Question 1

Question 1a

1a 1)

$\mathcal{N}(x)$ is the Cumulative Distribution Function for the Standard Normal Distribution, in this report we refer to $\mathcal{N}(x)$ as $\Phi(x)$ throughout.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy \qquad \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

1a 2)

$$d_1 = \frac{ln(S/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t} \qquad \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Show that:

$$\begin{aligned}
S\,\Phi'(d_1) &= K\, e^{-r(T-t)}\ \Phi'(d_2) \\
&= K\, exp(-r(T-t))\ \Phi'(d_1 - \sigma\sqrt{T-t}) \\
&= K\, exp(-r(T-t))\ \frac{1}{\sqrt{2\pi}}\ exp\left(-\frac{(d_1 - \sigma\sqrt{T-t})^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}}\ exp\left(ln(K) - r(T-t) - \frac{(d_1 - \sigma\sqrt{T-t})^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}}\ exp\left(ln(K) - r(T-t) - \frac{d_1^2 + \sigma^2(T-t)}{2} + d_1\sigma\sqrt{T-t}\right) \\
&= exp\left(ln(K) - r(T-t) - \frac{\sigma^2(T-t)}{2} + d_1\sigma\sqrt{T-t}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \\
&= exp\left(ln(K) - r(T-t) - \frac{\sigma^2(T-t)}{2} + ln(S) - ln(K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right)\ \Phi'(d_1) \\
&= exp\Big(ln(S)\Big)\ \Phi'(d_1) \\
&= S\,\Phi'(d_1) & (\therefore\ QED)
\end{aligned}$$

1a 3)

$$\frac{\partial d_1}{\partial S} = \frac{1}{\sigma S\sqrt{T-t}} \qquad \frac{\partial d_2}{\partial S} = \frac{1}{\sigma S\sqrt{T-t}}$$

1a 4a)

Across the field of economics certain derivatives are so commonly used that they have unique symbols, in this example we consider $\frac{\partial c}{\partial t}$ (Θ). We also introduce a variable ψ for brevity.

$$c = S\,\Phi(d_1) - K\, exp(-r(T-t))\,\Phi(d_2) \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy \qquad \psi = exp(-r(T-t))$$

Show that:

$$\frac{\partial c}{\partial t} = -rK\,exp(-r(T-t))\Phi(d_2) - S\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial t}\bigg(S\,\Phi(d_1) - K\,exp(-r(T-t))\,\Phi(d_2)\bigg)$$

$$= S\frac{\partial}{\partial t}\bigg(\Phi(d_1)\bigg) - K\frac{\partial}{\partial t}\bigg(\psi\Phi(d_2)\bigg)$$

$$\frac{\partial\psi\Phi(d_2)}{\partial t} = \psi\cdot\frac{\partial\Phi(d_2)}{\partial t} + \frac{\partial\psi}{\partial t}\cdot\Phi(d_2) \quad \text{where} \quad \frac{\partial\psi}{\partial t} = r\,\psi,\ \frac{\partial\Phi(d_2)}{\partial t} = \Phi'(d_2)\frac{\partial d_2}{\partial t} \quad \text{(Apply product rule)}$$

$$= S\frac{\partial}{\partial t}\bigg(\Phi(d_1)\bigg) - K(\psi\Phi'(d_2)\frac{\partial d_2}{\partial t} + r\psi\Phi(d_2))$$

$$= S\frac{\partial}{\partial t}\bigg(\Phi(d_1)\bigg) - S\Phi'(d_1)\frac{\partial d_2}{\partial t} - K\psi r\Phi(d_2) \qquad \text{(Recall that } \Phi'(d_1) = K\psi\Phi'(d_2))$$

$$= S\Phi'(d_1)\frac{\partial d_1}{\partial t} - S\Phi'(d_1)\frac{\partial d_2}{\partial t} - K\psi r\Phi(d_2)$$

$$= S\Phi'(d_1)\bigg(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\bigg) - K\psi r\Phi(d_2)$$

$$= S\Phi'(d_1)\bigg(\frac{-\sigma}{2\sqrt{T-t}}\bigg) - K\psi r\Phi(d_2)$$

$$= -rK\,exp(-r(T-t))\Phi(d_2) - S\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}} \qquad (\therefore\ QED)$$

1a 4b)

In addition to Θ, $\frac{\partial c}{\partial S}$ is a commonly used derivative, referred to as the Hedge Ratio (Δ).

