 2.7.2 Codes of minimum distance constrained to an energy value

Lets recapitulate a little more what we have done in the latter chapter. We define two binary sequences of excitations  $\vec{n}_i$ ,  $\vec{n}_j$ , and the vector of differences  $\vec{e}_{ij} = \vec{n}_i + \vec{n}_j$ , we will denote the distance between these two binary sequences by

$$\begin{aligned} d &= W(\vec{e}_{ij}) \rightarrow \text{Weight of } \vec{e}_{ij} \\ &= d_H(\vec{n}_i, \vec{n}_j) \rightarrow \text{Hamming distance.} \end{aligned} \quad (2.80)$$

we show that there is a specific distance  $d > L$  at which the state  $\hat{X}_{ij} = \text{Tr}_{N/L} (\langle \vec{n}_i | \vec{n}_j \rangle)$  is equal to zero  $\hat{X}_{ij}$ . This might sound quite familiar since it is connected to minimum distance codes and to see this more clearly consider a set of binary codewords

$$\mathcal{C} = \{ \vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(2^k)} \} \quad \vec{x} \in \{0, 1\}^N, \quad (2.81)$$

and size  $|\mathcal{C}| = 2^{NR}$ , with  $R = k/N$  as defined in (2.33). Analogously, consider the Hilbert space

$$\mathcal{H}_{\mathcal{C}} = \text{Span} \left( |\vec{x}^{(1)}\rangle, |\vec{x}^{(2)}\rangle, \dots, |\vec{x}^{(2^k)}\rangle \right). \quad (2.82)$$

It is easy to check that  $|\mathcal{H}_{\mathcal{C}}| = |\mathcal{C}| = 2^{NR}$ . From the results showed from section 2.6 we can assure that there exist a Hilbert space with dimension  $\dim |\mathcal{H}_{\mathcal{C}}| = 2^{N(1-\mathcal{H}(\ell))}$ , where  $\ell = L/N$ , the relative distance. More specifically,  $\forall |\psi\rangle \in \mathcal{H}_{\mathcal{C}}$ ,

$$|\psi\rangle = \sum_{\vec{n} \in \mathcal{C}} \psi(\vec{n}) |\vec{n}\rangle, \quad (2.83)$$

where the code  $\mathcal{C}$  refers to a code  $(N, k, \ell)$ ,  $d > \ell$ . So

$$\rho_L(\psi) = \text{Tr}_{N/L}(|\psi\rangle\langle\psi|) = \sum |\psi(\vec{n})|^2 |\vec{n}\rangle\langle\vec{n}|. \quad (2.84)$$

Thus, the states belonging to  $\mathcal{H}_C$  are like frozen states, meaning that

$$\frac{d\rho_L}{dt} = 0. \quad (2.85)$$

One of the most important question one can ask is what is the biggest code with relative distance  $\ell$  constrained with certain value of energy?. So if we denote by

$$S(\ell) = \bigcup_{\mathcal{C}, \delta > \ell} \mathcal{H}_C, \quad (2.86)$$

we would like to know how big indeed is this set and even more we would like to know its error exponent. For answering these questions we first define this problems in terms of random variables. Let  $X_i(\theta_k)$  be a random probability that takes the value 1 with probability  $p(\theta_k)$  and the value 0 with probability  $1 - p(\theta_k)$ , and let  $X = \sum_i X_i(\theta_k)$  be the sum of these random variables, or equivalently the number of errors. Then we ask ourselves about the probability of having certain number of errors. Particularly we ask the probability of having a quantity of error greater or equal than a certain quantity. To address this question we can make use of the Chernoff inequality<sup>15</sup>

$$\begin{aligned} P(e^{SX} \geq e^{Sd}) &\leq \min_S \langle e^{SX} \rangle e^{-Sd} \\ P(e^{-SX} \geq e^{-Sd}) &\leq \min_S \langle e^{-SX} \rangle e^{Sd}. \end{aligned} \quad (2.87)$$

We then compute the expected value  $\langle e^{SX} \rangle$  taking into account that we are working with independent variables

$$\langle e^{SX} \rangle = \prod_k E(e^{SX(\theta_k)}) = \prod_k (1 + p(\theta_k)(e^S - 1)) = e^{\sum_k \log(1 + p(\theta_k)(e^S - 1))} \equiv e^{-Nr(\delta)}, \quad (2.88)$$

where  $r(\delta)$  corresponds to the correspondent error exponent

$$r(\delta) = \min_S \frac{1}{N} (\log \langle e^{SX} \rangle - S\delta). \quad (2.89)$$

Since we are interested in the case when  $N \rightarrow \infty$ , the error exponent can be written as

$$r(\delta) \xrightarrow{N \rightarrow \infty} \min_S \oint \frac{d\theta}{2\pi} \log (1 + p(\theta)(e^S - 1)) - S\delta, \quad (2.90)$$

---

<sup>15</sup>In this part we do not specify what distribution is the one we are choosing, one might guess that it is related to the Fermi-Dirac distribution, but our result will be in terms of the this general distribution  $p(\theta_k)$ . In the next chapter we will provide an expression for the case of the XY model.

which even though can not be analytically solved, it can be numerically solved by deriving and obtaining

$$\delta = \oint \frac{d\theta}{2\pi} \frac{p(\theta)e^S}{1 - p(\theta) + p(\theta)e^S}. \quad (2.91)$$

Thus we conclude that the mean number of codes at distance  $d$  is given by

$$\langle S_C(d) \rangle = \binom{M}{2} 2^{-Nr(\delta)}, \quad (2.92)$$

with  $M \equiv 2^{NR}$ , we conclude that when ever we work with rates lower than  $r(\delta)/2$  the average number of pairs at a distance  $d$  goes to zero exponentially. Other wise the number of pairs that have minimum distance  $d$  are exponentially large

$$\langle S_C(d) \rangle \doteq 2^{N(2R-r(\delta))}. \quad (2.93)$$

With the latter equation we answer the two questions asked at the end of the last section. However we have not seen what happens in the case when we work with distances less than  $L$ . In the next section we will discuss a little bit more about it and we will provide the intuition behind this case.

### 2.7.3 Case when the numbers of errors is less than $L$

For this case we know that the term  $\hat{X}_{ij}$  was different than zero, however, we are going to show that nevertheless it is not necessarily zero, this quantity will be small. To study this we will consider the norm of  $\|\hat{X}_{ij}\|$ , if we take into account the relation found in (2.79) we will find that

$$\begin{aligned} \|\hat{X}_{ij}\|^2 &= \text{Tr} \left( \hat{X}_{ij} \hat{X}_{ij}^\dagger \right) = \frac{1}{2^L} \sum_{\vec{x}, \vec{y}} |f(\vec{x}, \vec{y})|^2 \\ &= \frac{1}{2^L} \sum_{\substack{\vec{x}, \vec{y} \\ \vec{x}'', \vec{y}''}} (\mathcal{U}_{\vec{x}\vec{x}'} \mathcal{U}_{\vec{x}\vec{x}''}) (\mathcal{V}_{\vec{y}\vec{y}'} \mathcal{V}_{\vec{y}\vec{y}''}) e^{\phi(\vec{x}, \vec{y}') \phi(\vec{x}'', \vec{y}'')} \delta_{\vec{n}_i + \vec{n}_j, \vec{x}' + \vec{y}'} \delta_{\vec{n}_i + \vec{n}_j, \vec{x}'' + \vec{y}''}. \end{aligned} \quad (2.94)$$

Our purpose will then be to bound the terms  $\sum_{\vec{x}} (\mathcal{U}_{\vec{x}\vec{x}'} \mathcal{U}_{\vec{x}\vec{x}''})$ , the reason to this is because if we can bound these terms by some quantity, this bound will also holds for the part containing the  $\vec{y}$ 's, thus for the rest of this work we will working with this quantity instead of working with (2.94). First, notice that the expression in (2.79) will not be too useful to us, since we are more interested in changing basis, for example we will be constantly changing from the spacial modes to the normal modes, so, by writing in this way the products of the operators we are not taking into account the geometric meaning of these terms. In the expansion of an operator, the resulting terms from  $\gamma(\vec{x}, \vec{y})$  have to be interpreted as  $p$ -forms, meaning that this will correspond to a volume generated by some vectors. So if we want to take into account the fact that volumes change over basis, we should have to understand that

the products of the operators in  $\gamma(\vec{x}, \vec{y})$  should transform in a particular way to take this into account. It is not trivial, but is simple to check that the way this products should change in order to transform as Grassmann variables and take into account the change on the volumes over transformation is via the antisymmetrizing operator. So when we see the products such as  $\gamma_1\gamma_2\gamma_3$  we have to understand it as  $\frac{1}{3!}\gamma_{[1}\gamma_2\gamma_{3]}$ . Generalising these ideas we consider a general  $p$ -form, lets say  $\gamma(\vec{\alpha})$

$$\gamma(\vec{\alpha}) = [\gamma_{\alpha_1}\gamma_{\alpha_2}\gamma_{\alpha_3} \dots \gamma_{\alpha_p}], \quad (2.95)$$

if each of these elements transform as  $\gamma_{\alpha_i} = O_{\alpha_i j}\tilde{\gamma}_j$ . The  $p$ - form will transform as

$$\gamma(\vec{\alpha}) = \det(O|_{\vec{\alpha}, \vec{\beta}}) \gamma_{\vec{\beta}}, \quad (2.96)$$

where the term  $\det(O|_{\vec{\alpha}, \vec{\beta}})$  refers to the minor of the matrix  $O$ . Turning back to our main problem look that the term of  $\sum_{\vec{x}} (\mathcal{U}_{\vec{x}\vec{x}'} \mathcal{U}_{\vec{x}\vec{x}''})$  can be written in terms of this determinants as

$$\sum_{\vec{x}} (\mathcal{U}_{\vec{x}\vec{x}'} \mathcal{U}_{\vec{x}\vec{x}''}) = \det \left[ (O \Pi O^T) |_{\vec{x}', \vec{x}''} \right]. \quad (2.97)$$

Therefore if we are able to find a bound for this determinant we could show that the quantity  $\|\hat{X}_{ij}\|$  is indeed small as we have been saying. Nonetheless, to bound this quantity in general is not an easy task and we can not show a general bound for this quantity, even though, we will explore how this quantity grows for the case of the one dimensional  $XY$  model. As we will show the quantity  $\|\hat{X}_{ij}\|$  for the case of this model is indeed small, and then it provide us an intuition that there must be some mechanism over the other models such that this quantity remains small for other Fermionic systems.

