Discrete Applied Mathematics 237 (2018) 97–108

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

(d, n)-packing colorings of infinite lattices

Danilo Korže [a], Aleksander Vesel [b,*]

[a] Faculty of Electrical Engineering and Computer Science, University of Maribor, SI-2000 Maribor, Slovenia
[b] Faculty of Natural Sciences and Mathematics, University of Maribor, SI-2000 Maribor, Slovenia

ARTICLE INFO

Article history:
Received 20 July 2017
Received in revised form 20 November 2017
Accepted 27 November 2017
Available online 20 December 2017

Keywords:
Coloring
S-packing
Infinite lattice

ABSTRACT

For a nondecreasing sequence of integers $S = (s_1, s_2, \dots)$ an S-packing k-coloring of a graph G is a mapping from $V(G)$ to $\{1, 2, \dots, k\}$ such that vertices with color $i \in \{1, 2, \dots, k\}$ have pairwise distance greater than s_i. A natural restriction of this concept obtained by setting $s_i = d + \lfloor \frac{i-1}{n} \rfloor$ is called a (d, n)-packing coloring of a graph G. The smallest integer k for which there exists a (d, n)-packing coloring of G is called the (d, n)-packing chromatic number of G. We study (d, n)-packing chromatic colorings of several lattices including the infinite square, hexagonal, triangular, eight-regular, octagonal and two-row square lattice.

1. Introduction

The frequency assignment problem asks for assigning frequencies to transmitters in a wireless network and includes a variety of specific subproblems. Many of them are variations of the classical graph coloring which involve graph distance, i.e. a condition is usually imposed on the vertices that are given the same color. In particular, Goddard et al. [5] introduced the concept of broadcast chromatic number. This model, described below, was latter to become known as the packing chromatic number.

A *k-coloring* of a graph G is a function f from $V(G)$ onto a set $C = \{1, 2, \dots, k\}$ (with no additional constraints). The elements of C are called *colors*. Let X_i denote the set of vertices with the image (color) i. Note that $X_1, \dots, X_k$ is a partition of the vertex set of G into disjoint (color) classes.

Let $X_1, \dots, X_k$ be a partition of the vertex set of G with respect to the following constraints: each color class X_i is a set of vertices with the property that any distinct pair $u, v \in X_i$ satisfies $d_G(u, v) > i$. Here $d_G(u, v)$ denotes the usual shortest path distance between u and v. Then X_i is said to be an *i-packing*, while such a partition is called a *packing k-coloring*. The smallest integer k for which there exists a packing k-coloring of G is called the *packing chromatic number of* G and it is denoted by $\chi_\rho(G)$.

A more general concept which can be seen already in [5] is later formally introduced in [7] as follows. For a nondecreasing sequence of integers $S = (s_1, s_2, \dots)$, an *S-packing k-coloring* is a k-coloring c of $V(G)$ such that for every i, with $1 \le i \le k$, $c^{-1}(i)$ is an s_i-packing. The *S-packing chromatic number* of G denoted by $\chi_\rho^S(G)$, is the smallest k such that G admits an S-packing k-coloring.

Gastineau et al. [4] proposed the variation of the S-packing coloring, where for integers n and d the sequence $S = (s_1, s_2, \dots)$ is given by $s_i = d + \lfloor \frac{i-1}{n} \rfloor$. In this setting, an S-packing k-coloring of a graph G is called a *(d, n)-packing k-coloring*, while the smallest integer k for which there exists a (d, n)-packing k-coloring of G is called the *(d, n)-packing chromatic number* and denoted by $\chi_\rho^{d,n}(G)$. Note that for $d = 1$ and sufficiently large n a (d, n)-packing k-coloring is the classical graph coloring with k colors, while for $(d, n) = (1, 1)$ we obtain a packing k-coloring.

* Corresponding author.
E-mail address: aleksander.vesel@um.si (A. Vesel).

https://doi.org/10.1016/j.dam.2017.11.036
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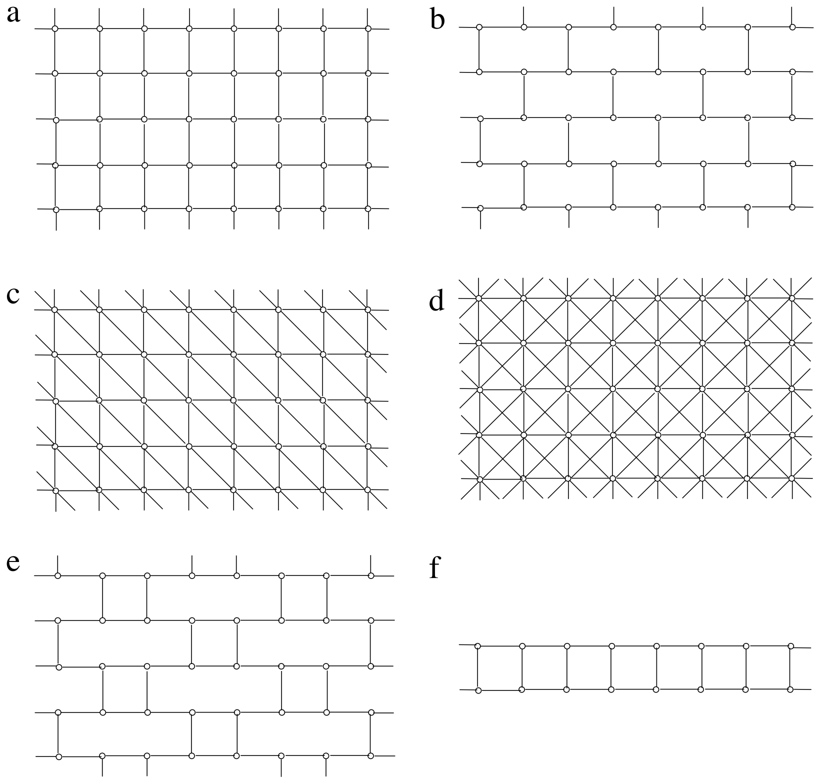

Fig. 1. a. Square lattice $\mathbb{Z}\Box\mathbb{Z}$, b. hexagonal lattice $\mathcal{H}$, c. triangular lattice $\mathcal{T}$, d. eight-regular lattice $\mathbb{Z}\boxtimes\mathbb{Z}$ e. octagonal lattice $\mathcal{O}$ f. two-row square lattice $P_2\Box\mathbb{Z}$.

In a cellular radio system, a land area to be supplied with radio service is often divided into nearly congruent polygonal cells, with each transceiver at the center of the cell it covers. These types of networks, with respect to the mutual position and the power of the transceiver, can be modeled by various infinite regular lattices. The square lattice $\mathbb{Z}\Box\mathbb{Z}$, the hexagonal lattice $\mathcal{H}$, the triangular lattice $\mathcal{T}$, the eight-regular lattice $\mathbb{Z}\boxtimes\mathbb{Z}$, the octagonal lattice $\mathcal{O}$ and the two-row square lattice $P_2\Box\mathbb{Z}$ are depicted in Fig. 1.

The research of the packing chromatic number started by investigating the packing chromatic number of the square lattice. Goddard et al. [5] determined the packing chromatic number for infinite subgraphs of the square lattice with up to five rows. In the same paper, the question of determining the packing chromatic number of the infinite square lattice was posed and bounds on the packing chromatic number of the square lattice were determined. These bounds were later improved by Soukal and Holub [14] and Ekstein et al. [1]. Very recently, Martin et al. [11] set the upper and the lower bound of the packing chromatic number of the square lattice to 15 and 13, respectively. The packing chromatic number of the hexagonal lattice was determined by Fiala et al. [2] and by Vesel and Korže [10]. S-packing chromatic numbers of the square, hexagonal, triangular and two-row square lattices have been studied by Goddard et al. [5], Goddard and Xu [6,7] and Gastineau et al. [4]. The interested reader is also referred to [4] for a detailed review of the related work.

