Free Fermions and Bosons

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SUMMARY

We will consider a one-dimensional, translationally invariant closed chain of N free fermions or bosons, with local (near-neighbor) interactions. Let ψ_x (resp. $\psi_x^\dagger$) be annihilation/creation operators at the site x ($x = 0, 1, \ldots, N-1, N \equiv 0$) and b_q (resp. $b_q^\dagger$) their counterparts in the plane-wave basis, obtained through

$$b_q = \frac{e^{i\eta_q}}{\sqrt{N}} \sum_{x=0}^{N-1} e^{-i\phi_q x} \psi_x \, , \tag{1}$$

and its hermitian conjugate, with $\phi_q = \frac{2\pi}{N} q$, where η_q is a phase to be adjusted for later convenience. In the plane-wave basis we take the Hamiltonian to be of the form

$$H_N = \sum_{q=0}^{N} \epsilon(\phi_q) b_q^\dagger b_q \, , \tag{2}$$

where we will assume that $\epsilon(\phi)$ has a Fourier expansion of the form

$$\epsilon(\phi) = \sum_{m=0}^{m_{max}} \epsilon_m \cos(m\phi) \tag{3}$$

with $m_{max} \sim O(1)$.

Likewise, we will consider a set of local plane wave modes for the portion of the chain of length L comprising the sites $x = 0, 1 \ldots L-1$,

$$\widetilde{b}_k = (-1)^k \frac{e^{-i\tilde{\phi}_k/2}}{\sqrt{L}} \sum_{x=0}^{L-1} e^{-i\widetilde{\phi}_k x} \psi_x \, , \tag{4}$$

where similarly $\widetilde{\phi}_k = \frac{2\pi}{L} k$. As in the case of the full chain of length N, the plain wave modes diagonalize any Hamiltonian of the same form (2), with N replaced by L. In particular, we will use the Hamiltonian

$$H_L = \sum_{k=0}^{L} \epsilon(\phi_k) \widetilde{b}_k^\dagger \widetilde{b}_k \, , \tag{5}$$

with the same function $\epsilon(\phi)$ as defined earlier.

We now expand the local operators $\widetilde{b}_k$ in terms of those of the full chain b_q, and choosing the phase η_q to be $\eta_q = \phi_q(L-1)/2$, the relation between the two becomes

$$\widetilde{b}_k = \frac{1}{\sqrt{NL}} \sum_{q=0}^{N-1} D_L(\phi_q - \widetilde{\phi}_k) \; b_q \, , \tag{6}$$

where

$$D_L(\phi) = \frac{\sin\left(\frac{\phi}{2} L\right)}{\sin\left(\frac{\phi}{2}\right)} \, . \tag{7}$$

is the Dirichlet kernel. The following two identities are easily verified

$$\frac{1}{N} \sum_{q=0}^{N-1} D_L(\phi_q - \widetilde{\phi}_k) D_L(\phi_q - \widetilde{\phi}_{k'}) = L\delta_{kk'} \tag{8}$$

and

$$\frac{1}{L}\sum_{k=0}^{L-1} D_L(\phi-\widetilde{\phi}_k)D_L(\phi'-\widetilde{\phi}_k) = D_L(\phi-\phi') \tag{9}$$

Now consider an excited state $|\mathbf{n}\rangle$ of the chain, described by a set of excitation numbers $\mathbf{n} \equiv (n_0, n_1, \ldots, n_{N-1})$ where $n_q \in \{0,1\}$ (fermions) or $n_q \in \mathbb{N}_0$ (bosons). In particular, the state satisfies

$$\langle \mathbf{n}|\, b_q^\dagger b_{q'}\, |\mathbf{n}\rangle = \delta_{q,q'} n_q\,, \qquad \langle \mathbf{n}|\, b_q\, |\mathbf{n}\rangle = \langle \mathbf{n}|\, b_q b_{q'}\, |\mathbf{n}\rangle = \langle \mathbf{n}|\, b_q^\dagger b_{q'}^\dagger\, |\mathbf{n}\rangle = 0\,, \tag{10}$$

for all q, q'. To any such state, we can associate a hermitian $L \times L$ matrix A, defined by the matrix elements

$$A_{kk'}(\mathbf{n}) = \langle \mathbf{n}|\, \widetilde{b}_k^\dagger \widetilde{b}_{k'}\, |\mathbf{n}\rangle = \frac{1}{NL}\sum_{q=0}^{N-1} D_L(\phi_q-\widetilde{\phi}_k)D_L(\phi_q-\widetilde{\phi}_{k'})n_q\,. \tag{11}$$

In the case of free fermions or bosons, the matrix A provides a full characterization of the matrix elements of all bilinear combinations of the operators $\widetilde{b}_k^\dagger$ and $\widetilde{b}_k$ in the state $|\mathbf{n}\rangle$, or equivalently, in the state

$$\rho_L(\mathbf{n}) = \mathrm{Tr}_{N-L}\, |\mathbf{n}\rangle\langle\mathbf{n}|\,, \tag{12}$$

the corresponding partial density matrix on the subchain.

