Simple Heuristics for Unit Disk Graphs [*]

M.V. Marathe [†] *H. Breu* [‡] *H.B. Hunt III* [§] *S.S. Ravi* [¶] *D.J. Rosenkrantz* [‖]

Abstract

Unit disk graphs are intersection graphs of circles of unit radius in the plane. We present simple and provably good heuristics for a number of classical NP-hard optimization problems on unit disk graphs. The problems considered include maximum independent set, minimum vertex cover, minimum coloring and minimum dominating set. We also present an on-line coloring heuristic which achieves a competitive ratio of 6 for unit disk graphs. Our heuristics do not need a geometric representation of unit disk graphs. Geometric representations are used only in establishing the performance guarantees of the heuristics. Several of our approximation algorithms can be extended to intersection graphs of circles of arbitrary radii in the plane, intersection graphs of regular polygons, and to intersection graphs of higher dimensional regular objects.

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[†] Department of Computer Science, University at Albany – SUNY, Albany, NY 12222. Research supported by NSF Grant CCR-89-03319. Current address: P.O. Box 1663, MS M986, Los Alamos National Laboratory, Los Alamos, NM 87544. Email: madhav@gardner.lanl.gov.

[‡] Department of Computer Science, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z4. Email: breu@cs.ubc.ca.

[§] Department of Computer Science, University at Albany – SUNY, Albany, NY 12222. Email: hbh@cs.albany.edu. Research supported by NSF Grant CCR-89-03319.

[¶] Department of Computer Science, University at Albany – SUNY, Albany, NY 12222. Email: ravi@cs.albany.edu. Research supported by NSF Grant CCR-89-05296.

[‖] Department of Computer Science, University at Albany – SUNY, Albany, NY 12222. Email: djr@cs.albany.edu. Research supported by NSF Grant CCR-90-06396.

1 Introduction, motivation and summary of results

Intersection graphs of geometric objects have been both widely studied and used to model many problems in real life [Ro78]. In this paper we consider intersection graphs of regular polygons, emphasizing intersection graphs of unit disks. A graph is a **unit disk graph** if and only if its vertices can be put in one to one correspondence with equisized circles in a plane in such a way that two vertices are joined by an edge if and only if the corresponding circles intersect. (It is assumed that tangent circles intersect.) Without loss of generality, the radius of each circle (disk) is assumed to be 1. Unit disk graphs have been used to model broadcast networks [Ha80, Ka84, YWS84] and optimal facility location [WK88]. For example, the problem of placing k facilities where proximity is undesirable can be modeled as the problem of finding an independent set of size k in a unit disk graph [WK88]. The problem of assigning distinct frequencies for transmitters with intersecting ranges corresponds to the minimum coloring problem for unit disk graphs. The minimum dominating set problem corresponds to selecting a minimum number of transmitters so that all the other stations are within the range of at least one of the chosen transmitters [Ha80]. As observed in [CCJ90], unit disk graphs are not perfect graphs. (An odd cycle of length five or more is a unit disk graph but not perfect.) Also, unit disk graphs are not planar. (A clique of size five or more is a unit disk graph but not planar.) Thus, in general, many of the known efficient algorithms for planar graphs and perfect graphs may not be applicable to unit disk graphs.

All of the problems mentioned above and many others remain NP-hard for unit disk graphs [CCJ90]. Motivated by the practical importance of these problems, we present simple heuristics with provably good performance guarantees for a number of such problems including the three mentioned above. Our heuristics do not need a geometric representation of a unit disk graph as part of the input. Geometric representations are used only in establishing some graph theoretic properties of unit disk graphs. These properties, in turn, are used in deriving the performance guarantees provided by the heuristics. Similar properties hold for the intersection graphs of other regular polygons and geometric objects in higher dimensions. Consequently, our heuristics can be extended to those intersection graphs as well.

Our results are summarized below. To provide a proper perspective, known results regarding performance guarantees for general graphs are also included.

We present a heuristic with a performance guarantee of 3/2 for the minimum vertex cover problem for unit disk graphs. (The best known heuristic for the vertex cover problem for general graphs provides a performance guarantee of 2 [GJ79, BE85]. It is also known that there is no polynomial time approximation scheme (PTAS) for the vertex cover problem for general graphs, unless $\mathbf{P} = \mathbf{NP}$[ALM+92].)

We present a simple heuristic with a performance guarantee of 3 for the maximum independent set problem for unit disk graphs. (The maximum independent set for general graphs is notoriously hard to approximate; unless $\mathbf{P} = \mathbf{NP}$, there is an $\epsilon > 0$ such that no polynomial time algorithm for the problem can provide a performance guarantee of $O(n^\epsilon)$ [ALM+92].)

