Mechanics of a system of particles

8/31/21

While ~~most~~ the idealization of a particle is useful and widely used, most objects have some internal structure and could be better described as a "collection of particles" or a "system of particles."

A natural connection between a "system of particles" concept and an object made out of matter is to consider each atom as a particle. If you wanted to get the dynamics of the system, you could just apply Newton's second law to each particle, e.g.

Eq. 1.19 Goldstein

$$\dot{\vec{P}}_i = \sum_j \vec{F}_{ji} + F_i^{(e)}$$

Force exerted by the external field.

Force exerted by particle j on part i, sum over j, all the particles that are not i

Notice that the nature of the interaction between internal and external fields can be different.

total momentum

For all particles, $\frac{d}{dt}\dot{\vec{P}} = \sum_i \sum_j F_{ji} + \sum_i F_i^{(e)}$

It is easy to cheat here by using Newton's third law. One form ~~easy~~ states it as $\vec{F}_{ij} = -\vec{F}_{ji}$

Also, that the force exerted by a particle on itself is zero

$$
\begin{aligned}
&\vec{F}_{11} + \vec{F}_{21} + \vec{F}_{31} + \dots + \vec{F}_{j1} + \\
&\vec{F}_{12} + \vec{F}_{22} + \vec{F}_{32} + \dots + \vec{F}_{j2} + \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
&F_{1j} + \vec{F}_{2j} + \vec{F}_{3j} + \dots + \vec{F}_{ij}
\end{aligned}
$$

with the $\vec{F}_{ii} = 0$, and Newton's third law

$$
\begin{aligned}
\sum_i \sum_j \vec{F}_{ij} = \; & 0 + \vec{F}_{21} + \vec{F}_{31} + \dots + \vec{F}_{j1} + \\
& -\vec{F}_{21} + 0 + \vec{F}_{32} + \dots + \vec{F}_{j2} + \\
& \vdots \\
& -\vec{F}_{j1} - \vec{F}_{j2} - \vec{F}_{j3} - \dots + 0 \qquad = 0
\end{aligned}
$$

so $$\frac{d}{dt} \sum_i \vec{p}_i = \cancel{\sum_i \sum_j \vec{F}_{ji}}^{\,0} + \sum_i \vec{F}_i^{(e)}$$

$$\frac{d}{dt} \sum_i m_i \frac{d}{dt} \vec{r}_i = \sum_i \vec{F}_i^{(e)} = \vec{F}^{(e)}$$

The <u>center of mass</u> is the weighted position average of the mass,

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \implies \sum_i m_i \vec{r}_i = \underbrace{M}_{\text{total mass}} \vec{R}$$

so $$\boxed{M \frac{d^2}{dt^2} \vec{R} = \vec{F}^{(e)}}$$ Eq. 1.22

$\boxed{\text{From Eq. 1.22 it follows that if the external force is zero, the momentum associated with the center of mass is constant}}$

Conservation theorem for the linear momentum of a system of particles.

weak form

The caviat is that this only holds if Newton's third law is true. Newton himself believed it was always true.

Weak $\longrightarrow$ Forces particles exert on each other are equal and opposite

E.g. both $\xrightarrow{F_{12}}$ $\xleftarrow{-F_{21}}$ $\uparrow F_{12}$ $\downarrow -F_{21}$ comply

★ useful. At least in principle we can describe the dynamics of anything in the Universe by describing the dynamics of each of its particles. One equation per particle, though. That's a lot of equations!

Now let's look at the total angular momentum of a system of particles. Since $\vec{L}_i = \vec{r}_i \times \vec{p}_i$,

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i$$

$\vec{r} \times \dot{\vec{p}}$

we showed before that $\frac{d}{dt}\vec{L} \equiv \dot{\vec{L}} = \vec{N} = \vec{r} \times \frac{d}{dt}(m\vec{v})$

Eq. 1.24 RHS

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\left[\sum_i \vec{L}_i\right] = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i)$$

so $\sum_i \vec{r}_i \times \dot{\vec{p}}_i = \vec{N}$

Remember that $\dot{\vec{p}}_i = \sum_j \vec{F}_{ji} + \vec{F}_i^{(e)}$, so

$$\vec{N} = \sum_i \vec{r}_i \times \left(\sum_j \vec{F}_{ji} + \vec{F}_i^{(e)} \right)$$

The cross product is distributive over addition, so

$$\vec{N} = \sum_i \vec{r}_i \times \sum_j \vec{F}_{ji} + \sum_i \vec{r}_i \times \vec{F}_i^{(e)}$$

Eq. 1.24 LHS

$$\vec{N} = \underbrace{\sum_i \sum_j \vec{r}_i \times \vec{F}_{ji}}_{\text{"spin"}} + \underbrace{\sum_i \vec{r}_i \times \vec{F}_i^{(e)}}_{\text{"angular momentum"}}$$

Looks cumbersome, but it is worth analyzing in detail
Let's write the terms of the "spin" explicitely