In the next chapter we are going to present the one dimensional  $XY$  model, its importance for our study as well as some calculations of the previous results to illustrate the behaviour of Ultra orthogonality for this specific case.

# Chapter 3

## Getting to understand Ultra Orthogonality in the $XY$ Model.

As discussed in the previous chapters, we are interested in solvable Fermionic systems. Indeed the one dimensional  $XY$  model is one of those systems [57]. The  $XY$  Hamiltonian model is a set of  $N$  spin  $1/2$  particles located on the sites of d-dimensional lattice. Nevertheless, whenever we talk about the  $XY$  model, we will be referring to the  $1D$   $XY$  model.

### 3.1 The $XY$ Model

A chain of  $N$  spins where each spin is able to interact with its nearest neighbours in the  $X$  and  $Y$  coordinate as well as an external magnetic field, will be described by the Hamiltonian of the form

$$H_{XY} = -\frac{1}{2} \sum_{l=0}^{N-1} \left( \frac{1+\gamma}{2} \sigma_l^x \sigma_{l+1}^x + \frac{1-\gamma}{2} \sigma_l^y \sigma_{l+1}^y + \lambda \sigma_l^z \right), \quad (3.1)$$

where  $\gamma$  is so-called the anisotropy parameter and represents the difference between the strength of the  $XX$  interaction and the  $YY$  interaction<sup>1</sup>,  $\lambda$  is the intensity of the external magnetic field and  $\sigma_l^i$  is the Pauli matrix ( $i = x, y, z$ ) acting over the  $l$  site of the chain.

The XY model is a model that has been widely studied for a variety of values of  $\lambda$  and  $\gamma$  and in some limits it has a correspondence to other models of interest in condensed matter [58–60]<sup>2</sup>.

#### 3.1.1 The spectrum

To find the spectrum of the Hamiltonian (3.1) we need to perform some special transformations.

---

<sup>1</sup>When we talk about interactions we mean interactions between spins.

<sup>2</sup>Some examples of this kind, are the Boson Hubbard model in the limit of hard Bosons. the case when  $\gamma = 1$  correspond to the Ising model, and the Kitaev chain is equivalent to the XY model under a proper identification of the parameters  $\mu$ ,  $t$  and  $\Delta$  with  $\gamma$  and  $\lambda$  [58, 59]

### 3.1.2 Jordan-Wigner Transformation

We first consider the non local transformation given by

$$\hat{b}_l = \left( \prod_{m < l} \sigma_m^z \right) \sigma_l^-, \quad \sigma_l^- = \frac{\sigma_l^x - i\sigma_l^y}{2}, \quad (3.2)$$

where these  $b_l$  represent spinless Fermionic operators, and its canonical anticommutation relation (CAR) is given by [61]

$$\left\{ \hat{b}_i^\dagger, \hat{b}_j^\dagger \right\} = \left\{ \hat{b}_i, \hat{b}_j \right\} = 0, \quad \left\{ \hat{b}_i^\dagger, \hat{b}_j \right\} = \delta_{i,j}. \quad (3.3)$$

So inverting the transformation we get

$$\begin{aligned} \sigma_l^z &= 1 - 2\hat{b}_l^\dagger \hat{b}_l \\ \sigma_l^x &= \left( \prod_{m < l} \left( 1 - 2\hat{b}_m^\dagger \hat{b}_m \right) \right) \left( \hat{b}_l^\dagger + \hat{b}_l \right) \\ \sigma_l^y &= i \left( \prod_{m < l} \left( 1 - 2\hat{b}_m^\dagger \hat{b}_m \right) \right) \left( \hat{b}_l^\dagger - \hat{b}_l \right). \end{aligned} \quad (3.4)$$

The terms of interaction in the Hamiltonian will look as

$$\begin{aligned} \hat{\sigma}_l^x \hat{\sigma}_{l+1}^x &= \left( \hat{b}_l^\dagger - \hat{b}_l \right) \left( \hat{b}_{l+1}^\dagger + \hat{b}_{l+1} \right) \\ \hat{\sigma}_l^y \hat{\sigma}_{l+1}^y &= - \left( \hat{b}_l^\dagger + \hat{b}_l \right) \left( \hat{b}_{l+1}^\dagger - \hat{b}_{l+1} \right), \end{aligned} \quad (3.5)$$

and the Hamiltonian will look like,

$$H_{XY} = -\frac{1}{2} \sum_l \left[ \left( \hat{b}_{l+1}^\dagger \hat{b}_l + \hat{b}_l^\dagger \hat{b}_{l+1} \right) + \gamma \left( \hat{b}_l^\dagger \hat{b}_{l+1}^\dagger - \hat{b}_l \hat{b}_{l+1} \right) \right] - \frac{\lambda}{2} \sum_l \left( 1 - 2\hat{b}_l^\dagger \hat{b}_l \right), \quad (3.6)$$

after this transformation. The term of  $-\lambda N/2$  is usually ignored since it cause only a gauge of the spectrum in the energy [61].

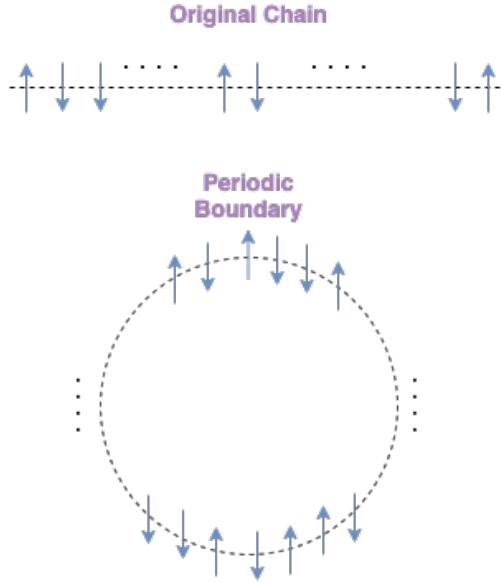
### 3.1.3 Fourier Transformation

It is possible to exploit an other symmetry in the system. It comes by considering periodic boundary conditions (PBC) [61]. This can be easily done by identifying the spin in the site  $N$  with the spin in the site 1. After imposing this condition, we have that the Fourier transform of the operator  $\hat{b}_l$  will look as

$$\hat{d}_k = \frac{1}{\sqrt{N}} \sum_{l=1}^N \hat{b}_l e^{-i\phi_k l}, \quad (3.7)$$

with  $\theta_k = \frac{2\pi}{N} k$ .

Since the Fourier transformation is unitary, the operators  $\hat{d}_k$  will preserve the CAR.



**Figure 3.1:** Illustration of what a boundary condition means in the case of our spin chain.

Thus, the Hamiltonian can be written in terms of the operators  $\hat{d}_k$  as

$$H_{XY} = \sum_{k=-(N-1)/2}^{(N-1)/2} (-\lambda + \cos \phi_k) \hat{d}_k^\dagger \hat{d}_k + \frac{i\gamma}{2} \sum_{k=-(N-1)/2}^{(N-1)/2} \sin \phi_k (\hat{d}_k \hat{d}_{-k} + h.c.), \quad (3.8)$$

where we have ignored an additional term which is proportional to  $1/N$  which will vanish for in the thermodynamic limit  $N \rightarrow \infty$  [59, 60], which is our case of interest.

### 3.1.4 Bogoliubov -Valantin Transformation

As mentioned before Fermionic quadratic Hamiltonians can be easily diagonalised via a Bogoliubov-Valantin transformation over the operators  $\hat{d}_k$

$$\tilde{d}_k = u_k \hat{d}_k^\dagger + i v_k \hat{d}_{-k}. \quad (3.9)$$

Since we want this transformation to preserve CAR, it is needed that  $u_k^2 + v_k^2 = 1$ , which implies that we can use the parametrization  $u_k = \cos(\psi_k/2)$  and  $v_k = \sin(\psi_k/2)$ , with

$$\cos \frac{\psi_k}{2} = \frac{-\lambda + \cos \phi_k}{\sqrt{(\lambda - \cos \phi_k)^2 + (\gamma \sin \phi_k)^2}}, \quad (3.10)$$

So finally our Hamiltonian will look as

$$H_{XY} = \sum_{-(N-1)/2}^{(N-1)/2} \tilde{\Lambda}_k \tilde{d}_k^\dagger \tilde{d}_k, \quad (3.11)$$

with

$$\tilde{\Lambda}_k := \sqrt{(\lambda - \cos \phi_k)^2 + (\gamma \sin \phi_k)^2}, \quad (3.12)$$

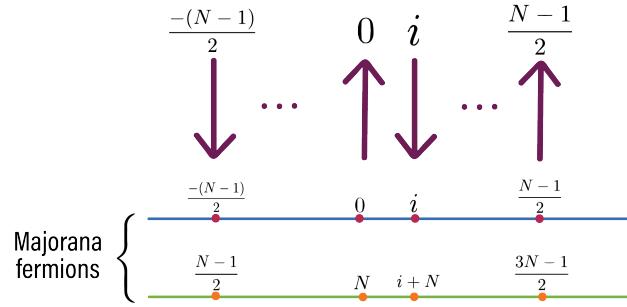
where the latter expression allow us to identify the critical regions of the model.