We show that the off-line minimum vertex coloring problem can be approximated to within a factor of 3 of the optimal value for unit disk graphs. We note that unless $\mathbf{P} \neq \mathbf{NP}$, there is no PTAS for the minimum coloring problem for unit disk graphs since the 3-coloring problem for unit disk graphs is NP-hard [CCJ90]. We also give an *on-line* [Ira90] coloring heuristic with a competitive ratio of 6 for unit disk graphs. (For general graphs, unless $\mathbf{P} = \mathbf{NP}$, there is an $\epsilon > 0$ such that no polynomial time algorithm for the off-line minimum coloring problem can provide a performance guarantee of $O(n^\epsilon)$ [LY93]. On-line coloring algorithms with constant competitive ratios have been obtained previously for several other classes of graphs [HMR+94, Sl94, MHR93, Ki91, GL88, Sl87, KT81]. It is also known that no on-line algorithm can provide a competitive ratio of $o(\log n)$ even for trees [Ira90].)

We observe that a very simple heuristic provides a performance guarantee of 5 for the minimum dominating set problem for unit disk graphs. This heuristic also provides a performance guarantee of 5 for the minimum independent domination problem. The heuristic can be modified to obtain another heuristic which provides a performance guarantee of 10 for both the minimum total domination and the minimum connected domination problems. (For general graphs, the minimum dominating set problem can be approximated to within a factor $O(\log n)$ of the optimal value [Jo74]; it is also known that no polynomial time algorithm can provide a performance guarantee of $o(\log n)$, unless every problem in $\mathbf{NP}$ can be solved in deterministic time $O(n^{\text{poly} \log n})$ [LY93]. For general graphs, it is also known that there is an $\epsilon > 0$ such that the minimum independent domination problem cannot be approximated to within a factor of $O(n^\epsilon)$ unless $\mathbf{P} = \mathbf{NP}$[Kan93, Ir91].)

For intersection graphs of circles of arbitrary radii in the plane, we present heuristics with performance guarantees of 5/3, 6 and 5 respectively for vertex cover, off-line coloring and independent problems respectively. These heuristics are obtained as extensions of the corresponding heuristics for unit disk graphs. We also show how the heuristics can be extended to intersection graphs of regular polygons.

The remainder of the paper is organized as follows. Section 2 contains the necessary definitions. In Section 3, we establish several properties of unit disk graphs. The approximation algorithms of Section 4 utilize these properties. Section 5 contains extensions of our results to other intersection graphs and Section 6 contains some concluding remarks.

2 Definitions

We defined unit disk graphs as the intersection graphs of sets of unit disks in the plane. Throughout this paper, it is assumed that tangent circles intersect. The model described above will be referred to as the **intersection model** [CCJ90]. Unit disk graphs can also be defined using the **proximity model** [CCJ90] in which the nodes of the graph are in one-to-one correspondence with a set of points in the plane, and two vertices are joined by an edge if and only if the distance between the corresponding points is at most some specified bound. When considering a geometric representation, we often do not distinguish between a vertex and its corresponding circle or point.

In the intersection model, we assume that the radius of each disk is 1. It is easily seen that two such disks in the plane intersect if and only if the distance between their centers is at most 2. Thus it is easy to translate a description of a given unit disk graph in the intersection model into a description in the proximity model and vice versa in linear time. The recognition problem for unit disk graphs was shown to be NP-hard in [BK93]. As already mentioned, none of our heuristics requires a geometric representation of a unit disk graph as part of the input.

An approximation algorithm for an optimization problem Π provides a **performance guarantee** of ρ if for every instance I of Π, the solution value returned by the approximation algorithm is within a factor ρ of the optimal value for I. A **polynomial time approximation scheme** (PTAS) for problem Π is a polynomial time approximation algorithm which given an instance I of Π and an $\epsilon > 0$, returns a solution which is within a factor $(1+\epsilon)$ of the optimal value for I.

For the sake of completeness, we now define several graph theoretic parameters. Given an undirected graph $G(V, E)$, a **minimum vertex cover** for G is a smallest cardinality subset V' of V such that for each edge (x, y) in E, at least one of x and y is in V'. A **maximum independent set** is a maximum cardinality subset V' of V such that there is no edge between any two vertices in V'. A **dominating set** is a subset V' of V such that each vertex in $(V - V')$ has at least one neighbor in V'. An **independent dominating set** is a set of nodes that is both an independent set and a dominating set. A subset V' of V is a **total dominating set** if every node in V has a neighbor in V' (i.e., in addition to the nodes in $V - V'$, each node in V' must also be dominated by another node in V'). A **connected dominating set** is a dominating set V' in which the vertex induced subgraph on V' is connected.

We end this section with some graph theoretic notation. We use $K_{p,q}$ to denote the complete bipartite graph with p and q nodes in the two sets of the bipartition. We also use K_r to denote the clique with r nodes. Given a graph $G(V, E)$ and a node v, we use $N(v)$ to denote the set of nodes adjacent to v; we refer to $N(v)$ as the **neighborhood** of v. For a graph $G(V, E)$ and a subset $V' \subseteq V$, we use $G(V')$ to denote the subgraph of G induced on V'. For the remainder of this paper, we assume that $|V| = n$ and $|E| = m$.

