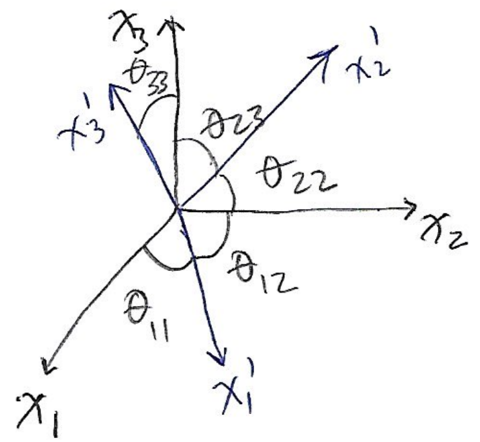
10/7/21

Orthogonal transformations

We will use x_1, x_2, x_3 instead x, y, z. This is notationally convenient, as you will see. Also, we will represent $\cos\theta_{ij}$ as just a_{ij}. Consider the following system:

i is the primed system



As before,

$$x_1' = \vec{r}\cdot\hat{i}' = x_1\cos\theta_{11} + x_2\cos\theta_{12} + x_3\cos\theta_{13}$$

$$x_2' = \vec{r}\cdot\hat{j}' = x_1\cos\theta_{21} + x_2\cos\theta_{22} + x_3\cos\theta_{23}$$

$$x_3' = \vec{r}\cdot\hat{k}' = x_1\cos\theta_{31} + x_2\cos\theta_{32} + x_3\cos\theta_{33}$$

we can write it more compactly

Eq. 4.12

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

This is an example of a linear transformation

the coefficients a_{ij} are independent of x and x'

Such transformations leave the length invariant.

To express it even more compactly, use the Einstein notation

The rule is, if the indices are repeated, it implies a sum over all values of index

$$x_i' = a_{ij}\,x_j \quad i = 1, 2, 3$$

For $i = 1$, repeated when $j = 1$

$x_1' = a_{1j}\,x_j$, but then sum over all possible values of j

$$x_1' = \sum_j a_{1j}x_j = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

For $i = 2$, repeated when $j = 2$

$$x_2' = a_{2j}\, x_j$$

but then sum over all possible values of j

$$x_2' = \sum_j a_{2j}\, x_j = a_{21} x_1 + a_{22} x_2 + a_{23} x_3$$

For $i = 3$

$$x_3' = a_{3j}\, x_j$$

Sum over j

$$x_3' = \sum_j a_{3j}\, x_3$$
$$= a_{31} x_1 + a_{32} x_2 + a_{33} x_3$$

Consider the case $x_i x_i$

it is always repeated, since $i = i$, so $\sum_i x_i^2 = x_1^2 + x_2^2 + x_3^2 \ldots$

so $x_i x_i = |\vec{r}|^2$ squared of the magnitude of a vector

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Since linear transformations are invariant, $x_i' x_i' = x_i x_i$

dummy variables

Therefore, $a_{ij}\, x_j\, a_{ik}\, x_k = x_i' x_i'$

If $j = k$, $\sum_j a_{ij} x_j \cdot \sum_j a_{ij} x_j = x_j x_j \cdot \sum_j a_{ij} \sum_j a_{ij}$

$$= x_j x_j \cdot a_{ij}\, a_{ik}$$

To keep the transformation ~~which is true~~ invariant, we need $\boxed{a_{ij}\, a_{ik} = 1}$

if $j \neq k$, $\sum_j a_{ij} x_j \cdot \sum_k a_{ik} x_k = x_j x_k \cdot \sum_j a_{ij} \sum_k a_{ik}$

$$= x_j x_k \cdot a_{ij}\, a_{ik}$$

These transformations should not be part of the sum since they can't be the square of the magnitude of a vector, so we need $\boxed{a_{ij}\, a_{ik} = 0}$

Mh... $a_{ij} a_{ik} = 1$ if $j = k$

God created the integers

$a_{ij} a_{ik} = 0$ if $j \neq k$ Kronecker delta

the rest is the work of man

Orthogonality condition

so $\boxed{a_{ij} a_{ik} = \delta_{jk} \quad \text{for } j,k = 1,2,3}$ Eq. 4.15

$a_{ij} = \cos\theta_{ij}$, so $\cos\theta_{ij} \cos\theta_{ik} = \delta_{jk}$ cf. Eq. 4.9

since i is repeated, $\sum_i \cos\theta_{ij} \cos\theta_{ik} = \delta_{jk}$

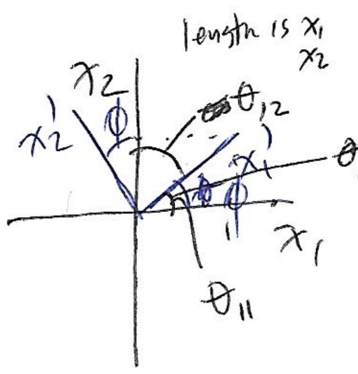
Evidently, $a_{ij} = \cos\theta_{ij}$ satisfies Eq. 4.15, but it is not unique. Any a_{ij} that satisfies the orthogonality condition and is a linear transformation is an <u>orthogonal transformation</u>.

