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ON COLORING j-UNIT SPHERE GRAPHS

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FEW 512

DEPARTMENT OF ECONOMICS
RESEARCH MEMORANDUM

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On Coloring j-Unit Sphere Graphs

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ABSTRACT

In this article we will discuss the chromatic number problem for j-unit sphere graphs. If $j = 1$ there is a linear algorithm that solves the problem, but if $j = 2$, even the 3-colorability problem is NP-complete.

1 Introduction

The chromatic number problem for graphs is the problem of determining the minimum number of colors needed to color the vertices of a graph so that adjacent vertices get different colors. Let $G = (V, E)$ be the graph considered, then this optimal value is denoted with $\chi(G)$. A subset $C \subseteq V$ is called a clique if any two of its vertices are adjacent. The size of the biggest clique of G is denoted with $\omega(G)$. Clearly $\chi(G) \geq \omega(G)$, because if C is a clique of G, all vertices in G must have different colors.

In this article we will discuss the chromatic number problem for j-unit sphere graphs. They are defined as follows:

Definition: $G = (V, E)$ is called a j-unit sphere graph if there is a one to one function f from V into Euclidean j-space so that for all vertices $v \neq w$ in V:

$$vw \in E \Leftrightarrow d[f(v), f(w)] \leq 1,$$

where $d[\alpha, \beta]$ is the Euclidean distance.

In this article we will identify a vertex with its corresponding point in the Euclidean space.

For $i = 1$ these graphs are called unit interval graphs or indifference graphs. These graphs were characterized by Roberts[5]. For $j = 2$ they are called unit disk graphs, a class of graphs introduced by Hale[3]. By definition, if G is a j-unit sphere graph and $k \geq j$, G is a k-unit sphere graph as well.

2 Unit Interval Graphs

For unit interval graphs there exists an easy linear coloring algorithm that gives a minimum coloring. Suppose $G = (V, E)$ is a unit interval graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ for which $v_1 \leq v_2 \leq \cdots \leq v_n$. Apply a first-fit coloring on this vertex sequence, that means, using the positive integers as the color set, assign to each successive vertex the least color possible subject to maintaining a proper coloring.

Claim: The coloring we find this way is minimum.

Proof: Suppose we used p colors (so $\chi(G) \leq p$) and let v_k be the first vertex that is colored with the p-th color. This vertex was colored with the p-th color because for $j = 1, 2, \ldots, p-1$ there was a vertex $v_{i(j)} \leq v_k$ adjacent to v_k colored with the j-th color. This means that $v_k - v_{i(j)} \leq 1$ for $j = 1, 2, \ldots, p-1$ and hence $|v_{i(j_1)} - v_{i(j_2)}| \leq 1$ for $j_1, j_2 \in \{1, 2, \ldots, p-1\}$ which means that the vertices $v_{i(1)}, \ldots, v_{i(p-1)}, v_k$ form a clique of size p. So $\chi(G) \geq p$ and hence $\chi(G) = p$. □

3 Unit Disk Graphs

It is proven in [2] that the 3-coloration problem for general graphs (3C), that means finding out whether $\chi(G) \leq 3$, is NP-complete. In this part we will prove that even if we restrict the question to unit disk graphs, the problem remains NP-complete.

Hale[3] already mentioned the NP-completeness of the graph coloring problem restricted to unit disk graphs, but without a proof or a reference. As a result it is not likely that there exists a polynomial time algorithm that solves the chromatic number problem for unit disk graphs. The 3-coloration problem for unit disk graphs (3CUD) is defined as follows:

Name: 3CUD.

Input: A unit disk graph $G(V, E)$.

Question: Is $\chi(G) \leq 3$?

Theorem: 3CUD is NP-complete.