3 Some properties of unit disk graphs

Many of our heuristics are based on a forbidden subgraph property of unit disk graphs. The proof of this property relies on a geometric observation concerning packing of unit disks in the plane. (The problems of Packing and Covering have been of interest to researchers for quite some time. See [CS88] for more on this subject.)

Lemma 3.1 *Let C be a circle of radius r and let S be a set of circles of radius r such that every circle in S intersects C and no two circles in S intersect each other. Then, $|S| \leq 5$.*

Proof: Suppose $|S| \geq 6$. Let s_i, $1 \leq i \leq 6$, denote the centers of any six circles in S. Let c denote the center of C. Denote the ray $\overrightarrow{c\,s_i}$ by r_i $(1 \leq i \leq 6)$. Since there are six rays emanating from c, there must at least one pair of rays r_j and r_k such that the angle between them is at most $60°$. Now, it can be verified that the distance between s_j and s_k is at most $2r$, which implies that circles centered at s_j and s_k intersect, contradicting our assumption. Thus $|S| \leq 5$. $\Box$

An immediate consequence of Lemma 3.1 is the following:

Lemma 3.2 *Let $G(V, E)$ be a unit disk graph. Then G cannot contain an induced subgraph isomorphic to $K_{1,6}$.* $\Box$

Lemma 3.2 indicates that in any unit disk graph, the size of a maximum independent set in the subgraph of G induced on the neighborhood of any vertex is at most 5. The following lemma, which can be proven in a manner similar to that of Lemma 3.1, points out that the neighborhoods of certain nodes in a unit disk graph have even smaller independent sets.

Lemma 3.3 *Let G be a unit disk graph, and let v be a vertex such that the unit disk corresponding to v (in some model for G) has the smallest X-coordinate. The size of a maximum independent set in $G(N(v))$ is at most 3.* $\Box$

Finally, the following property of unit disk graphs is useful in establishing the performance of our heuristic for on-line vertex coloring.

Lemma 3.4 *A unit disk graph G with maximum node degree Δ contains a clique of size at least $\lceil \Delta/6 \rceil + 1$.*

Proof: Consider a geometric representation of G in which each node of G corresponds to a circle of radius 1. Let v be a node of maximum degree Δ. The centers of the circles corresponding to the Δ neighbors of v are all within the circle C of radius 2 with center at the center of the circle corresponding to v. Thus at least $\lceil \Delta/6 \rceil$ of the neighbors of v must lie within some $60°$ sector of the circle C. Since the maximum distance between a pair of points in that segment is 2, it follows that the nodes corresponding to the points in that sector must be pairwise adjacent. Thus, these $\lceil \Delta/6 \rceil$ nodes, together with the node v, must form a clique of size $\lceil \Delta/6 \rceil + 1$. $\square$

4 Approximation Algorithms

We now proceed to give approximations to the various graph theoretic problems. We start with a heuristic for the minimum vertex cover problem.

4.1 Minimum Vertex Cover

The heuristic given here is essentially the same as the one given in in Bar-Yehuda and Even [BE82] for planar graphs but the analysis which leads to a performance guarantee of 1.5 is different. It is interesting that an algorithm which provides a performance guarantee of 1.5 for planar graphs provides the same performance guarantee for unit disk graphs as well.

Before presenting the details of the heuristic, we mention some results which are used in the design and analysis of the heuristic. The first result deals with the **Nemhauser-Trotter decomposition** (NT decomposition) of a graph. The properties of this decomposition are stated in the following result from [NT75, BE85].

Theorem 4.1 *[NT75, BE85] Given an undirected graph $G(V, E)$, there are two disjoint subsets P and Q of V such that the following properties hold.*

1. *There exists an optimum cover $VC^*(G)$ for G such that $P \subseteq VC^*(G)$.*
2. *If D is a vertex cover for $G(Q)$, then $D \cup P$ is a vertex cover for G.*
3. *For any optimum vertex cover $VC^*(G)$ of G, $|VC^*(G)| \geq |P| + |Q|/2$.*

Moreover, the sets P and Q can be found in polynomial time. $\square$

The second result that is used in the heuristic is a theorem due to Hochbaum [Ho83]. This theorem points out how a near-optimal vertex cover can be obtained in some cases starting from an NT decomposition. We have included the proof from [Ho83] since our heuristic uses the method given in the proof.

Theorem 4.2 *[Ho83] Let $G(V, E)$ be a node weighted graph and let P and Q denote the subsets obtained using an NT decomposition of G. If $G(Q)$ can be colored using k colors, then we can find a vertex cover whose cardinality is at most $2(1 - 1/k)$ times that of an optimal vertex cover.*

Proof: Color the nodes of $G(Q)$ using k colors. This procedure partitions Q into at most k color classes. Let S be a color class of largest cardinality. Clearly, $|S| \geq |Q|/k$. It is easy to see from Theorem 4.1 that the set $C = P \cup (Q - S)$ is a vertex cover for G. Further, $|C| \leq |P| + (k-1)|Q|/k$ $= \dfrac{2(k-1)}{k}\left[\dfrac{k|P|}{2(k-1)} + \dfrac{|Q|}{2}\right]$. By Theorem 4.1, any optimal vertex cover for G has cardinality at least $|P| + |Q|/2$. Thus, C is a vertex cover for G whose cardinality is no more than $2(1 - 1/k)$ times that of an optimal vertex cover. □

Finally, the heuristic uses the following property of triangle-free unit disk graphs. (A graph is **triangle-free** if it does not contain a subgraph isomorphic to K_3.)

