

# BCS-like modewise entanglement of fermion Gaussian states

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## Abstract

We show that with respect to any bipartite division of modes, pure fermion Gaussian states display the same type of structure in its entanglement of modes as that of the BCS wave function, namely, that of a tensor product of entangled two-mode squeezed fermion states. We show that this structure applies to a wider class of “isotropic” mixed fermion states, for which we derive necessary and sufficient conditions for mode entanglement.

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The nature of many-body entanglement in various solid-state models such as spin chains, superconductivity, and harmonic chains, has been a recent subject of intense investigation [1–12]. The motivation for this effort is two-fold: on the one hand, such systems exhibit a rich entanglement structure which could potentially be harnessed for quantum information processing [7–10]; on the other hand, this effort promises to deepen our understanding of certain universal features in many-body physics—such as quantum phase transitions—and their relation to entanglement [2–5,12].

In this Letter we investigate the properties of bipartite fermion mode entanglement [1] for a wide class of models describing correlated fermion systems, and show that this entanglement displays a simple, universal structure. As it is well known, under certain field transformations or after suitable approximations, a large class of interacting theories can be mapped to effective theories described by quadratic Hamiltonians of the generic form

$$H = \sum C_{ij} b_i^\dagger b_j + \sum (A_{ij} b_i^\dagger b_j^\dagger + \text{h.c.}), \quad (1)$$

which are then diagonalized through Bogoliubov–Valatin (i.e., canonical) transformations [24,25] to appropriate quasi-particle mode bases. Important classes of such theories include the Hubbard model and the BCS theory of superconductivity in the “mean field” or Hartree–Fock approximation, as well as certain

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exactly solvable spin–chain models after a Jordan–Wigner transformation [11,12]. As we shall shortly clarify, an important feature of the above class of Hamiltonians is that not only the ground states (the quasi-particle vacuum), but also all eigenstates describing a definite set of quasi-particle excitations, belong to the general class of *fermion Gaussian states*. The bipartite entanglement structure of the eigenstates of (1) is then given by the following result, which is applicable to all pure Gaussian fermion states, and which constitutes the main result of this Letter:

*Given a collection of  $N$ -fermion systems or “modes”, partitioned into two arbitrary sets,  $A = \{A_1, \dots, A_m\}$  and  $B = \{B_1, \dots, B_n\}$ , of sizes  $m$  and  $n = N - m$ , respectively, any pure fermion Gaussian state  $|\psi\rangle_{AB}$  of the modes may always be written as*

$$|\psi\rangle_{AB} = |\tilde{\psi}_1\rangle_{\tilde{A}_1\tilde{B}_1} |\tilde{\psi}_2\rangle_{\tilde{A}_2\tilde{B}_2} \cdots |\tilde{\psi}_s\rangle_{\tilde{A}_s\tilde{B}_s} |0\rangle_{\tilde{A}_F} |0\rangle_{\tilde{B}_F}. \quad (2)$$

Here,  $s \leq \min(m, n)$ , and  $\tilde{A} = \{\tilde{A}_1, \dots, \tilde{A}_m\}$ ,  $\tilde{B} = \{\tilde{B}_1, \dots, \tilde{B}_n\}$  are new sets of modes obtained from  $A$  and  $B$ , respectively, through local fermion canonical transformations. The states  $|\tilde{\psi}_k\rangle$  are two-mode fermion squeezed states of the form

$$|\tilde{\psi}_k\rangle_{\tilde{A}_k\tilde{B}_k} = \cos\theta_k |00\rangle_{\tilde{A}_k\tilde{B}_k} - \sin\theta_k |11\rangle_{\tilde{A}_k\tilde{B}_k}, \quad (3)$$

entangling the modes  $\tilde{A}_k$  and  $\tilde{B}_k$  for  $k \leq s$ , and  $|0\rangle_{\tilde{A}_F}$  and  $|0\rangle_{\tilde{B}_F}$  are products of vacuum states for the remaining modes in  $\tilde{A}$  and  $\tilde{B}$ , respectively.

