Eq. 3.63

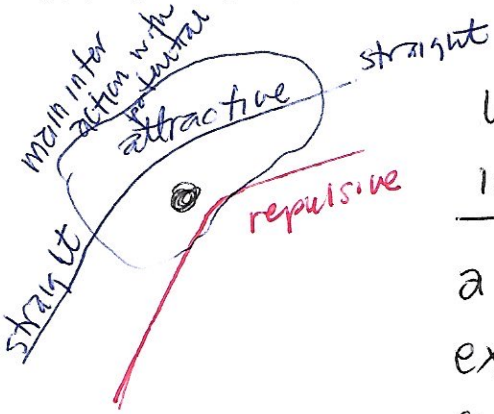
$\frac{l^2}{mk} = 1$ as well, so $a(1-e^2) = \frac{l^2}{mk}$

$$\Rightarrow l = \sqrt{amk\,(1-e^2)}$$

Scattering in a central force field

9/30/21

Consider a uniform beam of particles, whether electrons, α-particles, or planets is irrelevant! but they have the same mass and incident energy. They are also directed towards a center of force.

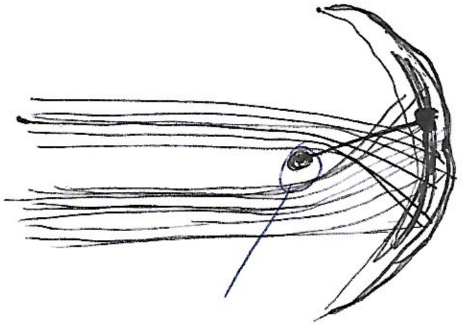


We will characterize the beam by its intensity or flux density. Remember that a flux is a quantity through an area, for example # of particles per unit area. A flux density is flux per unit time. So intensity $\frac{\frac{\#\text{ of particles}}{\text{unit area}}}{\text{unit time}}$

In general, the path of the incident beam will be different than that of the outgoing beam. The differential scattering cross section is given by

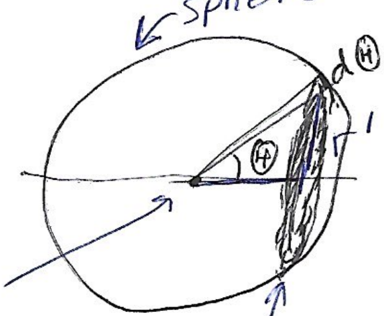
$$\sigma(\Omega)\,d\Omega = \frac{\#\text{ of particles scattered into solid angle } d\Omega/\text{time}}{\text{incident intensity}}$$

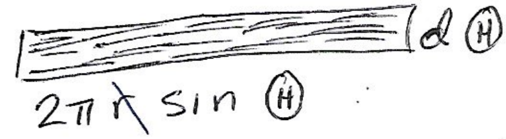
$d\Omega$ is an element of solid angle in the direction Ω.



Since the central force is spherically symmetric, $\cancel{d\Omega = 2\pi \sin\Theta\, d\Theta}$

$$d\Omega = 2\pi \sin\Theta\, d\Theta$$





or unit circle

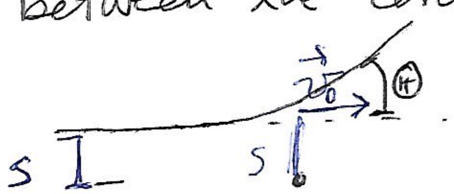
$r' = r\sin\Theta$

Because it is a solid angle flux, should be independent of r, the same for every r, so

$$d\Omega = 2\pi \sin\Theta\, d\Theta$$

Eq. 3.89

The impact parameter s is the perpendicular distance between the center of force and the incident velocity.



Just like in the case of Keplerian orbits, due to the spherical symmetry, angular momentum is conserved. and its magnitude is $l = |\vec{r} \times \vec{p}| = |\vec{r}||m\vec{v}_0|\sin\alpha_{r,p} = s m v_0 \sin(\frac{\pi}{2})$

$$l = m v_0 s = s\sqrt{2E m^{\cancel{2}}/\cancel{m}} = s\sqrt{2mE}$$

Eq. 3.90

Since the potential energy is zero at $\vec{r} = \pm\infty$, there is only kinetic energy, so $E = \frac{1}{2} m v_0^2 \Rightarrow v_0 = \sqrt{\frac{2E}{m}}$

In classical physics, if E and s are fixed, the angle of scattering Θ is determined uniquely. This is obviously not the case in quantum physics.