Show that:

$$\frac{\partial c}{\partial S} = \Phi(d_1)$$

$$\frac{\partial c}{\partial S} = \frac{\partial}{\partial S}\bigg(S\,\Phi(d_1) - K\,exp(-r(T-t))\,\Phi(d_2)\bigg)$$

$$= \frac{\partial}{\partial S}\bigg(S\,\Phi(d_1)\bigg) - K\psi\frac{\partial}{\partial S}\bigg(\Phi(d_2)\bigg)$$

$$\frac{\partial S\,\Phi(d_1)}{\partial S} = S\cdot\frac{\partial\Phi(d_1)}{\partial S} + \frac{\partial S}{\partial S}\cdot\Phi(d_1) \quad \text{where} \quad \frac{\partial S}{\partial S} = 1,\ \frac{\partial\Phi(d_1)}{\partial S} = \Phi'(d_1)\frac{\partial d_1}{\partial S} \quad \text{(Apply product rule)}$$

$$= S\Phi'(d_1)\frac{\partial d_1}{\partial S} + \Phi(d_1) - K\psi\frac{\partial}{\partial S}\bigg(\Phi(d_2)\bigg)$$

$$= S\Phi'(d_1)\frac{\partial d_1}{\partial S} + \Phi(d_1) - K\psi\Phi'(d_2)\frac{\partial d_2}{\partial S}$$

$$= S\Phi'(d_1)\frac{\partial d_1}{\partial S} + \Phi(d_1) - S\Phi'(d_1)\frac{\partial d_2}{\partial S} \qquad \text{(Recall that } \Phi'(d_1) = K\psi\Phi'(d_2))$$

$$= S\Phi'(d_1)\bigg(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}\bigg) + \Phi(d_1)$$

$$= S\Phi'(d_1)\big(0\big) + \Phi(d_1) \qquad \text{(Recall that } \tfrac{\partial d_1}{\partial S} = \tfrac{\partial d_2}{\partial S})$$

$$= \Phi(d_1) \qquad (\therefore\ QED)$$

1a 5)

Given that:

$$\frac{\partial^2 c}{\partial S^2} = \Phi'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$

And substituting this into the equation for a European stock option not paying dividends:

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc = 0$$

We can derive the formula for the Black Scholes (BS) call option as follows:

$$\begin{aligned}
0 &= -rK\,exp(-r(T-t))\Phi(d_2) - S\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}} + rS(\Phi(d_1)) + \frac{1}{2}\sigma^2 S^2\left(\Phi'(d_1)\frac{1}{S\sigma\sqrt{T-t}}\right) - rc \\
rc &= -rK\psi\Phi(d_2) - S\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}} + rS(\Phi(d_1)) + S\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}} \\
rc &= -rK\psi\Phi(d_2) + rS(\Phi(d_1)) \\
c &= S(\Phi(d_1)) - K\psi\Phi(d_2) \\
c &= S\,\Phi(d_1) - K\,exp(-r(T-t))\,\Phi(d_2)
\end{aligned}$$

Question 1b

To work on the largest possible complete options dataset from `FTSEOptionsData` (D), we can run a constrained optimisation problem to maximise the amount of contiguous daily data for each option i, in both put (p) and call (c) such that all options contain complete data for the same period ($s \rightarrow e$), we denote this series using the format ($D_{(c|p),i,s,e}$), with the optimisation shown in Equation 1.

$$\begin{aligned}
\underset{I,s,e}{\text{maximise}} \quad & \sum_{t=\{c,p\},\, i\in I} |D_{t,i,s,e}| \\
\text{subject to} \quad & D_{t,i,j,j} \neq \emptyset,\; j = 1,\ldots,e-s
\end{aligned} \tag{1}$$

The resulting 56 options, and 218 contiguous days were 12/02/2018 thru 12/12/2018 (days 1 - 218). As these options were JAN19 options, we can deduce that they expired on day 245 of our data set, as most stock options expire on the third Saturday of the month, however can only be traded on that Friday[1]. In addition to this `FTSEOPtionsData` provides us with 10 year Government bench mark bid yields, however these needed to be adjusted to calculate annual interest rate rather than 10 year period returns. We can achieve this by using the formula $r_{annual} = r^{0.1} - 1$.

