Discrete Applied Mathematics 198 (2016) 164–169

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On dominating sets of maximal outerplanar and planar graphs[☆]

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ARTICLE INFO

Article history:
Received 7 October 2013
Received in revised form 6 June 2015
Accepted 21 June 2015
Available online 7 July 2015

Keywords:
Dominating set
Domination number
Maximal planar graph
Maximal outerplanar graph

ABSTRACT

A set $D \subseteq V(G)$ of a graph G is a dominating set if every vertex $v \in V(G)$ is either in D or adjacent to a vertex in D. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. Campos and Wakabayashi (2013) and Tokunaga (2013) proved independently that if G is an n-vertex maximal outerplanar graph having t vertices of degree 2, then $\gamma(G) \leq \frac{n+t}{4}$. We improve their upper bound by showing $\gamma(G) \leq \frac{n+k}{4}$, where k is the number of pairs of consecutive 2-degree vertices with distance at least 3 on the outer cycle. Moreover, we prove that $\gamma(G) \leq \frac{5n}{16}$ for a Hamiltonian maximal planar graph G with $n \geq 7$ vertices.

1. Introduction

In this paper, only finite, undirected and simple graphs are considered. For a graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. By $\Delta(G)$ and $\delta(G)$ we denote the *maximum degree* and *minimum degree* of graph G, respectively. We denote by $|V(G)|$ the *order* of a graph G, and a graph of order n is said to be an n-vertex graph. For a vertex $v \in V(G)$, let $N_G(v)$ and $N_G[v]$ denote the *open neighborhood* and the *closed neighborhood* of v, respectively; thus $N_G(v) = \{u : uv \in E(G)\}$ and $N_G[v] = \{v\} \cup N_G(v)$. We denote by $d_G(v) = |N_G(v)|$ the *degree* of v in G. Moreover, when no confusion can arise, $N_G(v)$, $N_G[v]$, $d_G(v)$ are simplified by $N(v)$, $N[v]$, $d(v)$, respectively. A vertex with degree d is said to be a d-degree vertex. We denote by P_n a *path* of order n. A fan F_n is a graph of order $n + 1$ obtained by adding a vertex v to P_n with v adjacent to each vertex of P_n. The notations and terminologies not mentioned here can be found in [1].

For a graph $G = (V(G), E(G))$, a *dominating set* $D \subseteq V(G)$ of a graph G is a set such that every vertex $v \in V(G)$ is either in D or adjacent to a vertex in D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. *The dominating set problem* asks for finding the minimum k such that a given graph has dominating set of k vertices. Garey

☆ Supported by 973 Program of China 2013CB329600, National Natural Science Foundation of China Grant 61309015 and National Natural Science Foundation of China Special Equipment Grant 61127005.

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http://dx.doi.org/10.1016/j.dam.2015.06.024
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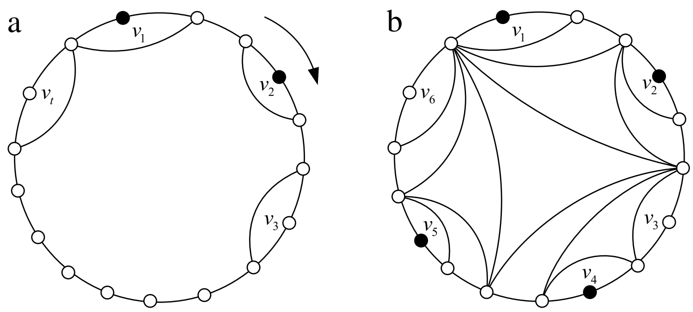


Fig. 1. A maximal outerplanar graph G on 17 vertices with 4 bad vertices.

and Johnson [4] showed that the dominating set problem is **NP**-complete even for planar graphs with maximum degree 3 and for planar 4-regular graphs.

A plane graph G is said to be a *triangulated disc* if all of its faces except the infinite face are triangles. A graph G is *outerplanar* if it has an embedding in the plane such that all vertices belong to the boundary of its outer face (the unbounded face). A planar (resp. outerplanar) graph G is *maximal* if $G + uv$ is not planar (resp. outerplanar) for any two nonadjacent vertices u and v of G.