GIBBS SAMPLING

Consider now sampling over the excited states $|\mathbf{n}\rangle$, where the occupations n_q are sampled independently according to the Boltzmann distribution, i.e.,

$$p(\mathbf{n}|\beta) = \prod_{q=0}^{N-1} p(n_q|\beta)\,, \qquad p(n_q|\beta) = \frac{e^{-\beta\epsilon(\phi_q)n_q}}{\sum_{n_q} e^{-\beta\epsilon(\phi_q)n_q}}\,. \tag{13}$$

The average number of excitations in the mode at angle ϕ_q is given by

$$f(\phi_q|\beta) \equiv \langle n_q\rangle_\beta = \frac{1}{e^{\beta\epsilon(\phi_q)} \pm 1} \tag{14}$$

where the upper sign applies to fermions and the lower one to bosons, while the variance in the number of excitattions is given by

$$v(\phi_q|\beta) \equiv \langle n_q^2\rangle_\beta - \langle n_q\rangle_\beta^2 = \frac{1}{e^{\beta\epsilon(\phi_q)} \pm 1} = f(\phi_q)\,(1 \mp f(\phi_q)) \tag{15}$$

With this method of sampling, the mean energy is

$$\langle E\rangle_\beta = \sum_q f_q(\phi_q|\beta)\epsilon_q(\phi_q) = N \oint_N \frac{d\phi}{2\pi} f(\phi|\beta)\epsilon(\phi)\,, \tag{16}$$

where here and henceforth $\oint_N$ will denote the Riemman sum approximation to the corresponding integral with N subdivisions. As we will consider the limit $N \to \infty$, in most cases one may replace the integral for $\oint_N$. Similarly, the energy variance of the sampled states is given by

$$\left\langle \Delta E^2\right\rangle_\beta = \sum_q v_q(\phi_q|\beta)\epsilon_q^2(\phi_q) = N \oint_N \frac{d\phi}{2\pi} v(\phi|\beta)\epsilon^2(\phi)\,. \tag{17}$$

Thus, Gibbs sampling provides a more-or-less even sampling of states within ΔE of the energy $\langle E\rangle$, where $\Delta E/E \sim O(N^{-1/2})$.

From standard statistical physics, the ensemble defined by Gibbs sampling is the canonical ensemble defined on the full chain, with thermal density matrix

$$\rho_T(\beta,N) = \sum_{\mathbf{n}} p(\mathbf{n}|\beta)\,|\mathbf{n}\rangle\langle\mathbf{n}| = \frac{e^{-\beta H_N}}{Z(\beta,N)}\,, \qquad \log Z(\beta,N) = \pm N\frac{d\phi}{2\pi}\oint_N \log\left(1\pm e^{-\beta\epsilon(\phi)}\right)\,. \tag{18}$$

This thermal state defines a reduced density matrix in the subchain of length L

$$\rho(\beta,N)|_L \equiv \mathrm{Tr}_{N-L}\,\rho_T(\beta,N)\,, \tag{19}$$

which may be expected to correspond to the local thermal state, i.e.,

$$\rho(\beta,N)|_L \simeq \rho_T(\beta,L) \equiv \frac{e^{-\beta H_L}}{Z(\beta,L)}\,, \tag{20}$$

for L sufficient larger than the correlation length so that boundary effects can be neglected. In any case, since in both the fermionic and bosonic case the thermal state $\rho_T(\beta,N)$ is Gaussian, the reduced state $\rho(\beta,N)|_L$ will also be Gaussian. Hence in both cases the reduced state $\rho(\beta,N)|_L$ will be uniquely characterized by its covariance matrix, and in our particular case of interest, by the matrix of correlations (in the local plane wave basis, say)

$$A^{\beta}{}_{kk'} = \mathrm{Tr}\left(\rho_T(\beta,N)\widetilde{b}_k^{\dagger}\widetilde{b}_{k'}\right) = \frac{1}{L}\oint_N D_L(\phi-\widetilde{\phi}_k)D_L(\phi-\widetilde{\phi}_{k'})f(\phi|\beta)\,. \tag{21}$$