Using this adjusted dataset, which we will denote as D_{all}, we have four of five parameters that can be used with the BS formula to predict option prices. The final required parameter is volatility (σ). Hull states that we can estimate volatility from historical stock prices (S) [1], using Equation 2:

$$\hat{\sigma} = \frac{s}{\sqrt{T-t}} \qquad s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(u_i - \bar{x})^2} \qquad u_i = ln\left(\frac{S_i}{S_{i-1}}\right) \tag{2}$$

The final invariants that we specify before estimating option prices are: 252 days in a trading year and a windows size (N) for calculating $\hat{\sigma}$ of 54 days preceding the option prediction date.

[1] https://investinganswers.com/articles/understanding-option-expiration-dates-and-cycles

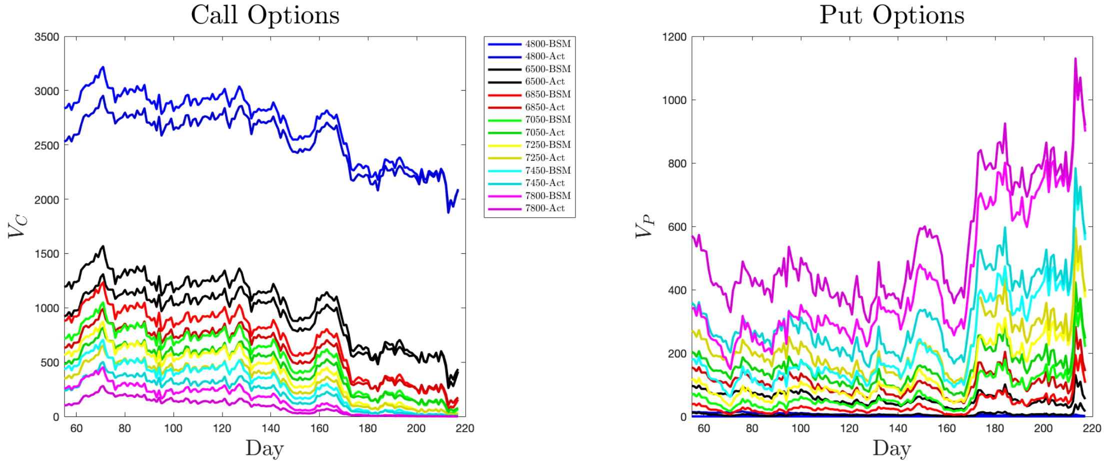

Figure 1: A comparison of BS option price estimates on D_{all} using a 54 day sliding window against actual option prices for the day.

As shown in Figure 1, whilst volatility estimations from a historical sliding window provided correct directionality for option pricing, the BS formula tended to systematically overestimate the value of call options and underestimate the value of put options, this error was amplified higher and lower options respectively. In addition to this, with greater values of τ (time to expiration), the put/call estimation was less accurate.

Question 1c

If we rearrange the BS formula, given S, K, r, τ and $V_{C|P}$, we can find the Implied Volatility σ of a an option at τ. We can produce a 2D projection of σ for a 30 day time frame, for a selection of K, in addition to presenting the historical volatility estimate proposed by Hull [1] using a 54 day sliding window, see Figure 2.