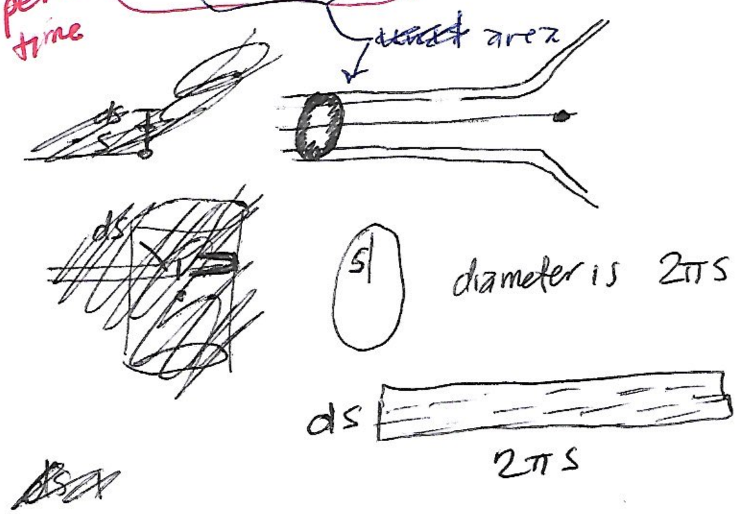
Assume that different values of s cannot produce the same scattering. This makes sense, e.g.

we can see that a larger s results in a smaller $d\Theta$. The ~~num~~ # of particles scattered into a solid angle $d\Omega$ between Θ and $\Theta + d\Theta$ is the same as that of the corresponding s and $s+ds$

number of particles incident between s and $s+ds$ per unit time

intensity # of particles per unit area per unit time

$$I \, 2\pi s \, |ds|$$



On the other side, we use previous diagram and get

$$d\Omega = 2\pi \sin\Theta \, d\Theta$$

$\sigma(\Omega)\,d\Omega$ was # of particles scattered into solid angle per unit time ~~per unit area~~ Intensity, so

$\sigma(\Theta) I \, 2\pi \sin\Theta \, d\Theta$ is # particles scattered per unit time.

so by conservation of particles,

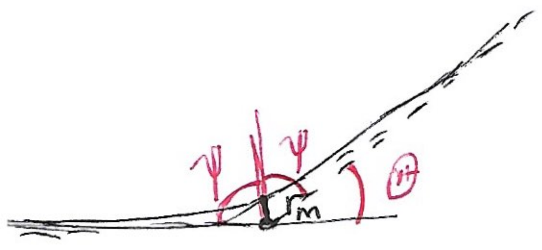
Eq. 3.91

$$\cancel{2\pi} \cancel{I} s\,|ds| = \cancel{2\pi}\, \sigma(\Theta)\, \cancel{I} \sin\Theta \, |d\Theta|$$

Absolute values used due to symmetry, and to keep number of particles positive

If ~~the~~ s is a function of the energy and scattering angle $s(\Theta, E)$, then $\sigma(\Theta) = \dfrac{s}{\sin\Theta}\left|\dfrac{ds}{d\Theta}\right|$

If ψ is the angle between incoming or outgoing asymptote and the direction of closest approach,



$$\Theta = \pi - 2\psi$$

In a Keplerian orbit

$$\int d\theta = \int \frac{\ell\, dr}{mr^2\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}$$

$$\theta = \int_{r_0}^{r} \frac{dr}{\cancel{m} r^2 \sqrt{\frac{m\,2}{\cancel{m}\,\ell^2}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}} + \theta_0$$

$$\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}} + \theta_0$$

we can get ψ by letting $r_0 = \infty$ and $\theta_0 = \pi$ (incoming)

or $r = r_m$ $\quad \theta = \pi - \psi$ along r_m ~~outgoing~~, so

$$\pi - \psi = \int_{\infty}^{r_m} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}} + \pi$$

$$\psi = \int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}}$$

with $\ell = s\sqrt{2mE}$, $\qquad 2mE = \ell^2/s^2$

$$\Psi = \int_{r_m}^{\infty} r \sqrt{\frac{r^2 \ell^2}{\ell^2 s^2} - }$$

$$\Theta(s) = \pi - 2\int_{r_m}^{\infty} \frac{s\,dr}{r\sqrt{r^2\left(1 - \frac{V(r)}{E}\right) - s^2}}$$

let $u = 1/r$

$$\Theta(s) = \pi - 2\int_{0}^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V(u)}{E} - s^2 u^2}}$$