Together with a similar decomposition obtained for boson Gaussian states in [13,14], the above result shows that the bipartite entanglement structure of pure multi-mode Gaussian states is of  $1 \times 1$ -entangled mode pairs independently of the statistics, the partition of modes, or the nature of the basis in which the partition is performed. The present result shows that in the fermion case, the mode-pairing is similar to that of the BCS wave function  $|\text{BCS}\rangle = \prod_k (\cos\theta_k + \sin\theta_k b_{k\uparrow}^\dagger b_{-k\downarrow}^\dagger) |\text{vac}\rangle$ , except that the vacuum state and the pairwise excitations in (2) are all relative to a quasi-particle spectrum determined by the multi-mode state  $|\psi\rangle$  and the choice of partition.

As in the boson case investigated in [13], it can be shown that the above theorem and the decomposition (2), follows for an arbitrary pure Gaussian Fermi state from the properties of the Schmidt decomposition, and from the comparison of the reduced density

matrices in their normal form, at sectors  $A$  and  $B$ . In this Letter we shall, however, follow a more general framework, which enables us to generalize this theorem to a certain class of “isotropic” mixed Gaussian states, characterized by a symmetry property of the covariance matrices. The remainder of the Letter is thus structured as follows: we first review fermion Gaussian states, canonical transformations, and define the fermion covariance matrix; next, we define isotropic Gaussian states and prove a general decomposition for isotropic states in the language of covariance matrices; finally we also derive a necessary and sufficient conditions for the entanglement of isotropic mixed states and close with some remarks on the applicability of our results.

Let a system of  $N$  fermion modes be described by a set of creation and annihilation operators  $b_i^\dagger, b_i$  satisfying anticommutation relations  $\{b_i^\dagger, b_j^\dagger\} = \{b_i, b_j\} = 0$ ,  $\{b_i, b_j^\dagger\} = \delta_{ij}$ . We define a fermion Gaussian state for such a system as any state  $\rho$  that, with a certain choice of mode basis  $\tilde{b}_i = u_i^j b_j + v_i^j b_j^\dagger$  preserving the fermion anticommutation algebra, acquires the form [15]

$$\rho = \bigotimes_{k=1}^N \tilde{\rho}_k, \quad \tilde{\rho}_k = \frac{1}{2} (1 - \lambda_i [\tilde{b}_i^\dagger, \tilde{b}_i]), \quad (4)$$

with  $|\lambda_i| \leq 1$  and  $|\lambda_i| = 1$  for pure states (note that  $\rho$  can also be written in the form  $\rho = Z^{-1} \exp(-\beta \tilde{b}_i^\dagger \tilde{b}_i)$ ). Fermion Gaussian states share with their boson counterparts the property that correlation functions for the creation/annihilation operators are completely determined by the two-point functions according to Wick’s theorem [16]. Moreover, since this property is extensible to correlation function pertaining to a reduced subset of the modes, it follows that any partial (reduced) density matrix obtained from  $\rho$  remains Gaussian [17].

The Gaussian nature of a state is preserved under any unitary transformation that induces a canonical linear transformation of the fermion variables, in other words, that preserves the anticommutation relations. To determine the group of canonical transformations, it becomes convenient to replace the  $N$  creation and annihilation operators by  $2N$  Hermitian “fermion quadrature” combinations

$$\gamma_{2i-1} = b_i + b_i^\dagger, \quad \gamma_{2i} = i(b_i - b_i^\dagger). \quad (5)$$

The anticommutation relations can then be seen to be a consequence of an  $\mathbb{R}^{2N}$  Clifford (or Dirac) algebra [21]

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta} \quad (6)$$

satisfied by the  $\gamma_\alpha$ . Inspection of (6) then shows that any linear transformation of the form  $\tilde{\gamma}_\alpha = O_{\alpha\beta}\gamma_\beta$ , where  $O \in O(2N)$  (the full orthogonal group in  $2N$  dimensions), preserves the Clifford algebra (6) and hence the anticommutation algebra of the fermion fields.

