Eigenvalue equation of oscillations

11/18/21

Eq. 6.8 leaves us in a familiar place: $\underbrace{T_{ij}\ddot{\eta}_j}_{} = \underbrace{-V_{ij}\eta_j}_{}$

It is Hooke's law! But in n dimensions

$F = ma = -k\Delta x$

Luckily, we derived in pgs. 142–143 what the solution looks like:

$x = e^{i\sqrt{k/m}\,t + C}$

with $\omega = \sqrt{k/m}$

$\eta_j = C a_j e^{i\omega t}$ ← imaginary, not index

Hence, $\dot{\eta}_j = \frac{d}{dt}\eta_j = \frac{d}{dt}\left[C a_j e^{i\omega t}\right] = C a_j \frac{d}{dt} e^{i\omega t}$

$$\dot{\eta}_j = C a_j e^{i\omega t} \cdot i\omega = i\omega\, \eta_j$$

and $\ddot{\eta}_j = \frac{d}{dt}\dot{\eta}_j = \frac{d}{dt}\left(C a_j i\omega e^{i\omega t}\right) = i\omega C a_j \frac{d}{dt} e^{i\omega t}$

$$\ddot{\eta}_j = C a_j e^{i\omega t} \cdot i^2\omega^2 = -\omega^2 C a_j e^{i\omega t} = -\omega^2 \eta_j$$

~~so~~ $\cancel{T_{ij}(-\omega^2 \eta_j}$

so $T_{ij}\left(-\omega^2 C a_j e^{i\omega t}\right) + V_{ij}\left(C a_j e^{i\omega t}\right) = 0$

$$\cancel{Ce^{i\omega t}}\left[V_{ij} a_j - \omega^2 T_{ij} a_j\right] = 0$$

Eq. 6.12 $V_{ij} a_j - \omega^2 T_{ij} a_j = 0$ System of n homogeneous equations

Good old matrix multiplication

Actually V_{ij} and T_{ij} are rank-2 tensors

Eq. 6.12 leaves us in another familiar place

$$\tilde{V}\vec{a} - \omega^2 \tilde{T}\vec{a} = 0$$

$$(\tilde{V} - \omega^2 \tilde{T})\vec{a} = 0$$

either $\vec{a} = 0$, or more interestingly,

$$|\tilde{V} - \omega^2 \tilde{T}| = 0$$

Luckily, we derived in pgs. 89-90 what the solution looks like.

$$\begin{vmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & \cdots \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & \cdots \\ V_{31} - \omega^2 T_{31} & & \\ \vdots & & \end{vmatrix} = 0$$

The roots of the characteristic equation, which is an algebraic equation of degree n for ω^2, are solutions to the equations of motion. They are the eigenfrequencies

We can rewrite Eq. 6.12 as $\tilde{V}\vec{a} = \omega^2 \tilde{T}\vec{a}$

$$\tilde{V}\vec{a} = \lambda \tilde{T}\vec{a} \quad \text{Eq. 6.14}$$

Each eigenfrequency ω_K is related to the eigenvalue λ_K and to the eigenvector $\vec{a}_K$. The matrix of eigenvectors is $\tilde{A}$. More generally, $\tilde{V}\vec{a}_K = \lambda_K \tilde{T}\vec{a}_K$

Eq. 6.15

In the most general case, ω_k, λ_k, $\vec{a}_k$ are complex

Take the transpose on both sides of Eq. 6.14

$$(\tilde{V}\vec{a}_k)^T = (\lambda_k \tilde{T}\vec{a}_k)^T$$

$$\vec{a}_k^T \tilde{V}^T = \lambda_k (\tilde{T}\vec{a}_k)^T = \lambda_k \vec{a}_k^T \tilde{T}^T$$

Properties of transpose

$$(\tilde{A}\tilde{B})^T = \tilde{B}^T \tilde{A}^T$$

$$(cA)^T = cA^T$$

Now take the complex conjugate, this is the adjoint or conjugate transpose

$$\overline{\vec{a}_k^T \tilde{V}^T} = \overline{\lambda_k \vec{a}_k^T \tilde{T}^T}$$

we know that the elements of $\tilde{V}$ are force constants which must be real, and $\tilde{T}$ are masses which must also be real. Furthermore, we showed before that $\tilde{V}$ and $\tilde{T}$ are symmetric. As in the case of the moment of inertia tensor (pg. 115), $\tilde{T}$ and $\tilde{V}$ are Hermitian. Hence,

$$\overline{\vec{a}_k^T} \tilde{V} = \overline{\lambda_k \vec{a}_k^T} \tilde{T} \Rightarrow \vec{a}_\ell^* \tilde{V} = \bar{\lambda}_\ell \vec{a}_\ell^* \tilde{T} \qquad \text{Eq. 6.16}$$

* denotes the adjoint, also called the conjugate transpose

The bar denotes the complex conjugate.