Thus, we have two equivalent statements. From the definition of the canonical density matrix on the chain, and of its reduced state on the subchain, the Gibbs average of the reduced partial density matrices $\rho_L(\mathbf{n})$ satisfies

$$\rho(\beta,N)|_L = \langle\rho_L(\mathbf{n})\rangle_\beta\,. \tag{22}$$

By the Gaussian nature of $\rho(\beta,N)|_L$, this relation is equivalent to a relation of correlation matrices,

$$A^{\beta}{}_{kk'} = \langle A_{kk'}(\mathbf{n})\rangle_\beta\,, \tag{23}$$

stating that the Gibbs average of the correlation matrices $A_{kk'}(\mathbf{n})$ is the matrix $A^{\beta}{}_{kk'}$. In what follows we want to explore precisely how the individual $A_{kk'}(\mathbf{n})$ concentrate around the ensemble average $A^{\beta}{}_{kk'}$. In the case of Fermions, for which the states $|\mathbf{n}\rangle$, and hence the reduced states $\rho_L(\mathbf{n})$, are Gaussian, statements of concentration of the correlation matrices $A_{kk'}(\mathbf{n})$ about $A^{\beta}{}_{kk'}$ translate to corresponding statements of concentration of the states $\rho_L(\mathbf{n})$ about the reduced thermal state $\rho(\beta,N)|_L$. In the bosonic case, the equivalence is less direct, as the individual states $\rho_L(\mathbf{n})$ are not Gaussian a priori. To establish a similar correspondence, we will need to furthermore argue that under suitable conditions, the reduced states $\rho_L(\mathbf{n})$ are essentially Gaussian. This will be done later.

Before proceeding, we note that a closed form can be given for the matrix elements (see sec. ?). For a given real periodic function $f(\phi) = \sum_{m=-\infty}^{\infty} f_m e^{im\phi}$, with Fourier components $f_{-m} = f_m^*$, define the complex analytic signal $\widehat{f}(\phi)$ associated with $f(\phi)$ as

$$\widehat{f}(\phi) = f_0 + 2\sum_{m=1}^{\infty} f_m e^{im\phi}\,, \tag{24}$$

and the operation $\mathcal{P}_M$, as the projection onto the fourier components with $|m|\le M$, i.e.,

$$\mathcal{P}_M\cdot f(\phi) = \sum_{m=-M}^{M} f_m e^{im\phi}\,. \tag{25}$$

then for $N\to\infty$, the matrix elements are given by

$$A^{\beta}{}_{kk'} = \mathcal{P}_L\cdot f(\widetilde{\phi}_k|\beta)\delta_{k,k'} - \frac{1}{2L}\mathrm{Im}\frac{\mathcal{P}_L\cdot\widehat{f}(\widetilde{\phi}_k|\beta) - \mathcal{P}_L\cdot\widehat{f}(\widetilde{\phi}_{k'}|\beta)}{\sin\left(\frac{\widetilde{\phi}_k-\widetilde{\phi}_{k'}}{2}\right)}\,, \tag{26}$$

where the value for $k=k'$ of the second term is obtained from the limit $\widetilde{\phi}_{k'}\to\widetilde{\phi}_k$ in the arguments. Let us now note that the function $f(\phi|\beta)$ defines, via its Fourier transform, the thermal field correlation function in the $N\to\infty$:

$$g(x-x') = \langle\psi_x^{\dagger}\psi_{x'}\rangle_\beta = \oint\frac{d\phi}{2\pi}e^{i\phi(x-x')}f(\phi|\beta)\,. \tag{27}$$

Suppose $f(\phi|\beta)$ is periodic and infinitely differentiable in $[0,2\pi]$. Then (?) the integral

$$\oint \frac{d\phi}{2\pi}|f^{(n)}(\phi|\beta)|^2$$

exists for every integer n. Then by Parseval's theorem the sum

$$\sum_{m=-\infty}^{\infty} m^{2n}|f_m|^2 = \oint \frac{d\phi}{2\pi}|f^{(n)}(\phi|\beta)|^2$$

converges, indicating that $|f_m|^2$ must decay faster than any power of m. Hence, f_m belongs to the Schwartz space $S(\mathbb{Z})$ of rapidle decreasing functions on $\mathbb{Z}$. Typically, the correlation function will decay as an exponential

$$g(x) \to e^{-|x|/\xi_\beta}\,, \tag{28}$$

for some absolute (independent of N) correlation length ξ_β. We shall say that in those cases $f(\phi|\beta)$ is *band-limited*, with bandwidth $\simeq \xi_\beta \sim O(1)$ (in N or L).

Whenever $f(\phi|\beta)$ is band-limited, with $L \gg \xi_\beta$, then

$$A^{\beta}{}_{kk'} = f(\widetilde{\phi}_k|\beta)\delta_{kk'} - \frac{1}{2L}\mathrm{Im}\frac{\widehat{f}(\widetilde{\phi}_k|\beta) - \widehat{f}(\widetilde{\phi}_{k'}|\beta)}{\sin\left(\frac{\widetilde{\phi}_k - \widetilde{\phi}_{k'}}{2}\right)}\,, \tag{29}$$

where the second term is $O(1/L)$. In such a case, the matrix $A^{\beta}{}_{kk'}$ coincides with the correlation matrix of the thermal state for the Hamiltonian H_L up to $O(1/L)$ corrections.