That the group  $O(2N)$  of fermion canonical transformations can be implemented unitarily on the Fock space (i.e.,  $\gamma'_\alpha = U^\dagger \gamma_\alpha U$ ), follows from the following: for any real vector  $v \in \mathbb{R}^{2N}$ , define  $\not{v} = \not{v}^\dagger = v \cdot \gamma$ , and  $|v|^2 = v \cdot v (= \not{v}^2)$  with the usual inner product. We then verify that any real vector  $a$  with  $|a| = 1$  defines a unitary operator  $U = \not{a}$ , the action of which is an inversion ( $\det(O) = -1$ ) in the orthogonal subspace to  $a$

$$\not{a}^\dagger \gamma_\mu \not{a} = [2a_\mu a_\nu - \delta_{\mu\nu}] \gamma_\nu, \quad (7)$$

in  $\mathbb{R}^{2n}$ , this inversion is equivalent to an  $SO(2N)$  rotation times a reflection along  $a$ . Since any orthogonal transformation can be generated from a product of a certain number of such reflections along different axes [23], it follows that any orthogonal transformation of the  $\gamma$ 's can be induced by a suitably chosen unitary operator of the form  $U = \not{a}_1 \not{a}_2 \not{a}_3 \dots \not{a}_p$  with  $|a_k| = 1$ .

The group of operations thus defined is the  $Pin(2N)$  group, the double cover of  $O(2N)$  [21–23]. In turn, the subgroup of  $Pin(2N)$  generated by an even number of reflections is the covering group of  $SO(2N)$ ,  $Spin(2N)$ , associated with the linear fermion canonical transformations (also known as the Bogoliubov–Valatin [24,25] or squeeze transformations [26]). More generally,  $Pin(2N)$  includes improper orthogonal transformations ( $\det(O) = -1$ ) generated by an odd number of reflections, and which may in fact be forbidden due to fermion number superselection rules in a fundamental fermion theory. This fact, however, does not affect the generality of the results presented here. Since an improper transformation is a  $SO(2N)$  rotation times some canonical reflection, an improper transformation is equivalent to a proper transformation but with the role of the creation and annihilation operators exchanged for at most one mode.

An important consequence of the  $O(2N)$  equivalence of fermion Gaussian states is that all multiparticle eigenstates obtained from a given vacuum are Gaussian. Indeed, if  $|\text{vac}\rangle$  is the ground state of a certain Hamiltonian, annihilated by the  $b_i$  operators in a given quasi-particle basis, then for each mode it follows that  $b_i^\dagger |\text{vac}\rangle = \gamma_{2i} |\text{vac}\rangle$ ; it therefore follows that multiparticle states are obtained from some  $U \in Pin(2N)$  acting on  $|\text{vac}\rangle$ . Since the vacuum state is Gaussian, the resulting state will also be Gaussian.

We next define for any Gaussian quantum state the fermion covariance matrix (FCM) as

$$M_{\alpha\beta} = \frac{1}{2i} \langle [\gamma_\alpha, \gamma_\beta] \rangle_\rho. \quad (8)$$

Being an antisymmetric  $2N \times 2N$  matrix, the FCM can always be brought to the block diagonal form

$$W = O M O^T = \bigoplus_{i=1}^N \lambda_i J_2, \quad J_2 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (9)$$

by means of a  $O(2N)$  transformation, and with the  $\lambda_i \geq 0$ . In terms of the creation/annihilation operators obtained from the transformed gamma's  $\tilde{\gamma} = O\gamma$ , the Gaussian state  $\rho$  then takes the form (4) with  $W$  as its FCM. By setting all the  $\lambda_i$  positive or zero, we obtain a unique characterization of a fermion Gaussian state (up to relabelling of the modes) analogous to the so-called Williamson normal form of the boson case [13, 18]. We shall therefore term the canonical form  $W$  as the fermion Williamson form, and the  $\{\lambda_i\}$  ( $\lambda_i \geq 0$ ) in  $W$  as the Williamson eigenvalues.