The paper is organized as follows. In the next section we give definitions and results needed in this paper. In Section 3, the main computer search methods applied in the paper are described. Finally, in Section 4, we summarize the results on the (d, n)-packing chromatic number for the lattices of interest. This section also contains results on the density of an i-packing in the eight-regular and octagonal infinite lattice.

2. Preliminaries

The *path* P_n is the graph whose vertices are $0, 1, \ldots, n-1$ and for which two vertices are adjacent precisely if their difference is ± 1. For an integer $n \geq 3$, the *cycle of length* n is the graph C_n whose vertices are $0, 1, \ldots, n-1$ and whose edges are the pairs $i, i+1$, where the arithmetic is done modulo n. A *walk* in a directed graph D is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_n$ such that $v_i v_{i+1} \in A(D)$ for $1, 2, \ldots, n-1$. If $v_1 = v_n$ we say it is a *closed walk*.

The *Cartesian product* of graphs G and H is the graph $G\Box H$ with vertex set $V(G) \times V(H)$ and $(x_1, x_2)(y_1, y_2) \in E(G\Box H)$ whenever $x_1y_1 \in E(G)$ and $x_2 = y_2$, or $x_2y_2 \in E(H)$ and $x_1 = y_1$.

The *strong product* of graphs G and H is the graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$ and $(x_1, x_2)(y_1, y_2) \in E(G \boxtimes H)$ whenever $x_1y_1 \in E(G)$ and $x_2 = y_2$, or $x_2y_2 \in E(H)$ and $x_1 = y_1$, or $x_1y_1 \in E(G)$ and $x_2y_2 \in E(H)$.

The Cartesian product and the strong product are commutative and associative, having the trivial graph as a unit, cf. [8].

As suggested in Fig. 1, the square lattice $\mathbb{Z}\Box\mathbb{Z}$, the hexagonal lattice $\mathcal{H}$, the triangular lattice $\mathcal{T}$, the eight-regular lattice $\mathbb{Z}\boxtimes\mathbb{Z}$ and the octagonal lattice $\mathcal{O}$ can be viewed as graphs with vertex set $\mathbb{Z}\times\mathbb{Z}$, where $\mathbb{Z}$ denotes the two-way infinite path. In particular, the square lattice (resp. the eight-regular lattice) can be viewed as the Cartesian product (resp. strong product) of two 2-way infinite paths. A row (resp. column) of a lattice is its subgraph induced by the set of vertices $\{i\}\times\mathbb{Z}$ (resp. $\mathbb{Z}\times\{i\}$).

Lower bounds on the S-packing chromatic number can be established by the following proposition.

Proposition 1. *If H is a subgraph of G, then $\chi_\rho^S(G) \geq \chi_\rho^S(H)$.*

In order to obtain an upper bound, observe a subgraph of a lattice induced by the intersection of n consecutive rows and m consecutive columns. This subgraph will be denoted by $P_n\Box P_m$, $\mathcal{H}_P^{n,m}$, $\mathcal{T}_P^{n,m}$, $P_n\boxtimes P_m$, and $\mathcal{O}_P^{n,m}$, if the graph of interest is the square, hexagonal, triangular, eight-regular, or octagonal lattice, respectively.

Since $P_n\Box P_m$ (resp. $P_n\boxtimes P_m$) can be viewed as a subgraph of $\mathbb{Z}\Box\mathbb{Z}$ (resp. $\mathbb{Z}\boxtimes\mathbb{Z}$) induced by the intersection of n consecutive rows and m consecutive columns, we may say that the graph $C_n\Box C_m$ (resp. $C_n\boxtimes C_m$) is obtained from $P_n\Box P_m$ (resp. $P_n\boxtimes P_m$) by joining its vertices of the first and the last row as well as of the first and the last column in the natural way. In an analogous manner, we obtain the graphs $\mathcal{H}_C^{2n,2m}$, $\mathcal{T}_C^{n,m}$ and $\mathcal{O}_C^{2n,4m}$ from the graphs $\mathcal{H}_P^{2n,2m}$, $\mathcal{T}_P^{n,m}$ and $\mathcal{O}_P^{2n,4m}$, respectively.

The proof of the following result can be easily obtained.

Proposition 2. *Let $G \in \{C_n\Box C_m, C_n\boxtimes C_m, \mathcal{T}_C^{n,m}\}$ and $\mathcal{L}_G$ the lattice that corresponds to G.*

(i) If $\chi_\rho^S(G) \leq k$ and $s_k \leq n \leq m$, then $\chi_\rho^S(\mathcal{L}_G) \leq \chi_\rho^S(G)$.

(ii) If $\chi_\rho^S(\mathcal{H}_C^{2n,2m}) \leq k$ and $s_k \leq 2n \leq 2m$, then $\chi_\rho^S(\mathcal{H}) \leq \chi_\rho^S(\mathcal{H}_C^{2n,2m})$.

(iii) If $\chi_\rho^S(\mathcal{O}_C^{2n,4m}) \leq k$ and $s_k \leq \min\{2n,4m\}$, then $\chi_\rho^S(\mathcal{O}) \leq \chi_\rho^S(\mathcal{O}_C^{2n,4m})$.

Let $G=(V,E)$ be a graph and let k be a positive integer. For a vertex v of $V(G)$, the *ball of radius k centered at v* is the set $B_k(v)=\{u\in V(G)\mid d_G(u,v)\leq k\}$ and the *sphere of radius k centered at v* is the set $\partial B_k(v)=\{u\in V(G)\mid d_G(u,v)=k\}$. The *density of a set of vertices* $X\subset V(G)$ is $d(X)=\limsup\limits_{l\to\infty}\max\limits_{v\in V(G)}\{\frac{|X\cap B_l(v)|}{|B_l(v)|}\}$.

The notion of k-area was introduced by Fiala et al. [2] and modified by Gastineau et al. [4] as follows.

Let G be a graph, $x\in V(G)$, and let k be a positive integer. For a vertex $u\in V(G)$, let $N(u)$ denote the set of vertices adjacent to u in G. The *k-area* $A(x,k)$ assigned to G is defined by:

$$A(x,k)=\begin{cases}|B_{k/2}(x)|, & k \text{ even}\\ |B_{\lfloor k/2\rfloor}(x)| + \displaystyle\sum_{u\in\partial B_{\lceil k/2\rceil}(x)} \frac{|N(u)\cap B_{\lfloor k/2\rfloor}(x)| + |N(u)\cap\partial B_{\lceil k/2\rceil}(x)|/2}{deg(u)}, & k \text{ odd.}\end{cases}$$

For vertex-transitive graphs, a k-area is denoted by $A(k)$ since it is the same for all vertices.

Lemma 1 ([4]). *Let G be a vertex transitive graph with finite degree and i be a positive integer. If X_i is an i-packing, then*

$$d(X_i)\leq\frac{1}{A(i)}.$$

Lemma 2 ([4]). *Let G be a vertex transitive graph with finite degree. If G has a finite S-packing chromatic number, then*

$$\sum_{i=1}^{\infty}\frac{1}{A(s_i)}\geq 1.$$

The next two corollaries are easy consequences of the above result.

Corollary 1. *Let G be a vertex transitive graph with finite degree. If G has a finite (d,n)-packing chromatic number, then*

$$\sum_{i=1}^{\infty}\frac{1}{A(d+\lfloor\frac{i-1}{n}\rfloor)}\geq 1.$$

Corollary 2. *Let G be a vertex transitive graph with finite degree. If d and n are positive integers, then*

$$\chi_\rho^{d,n}(G)\geq A(d).$$

Note that both of the above corollaries provide a lower bound for the S-packing k-coloring number of a vertex transitive graph.