In 1996, Matheson and Tarjan [11] proved that any triangulated disc G with n vertices satisfies $\gamma(G) \leq \frac{n}{3}$. They constructed an infinite class of outerplane graphs which achieve this bound. Furthermore, they conjectured that the upper bound would decrease to $\frac{n}{4}$ if the disc G is bounded by a triangle, namely G is a maximal planar graph.

Conjecture 1.1 (*Matheson and Tarjan [11]*). *For every n-vertex maximal planar graph G with sufficiently large n, $\gamma(G) \leq \frac{n}{4}$.*

They exhibited an infinite class of n-vertex maximal planar graphs whose domination number is $\frac{n}{4}$, where n is divided by four. Hence this bound is the best possible. These graphs are constructed from $\frac{n}{4}$ copies of K_4 drawn in the plane, with selected edges added to the outer face to create a maximal planar graph. Recently, King and Pelsmajer [9] confirmed Conjecture 1.1 for maximal planar graphs with maximum degree 6.

The dominating set problem for general graphs has been intensively studied [6,7]. In particular, bounds on the domination number of a graph in terms of its order and minimum degree have attracted much attention [12,13,16]. Motivated with the applications of the dominating set problem and Conjectures of domination numbers of planar graphs, bounds on the domination number of planar graphs of small diameter [3,5,10] have received much attention. In addition, Honjo, Kawarabayashi and Nakamoto [8] extended Matheson and Tarjan's bound of $\frac{n}{3}$ to triangulations of other surfaces.

For an n-vertex maximal outerplanar graph having t vertices of degree 2, Campos and Wakabayashi [2] and Tokunaga [14] proved independently that $\gamma(G) \leq \frac{n+t}{4}$, and the former showed that this upper-bound is tight for all $t \geq 2$. In this paper, we improve this upper bound for maximal outerplanar graphs and prove that $\gamma(G) \leq \frac{5n}{16}$ for a Hamiltonian maximal planar graph G with $n \geq 7$ vertices.

2. Improved upper bound of maximal outerplanar graphs

Let G be a maximal outerplanar graph, then there is an embedding of G in the plane such that all of its vertices are on the outer cycle C which is the boundary of the outer face and each inner face is a triangle. For an inner face f of G, f is said to be an *internal triangle* if it is not adjacent to the outer face. A maximal outerplanar graph G is called *striped* if it has no internal triangles. We may use the term triangle to refer to an inner face or to a subgraph that is isomorphic to K_3.

Lemma 2.1 (*Campos and Wakabayashi [2]*). *Let G be a maximal outerplanar graph of order $n \geq 3$. Then G has at least two vertices of degree 2. Furthermore, G has $k + 2$ vertices of degree 2 if it has k internal triangles.*

For a maximal outerplanar graph G on n vertices and its outer cycle C, let t be the number of vertices with degree two and k be the number of pairs of consecutive 2-degree vertices with distance at least 3 on C. It can be seen that $k \leq t$ and there exist many maximal outerplanar graphs with $k \ll t$. Campos and Wakabayashi [2] and Tokunaga [14] proved that the domination number of G is bounded by $\frac{n+t}{4}$, while we show that the domination number of G is bounded by $\frac{n+k}{4}$. In order to describe simply the parameter k, we first introduce the concept of a bad vertex. Let $v_1, v_2, \ldots, v_t$ be all the vertices with degree 2 which appear in the clockwise direction on C (see Fig. 1(a)). If the distance between v_i and v_{i+1} on C is at least 3, then v_i is called a *bad vertex* of G (see Fig. 1(a)), where the subscript is taken modulo t and $i \in \{1, 2, \ldots, t\}$. Fig. 1(b) shows a maximal outerplanar graph G on 17 vertices with 4 bad vertices v_1, v_2, v_4, v_5. It can be seen that the number of bad vertices of G is equal to k.