### 3.1.5 Fermionic Covariance Matrix for the XY model

As we mentioned before, sometimes it turns out to be better, and useful to work directly with the Covariance matrix. To be able to do so, we need to express the Hamiltonian (3.1) in terms of Majorana fermions. This can be done by using an analogous of the Jordan Wigner transformation use to diagonalised the XY Hamiltonian but now we apply it to the  $2N$  Majorana fermions

$$\hat{\gamma}_l = \left( \prod_{m < l} \hat{\sigma}_m^z \right) \hat{\sigma}_l^x, \quad \hat{\gamma}_{l+N} = \left( \prod_{m < l} \hat{\sigma}_m^z \right) \hat{\sigma}_l^y, \quad (3.13)$$

where again  $l = 1, 2 \dots N-1$ ,



**Figure 3.2:** Illustration of how the spins in the chain are mapped to the Majorana fermions.

and similarly as before we have that

$$\hat{\gamma}_l \hat{\gamma}_{l+N} = \left( \prod_{m < l} \hat{\sigma}_m^z \right) \left( \prod_{m < l} \hat{\sigma}_m^z \right) \hat{\sigma}_l^x \hat{\sigma}_l^y = i \hat{\sigma}_l^z, \quad (3.14)$$

$$\hat{\gamma}_{l+N} \hat{\gamma}_{l+1} = \left( \prod_{m < l} \hat{\sigma}_m^z \right) \hat{\sigma}_l^y \left( \prod_{m < l+1} \hat{\sigma}_m^z \right) \hat{\sigma}_{l+1}^x = \hat{\sigma}_l^y \hat{\sigma}_l^z \hat{\sigma}_{l+1}^x = i \hat{\sigma}_l^x \hat{\sigma}_{l+1}^x, \quad (3.15)$$

$$\hat{\gamma}_l \hat{\gamma}_{l+N+1} = \left( \prod_{m < l} \hat{\sigma}_m^z \right) \hat{\sigma}_l^x \left( \prod_{m < l+1} \hat{\sigma}_m^z \right) \hat{\sigma}_{l+1}^y = \hat{\sigma}_l^x \hat{\sigma}_l^z \hat{\sigma}_{l+1}^y = -i \hat{\sigma}_l^y \hat{\sigma}_{l+1}^y. \quad (3.16)$$

Which coincide, up to constant factors, with the three terms in the Hamiltonian (3.1). This will lead us to a Hamiltonian of the form [36, 62]

$$H_{XY} = \frac{i}{4} \sum_{\alpha, \beta=0}^{2N} \Omega_{\alpha\beta} [\hat{\gamma}_\alpha, \hat{\gamma}_\beta], \quad (3.17)$$

where  $\Omega$  is the antisymmetric matrix of the form

$$\Omega = \left[ \begin{array}{c|c} 0 & \tilde{\Omega} \\ \hline -\tilde{\Omega}^T & 0 \end{array} \right], \quad (3.18)$$

with

$$\tilde{\Omega} = \begin{pmatrix} \lambda & \frac{1-\gamma}{2} & 0 & 0 & \dots & 0 & \frac{1+\gamma}{2} \\ \frac{1+\gamma}{2} & \lambda & \frac{1-\gamma}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1+\gamma}{2} & \lambda & \frac{1-\gamma}{2} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots & \vdots \\ \frac{1-\gamma}{2} & 0 & 0 & 0 & \dots & \frac{1+\gamma}{2} & \lambda \end{pmatrix}. \quad (3.19)$$

In general (3.18) can be diagonalised via an orthogonal transformation  $O$  [36, 62]<sup>3</sup>

$$\Omega = O \left[ \begin{array}{c|c} 0 & \omega \\ \hline \omega & 0 \end{array} \right] O^T, \quad (3.20)$$

where  $O \in O(2N)$ . Writing it in terms of two smaller orthogonal matrices will look as

$$O = \left[ \begin{array}{c|c} O_1 & 0 \\ \hline 0 & O_2 \end{array} \right], \quad (3.21)$$

and  $\omega$  is a diagonal matrix of size  $N \times N$  which holds excitation numbers  $-1/2 + n$ . By doing the product of matrices in (3.20) we can easily see that

$$\tilde{\Omega} = O_1 \omega O_2^T, \quad (3.22)$$

which is nothing but the singular value decomposition of the matrix  $\tilde{\Omega}$ . The latter result tell us that a fast way to construct the matrix  $O$ , which diagonalise  $\Omega$ , is to focus on  $\tilde{\Omega}$ .

A fact that we can exploit is that , the matrix described in equation (3.19)  $\tilde{\Omega}$  is a circulant real matrix, meaning that it can be easily diagonalised by means of a Fourier transform. So we can write

$$\tilde{\Omega}_{mn} = \frac{1}{N} \sum_{\theta_k \in (-\pi, \pi)} \omega(\theta_k) e^{\phi(\theta_k)} e^{i(m-n)\theta_k}. \quad (3.23)$$

where  $\omega(\theta_k) = \omega(\theta_k)^* = \omega(-\theta_k)$ ,  $\phi(\theta_k) = -\phi(\theta_k)$  and are given by

$$\omega^2(\theta_k) := (\lambda - \cos \theta_k)^2 + \gamma^2 \sin^2 \theta_k, \quad (3.24)$$

and

$$\phi(\theta_k) := \arctan \left( \frac{\lambda - \cos \theta_k}{-\gamma \sin \theta_k} \right). \quad (3.25)$$

---

<sup>3</sup>This special relation provide us a way to transform from spacial modes to excitation in the chain, so that we can either excite the chain and see what the spatial modes are or the other way.

So expanding the equation (3.23), we get

$$\begin{aligned}\tilde{\Omega}_{mn} &= \frac{1}{N} \left[ \omega(0) + (-1)^{m-n} \omega(\pi) + 2 \sum_{0 < \theta_k < \pi} \omega(\theta_k) \cos(\theta_k(m-n) + \phi(\theta_k)) \right] \\ &= \frac{\omega(0)}{N} + (-1)^{m-n} \frac{\omega(\pi)}{N} + \sum_{0 < \theta_k \leq \pi} \omega(\theta_k) (u_m^c(\theta_k) v_n^c(\theta_k) + u_m^s(\theta_k) v_n^s(\theta_k)),\end{aligned}\quad (3.26)$$

where

$$u_m^c(\theta_k) = \sqrt{\frac{2}{N}} \cos(m\theta_k + \phi(\theta_k)), \quad u_m^s(\theta_k) = \sqrt{\frac{2}{N}} \sin(m\theta_k + \phi(\theta_k)), \quad (3.27)$$

$$v_n^c(\theta_k) = \sqrt{\frac{2}{N}} \cos(n\theta_k), \quad v_n^s(\theta_k) = \sqrt{\frac{2}{N}} \sin(n\theta_k). \quad (3.28)$$

Now defining  $u^s(0) = v^s(\pi) = 0$ ,  $u^c(0) = v^c(\pi) = \frac{1}{\sqrt{N}}$ , we have that  $\tilde{\Omega}_{m,n}$  can be written as

$$\tilde{\Omega}_{mn} = \sum \omega(\theta_k) (u_m^c(\theta_k) v_n^c(\theta_k) + u_m^s(\theta_k) v_n^s(\theta_k)). \quad (3.29)$$

Therefore the upper part of the Hamiltonian reads

$$H = \sum_{m,n=0}^{N-1} \frac{i}{4} \sum_{\theta_k=0}^{\pi} \omega(\theta_k) (u_m^c(\theta_k) v_n^c(\theta_k) + u_m^s(\theta_k) v_n^s(\theta_k)) [\hat{\gamma}_n, \hat{\gamma}_{m+N}], \quad (3.30)$$

$$H = \sum_{\theta_k=0}^{\pi} \omega(\theta_k) (\underbrace{[\hat{\gamma}_k^c, \hat{\gamma}_{k+N}^c]}_{1-2\sigma_k^z} + \underbrace{[\hat{\gamma}_k^s, \hat{\gamma}_{k+N}^s]}_{1-2\sigma_k^z}), \quad (3.31)$$

where

$$\hat{\gamma}_k^{c,s} := \sum_n u_n^{c,s}(\theta_k) \hat{\gamma}_n, \quad \hat{\gamma}_{k+N}^{c,s} := \sum_n v_n^{c,s}(\theta_k) \hat{\gamma}_{n+N}. \quad (3.32)$$

