Tensors and other beasts   10/28/21

The kinetic energy is given by $T = \frac{1}{2} m_i v_i^2$

Consider the situation again in which the rigid body moves with one point stationary, so it rotates about that point and has no translation.

By applying Eq. 4.86, we showed that in this situation $\boxed{\vec{v}_i = \vec{\omega} \times \vec{r}_i}$. The kinetic energy is then

$$T = \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2} m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i)$$

The scalar triple product is unchanged under a circular shift $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

so $T = \frac{1}{2} m_i \, \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) = \frac{\vec{\omega}}{2} \cdot m_i (\vec{r}_i \times \vec{v}_i)$

since $\vec{L} = m_i (\vec{r}_i \times \vec{v}_i)$, $T = \frac{\vec{\omega}}{2} \cdot \vec{L}$

and since $\vec{L} = \tilde{I} \vec{\omega}$, $\boxed{T = \frac{\vec{\omega} \cdot \hat{I} \cdot \vec{\omega}}{2}}$ Eq 5.16

Let $\hat{n}$ be a unit vector in the direction of $\vec{\omega}$, then $\vec{\omega} = \omega \hat{n}$ and $T = \frac{\omega^2}{2} \hat{n} \cdot \tilde{I} \cdot \hat{n}$

The double dot product with the tensor in the middle is termed a contraction *What is a contraction?

☆ What is a tensor? ~~vector?~~

A tensor is a geometric object that is independent of a basis and is used to store numbers and mathematical operations, etc. In a consistent way. There are also consistent rules for how tensors of different rank can interact. that often describes physical systems.

	Degree 0	Degree 1	Degree 2	Degree 3
Geometric object	a	a_i	a_{ij}	a_{ijk}
	Scalar / point 0-D	vector / ~~plane~~ line 1D	matrix / plane 2-D	cube 3-D
		one index	two indices	three indices

Basis independent — A basis, for example the ~~i, j, k~~ Cartesian system, defines coordinates. A vector looks different in Cartesian vs spherical coordinates, but it is the same magnitude and same direction

Storage — The indices indicate the location of what we are looking for

T_{ij} gives a square

T_{ijk} gives a cube

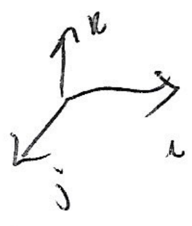
$$\begin{array}{c|c} & j \rightarrow \\ \hline i \downarrow & \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \end{array}$$

scalars

derivatives

Jacobian

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$



Notice that the number of components is dimensionality$^{\text{rank}}$

consistent rules of engagement.

For example, when a tensor undergoes an orthogonal transformation of coordinates, the rule is

each index is one row/column

there is one coefficient with indices for each index ~~coeffici~~ of the tensor

$$\underbrace{T'_{ijk}}_{\text{number of dimensions}} = a_{il} \, \underbrace{a_{jm}}_{\text{2nd}} \, \underbrace{a_{kn}}_{\text{3rd}} \cdots \underbrace{T_{lmn}}_{\text{and so on}}$$

1st index primed

1st index primed

Applying this rule for a tensor of rank 1, we get

$$T'_i = a_{ij} T_j$$

For 3-D, $i = 1, 2, 3$, $j = l$

Completely equivalent to a vector

This is the same as Eq 4.12, the transformation Eqs. for a vector

Applying this rule to a tensor of rank 2, we get

$$T'_{ij} = \underbrace{a_{ik} a_{jl}}_{\text{A matrix in Einstein notation}} T_{kl}$$

For 3-D $i = 1, 2, 3$
$j = 1, 2, 3$

Compare to Eq. 4.29

A tensor is defined by its transformation properties under orthogonal transformations

There is no restrictions on the types of transformations that square matrices may undergo

Tensors are a subset of all matrices

The mechanics are the same

Finally, for a tensor of rank 0, $T' = T$

obviously invariant to orthogonal transformation

Let $\tilde{A}$ be an operator acting on vector $\vec{F}$ to produce vector $\vec{G} = \tilde{A}\vec{F}$

If the coordinate system is transformed by $\tilde{B}$, the components of $\vec{G}$ in the new system will be $\tilde{B}\vec{G} = \tilde{B}\tilde{A}F$. Since $\tilde{B}\tilde{B}^{-1} = \tilde{I}$, $\tilde{B}\vec{G} = \tilde{B}\tilde{A}\tilde{B}^{-1}\tilde{B}\vec{F}$

Comparing with the original equation $\vec{G}_B = \tilde{A}'\vec{F}_B$

The operator $\tilde{A}' = \tilde{B}\tilde{A}\tilde{B}^{-1}$ is expressed in the new system

To produce a transformation on a vector, $\tilde{B}\vec{G}$

" " " " " an operator, $\tilde{B}\tilde{A}\tilde{B}^{-1}$

(similarity transformation)

For orthogonal transformations, $\tilde{B}^{-1} = \tilde{B}^{T}$, thus

← assume matrix

$$\tilde{T}' = \tilde{B}\tilde{T}\tilde{B}^{T} \Rightarrow T'_{ij} = a_{ik}T_{kl}a_{jl} \quad \text{Eq. 5.13}$$

$$= \underbrace{a_{ik}a_{jl}T_{kl}}$$

We can apply the terminology and operations of matrix algebra to tensors.