Theorem 2.2. *Let G be a maximal outerplanar graph of order n. If G has no bad vertices, then $\gamma(G) = \lceil \frac{n}{4} \rceil$.*

Proof. Let C be the outer cycle of G and $v_1, v_2, \ldots, v_t$ be a cyclic clockwise order of its t vertices of degree 2. Since G has no bad vertices, the distance between v_i and v_{i+1} on C is exact two. Thus, $n = 2t$. For any minimum dominating set D of G, each vertex of D dominates at most two vertices in $\{v_1, v_2, \ldots, v_t\}$. So $|D| \geq \lceil \frac{t}{2} \rceil = \lceil \frac{n}{4} \rceil$. We consider a set D' with

$$D' = \begin{cases} \{w_i : w_i \in N(v_i) \cap N(v_{i+1}), i = 1, 3, \ldots, t-1\}, & \text{if } t \text{ is even;} \\ \{v_t\} \cup \{w_i : w_i \in N(v_i) \cap N(v_{i+1}), i = 1, 3, \ldots, t-2\}, & \text{otherwise.} \end{cases}$$

Then D' is a dominating set of G and $|D'| = \lceil \frac{n}{4} \rceil$, which completes the proof. $\square$

Theorem 2.3. *Let G be an n-vertex maximal outerplanar graph. If G has $k > 0$ bad vertices, then $\gamma(G) \leq \frac{n+k}{4}$.*

Proof. The proof is by induction on $n + k$. It is easy to see that the result is true for $n \leq 6$. We denote by $V_2(G)$ the set of all vertices of degree 2 in G, then $G - V_2(G)$ is also a maximal outerplanar graph. Let u be a vertex of degree 2 in $G - V_2(G)$. Then $3 \leq d_G(u) \leq 4$. Now we consider the following two cases.

Case 1: $d_G(u) = 4$.

In this case, there exist two vertices $v_1, v_2 \in N_G(u)$ such that $d_G(v_1) = d_G(v_2) = 2$. Let $N_G(u) = \{v_1, v_2, u_1, u_2\}$. Then u_1 and u_2 are adjacent. Now we construct a graph G' by removing the vertices v_1, v_2 and u, and contracting the edge u_1u_2. The new vertex resulting from the contraction of edge u_1u_2 is denoted by u'. Note that G' is an $(n-4)$-vertex maximal outerplanar graph with at most k bad vertices. By the induction hypothesis, $\gamma(G') \leq \frac{n-4+k}{4}$. Let D' be a minimum dominating set of G'. We consider a set D of G with

$$D = \begin{cases} D' \setminus \{u'\} \cup \{u_1, u_2\}, & \text{if } u' \in D; \\ D' \cup \{u\}, & \text{otherwise.} \end{cases}$$

Then D is a dominating set of G and

$$\gamma(G) \leq |D'| + 1 = \gamma(G') + 1 \leq \frac{n-4+k}{4} + 1 = \frac{n+k}{4}.$$

Case 2: $d_G(u) = 3$.

In this case, there exists exactly one vertex $v \in N_G(u)$ with degree two. Let $u_1 \in N_G(u) \cap N_G(v)$ and $N_G(u) = \{v, u_1, u_2\}$. Then we have u_1 and u_2 are adjacent. We assume w.l.o.g. that u precedes v in the cyclic clockwise order on C. So v is a bad vertex of G.

Let v_1 be the succeeding 2-degree vertex of v on C. If v_1 is a bad vertex of G, then we construct a graph G' by removing the vertices u and v, and contracting the edge u_1u_2, in which the new vertex resulting from the contraction of edge u_1u_2 is denoted by u'. Clearly, G' is an $(n-3)$-vertex maximal outerplanar graph with at most $k-1$ bad vertices. By the induction hypothesis, $\gamma(G') \leq \frac{n-3+k-1}{4}$. For any minimum dominating set D' of G', we consider a set D with

$$D = \begin{cases} (D' \setminus \{u'\}) \cup \{u_1, u_2\}, & \text{if } u' \in D; \\ D' \cup \{u_1\}, & \text{otherwise.} \end{cases}$$

Then D is a dominating set of G and

$$\gamma(G) \leq |D'| + 1 = \gamma(G') + 1 \leq \frac{n-3+k-1}{4} + 1 \leq \frac{n+k}{4}.$$