3. Computing the S-packing coloring problem

3.1. Reducing to SAT instance

The questions of packing coloring for various finite and infinite graphs have been reduced to SAT problems already in [13] and [11].

We consider herein a graph $G = (V, E)$, a positive integer k and $S = (s_1, s_2, \dots, s_k)$. For every $v \in V$ and every $i \in \{1, 2, \dots, k\}$, introduce an atom $x_{v,i}$. Intuitively, this atom expresses that vertex v is assigned color i. We have the following propositional formulas:

$$\vee_{i=1}^{k} x_{v,i} \qquad (v \in V(G)) \tag{1}$$

$$\neg x_{v,i} \vee \neg x_{u,i} \qquad (d(v,u) \le s_i, \; v, u \in V(G),\ 1 \le i \le k). \tag{2}$$

The above propositional formulas transform an S-packing k-coloring problem into a propositional satisfiability test (SAT). We can check that a SAT instance is satisfiable if and only if G has a feasible packing k-coloring with colors $\{1, 2, \dots, k\}$.

We used the SAT-solver Glucose Syrup 4.1 [15] in order to find the solutions of the above clauses.

3.2. Simulated annealing

Soukal and Holub presented a Simulated Annealing algorithm which was used to improve the upper bound on the packing chromatic number of $\mathbb{Z} \times \mathbb{Z}$ to 17 [14]. We adapted this algorithm for the search of the S-packing coloring of the lattices of interest in a natural way, therefore we do not give the details herein. It is worth to mentioning, however, that some of the vertices of the input graph could be precolored before the algorithm begins. The following constants have been used for computations: the initial temperature $t_{max} = 2$, the final temperature $t_{min} = 0.001$, the exponential decrease factor $q = 0.3$ and the number of iterations $k \in \{400, 500\}$. The adjusted algorithm proved to be an effective tool for improving the upper bounds on the S-packing chromatic number. Moreover, the colorings computed by the algorithm have also been used as a basis for the SAT instances. In particular, for some small value $t < k$ the color classes $X_i, \dots, X_t$ of the k-coloring provided by the Simulated Annealing algorithm can be preset to the objects (i.e. partially pre-colored graphs) given to the SAT-solver.

3.3. Dynamic algorithm

The idea is introduced in [9] in a very general framework. In this paper, the concept is narrowed searching for the (d, n)-packing chromatic number of $P_2 \square \mathbb{Z}$.

For given integers d, n and k we set $t = d + \lfloor \frac{k-1}{n} \rfloor$ and define a directed graph $D^k_{d,n}$ as follows. The vertices of $D^k_{d,n}$ are all (d, n)-packing k-colorings of $P_2 \square P_t$. Let u and v be two distinct vertices of $D^k_{n,d}$. Then f_{uv} denotes a k-coloring of $P_2 \square P_{2t}$ such that u and v compose the respective (d, n)-packing k-coloring of the first and the second copy of $P_2 \square P_t$ in $P_2 \square P_{2t}$. Note that f_{uv} need not be a (d, n)-packing k-coloring of $P_2 \square P_{2t}$. However, uv is an arc in $D^k_{d,n}$ if and only if f_{uv} is a (d, n)-packing k-coloring of $P_2 \square P_{2t}$.

The following result is the basis for the dynamic computation of the (d, n)-packing chromatic number of $P_2 \square \mathbb{Z}$.

Lemma 3. *Let n, d, and k be integers. Then $P_2 \square \mathbb{Z}$ admits a (d, n)-packing k-coloring if and only if $D^k_{d,n}$ contains a closed directed walk.*

Proof. Let $t = d + \lfloor \frac{k-1}{n} \rfloor$. Suppose first that $D^k_{d,n}$ contains a closed directed walk P. Let u, v and w be consecutive vertices of P. It follows that f_{uv} and f_{vw} are both (d, n)-packing k-colorings of $P_2 \square P_{2t}$. Let f_{uvw} be a k-coloring of $P_2 \square P_{3t}$ such that u, v and w compose the respective (d, n)-packing k-colorings of the consecutive copies of $P_2 \square P_t$ in $P_2 \square P_{3t}$. Let x (resp. y) be a vertex of a copy of $P_2 \square P_t$ that corresponds to u (resp. w). Since the distance between x and y in the graph $P_2 \square P_{3t}$ that corresponds to f_{uvw} is at least $t + 1$, f_{uvw} is a (d, n)-packing k-coloring of $P_2 \square P_{3t}$ and this case is settled.

Assume now that $P_2 \square \mathbb{Z}$ admits a (d, n)-packing k-coloring denoted by f. Let G_ℓ, $\ell \ge 1$, be a copy of $P_2 \square P_{\ell t}$ in $P_2 \square \mathbb{Z}$. By the definition of $D^k_{d,n}$, the restriction of f to G_ℓ corresponds to a walk of length ℓ in $D^k_{d,n}$. Since ℓ can be chosen arbitrarily and the cardinality of $V(D^k_{d,n})$ is finite, $D^k_{d,n}$ cannot be acyclic. This completes the proof. $\square$

4. Results

The results of this section, (d, n)-packing chromatic numbers and bounds for lattices, are mostly obtained by extensive computations which are based on the methods presented in Section 3. When the existence (resp. nonexistence) of a S-packing k-coloring of a graph from the set $\{C_n \square C_m, C_n \boxtimes C_m, \mathcal{H}_C^{2n,2m}, \mathcal{T}_C^{n,m}, \mathcal{O}_C^{2n,4m}\}$ (resp. $\{P_n \square P_m, P_n \boxtimes P_m, \mathcal{H}_P^{2n,2m}, \mathcal{T}_P^{n,m}, \mathcal{O}_P^{2n,4m}\}$) is established, Proposition 2 (resp. Proposition 1) is applied in order to obtain an upper (resp. lower) bound on the S-packing k-chromatic number of the corresponding lattice.

In addition, we applied Corollary 1 or Corollary 2 (for $n \ge A(d)$) to obtain lower bounds.

The obtained colorings as well as the SAT clauses used in the proofs herein can be obtained by the authors upon request or at the web page http://omr.fnm.um.si/wp-content/uploads/2017/06/AllLattices.pdf.