The transformation of a square matrix complies with the definition of tensor, so it is a tensor.

Vectors can be used to construct tensors, or more riguruosly speaking, tensors of a given rank can be used to construct tensors of a higher rank.

Let $\vec{A}$ have components A_i, $\vec{B}$ components $\vec{B}_j$

together they construct tensor $\tilde{T}_{ij} = \vec{A}_i \vec{B}_j$.

For 3 dimensions, $i = 1,2,3$ $\quad \tilde{T} = \vec{A} \otimes \vec{B}$ Tensor product
$j = 1,2,3$

$$\tilde{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} = \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix}$$

Transforming the whole matrix would take too long, but we can check an element, for example T_{xy}

Definition

Each individual vector transforms according to

$\boxed{T_i' = a_{ij} T_j}$, so $A_x' = a_{xj} A_j$ and $B_y' = a_{yj} B_j$

So $\boxed{T_{xy}' = \sum_i \sum_j a_{xi} a_{yj} T_{ij} =}$

Definition

Einstein

$$T_{xy}' = \cancel{\sum_i \sum_j} a_{xi} a_{yj} A_i B_j$$

$$= \cancel{\sum_i \sum_j} a_{xi} A_i \, a_{yj} B_j = A_x' B_y'$$

The dot product with the tensor before a vector is $\vec{D} = \tilde{T} \cdot \vec{C}$ with $D_i = \sum_j T_{ij} C_j = T_{ij} C_j$

if the vector is before the tensor,

$$\vec{E} = \vec{F} \cdot \tilde{T} \quad \text{with} \quad E_i = \sum_j F_j T_{ij} = F_j T_{ji}$$

The result of a double dot product is

$$S = \vec{F} \cdot \tilde{T} \cdot \vec{C} = \sum_i \sum_j F_i T_{ij} C_j = F_i T_{ij} C_j$$

and it is called a contraction. Finally, for ~~vector~~ tensor $\tilde{T}$ constructed from $\vec{A}$ and $\vec{B}$, $\tilde{T} = \vec{A} \otimes \vec{B}$

$$\tilde{T} \cdot \vec{C} = \vec{A}(\vec{B} \cdot \vec{C}) \qquad \vec{F} \cdot \tilde{T} = (\vec{F} \cdot \vec{A})\vec{B}$$

Where were we? Ah, yes! $T = \frac{\omega^2}{2} \hat{n} \cdot \tilde{I} \cdot \hat{n}$

$$T = \frac{\vec{\omega} \cdot \tilde{I} \cdot \vec{\omega}}{2} = \frac{1}{2} [\omega_x \; \omega_y \; \omega_z] \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\hat{n} = \frac{\vec{\omega}}{|\omega|}$$

$$T = \frac{1}{2} [\omega_x \; \omega_y \; \omega_z] \begin{bmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix}$$

In Einstein notation, $T = \frac{1}{2} \omega_j I_{jk} \omega_k$

$$T = \frac{1}{2} \left[I_{xx}\omega_x\omega_x + I_{xy}\omega_y\omega_x + I_{xz}\omega_x\omega_z + I_{yx}\omega_x\omega_y + I_{yy}\omega_y\omega_y + \dots \right]$$

$$T = \frac{1}{2}\Big[I_{xx}\,\omega_x^2 + I_{yy}\,\omega_y^2 + I_{zz}\,\omega_z^2 + 2I_{xy}\,\omega_x\omega_y$$
$$+ 2I_{yz}\,\omega_y\omega_z + 2I_{xz}\,\omega_x\omega_z\Big]$$

without loss of generality, align $\hat{n}$ with ω_z

$\vec{\omega} = \omega_z\hat{n}$. All the terms are zero except for $I_{zz}\,\omega_z^2$

$$T = \frac{1}{2} I_{zz}\,\omega_z^2$$

axis of rotation

$I = \hat{n}\cdot\tilde{I}\cdot\hat{n}$ is known as the moment of inertia about the axis of rotation. A scalar!

$\hat{n}\cdot\tilde{I}\cdot\hat{n} = m_i\Big[r_i^2 - (r_i\cdot\hat{n})$