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On minimum clique partition and maximum independent set on unit disk graphs and penny graphs: complexity and approximation

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Abstract

A graph G is a *unit disk graph* if it is the intersection graph of a family of unit disks in the euclidean plane. If the disks do not overlap, then G is also a *unit coin graph or penny graph.* In this work we establish the complexity of the minimum clique partition problem and the maximum independent set problem for penny graphs, both NP-complete, and present two approximation algorithms for finding clique partitions: a 3-approximation algorithm for unit disk graphs and a $\frac{3}{2}$-approximation algorithm for penny graphs.

Keywords: unit disk graph, unit coin graph, penny graph, minimum clique partition, approximation algorithms

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1 Introduction

Given a family $\mathscr{F}$ of geometric objects, the *intersection graph* of $\mathscr{F}$ is the graph whose vertices are in a one-to-one correspondence with the objects of $\mathscr{F}$ in such a way that there exists an edge joining two vertices if and only if the corresponding objects intersect. A *unit disk* is a disk of radius one in the euclidean plane. Two unit disks *intersect* if the distance between their centers is less than or equal to 2. Two unit disks *overlap* if the distance between their centers is strictly less than 2. A graph $G = (V, E)$ is a *unit disk graph*, or *UD graph*, if it is the intersection graph of a family of unit disks. If the disks do not overlap, then G is also a *unit coin graph* or *penny graph*. A *realization* of a UD graph (resp. penny graph) G is a family $\mathscr{F}$ of unit disks (resp. non-overlapping unit disks) such that G is the intersection graph of $\mathscr{F}$. (Assume without loss of generality that $\mathscr{F}$ uses $O(|V|^2)$ area units.)

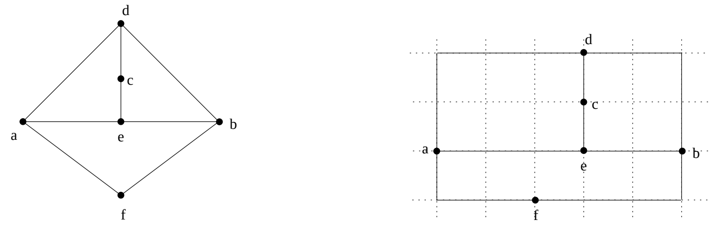
Intersection graphs of geometric objects have received much attention since the 70's. In this context, UD graphs and penny graphs appear in the modelling of several problems [2,3,4,5,9,10,11]. In [4], Clark *et al.* proved that the maximum independent set problem is NP-complete when restricted to UD graphs. This intractability result motivated the search for approximation algorithms and polynomial-time approximation schemes. In [10], Marathe *et al.* present a simple greedy 3-approximation algorithm for finding independent sets in UD graphs running in $O(|V|^5)$ time. In [9], Hunt III *et al.* present a polynomial-time approximation scheme for finding independent sets in UD graphs which makes use of the "shifting strategy" [1]. The algorithm takes a realization as input, and produces a solution with size at least $k/(k+1)$ times the optimal size, where k is the smallest integer such that $(k/(k+1))^2 \geq 1-\varepsilon$, for a given $\varepsilon > 0$. The running time is $|V|^{O(k)}$.

We now summarize the contributions of this work. In Section 2, we extend Clark's result [4] by proving that MAXIMUM INDEPENDENT SET ([7], p. 194) is NP-complete even when restricted to the class of penny graphs. In Section 3, we also establish the NP-completeness of MINIMUM CLIQUE PARTITION ([7], p. 193) when restricted to penny graphs. Finally, in Section 4, we present two approximation algorithms for finding clique partitions: a 3-approximation algorithm for UD graphs and a $\frac{3}{2}$-approximation algorithm for penny graphs.

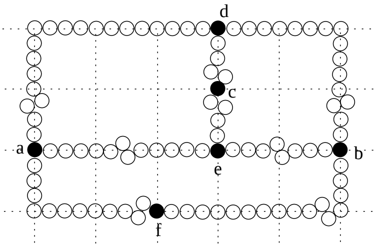
2 MAXIMUM INDEPENDENT SET restricted to penny graphs

Let $G = (V, E)$. Clearly, $V' \subseteq V$ is a vertex cover of G (a subset of vertices containing at least one extreme of each edge in E) if and only if $V \backslash V'$ is an independet set of G. Thus, we show first that VERTEX COVER ([7], p. 190) restricted to penny graphs is NP-complete.