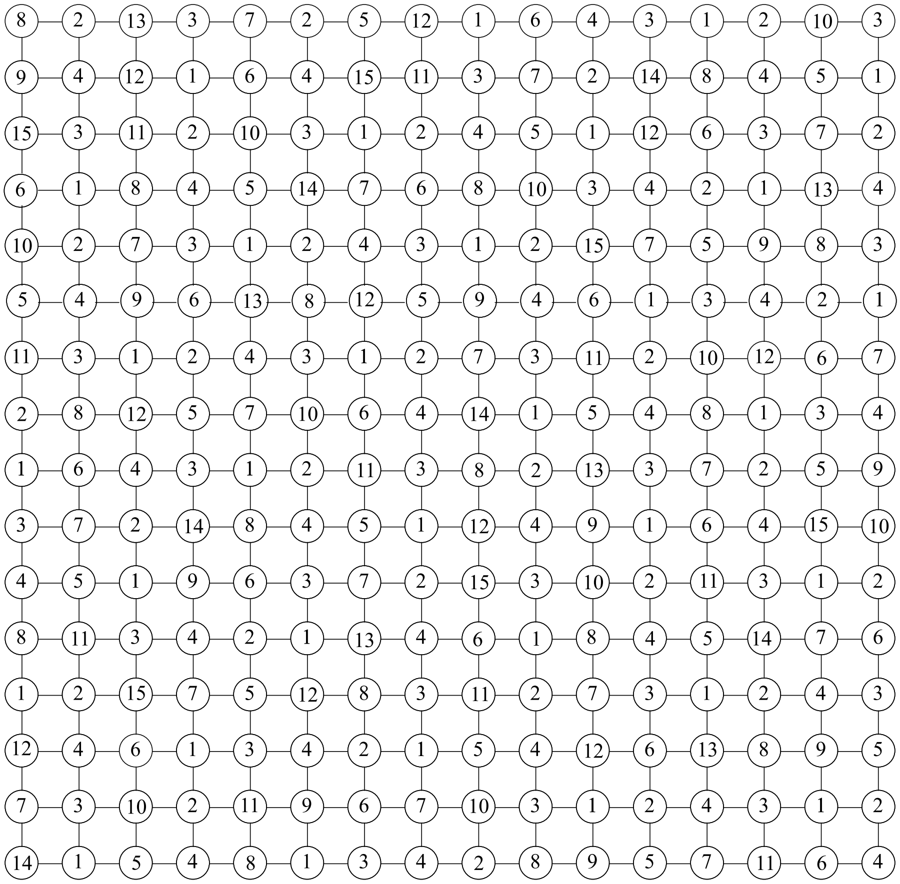

Fig. 2. A $(3, 4)$-packing 15-coloring of $C_{16}\Box C_{16}$.

4.1. The infinite square lattice

We need the following result from [4].

Proposition 3. *Let $A(k)$ denote the k-area of $\mathbb{Z}\Box\mathbb{Z}$. If $k \geq 0$, then*
(i) $A(2k) = 2k^2 + 2k + 1$,
(ii) $A(2k - 1) = 2k^2$.

The k-areas of interest are: $A(1) = 2, A(2) = 5, A(3) = 8, A(4) = 13, A(5) = 18$ and $A(6) = 25$.

All known results on (d, n)-packing chromatic number of the square lattice are summarized in Table 1.

We showed with our SAT solver that the following colorings cannot be obtained: a $(2, 2)$-packing 11-coloring of $P_{12}\Box P_{12}$, a $(2, 3)$-packing 7-coloring of $P_{12}\Box P_{12}$, and a $(3, 5)$-packing 10-coloring of $P_{12}\Box P_{12}$. These results give the corresponding lower bounds on the $(2, 2)$-packing, $(2, 3)$-packing and $(3, 5)$-packing coloring number.

The SAT solver also provided the following colorings: a $(3, 4)$-packing 15-coloring of $C_{16}\Box C_{16}$ (see Fig. 2), a $(3, 5)$-packing 12-coloring of $C_{12}\Box C_{12}$, and a $(3, 8)$-packing 8-coloring of $C_{16}\Box C_{16}$.

The other new upper bounds were obtained by using the Simulated Annealing algorithm. In particular, we obtained the following constructions: a $(3, 3)$-packing 26-coloring of $C_{20}\Box C_{20}$, a $(3, 5)$-packing 12-coloring of $C_{12}\Box C_{12}$, a $(3, 6)$-packing 10-coloring of $C_{40}\Box C_{40}$, a $(4, 4)$-packing 45-coloring of $C_{26}\Box C_{26}$, a $(4, 5)$-packing 31-coloring of $C_{24}\Box C_{24}$, a $(4, 6)$-packing 26-coloring of $C_{24}\Box C_{24}$, a $(4, 13)$-packing 13-coloring of $C_{13}\Box C_{13}$, a $(5, 5)$-packing 59-coloring of $C_{24}\Box C_{24}$, a $(5, 6)$-packing 48-coloring of $C_{24}\Box C_{24}$, a $(5, 18)$-packing 18-coloring of $C_{18}\Box C_{18}$, and a $(6, 25)$-packing 25-coloring of $C_{25}\Box C_{25}$.

4.2. The infinite hexagonal lattice

The k-areas of the infinite hexagonal lattice are obtained in [4]

Table 1
Values and bounds on $\chi_\rho^{d,n}(\mathbb{Z}\square\mathbb{Z})$.

d	n						$n \geq A(d)$
	1	2	3	4	5	6	
1	13[a]–15[a]	2	2	2	2	2	2
2	∞	**12**–20	**8**	6[b]	5[b]	5	5
3	∞	∞	16-**26**	12–**15**	**11–12**	**10**	**8**
4	∞	∞	44–?	25–**45**	20–**31**	18–**26**	**13**
5	∞	∞	199–?	50–?	35–**59**	29–**48**	**18**
6	∞	∞	∞	?	?	?	**25**

Key to Table 1.
Regular font. Obtained in [4].
Bold font. New results obtained by computer search methods.
[a] Obtained in [11].
[b] Obtained in [7].

Table 2
Computed values for $\chi_\rho^{d,n}(\mathcal{H})$.

d	n						$n \geq A(d)$
	1	2	3	4	5	6	
1	7[a]	2	2	2	2	2	2
2	∞	8[b]	5	4	4	4	4
3	∞	15–**32**	**11**[b]	8[b]	8[b]	6	6
4	∞	61–?	20–**46**	15–**25**	13–**20**	12–**17**	**11**
5	∞	∞	40–?	25–**49**	21–**37**	19–**31**	**14**
6	∞	∞	?	?	?	?	19–**20**
7	∞	∞	?	?	?	?	**24**

Key to Table 2.
Regular font. Obtained in [4].
Bold font. New results obtained by computer search methods.
[a] Obtained in [2,10].
[b] Obtained by Shao [12].

Proposition 4. *Let $A(k)$ denote the k-area of $\mathcal{H}$. If $k \geq 1$, then*
(i) $A(2k) = \frac{3}{2}k^2 + \frac{3}{2}k + 1$,
(ii) $A(4k-3) = 6k^2 - 6k + 2$,
(iii) $A(4k-1) = 6k^2$.

The k-areas of interest are: $A(1) = 2, A(2) = 4, A(3) = 6, A(4) = 10, A(5) = 14, A(6) = 19$ and $A(7) = 24$.

All known results on (d, n)-packing chromatic number of the hexagonal lattice are summarized in Table 2. The bold numbers represent our new results.

We showed with the SAT solver that the following colorings cannot be obtained: a $(2, 3)$-packing 7-coloring of $\mathcal{H}_P^{12,12}$, a $(3, 3)$-packing 10-coloring of $\mathcal{H}_P^{8,8}$, a $(3, 5)$-packing 12-coloring of $\mathcal{H}_P^{8,8}$, and a $(4, 11)$-packing 10-coloring of $\mathcal{H}_P^{22,22}$.

In order to provide upper bounds, we found: a $(3, 2)$-packing 32-coloring of $\mathcal{H}_C^{24,24}$, a $(3, 3)$-packing 11-coloring of $\mathcal{H}_C^{24,24}$, a $(3, 4)$-packing 8-coloring of $\mathcal{H}_C^{24,24}$, a $(3, 5)$-packing 8-coloring of $\mathcal{H}_C^{24,24}$, a $(4, 3)$-packing 46-coloring of $\mathcal{H}_C^{24,24}$, a $(4, 4)$-packing 25-coloring of $\mathcal{H}_C^{20,20}$, a $(4, 5)$-packing 20-coloring of $\mathcal{H}_C^{20,20}$, a $(4, 6)$-packing 17-coloring of $\mathcal{H}_P^{12,12}$, a $(4, 11)$-packing 11-coloring of $\mathcal{H}_P^{22,22}$, a $(5, 4)$-packing 49-coloring of $\mathcal{H}_P^{24,24}$, a $(5, 5)$-packing 37-coloring of $\mathcal{H}_P^{24,24}$, a $(5, 6)$-packing 31-coloring of $\mathcal{H}_P^{24,24}$, a $(5, 14)$-packing 14-coloring of $\mathcal{H}_P^{14,14}$, a $(6, 20)$-packing 20-coloring of $\mathcal{H}_P^{20,20}$, and a $(7, 24)$-packing 24-coloring of $\mathcal{H}_P^{24,24}$. Most of the above results were obtained as a combination of the SAT reduction and Simulated Annealing approach.