Lemma 4.1 *Any triangle-free unit disk graph can be colored using 4 colors.*

Proof: We first observe that every triangle-free unit disk graph has a node with degree at most 3. To see this, let G be a triangle-free unit disk graph. Since G is triangle-free, the neighborhood of each node is an independent set. Thus, by Lemma 3.3, G must have a node of degree at most 3.

From the above observation and the fact that any induced subgraph of a triangle-free unit disk graph is also a triangle-free unit disk graph, it is easy to see that the following is an efficient algorithm for 4-coloring such graphs: Given a triangle-free unit disk graph with n nodes, we find and remove a node v of degree at most 3, recursively 4-color the resulting graph containing $n - 1$ nodes and then assign a color to v that has not been used for any of the at most three neighbors of v. □

We note that the algorithm presented in the proof of Lemma 4.1 does not require a geometric representation of the unit disk graph.

We are now ready to discuss the heuristic for vertex cover. The details of this heuristic, which we call VCover, are given in Figure 1. The heuristic begins by removing all triangles from the graph G. Every time a triangle is found, we add all the three vertices of the triangle to the vertex cover (V_1) obtained so far. To the resulting triangle-free graph, we apply NT decomposition and obtain the sets P and Q. In the next step we 4-color the graph $G(Q)$. (This is possible because of Lemma 4.1.) We then remove the vertices in a color class S of maximum cardinality and return the set $V_1 \cup P \cup (Q - S)$ as the approximate vertex cover. To determine the performance guarantee of this heuristic, we need the following result from [BE85].

1. $V_1 = \emptyset; V' = V$.
2. **while** $G(V')$ contains triangles **do**
3. **begin**
   (a) Let $X \subseteq V'$ be such that $G(X)$ is a triangle.
   (b) $V_1 = V_1 \cup X$.
   (c) $V' = V' - X$.
4. **end**
5. Obtain the sets P and Q by applying Nemhauser-Trotter's algorithm on $G(V')$.
6. Color the vertices of $G(Q)$ using 4 colors. Let S denote a color class containing the largest number of nodes.
7. **output** $VC(G) = V_1 \cup P \cup (Q - S)$ as the approximate vertex cover for $G(V, E)$.

Figure 1: Details of Heuristic VCover

Lemma 4.2 *[BE85] Let r_1 and r_2 denote the local ratios of two heuristics for the vertex cover problem. If G_1 denotes the graph obtained after applying the first heuristic to a graph G, then the performance of the algorithm which applies the two heuristics in succession is at most* $\max\{r_1(G), r_2(G_1)\}$. □

Theorem 4.3 *Let G be a unit disk graph. Let $VC^*(G)$ denote an optimal vertex cover for G and let $VC(G)$ denote the vertex cover produced by Heuristic VCover. Then $|VC(G)| \leq 1.5|VC^*(G)|$.*

Proof: Let us think of Heuristic VCover (Figure 1) as operating in two phases, where the first phase consists of Steps 1 through 4 and the second phase consists of the remaining steps. The first phase of the algorithm guarantees a local ratio of 1.5, because any optimal algorithm needs to pick at least two vertices from each triangle and the heuristic picked all three. From Theorem 4.1, Lemma 4.1, and Theorem 4.2, it follows that the second phase of the algorithm also guarantees a local ratio of 1.5. The result now follows from Lemma 4.2. □

The time complexity of Heuristic VCover is dominated by the time it takes to execute the Nemhauser-Trotter algorithm.

4.2 Minimum Vertex Coloring

4.2.1 Off-line Coloring

We present below a heuristic for coloring the vertices of a unit disk graph such that the number of colors used is no more than 3 times the number of colors used by an optimal algorithm. We use the following result due to Hochbaum [Ho83].

Theorem 4.4 *[Ho83] Let $G = (V, E)$ be a graph and let $\delta(G)$ denote the largest δ such that G contains a subgraph in which every vertex has a degree at least δ. Then G can be colored using $\delta(G) + 1$ colors. Moreover, such a coloring can be obtained in $O(|V| + |E|)$ time.* □

Our heuristic for off-line coloring simply colors the given unit disk graph G with $\delta(G)+1$ colors using the algorithm mentioned in the above theorem. We now prove the performance guarantee of this heuristic.

Theorem 4.5 *Let G be a unit disk graph. Let* OFFCOL(G) *and* OPT(G) *denote the number of colors used by the above heuristic and that used by an optimal algorithm respectively. Then* OFFCOL(G) ≤ 3 OPT(G).

