$\delta I = 0$ means that the variation of I is zero. ~~The rate of change of I at a point requires every possible direction~~

A function (for example I) has a stationary value at a certain point if the rate of change of the function in every possible direction from that point vanishes.

Notice that this does not imply an extremum, it can also be a saddlepoint.

For stability problems, this will be a minimum (must minimize potential energy). Otherwise it will not.

Consider a function of an arbitrary number of variables

$$F = F(u_1, u_2, \ldots, u_n)$$

F is continuous and differentiable in the variables u_k.

We will use variations to determine if ~~certain points~~ F has stationary values at certain points.

This is not actual change, for example $\frac{\partial F}{\partial u_i}$

Rather, it is a mathematical experiment: a virtual and infinitesimal change in the position. Lagrange introduced the δ notation to contrast with d.

Infinitesimal virtual changes in the coordinates are written $\delta u_1, \delta u_2, \ldots, \delta u_n$

The corresponding change in the function is

$$\delta F = \frac{\partial F}{\partial u_1}\delta u_1 + \frac{\partial F}{\partial u_2}\delta u_2 + \ldots + \frac{\partial F}{\partial u_n}\delta u_n$$

In order to deal with finite quantities, let

$$\delta u_1 = \epsilon a_1 ; \quad \delta u_2 = \epsilon a_2 , \ \ldots , \quad \delta u_n = \epsilon a_n$$

where $a_1, a_2, \ldots, a_n$ are the components of the virtual displacement in directions $1, 2, \ldots, n$. ϵ is a parameter that tends to zero. Hence

$$\frac{\delta F}{e} = \frac{\partial F}{\partial u_1} a_1 + \frac{\partial F}{\partial u_2} a_2 + \ldots + \frac{\partial F}{\partial u_n} a_n$$

if F has a stationary value at point $[u_1, u_2, \ldots, u_n]$,

then $\displaystyle\frac{\delta F}{e} = \sum_K \frac{\partial F}{\partial u_K} a_K = 0$ First variation

The virtual displacement a is arbitrary, so to hold in general, each term of the sum must independently be zero

$$\frac{\partial F}{\partial u_K} = 0 \quad \text{for } K = 1, 2, \ldots, n$$

Condition for stationary value

The Taylor expansion for more than one variable is given by: about $u_1, u_2, \ldots$

$$F(u_1 + \epsilon a_1, u_2 + \epsilon a_2, \ldots, u_n + \epsilon a_n) =$$

$$F(u_1, u_2, \ldots, u_n) + \sum_{j=1}^{n} \frac{\partial F}{\partial u_j} (\cancel{u_j} + \epsilon a_j - \cancel{u_j})$$

$$+ \frac{1}{2!} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 F}{\partial u_j \partial u_k} (\cancel{u_j} + \epsilon a_j - \cancel{u_j})(\cancel{u_k} + \epsilon a_j - \cancel{u_k})$$

$$+ \frac{1}{3!} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\partial^3 F}{\partial u_j \partial u_k \partial u_\ell} (\cancel{u_j} + \epsilon a_j - \cancel{u_j})(\cancel{u_k} + \epsilon a_k - \cancel{u_k})(\cancel{u_\ell} + \epsilon a_\ell - \ldots$$

Let $F(u_1 + \epsilon a_1, u_2 + \epsilon a_2, \ldots, u_n + \epsilon a_n) - F(u_1, u_2, \ldots, u_n) = \Delta F$,

then

$$\Delta F = \epsilon \sum_{j} \frac{\partial F}{\partial u_j} a_j + \frac{1}{2} \epsilon^2 \sum_{j} \sum_{k} \frac{\partial F}{\partial u_j \partial u_k} a_j a_k$$

$$+ \frac{1}{6} \epsilon^3 \sum_{j} \sum_{k} \sum_{\ell} \frac{\partial F}{\partial u_j \partial u_k \partial u_\ell} a_j a_k a_\ell + \ldots$$

If we are at a stationary value $\frac{\partial F}{\partial u_j} = 0 \quad \forall j$

Also, ϵ is arbitrary and goes to zero, so we can use a sufficiently small one to get rid of terms higher than ϵ^2 using δ notation,

second variation

$$\Delta F = \frac{1}{2} \delta^2 F = \frac{1}{2} \epsilon^2 \sum_{j} \sum_{k} \frac{\partial F}{\partial u_j \partial u_k} a_j a_k$$

★ if $\delta^2 F$ is always positive for all possibee values of $a_1, a_2, \dots, a_n$ then F is at a minimum.

★ if $\delta^2 F$ is always negative, F is at a maximum.

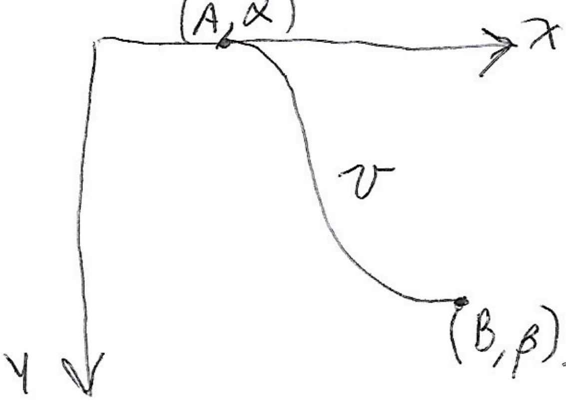
★ if $\delta^2 F$ is positive for some values of $a_1, a_2, \dots, a_n$, and negative for others, then F is not at either a minimum of maximum.