Proof: We show that 3C is polynomially reducible to 3CUD (3C$\propto$3CUD). The proof of this theorem is essentially the same as the proof of the NP-completeness of the 3-coloration problem for planar graphs (3CP), see Even[1] pp. 221-223. There it is proved that 3C$\propto$3CP by constructing a planar graph $f(I)$ for a given input $I = G(V, E)$ such that $f(I)$ is 3-colorable if and only if I is 3-colorable. That proof is based on the work of Stockmeyer[6] and Garey, Johnson and Stockmeyer[2]. Let $G(V, E)$ be the input I to the 3C problem. We construct a unit disk graph $f(I)$ for which two vertices are adjacent if and only if their distance is at most 2. This construction is only slightly different from the one to prove the NP-completeness of the 3-coloration problem for planar graphs. The arguments why this construction works, that means, proves the theorem, are exactly the same.

Consider the diamond unit disk graph D of Figure 1. It is up to the reader to check that two vertices are adjacent if and only if their distance is at most 2. Any valid 3-coloring of D gives the same color to u and u', and the same color to v and v', The color of u and v may or may not be the same. Thus, D, effectively performs a crossover of the coloring of u to u', of v to v', while no constraint as to the equality or inequality of the colors of u and v is introduced.

We construct $f(I)$ in two steps. First, construct a general layout as demonstrated

