Forced heavy symmetric top continued — 11/11/21 — (134)

In the system we are investigating, the torque due to gravity is $\vec{N} = \vec{R} \times (-M\vec{g}) = M\vec{g} \times \vec{R}$

with $\vec{R} = l\hat{k}$ and $\vec{g} = g\hat{k}'$, $\vec{N} = Mgl\,(\hat{k}' \times \hat{k})$

$\hat{k}'$ is the vertical and $\hat{k}$ is along the axis of rotation, so the torque is normal to the zy-plane. This is the line of nodes direction. In body coordinates,

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = N_1$$

$$I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) = 0$$

$$I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$$

The system has the symmetry $I_1 = I_2 \neq I_3$, so the last equation is $I_3\dot{\omega}_3 = 0 \Rightarrow \omega_3$ is constant and positive in general. Let's assume $0 = \dot{\omega}_1 = \dot{\omega}_2$ ω_1, ω_2 are ~~is~~ initially zero, then.

$$I_1\dot{\omega}_1 + \boxed{\cancel{\boxed{\omega_2}\omega_3(I_3 - I_1)}} = N_1$$

$\leftarrow$ $\dot{\omega}_1 \neq 0$, so $\omega_1 \neq 0$ in the next instant

$$I_2\dot{\omega}_2 + \boxed{\cancel{\omega_3\boxed{\omega_1}(I_1 - I_3)}} = 0$$

$$I_3\dot{\omega}_3 + \boxed{\cancel{\omega_1\omega_2\boxed{(I_1 - I_1)}}} = 0$$

In order to keep the second equation, $\dot{\omega}_2 \neq 0$ so $\omega_2 \neq 0$ in the next instant.

! Notice, however, that $\dot{\omega}_3$ is unaffected.

Forced heavy symmetric top continued

The Lagrangian of the system is

$$\frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 - Mgl\cos\theta$$

$$\mathcal{L} = \frac{I_1}{2}\left(\dot\theta^2 + \dot\varphi^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi + \dot\varphi\cos\theta\right)^2 - Mgl\cos\theta$$

φ and ψ are cyclic coordinates, so

we previously derived $\omega_3 = \dot\psi + \dot\varphi\cos\theta$

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot\psi} = \frac{\partial}{\partial \dot\psi}\left[\left(\dot\psi^2 + 2\dot\psi\dot\varphi\cos\theta + \dot\varphi^2\cos^2\theta\right)\frac{I_3}{2}\right]$$

$$= \left(2\dot\psi + 2\dot\varphi\cos\theta\right)\frac{I_3}{2} = I_3\left(\dot\psi + \dot\varphi\cos\theta\right)$$

~~Notice that only the second term on the r.h.s is affected and~~

$$\cancel{p_\psi = \frac{\partial \mathcal{L}}{\partial \omega_3} = \frac{\partial}{\partial \dot\psi}\left[\frac{1}{2} I_3 \omega_3^2\right] = \frac{\partial}{\partial \dot\psi}\left[\frac{1}{2} I_3 \dot\varphi^2\right]} \quad \cancel{\omega_3 = \dot\psi + \dot\varphi\cos\theta}$$

must be since I_3 is constant

$$\cancel{\frac{1}{2} I_3 \dot\varphi} = I_3\omega_3 = \underline{I_1 a}$$

For convenience we will see why

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot\varphi} = \frac{\partial}{\partial \dot\varphi}\left[\frac{I_1}{2}\dot\varphi^2\sin^2\theta + \frac{I_3}{2}\left(\dot\psi^2 + 2\dot\psi\dot\varphi\cos\theta + \dot\varphi^2\cos^2\theta\right)\right]$$

$$= \left(2\dot\varphi\sin^2\theta\right)\frac{I_1}{2} + \left(2\dot\psi\cos\theta + 2\dot\varphi\cos^2\theta\right)\frac{I_3}{2}$$

$$= I_1\dot\varphi\sin^2\theta + \dot\psi\cos\theta I_3 + \dot\varphi\cos^2\theta I_3$$

can we do this?

$$= \left(I_1\sin^2\theta + I_3\cos^2\theta\right)\dot\varphi + I_3\dot\psi\cos\theta = \underline{\underline{I_1 b}}$$

ditto

The system does not explicitely depends on time, so the total energy is conserved

$$E = T + V = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) + \frac{I_3}{2}\omega_3^2 + Mgl\cos\theta$$

~~Notice that $\omega_3 = \dot{\psi}$ is a constant and the only variable that matters in the 2nd term~~ Nop r.h.s.