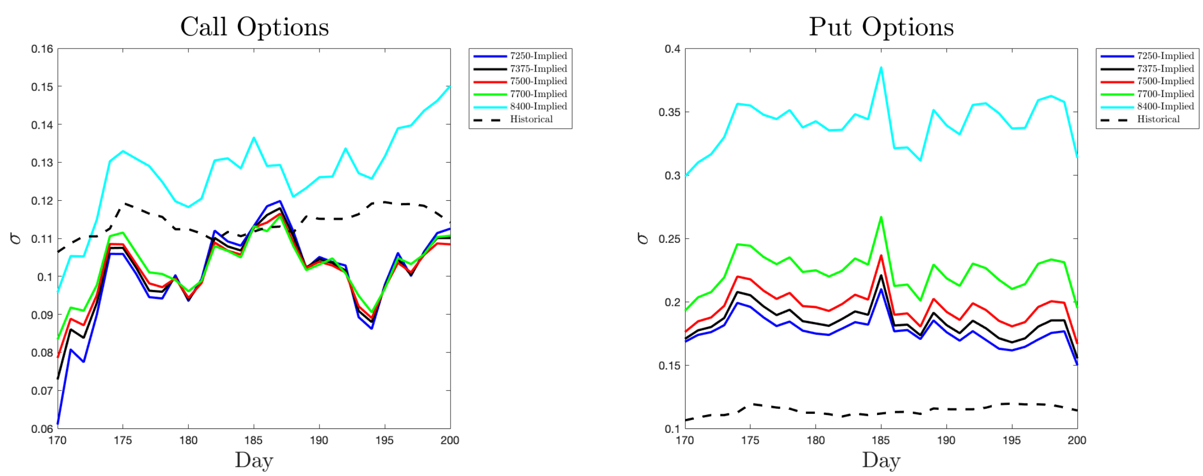

Figure 2: Variation of σ implied from the BS formula and associated call/put options, and the associated historical estimates using a 54 day sliding window.

As shown above, historical data can provide a rough idea of option volatility, however normally volatility will be very different for different K; using stock price history will not necessarily reflect future option volatility. We can examine this relationship by projecting D_{all} for days 170-200 into 3 dimensions, as per Figure 3.

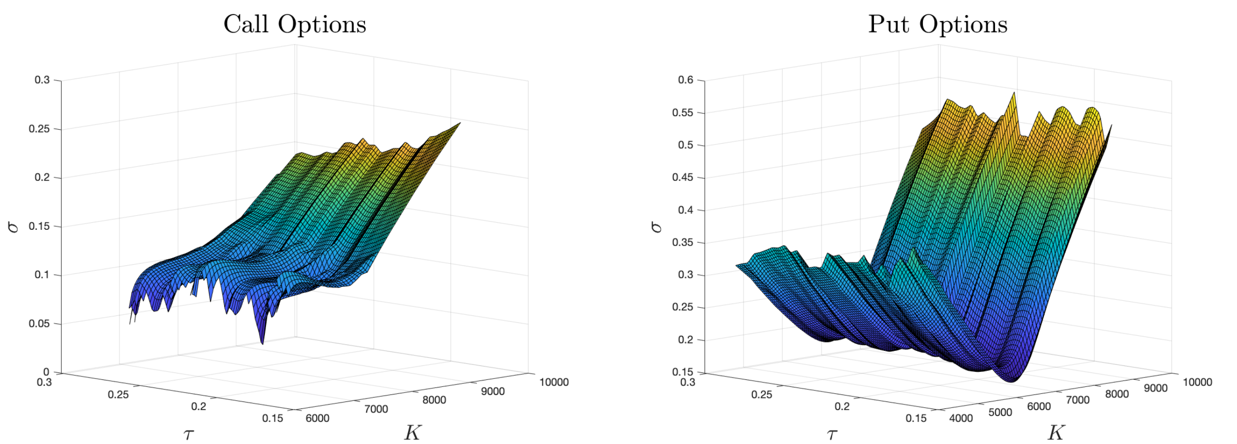

Figure 3: Implied Volatility surfaces for FTSE 100 options in D_{all}.

As shown in Figure 3, out-of-the-money and in-the-money options generally exhibit higher σ than at-the-money strike prices, known as a 'volatility smile' [2] this pattern was especially prominent in the case of put options. In theory this should not be the case as, as it predicts a flat σ across K, however in practice BS fails to account for other factors that impact σ, including liquidity and demand [3].

Question 1d

An alternative approach for options pricing is to use a binomial lattice. Under such an approach we can, for i intervals in τ, generate two paths, representing an increase in stock price u% and a decrease in stock price d%. Following this we calculate the option price at t via payoff weighted backtracking given r, K, σ.