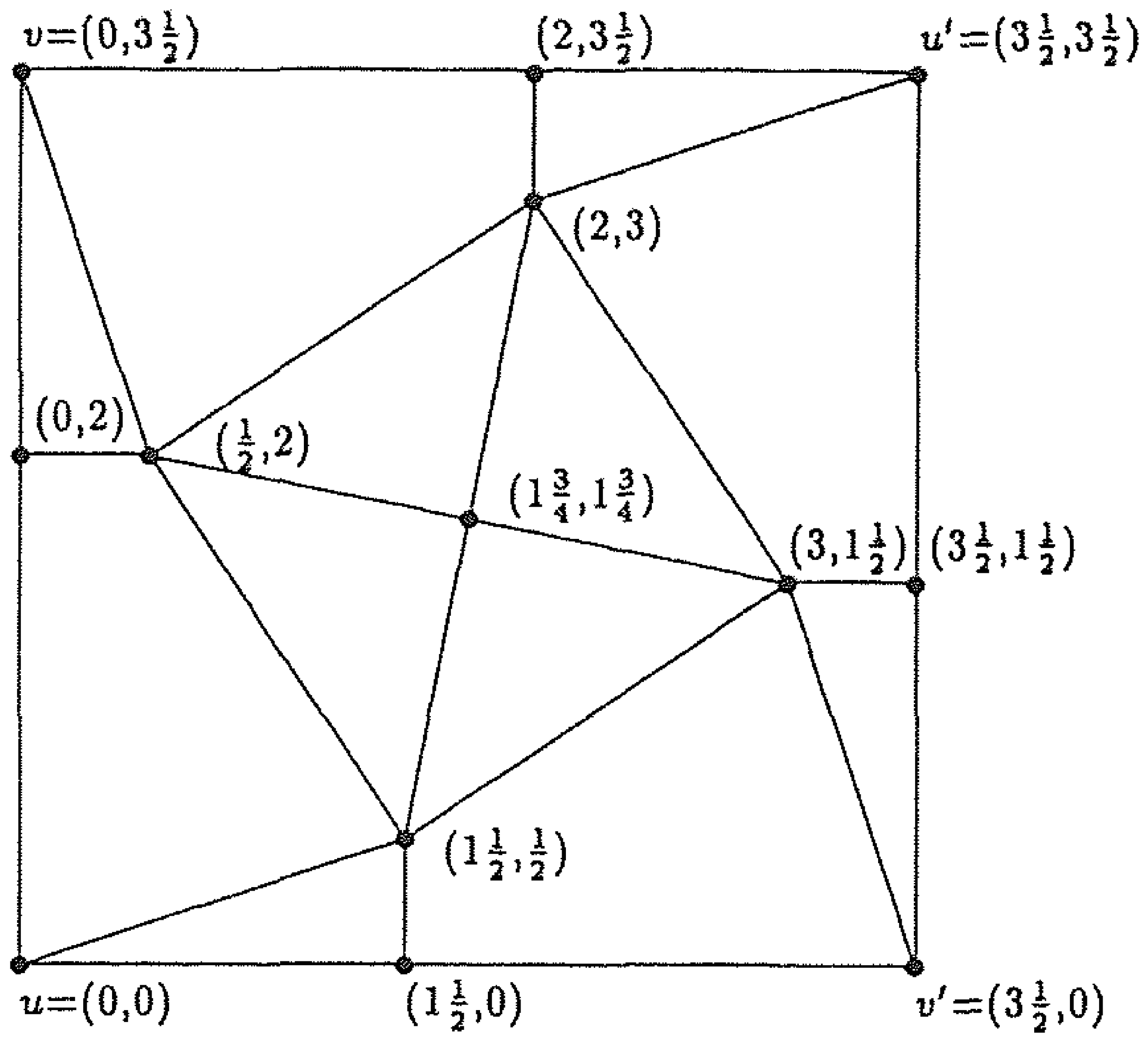

Figure 1: The graph D

in Figure 2 for the case of $n = 4$. This general layout depends only on n, and not on E. This layout has $n-1$ main layers of vertices and n main columns of vertices; these vertices are denoted u_{ij}, where $1 \le i \le n-1$ and $1 \le j \le n$. The distance between two neighbour vertices is taken $7\frac{1}{2}$. If $i+j$ is even then a copy of D is included and connected with the vertices $u_{ij}, u_{i(j+1)}, u_{(i+1)j}$ and $u_{(i+1)(j+1)}$ as is shown in Figure 2. If $i < n-1$ and even then u_{i1} and $u_{(i+1)1}$ are connected via 5 new vertices as shown in Figure 2 in the case of u_{21} and u_{31}. If $i < n-1$ and $i+n$ is even then u_{in} is connected, similarly, to $u_{(i+1)n}$; see u_{24} and u_{34} in Figure 2. This completes the construction of the layout. It is easy to see that this graph is a unit disk graph. The number of vertices is altogether: $n(n-1) + 21/2(n-1)(n-2) + 5(n-2)$.

Assume all vertices are colored using only three colors. Clearly, $u_{11}, u_{22}, \ldots, u_{(n-1)(n-1)}$ are all colored identically. If $i > 1$ and odd then $u_{1i}, u_{2(i+1)}, \ldots, u_{(n-i+1)n}, u_{(n-i+2)n}, u_{(n-i+3)(n-1)}, \ldots, u_{(n-1)(n-i+3)}$ are all colored identically. Also, if n is even, $u_{1n}, u_{2(n-1)}, \ldots, u_{(n-1)2}$ is such a track, and if $i < n$ and even then $u_{1i}, u_{2(i-1)}, \ldots, u_{i1}, u_{(i+1)1}, u_{(i+2)2}, \ldots, u_{(n-1)(n-i-1)}$ is such a track. Let us call the track that begins at u_{1i}, the i-th track.

Now, for every two tracks there is at least one $1 \le i \le n-1$ and $1 \le j < n$ such that u_{ij} is on one of these tracks and $u_{i(j+1)}$ is on the other. For the proof we refer to Even[1]. If this happens, we say that these two tracks are adjacent at the i-th level.

Now we turn to the second part of the reduction. If v_i and v_j are adjacent in G, make a construction using 6 new vertices to connect u_{kl} and $u_{k(l+1)}$ in the layout, as is shown in Figure 3, where u_{kl} is on the i-th track (or the j-th track) and $u_{k(l+1)}$ is on the j-th track (or the i-th track). Such a k and l can be found since every two tracks are adjacent somewhere. Notice that, by this construction, in case of a 3-coloring, u_{kl} and $u_{k(l+1)}$ don't have the same color. The new vertices can be added to the layout so that two vertices are adjacent if and only if their distance is at most 2, see Figure 4,