How do these elements interact when F is a definitive integral?

One of the first problems to be addressed with calculus of variations (1696)

Consider the Brachistochrone problem, by Johann Bernoulli

"Given two points A and B in a vertical plane, what is the curve traced out by a point acted only by gravity, which starts at A and reaches B in the shortest time?"



Since $v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v}$

$$\int_A^B dt = t_{AB} = \int_A^B \frac{ds}{v}$$

The velocity is given by energy conservation $\frac{1}{2}\cancel{m}v^2 = \cancel{m}gy$

$$\Rightarrow v = \sqrt{2gy}$$

Also, $(ds)^2 = (dx)^2 + (dy)^2 =$

$$\frac{1}{1} + \frac{(dy)^2}{(dx)^2} = \frac{(dx)^2 + (dy)^2}{(dx)^2} \Rightarrow ds^2 = dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

Let $\dot{y} \equiv \frac{dy}{dx}$

so $ds = \sqrt{1 + \dot{y}}\, dx$

Putting everything togheter,

$$t_{AB} = \int_A^B \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\, dx$$

To find the minimum, take first variation, make equal to zero.

Since the integral is a function of y, $\dot{y}$, and x, we can write it as $I = \int_A^B F(y, \dot{y}, x)\, dx$

An integral can be approximated by a sum. The interval between $\overset{x=A}{A}$ and $\overset{x=B}{B}$ is divided into smaller, equal intervals

$$x_0 = A,\ x_1,\ x_2,\ \dots,\ x_n,\ x_{n+1} = B$$

the corresponding "heights" are

$$y_0 = \alpha,\ y_1,\ y_2,\ \dots,\ y_n,\ y_{n+1} = \beta \qquad y_k = f(x_k)$$

The derivative $\dot{y} = \frac{dy}{dx}$ can be replaced by

$$z_k = \frac{\Delta y}{\Delta x} = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}$$

So $I \approx S = \sum_{k=0}^{n} F(y_k, z_k, x_k)(x_{k+1} - x_k)$

S can get arbitrarily close to I, when $n \to \infty$, $\Delta x \to 0$

To make things easier to compute, replace y_k with y_{k+1}

This is allowed if y_k and y_{k+1} are arbitrarily close

$$S' = \sum_{j=0}^{n} F(y_{j+1}, z_j, x_j)(x_{j+1} - x_j)$$

Using the first variation $\delta S' = \epsilon \sum_{k} \frac{\partial S'}{\partial u_k} a_k$

Let $u_k = y_{k+1}$, then

variable

specific point

comes from chain rule

$$\frac{\partial S'}{\partial y_{k+1}} = \left(\frac{\partial F}{\partial y}\right)_{x=x_k} (x_{k+1} - x_k)$$

$$+ \left(\frac{\partial F}{\partial \dot{y}}\right)_{x=x_k} \frac{1}{\cancel{x_{k+1} - x_k}} \cancel{(x_{k+1} - x_k)}$$

$$- \left(\frac{\partial F}{\partial \dot{y}}\right)_{x=x_{k+1}} \frac{1}{\cancel{x_{k+1} - x_k}} \cancel{(x_{k+1} - x_k)}$$

from definition of z_k

This one will appear in two terms

$$\frac{\partial S'}{\partial y_{k+1}} = \left(\frac{\partial F}{\partial y}\right)_{x=x_k} (x_{k+1} - x_k) + \left(\frac{\partial F}{\partial \dot{y}}\right)_{x=x_k} - \left(\frac{\partial F}{\partial \dot{y}}\right)_{x=x_{k+1}}$$

with $-\Delta\left(\frac{\partial F}{\partial \dot{y}}\right) = \left(\frac{\partial F}{\partial \dot{y}}\right)_{x=x_k} - \left(\frac{\partial F}{\partial \dot{y}}\right)_{x=x_{k+1}}$

and dividing by $\Delta x_k = x_{k+1} - x_k$

$$\frac{\partial S'}{\partial y_{k+1}} = \cancel{\frac{\Delta x_k}{\Delta x_k}} \left(\frac{\partial F}{\partial y}\right)_{x=x_k} - \frac{\Delta}{\Delta x_k}\left(\frac{\partial F}{\partial \dot{y}}\right)_{x=x_k} = 0$$

$$\frac{\partial S'}{\partial y_{K+1}} = \left[\frac{\partial F}{\partial y} - \frac{\Delta}{\Delta x}\left(\frac{\partial F}{\partial \dot{y}}\right)\right]_{x=x_K} = 0 \quad \forall\ K$$

In the limit $\Delta x \to 0$, we recover the differential Eq.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0$$

Compare to Eq. 2.11

This approach is due to Euler