Now, the Williamson spectrum can be obtained from the doubly degenerate spectrum of the matrix  $-M^2$ . This provides a simple characterization of pure Gaussian states in terms of their FCM; since such states are unitarily equivalent to pure states of the form (4), with  $\lambda_i = 1$ ,  $\forall i$ , the pure FCMs satisfy the condition  $-M^2 = 1$ . Pure Gaussian states can then be viewed as the extremal states in a more general class of *isotropic* Gaussian states, for which the square of their FCM's remains invariant under any  $O(2N)$  transformation:

$$M^2 = -\lambda_0^2 1, \quad \lambda_0 \leq 1. \quad (10)$$

It is then possible to show that if  $M$  is the FCM of an isotropic Gaussian state, and if the modes are separated into two sectors  $A$  and  $B$  ( $\gamma = \gamma_A \oplus \gamma_B$ ),

then there exist *local* orthogonal transformations  $\tilde{\gamma}_A = O_A \gamma_A$  and  $\tilde{\gamma}_B = O_B \gamma_B$  such that the upon reordering the modes the FCM takes the form

$$\tilde{M} = \tilde{M}_{\tilde{A}_1 \tilde{B}_1} \oplus \tilde{M}_{\tilde{A}_2 \tilde{B}_2} \oplus \cdots \oplus \tilde{M}_{\tilde{A}_s \tilde{B}_s} \oplus \tilde{M}_{\tilde{A}_F} \oplus \tilde{M}_{\tilde{B}_F}, \quad (11)$$

where  $\tilde{M}_{\tilde{A}_i \tilde{B}_i}$  is isotropic for the entangled pair sector  $\gamma_{\tilde{A}_i \tilde{B}_i} = \gamma_{\tilde{A}_i} \oplus \gamma_{\tilde{B}_i}$  and of the form

$$\tilde{M}_{\tilde{A}_i \tilde{B}_i} = \begin{pmatrix} 0 & -\lambda_i & 0 & \kappa_i \\ \lambda_i & 0 & \kappa_i & 0 \\ 0 & -\kappa_i & 0 & -\lambda_i \\ -\kappa_i & 0 & \lambda_i & 0 \end{pmatrix}, \quad \kappa_i^2 + \lambda_i^2 = \lambda_0^2, \quad (12)$$

and  $\tilde{M}_{\tilde{A}_F}$  and  $\tilde{M}_{\tilde{B}_F}$  are isotropic FCM's with Williamson eigenvalue  $\lambda_0$  for the remaining modes in the  $A$  and  $B$  sectors, respectively. From the correspondence between FCMs and Gaussian mixed states, it then follows that in the new mode basis, the isotropic fermion Gaussian state  $\rho_{AB}^{(0)}$  takes the form

$$\rho^{(0)} = \tilde{\rho}_{\tilde{A}_1 \tilde{B}_1} \otimes \tilde{\rho}_{\tilde{A}_2 \tilde{B}_2} \otimes \cdots \otimes \tilde{\rho}_{\tilde{A}_s \tilde{B}_s} \otimes \tilde{\rho}_{\tilde{A}_F}^{(0)} \otimes \tilde{\rho}_{\tilde{B}_F}^{(0)}, \quad (13)$$

where  $\tilde{\rho}_{\tilde{A}_i \tilde{B}_i}$  are two-mode fermion squeezed states with FCM of the form (12), and  $\tilde{\rho}_{\tilde{A}_F}^{(0)}$  and  $\tilde{\rho}_{\tilde{B}_F}^{(0)}$  are local isotropic states for the remaining modes in  $\tilde{A}$  and  $\tilde{B}$ . We note that for an FCM of the form (12), the corresponding two-modes squeezed state may be written as