Lemma 1 (Valiant, 1981 [12]) *A planar graph G with maximum degree 4 can be embedded in the plane using $O(|V|^2)$ area units in such a way that its vertices are at integer coordinates and its edges are drawn so that they are made up of line segments of the form $x = i$ or $y = j$, for integers i and j.*



a) Planar graph G with maximum degree 3 b) Drawing of G according to Lemma 1



c) Realization of G'

Figure 1. VERTEX COVER restricted to penny graphs.

Theorem 2 VERTEX COVER *restricted to penny graphs is NP-complete.*

Proof. The problem is clearly in NP. The reduction is done from VERTEX COVER restricted to planar graphs with maximum degree 3, which was shown to be NP-complete in [6]. From a planar graph G with maximum degree 3, we construct a penny graph G' such that there is a vertex cover S for G satisfying $|S| \leq k$ if and only if there is a vertex cover S' for G' satisfying $|S'| \leq k'$, where k' is specified in the sequel. First, draw G in the plane using Lemma 1. Next, construct a realization of G' (see Figure 1) by replacing each edge $(x, y) \in E$ by an even path consisting of $4k_{xy}$ white disks, where k_{xy} is the length of an edge between the vertices x and y. In order to achieve the value $4k_{xy}$, local displacements in the disks can be made, as shown in Figure 2. The number of vertices of G' is $|V| + \sum_{(x,y)\in E} 4k_{xy}$. Construct S' adding to S $2k_{xy}$ vertices corresponding to alternating white disks for each edge $(x, y) \in E$, starting from a disk corresponding to an extreme of (x, y) belonging to S. Therefore, S is a vertex cover for G satisfying $|S| \leq k$ if and only if S' is a vertex cover for G' satisfying $|S'| \leq k + \sum_{(x,y)\in E} 2k_{xy} = k'$. □

a) Odd number of disks   b) Achieving an even number of disks

Figure 2. Local displacements in the disks.

Corollary 3 MAXIMUM INDEPENDENT SET *restricted to penny graphs is NP-complete.*

3 MINIMUM CLIQUE PARTITION problem restricted to penny graphs

In this section we establish the NP-completeness of the minimum clique partition problem when restricted to penny graphs. The hardness proof is done from the following problem:

PLANAR 3SAT WITH AT MOST 3 OCCURRENCES PER VARIABLE (3-PSAT$_{\overline{3}}$)
Instance: A set of variables U and a collection of clauses $E = \{E_1, \ldots, E_m\}$ such that each clause contains at most three literals, each variable occurs at most three times, and the undirected graph $G = (N, M)$ is planar, where $N = U \cup E$ and $M = \{(u_i, E_j) \mid u_i \in E_j \text{ or } \overline{u}_i \in E_j\}$.
Question: Is there a truth assignment for U satisfying E?

Theorem 4 3-PSAT$_{\overline{3}}$ *is NP-complete.*

Theorem 5 MINIMUM CLIQUE PARTITION *restricted to penny graphs is NP-complete.*

Proofs of Theorems 4 and 5 will appear in the complete version of this work.

4 Approximation algorithms

Since UD graph recognition is NP-Hard [3] and the complexity of constructing a realization of a UD graph is still open, most algorithms demand a realization as input. The algorithms described in this section admit that a realization of a UD graph is given as input. Assume $|V| = n$ and $|E| = m$.

4.1 Approximation algorithm for finding clique partitions in UD graphs

The approximation algorithm presented in this subsection uses as subroutine an exact algorithm for finding optimal clique partitions in *k-strip graphs* for $k = \sqrt{3}$. A UD graph G is a k-strip graph if the centers of the disks in a realization of G are contained in the region $S_k = \{(x, y) \in \Re^2 \mid 0 \le y < k\}$. When $k \le \sqrt{3}$, G is also a cocomparability graph [2]. Therefore, it is easy in this case to find a minimum clique partition of G by simply coloring its complement $\overline{G}$ in time $O(n + \overline{m})$, where $\overline{m}$ is the number of edges of $\overline{G}$ [8].