k and ℓ are columns of $\tilde{A}$, in principle they could be the same

Eq. 6.16
Multiply from the right times $\vec{a}_k$, so

$$\underline{\vec{a}_\ell^{\,*} \tilde{V} \vec{a}_k} = \bar{\lambda}_\ell \vec{a}_\ell^{\,*} \tilde{T} \vec{a}_k$$

Multiply Eq. 6.15 from the left times $\vec{a}_\ell^{\,*}$, so

$$\underline{\vec{a}_\ell^{\,*} \tilde{V} \vec{a}_k} = \vec{a}_\ell^{\,*} \lambda_k \tilde{T} \vec{a}_k$$

Algebra says that if two quantities are equal to a third one, the first two are equal. Hence,

$$\lambda_k \underbrace{\vec{a}_\ell^{\,*} \tilde{T} \vec{a}_k} = \bar{\lambda}_\ell \underbrace{\vec{a}_\ell^{\,*} \tilde{T} \vec{a}_k}$$

$$\Rightarrow (\lambda_k - \bar{\lambda}_\ell) \vec{a}_\ell^{\,*} \tilde{T} \vec{a}_k = 0$$

When $\ell = k$, $\left[(\lambda_k - \bar{\lambda}_k) \vec{a}_k^{\,*} \tilde{T} \vec{a}_k \right]^T = 0$

take the transpose $(\lambda_k - \bar{\lambda}_k) \left[\vec{a}_k^{\,*} \tilde{T} \vec{a}_k \right]^T = (\lambda_k - \bar{\lambda}_k)(\tilde{T}\vec{a}_k)^T \vec{a}_k^{\,*T}$

$$= \underline{(\lambda_k - \bar{\lambda}_k) \vec{a}_k^{\,T} \tilde{T}^T \vec{a}_k^{\,*T}}$$

$$= (\lambda_k - \bar{\lambda}_k) \vec{a}_k^{\,T} \tilde{T}^T \overline{\vec{a}_k}$$

then the conjugate $\overline{(\lambda_k - \bar{\lambda}_k)}\, \overline{\vec{a}_k^{\,T}}\, \overline{\tilde{T}^T}\, \overline{\overline{\vec{a}_k}} = (\bar{\lambda}_k - \lambda_k) \vec{a}_k^{\,*} \tilde{T}^* \vec{a}_k$

Both quantities are equal to zero, so they are equal; $\tilde{T}$ is Hermitian, so $\tilde{T}^* = T$; Hence

$$(\lambda_k - \overline{\lambda_k})\vec{a}_k^{\,*}\tilde{T}\vec{a}_k = (\overline{\lambda_k} - \lambda_k)\vec{a}_k^{\,*}\tilde{T}\vec{a}_k$$

we can see that the matrix product $\vec{a}_k^{\,*}\tilde{T}\vec{a}_k$ is Hermitian, so it is real. Also

$$\vec{a}_k = \alpha_k + i\beta_k \ ; \ \vec{a}_k^{\,T} = \alpha_k^T + i\beta_k^T \ ;$$

$$\overline{\vec{a}_k^{\,T}} = \overline{\alpha_k^T + i\beta_k^T} = \overline{\alpha_k^T} - i\overline{\beta_k^T} = \alpha_k^T - i\beta_k^T = \vec{a}_k^{\,*}$$

so the matrix product is

$$(\alpha_k^T - i\beta_k^T)(\tilde{T})(\alpha_k + i\beta_k) = (\alpha_k^T\tilde{T} - i\beta_k^T\tilde{T})(\alpha_k + i\beta_k)$$

$$= \alpha_k^T\tilde{T}\alpha_k - i\beta_k^T\tilde{T}\alpha_k + i\alpha_k^T\tilde{T}\beta_k - i^2\beta_k^T\tilde{T}\beta_k$$

$$= \alpha_k^T\tilde{T}\alpha_k + \beta_k^T\tilde{T}\beta_k + \cancel{i(\alpha_k^T\tilde{T}\beta_k - \beta_k^T\tilde{T}\alpha_k)}$$ Eq. 6.19

Since $\tilde{T}^T = \tilde{T}$ symmetric,

$$\alpha_k^T\tilde{T}^T\beta_k$$

$$\left[(\tilde{T}\alpha_k)^T\beta_k\right]^T$$

$$\beta_k^T\tilde{T}\alpha_k$$

So yeah, the matrix product is real

Since $T = \frac{1}{2}\dot{\eta}^T \tilde{T} \dot{\eta}$, if $\dot{\eta}_k = \alpha_k$ $\quad \dot{\eta}_k = \beta_k$

$$T = \frac{1}{2}\alpha_k^T \tilde{T} \alpha_k \qquad T = \frac{1}{2}\beta_k^T \tilde{T} \beta_k$$

So the matrix product $\vec{a}_k^* \tilde{T} \vec{a}_k = 2T + 2T = 4T$

~~The kinetic energy can't be negative, so the matrix product is real and ~~positive~~ greater than or equal to zero.~~

If the amplitudes are not zero, the kinetic energy is not zero, and it can't be negative. Hence the matrix product is real and positive.