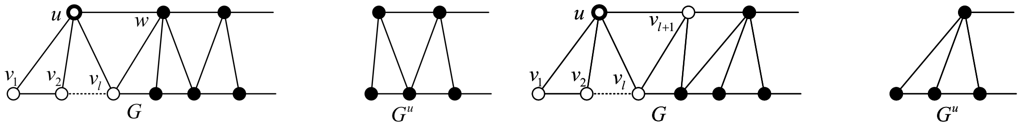
If v_1 is not a bad vertex of G, then we construct a graph G' by removing the vertices u, v and v_1, and contracting the edge u_1u_2. Clearly, G' is an $(n-4)$-vertex maximal outerplanar graph with at most k bad vertices. By the induction hypothesis, $\gamma(G') \leq \frac{n-4+k}{4}$. For any minimum dominating set D of G, let $D = (D' \setminus \{u'\}) \cup \{u_1, u_2\}$ if $u' \in D$; else $D = D' \cup \{u_1\}$. Then D is a dominating set of G and

$$\gamma(G) \leq |D| + 1 = \gamma(G') + 1 \leq \frac{n+k}{4}. \quad \square$$

Now we turn to study the domination number of a striped maximal outerplanar graph.

Lemma 2.4. *Let G be a striped maximal outerplanar graph of order $n \geq 4$. Then for any vertex $v \in V(G)$ of degree 2, the degrees of two neighbors of v are 3 and $\ell (\ell \geq 4)$, respectively.*

For a striped maximal outerplanar graph G of order $n \geq 6$ and a vertex $u \in V(G)$ adjacent to a 2-degree vertex and a 3-degree vertex, we define a graph G^u and the DELETED VERTEX SEQUENCE $(v_1, v_2, \ldots)$ as follows.



(a) The case of G_1 with $n-\ell$ vertices. (b) The case of G_1 with $n-\ell-1$ vertices.

Fig. 2. Sketch of G and G_1.

Procedure CREATE_GRAPH(G, u);
input: a graph G and a vertex $u \in V(G)$.
begin
  $i := 1$; $S := \emptyset$;
  $T := \{w : w \in N[u], d_{G-S}(w) = 2\}$;
  while ($T \neq \emptyset$)
  begin
  select a vertex $v \in T$;
  $S := S \cup \{v\}$; $v_i := v$; $i := i + 1$;
  $T := \{w : w \in N[u], d_{G-S}(w) = 2\}$;
  end
  $G^u := G - S$;
end.

In fact, G^u is a subgraph of G obtained by repeatedly removing vertices of degree 2 in $N[u]$ from G, and v_i is the ith removal of the procedure CREATE_GRAPH. See Fig. 2(a) and (b). Let $d(u) = \ell$. If $n = \ell + 1$, then G is a ℓ-fan, G^u is isomorphic to K_2 and $\gamma(G) = 1$. If $n \geq \ell + 2$, by Lemmas 2.1 and 2.4, it can be seen that $u = v_{\ell-1}$ and the DELETED VERTEX SEQUENCE is $(v_1, v_2, \ldots, v_{\ell-1}(=u), v_\ell)$ or $(v_1, v_2, \ldots, v_{\ell-1}(=u), v_\ell, v_{\ell+1})$. Thus G^u has $n-\ell$ or $n-\ell-1$ vertices. Furthermore, if $n = \ell + 2$ or $\ell + 3$, then G^u is isomorphic to K_2. If $n \geq \ell + 4$, then G^u is still a striped maximal outerplanar graph.

Theorem 2.5. *Let G be a striped maximal outerplanar graph of order $n \geq 8$ and u be a vertex of G that adjacent to a 2-degree vertex and a 3-degree vertex. Then $\gamma(G) = \gamma(G^u) + 1$.*

Proof. It is easy to see that the result is true for $n \leq d(u)+3$. If $n \geq d(u)+4$, let $d(u) = \ell$ and $N(u) = \{v_1, v_2, \ldots, v_\ell\}$. First note that for any minimum dominating set D_1 of G^u, $D_1 \cup \{u\}$ is a dominating set of G. Therefore, $\gamma(G) \leq |D_1|+1 = \gamma(G^u)+1$. Now we prove $\gamma(G) \geq \gamma(G^u) + 1$.

Let $(v_1, v_2, \ldots)$ be the DELETED VERTEX SEQUENCE of the procedure CREATE_GRAPH and D be a minimum dominating set of G. Because D contains at least one vertex of $N[v_1]$ and $N[v_i] \subseteq N[u]$ for any $i \in \{1, 2, \ldots, \ell-2\}$, $(D \setminus \{v_1, v_2, \ldots, v_{\ell-2}\}) \cup \{u\}$ is also a minimum dominating set of G. We assume w.l.o.g. that $u \in D$ and $v_i \notin D$ for any $i \in \{1, 2, \ldots, \ell-2\}$.