Remember the angular frequency of precession for the torque-free motion of a rigid body with $I_1 = I_2 \neq I_3$

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3 \Rightarrow I_1\Omega = (I_3 - I_1)\omega_3 = I_3\omega_3 - I_1\omega_3$$

$$I_3\omega_3 = I_1\Omega + I_1\omega_3 = I_1(\Omega + \omega_3)$$

Motivate the $I_1 a$ term but not necessary

From p_ψ, $\boxed{I_3(\dot{\psi} + \dot{\varphi}\cos\theta) = I_1 a} = I_3\omega_3$

possible since I_1, a, and p_ψ are constant.

$$I_3\dot{\psi} = I_1 a - I_3\dot{\varphi}\cos\theta$$

Possible since I_1, b, and p_φ are constant

From p_φ, $\boxed{(I_1\sin^2\theta + I_3\cos^2\theta)\dot{\varphi} + I_3\dot{\psi}\cos\theta = I_1 b}$

$$I_1\dot{\varphi}\sin^2\theta + I_3\dot{\varphi}\cos^2\theta + (I_1 a - I_3\dot{\varphi}\cos\theta)\cos\theta = I_1 b$$

$$I_1\dot{\varphi}\sin^2\theta + \cancel{I_3\dot{\varphi}\cos^2\theta} + I_1 a\cos\theta - \cancel{I_3\dot{\varphi}\cos^2\theta} = I_1 b$$

$$I_1\dot{\varphi}\sin^2\theta + I_1 a\cos\theta = I_1 b$$

Eq. 5.57

$$\text{so } \dot{\varphi} = \frac{I_1 b - I_1 a\cos\theta}{I_1\sin^2\theta} = \frac{b - a\cos\theta}{\sin^2\theta}$$

Hence, $I_3 \dot{\psi} = I_1 a - I_3 \left(\frac{b - a\cos\theta}{\sin^2\theta} \right) \cos\theta$

Eq. 5.58

$$\dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta \left(\frac{b - a\cos\theta}{\sin^2\theta} \right) \frac{\cancel{I_3}}{\cancel{I_3}}$$

Finally,

$$E = \frac{I_1}{2} \left[\dot{\theta}^2 + \left(\frac{b - a\cos\theta}{\sin^2\theta} \right)^2 \sin^2\theta \right] + \frac{I_3}{2} \left[\frac{I_1 a}{I_3} - \cancel{\cos\theta \left(\frac{b - a\cos\theta}{\sin^2\theta} \right)} + \cancel{\left(\frac{b - a\cos\theta}{\sin^2\theta} \right) \cos\theta} \right]^2 - Mgl\cos\theta$$

$$E = \frac{I_1 \dot{\theta}^2}{2} + \frac{I_1}{2} \frac{(b - a\cos\theta)^2}{\sin^{\cancel{4}\,2}\theta} \cancel{\sin^2\theta} + \frac{\cancel{I_3} I_1^2 a^2}{2 I_3^{\cancel{2}}} - Mgl\cos\theta$$

$$E = \frac{I_1 \dot{\theta}^2}{2} + \frac{I_1}{2} \left(\frac{b - a\cos\theta}{\sin\theta} \right)^2 + Mgl\cos\theta + \frac{I_1^2 a^2}{2 I_3}$$

The last term on the r.h.s. is constant, so let's consider instead $E' = E - I_1^2 a^2 / 2 I_3$, which is also constant. The first term on the r.h.s. is a velocity squared, so a kinetic energy. The second and third terms on the r.h.s. depend on position only, so it is an effective potential.