$$\tilde{\rho}_{\tilde{A}_i \tilde{B}_i} = \frac{1}{4} T_i (1 - \lambda_0 [\tilde{b}_{\tilde{A}_i}^\dagger, \tilde{b}_{\tilde{A}_i}]) (1 - \lambda_0 [\tilde{b}_{\tilde{B}_i}^\dagger, \tilde{b}_{\tilde{B}_i}]) T_i^\dagger \quad (14)$$

with  $T_i$  an entangling unitary operator

$$T_i = \exp[-(\tilde{b}_{\tilde{A}_i}^\dagger \tilde{b}_{\tilde{B}_i}^\dagger + \tilde{b}_{\tilde{A}_i} \tilde{b}_{\tilde{B}_i}) \theta_i], \quad \tan 2\theta_i = \frac{\kappa_i}{\lambda_i}. \quad (15)$$

Thus, in the pure case  $\lambda_0 = 1$ , the two-mode entangled states become projection operators onto two-mode squeezed states  $T_i |00\rangle_{\tilde{A}_i \tilde{B}_i}$  which expand out to the form (2), whereas the unentangled states become projection operators onto the vacuum states of the remaining modes.

To prove the decomposition (11), first perform local orthogonal transformations  $\tilde{\gamma}_A \oplus \tilde{\gamma}_B = (O_A \oplus O_B) \gamma_A \oplus \gamma_B$  bringing each of the local FCMs  $M_A =$

$\frac{1}{2i} \langle [\gamma_A, \gamma_A^\dagger] \rangle$ ,  $M_B = \frac{1}{2i} \langle [\gamma_B, \gamma_B^\dagger] \rangle$  into the canonical Williamson form (9), with all  $\lambda_i \geq 0$ . The total FCM thus obtained may be written as

$$\tilde{M} = \frac{1}{2i} \langle [\tilde{\gamma}, \tilde{\gamma}^\dagger] \rangle = \begin{pmatrix} W_A & \tilde{K} \\ -\tilde{K}^\dagger & W_B \end{pmatrix}, \quad (16)$$

with  $W_A = \bigoplus_{i=1}^m \lambda_{\tilde{A}_i} J_2$  and  $W_B = \bigoplus_{i=1}^n \lambda_{\tilde{B}_i} J_2$ . Substituting into the definition of the isotropic matrix  $M^2 = -\lambda_0^2$ , the following equation is obtained from (16)

$$W_A \tilde{K} + \tilde{K}^\dagger W_B = 0. \quad (17)$$

Consider then a  $2 \times 2$  sub-block  $\tilde{K}_{ij} \equiv -i \langle \gamma_{\tilde{A}_i} \gamma_{\tilde{B}_j}^\dagger \rangle$  of  $\tilde{K}$  connecting the modes with eigenvalues  $\lambda_{\tilde{A}_i}$  and  $\lambda_{\tilde{B}_j}$ . From Eq. (17) and using  $J_2^2 = -1_2$ , we find that

$$\lambda_{\tilde{A}_i} \tilde{K}_{ij} = \lambda_{\tilde{B}_j} J_2 \tilde{K}_{ij} J_2. \quad (18)$$

For  $\lambda_{\tilde{A}_i} \neq \lambda_{\tilde{B}_j}$ , this equation has no solution other than  $\tilde{K}_{ij} = 0$ , meaning that modes in  $A$  and modes  $B$  with different Williamson eigenvalues decouple.

Thus, let  $\tilde{\gamma}_{\tilde{A}_\lambda}$  and  $\tilde{\gamma}_{\tilde{B}_\lambda}$  stand for the modes in  $A$  and  $B$  with the same local Williamson eigenvalue  $\lambda$ , and group the modes according to their eigenvalues so that  $\tilde{M}$  takes the Jordan form  $\tilde{M} = \bigoplus_\lambda \tilde{M}_\lambda$  where each block  $\tilde{M}_\lambda$  is the FCM for the modes in  $A$  and  $B$  with a common local Williamson eigenvalue  $\lambda$ . Concentrating on a given  $\lambda$ , let  $g_A$  and  $g_B$  be the degeneracies of  $\lambda$  in the Williamson spectra of  $W_A$  and  $W_B$ , respectively, so that  $\tilde{M}_\lambda$  may be written as

$$\tilde{M}_\lambda = \begin{pmatrix} \lambda J_{2g_A} & \tilde{K}_\lambda \\ -\tilde{K}_\lambda^\dagger & \lambda J_{2g_B} \end{pmatrix}, \quad J_{2g} \equiv \bigoplus_{i=1}^g J_2. \quad (19)$$