CONCENTRATION OF THE $A(\mathbf{n})$ IN THE GIBBS ENSEMBLE

To study the statistics of the correlation matrix $A(\mathbf{n})$ let us first define the deviation from the sample average

$$\Delta A_{kk'}(\mathbf{n}) = A_{kk'}(\mathbf{n}) - A^{\beta}_{kk'}\,. \tag{30}$$

As each element $A_{kk'}$ is a sum of the independent random variables n_q, one expects that in the limit $N\to\infty$ the matrix A will be Gaussian-distributed with a covariance matrix of the elements scaling with N^{-1}. To show this, consider the characteristic function of the scaled matrix elements $\sqrt{N}\Delta A_{kk'}$:

$$\phi(S|\beta) = \langle e^{i\sum_{k\le k'}\sqrt{N}S_{kk'}\Delta A_{kk'}}\rangle_\beta\,, \tag{31}$$

where $S_{kk'}$ is the Fourier variable conjugate to $A_{kk'}$. From the definition of $\Delta A_{kk'}$ we can write

$$\sum_{k\le k'}\sqrt{N}S_{kk'}\Delta A_{kk'} = \frac{1}{\sqrt{N}L}\sum_q s(\phi_q)\Delta n_q \tag{32}$$

where

$$s(\phi_q) \equiv \sum_{k\le k'} S_{kk'}D_L(\phi_q - \widetilde{\phi}_k)D_L(\phi_q - \widetilde{\phi}_{k'})\,. \tag{33}$$

Thus

$$\begin{aligned}
\phi(S|\beta,N) &= \prod_q \sum_{n_q} p(n_q|\beta)e^{i\frac{s(\phi_q)\Delta n_q}{\sqrt{N}L}} && (34)\\
&= \exp\left[i\frac{1}{\sqrt{N}L}\sum_q s(\phi_q)f(\phi_q|\beta) \pm \sum_q \log\left(\frac{1\pm e^{-\beta\epsilon(\phi_q)+i\frac{s(\phi_q)}{\sqrt{N}L}}}{1\pm e^{-\beta\epsilon(\phi_q)}}\right)\right] && (35)\\
&= \exp\left[\sum_{k=2}^{\infty}\frac{1}{k!(\sqrt{N}L)^k}\sum_q i^k(s(\phi_q))^k c(\phi_q,k|\beta)\right]\,, && (36)
\end{aligned}$$

where $c(\phi_q, k|\beta)$ is the $k'th$ cumulant of $p(n_q|\beta)$. Thus,

$$\phi(S|\beta, N) = \exp\left[-\frac{1}{2L^2}\oint_N \frac{d\theta}{2\pi} s(\phi)v(\phi|\beta) + \sum_{k=3}^{\infty} \frac{i^k}{k!N^{k/2-1}L^k}\oint_N \frac{d\theta}{2\pi}(s(\phi))^k c(q,k|\beta)\right] \tag{37}$$

and therefore we have the limit

$$\lim_{N\to\infty} \phi(S|\beta, N) = \exp\left[-\frac{1}{2L^2}\oint \frac{d\theta}{2\pi} s^2(\phi)v(\phi|\beta)\right] \tag{38}$$

$$= \exp\left[-\frac{1}{2}\sum_{k_1,k_2,k_3,k_4} K(k_1,k_2,k_3,k_4) S_{k_1,k_2} S_{k_3,k_4}\right] \tag{39}$$

where

$$K(k_1,k_2,k_3,k_4) \equiv N\langle \Delta A_{k_1k_2}(\mathbf{n})\Delta A_{k_3k_4}(\mathbf{n})\rangle_\beta = \frac{1}{L^2}\oint \frac{d\phi}{2\pi}\left[\prod_{i=1}^{4} D_L(\phi - \tilde{\phi}_{k_i})\right] v(\phi|\beta)\,. \tag{40}$$

Supposing that $v(\phi|\beta)$ is band-limited, these scaled moments will follow a hierarchy of scales depending on the number of coincidences in k_i. In general, we can say $K(k_1,k_2,k_3,k_4)$ will be $O(L)$ when all of the k_i are within $O(1)$ of each other; otherwise, the scale of correlations will be at least an order lower in L. In other words, fluctuations in ΔA will be significant only within a band of width $O(1)$ about the diagonal, and correlations between elements in this band will be significant only for pairs of elements a distance $O(1)$ from each other.

Error

As a measure of deviation of $\Delta A(\mathbf{n})$ from its thermal average, consider the quantity

$$Q = \mathrm{Tr}(\Delta A^2) = ||\Delta A||^2_{HS}\,. \tag{41}$$

Using (9), one finds that

$$Q = \frac{1}{N^2}\sum_q \sum_{q'} D_L^2(\phi_q - \phi_{q'})\Delta n_q \Delta n_{q'}\,. \tag{42}$$

Its expectation value in the Gibbs ensemble is

$$\langle Q\rangle_\beta = \frac{L^2}{N^2}\sum_q \langle \Delta n_q^2\rangle_\beta = \frac{L^2}{N}\oint_N \frac{d\phi}{2\pi} v(\phi|\beta)\,. \tag{43}$$