4.3. *The infinite triangular lattice*

In order to obtain lower bounds of $\chi_\rho^{n,d}(\mathcal{T})$ for $n \geq A(d)$, we need the following result on k-areas of the infinite triangular lattice provided in [4].

Proposition 5. *Let $A(k)$ denote a k-area of $\mathcal{T}$. If $k \geq 1$, then*
(i) $A(2k) = 3k^2 + 3k + 1$,
(ii) $A(2k-1) = 3k^2$.

We need the following k-areas: $A(1) = 3, A(2) = 7, A(3) = 12, A(4) = 19, A(5) = 27$, and $A(6) = 37$.

Table 3
Computed values for $\chi_\rho^{d,n}(\mathcal{T})$.

d	n						$n \geq A(d)$
	1	2	3	4	5	6	
1	∞[a]	**6**	3	3	3	3	3
2	∞	127–?	14–**29**	**12–14**	**10–11**	**9**	**7**
3	∞	∞	81–?	28–**65**	20–**34**	17–**24**	**12**
4	∞	∞	∞	104–?	49–?	36–?	**19**
5	∞	∞	∞	∞	?	?	**27**

Key to Table 3.
Regular font. Obtained in [4].
Bold font. New results obtained by computer search methods.
[a] Obtained in [3].

Table 4
Computed values for $\chi_\rho^{d,n}(\mathbb{Z} \boxtimes \mathbb{Z})$.

d	n								$n \geq (d+1)^2$
	1	2	3	4	5	6	7	8	
1	∞	11–**21**	**6**	**4**	**4**	**4**	**4**	**4**	**4**
2	∞	∞	42–?	18–**31**	14–**19**	**12**	**11**	**11**	**9**
3	∞	∞	∞	105–?	43–?	31–?	26–**59**	23–**45**	**16**
4	∞	∞	∞	∞	213–?	83–?	58–?	48–?	**25**
5	∞	∞	∞	∞	∞	377–?	144–?	99–?	**36**
6	∞	∞	∞	∞	∞	∞	610–?	229–?	**49**

Key to Table 4.
Regular font. Obtained by Proposition 7 and Corollary 1.
Bold font. Obtained by computer search methods.

The results on the (d, n)-packing chromatic number of the triangular lattice are summarized in Table 3. The bold numbers represent new results.

We showed with the SAT solver that the following colorings cannot be obtained: a $(1, 2)$-packing 5-coloring of $\mathcal{T}_P^{12,12}$, a $(2, 4)$-packing 11-coloring of $\mathcal{T}_P^{14,14}$, a $(2, 5)$-packing 9-coloring of $\mathcal{T}_P^{12,12}$, and a $(2, 6)$-packing 8-coloring of $\mathcal{T}_P^{12,12}$.

The upper bounds are provided by the following constructions: a $(2, 3)$-packing 29-coloring of $\mathcal{T}_C^{28,28}$, a $(2, 4)$-packing 14-coloring of $\mathcal{T}_C^{14,14}$, a $(2, 5)$-packing 11-coloring of $\mathcal{T}_C^{14,14}$, a $(2, 6)$-packing 9-coloring of $\mathcal{T}_C^{22,22}$, a $(2, 7)$-packing 7-coloring of $\mathcal{T}_C^{14,14}$, a $(3, 4)$-packing 65-coloring of $\mathcal{T}_C^{24,24}$, a $(3, 5)$-packing 34-coloring of $\mathcal{T}_C^{24,24}$, a $(3, 6)$-packing 24-coloring of $\mathcal{T}_C^{24,24}$, a $(3, 12)$-packing 12-coloring of $\mathcal{T}_C^{12,12}$, a $(4, 19)$-packing 19-coloring of $\mathcal{T}_C^{19,19}$, and a $(5, 27)$-packing 27-coloring of $\mathcal{T}_C^{27,27}$.

4.4. The infinite eight-regular lattice

The proof of the following result can be easily obtained.

Proposition 6. *Let v be a vertex in the eight-regular lattice $\mathbb{Z} \boxtimes \mathbb{Z}$ and $k \geq 1$. Then*
(i) $|\partial B_k(v)| = 8k$,
(ii) $|B_k(v)| = (2k+1)^2$.

Proposition 7. *Let $A(k)$ denote a k-area of $\mathbb{Z} \boxtimes \mathbb{Z}$. If $k \geq 1$, then*

$$A(k) = (k+1)^2.$$

Proof. If k is even, then the assertion follows from Proposition 6.

Let $k = 2\ell + 1$. By Proposition 6, for an arbitrary vertex v of $\mathbb{Z} \boxtimes \mathbb{Z}$ we have $|\partial B_{\ell+1}(v)| = 8\ell + 8$. By the definition of $A(k)$, we have to observe the vertices of $\partial B_{\ell+1}(v)$. Note that $\partial B_{\ell+1}(v)$ admits 4 vertices with one neighbor in $B_\ell(v)$ and $8\ell + 4$ vertices u with three neighbors in $B_\ell(v)$. It follows that the contribution of these vertices with respect to the vertices of $B_\ell(v)$ is $(3(8\ell+4)+4)/8 = 3\ell + 2$. Since every vertex of $B_{\ell+1}(v)$ admits exactly two neighbors in $B_{\ell+1}(v)$, the contribution of these vertices with respect to the vertices of $B_{\ell+1}(v)$ is $2(8\ell+8)/2/8 = \ell + 1$. We finally obtain $A(2\ell+1) = (2\ell+1)^2 + 3\ell + 2 + \ell + 1 = (2\ell+2)^2$. □

Almost all lower bounds from Table 4 follows from Proposition 7 and Corollaries 1 and 2. The exception is provided by our SAT solver, which determined that a $(2, 8)$-packing 10-coloring of $P_{12} \boxtimes P_{12}$ cannot be obtained.

We found the following colorings which provide the upper bounds: a $(1, 2)$-packing 21-coloring of $C_{48} \boxtimes C_{48}$, a $(1, 3)$-packing 6-coloring of $C_{12} \boxtimes C_{12}$, a $(1, 4)$-packing 4-coloring of $C_{18} \boxtimes C_{18}$, a $(2, 4)$-packing 31-coloring of $C_{24} \boxtimes C_{24}$, a $(2, 5)$-packing 19-coloring of $C_{18} \boxtimes C_{18}$, a $(2, 6)$-packing 12-coloring of $C_{12} \boxtimes C_{12}$, a $(2, 7)$-packing 11-coloring of $C_{12} \boxtimes C_{12}$,

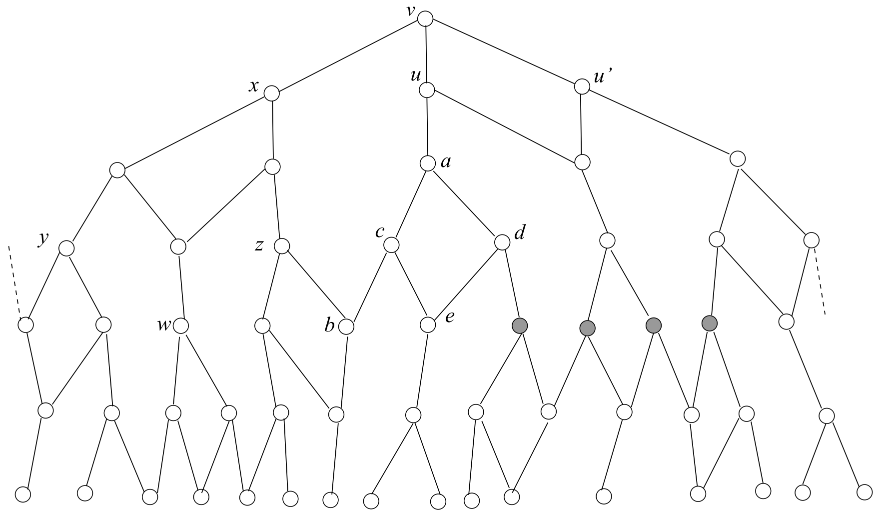

Fig. 3. Octagonal lattice.

a (2, 9)-packing 9-coloring of $C_{12} \boxtimes C_{12}$, a (3, 7)-packing 59-coloring of $C_{18} \boxtimes C_{18}$, a (3, 8)-packing 45-coloring of $C_{18} \boxtimes C_{18}$, a (3, 9)-packing 39-coloring of $C_{18} \boxtimes C_{18}$, a (3, 16)-packing 16-coloring of $C_{16} \boxtimes C_{16}$, a (4, 25)-packing 25-coloring of $C_{25} \boxtimes C_{25}$.

