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Hamiltonian Dynamics.

Probably the first "formal" physics you studied involved learning about position, velocity, acceleration. Motion. You can add a few more ingredients, like mass and charge, but at its most fundamental level, physics is about motion.

High energy	~ GeV
Heavy ions/ nuclear	~ MeV
AMO	~ keV
Chemistry	~ eV
CM hard/soft	~ meV
CMB	~ μeV

More particles can agglomerate at lower energies resulting in complicated and even complex behavior.

Reductionism starts to fail.

* Newton came up with the first "correct" description of motion, but he realized it had some issues, he had some understanding that chaos was around. God fixed it.

* Lagrange and Euler came up with a different formulation, but still equivalent to Newton. Laplace "didn't need that theory."

* There are other formulations, like the Hamiltonian and the Koopman-von Neumann formulations, what makes them distinct is the space ~~in which~~ they use to describe motion.

Newtonian	Physical
Lagrangian	Configuration
Hamiltonian	Phase
Koopman-von Neumann	Hilbert

* Just like vectors exist independently of the system you use to describe them, mechanics exist independently of the formulation you use

* Just like you can use transformations to describe the same vector with different systems, you can move from one formulation to another.

Newtonian - Newton's second law

Lagrangian - Euler-Lagrange Equation

Hamiltonian - Hamilton ~~Equations of~~ Equations ~~Motion~~

Koopman-Von Neumann Schrödinger-like Equation

Legendre transform

scalar product defined as an integration of the points in phase space.

* It is not surprising that humans first discovered Newtonian mechanics since it is the one that is most consistent with what we percieve as reality. But since all the formulations are equivalent, what is the true nature of reality?

Configurational vs Phase space

Why yet another mechanics formulation? Convenience.

Lagrangian - $\mathcal{L}(q, \dot{q}, t) \Rightarrow \quad p_i = \dfrac{\partial \mathcal{L}(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}$

since $\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \dfrac{\partial \mathcal{L}}{\partial q_i} = 0$, $\quad \dot{p}_i = \dfrac{\partial \mathcal{L}(q_j, \dot{q}_j, t)}{\partial q_i}$

Hamiltonian - $\mathcal{H}(q, p, t)$

we go from one to the other by switching one variable

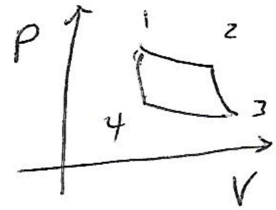
~~Legendar~~ Legendre transform

temp, not kinetic energy

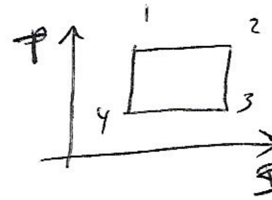
Consider the first law of thermodynamics for gas in a reversible process

$$dU = dQ - dW \qquad dU = T\,dS - P\,dV$$

internal energy   heat   work



so $U(S,V)$



$$d(TS) = T\,dS + S\,dT,$$

so

$$dU = d(TS) - S\,dT - p\,dV$$

Let $dF = d(U - TS) = dU - d(TS) = -S\,dT - p\,dV$

we see that $F(T, V)$

If we have a function of the form $df = u\,dx + v\,dy$

where $u = \dfrac{\partial f}{\partial x}$ and $v = \dfrac{\partial f}{\partial y}$, so $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

we can change the description from x, y to u, y by

considering $g = f - ux$

Eq. 8.5

$$dg = df - d(ux) = df - u\,dx - x\,du$$

$$dg = \cancel{u\,dx} + v\,dy - \cancel{u\,dx} - x\,du$$

$$dg = v\,dy - x\,du$$

Eq. 8.6

$$x = -\frac{\partial g}{\partial u} \text{ and } v = \frac{\partial g}{\partial y} \qquad \text{c.f. } u = \frac{\partial f}{\partial x} \text{ and } v = \frac{\partial f}{\partial y}$$

We derived the Hamiltonian before, we called ~~the~~ it the energy function

Eq. 2.53

$$h(q_1,\dots,q_n;\dot q_1,\dots,\dot q_n;t) = \sum_j \dot q_j \frac{\partial \mathcal{L}}{\partial \dot q_j} - \mathcal{L}$$

Einstein notation

Since $p_j = \dfrac{\partial \mathcal{L}}{\partial \dot q_j}$, rewrite as $\dot q_i p_i - \mathcal{L}(q,\dot q,t)$

so 1 equation per degree of freedom

We used h to express the Hamiltonian in configuration space, but in phase space we use $\mathcal{H}(q,p,t)$

~~$d\mathcal{H} = d(\dot q_i p_i) - d\mathcal{L}(q,\dot q,t)$~~

~~$d\mathcal{H} = \dot q_i dp_i$~~

total derivative

$$d\mathcal{H} = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt = d(\dot q p) - d\mathcal{L}$$

$$= \dot q\, dp + \cancel{p\, d\dot q} - \frac{\partial \mathcal{L}}{\partial q}dq - \cancel{\frac{\partial \mathcal{L}}{\partial \dot q}d\dot q} - \frac{\partial \mathcal{L}}{\partial t}dt$$

$p\, d\dot q$

so

$$\frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt$$

$$= -\frac{\partial \mathcal{L}}{\partial q}dq + \dot q\, dp - \frac{\partial \mathcal{L}}{\partial t}dt$$

so $\dot{q}_i = \frac{\partial H}{\partial p_i}$

$$+\frac{\partial H}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} = -\dot{p}_i$$

$$\frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial H}{\partial q_i} \\ -\frac{\partial \mathcal{L}}{\partial t} &= \frac{\partial H}{\partial t} \end{aligned}\right\}$$

Eq 8.18

~~2N+1~~

Eq. 8.19

2N +1

Canonical equations of Hamilton

2N first order equations replace N second order Lagrange equations