This is consistent with our previous analysis of the fluctuations of A: the error is due to what effectively are $\sim L$ independent components in an $O(1)$ band around the diagonal of A, each component with variance $\sim L/N$. Similarly, the variance in Q is given by

$$\langle \Delta Q^2\rangle = \frac{2}{N^2}\oint_N \frac{d\phi}{2\pi}\oint_N \frac{d\phi'}{2\pi} D_L^4(\phi - \phi')v(\phi|\beta)v(\phi'|\beta) + \frac{L^4}{N^3}\oint_N \frac{d\phi}{2\pi} c(\phi, 4|\beta) \tag{44}$$

The quantity in the first integral is $O(L^3)$, and hence for large N, $\langle \Delta Q^2\rangle$ is $O(L^3/N^2)$. The relative fluctuations $\Delta Q/Q$ will be $O(1/\sqrt{L})$, which is again consistent with our interpretation of Q arising from the sum of effectively $\sim L$ independent variables.

for some absolute constants C, $c > 0$.

State Concentration in the Fermionic case

The Gibbs ensemble involves an independent sampling of N excitations n_q. This independence leads to strong concentration of quantities around their ensemble average, *even when the ensemble averages differ from their expected* *thermal counterparts, as in the critical points.* In the Fermionic case, the bounded nature of the n_q allows for the use of powerful concentration inequalities. In particular, we have McDiarmid's inequality:

If $X_1 \dots X_n$ are independent random variables in ranges $R_1, R_2, \dots R_N$, and $F(X_1, X_2, \dots X_N)$ a function from $R_1, R_2, \dots R_N$ to C, with the property that if one freezes all but the i'th coordinate then F fluctuates by at most c_i, i.e.,

$$|F(x_1, \dots x_{i-1}, x_i, x_{i+1} \dots x_N) - F(x_1, \dots x_{i-1}, x_i', x_{i+1} \dots x_N)| \leq c_i$$

then , for any $\lambda > 0$

$$P(|F - \langle F \rangle| \geq \epsilon) \leq \exp(-2\frac{\epsilon^2}{\sum_i c_i^2})$$

The concentration can be readily applied to the error function $Q(\mathbf{n})$ defined earlier. Let

$$F(\mathbf{n}) = Q(\mathbf{n})\,.$$

Then

$$\begin{aligned}
|F(n_1, \dots n_{i-1}, n_i, n_{i+1} \dots n_N) - F(n_1, \dots n_{i-1}, n_i', n_{i+1} \dots n_N)| &\leq \left| \frac{L^2}{N^2}(n_i - n_i')^2 + (n_i - n_i')\frac{2}{N^2}\sum_{j\neq i} D_L^2(\phi_i - \phi_j)\Delta n_j \right| \\
&\leq \frac{L^2}{N^2} + \frac{2}{N^2}\sum_j D_L^2(\phi_i - \phi_j) \\
&= \frac{L^2}{N^2} + 2\frac{L}{N} \\
&\leq 3\frac{L}{N}
\end{aligned}$$

Hence,

$$P(|\Delta Q| \geq \epsilon) \leq e^{-\frac{2}{9}\frac{N}{L^2}\epsilon^2}\,. \tag{45}$$

While comparison with the previous expression for $\langle \Delta Q^2 \rangle$ shows that the above inequality is not very tight, it nevertheless guarantees that the the distribution of Q is subgaussian.

Next, we consider the devition of the density matrix $\rho_L(\mathbf{n})$ from its average, and we define the function

$$F(\mathbf{n}) = ||\rho_L(\mathbf{n}) - \rho(\beta, N)_L||_1 \tag{46}$$

We have the inequalities:

$$\begin{aligned}
|F(\mathbf{n}) - F(\mathbf{n}')| &\leq ||\rho_L(\mathbf{n}) - \rho_L(\mathbf{n}')||_1 &(47)\\
&\leq \sqrt{Tr\left(\rho_L(\mathbf{n}) - \rho_L(\mathbf{n}')\right)^2}\,. &(48)
\end{aligned}$$

Now consider all excitations n_j equal, except n_i, in which case we can take without loss of generality $n_=1$ and $n_i' = 0$. Then

$$\rho_L(\mathbf{n}) - \rho_L(\mathbf{n}') = \mathrm{Tr}_{L-N}\left(|n_1\rangle\langle n_1| \otimes \dots \otimes |n_{i-1}\rangle\langle n_{i-1}| \otimes (|1_i\rangle\langle 1_i| - |0_i\rangle\langle 0_i|) \otimes |n_{i+1}\rangle\langle n_{i+1}| \otimes \dots \otimes |n_N\rangle\langle n_N|\right) \tag{49}$$