Proof: Let $\delta(G)$ be as defined in Theorem 4.4. Thus,

$$OFFCOL(G) = \delta(G) + 1. \tag{1}$$

We now prove the following claim which relates $OPT(G)$ and $\delta(G)$.

Claim 1: $OPT(G) \geq \delta(G)/3 + 1$.

Proof: Let H be a subgraph of G such that every vertex in H has a degree at least $\delta(G)$. Consider a unit disk representation of H, and let $v^* \in H$ be a vertex such that the center of the unit disk corresponding to v^* has the leftmost X coordinate. Let $N_H(v^*)$ denote the neighborhood of v^* in H. By our choice of H,

$$|N_H(v^*)| \geq \delta(G). \tag{2}$$

Recall from Lemma 3.3 that the subgraph induced on the nodes in $N_H(v^*)$ has an independent set of size at most 3. Consider any valid coloring of the graph G. Since all the vertices colored with the same color form an independent set, in any valid coloring of the subgraph induced on the set of nodes in $N_H(v^*)$, no more than 3 vertices can belong to the same color class. Therefore, any valid coloring of the subgraph induced by the set $N_H(v^*) \cup \{v^*\}$ must use at least $|N_H(v^*)|/3 + 1$ colors. In other words, $OPT(G) \geq |N_H(v^*)|/3 + 1$. Now, from Equation (2), it follows that

$$OPT(G) \geq \delta(G)/3 + 1. \tag{3}$$

The theorem now follows from Equations (1) and (3). □

4.2.2 On-Line Coloring

An instance of an off-line coloring problem consists of all the nodes and edges of a graph. In the *on-line* version of the coloring problem, vertices of a graph are presented one at a time. When a vertex v is presented, all the edges from v to the vertices presented earlier are given. An on-line coloring algorithm assigns a color to the current vertex before the next vertex is presented. The color assigned to the vertex must be different from the colors assigned to its neighbors presented earlier. Once a color is assigned to a vertex, the algorithm is not allowed to change the color at a future time. The performance guarantee provided by a heuristic is defined as the worst-case ratio of the number of colors used by the heuristic to the number of colors needed in an optimal off-line coloring. In the context of on-line problems, the performance guarantee provided by a heuristic is referred to as its **competitive ratio**.

In this section, we observe that given a unit disk graph $G(V, E)$, there is an on-line coloring heuristic which achieves a competitive ratio of 6. The heuristic simply colors the vertices using the following greedy strategy:

Heuristic Greedy: Color each vertex with the first available color.

The following lemma provides an upper bound on the number of colors used by the greedy strategy for an arbitrary graph. The lemma can be proved easily by induction on the number of vertices.

Lemma 4.3 *Let G be an arbitrary graph with maximum node degree $\Delta(G)$. Let $Greedy(G)$ denote the number of colors used by Greedy to color G. Then $Greedy(G) \leq \Delta(G) + 1$.* □

Lemma 3.4 points out that for coloring any unit disk graph G with a maximum node degree of $\Delta(G)$ at least $\lceil \Delta(G)/6 \rceil$ colors are required. Therefore, the following theorem is an easy consequence of Lemmas 4.3 and 3.4.

Theorem 4.6 *Heuristic Greedy achieves a competitive ratio of at most 6.* □

4.3 Maximum Independent Set

Since unit disk graphs are $K_{1,6}$ free, one can use the algorithm in [Ho83] to obtain a performance guarantee of 5. However, we can do better for unit disk graphs because of the additional geometric structure they possess.

Our heuristic, which we call IS, provides a performance guarantee of 3. This heuristic is based on Lemma 3.3 which points out that every unit disk graph G has a node v such that size of any independent set in the subgraph induced on the neighborhood $N(v)$ of v is at most 3. Given the nodes and edges of a unit disk graph, such a node can be found in polynomial time. (Obviously, $O(n^5)$ time suffices.) This observation in conjunction with the fact that any induced subgraph of a unit disk graph is also a unit disk graph leads to the following simple heuristic for finding a near-optimal independent set in a unit disk graph. The heuristic repeatedly finds a node v such that the maximum independent set in the subgraph induced on $N(v)$ is at most 3, adds v to the independent set, and deletes v and $N(v)$ from the graph. These steps are repeated until the graph becomes empty. We now show that heuristic IS provides a performance guarantee of 3.