Above the diagonal

$$\vec{r}_1 \times \cancel{\vec{F}_{11}}^{0} + \vec{r}_1 \times \vec{F}_{21} + \vec{r}_1 \times \vec{F}_{31} + \dots + \vec{r}_1 \times \vec{F}_{j1} +$$

$$\vec{r}_2 \times \vec{F}_{12} + \vec{r}_2 \times \cancel{\vec{F}_{22}}^{0} + \vec{r}_2 \times \vec{F}_{32} + \dots + \vec{r}_2 \times \vec{F}_{j2} +$$

$$\vdots$$

$$\vec{r}_i \times \vec{F}_{1i} + \vec{r}_i \times \vec{F}_{2i} + \vec{r}_i \times \vec{F}_{3i} + \dots + \vec{r}_i \times \cancel{\vec{F}_{ji}}^{0} \quad jj$$

Below the diagonal

Using Newton's 3rd law $\vec{F}_{12} = -\vec{F}_{21}$, we can rewrite the elements below the diagonal: $\vec{r}_2 \times \vec{F}_{12} = -\vec{r}_2 \times \vec{F}_{21}$ and so on. Then we can factorize: $\vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12}$

$$(\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21} = \Longleftarrow = \vec{r}_1 \times \vec{F}_{21} - \vec{r}_2 \times \vec{F}_{21} =$$

Eq. 1.25

More generally, $\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}$

$= \vec{r}_{ij} \times \vec{F}_{ji}$

so

$$\vec{N} = \sum_i \sum_{j>i} (\vec{r}_{ij} \times \vec{F}_{ji}) + \sum_i \vec{r}_i \times \vec{F}_i^{(e)}$$

Notice that if we only had 1 particle, $i = j = 1$

The equation reduces to $\vec{N} = \vec{r} \times \vec{F} = \dot{\vec{L}}$

No net torque implied conservation of angular momentum, but here we have an extra term to take care of. One way to achieve this is by ensuring that the angle between the position vector and force vector is zero for every pair of particles.

$$\vec{r}_{ij} \times \vec{F}_{ji} = |\vec{r}_{ij}||\vec{F}_{ji}| \sin\theta_{r,F}$$

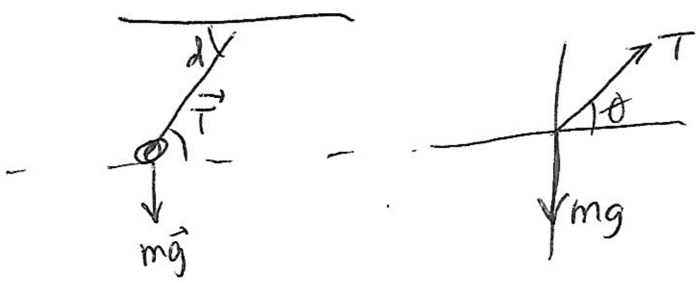
This is a stronger statement of the 3rd law

Strong → Forces particles exert on each other are <u>equal</u> and <u>opposite</u> and in the direction of the line joining the two particles

Many forces of interest are <u>central</u>, ~~which means that~~ a force exerted on an object is directed towards or away from the center of force, often another particle. Central forces comply with the strong law of action and reaction, so angular momentum is conserved in the absence of torque; a good thing for humans.

1.3 Constraints

Newton's laws are critical if we want to calculate Equations of motion, but so are constrains, which limit the motion of the system. We are familiar with constraints, for example for a pendulum



$$\sum F_x = T\cos\theta = ma_x$$
$$\sum F_y = -mg + T\sin\theta = ma_y$$

Voilà

Assume 2 dimensions rather than 3

Reduce degrees of freedom

Free-body diagram, align one of the forces with an axis

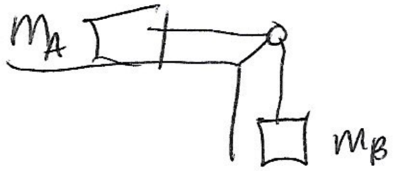
$$T = \frac{ma_x}{\cos\theta} \quad ; \quad T = \frac{ma_y + mg}{\sin\theta} \Rightarrow \frac{\cancel{m}a_x}{\cos\theta} = \frac{\cancel{m}(a_y+g)}{\sin\theta}$$

$$a_x \tan\theta = a_y + g$$

θ can be related to d, the length of the pendulum.

The tension in this case is a force of constraint. It gives you the line (1-Dimensional) in the xy-plane along which the pendulum can move. Constraints that can be expressed as equations connecting the coordinates of the particles and perhaps time, they are called holonomic constraints.

Sometimes the constraints are rather explicit even in intro mechanics, for example the "acceleration constraint" in the case of pulleys



$$\Sigma F_{Ax} = m_A a_{Ax}$$

$$\Sigma F_{By} = m_B a_{B,y}$$

$$a_{A,x} = -a_{B,y} = a$$

Here it is easier to see that each constraint, since it is 1 equation, reduces the <u>degrees of freedom</u> by 1, the dimensionality of the solution space.

A dramatic but boring example is rigid body.

In a gas, each particle can move in 3-D, so we need 3N equations to constrain the system. But in a solid we can define a point, for example the center of mass, and use it to specify the position of each of the particles. This adds 3N constraints, although now you have 3 translations for the newly defined center of mass, and 3 rotations. You go from $3N \rightarrow 6$ degrees of freedom.

Adding a new particle induces 3 degres of freedom and 3 const. for a net gain of <u>zero</u>