But note that

$$\begin{aligned}
|1_i\rangle\langle 1_i| - |0_i\rangle\langle 0_i| &= -2\frac{\partial}{\partial\mu}e^{-\mu(N_i - 1/2)}\Big|_{\mu=0} &(50)\\
&= -4\frac{\partial}{\partial\mu}\frac{e^{-\mu N_i}}{1+e^{-\mu}}\Big|_{\mu=0} &(51)
\end{aligned}$$

where $N_i = |1_i\rangle\langle 1_i|$. Hence,

$$\rho_L(\mathbf{n}) - \rho_L(\mathbf{n}') = -4\frac{\partial}{\partial\mu}\rho_L(\mathbf{n}, \mu)\Big|_{\mu=0} \tag{52}$$

where $\rho_L(\mathbf{n},\mu)$ is the reduced density matrix of a Gaussian state in which the $i'th$ excitation is in the mixed state $\propto e^{-\mu N_i}$ and all others in their respective pure states. Using this we have the bound:

$$|F(\mathbf{n}) - F(\mathbf{n}')| \leq 4 \sqrt{\frac{\partial^2}{\partial\mu\partial\mu'} \mathrm{Tr}\left(\rho_L(\mathbf{n},\mu)\rho_L(\mathbf{n},\mu')\right)}\Bigg|_{\mu=\mu'=0} \tag{53}$$

For tow Gaussian states ρ and ρ' with Majorana covariance matrices M and M',

$$\mathrm{Tr}\left(\rho\rho'\right) = \frac{1}{2^n}\mathrm{Pf}(M)\mathrm{Pf}(M' - M^{-1})\,. \tag{54}$$

When both M and M' have the structure

$$M = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} \tag{55}$$

where S is a symmetric matrix, then

$$\mathrm{Pf}(M)\mathrm{Pf}(M' - M^{-1}) = \det(SS' + 1)\,, \tag{56}$$

and in terms of the matrix A previously defined, $S = (1 - 2A)$ and hence

$$\mathrm{Tr}\left(\rho\rho'\right) = \det(1 + 2AA' - A - A') \tag{57}$$

For the matrix $\rho_L(\mathbf{n},\mu)$, let us define

$$A(\mathbf{n},\mu) = \widetilde{A} - \frac{L}{2N}\coth(\mu/2)\,|v\rangle\langle v| \tag{58}$$

where

$$\widetilde{A}_{k,k'} = \frac{1}{NL}\sum_{q_j\neq i} D_L(\phi_q - \tilde{\phi}_k)D_L(\phi_q - \tilde{\phi}_{k'}) + \frac{1}{2}\frac{1}{NL}D_L(\phi_i - \tilde{\phi}_k)D_L(\phi_i - \tilde{\phi}_{k'}) \tag{59}$$

and $|v\rangle$ is a unit vector with components

$$v_k = \frac{1}{L}D_L(\phi_i - \tilde{\phi}_k)\,. \tag{60}$$

After some algebra one can show that

$$\mathrm{Tr}\left(\rho_L(\mathbf{n},\mu)\rho_L(\mathbf{n},\mu')\right)\big|_{\mu=\mu'=0} = \left(\frac{L}{4N}\right)^2 \det(B)\,\langle v|\,B^{-1}\,|v\rangle \tag{61}$$

$$\times\left[1 + \frac{\langle v|\,(2\widetilde{A}-1)B^{-1}\,|v\rangle^2}{\langle v|\,B^{-1}\,|v\rangle} - \langle v|\,(2\widetilde{A}-1)^2B^{-1}\,|v\rangle\right] \tag{62}$$

where $B = 1 + 2\widetilde{A}^2 - 2\widetilde{A}$. This quantity can be bound by

$$\mathrm{Tr}\left(\rho_L(\mathbf{n},\mu)\rho_L(\mathbf{n},\mu')\right)\big|_{\mu=\mu'=0} \leq 2\left(\frac{L}{4N}\right)^2 \tag{63}$$

in which case

$$|F(\mathbf{n}) - F(\mathbf{n}')| \leq \sqrt{2}\frac{L}{N}\,. \tag{64}$$

Using McDiarmid's inequality we then have

$$P(|F - \langle F\rangle| > \epsilon) \leq \exp(-\frac{N}{L^2}\epsilon^2) \tag{65}$$

Formulas

We will need a general expression for the following integral

$$I(k_1,k_2,\ldots,k_r)[f] = \oint \frac{d\phi}{2\pi}\left[\prod_{i=1}^{r} D_L(\phi-\tilde{\phi}_k)\right] f(\phi) \tag{66}$$

for smooth real L_2 functions $f(\phi)$, 2π-periodic in ϕ, and where we assume that p is even. First, let

$$f(\phi) = \sum_{m=-\infty}^{\infty} f_m e^{im\phi}\,, \tag{67}$$

and define

$$I_m(k_1,k_2,\ldots,k_l) = \oint \frac{d\phi}{2\pi}\left[\prod_{i=1}^{l} D_L(\phi-\tilde{\phi}_k)\right] e^{im\phi}\,. \tag{68}$$