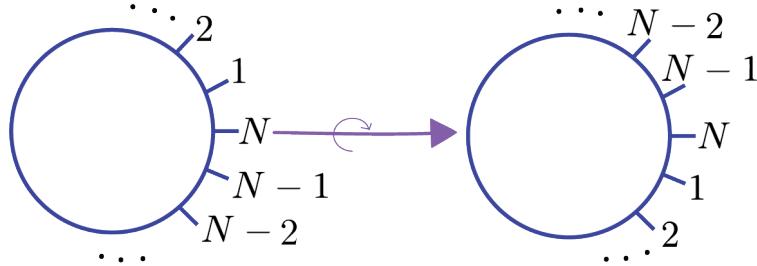
Now we look back on the fact that the Fermionic covariance matrix, defined by  $\Gamma_{\alpha\beta} = \frac{1}{2i} \text{tr}(\rho [\gamma_\alpha, \gamma_\beta]) = \frac{1}{2i} \langle [\gamma_\alpha, \gamma_\beta] \rangle$ , that brings  $\Omega$  into its Williamson form, does the same on the Fermionic covariance matrix. Thus for a state  $|\vec{n}\rangle$ , consider an eigenstate of the base  $(c, s, \theta_k)$ , where  $m^{c,s}(\theta_k) = 1/2$ , with  $n^{c,s}(\theta_k)$  the occupation number of cosine, sine in the  $k$ -mode. We get

$$\begin{aligned}\tilde{\Gamma}_{mn} &= \sum_{\theta_k}^{\pi} [m^c(\theta_k) u_m^c(\theta_k) v_n^c(\theta_k) + m^s(\theta_k) u_m^s(\theta_k) v_n^s(\theta_k)] \\ &= \sum_{\theta_k}^{\pi} \left( \frac{m^c(\theta_k) + m^s(\theta_k)}{2} \right) (u_m^c(\theta_k) v_n^c(\theta_k) + u_m^s(\theta_k) v_n^s(\theta_k)) \\ &\quad + \sum_{\theta_k}^{\pi} \left( \frac{m^c(\theta_k) - m^s(\theta_k)}{2} \right) (u_m^c(\theta_k) v_n^c(\theta_k) - u_m^s(\theta_k) v_n^s(\theta_k)).\end{aligned}\quad (3.33)$$

by defining  $m^\pm(\theta_k) = \frac{m^c(\theta_k) \pm m^s(\theta_k)}{2}$  and inverting the transformations done above, we finally get that

$$\tilde{\Gamma}_{mn} = \underbrace{\sum_{\theta_k}^{\pi} m^+(\theta_k) e^{i\phi(\theta_k)} e^{i(n-m)\theta_k}}_{\tilde{\Gamma}_{mn}^+} + \underbrace{\sum_{\theta_k}^{\pi} m^-(\theta_k) e^{i\phi(\theta_k)} e^{i(n+m)\theta_k}}_{\tilde{\Gamma}_{mn}^-}. \quad (3.34)$$

We notice that  $\tilde{\Gamma}_{mn}^+$  is circulant, whereas  $\tilde{\Gamma}_{mn}^-$  is not, nevertheless, observe that  $\tilde{\Gamma}_{mn}^+ = \tilde{\Gamma}_{mn'}$ , with  $n'$  a change on the index  $n \rightarrow -n'$ , which can be interpreted as a rotation over the circle.



**Figure 3.3:** Meaning of the relabel done in the circulant matrix, which can be seen as a reflection over the circle.

Explicitly we can write that if  $\tilde{\Gamma}_{mn}^+$  has the shape

$$\begin{pmatrix} a_0 & a_{-1} & \cdots & a_2 & a_1 \\ a_1 & a_0 & \cdots & a_3 & a_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{-1} & a_{-2} & a_{-3} & \cdots & a_0 \end{pmatrix}, \quad (3.35)$$

then  $\tilde{\Gamma}_{mn}^-$  will be given by

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{-2} & a_{-1} \\ a_1 & a_2 & \cdots & a_{-1} & a_0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{-1} & a_0 & a_1 & \cdots & a_{-2} \end{pmatrix}. \quad (3.36)$$

So we can spot 3 things. First, the FCM always can be written as a circulant matrix plus an anticirculant matrix. Second, in the Ground state, the FCM is circulant only, since the fermion occupation numbers  $n^c(\theta_k) = n^s(\theta_k) = 0, \forall k$ . Third, for a generic state, we have that in average the FCM matrix is always circulant, because  $\langle n^c(\theta_k) \rangle = \langle n^s(\theta_k) \rangle$ .

Now that we have showed a full characterization of the XY model, we change gears and start

to board our main problem. In the next section we are going to show how this is possible to find a bound in the case of the  $XY$  model and even more, we will show that the mechanism for Ultra Orthogonality in this system, is indeed related with the structure of the system.

## 3.2 Exploring “Ultra Orthogonality”

Now having all the tools we need we can apply what we discussed in section 2. First we are going to look at the error exponents for the  $XY$  model, we will compute the probabilities of having errors and show some numerical results. Then we move to our next problem, we will show for the  $XY$  model is possible to bound the determinant and even more we find an analytical result for this determinant based on a generalisation of Szegő limit theorems.

### 3.2.1 Error exponent

First we recall the fact that in order to sample states at a given temperature we make use of a very well known technique, named Gibbs sampling. This technique is a special case of a Markov chain Monte Carlo algorithm. It is well known for obtaining sequences of observations which are approximated from a specified multivariate probability distribution. It is a very useful tool for simulations in Markov processes for which the transition from which transition matrix cannot be formulated explicitly because the state space is too large [63–65]. Our goal is then to sample over excited states  $|\vec{n}\rangle$ , where the occupations  $n_q$  will be sampled independently according to the Boltzmann distribution

$$p(\vec{n} \mid \beta) = \prod_{q=0}^{N-1} p(n_q \mid \beta), \quad p(n_q \mid \beta) = \frac{e^{-\beta\epsilon(\theta_q)n_q}}{\sum_{n_q} e^{-\beta\epsilon(\theta_q)n_q}}, \quad (3.37)$$

where we explicitly put the dependence of the energy with certain angle in (3.37) having in mind the case of the  $XY$  model<sup>4</sup>. The goal of using this technique that the size of the Hilbert space is to find the probability distribution with a given temperature  $\beta$ . It is well known that in the Fermionic case, the average number of excitations in the mode at angle  $\theta$  is given by

$$f(\theta_q \mid \beta) \equiv \langle n_q \rangle_\beta = \frac{1}{e^{\beta\epsilon(\theta_q)} + 1}, \quad (3.38)$$

while the variance in the number of excitations is given by

$$v(\theta_q \mid \beta) \equiv \langle n_q^2 \rangle_\beta - \langle n_q \rangle_\beta^2 = \frac{1}{e^{\beta\epsilon(\theta_q)} + 1} = f(\theta_q)(1 - f(\theta_q)). \quad (3.39)$$

---

<sup>4</sup>However, the  $XY$  model is not the only model in which the energy can be parametrized in terms of an angle, in general it has been shown that every one-dimentional translationally invariant closed chain of free Fermions/Bosons will have this property [37, 57, 58, 62, 66].

With this method of sampling, the mean energy is

$$\langle E \rangle_\beta = \sum_q f_q(\theta_q | \beta) \epsilon_q(\theta_q) = N \oint_N \frac{d\theta}{2\pi} f(\theta | \beta) \epsilon(\theta), \quad (3.40)$$

where  $\oint$  denote the Riemann sum approximation to the respective integral with  $N$  subdivisions. As we mentioned before we are interested in the limit  $N \rightarrow \infty$ , we will replace the sums by its correspondent integral. Similarly we can compute the energy variance of the sampled states

$$\langle \Delta E^2 \rangle_\beta = \sum_q v_q(\theta_q | \beta) \epsilon_q^2(\theta_q) = N \oint_N \frac{d\theta}{2\pi} v(\theta | \beta) \epsilon^2(\theta). \quad (3.41)$$

Thus Gibbs sampling provides an even sampling of states within  $\Delta E$  of the energy  $\langle E \rangle$ , where  $\Delta E / \langle E \rangle \sim O(N^{-1/2})$ .

Thinking back on the case of the  $XY$  model, the ensemble defined by Gibbs sampling is nothing but the canonical ensemble defined on the full chain, with thermal density matrix

$$\rho_T(\beta, N) = \sum_{\vec{n}} p(\vec{n} | \beta) |\vec{n}\rangle \langle \vec{n}| = \frac{e^{-\beta H_N}}{Z(\beta, N)}, \quad \log Z(\beta, N) = N \oint_N \frac{d\theta}{2\pi} \log (1 \pm e^{-\beta \epsilon(\theta)}). \quad (3.42)$$

This thermal state defines a reduced density matrix in the subchain of length  $L$

$$\rho(\beta, N)|_L \equiv \text{Tr}_{N-L} \rho_T(\beta, N), \quad (3.43)$$

which is expected to correspond to the local thermal state,

$$\rho(\beta, N)|_L \simeq \rho_T(\beta, L) \equiv \frac{e^{-\beta H_L}}{Z(\beta, L)}, \quad (3.44)$$

for  $L$  sufficiently larger in comparison to the correlation length, in which the boundary effects can be neglected. Since we know this states conserve Gaussianity under partial traces, the reduced state  $\rho(\beta, N)|_L$  will be also Gaussian, meaning that the state will be uniquely characterised by its covariance matrix. This arguments lead us to the conclusion that the Gibbs average of the reduced partial density matrices  $\rho_L(\vec{n})$  satisfies

$$\rho(\beta, N)|_L = \langle \rho_L(\vec{n}) \rangle_\beta. \quad (3.45)$$

The latter series of arguments where need to understand why this method of sampling is appropriate for our purpose. Even more, it provide us an expression to the probability of having and excitation (a 1) or not (0). Having said that, we move gears to the computing the probability of having an error over two independent sequences. We can write this probability in terms of our energy as and a given  $\theta_k$

$$n(\theta_k) = \frac{1}{1 + e^{\beta(\Omega(\theta_k) - \Omega^*)}}, \quad (3.46)$$

where  $\Omega(\theta_k)$  is given by equation (3.12), and  $\Omega^*$  correspond to the minimum of energy<sup>5</sup>. Then the probability of two sequences not having an error in the position  $k$  will be given by

$$1 - p(\theta_k) = n(\theta_k)^2 + (1 - n(\theta_k))^2 = \frac{1}{1 + \operatorname{sech}(\beta(\Omega(\theta_k) - \Omega^*))}, \quad (3.47)$$