Algorithm 1 - Finds an approximate clique partition in a UD graph

1: divide the plane into horizontal strips of width $\sqrt{3}$
2: associate to each strip i a subgraph G_i induced by the vertices of G corresponding to the disks whose centers lie in strip i
3: find an exact clique partition Z_i for each G_i
4: return a clique partition $Z = \cup Z_i$

Step 1 divides the plane into ℓ strips of width $\sqrt{3}$ each, where $\ell = \lceil \frac{\overline{y}}{\sqrt{3}} \rceil$ and $\overline{y}$ is the maximum value of an y-coordinate considering the centers of the disks in the realization of G. Every disk whose center $c = (x, y)$ satisfies $(i-1)\sqrt{3} \le y < i\sqrt{3}$ belongs to strip i, $1 \le i \le \ell$.

Let Z^* be a minimum clique partition of G, and let Z_i^* be the restriction of Z^* to G_i, that is, $Z_i^* = \{X \ne \emptyset \mid X = C \cap V(G_i) \text{ and } C \in Z^*\}$. Since Step 3 covers G_i optimally, $|Z_i| \le |Z_i^*|$. Now, let C be a clique in Z^*. A simple geometric argument shows that the centers of the disks associated to the vertices of C are contained in a region of the plane distributed along at most three strips. This implies that C is the union of at most three disjoint cliques belonging to Z_{j-1}^*, Z_j^* and Z_{j+1}^* for some j. That is, $|Z| = \sum_{i=1}^{\ell} |Z_i| \le \sum_{i=1}^{\ell} |Z_i^*| \le 3|Z^*|$. Hence Algorithm 1 is a 3-approximation algorithm. Its overall running time is $O(n + \overline{m})$.

4.2 Approximation algorithm for finding clique partitions in penny graphs

Let x_v denote the x-coordinate of the center of the disk associated to vertex v in the realization of G, and let $N(v)$ denote the set of neighbors of v. The algorithm in this subsection is a simple greedy heuristic based on the following straightforward facts:

Fact 1. if v is a vertex such that x_v is minimum, then $|N(v)| \leq 4$.
Fact 2. if C is a clique in a penny graph, then $|C| \leq 3$.

Assume that the input graph G is connected (if G is not connected, the algorithm can be applied to each connected component separately). The notation $G[v \cup N(v)]$ stands for the subgraph of G induced by v and its neighbors.

Algorithm 2 - Finds an approximate clique partition in a penny graph

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1: let $v_1, \ldots, v_n$ be an ordering of the vertices of G such that $i < j$ implies $x_{v_i} \leq x_{v_j}$
2: $Z := \emptyset$
3: mark all the vertices as *uncovered*
4: **repeat**
5:    let v be the leftmost uncovered vertex
6:    add to Z the largest clique C containing v in the subgraph $G[v \cup N(v)]$
7:    mark the elements of C as *covered*
8: **until** all the vertices are covered

Let $|Z^*|$ be a minimum clique partition of G. By Fact 2, $|Z^*| \geq \lceil \frac{n}{3} \rceil$. Since G is connected, the size of C in Step 6 is at least 2, and therefore $|Z| \leq \lfloor \frac{n}{2} \rfloor$. The approximation performance of Algorithm 2 is thus $\frac{|Z|}{|Z^*|} \leq \frac{\lfloor n/2 \rfloor}{\lceil n/3 \rceil} \leq \frac{3}{2}$.

Step 1 takes $O(n \log n)$ time. Steps 2 and 3 take $O(n)$ time. The **repeat** command (Steps 4 – 8) takes $O(n)$ total time, because Fact 1 guarantees that we can compute C (Step 6) in constant time, and Fact 2 guarantees that Step 7 can also be executed in constant time. The running time is dominated by Step 1, therefore the total running time of Algorithm 2 is $O(n \log n)$.

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