By the reality of the Dirichlet kernel we need only consider this integral for $m \geq 0$. Let $\eta_1 = e^{-i\tilde{\phi}_k}$ and $z = e^{i\phi}$. Then for $\tilde{\phi}_k = \frac{2\pi k}{L}$ (as shall be assumed henceforth), the dirichlet kernel may be written as

$$D_L(\phi-\tilde{\phi}_k) = \frac{\eta_i^{1/2}}{z^{(L-1)/2}}\frac{1-z^L}{1-z\eta_i}\,. \tag{69}$$

Hence, the integral $I_m(k_1,k_2,\ldots,k_l)$ can be written as the contour integral

$$I_m(k_1,k_2,\ldots,k_l) = \frac{1}{2\pi i}\oint_{|z|<1} dz \frac{(1-z^L)^p}{z^{(L-1)p/2-m+1}}\prod_{i=1}^{p}\frac{\eta_i^{1/2}}{1-z\eta_i} \tag{70}$$

$$= \sum_{q=0}^{p}(-1)^q\binom{p}{q}\frac{1}{2\pi i}\oint_{|z|<1} dz \frac{1}{z^{(L-1)p/2-Lq-m+1}}\prod_{i=1}^{p}\frac{\eta_i^{1/2}}{1-z\eta_i}\,. \tag{71}$$

The integral in the sum vanishes whenever $m > L(\frac{p}{2}-q)-\frac{p}{2}$. For the non- vanishing values, we use the iversion $z \to 1/z$, to obtain

$$I_m(k_1,k_2,\ldots,k_l) = \sum_{q:m\leq L(\frac{p}{2}-q)-\frac{p}{2}}(-1)^q\binom{p}{q}\frac{1}{2\pi i}\oint_{|z|>1} dz \frac{z^{(L+1)p/2-Lq-m-1}}{\prod_{i=1}^{p}\eta_i^{-1/2}(z-\eta_i)}\,, \tag{72}$$

and finally,

$$I_m(k_1,k_2,\ldots,k_l) = \sum_{q:m\leq L(\frac{p}{2}-q)-\frac{p}{2}}(-1)^q\binom{p}{q}\sum_{i=1}^{p}\mathrm{Res}_{z=\eta_i}\frac{z^{(L+1)p/2-Lq-m-1}}{\prod_{j=1}^{p}\eta_j^{-1/2}(z-\eta_j)}\,. \tag{73}$$

When all the k_i are different, we have

$$I_m(k_1,k_2,\ldots,k_l) = \left[\sum_{q:m\leq L(\frac{p}{2}-q)-\frac{k}{2}}(-1)^q\binom{p}{q}\right]\sum_{i=1}^{p}\frac{\eta_i^{(p-1)/2-m}}{\prod_{j\neq i}^{p}\eta_j^{-1/2}(\eta_i-\eta_j)} \tag{74}$$

$$= \frac{1}{(-2i)^{p-1}}\left[\sum_{q:m\leq L(\frac{p}{2}-q)-\frac{p}{2}}(-1)^q\binom{p}{q}\right]\sum_{i=1}^{p}\frac{e^{im\widetilde{\phi}_k}}{\prod_{j\neq i}\sin\left(\frac{\widetilde{\phi}_{k_i}-\widetilde{\phi}_{k_j}}{2}\right)}\,. \tag{75}$$

To proceed, recall the previously defined operator $\mathcal{P}_M$ and complex analytic signal $\widehat{f}(\phi)$ for a given $f(\phi)$ Then, with all the k_i, different

$$I(k_1,k_2,\ldots,k_r)[f] = -\frac{(-1)^{\frac{p}{2}}}{2^{(p-1)}}\sum_{i=1}^{p}\frac{1}{\prod_{j\neq i}\sin\left(\frac{\widetilde{\phi}_{k_i}-\widetilde{\phi}_{k_j}}{2}\right)}\sum_{q=0}^{\frac{p}{2}-1}(-1)^q\binom{p}{q}\,\mathrm{Im}\,\mathcal{P}_{L(\frac{p}{2}-q)-\frac{p}{2}}\cdot\widehat{f}(\widetilde{\phi}_k) \tag{76}$$

The expression simplifies greatly when $f(\phi)$ is band-limited with $m_{max} < L - p/2$. Then the integral becomes

$$I(k_1, k_2, \ldots, k_r)[f] = -\frac{1}{2^p}\binom{p}{\frac{p}{2}}\sum_{i=1}^{p}\frac{\mathrm{Im}\widehat{f}(\widetilde{\phi}_{k_i})}{\prod_{j\neq i}\sin\left(\frac{\widetilde{\phi}_{k_i}-\widetilde{\phi}_{k_j}}{2}\right)}\,. \tag{77}$$