Note that if $n \geq (d+1)^2$, it is not difficult to construct a (d, n)-packing $(d+1)^2$-coloring of $\mathbb{Z} \boxtimes \mathbb{Z}$. Thus, by Corollary 2 and Proposition 7, for every $n \geq (d+1)^2$ we have $\chi_\rho^{d,n}(\mathbb{Z} \boxtimes \mathbb{Z}) = (d+1)^2$.

4.5. The infinite octagonal lattice

In the first part of this subsection we present results on k-areas of the infinite octagonal lattice.

Let v be a vertex of the octagonal lattice $\mathcal{O}$. If $x \in \partial B_k(v)$ admits two neighbors, say y and z, in $\partial B_{k+1}(v)$ such that x, y, z lie in a common 4-cycle (resp. 8-cycle), then we say that a 4-cycle (resp. 8-cycle) *is started in x*. Note, for example, that two 8-cycles and one 4-cycle are started in v. Analogously, if $x \in \partial B_{k+1}(v)$ admits two neighbors, say y and z, in $\partial B_k(v)$ such that x, y, z lie in a common 4-cycle (resp. 8-cycle), then we say that a 4-cycle (resp. 8-cycle) *is finished in x*.

Let h_i and f_i denote the number of 8-cycles and 4-cycles which are started in $\partial B_i(v)$, respectively. Note that $h_0 = 2$ and $f_0 = 1$.

Proposition 8. *If* $i \geq 0$, *then*

(i) $h_i = 2\lfloor \frac{i}{3} \rfloor + 2$

(ii) $f_i = \begin{cases} 2\lfloor \frac{i}{3} \rfloor + 2, & i \equiv 2 \pmod 3, \\ 2\lfloor \frac{i}{3} \rfloor + 1, & \text{otherwise.} \end{cases}$

Proof. We show first that for $i \geq 5$ the following recurrence relations hold:
$f_i = 2f_{i-2} + f_{i-3} - h_{i-4} - h_{i-5}$ and
$h_i = 2f_{i-1} - h_{i-4}$.

In order to establish the relation for f_i, observe first that one 4-cycle is started in $x \in \partial B_i(v)$. We can see that two vertices of $\partial B_{i+1}(v)$ adjacent to x have together three neighbors in $\partial B_{i+2}(v)$. Note that at most two of them can start a 4-cycle, while the third one has exactly one neighbor in $\partial B_{i+3}(v)$ and this neighbor starts a 4-cycle. As an example, observe the vertex x in Fig. 3. Since x starts a 4-cycle in $\partial B_1(v)$, two 4-cycles are started in $\partial B_3(v)$ (in y and z), furthermore, one 4-cycle is started in $w \in \partial B_4(v)$. If the described situation occurs, we say that x *makes* two 4-cycles in $\partial B_3(v)$ and one in $\partial B_4(v)$. It follows that the value of f_i is bounded above by $2f_{i-2} + f_{i-3}$. In order to see that the obtained sum has to be subtracted by the value of $h_{i-4} + h_{i-5}$, observe the vertex b in Fig. 3. Note that b finishes an 8-cycle started in v, say C. Obviously, b does not start a 4-cycle. In other words, the vertex a makes only one 4-cycle in $\partial B_4(v)$. Moreover, the 8-cycle C causes that the vertex z makes only one 4-cycle in $\partial B_5(v)$. By this observation, the recurrence for f_i is settled.

The proof to establish the relation for h_i is analogous. If a 4-cycle is started in a vertex of $\partial B_i(v)$, then it can make two 8-cycles in $\partial B_{i+1}(v)$ (see for example the vertex x in Fig. 3). Thus, the value of h_i is bounded above by $2f_{i-1}$. This value has to be subtracted by the value of h_{i-4} as can be shown in Fig. 3, where the vertex z makes only one 8-cycle in $\partial B_4(v)$, since the vertex b finishes the 8-cycle C started in v.

The rest of the proof is by induction on i. The values of h_i and f_i, $i = 0, 1, 2, 3, 4$, can be easily established (e.g. from Fig. 3). Let then $i = 3k - 1$, $k \geq 2$, and suppose that the proposition holds for the values smaller than i. We have $f_{3k-1} = 2f_{3(k-1)} + f_{3(k-1)-1} - h_{3(k-1)-2} - h_{3(k-2)}$ and $h_{3k-1} = 2f_{3k-2} - h_{3(k-1)-1}$. By the inductive hypothesis we get $f_{3k-1} = 4(k-1) + 2 + 2(k-2) + 1 - 2(k-2) - 2 - 2(k-2) - 2 = 2k = 2(k-1) + 2$ and $h_{3k-1} = 2k = 2(k-1) + 2$. The proof for $i = 3k$ and $i = 3k + 1$ is analogous. □

For $i \geq 0$ set $b_i := |\partial B_i(v)|$.

Proposition 9. *If $i \geq 0$, then*

$$|\partial B_i(v)| = \begin{cases} 8\dfrac{i}{3}, & i \equiv 0 \pmod 3, \\ 8\dfrac{i+1}{3} - 3, & i \equiv 2 \pmod 3, \\ 8\dfrac{i+2}{3} - 5, & i \equiv 1 \pmod 3. \end{cases}$$

Proof. We first show that for $i \geq 5$ we have

$$b_i = 2b_{i-1} - f_{i-2} - f_{i-3} - h_{i-4} - h_{i-5}.$$

We can easily deduce (see also Fig. 3) that for $i \geq 1$ every vertex of $\partial B_i(v)$ admits at most two neighbors in $\partial B_{i+1}(v)$ (this contributes $2b_{i-1}$ to the value of b_i). A vertex of $\partial B_i(v)$ admits exactly one neighbor in $\partial B_{i+1}(v)$ if and only if it finishes a 4-cycle or an 8-cycle, see for example vertices b and e in Fig. 3 (this contributes $-f_{i-3}$ and $-h_{i-5}$ to the value of b_i). Moreover, the two vertices of $\partial B_i(v)$ that lie on a common 4-cycle, have together exactly three neighbors in $\partial B_{i+1}(v)$, see for example vertices c and d in Fig. 3 (this contributes $-f_{i-2}$ to the value of b_i). Finally, observe again a pair vertices of $\partial B_i(v)$ that lie on a common 4-cycle (see for example vertices u and u' in Fig. 3). Note that two 8-cycles, say C and C', are started in such a pair. The four vertices of C and C' which are adjacent to the vertices that end C and C' (see gray vertices in Fig. 3) have altogether five neighbors in $\partial B_{i+1}(v)$ (this contributes $-h_{i-4}$ to the value of b_i).