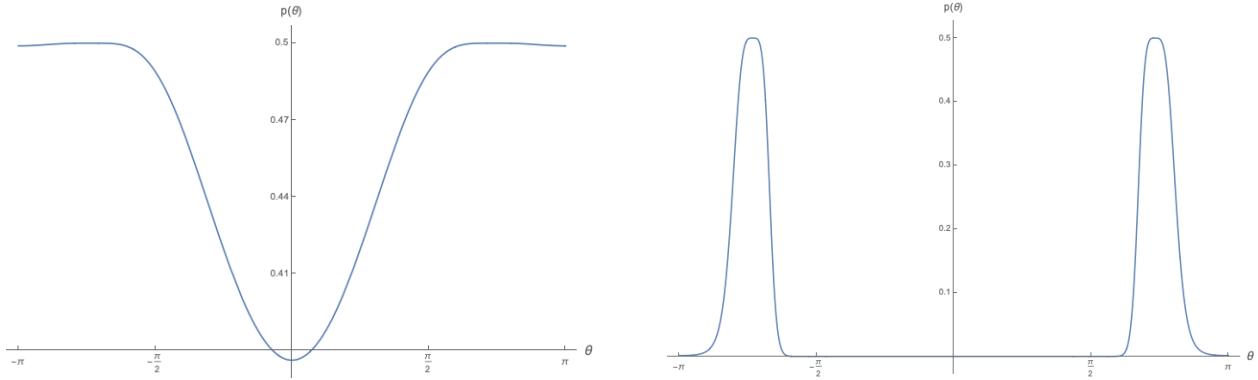
and then the probability of having an error will be given by

$$p(\theta_k) = \frac{1}{1 + \cosh(\beta(\Omega(\theta_k) - \Omega^*))}, \quad (3.48)$$

So the expected value of the number of errors  $X = \sum_k X(\theta_k)$  is given by

$$\mu = \sum p(\theta_k) \xrightarrow{N \rightarrow \infty} \frac{N}{2\pi} \oint d\theta \frac{1}{1 + \cosh(\beta(\Omega(\theta) - \Omega^*))}. \quad (3.49)$$

In Figure 3.4, we show the behaviour of the probability distribution as a function of the angle.



(a) Probability distributions of errors as a function of the position in the chain. this plot correspond to a value of  $\beta = 1$ .

(b) Probability distributions of errors as a function of the position in the chain. this plot correspond to a value of  $\beta = 80$ .

**Figure 3.4:** An illustration of how the probability distribution  $p(\theta_k)$  behaves as a function of the angle  $\theta$ . As we can see in both figures there are regions of the chain in which the probability of having an error is smaller.

As we would like to study a little deeper the behaviour of the expected value as a function of the temperature we could take the obtain some expressions for some limit cases. We can expand (3.49) when  $\beta \rightarrow 0$ .

<sup>5</sup>It is not hard to check that the minimum of energy happens at  $\theta^* = \pm \arccos\left(-\frac{\lambda}{1-\gamma^2}\right)$  with value

$$\sqrt{\frac{\gamma^2(\gamma^2 - 1 + \lambda^2)}{(\gamma^2 - 1)}}.$$

$$\begin{aligned}\mu &\approx \frac{N}{2\pi} \left( \frac{2\pi}{2} - \beta^2 \int_{-\pi}^{\pi} \frac{(\Omega(\theta) - \Omega^*)^2}{8} d\theta \right) \\ &= \frac{N}{2} - \frac{N\beta^2}{16} \int_{-\pi}^{\pi} (\Omega(\theta) - \Omega^*)^2 d\theta,\end{aligned}\quad (3.50)$$

which means that in the limit of  $\beta \rightarrow 0$  (High temperatures), the expected value  $\mu$  will tend to  $N/2$ . From other side, if we look at the case when  $\beta \rightarrow \infty$ , the integral (3.49) can be approximated as

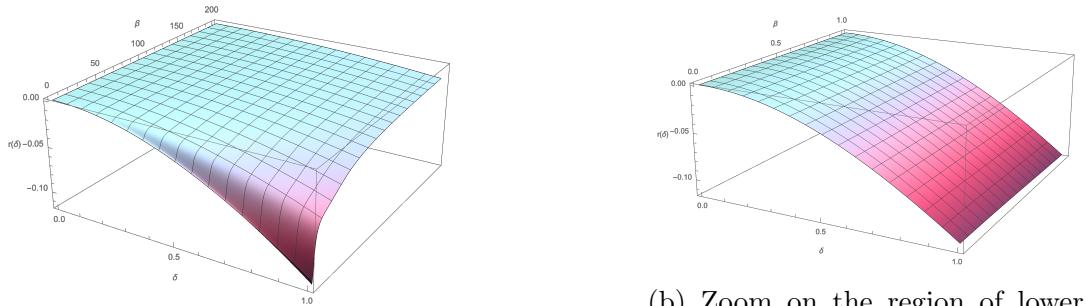
$$\mu \rightarrow \frac{N}{2\pi} \oint e^{-\beta(\Omega(\theta) - \Omega^*)} d\theta \rightarrow 0, \quad (3.51)$$

which is agreement with the fact that at zero temperature, the possible sequences are those which have no excitations, and therefore will have only zeros.

We can now compute the error exponent associated with this model by just replacing the probability in (3.48) into the expression found in (2.90), to obtain

$$r(\delta) = \min_S \oint \frac{d\theta}{2\pi} \log \left( 1 + \frac{1}{1 + \cosh(\beta(\Omega(\theta) - \Omega^*))} (e^S - 1) \right) - S\delta. \quad (3.52)$$

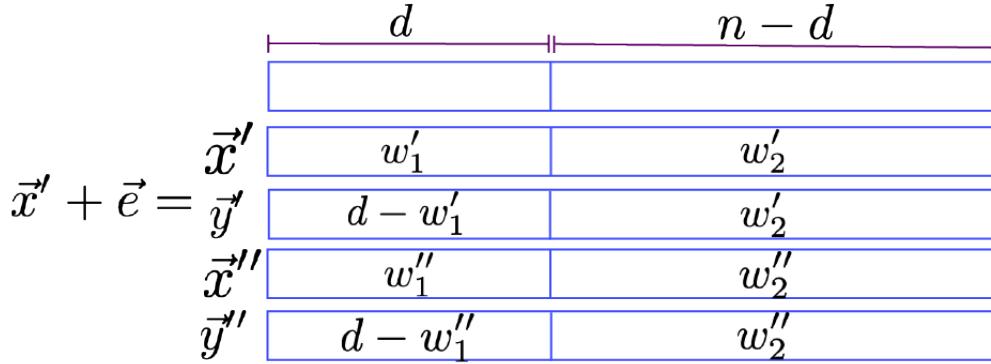
Even though we are not able to find an exact solution to this, we can do a numerical exercise and see how this behaves as a function of the temperature. In figure 3.5 our results are shown.



(a) Error exponent function for the  $XY$  model, in the figure the behaviour as a function of  $\beta$  is shown. The equation(3.52) is solved for values of  $\beta$  running from 0.01 to 200  
(b) Zoom on the region of lower  $\beta$ 's. The equation(3.52) is solved for values of  $\beta$  running from 0.01 to 1

**Figure 3.5:** Plot of the solution of equation (3.52), the behaviour of  $r(\delta)$  is shown as a function of  $\beta$ .

Now we go back the problem we left behind and start analysing the case when the number of errors is less than  $L$ .



**Figure 3.6:** Graphical description of the situation in which we order the modes such that the first  $d$  elements correspond to the errors and the rest  $N - d$  to the coincidences. This picture illustrate the relation between the weights of the sequences.

### 3.2.2 Finding Bounds for $\hat{X}_{ij}$

In Chapter two we discussed that if we are able to show that the operator  $\hat{X}_{ij}$  in norm is small, we could assert that Super Orthogonality will hold. First we want to find a bound to this quantity in terms of the transformations  $\mathcal{U}$  and  $\mathcal{V}$ . As a first approach we could reorder our sequences and divide them in two, one part of length  $d$  which will have all the errors and the rest  $N - d$ . Note that if we do so, for the region of the errors, one can check that if we have in the  $\vec{x}'$  a 0 then in the sequence  $\vec{y}'$  there must be a 1 in the same position, and for the region of the  $N - d$  we need that whenever we have a 0 in the  $\vec{x}'$  sequence, there must be a 0 in the  $\vec{y}'$ . Indeed it holds also for  $\vec{x}''$  and  $\vec{y}''$ , which in general provides a relation with the weights of each pair of sequences. In figure 3.6, there is a graphical description to illustrate this situation.