If $|V(G^u)| = |V(G)| - \ell$, then there exists a vertex $w \in N(u)$ that belongs to $V(G^u)$ (see Fig. 2(a)). In this case, the degree of v_ℓ in $G - \{v_1, v_2, \ldots, v_{\ell-2}, u\}$ is 2 and $N[u] \cup N[v_\ell] \subseteq N[u] \cup N[w]$. We consider a set D' with

$$D' = \begin{cases} D, & \text{if } v_\ell \notin D; \\ (D \setminus \{v_\ell\}) \cup \{w\}, & \text{otherwise.} \end{cases}$$

It can be seen that D' is a minimum dominating set of G and $D' \setminus \{u\} \subseteq V(G^u)$. Let x be the common neighbor of v_ℓ and w in $G - \{v_1, v_2, \ldots, v_{\ell-2}, u\}$. Then x is a 2-degree vertex of $G - \{v_1, v_2, \ldots, v_{\ell-2}, u, v_\ell\}$. One can see that a vertex $y \in D' \setminus \{u\}$ dominates w if y dominates x. Thus $D' \setminus \{u\}$ is a dominating set of G^u. If $|V(G^u)| = |V(G)| - \ell - 1$, then the degrees of both v_ℓ in $G - \{v_1, v_2, \ldots, v_{\ell-2}, u\}$ and $v_{\ell+1}$ in $G - \{v_1, v_2, \ldots, v_{\ell-2}, u, v_\ell\}$ are 2 (see Fig. 2(b)). Let x be the common neighbor of v_ℓ and $v_{\ell+1}$ in $G - \{v_1, v_2, \ldots, v_{\ell-2}, u\}$. We consider a set D' with

$$D' = \begin{cases} D, & \text{if } v_\ell, v_{\ell+1} \notin D; \\ (D \setminus \{v_\ell, v_{\ell+1}\}) \cup \{x\}, & \text{otherwise.} \end{cases}$$

Then D' is a minimum dominating set of G and $D' \setminus \{u\} \subseteq V(G^u)$. Thus $D' \setminus \{u\}$ is a dominating set of G^u. Therefore, $\gamma(G^u) \leq \gamma(G) - 1$. □

Based on Theorem 2.5, one can give a linear time algorithm for computing the domination number of a striped maximal outerplanar graph.

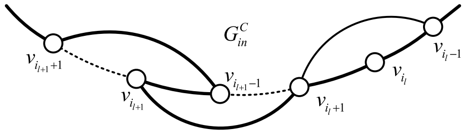


Fig. 3. The case of v_{i_2-1} is a bad vertex of G_{out}^C.

3. Upper bound of Hamiltonian maximal planar graphs

In this section, we consider the domination number of maximal planar graphs which have a Hamilton cycle. Let G be a Hamiltonian maximal planar graph and C be a Hamilton cycle of G. Then G can be partitioned into two maximal outerplanar graphs by C. Denote by G_{in}^C the maximal outerplanar graph consists of C and all edges inside of C and by G_{out}^C the maximal outerplanar graph consists of C and all edges outside of C.

Lemma 3.1. *Let G be a Hamiltonian maximal outerplanar graph of order n. Then there exists a Hamilton cycle C of G such that G_{in}^C or G_{out}^C has at most $\frac{n}{4}$ bad vertices.*

Proof. First we choose a Hamilton cycle C of G such that the number k of bad vertices of G_{in}^C is minimum. Let $v_1, v_2, \dots, v_n$ be all vertices of G and $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ be all bad vertices of G_{in}^C on C in the clockwise direction. In what follows, all of the subscripts of vertices are taken modulo n and in $\{1, 2, \dots, n\}$, and all of the sub-subscripts of bad vertices are taken modulo k and in $\{1, 2, \dots, k\}$. If the distance between every two consecutive bad vertices on C is at least 4, then $k \le \frac{n}{4}$.