Theorem 4.7 *Let G be a unit disk graph, and let* $\mathrm{IS}(G)$ *and* $\mathrm{OPT}(G)$ *denote respectively the approximate independent set produced by* IS, *and an optimal independent set for G. Then* $|\mathrm{IS}(G)| \geq |\mathrm{OPT}(G)|/3$. □

Proof: Define the (closed) *r-neighborhood* of a vertex v in $IS(G)$ to be the set $N(v) \cup \{v\}$, where $N(v)$ is the neighborhood of v when the algorithm adds v to $IS(G)$. By construction, every vertex in the graph, and therefore in $OPT(G)$, is in the r-neighborhood of at least one vertex in $IS(G)$. Also by construction, the size of a maximum independent set in every r-neighborhood is 3. Therefore, since $OPT(G)$ is independent, every r-neighborhood contains at most 3 vertices from $OPT(G)$. It follows that $|OPT(G)| \leq 3|IS(G)|$. □

The running time of the heuristic can be reduced significantly when a geometric representation of the unit disk graph is available. Recall from the proof of Lemma 3.3 that a node v corresponding to a unit disk whose center has the least X-coordinate satisfies the required property. Therefore, when a geometric representation of the unit disk graph is given, we can implement the above heuristic in the following manner. Form a list L of the centers of the unit disks in increasing order of their X-coordinates. At each step, add the first item v from L into the independent set and delete from L both v and the nodes which are adjacent to v. The algorithm terminates when L becomes empty. It is easy to see that the algorithm can be implemented to run in $O(n^2)$ time by first computing for each unit disk v a list of disks that intersect v.

4.4 Domination Problems

In this section we give efficient heuristics for various domination problems for unit disk graphs. The problems considered in this section are minimum dominating set, minimum independent dominating set, minimum connected dominating set, and minimum total dominating set.

4.4.1 Dominating Set and Independent Dominating Set

An independent set S is **maximal** if no proper superset of S is also an independent set. It is well known and easy to see that for any graph, any maximal independent set is also a dominating set. Thus, one method of finding a dominating set (which is also an independent dominating set) in a graph is to find a maximal independent set. A straightforward method of finding a maximal independent set is to repeat the following steps until the graph becomes empty: Select an arbitrary node v, add v to the independent set and delete v and $N(v)$ from the graph. Obviously, this method can be implemented to run in linear time.

We now observe that for unit disk graphs, the size of any maximal independent set is within a factor of 5 of the size of a minimum dominating set.

Theorem 4.8 *Let G be a unit disk graph. Let D^* be a minimum dominating set for G and let D be any maximal independent set for G. Then $|D| \leq 5\,|D^*|$.*

Proof: Since D is an independent set, by Lemma 3.2, no vertex in D^* can dominate more than 5 vertices in D. Hence, $|D^*| \geq |D|/5$ and the theorem follows. □

Clearly, the same performance guarantee holds for independent domination because D is also an independent set.

In general, given a graph which is $K_{1,p}$ free, the cardinality of any maximal independent set is at most $p-1$ times that of a minimum (independent) dominating set.

4.4.2 Connected Domination and Total Domination

We assume that G has no isolated nodes; otherwise, G has neither a connected dominating set nor a total dominating set. If G consists of two or more connected components each containing two or more nodes, then G does not have a connected dominating set. However, in this case, we can find a near-optimal total dominating set as follows. We first find a maximal independent set X for G. Then, for each node $x \in X$, we choose a node $y \in V - X$ such that x is adjacent to y. Clearly, this procedure leads to a total dominating set of size at most $2|X|$. Hence from Theorem 4.8, the size of the resulting total dominating set is at most 10 times that of a minimum dominating set.

In view of the above, we assume for the remainder of this section that G is connected. For any connected graph, a connected dominating set is also a total dominating set. Therefore, we focus on finding a connected dominating set.

We begin with an overview of our heuristic for connected domination. The heuristic first selects an arbitrary vertex v from the graph and constructs the breadth-first spanning tree T of G rooted at v. For any node v_j, let $l(v_j)$ denote the number of edges in the path from v to v_j in T. We

1. Arbitrarily pick a vertex $v \in V$.
2. Construct the breadth-first spanning tree T of G rooted at v.
3. Let k be the depth of T.
4. Let S_i denote the nodes at level i in T, for all $0 \leq i \leq k$.
5. Set $IS_0 = \{v\}$; $NS_0 = \emptyset$.
6. **for** $i = 1$ **to** k **do begin**
7. $\quad DS_i = \{v | v \in S_i$ and v is dominated by some vertex in $IS_{i-1}\}$.
8. $\quad$ Pick a maximal independent set IS_i in $G(S_i - DS_i)$.
9. $\quad NS_i = \{u \,|\, u$ is the parent (in T) of some $v \in IS_i\}$. (Note that $NS_i \subseteq S_{i-1}$.)
10. **end**
11. **output** $(\cup_{i=0}^{k} IS_i) \cup (\cup_{i=0}^{k} NS_i)$ as the connected dominating set.

Figure 2: Details of Heuristic CDOM

will refer to $l(v_j)$ as the **level** of v_j. Let k denote the maximum level in G. Thus, the tree T partitions the vertices of G into sets S_i, $0 \leq i \leq k$, where S_i is the set of nodes at level i. The connected dominating set produced by the heuristic is the union of two sets of nodes. The first set is a maximal independent set for G obtained by appropriately selecting an independent set IS_i from each graph $G(S_i)$, $1 \leq i \leq k$. The second set of nodes is used to ensure connectivity. The details of this heuristic, which we call CDOM, appear in Figure 2. We are now ready to prove the correctness of the heuristic and its performance guarantee.