Explicitly, if we mane  $w'_x$ ,  $w'_y$ ,  $w''_x$  and  $w''_y$  to the total weight of the sequences  $\vec{x}'$ ,  $\vec{y}'$ ,  $\vec{x}''$ ,  $\vec{y}''$  respectively,

$$\begin{aligned} w'_x &= w'_1 + w'_2, \\ w'_y &= d - w'_1 + w'_2, \\ w''_x &= w''_1 + w''_2, \\ w''_y &= d - w''_1 + w''_2. \end{aligned} \tag{3.53}$$

Nonetheless, from the way our problem is structured, we know that the transformations we are dealing with ( $\mathcal{U}_{\vec{x}\vec{x}'}$  and  $\mathcal{V}_{\vec{y}\vec{y}'}$ ) have an special structure which as we showed in section 3.1.5 separates the modes of  $x$  from the ones of  $y$ . This implies that the weight of the sequences have to, which is related to the degree of the  $p$ -form, has to be the same for  $\vec{x}'$  and  $\vec{x}''$ , and

therefore implying that

$$w'_1 = w''_1 \equiv w_1, \quad w'_2 = w''_2 \equiv w_2 \quad (3.54)$$

where  $w_1$  and  $w_2$  are restricted to the conditions. In figure ?? we illustrate the region we work with

$$\begin{aligned} 0 &\leq w_1 \leq d \\ 0 &\leq w_2 \leq N - d \\ 0 &\leq w_1 + w_2 \leq L \\ 0 &\leq d - w_1 + w_2 \leq L, \end{aligned} \quad (3.55)$$

Then, notice that  $\|\hat{X}_{ij}\|$  would reads<sup>6</sup>

$$\|X_{ij}\|_2^2 \leq \frac{1}{2^L} \sum_{w_1, w_2} \binom{d}{w_1}^2 \binom{n-d}{w_2}^2 \left[ \frac{2}{N}(w_1 + w_2) \right]^{w_1+w_2} \left[ \frac{2}{N}(d - w_1 + w_2) \right]^{d-w_1+w_2}. \quad (3.56)$$

In order to find a bound we suppose that equation (3.56) follows a law of large deviations, therefore we can assert that

$$\|X_{ij}\|_2^2 \leq e^{Nf(w_1, w_2, L, d, N)}, \quad (3.57)$$

where  $f(w_1, w_2, L, d, N)$  will be given by

$$\begin{aligned} f(w_1, w_2, L, d, N) &= 2 \frac{d}{N} H_2 \left( \frac{w_1}{2} \right) + 2 \left( 1 - \frac{d}{N} \right) H_2 \left( \frac{w_2}{N-d} \right) + \frac{w_1 + w_2}{N} \log_2 \left( 2 \frac{w_1 + w_2}{N} \right) \\ &\quad + \frac{d - w_1 + w_2}{N} \log_2 \left( 2 \frac{d - w_1 + w_2}{N} \right) - \frac{L}{N}. \end{aligned} \quad (3.58)$$

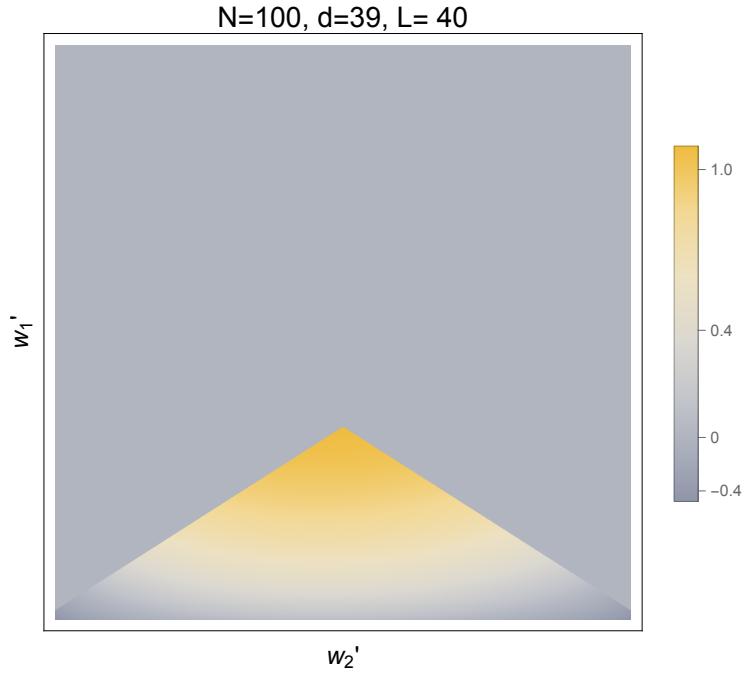
Our goal is now study the behaviour of this function in the region described in (3.55). Notice that if the function is negative in the all the region, we can assure that the quantity  $\|\hat{X}_{ij}\|$  will be bounded by a quantity which is exponentially small in  $N$  (The size of the system). In figures 3.7 and 3.8 we can see the behaviour of the function  $f$  is such that over the region, it will be positive. This is,  $\|\hat{X}_{ij}\|$  will be bounded by a quantity which is exponentially large, which is quite the opposite we wanted to proof. However, this approach was quite naïve and we never took into account the structure of the  $XY$  model. As we will show briefly, to

---

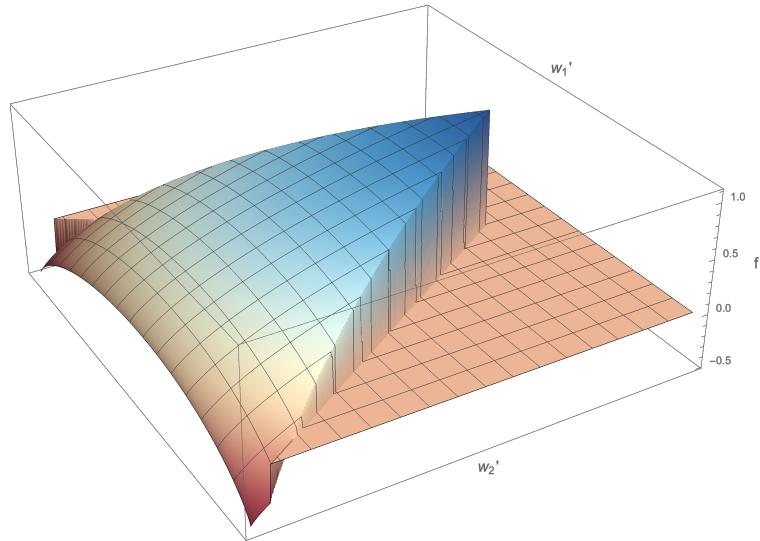
<sup>6</sup>Here we simply count the number of possible sequences that are restricted for the conditions mentioned before. Additionally to this, we use the Haddamard bound for bound the determinant. This bound establishes that if the entries of an  $n$  by  $n$  matrix  $M$ , are bounded by a quantity  $B$ ,

$$|\det(M)| \leq B^n n^{n/2}.$$

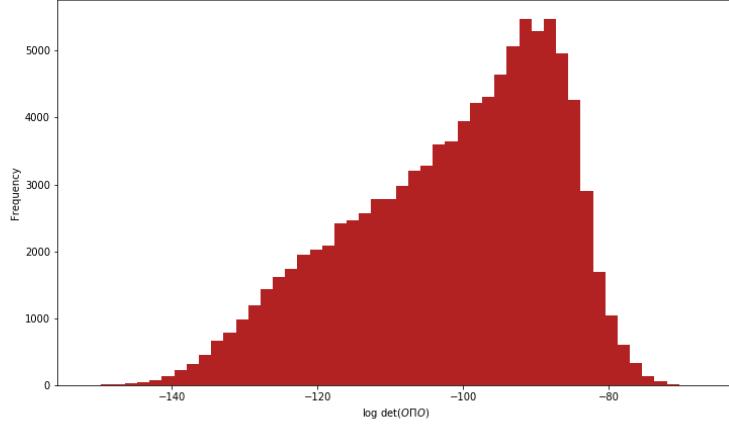
For our particular case notice that the transformations  $\mathcal{U}, \mathcal{V}$  are bounded by  $1/\sqrt{N}$



**Figure 3.7:** Illustration of the behaviour of the function  $f$  in the region described in (3.55), for values of  $N = 100$ ,  $d = 39$  and  $L = 40$ . As we can see most of the values are positive in the region.



**Figure 3.8:** Illustration in 3D of the behaviour of the function  $f$  in the region described in (3.55), for values of  $N = 100$ ,  $d = 39$  and  $L = 40$ . In this plot we can see clearer that only a small part of the whole region is negative.



**Figure 3.9:** Histogram of the log of the absolute value of the determinant in (2.97) for the case of the  $XY$  model. The parameters here were  $N = 200$  and  $L = 50$ .

bound this quantity and proof what we want it is needed to consider the structure of the  $XY$  model.

First to convince us that this quantity is indeed quite small we first do a numerical experiment. We are going to compute and ensemble of determinants in (2.97) for the case of the  $XY$  model. To do so, we first generate the transformations  $O$  as we described in section 3.1.5, we take the projection to at the fist  $L$  modes and by randomly generating the sequences  $\vec{x}', \vec{x}''$  and computing the determinant of the correspondent minor. The results for a particular value of  $L$  and  $N$  are shown in figure 3.9.

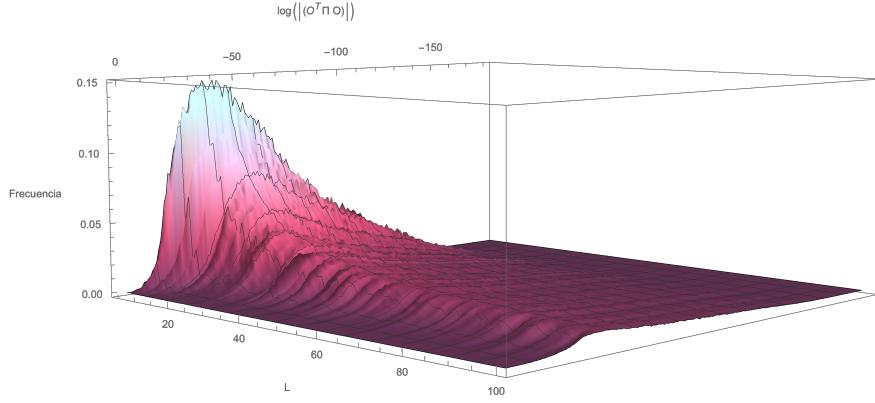
Even more, if we change  $L$  for a given  $N$ , we could build a surface with the histograms at different length  $L$ . This is shown in figure

Having seen these results we then have an insight that indeed  $\|\hat{X}_{ij}\|$  must be quite small. To understand why this is happening, we will find an expression for  $O^T \Pi_L O$ . Notice that the rows in our transformations for the  $XY$  are cosine functions, which in terms of the angle  $\theta_k$  reads,

$$u^k(x) = \sqrt{\frac{2}{N}} \cos \theta_k x. \quad (3.59)$$

Since our interest is to find the an explicit form for the sum of the product of two of these functions, we have than that

$$\sum_{\vec{x}} O_{\vec{x}, \vec{x}'} O_{\vec{x}, \vec{x}''} \equiv A \rightarrow A_{i,j} = \sum_{x=-\frac{L-1}{2}}^{\frac{L-1}{2}} u^i(x) u^j(x). \quad (3.60)$$