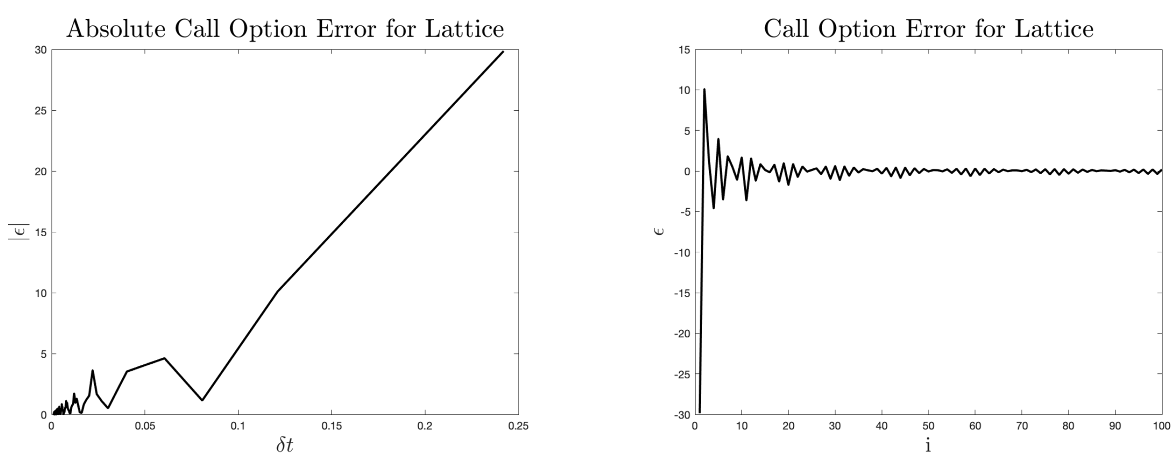

Figure 4: A comparison between option price calculated by BS and using varied steps in a binomial lattice for a European call option with the parameters $S = 7013.90$, $K = 7300$, $\tau = 0.25$, $r = 3.31\%$, $\sigma = 11.33\%$.

In this question we consider the error ϵ and absolute error $|\epsilon|$ exhibited between the lattice method for a European call option. To investigate this we use the following parameters $S = 7013.90$, $K = 7300$, $\tau = 0.25$, $r = 3.31\%$, $\sigma = 11.33\%$ from `FTSEOptionsData`.

In Figure 4 we can see that as δt decreases, $\lim_{\delta t \to 0} |\epsilon| = 0$; thus for a European call option the lattice is equivalent to pricing via BS. Interestingly, to do this, as i increases the ϵ oscillates until convergence.

The following `MATLAB` code for pricing an American option from [4].

```
for tau=1:N
    for i= (tau+1):2:(2*N+1-tau)
        hold = p_u*PVals(i+1) + p_d*PVals(i-1);
        PVals(i) = max(hold, K-SVals(i));
    end
end
```

For an American put option we will sell for price K at some time t. As this is an American option we can exercise at any t from $0 \to T$; if $S_t > K$ it is not desirable to exercise the put option.

In the `MATLAB` excerpt we can assume that we have already generated the lattice, storing put values in `PVals`, and stock values in `SVals`. This excerpt specifically presents the backwards propagation step for an American put option. Given N steps, at at each time step τ we have the right to exercise the option; evaluate whether it is better to hold the option given expected future payoff or sell, in which case we want $K - S_t$ to provide greater returns than the former.

With regards to looping convention, this excerpt stores a binary tree in a flat list, calculate the put options based on the previous two; walking backwards in the lattice. Therefore the final value for the American put option is stored in the middle element of `PVals`, (N+1).

To make this except cater to an American call option we would have to modify the `max` statement to be `max(hold, SVals(i)-K)`, as for a call option we are looking to buy the option, and would want to purchase the option at $K < S_t$.

Question 2

Question 2a

In this question we investigate the approximation of option prices using nonparametric neural network type models, as per Hutchinson et al. [5].

To ensure a realistic simulation of a stock, we consider the values observed in `FTSEOptionsData`. For this investigation we will use the same set of K as used in Question 1b in addition to the constants $\sigma = 20\%, r = 4\%, S_0 = 7000$, as per Hutchinson et al. [5].

In addition to the above, we have used a seed of 9 for `MATLAB`'s random number generator for reproducability.

To generate a set of stock prices, from which option prices can be created using the BS formula, we adopt the same approach as Hutchinson et al. [5] and use the Geometric Brownian Motion equation to generate 6 months (126 trading days) of S, and an annual return $\mu = 10\%$, this can be seen in Equation 3. All options generated from this distribution expire on day 127; enabling the last day to be tradeable.

$$S_t = S_0 \cdot exp\left[\sum_{i=1}^{N} z_i\right]\ ,\ Z \sim \mathcal{N}(\mu/252, \sigma^2/252) \tag{3}$$

Question 2b

To fit the RBF we consider a training phase of 40 days from the 6 month period. The first function we used in creating the RBF is `fitgmdist`. This function enables us to generate four Gaussian distributions, and return their means and co-variance matrices. Using these distributions we can create an RBF model by generating x for the 40 training points.