Lemma 4.4 *The output of Heuristic CDOM is a connected dominating set.*

Proof: We prove by induction on k that $\cup_{i=0}^{k} IS_i$ dominates the graph $G(\cup_{i=0}^{k} S_i)$, and that $G((\cup_{i=0}^{k} IS_i) \cup (\cup_{i=0}^{k} NS_i))$ is connected.

Basis: For $k = 0$ we have $G(\cup_{i=0}^{k} S_i) = (\{v\}, \emptyset)$. Furthermore, $(\cup_{i=0}^{k} IS_i) \cup (\cup_{i=0}^{k} NS_i) = IS_0 \cup NS_0 = \{v\}$, so the lemma holds.

Inductive Step: Assume the statment is true for some $k \geq 0$ and consider $k+1$. By the inductive hypothesis, $\cup_{i=0}^{k} IS_i$, and therefore $\cup_{i=0}^{k+1} IS_i$, dominates $G(\cup_{i=0}^{k} S_i)$. We therefore only need to show that S_{i+1} is also dominated. By Step 7, all vertices in DS_{k+1} are dominated by IS_k, and by Step 8, all vertices in $S_{k+1} - DS_{k+1}$ are dominated by IS_{k+1} (since IS_{k+1} is a maximal independent set in $G(S_{k+1} - DS_{k+1})$). That is, S_{k+1} is dominated by $IS_k \cup IS_{k+1}$, and therefore by $\cup_{i=0}^{k+1} IS_i$.

Now, by the inductive hypothesis, $G((\cup_{i=0}^{k} IS_i) \cup (\cup_{i=0}^{k} NS_i))$ is connected. By Step 9, $NS_{k+1} \subseteq S_k$ since $IS_{k+1} \subseteq S_{k+1}$. By the inductive hypothesis, S_k, and therefore NS_{k+1}, is dominated by $(\cup_{i=0}^{k} IS_i)$. Therefore $G((\cup_{i=0}^{k} IS_i) \cup (\cup_{i=0}^{k+1} NS_i))$ is connected. Finally, by Step 9, every vertex in

IS_{k+1} is adjacent to NS_{k+1}. Therefore $G((\cup_{i=0}^{k+1} IS_i) \cup (\cup_{i=0}^{k+1} NS_i))$ is connected. $\square$

Theorem 4.9 *Given a unit disk graph G, Heuristic CDOM computes a connected dominating set whose size is no more than 10 times that of an optimal connected dominating set for G. The same heuristic also yields a total dominating set of size no more than 10 times that of a minimum total dominating set.*

Proof: Let $IS = \cup_{i=0}^{k} IS_i$ and $NS = \cup_{i=0}^{k} NS_i$. It is easy to verify that IS is a maximal independent set. Therefore, by Theorem 4.8, $|IS|$ is no more than 5 times the size of a minimum dominating set (OPT), and hence also within 5 times the size of minimum connected dominating set. By Steps 5 and 9, $|NS_i| \leq |IS_i|$ for all $0 \leq i \leq k$, so $|NS \cup IS| \leq 2|IS| \leq 10|OPT|$, and the bound of 10 follows. Since every connected dominating set is also a total dominating set, the bound with respect to a minimum total dominating set is also 10. $\square$

In general, given a $K_{1,p}$ free graph (i.e., a graph which has no subgraph isomorphic to $K_{1,p}$), the above heuristic obtains a solution which is no more than $2(p-1)$ times any optimal connected/total dominating set.

5 Extensions to other intersection graphs

Thus far, we have considered intersection graphs of circles with the the same radius. In this section, we show that the ideas presented in the previous sections can be extended to obtain constant performance guarantees for several optimization problems for intersection graphs of circles of arbitrary radii in the plane and intersection graphs of regular polygons.

5.1 Circle intersection graphs

We use the term **circle intersection graphs** to refer to intersection graphs of circles of arbitrary radii in the plane. A graph G is a circle intersection graph if the vertices of G can be placed in one-to-one correspondence with a set of circles in the plane such that two vertices of G are adjacent if and only if their corresponding circles intersect. As before, tangential circles are assumed to intersect. We note that the class of circle intersection graphs is different from the class of **coin graphs** considered in [Th91]. This is because in the case of coin graphs, two vertices are adjacent if and only if their corresponding circles *touch*. It is known that the class of coin graphs coincides with the class of planar graphs [Th91].

We now show how to extend the heuristics for unit disk graphs to circle intersection graphs. We begin with a lemma which is similar to Lemma 3.2.

Lemma 5.1 *Let G be a circle intersection graph. Then, G has a node v such that the graph $G(\{v\} \cup N(v))$ does not contain an induced subgraph isomorphic to $K_{1,6}$.*

Proof: Consider a circle intersection model for G. Let v be a node whose corresponding circle C has the smallest radius. We can now use the argument given in the proof of Lemma 3.2 to obtain the result. □

The above lemma indicates that every circle intersection graph G contains a node v such that the size of any independent set in $G(N(v))$ is at most 5. Using this observation and a proof similar to that of Lemma 4.1, it is easy to prove the following result.