If there exist consecutive pairs of bad vertices such that the distance between each pair of bad vertices on C is 3, then for any consecutive pairs of bad vertices $v_{i_\ell}, v_{i_{\ell+1}}, v_{i_\ell+1}$ and $v_{i_{\ell+1}-1}$ are adjacent on C.

Claim. G_{out}^C has no bad vertices in $\{v_{i_\ell}, v_{i_\ell+1}, v_{i_{\ell+1}-1}, v_{i_{\ell+1}}\}$.

Since $\delta(G) \ge 3$, v_{i_j} is not the bad vertex of G_{out}^C, where $j = 1, 2, \dots, k$. If one of $v_{i_\ell+1}, v_{i_{\ell+1}-1}$ is a bad vertex of G_{out}^C, say $v_{i_{\ell+1}-1}$, then $v_{i_\ell+1}$ and $v_{i_{\ell+1}}$ are adjacent in G_{out}^C. See Fig. 3. Let $C' = C - \{v_{i_\ell+1}v_{i_{\ell+1}-1}, v_{i_{\ell+1}}v_{i_{\ell+1}+1}\} + \{v_{i_\ell+1}v_{i_{\ell+1}}, v_{i_{\ell+1}-1}v_{i_{\ell+1}+1}\}$, then $G_{in}^{C'} = G_{in}^C - v_{i_{\ell+1}}v_{i_{\ell+1}+1} + v_{i_\ell+1}v_{i_{\ell+1}}$ and $v_{i_{\ell+1}}$ is a vertex of degree 2 in $G_{in}^{C'}$. Since the distance between v_{i_ℓ} and $v_{i_{\ell+1}}$ is 2, v_{i_ℓ} is not a bad vertex of $G_{in}^{C'}$. Therefore, the number of bad vertices of $G_{in}^{C'}$ is less than k, contradicting the choice of C. The Claim is established.

A vertex $u \in V(G_{in}^C)$ is called a *free vertex* of G_{in}^C if $u \notin \bigcup_{\ell=1}^{k} N_{G_{in}^C}[v_{i_\ell}]$. Suppose that $k > \frac{n}{4}$. For any two bad vertices v_{i_j}, v_{i_ℓ}, we have $N_{G_{in}^C}(v_{i_j}) \cap N_{G_{in}^C}(v_{i_\ell}) = \emptyset$. So

$$\left|\bigcup_{\ell=1}^{k} N[v_{i_\ell}]\right| = \left|\bigcup_{\ell=1}^{k} \{v_{i_\ell}\}\right| + \left|\bigcup_{\ell=1}^{k} N(v_{i_\ell})\right| > \frac{n}{4} + \frac{n}{2} = \frac{3n}{4}.$$

Therefore, the number of free vertices of G_{in}^C is less than $\frac{n}{4}$.

Let $A_\ell = \{v_{i_\ell+1}, \dots, v_{i_{\ell+1}-1}\}$, $\ell = 1, 2, \dots, k$. From Claim we know that every bad vertex of G_{out}^C must belong to the set

$$\bigcup_{\ell=1}^{k} (A_\ell : A_\ell \text{ contains at least one free vertices of } G_{in}^C).$$

For a set A_ℓ, $\ell \in \{1, 2, \dots, k\}$, if A_ℓ contains exactly one free vertex of G_{in}^C, then there is at most one bad vertex of G_{out}^C in A_ℓ. If A_ℓ contains $s \ge 2$ bad vertices of G_{out}^C, then A_ℓ contains at least $3s-4$ free vertices of G_{in}^C. Because $3s-4 \ge s$ for any $s \ge 2$, the number of bad vertices of G_{out}^C is less than the number of free vertices of G_{in}^C. Thus, the number of bad vertices of G_{out}^C is less than $\frac{n}{4}$. $\square$

There exist a Hamiltonian maximal planar graph G and a Hamiltonian cycle C of G such that both G_{in}^C and G_{out}^C have exactly $n/4$ bad vertices. For example, Fig. 4 is such a graph, where v_1, v_2, v_3 are bad vertices of G_{in}^C and u_1, u_2, u_3 are bad vertices of G_{out}^C. But it is difficult to find a Hamiltonian maximal planar graph G such that for every Hamiltonian cycle C of G, both G_{in}^C and G_{out}^C have exactly $n/4$ bad vertices.

The following result is obtained by Theorem 2.3 and Lemma 3.1.