$$\hat{C} = \sum_{i=1}^{4} \lambda_i \phi(\boldsymbol{x}) + \boldsymbol{w}^T \boldsymbol{x} + w_0 \qquad \boldsymbol{x} = \left[S/K\ \tau\right]^T \qquad \phi(\boldsymbol{x}) = \sqrt{(\boldsymbol{x} - \boldsymbol{\mu}_i)^T \Sigma_i (\boldsymbol{x} - \boldsymbol{\mu}_i)} \tag{4}$$

$\phi(x)$ in this instance represents the Mahalanobis distance, returning the number of standard deviations a point is from μ. To determine the weights of the RBF; train the model we use the `LinearModel.fit` function of `MATLAB`, passing the call options generated by BS for the 40 training points as target points. The resulting parameters used to build the RBF model were:

$$\lambda_1 = -2.9562,\ \lambda_2 = 2.5033,\ \lambda_3 = 2.3187,\ \lambda_4 = 1.3908,\ w_0 = -0.5640$$

$$\boldsymbol{w} = \begin{bmatrix} 0.5662 \\ 0.0892 \end{bmatrix},\ \boldsymbol{\mu}_1 = \begin{bmatrix} 1.0233 \\ 0.3670 \end{bmatrix},\ \boldsymbol{\mu}_2 = \begin{bmatrix} 0.9585 \\ 0.4317 \end{bmatrix},\ \boldsymbol{\mu}_3 = \begin{bmatrix} 0.9714 \\ 0.4855 \end{bmatrix},\ \boldsymbol{\mu}_4 = \begin{bmatrix} 1.0501 \\ 0.4213 \end{bmatrix}$$

$$\Sigma_1 = \begin{bmatrix} 0.0031 & -0.0002 \\ -0.0002 & 0.0002 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 0.0023 & -0.0004 \\ -0.0004 & 0.0009 \end{bmatrix},\ \Sigma_3 = \begin{bmatrix} 0.0031 & 0.0001 \\ 0.0001 & 0.0001 \end{bmatrix},\ \Sigma_4 = \begin{bmatrix} 0.0433 & -0.0010 \\ -0.0010 & 0.0018 \end{bmatrix}$$

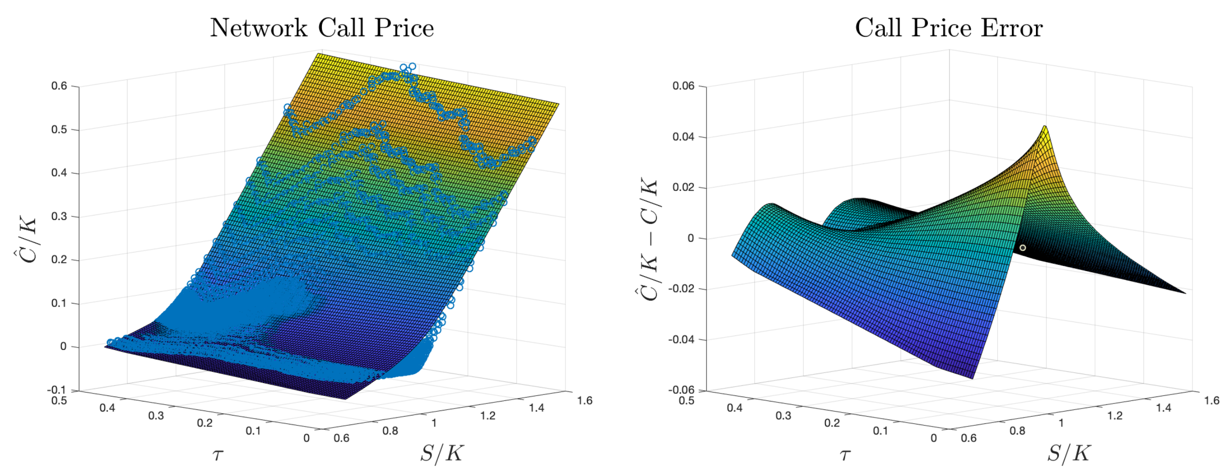

Figure 5: Calculating the normalized call option price ($\frac{\hat{C}}{K}$) using the RBF model. Network Call Price shows how the result of the RBF (surface) compares with the results evaluated from BS (points). Delta Error plots the difference between the surface and these points as a surface.