Lemma 5.2 *Any triangle-free circle intersection graph can be colored using 6 colors.* □

Lemma 5.2 enables us to obtain a heuristic for vertex cover with a performance guarantee of 5/3 for circle intersection graphs. The heuristic is the same as Heuristic VCover in Figure 1 except that in Step 6, the graph $G(Q)$ is colored using 6 colors (instead of 4). The performance guarantee of 5/3 follows from Theorem 4.2 and Lemma 4.2.

The heuristic given for off-line coloring of unit disk graphs (Section 4.2.1) can be easily shown to provide a performance guarantee of 6 for circle intersection graphs. The proof is very similar to that of Theorem 4.5. The only difference is the following. We prove that $OPT(G) \geq \delta(G)/6 + 1$ by considering the vertex v^* to be a vertex whose corresponding circle has the smallest radius and using Lemma 5.1.

It is also possible to obtain a heuristic with a performance guarantee of 5 for the maximum independent set problem for circle intersection graphs. This heuristic is identical to that for unit disk graphs (Section 4.3) except that at each step, we select a node v such that no independent set in the subgraph $G(N(v))$ is of size 6 or more. Lemma 5.1 enables us to conclude that the heuristic provides a performance guarantee of 5.

The following theorem summarizes the above discussion.

Theorem 5.1 *There are polynomial time heuristics that provide performance guarantees of 5/3, 6, and 5 respectively for minimum vertex cover, minimum off-line vertex coloring, and maximum independent set problems for circle intersection graphs.* □

5.2 Intersection graphs of regular polygons

The ideas in the previous sections can also be extended to intersection graphs of regular polygons. Let us call a p sided polygon **a unit regular polygon** if the polygon is inscribed in a circle of radius 1 and the sides of the polygon are all equal. Each such polygon can be uniquely specified up to rotation by the number of sides and the coordinates of the center of the polygon. Now, for any graph which is an intersection graph of p-sided unit regular polygons, we can derive a (crude) bound on the size of a maximum independent set in the subgraph induced on the neighborhood of any node. This bound depends only on the number of sides p of the polygon.

Lemma 5.3 *Let P be a p-sided unit regular polygon, and let S be a set of p-sided unit regular polygons such that every polygon in S intersects P and no two polygons in S intersect each other. There is an integer t, depending only on p, such that $|S| \leq t$.*

Proof: It is easy to verify that the area α_p of a p-sided unit regular polygon is given by $\alpha_p = p \sin(2\pi/p)/2$. It is also easy to see that any p-sided unit regular polygon which intersects P must lie completely within the circle C of radius 3 with center at the center of P. The area of this circle is 9π. Thus, the maximum number of p-sided unit regular polygons which lie completely within C and which do not intersect each other is $\lceil \text{Area of } C/\alpha_p \rceil$ which is equal to $\lceil 18\pi/(p \sin(2\pi/p)) \rceil$. Obviously, this bound depends only on p. □

Thus, for any *fixed* p, there is a fixed integer t such that the intersection graphs of p-sided unit regular polygons do not contain $K_{1,t+1}$ as an induced subgraph. Therefore, we can obtain polynomial time heuristics with constant performance guarantees for most of the problems considered in Section 4 for intersection graphs of p-sided unit regular polygons. The following theorem provides a formal statement of our results for intersection graphs of p-sided unit regular polygons.

Theorem 5.2 *For intersection graphs of p-sided unit regular polygons, there are heuristics with constant performance guarantees for the following optimization problems: minimum vertex cover, maximum independent set, minimum off-line coloring, minimum dominating set, minimum independent dominating set, minimum connected dominating set and minimum total dominating set.* □

In a similar fashion, we can also extend the results in the previous sections to obtain heuristics with constant performance guarantees for intersection graphs of unit balls in higher dimensions.

6 Concluding remarks

We have given efficient approximations for a variety of standard problems on unit disk graphs. These heuristics have been designed using the underlying geometric structure of unit disk graphs. We observed that our heuristics can be extended to intersection graphs of circles in the plane and to intersection graphs of regular polygons.

The heuristics presented in this paper do not require a geometric representation of the input graph. In [HMR+94], we have shown that when a geometric representation of a unit disk graph is available as part of the input, it is possible to obtain polynomial time approximation schemes for maximum independent set, minimum vertex cover and minimum dominating set problems. These results are obtained using ideas from [Ba83] and [HM85].

Our heuristics for the maximum independent set problem for unit disk graphs and circle intersection graphs also provide a slight generalization of a result in [Ho83]. In that paper, Hochbaum observed that for any graph which does not contain $K_{1,r}$ as an induced subgraph, there is a heuristic with a performance guarantee of $r-1$ for the maximum independent set problem. Our results point out that if a graph G has a node v such that $G(\{v\} \cup N(v))$ does not contain $K_{1,r}$ as an induced subgraph and this property holds for all induced subgraphs of G, then there is a heuristic with a performance guarantee of $r-1$ for the maximum independent set problem for G.

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