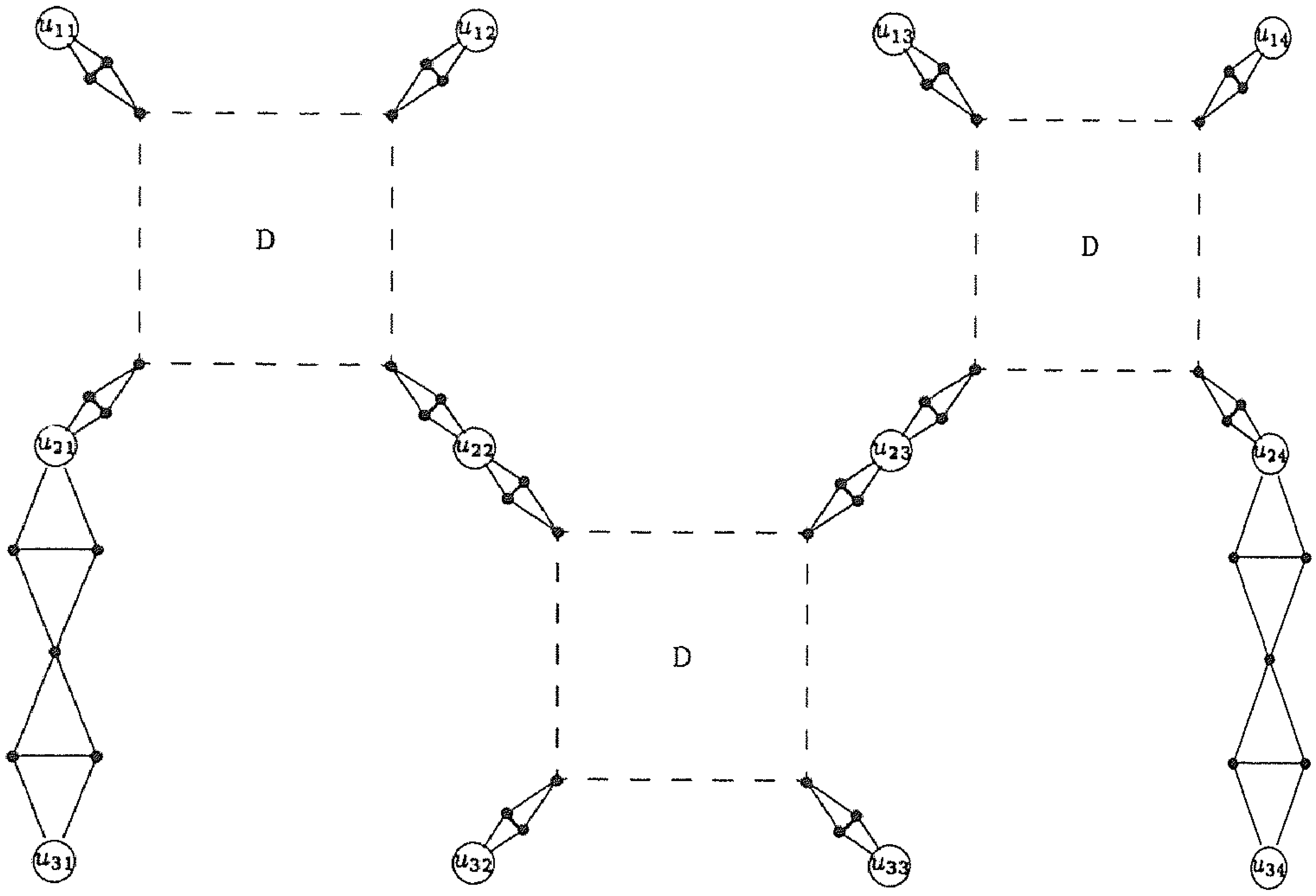

Figure 2:

Figure 3:

where the big dots represent two vertices with the same coordinates.

Now, if G is 3-colorable, color all the vertices of the i-th track with the color of v_i. Clearly the remaining vertices of $f(I)$ can also be legally colored. Conversely, if $f(I)$ is 3-colorable, each track i is uniformly colored, and we can assign v_i, in G, the same color. No two adjacent vertices in G get the same color, because of the construction connecting the two corresponding tracks in $f(I)$, and since $f(I)$ is legally colored, the colors of these two tracks are different. $\square$

Notice that the graph we have constructed is a planar graph as well, so in fact we have proved the stronger result that the 3-colorability problem for a planar unit disk graph is NP-complete.