Now define  $\kappa_\lambda = \sqrt{\lambda_0^2 - \lambda^2}$  and note that  $\tilde{M}_\lambda$  is also an isotropic FCM with Williamson eigenvalue  $\lambda_0$ . Thus,  $\tilde{M}^2 = -\lambda_0^2$  yields

$$\tilde{K}_\lambda \tilde{K}_\lambda^\dagger = \kappa_\lambda^2 1_{2g_A}, \quad \tilde{K}_\lambda^\dagger \tilde{K}_\lambda = \kappa_\lambda^2 1_{2g_B}, \quad (20a)$$

$$J_{2g_A} \tilde{K}_\lambda J_{2g_B} = \tilde{K}_\lambda, \quad (20b)$$

where we have used the fact that  $J_{2g}^2 = -1_{2g}$ . Taking the trace of both equations in (20a) and using  $\text{Tr}_A[K_\lambda K_\lambda^\dagger] = \text{Tr}_B[K_\lambda^\dagger K_\lambda]$ , we find that  $\kappa_\lambda^2 (g_A - g_B) = 0$ , proving that for  $\lambda \neq \lambda_0$ , the degeneracies of  $\lambda$  in the local FCM's are the same. Thus, let

$g_A = g_B = g$  and express  $\tilde{K}$  in terms of an unspecified matrix  $Q_\lambda$  according to

$$\tilde{K}_\lambda \equiv \kappa Q_\lambda \beta, \quad \beta \equiv \bigoplus_{i=1}^N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (21)$$

Noting that  $\beta^2 = 1_{2g}$  and  $\{\beta, J_{2g}\} = 0$ , we find from Eqs. (20) that  $Q_\lambda$  satisfies

$$Q_\lambda Q_\lambda^T = Q_\lambda^T Q_\lambda = 1_{2g}, \quad Q_\lambda J_{2g} Q_\lambda^{-1} = J_{2g}, \quad (22)$$

indicating that  $Q_\lambda$  is an orthogonal symplectic transformation. Hence,  $\tilde{K}$  can be brought to the block diagonal form  $\tilde{K}'_\lambda = \kappa_\lambda \beta$  by means of a one-sided orthogonal transformation  $O_A = Q_\lambda^T$ , leaving the upper diagonal block of (19) intact. Grouping the resulting modes in each degenerate space as  $\tilde{\gamma}_\lambda = \bigoplus_{i=1}^g (\tilde{\gamma}_{\tilde{A}_i} \oplus \tilde{\gamma}_{\tilde{B}_i})$ , we achieve the pair-wise decomposition for  $\lambda \neq \lambda_0$ , with each pair described by a covariance matrix of the form (12). For  $\lambda = \lambda_0$ , the degeneracies on each side are not restricted, and Eqs. (20) imply that  $\tilde{K}_{\lambda_0} \tilde{K}_{\lambda_0}^T = 0 \Rightarrow \tilde{K}_{\lambda_0} = 0$ . Therefore, local modes with Williamson eigenvalue  $\lambda_0$  decouple, as may be expected for the pure case  $\lambda_0 = 1$ .

