# Conservation theorems and symmetry properties

"EOM"

In general, the Equations of Motion of a system with $\underline{\underline{n}}$ degrees of freedom will consist of $\underline{\underline{n}}$ differential equations that are second order in time. Performing the appropriate integrations will result in $\underline{\underline{2n}}$ constants of integration, which can be found from the initial values of the $q_j$'s and $\dot{q}_j$'s (n from the $q_j$'s and another n from the $\dot{q}_j$'s equals 2n).

Homogeneous, so equal to zero

$$\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y = 0$$

Sometimes, the EOMs will be integrable, but not always. These integrals can be tricky!

Often, we will try to extract as much information about the system <u>without</u> completing all the integrals. Conserved quantities are particularly useful.

Consider a system of $i$ particles that interact with potentials that are a function of position <u>only</u>, $V(x, \cancel{\dot{x}})$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \equiv \frac{\partial}{\partial \dot{x}_i}(T-V) = \frac{\partial T}{\partial \dot{x}_i} - \cancel{\frac{\partial V}{\partial \dot{x}_i}} = \frac{\partial}{\partial \dot{x}_i}\sum_{j}\frac{1}{2}m_j(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2)$$

After Eq. 2.43, use different index

Derivatives along orthogonal directions are zero.

(when $i = j$)

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \cdot \frac{1}{2} m_i \dot{x}_i^2 = \cancel{\frac{1}{2}} \cdot \cancel{2} m_i \dot{x}_i = p_{ix}$$

$p_{ix}$ is the $x$-component of the linear momentum $p$ of the $i^{th}$-particle.

we use this as the basis to define the generalized momentum associated with generalized coordinate $q_j$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$$

Also called conjugate momentum and canonical momentum.

Caveat 1. If $q_j$ is not a Cartesian coordinate, the units will in general not be the units of linear momentum kg m/s

Caveat 2. If the potential depends on the velocity along $q_j$ such that $V(q_j, \dot{q}_j)$, the generalized momentum will not be identical to the mechanical momentum.

E.g., for particles in an electromagnetic field,

$$\mathcal{L} = T - V = \sum_i \frac{1}{2} m_i \dot{r}_i^2 - \sum_i \underbrace{e_i}_{\text{charge}} \underbrace{\phi(x_i)}_{\text{scalar potential}} + \sum_i e_i \underbrace{\vec{A}(x_i)}_{\text{vector potential}} \cdot \dot{\vec{r}}_i$$

$$p_{ix} = \frac{\partial}{\partial \dot{x}_i} \mathcal{L} = m_i \dot{x}_i - 0 + e_i \vec{A}(x_i) \cdot \frac{d}{d\dot{x}_i} \dot{x}_i = \underbrace{m_i \dot{x}_i}_{\text{mechanical momentum}} + \underbrace{e_i A_x}_{\text{other}}$$

Caveat 3. If the Lagrangian does not depend on a particular coordinate $q_j$, even if it does depend on $\dot{q}_j$, the EOM reduces to

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \cancel{\frac{\partial \mathcal{L}}{\partial q_j}}^{0} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) = 0$$

$$\boxed{\frac{d}{dt} p_j = 0} \quad \cancel{\int dp_j = \int 0\, dt}$$

Eq. 2.46

$\Rightarrow p_j$ is constant

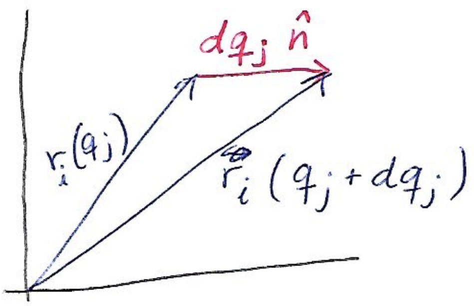
Definition: if generalized coordinate $q_j$ is absent from the Lagrangian, the coordinate is said to be cyclic or ignorable.

Note that derivation above assumes $q_j$'s to be linearly independent, so need to implement any constraints. (If there are constraints, $q_j$'s are not linearly independent.