Theorem 3.2. *Let G be a Hamiltonian maximal planar graph of order n. Then*

$$\gamma(G) \le \begin{cases} 2, & \text{if } n \le 6; \\ \dfrac{5n}{16}, & \text{otherwise.} \end{cases}$$

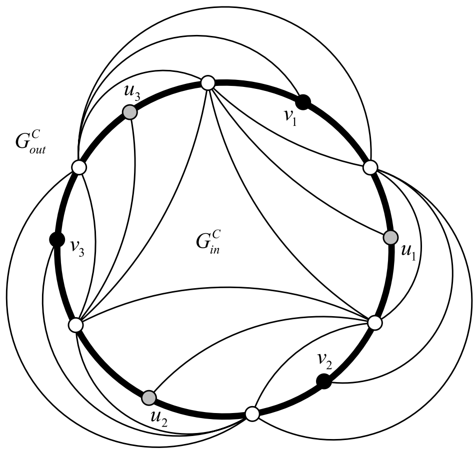


Fig. 4. A Hamiltonian maximal planar graph G with a Hamiltonian cycle C such that both G^C_{in} and G^C_{out} have exactly $n/4$ bad vertices.

Proof. It is easy to check that $\gamma(G) \leq 2$ for $n \leq 6$. If $n \geq 7$, by Lemma 3.1, we can choose a Hamilton cycle C of G such that G^C_{in} or G^C_{out} has at most $\frac{n}{4}$ bad vertices. We assume w.l.o.g. that G^C_{in} has $k \leq \frac{n}{4}$ bad vertices. If $k = 0$, then by Theorem 2.2, we have $\gamma(G) = \lceil \frac{n}{4} \rceil < \frac{5n}{16}$. (In this case n is even.) If $k > 0$, then by Theorem 2.3, we have $\gamma(G) \leq \frac{n+k}{4} \leq \frac{5n}{16}$. $\square$

Since every 4-connected maximal planar graph is Hamiltonian by Whitney [15]. So we have the following result.

Corollary 3.3. *Let G be a 4-connected maximal planar graph of order $n \geq 7$. Then $\gamma(G) \leq \frac{5n}{16}$.*

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, 2008.
[2] C.N. Campos, Y. Wakabayashi, On dominating sets of maximal outerplanar graphs, Discrete Appl. Math. 161 (2013) 330–335.
[3] M. Dorfling, W. Goddard, M.A. Henning, Domination in planar graphs with small diameter. II, Ars Combin. 78 (2006) 237–255.
[4] M.R. Garey, D.S. Johnson, Computers and intractability, in: A Series of Books in the Mathematical Sciences, in: A Guide to The Theory of NP-Completeness, W. H. Freeman and Co, San Francisco, Calif, 1979.
[5] W. Goddard, M.A. Henning, Domination in planar graphs with small diameter, J. Graph Theory 40 (1) (2002) 1–25.
[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[7] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.
[8] T. Honjo, K.-i. Kawarabayashi, A. Nakamoto, Dominating sets in triangulations on surfaces, J. Graph Theory 63 (1) (2010) 17–30.
[9] E.L. King, M.J. Pelsmajer, Dominating sets in plane triangulations, Discrete Math. 310 (2010) 2221–2230.
[10] G. MacGillivray, K. Seyffarth, Domination numbers of planar graphs, J. Graph Theory 22 (3) (1996) 213–229.
[11] L.R. Matheson, R.E. Tarjan, Dominating sets in planar graphs, European J. Combin. 17 (1996) 565–568.
[12] B. Reed, Paths, stars and the number three, Combin. Probab. Comput. 5 (1996) 277–295.
[13] M.Y. Sohn, X. Yuan, Domination in graphs of minimum degree four, J. Korean Math. Soc. 46 (4) (2009) 759–773.
[14] S. Tokunaga, Dominating sets of maximal outerplanar graphs, Discrete Appl. Math. (2013). http://dx.doi.org/10.1016/j.dam.2013.06.025.
[15] H. Whitney, A theorem on graphs, Ann. of Math. 32 (1931) 378–390.
[16] H.M. Xing, L. Sun, X.G. Chen, Domination in graphs of minimum degree five, Graphs Combin. 22 (1) (2006) 127–143.