The rest of the proof is by induction on i. The values of b_i for $i = 0, 1, 2, 3, 4$ can be easily established (e.g. from Fig. 3). Let then $i = 3k - 1$, $k \geq 2$, and suppose that the proposition holds for the values smaller than i. We have $b_{3k-1} = 2b_{3k-2} - f_{3(k-1)} - f_{3(k-1)-1} - h_{3(k-1)-2} - h_{3(k-2)} = 2(8k-5) - 2(k-1) - 1 - 2(k-2) - 2 - 2(k-2) - 2 - 2(k-2) - 2 = 8k - 3$. The proof for $i = 3k$ and $i = 3k + 1$ is analogous. □

Let for an integer k define

$$k_r := \begin{cases} 0, & k \equiv 0 \pmod 3 \\ 3, & k \equiv 1 \pmod 3 \\ 5, & i \equiv 2 \pmod 3 \end{cases}$$

By Proposition 9, we have $b_i = 8\lfloor \frac{i}{3} \rfloor + i_r$ for $i \geq 3$.

Corollary 3. *Let i and k be integers. Then*

$$b_{i+k} = \begin{cases} b_i + b_k + 1, & i \equiv 2 \pmod 3 \text{ and } k \equiv 2 \pmod 3 \\ b_i + b_k - 1, & i \equiv 1 \pmod 3 \text{ and } k \equiv 1 \pmod 3 \\ b_i + b_k, & \text{otherwise}. \end{cases}$$

Proof. Note that $b_{i+k} = 8\lfloor \frac{i+k}{3} \rfloor + (i+k)_r$. If $i \equiv 2 \pmod 3$ and $k \equiv 2 \pmod 3$, then $b_{i+k} = 8\lfloor \frac{i+k}{3} \rfloor + 3 = 8\lfloor \frac{i}{3} \rfloor + 8\lfloor \frac{k}{3} \rfloor + 8 + 3 = b_i + b_k + 1$.

The proof for other cases is analogous. □

Proposition 10. *If $k \geq 1$, then*

(i) $b_{\binom{3k}{2}} = 12k^2 - 4k$,

(ii) $b_{\binom{3k-1}{2}} = 12k^2 - 12k + 3$,

(iii) $b_{\binom{3k-2}{2}} = 12k^2 - 20k + 8$.

Proof. (i) If $k = 2t$, then $b_{\binom{6t}{2}} = b_{3t(6t-1)} = 8t(6t-1) = 12k^2 - 4k$.

If $k = 2t - 1$, then $b_{\binom{6t-3}{2}} = b_{3(2t-1)(3t-1)} = 8(2t-1)(3t-1) = 12k^2 - 4k$.

The proof for (ii) and (iii) is analogous. □

Proposition 11. *Let v be a vertex in the octagonal lattice $\mathcal{O}$ and $k \geq 1$. Then*

$$|B_k(v)| = b_{\binom{k+1}{2}} + 1.$$

Proof. We have to prove that $b_1 + b_2 + \cdots + b_k = b_{\binom{k+1}{2}}$. The proof is by induction on k. It can be easily verified that the claim holds for $k = 1$. Let $k \geq 2$ and assume now that the claim holds for $k-1$. By the inductive hypothesis, $b_1 + b_2 + \cdots + b_k = b_{\binom{k}{2}} + b_k$. From Corollary 3 immediately follows that $b_{\binom{k}{2}} + b_k = b_{\binom{k+1}{2}}$. □

Let v be a vertex of the octagonal lattice $\mathcal{O}$. Then e_i, $i \geq 1$, denotes the number of edges between the vertices of $\partial B_{i-1}(v)$ and $\partial B_i(v)$.

Proposition 12. *If $i \geq 1$, then*

$$e_i = 3\lfloor \frac{4i-1}{3} \rfloor.$$

Proof. We claim that

$$e_i = \begin{cases} 4i-3, & i \equiv 0 \,(\text{mod } 3) \\ 4i-2 & i \equiv 2 \,(\text{mod } 3) \\ 4i-1, & i \equiv 1 \,(\text{mod } 3). \end{cases}$$

The proof is by induction on i. It is easy to see that the proposition holds for $i \leq 2$. Let then $i \geq 3$.

Since $e_i + e_{i+1} = 3b_i$, we have $e_{i+1} - e_{i-1} = 3(b_i - b_{i-1})$. If $i \equiv 0 \,(\text{mod } 3)$, then by the inductive hypothesis we have

$$e_{i+1} = 4i - 6 + 9 = 4i + 3.$$

If $i \equiv 1 \,(\text{mod } 3)$ or $i \equiv 2 \,(\text{mod } 3)$, the proof is analogous. □

Proposition 13. *Let $A(k)$ denote a k-area of $\mathcal{O}$. If $k \geq 1$, then*
(i) $A(2k) = b_{\binom{k+1}{2}} + 1$
(ii) $A(2k-1) = b_{\binom{k}{2}} + 1 + \frac{e_k}{3}$.

Proof. For (i) use the definition of $A(k)$ and Proposition 11.

For (ii) note that since $\mathcal{O}$ is bipartite, the set of edges of the subgraph of $\mathcal{O}$ induced by the vertices of $B_k(v)$ is empty. It follows that $|N(u) \cap \partial B_k(v)| = 0$ for every $u \in B_k(v)$. Since

$$\sum_{u \in \partial B_k(v)} \frac{|N(u) \cap B_{k-1}(v)|}{deg(u)} = \frac{e_k}{3},$$

the assertion follows. □

Corollary 4. *Let $k \geq 1$. Then*
(i) $A(6k) = 12k^2 + 4k + 1$,
(ii) $A(6k-1) = 12k^2$,
(iii) $A(6k-2) = 12k^2 - 4k + 1$,
(iv) $A(6k-3) = 12k^2 - 8k + 2$,
(v) $A(6k-4) = 12k^2 - 12k + 4$,
(vi) $A(6k-5) = 12k^2 - 16k + 6$.

The following k-areas are of interest: $A(1) = 2$, $A(2) = 4$, $A(3) = 6$, $A(4) = 9$, $A(5) = 12$, $A(6) = 17$ and $A(7) = 22$.

The results on (d, n)-packing chromatic number of the octagonal lattice are summarized in Table 5.

Most of the lower bounds from Table 5 follow from Corollaries 1, 2 and 4. The others are provided with our SAT solver which established that the following colorings cannot be obtained: a packing 6-coloring of $\mathcal{O}_P^{12,12}$, a (2,2)-packing 7-coloring of $\mathcal{O}_P^{20,20}$, a (2,3)-packing 4-coloring of $\mathcal{O}_P^{16,16}$, a (2,4)-packing 3-coloring of $\mathcal{O}_P^{16,16}$, a (3,3)-packing 10-coloring of $\mathcal{O}_P^{18,18}$, a (3,4)-packing 7-coloring of $\mathcal{O}_P^{16,16}$, a (3,5)-packing 7-coloring of $\mathcal{O}_P^{16,16}$, a (3,6)-packing 5-coloring of $\mathcal{O}_P^{16,16}$, and a (4,10)-packing 9-coloring of $\mathcal{O}_P^{20,20}$.