☆ The conservation of linear momentum and conservation of angular momentum theorems that we derived before from Newtonian mechanics can be derived from $\frac{dp_j}{dt} = 0$, but it is more general since it inherently includes cases in which Newton's 3rd law is violated, e.g., EM. Remember that Newton struggled here.

$$\frac{d}{dt} p_x = \frac{d}{dt}\left(m\dot{x} + eA_x\right) = 0 \quad \Rightarrow p_x = m\dot{x} + eA_x \text{ is constant}$$

Consider generalized coordinate $q_j$ such that $dq_j$ is a translation of the system as a whole. This can be interpreted as a shift in the origin. We further limit ourselves to conservative systems as we did before when deriving the Newtonian conservation theorems. The EOM is




conservative forces

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{d}{dt}\left(\cancel{\frac{\partial V}{\partial \dot{q}_j}}^{\,0}\right) - \underbrace{\cancel{\frac{\partial T}{\partial q_j}}}_{}^{\,0\text{ kinetic energy not affected by shifts}} + \frac{\partial V}{\partial q_j} = 0$$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \cancel{\frac{\partial V}{\partial \dot{q}_j}}^{\,0}, \text{ so}$$

generalized force

$$\frac{d}{dt} p_j + \frac{\partial V}{\partial q_j} = 0 \Rightarrow \dot{p}_j = -\frac{\partial V}{\partial q_j} \equiv Q_j$$

with $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

From the definition of derivative, $\frac{\partial \vec{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\vec{r}_i(q_j + dq_j) - \vec{r}_i(q_j)}{dq_j}$

$$= \frac{dq_j}{dq_j}\hat{n} = \hat{n}$$

Hence $Q_j = \sum_i \vec{F}_i \cdot \hat{n} = \hat{n} \cdot \vec{F}$

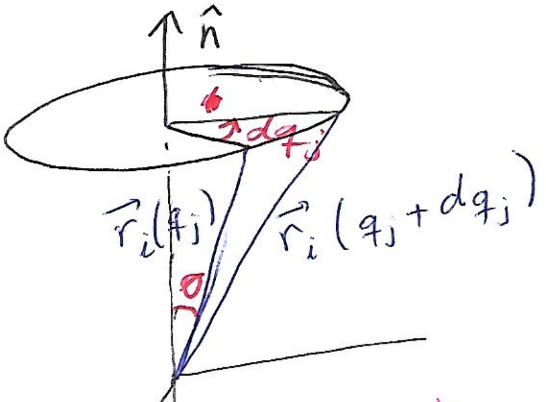
component of the total force along $\hat{n}$

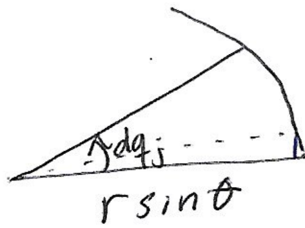
$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \hat{n} = \hat{n} \cdot \sum_i m_i \vec{v}_i$$

Component of total momentum along $\hat{n}$

$\boxed{\text{If } q_j \text{ is cyclic, } Q_j = 0 \text{ and linear momentum is conserved!}}$

Now consider the change in the position vector due to a rotation of the system.




direction is orthogonal to $\hat{n}$ and $\vec{r}_i$ ← magnitude

$$\Rightarrow \left| \frac{\partial r_i}{\partial q_j} \right| = r_i \sin\theta$$

$$dr_i = r_i \sin\theta \, dq_j$$

~~The infinitesimal displacement vector in cylindrical coordinates is~~ $\cancel{d\vec{l} = \hat{r}\,dr + \hat{\varphi}\,r\,d\varphi + \hat{z}\,dz}$ ~~with~~

$\theta$ is the angle between the position vector and the normal of the rotation plane. Since the magnitude of $r_i$ is constant, $\theta$ is constant.

In vector form, $\dfrac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \times \vec{r}_i$

Hence $$Q_j = \sum_i \vec{F}_i \cdot (\hat{n} \times \vec{r}_i)$$

Using vector formula $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{C} \times \vec{A}) \cdot \vec{B}$

$$Q_j = \sum_i \underbrace{(\vec{r}_i \times \vec{F}_i)}_{\text{Torque } \vec{N} = \vec{r} \times \vec{F}} \cdot \hat{n} = \hat{n} \cdot \sum_i \vec{N}_i = \hat{n} \cdot \vec{N}$$

Component of the total torque along $\hat{n}$

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot (\hat{n} \times \vec{r}_i)$$

$$p_j = \sum_i \underbrace{(\vec{r}_i \times m_i \vec{v}_i)}_{\substack{\text{angular momentum} \\ \vec{L} = \vec{r} \times \vec{p}}} \cdot \hat{n} = \hat{n} \cdot \sum_i \vec{L}_i = \hat{n} \cdot \vec{L}$$

Component of the total angular momentum along $\hat{n}$.

$$\boxed{\text{If } q_j \text{ is cyclic, } Q_j = 0 \text{ and the angular momentum is conserved!}}$$