More on the Calculus of variations

Consider again a definitive integral of the form

$$I = \int_A^B F(y, \dot{y}, x)\,dx$$

We showed that, using differential calculus, in the limit $\Delta x \to 0$, we can deal with variations. Now let's look at the more direct method.

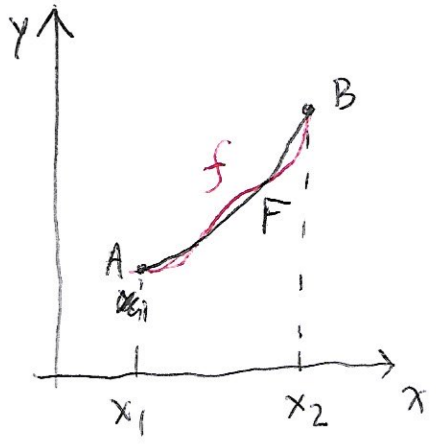
Instead of F, consider a new function

infinitesimal parameter

$$f(y(x,\alpha), \dot{y}(x,\alpha), x) = F(y, \dot{y}, x) + \alpha\,\eta(x)$$

f is a path that is infinitely close to F

any continuos + differentiable fn that complies with boundary conditions.



Eq. 2.4

Notice that $y(x,\alpha) = y(x) + \alpha\,\eta(x)$

then $\dot{y}(x,\alpha) = \dot{y}(x) + \alpha\dot{\eta}(x)$

and $f(y(x,\alpha), \dot{y}(x,\alpha), x) = F(y + \alpha\eta, \dot{y} + \alpha\dot{\eta}, x)$

$$F(y+\alpha\eta, \dot{y}+\alpha\dot{\eta}, x) - F(y, \dot{y}, x) = \delta F(y, \dot{y}, x)$$
$$= \alpha\eta(x)$$

so this term is the variation!

From the definition of first variation $\delta F = \epsilon \sum_k \frac{\partial F}{\partial u_k} a_k$

The infinitesimal factor is now α, the variables are u_k {$y, \dot{y}, x$} although $\frac{\partial F}{\partial x} = 0$ by definition. The virtual directions a_k are {$\eta, \dot{\eta}$}. So

$$\delta F = \alpha \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial \dot{y}}\dot{\eta} \right]$$

$$\delta \int_A^B F(y, \dot{y}, x)\,dx = \int_A^B \delta F(y, \dot{y}, x)\,dx = \alpha \int_A^B \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial \dot{y}}\dot{\eta} \right) dx$$

we have an expression that allows us to evaluate <u>directly</u> the variation!

Integration by parts: $\int u\,dv = uv - \int v\,du$

Let $u = \frac{\partial F}{\partial \dot{y}}$ $\quad dv = \dot{\eta}\,dx$

$du = \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right)dx \quad v = \int \dot{\eta}\,dx = \eta$

Integrate second term by parts:

$$\int_A^B \frac{\partial F}{\partial \dot{y}}\dot{\eta}\,dx = \cancel{\frac{\partial F}{\partial \dot{y}}\eta \Big|_A^B}^{\,0} - \int_A^B \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right)\eta\,dx$$

since $\eta(A) = \eta(B) = 0$

Rewrite
$$\frac{\delta I}{\alpha} = \int_A^B \left[\frac{\partial F}{\partial y}\eta - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right)\eta \right] \eta\, dx$$

Let
$$M(x) = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right)$$

$$\frac{\delta I}{\alpha} = \int_A^B M(x)\eta(x)\,dx = 0$$

If we want the variation to be zero, $M(x)$ must be zero at every point between A and B, since $\eta(x)$ is an arbitrary Function

This is know as the fundamental lemma of the calculus of variations

So, as before, we need $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0$ everywhere for the variation to be zero. Now we got it directly rather than in the limit, though. Compare to section 2.2. Goldstein

Hamilton's principle actually stated that

$$\delta I = \int_{t_1}^{t_2} \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\,dt = 0$$

Let's adapt our notation

$y \rightarrow q_k$ (generalized coordinate)

$\dot{y} \rightarrow \dot{q}_k$ rate of change of gen coord

$x \rightarrow t$ independent variable

$F \rightarrow \mathcal{L}$

we apply the notation transformation for each q_K independently. This will produce the following n differential equations

$$+\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_K}\right) = +\frac{\partial \mathcal{L}}{\partial q_K}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_K}\right) - \frac{\partial \mathcal{L}}{\partial q_K} = 0 \qquad (K=1,2,\ldots,n)$$

These are called the Euler-Lagrange differential equations, or the Lagrange equations of motion.

2.4 Lagrange multipliers

Consider the function $F(u_1, u_2, \ldots, u_n)$

if the u_K are all independent, F is at a stationary value if its 1st variation is

$$\delta F = \underbrace{\frac{\partial F}{\partial u_1}\delta u_1}_{\text{each term is independently zero}} + \frac{\partial F}{\partial u_2}\delta u_2 + \ldots + \frac{\partial F}{\partial u_n}\delta u_n = 0$$

But this <u>IS NOT</u> the case if there are constraints.
<u>Holonomic</u> constrains in particular, of the form

$$f(u_1, u_2, \ldots, u_n) = 0$$

each decreases the number of independent variables. each term not independent no more

It is possible to get u_n in terms of the other $n-1$ variables and eliminate, but this is usually cumbersome and non-trivial. Luckily, Lagrange found a way to achieve this that is general and easier.