**Figure 3.10:** Surface form by the histograms of the log of the absolute value of the determinant in (2.97) for the case of the  $XY$  model. Here the dependence with the size of the subsystem  $L$ . The parameters here were  $N = 200$  and varying from  $L = 7$  to  $90$ .

with a little of algebra and a property of the sum of cosines<sup>7</sup> we can show that

$$A_{i,j} = \frac{1}{N} D_L(\theta_j - \theta_i), \quad (3.61)$$

where  $D_L(\theta_j - \theta_i)$  is the Dirichlet kernel given by

$$D_L(x) = \frac{\sin\left(x\frac{L}{2}\right)}{\sin\left(\frac{x}{2}\right)}. \quad (3.62)$$

So in terms of the sequences  $\vec{x}', \vec{x}''$ , the matrix  $O^T \Pi_L O$  can be written as

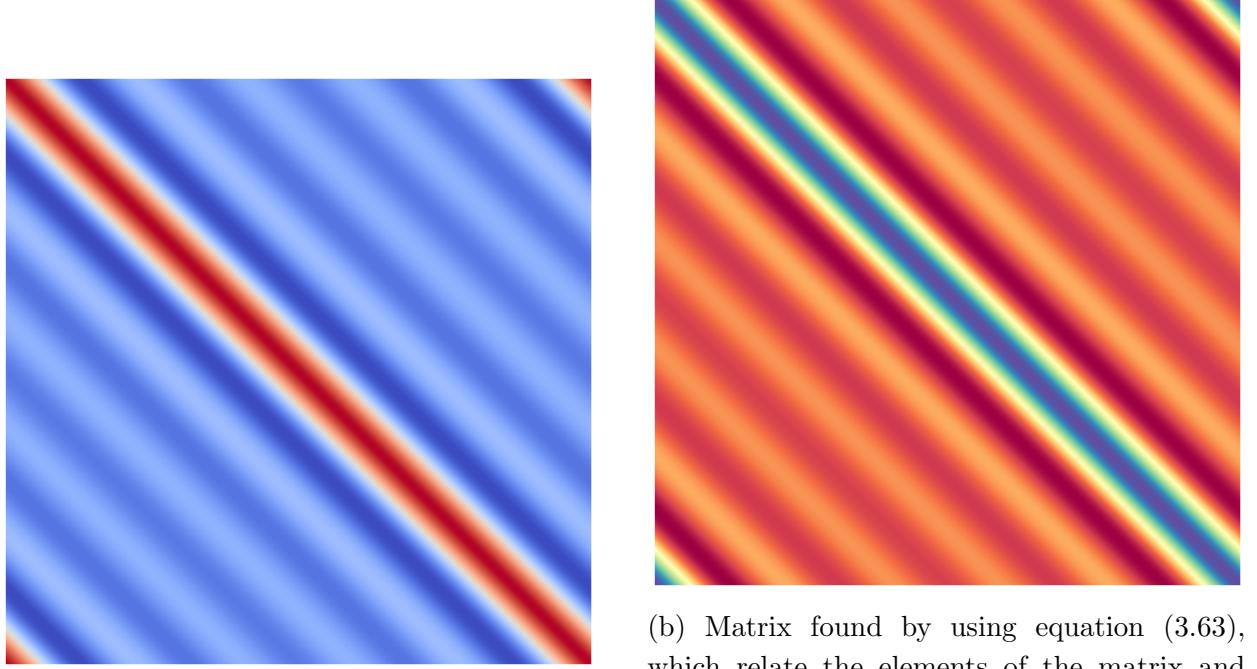
$$O^T \Pi_L O |_{\vec{x}', \vec{x}''} = \frac{1}{N} D_L \left( \frac{2\pi}{N} (\vec{x}' - \vec{x}'') \right). \quad (3.63)$$

Therefore in order to bound the determinant of the minor we could do two things. One is that we could try to find one bound by force, but this technique was proved previously to do not work really well. Or the other is to exploit the fact that the matrix  $O^T \Pi_L O$  belongs to a particular kind of matrices named Toeplitz matrices. As we show in figure 3.11, this kind of matrix is known to have fixed values over the diagonals, being one kind of sparse matrix.

---

<sup>7</sup>The exact relation used to proof this is given by

$$\cos \theta_i x \cos \theta_j x = \frac{1}{2} (\cos(x(\theta_j - \theta_k)) + \cos(x(\theta_j + \theta_k)))$$



(a) Plot of the matrix  $O^T \Pi_L O$ , with  $N = 8000$  and  $L = 30$

(b) Matrix found by using equation (3.63), which relate the elements of the matrix and Dirichlet kernel  $\frac{1}{N} D_L \left( \frac{2\pi}{N} (\vec{x}' - \vec{x}'') \right)$ , with  $N = 8000$  and  $L = 30$

**Figure 3.11:** Comparison between the two matrices, this illustrates first, the fact that indeed the matrix  $O^T \Pi_L O$  is a Toeplitz matrix and second, that for the case of the  $XY$  model the matrix  $O^T \Pi_L O$  can be written in terms of the Dirichlet kernel.

This kind of matrices have been a subject of great study this is because this kind of matrices appear in many physical systems [67–69]. It turns out that the computation of this kind of determinants has already been studied and indeed, it this computation is a case of use of the Frobenius-Schur duality theorem [68]. Particularly, it is possible to use a generalisation of Szegö theorem [70] provided by Bump and Diaconis [71]. In this theorem, as in the case of Szegö theorem, it is possible to find an asymptotic expression for the determinant of Toeplitz minors. Astonishingly, the character of the symmetric group ( $U(N)$ ) emerges in this expression.

Let  $\lambda$  and  $\mu$  be partitions of length  $\leq N$  [68],

$$D_{n-1}^{\lambda, \mu}(f) = \det(d_{\lambda_i - \mu_j - i + j}), \quad (3.64)$$

where  $d_i$  are the elements of the Toeplitz matrix. It is not hard to check that indeed the expression in (3.64) is the minor in a large Toeplitz matrix [68].

**Theorem 3.2.1** (Heine, Szegö, Bump, Diaconis). *Let  $f \in L^1(\mathbb{T})$  be given, with  $f(t) = \sum_{n=-\infty}^{\infty} d_n t^n$ , the symbol of the Toeplitz matrix. Let  $\lambda$  and  $\mu$  be partitions of length  $\leq N$ . Define a function  $\Phi_{n,f}(g)$  on  $U(n)$  by  $\Phi_{n,f}(g) = \prod_{i=1}^n f(t_i)$ , where  $t_i$  are the eigenvalues of*

$g \in U(n)$ . Then

$$D_{n-1}^{\lambda, \mu}(f) = \int_{U(n)} \Phi_{n,f}(g) \overline{\chi_\lambda(g)} \chi_\mu(g) dg, \quad (3.65)$$

with  $\chi_i$  the character of  $U(n)$  [68, 71].

Note that if  $\lambda$  and  $\mu$  are trivial, the last expression will become *Heine-Szegő* identity. With the latter theorem we have then an analytical expression to compute determinant of the Toeplitz minor and find the exact value of (2.97) for the case of the  $XY$  model.

# Bibliography

- [1] Ludwig Boltzmann. *Über die mechanische Bedeutung des zweiten Hauptsatzes der Wärmetheorie: (vorgelegt in der Sitzung am 8. Februar 1866)*. Staatsdruckerei. Google-Books-ID: 8Hc5AAAAcAAJ. iv, 7
- [2] Sandu Popescu, Anthony J. Short, and Andreas Winter. The foundations of statistical mechanics from entanglement: Individual states vs. averages. 1, 6, 16, 17, 19
- [3] Michał Horodecki, Jonathan Oppenheim, and Andreas Winter. Partial quantum information. 436(7051):673–676. 1
- [4] Vitali D. Milman and Gideon Schechtman. *Asymptotic Theory of Finite Dimensional Normed Spaces: Isoperimetric Inequalities in Riemannian Manifolds*. Springer. Google-Books-ID: q-RqCQAAQBAJ. 1
- [5] Michel Ledoux. *The Concentration of Measure Phenomenon*. American Mathematical Society. 1
- [6] J. Gemmer, M. Michel, and G. Mahler. *Quantum Thermodynamics: Emergence of Thermodynamic Behavior Within Composite Quantum Systems*, volume 657 of *Lecture Notes in Physics*. Springer Berlin Heidelberg. 1, 5
- [7] Sheldon Goldstein, Joel L. Lebowitz, Roderich Tumulka, and Nino Zanghì. Canonical typicality. 96(5):050403. 1, 5, 6
- [8] Noah Linden, Sandu Popescu, Anthony J. Short, and Andreas Winter. Quantum mechanical evolution towards thermal equilibrium. 79(6):061103. 1, 6, 20, 21, 22, 24, 25
- [9] Noah Linden, Sandu Popescu, Anthony J. Short, and Andreas Winter. On the speed of fluctuations around thermodynamic equilibrium. 12(5):055021. 1
- [10] Artur S. L. Malabarba, Luis Pedro García-Pintos, Noah Linden, Terence C. Farrelly, and Anthony J. Short. Quantum systems equilibrate rapidly for most observables. 90(1):012121. 1