We now consider the case where there are coincidences. Suppose that in the integral a set of d distinct k_i appear, each a number n_i of times. Then from the residue formula we have

$$I_m(k_1^{\times n_1}, k_2^{\times n_2}, \ldots k_d^{\times n_d}) = \prod_{i=1}^{d}\eta_i^{n_i/2}\sum_{i=1}^{d}\frac{1}{(n_i-1)!}\left(\frac{\partial}{\partial\eta_i}\right)^{n_i-1}\frac{\sum_q^*(-1)^q\binom{p}{q}\eta_i^{L(\frac{p}{2}-q)+\frac{p}{2}-m-1}}{\prod_{j\neq i}(\eta_i-\eta_j)^{n_j}} \tag{78}$$

where the sum over q is restricted as before ($m \leq L(\frac{p}{2}-q)-\frac{p}{2}$). From here it is easily seen that the dominant power of L in the integral is the largest multiplicity of the n_i. If the $k_i's$ are ordered in non-increasing order of their multiplicities, and supposing that $k_1 \ldots k_r$ all have the maximum multiplicity n_1, then then dominant term in the integral, of order L^{n_1-1}, will be

$$I_m(k_1^{\times n_1}, k_2^{\times n_1}, \ldots k_r^{\times n_1}, k_{r+1}^{\times n_2}, \ldots k_d^{\times n_d}) = i^{-n_1}L^{n_1-1}C(p,n_1)\sum_{i=1}^{r}\frac{e^{im\phi_{k_i}}}{\prod_{j\neq i}\sin\left(\frac{\widetilde{\phi}_{k_i}-\widetilde{\phi}_{k_j}}{2}\right)^{n_j}} + O(L^{n_1-2}) \tag{79}$$

where

$$C(p,n_1) = \frac{(-1)^{p/2}}{2^p(n_1-1)!}\sum_q^*(-1)^q\binom{p}{q}\left(\frac{p}{2}-q\right)^{n_1-1} \tag{80}$$

Cases in which there are coincidences between the k_i must be treated separately, and we will only pursue the cases $p = 2$ and $p = 4$ and assuming $f(\phi)$ is band-limited with $m_{max} < L - p/2$. Assuming a band-limited $f(\phi)$ with $m_{max} < L$, the dominant term in the integral becomes

$$I(k_1^{\times n_1}, k_2^{\times n_1}, \ldots k_r^{\times n_1}, k_{r+1}^{\times n_2}, \ldots k_d^{\times n_d})[f] = L^{n_1-1}C(p,n_1)\sum_{i=1}^{r}\frac{\mathrm{Re}\left(i^{-n_1}\hat{f}(\phi)\right)}{\prod_{j\neq i}\sin\left(\frac{\widetilde{\phi}_{k_i}-\widetilde{\phi}_{k_j}}{2}\right)^{n_j}} + O(L^{n_1-2}) \tag{81}$$

In the case $p = 2$, the only possible value of q is $q = 0$. For $k_1 = k_2 = k$, and $m > 0$,

$$I_m(k,k) = \mathrm{Res}_{z=\eta_1}\frac{z^{L-m}}{\eta^{-1}(z-\eta)^2} = (L-m)\eta^{L-m} = (L-m)e^{im\phi_k}\,. \tag{82}$$

Thus we have

$$I(k,k)[f] = Lf(\phi_k) - \mathrm{Im}\frac{\partial\widehat{f}}{\partial_\phi}\,.(\widetilde{\phi}_k) \tag{83}$$

They can be classified as follows (assume in the classification that all k_i are distinct):

a) $K(k_1, k_2, k_3, k_4) \sim O(1/L^2)$ unless

b) $K(k_1, k_1, k_2, k_3) \sim O(1/L)$, with $|k_2 - k_1| \bmod L \sim O(L)$ and $|k_3 - k_1| \bmod L \sim O(L)$.

c) $K(k_1, k_1, k_2, k_3) \sim O(1)$, with $|k_2 - k_1| \bmod L \sim O(1)$ and $|k_3 - k_1| \bmod L \sim O(L)$

d) $K(k_1, k_1, k_2, k_3) \sim O(L)$, with $|k_2 - k_1| \bmod L \sim O(1)$ and $|k_3 - k_1| \bmod L \sim O(1)$

e) $K(k_1, k_1, k_2, k_2) \sim O(1/L)$, with $|k_2 - k_1| \bmod L \sim O(L)$

f) $K(k_1, k_1, k_2, k_2) \sim O(L)$, with $|k_2 - k_1| \bmod L \sim O(1)$

g) $K(k_1, k_1, k_1, k_2) \sim O(1/L)$, with $|k_2 - k_1| \bmod L \sim O(L)$

h) $K(k_1, k_1, k_1, k_2) \sim O(L)$, with $|k_2 - k_1| \bmod L \sim O(1)$

h) $K(k_1, k_1, k_1, k_1) \sim O(L)$

In short, $K(k_1, k_2, k_3, k_4)$ will be $O(L/N)$ when at least two of the k_i ar the same and the other two are within $O(1)$ of the coincident k_i; otherwise, the scale of correlations will be an order lower in L.