The variation of the constraint is:

$$\delta f = \frac{\partial f}{\partial u_1}\delta u_1 + \frac{\partial f}{\partial u_2}\delta u_2 + \ldots + \frac{\partial f}{\partial u_n}\delta u_n = \delta 0 = 0$$

If we multiply by some undetermined factor λ, the variation of the constraint times λ is still zero

$$\lambda \delta f = \lambda\left(\frac{\partial f}{\partial u_1}\delta u_1 + \frac{\partial f}{\partial u_2}\delta u_2 + \ldots + \frac{\delta f}{\partial u_n}\delta u_n\right) = 0$$

so we can add it to the variation of F,

$$\delta F = \frac{\partial F}{\partial u_1}\delta u_1 + \frac{\partial F}{\partial u_2}\delta u_2 + \ldots + \frac{\partial F}{\partial u_n}\delta u_n + \lambda\left(\frac{\partial f}{\partial u_1}\delta u_1 + \ldots + \frac{\partial f}{\partial u_n}\delta u_n\right) = 0$$

undetermined factor

so $$\sum_K^n \left(\frac{\partial F}{\partial u_K} + \lambda \frac{\partial f}{\partial u_K}\right)\delta u_K = 0$$

if you want to eliminate u_n, then just find the value of λ that eliminates it, $\frac{\partial F}{\partial u_n} + \lambda \frac{\partial f}{\partial u_n} = 0$. This results in

factor has been determined

$$\sum_K^{n-1} \left(\frac{\partial F}{\partial u_K} + \lambda \frac{\partial f}{\partial u_K}\right)\delta u_K = 0$$

now each term independently zero since variables independent if instead of 1 constraint, there are m constraints

$$f_1(u_1, u_2, \dots, u_n) = 0$$
$$\vdots \qquad \vdots$$
$$f_m(u_1, u_2, \dots, u_n) = 0$$

we apply the same steps for each constraint

$$\lambda_1 \delta f_1 = \lambda_1 \left(\frac{\partial f_1}{\partial u_1} \delta u_1 + \frac{\partial f_1}{\partial u_2} \delta u_2 + \dots + \frac{\partial f_1}{\partial u_n} \delta u_n \right) = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\lambda_m \delta f_m = \lambda_m \left(\frac{\partial f_m}{\partial u_1} \delta u_1 + \frac{\partial f_m}{\partial u_2} \delta u_2 + \dots + \frac{\partial f_m}{\partial u_m} \delta u_m \right) = 0$$

Since each one is zero, add them to variation of F

$$\delta F = \frac{\partial F}{\partial u_1} \delta u_1 + \dots + \frac{\partial F}{\partial u_n} + \lambda_1 \left(\frac{\partial f_1}{\partial u_1} \delta u_1 + \dots + \frac{\partial f_1}{\partial u_n} \delta u_n \right) + \dots + \lambda_m \left(\frac{\partial f_m}{\partial u_1} \delta u_1 + \dots + \frac{\partial f_m}{\partial u_n} \delta u_n \right) = 0$$

$$\delta F = \sum_{K}^{n} \left(\frac{\partial F}{\partial u_K} + \lambda_1 \frac{\partial f_1}{\partial u_K} + \dots + \lambda_m \frac{\partial f_m}{\partial u_K} \right) \delta u_K = 0$$

we can eliminate the last m variables:

$$\frac{\partial F}{\partial u_K} + \lambda_1 \frac{\partial f_1}{\partial u_K} + \dots + \lambda_m \frac{\partial f_m}{\partial u_K} = 0 \quad \text{for } n-m+1 \le K \le n$$

system of equations, m equations and m variables we get

since variables are independent, each K-term equal to zero.

$$\delta F = \sum_{k}^{n-m}\left(\frac{\partial F}{\partial u_K} + \lambda_1 \frac{\partial f_1}{\partial u_K} + \dots + \lambda_m \frac{\partial f_m}{\partial u_K}\right)\delta u_K = 0$$

Notice that we can write the equation above as

$$\delta F + \lambda_1 \delta f_1 + \dots + \lambda_m \delta f_m = 0$$

The variation is distributive, so

$$\delta\left(F + \lambda_1 f_1 + \dots + \lambda_m f_m\right) = 0 \qquad \bar{F} = F + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha$$

Let $\bar{F} = F + \lambda_1 f_1 + \dots + \lambda_m f_m$, then $\boxed{\delta \bar{F} = 0}$

☆ When there are constraints, rather than the stationary value of F, we want the stationary value of $\bar{F}$ in which constraints have been eliminated and all remaining variables are independent.

In ~~the~~ case ~~of~~ there are additional constrants, we modify the integral from Hamilton's principle.

as above

~~Let~~ $$I = \int_A^B \bar{\mathcal{L}}\, dt = \int_A^B \left(\mathcal{L} + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha\right) dt \qquad \text{Eq 2.20}$$

typo in Goldstein Eq. 2.21

$$\delta I = \int_A^B dt \left(\cancel{\sum_{i=1}^{n}} \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i}\right)\delta q_i\right) = 0$$

not all n are linearly independent

By appropriately chosing the λ_α s, eliminate variables

will get m equations of the form Eq 2.22

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} = 0 \quad \text{for } n-m+1 \leq k \leq n$$

find λ_α that makes equation zero.

will get n-m equations of the SAME form. In this case they are zero because of the virtual displacement δq_i

Notice that Eq. 2.23

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = -\sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} = Q_k$$

generalized force

The Q_k have the magnitudes of the forces need to produce individual constraints.