We briefly comment on the quantification of bipartite entanglement as obtained from the modewise decomposition. In a fundamental fermion theory, the characterization of entanglement is somewhat delicate, as one needs to take into account the absence of a natural tensor-product decomposition of the fermionic Hilbert space, as well as restrictions on the set of possible local operations posed by the anticommutativity of fermion operators [27] and superselection rules due to fermion number conservation [28]. Two measures are available: the explicit entanglement content of the state with respect to the second quantized mode Fock space—quantified by the “entanglement of modes”  $E_M$  [1], and the operationally accessible bipartite entanglement of the state—quantified by the “entanglement of particles”  $E_P$ , with  $E_P \leq E_M$  [28]. For the pure case of Eq. (2),  $E_M$  will simply be the sum of the von Neumann entropies of the partial density matrices obtained from the Schmidt coefficients of the individual entangled mode pairs in Eq. (3). On the other hand, a characterization in terms of  $E_P$  is less direct, as the measure is superadditive and will depend on the fundamental fermion content of the mode bases in the decomposition. Nevertheless, since  $E_P$  and  $E_M$  coincide asymptotically [28], one may expect that for

entangled states involving a large number of fermions,  $E_M$  should give a reasonable indication of the usable entanglement.

If mixed fermion states are analyzed according to the  $E_M$  criterion, it is possible to assess the separability of isotropic Gaussian states via the Peres–Horodecki partial transpose (PT) criterion [19]. Since a two-mode mixed state (14) is equivalent to a two-qubit mixed state, PT-negativity of at least one of the states  $\tilde{\rho}_{\tilde{A}_s \tilde{B}_s}$  in Eq. (13) will then be sufficient [20] to ascertain the inseparability of the isotropic state  $\rho^{(0)}$ . Using the fact that the entangling operator  $T_i$  in (15) performs a rotation between  $|00\rangle_{\tilde{A}_i \tilde{B}_i}$  and  $|11\rangle_{\tilde{A}_i \tilde{B}_i}$  while leaving  $|01\rangle_{\tilde{A}_i \tilde{B}_i}$  and  $|10\rangle_{\tilde{A}_i \tilde{B}_i}$  unaltered, the matrix representation of  $\tilde{\rho}_i$  can easily be shown to take the block form:

$$[\tilde{\rho}_i] = \frac{1}{4} \begin{pmatrix} (1 + \lambda_i)^2 + \kappa_i^2 & 2\kappa_i \\ 2\kappa_i & (1 - \lambda_i)^2 + \kappa_i^2 \end{pmatrix} \oplus \begin{pmatrix} 1 - \lambda_0^2 & 0 \\ 0 & 1 - \lambda_0^2 \end{pmatrix}, \quad (23)$$

where the first block refers to the  $|00\rangle_{\tilde{A}_i \tilde{B}_i}$  and  $|11\rangle_{\tilde{A}_i \tilde{B}_i}$  vectors and the second to the  $|01\rangle_{\tilde{A}_i \tilde{B}_i}$  and  $|10\rangle_{\tilde{A}_i \tilde{B}_i}$  vectors. In this decomposition, the partial transpose operation amounts to swapping the off-diagonal elements between the two blocks. The negative partial transpose criterion for  $\tilde{\rho}_i$  then becomes  $\lambda_0 \geq \kappa_i > \frac{1}{2}(1 - \lambda_0^2)$ . Note that this condition cannot be satisfied for any  $\kappa_i$  if  $\lambda_0 < \sqrt{2} - 1$  and is satisfied for all non-zero  $\kappa_i$  in the pure case  $\lambda_0 = 1$  as expected.

In conclusion, we have studied fermion mode entanglement and showed that under an arbitrary bipartite division of a set of fermion modes, Gaussian mode entanglement has a universal structure of a generalized BCS-like wave-function. Interestingly, unlike the Gaussian boson case, our result applies not only to the ground state, but also to all definite excitation eigenstates of quadratic fermion Hamiltonians. Our result can be used for computing the bipartite mode entanglement of a variety of states in a wide class of models, such as certain spin-chain models, and fermion liquids and superconductivity in the mean-field approximation, we believe that the present results can be helpful in connection with information processing applications involving coupled many-particle fermion systems or superconductors. For instance, when two such systems are coupled by a quadratic tunnelling interaction [29], our result im-



plies that under a proper choice of “local operators”, the total wave function factorizes to a product of non-maximal EPR-like states.

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