! Because of the symmetries, this problem simplifies to a 1-D problem.

$$I_1 a = I_3 \omega_3 \text{, so } \frac{I_1^2 a^2}{2I_3} = \frac{I_3^{\cancel{2}} \omega_3^2}{2\cancel{I_3}}$$

constant

Let $\alpha = \frac{2E - I_3\omega_3^2}{I_1}$

$$E' = E - \frac{I_3\omega_3^2}{2} \Rightarrow 2E' = 2E - I_3\omega_3^2$$

Define

$$\frac{2E'}{I_1} = \frac{2}{I_1}\left[\frac{I_1\dot{\theta}^2}{2} + \frac{I_1}{2}\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta\right]$$

$$\underline{\alpha} = \frac{2E'}{I_1} = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \frac{2Mgl\cos\theta}{I_1}$$

let $\beta = \frac{2Mgl}{I_1}$ (constant) $\alpha = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta$

Notice: $a = \frac{p_\psi}{I_1}$; $b = \frac{p_\varphi}{I_1}$ (Also constants)

Let $u = \cos\theta$, $\cos^2\theta - 1 = \sin^2\theta = 1 - u^2$

$\theta = \arccos u$

$\cancel{(1-u^2)(\alpha - \beta u) = \alpha - \beta u - \alpha u^2 + \beta u^3}$

$$\dot{\theta}^2 = \left[\frac{d}{dt}\arccos u\right]^2$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \text{, so}$$

$$\frac{d}{dt}\arccos u = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dt} = -\frac{\dot{u}}{\sqrt{1-u^2}}$$

$$\dot{\theta}^2 = \left[-\frac{\dot{u}}{\sqrt{1-u^2}}\right]^2 = \frac{\dot{u}^2}{1-u^2}$$

Hence $\alpha = \frac{\dot{u}^2}{1-u^2} + \frac{(b - au)^2}{1-u^2} + \beta u$

$$\left[\alpha - \frac{(b-au)^2}{1-u^2} - \beta u\right](1-u^2) = \dot{u}^2$$

$$\dot{u}^2 = (1-u^2)(\alpha-\beta u) - (b-au)^2 \qquad \text{Eq. 5.62}$$

Evidently, $\dfrac{du}{dt} = \sqrt{(1-u^2)(\alpha-\beta u) - (b-au)^2}$, so

$$t = \int_{u(0)}^{u(t')} \frac{du}{\sqrt{(1-u^2)(\alpha-\beta u)-(b-au)^2}}$$

Also, $\dot{u}^2 = \alpha - \beta u - \alpha u^2 + \beta u^3 - b^2 + 2abu - a^2u^2$

Let's explore this equation

$$= \beta u^3 - u^2(\alpha + a^2) - u(\beta - 2ab) + \alpha - b^2$$

Let $\dot{u}^2 \equiv f(u)$, then $f(u) = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2)$

Since β represents the torque term, $\beta = 0$ is the torque-free system, althoug with a fixed point, so it describes the gyroscope. In this case $f(u) = -(\alpha + a^2)u^2 + 2abu + \alpha - b^2$ is a quadratic equation. If the top is supported, $\beta > 0$.

The roots of the equation indicate the angle θ at which $\dot{\theta}$ changes signs, the "turning points" or turning angles. Just like with parabolic motion.

* On a horizontal surface, $\alpha > 0$ and $\beta > 0$.

* If it looks like this: then $\beta > 0$, but α could be positive OR negative.