Overall, whilst the model got progressively worse at determining $\frac{\hat{C}}{K}$ as τ decreased, most likely as a result of exiting the training phase from where it has learnt values, the magnitude of absolute error is not that significant; still providing a good fit near $\tau = 0$ for approximating the BS call option price. However, given the shape and exponential nature of the error plane, one could reasonably assume that if the training period has been shorter, the fit would have been much worse nearer $\tau = 0$.

Question 2c

In this section we look at attempting to find the Hedge Ratio (Δ), an important parameter as it allows traders to balance portfolio risk [6]. To calculate Δ using the RBF model that we created, we can rearrange the BS formula to get:

$$\Delta_{RBF} = \frac{\frac{\hat{C}}{K} + e^{-r\tau}\Phi(d_2)}{\frac{S}{K}} \qquad d_2 = \frac{ln\left(\frac{S}{K}\right) + \tau \cdot (r + \frac{\sigma^2}{2})}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau} \tag{5}$$

All of these values can be deduced from known option data or calculated from the RBF.

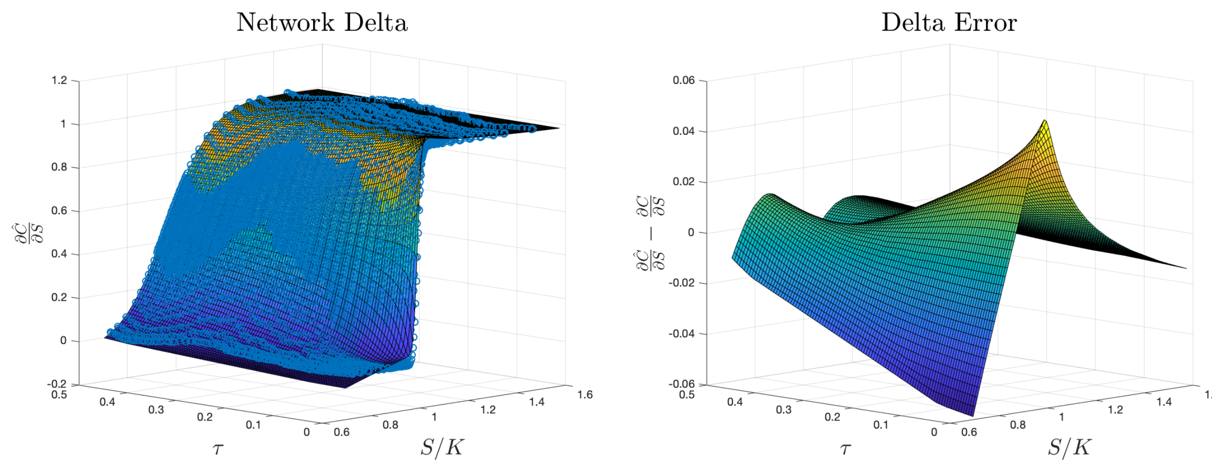

Figure 6: Calculating the Hedge Ratio (Δ) using the RBF generated in Question 2b. Network Delta shows how the result of the RBF (surface) compares with results evaluated from BS (points). Delta Error plots the difference between the surface and these points.

As shown in Figure 6 the RBF creates a good approximation of the Hedge Ratio. Interestingly, in this case the error shows a similar pattern to that of the call option price, however upon further inspection this is no surprise given $\Delta_{RBF} \propto \frac{\hat{C}}{K}$ given deduction from Equation 5.

References

[1] J. C. Hull, *Options futures and other derivatives*. Pearson Education India, 2003.

[2] M. Rubinstein, "Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active cboe option classes from august 23, 1976 through august 31, 1978," *The Journal of Finance*, vol. 40, no. 2, pp. 455–480, 1985.

[3] E. Ghysels, A. C. Harvey, and E. Renault, “5 stochastic volatility,” *Handbook of statistics*, vol. 14, pp. 119–191, 1996.

[4] P. Brandimarte, *Numerical methods in finance and economics: a MATLAB-based introduction*. John Wiley & Sons, 2013.

[5] J. M. Hutchinson, A. W. Lo, and T. Poggio, “A nonparametric approach to pricing and hedging derivative securities via learning networks,” *The Journal of Finance*, vol. 49, no. 3, pp. 851–889, 1994.

[6] L. Montesdeoca and M. Niranjan, “Extending the feature set of a data-driven artificial neural network model of pricing financial options,” in *2016 IEEE Symposium Series on Computational Intelligence (SSCI)*, IEEE, 2016, pp. 1–6.