The upper bounds are given by the following constructions: a packing 7-coloring of $\mathcal{O}_C^{12,12}$, a (1,2)-packing 2-coloring of $\mathcal{O}_C^{12,12}$, a (2,2)-packing 8-coloring of $\mathcal{O}_C^{12,12}$, a (2,3)-packing 5-coloring of $\mathcal{O}_C^{12,12}$, a (2,4)-packing 4-coloring of $\mathcal{O}_C^{12,12}$, a (3,2)-packing 31-coloring of $\mathcal{O}_C^{20,20}$, a (3,3)-packing 11-coloring of $\mathcal{O}_C^{12,12}$, a (3,4)-packing 8-coloring of $\mathcal{O}_C^{12,12}$, a (3,6)-packing 6-coloring of $\mathcal{O}_C^{12,12}$, a (4,3)-packing 35-coloring of $\mathcal{O}_C^{28,28}$, a (4,4)-packing 20-coloring of $\mathcal{O}_C^{20,20}$, a (4,5)-packing 17-coloring of $\mathcal{O}_C^{20,20}$, a (4,6)-packing 15-coloring of $\mathcal{O}_C^{20,20}$, a (4,10)-packing 10-coloring of $\mathcal{O}_C^{20,20}$, a (5,4)-packing 38-coloring of $\mathcal{O}_C^{24,24}$, a (5,5)-packing 29-coloring of $\mathcal{O}_C^{28,28}$, a (5,6)-packing 25-coloring of $\mathcal{O}_C^{28,28}$, a (5,14)-packing 14-coloring of $\mathcal{O}_C^{28,28}$, a (6,19)-packing 19-coloring of $\mathcal{O}_C^{76,76}$, and a (7,24)-packing 24-coloring of $\mathcal{O}_C^{12,12}$.

Table 5
Computed values for $\chi_\rho^{d,n}(\mathcal{O})$.

d	n						$n \geq A(d)$
	1	2	3	4	5	6	
1	**7**	**2**	**2**	**2**	**2**	**2**	**2**
2	∞	**8**	**5**	**4**	**4**	**4**	**4**
3	∞	12–**31**	**11**	**8**	**8**	**6**	**6**
4	∞	32–?	15–**35**	12–**20**	11–**17**	10–**15**	**10**
5	∞	∞	28–?	20–**38**	17–**29**	16–**25**	12–**14**
6	∞	∞	?	?	?	?	17–**19**
7	∞	∞	?	?	?	?	22–**24**

Key to Table 5.
Regular font. Obtained by Corollaries 1 and 4.
Bold font. Obtained by computer search methods.

Table 6
Computed values for $\chi_\rho^{d,n}(\mathbb{Z}\square P_2)$.

d	n						$n \geq 2d$
	1	2	3	4	5	6	
1	5[a]	2[a]	2[a]	2[a]	2[a]	2[a]	2[a]
2	**12**	**6**	**5**	4[a]	4[a]	4[a]	4[a]
3	17–**19**	**10**	**8**	**7**	**7**	**6**	**6**
4	23–**27**	13–**16**	**11**	**10**	**9**	**9**	**8**
5	29–**34**	16–**19**	13–**15**	12–**14**	12–**13**	11–**12**	**10**
6	36–**42**	19–**23**	16–**19**	15–**17**	14–**16**	14–**15**	**12**

Key to Table 6.
Regular font. Obtained by Corollary 1 and Proposition 14.
Bold font. Obtained by computer search methods.
[a] Obtained in [7].

4.6. The infinite two-row lattice

It is easy to see that the following holds, e.g. [7].

Proposition 14. *Let $A(k)$ denote the k-area of $\mathbb{Z}\square P_2$. If $k \geq 0$, then $A(k) = 2k$.*

The results on (d, n)-packing chromatic number of the infinite two-row lattice are given in Table 6. Note that the result for $(d, n) = (1, 1)$ has been presented already in [5]. Since a (1,2)-packing 2-coloring of $C_2\square P_2$ is trivial to obtain, we show for the rest of the values that they are obtained as described below.

The lower bounds are mostly from Proposition 14 and Corollaries 1 and 2. Two exceptions are given with our SAT solver which established that a (2,1)-packing 11-coloring of $P_2\square P_{52}$ and a (3,2)-packing 9-coloring of $P_2\square P_{120}$ cannot be obtained.

In order to provide the upper bounds, we found the following constructions with a combination of SAT reduction and Simulated Annealing: a (2,1)-packing 12-coloring of $P_2\square C_{40}$, a (2,2)-packing 6-coloring of $P_2\square C_{16}$, a (2,3)-packing 5-coloring of $P_2\square C_{12}$, a (2,4)-packing 4-coloring of $P_2\square C_4$, a (3,1)-packing 19-coloring of $P_2\square C_{120}$, a (3,2)-packing 10-coloring of $P_2\square C_{120}$, a (3,3)-packing 8-coloring of $P_2\square C_{100}$, a (3,6)-packing 6-coloring of $P_2\square C_6$, a (4,1)-packing 27-coloring of $P_2\square C_{160}$, a (4,3)-packing 11-coloring of $P_2\square C_{144}$, a (4,4)-packing 10-coloring of $P_2\square C_{160}$, a (4,8)-packing 8-coloring of $P_2\square C_8$, a (5,1)-packing 34-coloring of $P_2\square C_{200}$, a (5,2)-packing 19-coloring of $P_2\square C_{120}$, a (5,3)-packing 15-coloring of $P_2\square C_{120}$, a (5,4)-packing 14-coloring of $P_2\square C_{120}$, a (5,5)-packing 13-coloring of $P_2\square C_{120}$, a (5,6)-packing 12-coloring of $P_2\square C_{120}$, a (5,10)-packing 10-coloring of $P_2\square C_{10}$, a (6,1)-packing 42-coloring of $P_2\square C_{232}$, a (6,2)-packing 23-coloring of $P_2\square C_{120}$, a (6,3)-packing 19-coloring of $P_2\square C_{120}$, a (6,4)-packing 17-coloring of $P_2\square C_{120}$, a (6,5)-packing 16-coloring of $P_2\square C_{120}$, a (6,6)-packing 15-coloring of $P_2\square C_{120}$, and a (6,12)-packing 12-coloring of $P_2\square C_{12}$.

With the dynamic algorithm approach we found a (4,5)-packing 9-coloring of $P_2\square C_{144}$ and a (3,4)-packing 7-coloring of $P_2\square C_{16}$.

If $n \geq 2d$, it is not difficult to construct a (d, n)-packing $2d$-coloring of $\mathbb{Z}\square P_2$. Thus, by Corollary 2 and Proposition 14, for every $n \geq 2d$ we have $\chi_\rho^{d,n}(\mathbb{Z}\square P_2) = 2d$.

5. Conclusion

In this paper, several new results on the (d, n)-packing chromatic number for the infinite square, hexagonal, triangular, eight-regular, octagonal and two-row square lattice are presented.

However, the problem of finding the packing chromatic number of the infinite square lattice, remains unsolved. The largest finite square lattice for which we found a 13-packing was $P_{16}\square P_{16}$, while the largest lattice for which we found a 14-packing was $P_{22}\square P_{22}$. Many attempts of coloring larger finite grids with 13 and 14 colors by using the SAT reduction have also been made, but none of these computations finished although computer programs had being run for several weeks. On the other hand, a 15-packing of $C_{72}\square C_{72}$ could be obtained after couple of seconds. Based on these experiences, we conjecture that the packing chromatic number of the infinite square lattice is 15.

Our results show that the (d, n)-packing chromatic number of an infinite lattice could be easier to compute when $n \geq A(d)$. For the infinite eight-regular and two-row square lattice we even provide closed formulas in these cases. It seems that these numbers also follow a known sequence for the infinite square and triangular lattice. Thus, we state the following

Question 1. Let $n \geq A(d)$. Is it true that $\chi_\rho^{d,n}(\mathbb{Z}\square\mathbb{Z}) = \lceil \frac{(d+1)^2}{2} \rceil$?

Question 2. Let $n \geq A(d)$. Is it true that $\chi_\rho^{d,n}(\mathcal{T}) = \lceil \frac{3(d+1)^2}{4} \rceil$?

Acknowledgments

We thank anonymous referees for helpful comments and suggestions. This work was supported by the Slovenian Research Agency under the grants P1-0297, J1-7110 and J2-7357.

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