- [11] Sandu Popescu, Anthony J. Short, and Andreas Winter. Entanglement and the foundations of statistical mechanics. 2(11):754–758. 5, 6, 16, 17, 19
- [12] L. D. Landau and E. M. Lifshitz. *Statistical Physics: Volume 5*. Butterworth-Heinemann, 3 edition edition. 5
- [13] J. Uffink. *Handbook for Philosophy of Physics 924-1074*. Elsevier, Amsterdam. 6, 10
- [14] Ludwig Boltzmann. Zur priorität der auffindung der beziehung zwischen dem zweiten hauptsatze der mechanischen wärmetheorie und dem principe der kleinsten wirkung. 219(6):211–230. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.18712190606>. 7
- [15] Ludwig Eduard Boltzmann. *Einige allgemeine sätze über Wärmegleichgewicht*. K. Akad. der Wissensch. 7
- [16] Josiah Willard Gibbs. *Elementary Principles in Statistical Mechanics: Developed with Especial Reference to the Rational Foundation of Thermodynamics*. C. Scribner's Sons. 8
- [17] Paul Ehrenfest. *The conceptual foundations of the statistical approach in mechanics, by Paul and Tatiana Ehrenfest. Translated by Michael J. Moravcsik*. Ithaca, N.Y., Cornell University Press [1959]. 9, 10
- [18] César R. de Oliveira and Thiago Werlang. Ergodic hypothesis in classical statistical mechanics. 29(2):189–201. Publisher: Sociedade Brasileira de Física. 10
- [19] Larry Engelhardt, Marie Lopez del Puerto, and Norman Chonacky. Simple and synergistic ways to understand the boltzmann distribution function. 83(9):787–793. Publisher: American Association of Physics Teachers. 10
- [20] Giovanni Gallavotti. *Statistical Mechanics: A Short Treatise*. Theoretical and Mathematical Physics. Springer-Verlag. 10
- [21] Sheldon Goldstein. Boltzmann's approach to statistical mechanics. In Jean Bricmont, Giancarlo Ghirardi, Detlef Dürr, Francesco Petruccione, Maria Carla Galavotti, and Nino Zanghi, editors, *Chance in Physics: Foundations and Perspectives*, Lecture Notes in Physics, pages 39–54. Springer. 10
- [22] Kerson Huang. *Statistical Mechanics, 2nd Edition*. John Wiley & Sons, 2 edition edition. 11, 12
- [23] E. Schrödinger. Discussion of probability relations between separated systems. 31(4):555–563. Publisher: Cambridge University Press. 13

- [24] Artur Ekert, Patrick Hayden, and Hitoshi Inamori. Basic concepts in quantum computation. version: 1. 14
- [25] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press. Google-Books-ID: 65FqEKQOfP8C. 14
- [26] K. W. Edwin and R. J. Taud. Optimal operating system composition using integer programming and statistic reserve determination. In *Electrical Load-Curve Coverage*, pages 13–30. Pergamon. 14
- [27] John Von Neumann. *Mathematical foundations of quantum mechanics*. Princeton, N.J. : Princeton University Press. 14
- [28] David J. C. MacKay. *Information Theory, Inference and Learning Algorithms*. Cambridge University Press, 1 edition edition. 15, 37, 39, 43
- [29] C. E. Shannon. A mathematical theory of communication. 27(3):379–423. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/j.1538-7305.1948.tb01338.x>. 15, 36
- [30] Martin B. Plenio and S. Virmani. An introduction to entanglement measures. 16
- [31] J. Gemmer and G. Mahler. Distribution of local entropy in the hilbert space of bi-partite quantum systems: origin of jaynes' principle. 31(2):249–257. 16
- [32] Don N. Page. Average entropy of a subsystem. 71(9):1291–1294. 16
- [33] Paolo Facchi and Giancarlo Garnero. Quantum thermodynamics and canonical typicality. 14(8):1740001. 19
- [34] C.A. Fuchs and J. van de Graaf. Cryptographic distinguishability measures for quantum-mechanical states. 45(4):1216–1227. Conference Name: IEEE Transactions on Information Theory. 24
- [35] Giancarlo Camilo, Gabriel T. Landi, and Sebas Eliëns. On the strong subadditivity of the  $r\backslash$ 'enyi entropies for bosonic and fermionic gaussian states. 99(4):045155. 24
- [36] Alonso Botero and Benni Reznik. BCS-like modewise entanglement of fermion gaussian states. 331(1):39 – 44. 28, 29, 56, 57
- [37] Eduardo Fradkin. *Field Theories Of Condensed Matter Systems Vol#82 On-demand Printing Of #52253*. CRC Press, 1 edition edition. 29, 60
- [38] F. A. Berezin. *The method of second quantization*. Number v. 24 in Pure and applied physics. Academic Press. 29

- [39] N. N. Bogoljubov. On a new method in the theory of superconductivity. 7(6):794–805. 29
- [40] B. Westwanski and A. Pawlikowski. On a general statistical wick theorem. 43(2):201–202. 29, 30
- [41] Luca Guido Molinari. Notes on wick’s theorem in many-body theory. 29
- [42] Martin Leijnse and Karsten Flensberg. Introduction to topological superconductivity and majorana fermions. 27(12):124003. 29
- [43] Siew-Ann Cheong and Christopher L. Henley. Many-body density matrices for free fermions. 30
- [44] Christina V. Kraus, Michael M. Wolf, J. Ignacio Cirac, and Geza Giedke. Pairing in fermionic systems: A quantum information perspective. 79(1):012306. 31
- [45] E. Knill, R. Laflamme, and G. J. Milburn. A scheme for efficient quantum computation with linear optics. 409(6816):46–52. Number: 6816 Publisher: Nature Publishing Group. 32
- [46] Barbara M. Terhal and David P. DiVincenzo. Classical simulation of noninteracting-fermion quantum circuits. 65(3):032325. 32
- [47] E. Knill. Fermionic linear optics and matchgates. 32
- [48] Density operators for fermions. In *Quantum Theory of Optical Coherence*, pages 595–631. John Wiley & Sons, Ltd. Section: 16 eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/9783527610075.ch16>. 32
- [49] D. E. Soper. Construction of the functional-integral representation for fermion green’s functions. 18:12. Institution: Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403. 33
- [50] Sergey Bravyi. Classical capacity of fermionic product channels. version: 1. 33
- [51] Sergey Bravyi. Lagrangian representation for fermionic linear optics. 34
- [52] E. T. Jaynes. *Probability Theory: The Logic of Science*. 37, 43
- [53] A. Barg and G.D. Forney. Random codes: minimum distances and error exponents. 48(9):2568–2573. Conference Name: IEEE Transactions on Information Theory. 43, 44
- [54] R. Gallager. Low-density parity-check codes. 44
- [55] Robert G. Gallager. *Information Theory and Reliable Communication*. 44

- [56] R. Gallager. The random coding bound is tight for the average code (corresp.). 19(2):244–246. 44
- [57] Elliott Lieb, Theodore Schultz, and Daniel Mattis. Two soluble models of an antiferromagnetic chain. 16(3):407–466. 53, 60
- [58] Shigetoshi Katsura. Statistical mechanics of the anisotropic linear heisenberg model. 127(5):1508–1518. 53, 60
- [59] Eytan Barouch and Barry M. McCoy. Statistical mechanics of the \$XY\$ model. II. spin-correlation functions. 3(2):786–804. 53, 55
- [60] Eytan Barouch, Barry M. McCoy, and Max Dresden. Statistical mechanics of the \$\mathcal{X}\mathcal{Y}\$ model. i. 2(3):1075–1092. 53, 55
- [61] A. F. Reyes-Lega. Some aspects of operator algebras in quantum physics. pages 1–74. 54
- [62] J. I. Latorre, E. Rico, and G. Vidal. Ground state entanglement in quantum spin chains. 4(1):48–92. 56, 57, 60
- [63] Christian P. Robert and George Casella. The multi-stage gibbs sampler. In Christian P. Robert and George Casella, editors, *Monte Carlo Statistical Methods*, Springer Texts in Statistics, pages 371–424. Springer. 60
- [64] Markov chain monte carlo in practice. 60
- [65] Gibbs sampler - an overview | ScienceDirect topics. 60
- [66] J. Eisert, M. Cramer, and M. B. Plenio. Area laws for the entanglement entropy - a review. 82(1):277–306. version: 4. 60
- [67] Albrecht Böttcher and Bernd Silbermann. *Introduction to Large Truncated Toeplitz Matrices*. Universitext. Springer-Verlag. 69
- [68] Daniel Bump. *Lie Groups*. 69, 70
- [69] The fisher-hartwig conjecture and toeplitz eigenvalues - ScienceDirect. 69
- [70] G. Szegö. Ein grenzwertsatz über die toeplitzschen determinanten einer reellen positiven funktion. 76(4):490–503. 69
- [71] Daniel Bump and Persi Diaconis. Toeplitz minors. 97:252–271. 69, 70
- [72] Seth Lloyd. Quantum information ProcessingQuantum information processing. In Robert A. Meyers, editor, *Computational Complexity: Theory, Techniques, and Applications*, pages 2496–2533. Springer.

- [73] Ingo Peschel. Calculation of reduced density matrices from correlation functions. 36(14):L205–L208.
- [74] *Field States*, chapter 11, pages 291–319. John Wiley & Sons, Ltd, 2005.
- [75] Philippe Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag.
- [76] Neri Merhav. Error exponents of typical random codes.
- [77] Yuval Domb, Ram Zamir, and Meir Feder. The random coding bound is tight for the average linear code or lattice. 62(1):121–130. Conference Name: IEEE Transactions on Information Theory.
- [78] Ming-Chiang Chung and Ingo Peschel. Density-matrix spectra of solvable fermionic systems. 64(6):064412.
- [79] Yuezhen Niu, Suk Bum Chung, Chen-Hsuan Hsu, Ipsita Mandal, S. Raghu, and Sudip Chakravarty. Majorana zero modes in a quantum ising chain with longer-ranged interactions. 85(3):035110.