If we order the vertices of a unit disk graph G lexicographically according to their coordinates in the plane and apply a first-fit algorithm as we did for unit interval graphs, we won't get arbitrary bad results. More precisely: if $p(G)$ is the number of colors we use applying the algorithm, the following inequality holds:

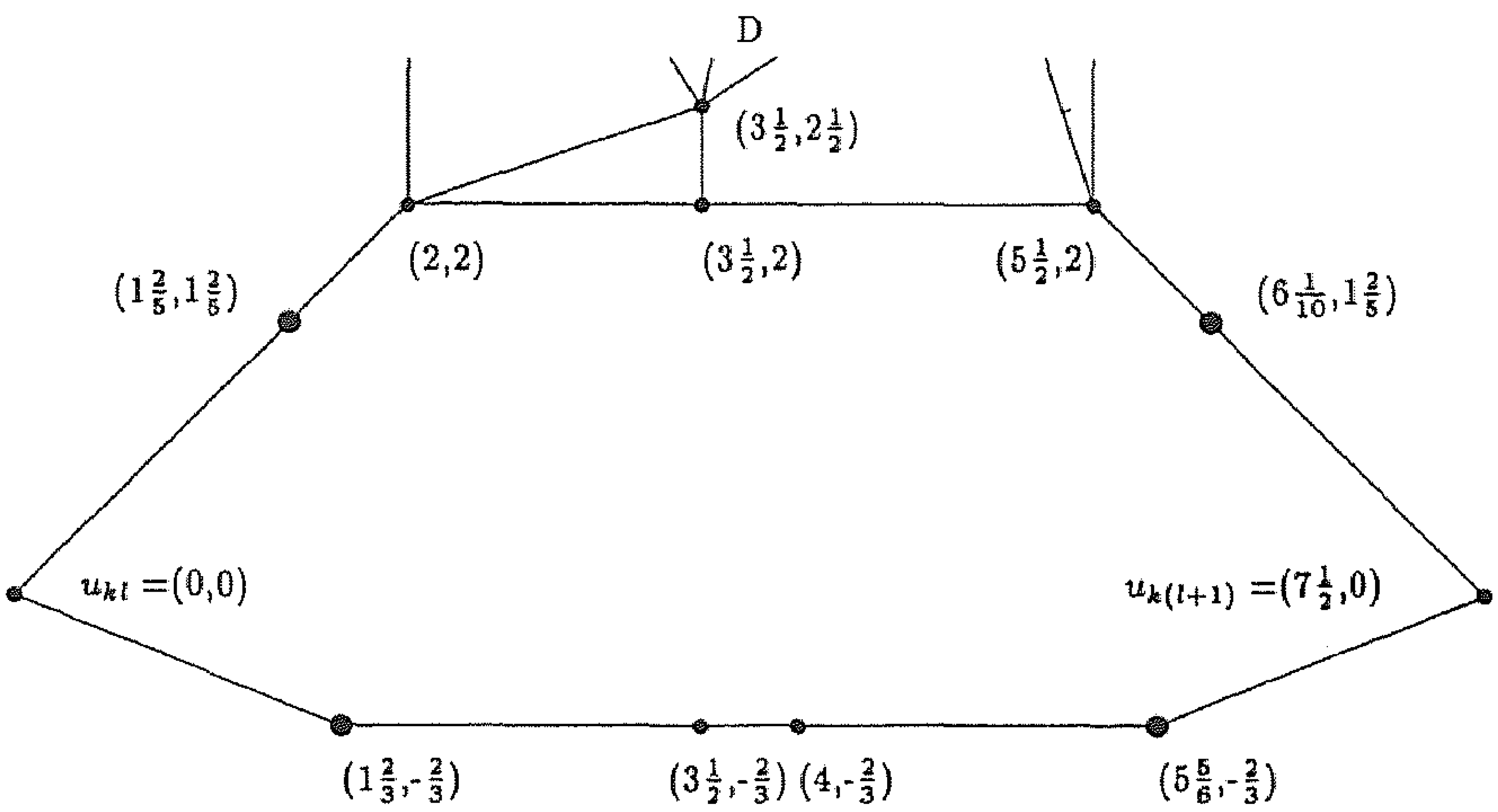

Figure 4:

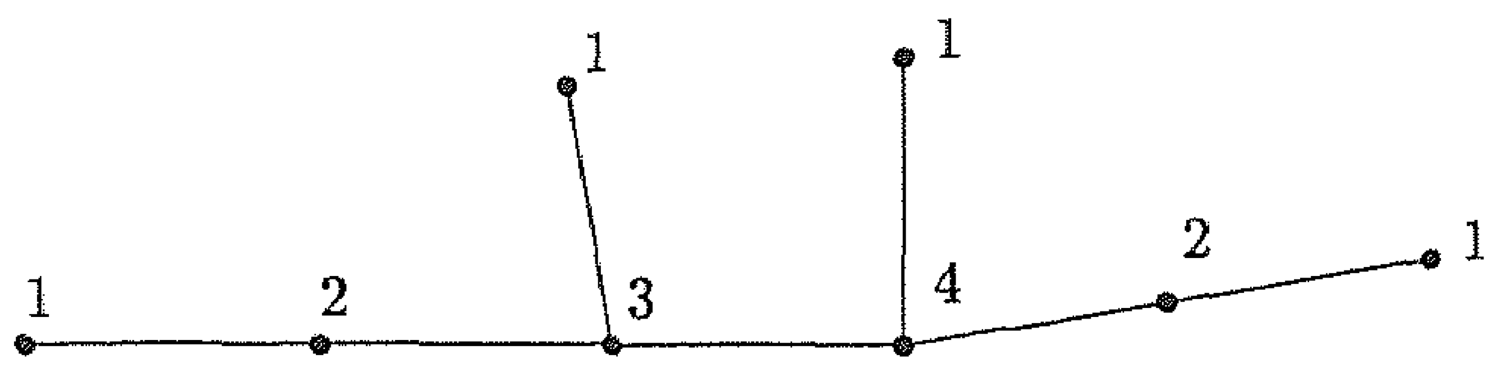

Figure 5:

Lemma:

$$p(G) \leq 3\omega(G) - 2$$

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of G in lexicographic order and let v_k be a vertex that is assigned the $p(G)$-th color. Look at all the vertices v_i adjacent to v_k with $i < k$. They are cituated on one half of a disk with radius one and center v_k. We can divide this sector in 3 sectors of size $60°$. The distance between two points from the same sector is always at most 1. This means that the vertices from the same sector together with v_k form a clique. Thus, there were at most $3(\omega(G) - 1)$ forbidden colors for v_k. □

Figure 3 shows a unit disk graph for which we find equality. The numbers represent the colors assigned by a first-fit algorithm.

For generalizing this result to higher dimensions we have to know in how many parts with diameter at most 1 one half of a j-dimensional unit sphere with radius one can be partitioned. Because we want our bound to be as sharp as possible, we are interested in the smallest cardinality possible of such a partition. These numbers, however, are not known in general, but grow rapidly for higher dimensions as can be seen in [4]. So, if we apply a first-fit coloring algorithm on j-unit sphere graphs, we may expect worse results for higher j.

References

[1] S. Even, *Graph Algorithms*, Pitman, (1979).

[2] M.R. Garey, D.S. Johnson and L.J. Stockmeyer, Some Simplified NP-Complete Graph Problems, *Theor. Comput. Sci.*, 1 (1976) 237-267.

[3] W.K. Hale, Frequency Assignment: Theory and Applications, *Proc. IEEE*, 68 (1980) 1497-1514.

[4] A.M. Odlyzko and N.J.A. Sloane, New Bounds on the Number of Unit Spheres that can touch a Unit Sphere in n Dimensions, *Journal of Combinatorial Theory (A)*, 26 (1979) 210-214.

[5] F.S. Roberts, Indifference Graphs, *Proof Techniques in Graph Theory*, (ed. F. Harary) Academic Press, New York (1969) 139-146.

[6] L.J. Stockmeyer, Planar 3-Colorability is NP-Complete, *SIGACT News*, 5 #3 (1973) 19-25.

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