~~f(u)~~

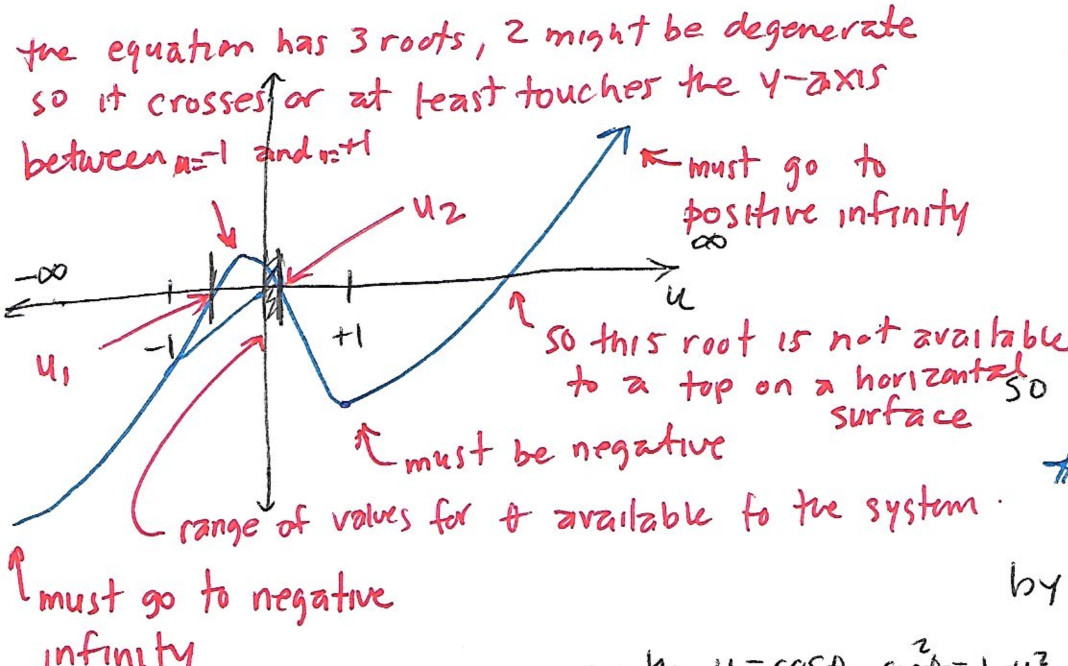
* Since $u = \cos\theta$, $-1 \le u \le 1$

* In the $\lim_{u \to \pm\infty} f(u) = \pm\infty$ since the cubic term dominates

$$f(1) = \cancel{\beta} - \cancel{\alpha} - a^2 - \cancel{\beta} + 2ab + \cancel{\alpha} - b^2 = -(a-b)^2$$

$$f(-1) = \cancel{-\beta} \cancel{-\alpha} - a^2 \cancel{+\beta} - 2ab \cancel{+\alpha} - b^2 = -(a+b)^2$$

so both $f(1)$ and $f(-1)$ are negative

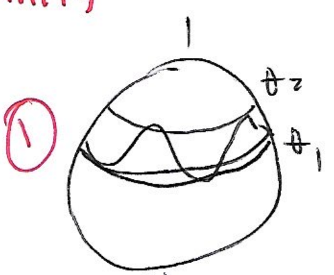


* On a horizontal surface, θ, which is the angle between the vertical and the axis of rotation $0 \le \theta \le 90$. so $u = \cos\theta \Rightarrow 1 \le u \le 0$

* Remember that the precession is given by φ, and $\dot{\varphi} = \frac{b - a\cos\theta}{\sin^2\theta}$.

with $u = \cos\theta$, $\sin^2\theta = 1 - u^2$, $\dot{\varphi} = \frac{b - au}{1 - u^2}$. The root is b−au. Let $u' = b/a$. if $u' > u_2$, the sign is the same between θ_1 and θ_2

This precession includes nutation, which didn't happen in the case of the torque-free rotation

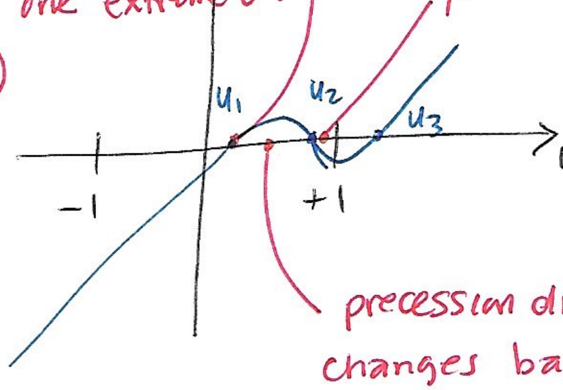


+nutation $\dot{\theta}$
precesion $\dot{\varphi}$

precession must vanish in one extreme θ.

③

same direction precession + nutation ①



precession direction changes back and forth + nutation ②

② 

③