The system of equations in Eq. 4.12 can be written in matrix form as

I will call this $\tilde{A}$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\text{transformation matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let's consider the 2-dimensional case



$x_1' = x_1 \cos\theta_{11} + x_2 \cos\theta_{12}$ ~~$+ x_3 \cos\theta_{13}$~~

with $\phi = \theta_{11}$

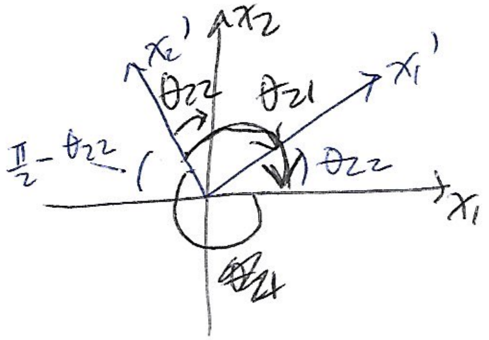
$\cos\theta_{12} = \sin(\pi/2 - \theta_{12})$ $= \sin\phi$

~~$\sin(\pi/2 - \theta_{12}) = \cos$~~

$\theta_{12} = \pi/2 - \phi$

$\Rightarrow \phi = \pi/2 - \theta_{12}$

$x_1' = x_1 \cos\phi + x_2 \sin\phi$



$$x_2' = x_1 \cos\theta_{21} + x_2 \cos\theta_{22}$$

we can see that $\phi = \theta_{22} = \theta_{11}$

$$\pi/2 - \phi + \theta_{21} = \pi$$
$$-\phi + \theta_{21} = \pi/2$$
$$-\phi = \pi/2 - \theta_{21}$$

$$\cos\theta_{21} = \sin(\pi/2 - \theta_{21})$$
$$= \sin(-\phi) = -\sin\phi$$

so $x_2' = -x_1 \sin\phi + x_2 \cos\phi$

we can write the transformation matrix as $\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

★ How many independent parameters are needed to specify the transformation

Answer: 1, just ϕ

★ Can we derive this from the orthogonality condition

In 2-D, $a_{ij} a_{ik} = \delta_{jk}$ for $j,k = 1,2$ $\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\delta_{11} = 1 = a_{11} a_{11} + a_{21} a_{21}$$
$$\boxed{\begin{aligned} \delta_{12} &= 0 = a_{11} a_{12} + a_{21} a_{22} \\ \delta_{21} &= 0 = a_{12} a_{11} + a_{22} a_{21} \end{aligned}}$$ same equation!
$$\delta_{22} = 1 = a_{12} a_{12} + a_{22} a_{22}$$

Answer: we have 4 unknowns and three equations, so we need 1 independent parameter

In 3-D we have 9 unknowns and 6 distinct equations, that's why we need 3 independent parameters

How do the orthogonality conditions look like in 2-D?

$$\cos\phi\cos\phi + (-\sin\phi)(-\sin\phi) = 1$$

$$\cos^2\phi + \sin^2\phi = 1$$

$$\cos\phi\sin\phi + (-\sin\phi)\cos\phi = 0$$

$$\cos\phi\sin\phi - \cos\phi\sin\phi = 0$$

$$\sin\phi\sin\phi + \cos\phi\cos\phi = 1$$

$$\sin^2\phi + \cos^2\phi = 1$$

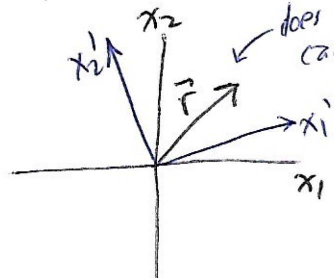
Notice that, even though the algebra remains exactly the same, the transformation matrix $\tilde{A}$ has two interpretations uses

$\tilde{A}$ can be an operator that, operating on the unprimed system, transforms it into the primed system. The vector is unchanged.

same vector involved

$$(\vec{r}) = \tilde{A}\vec{r}$$

does not change, cartesian system does

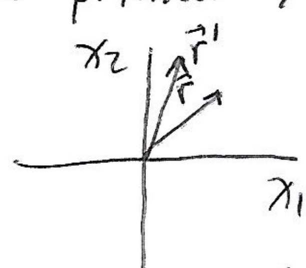


$r = x\hat{i} + y\hat{j}$

$r = x'\hat{i}' + y'\hat{j}'$

Also, $\tilde{A}$ operates on the vector $\vec{r}$ and rotates it with respect to the primed system, still using the primed system. Vector changes.

$$\vec{r}' = \tilde{A}\vec{r}$$



$r = x\hat{i} + y\hat{j}$

$\cancel{r = x\cos\phi\,\hat{i} + y\sin\phi\,\hat{j}}$

$r = (x\cos\phi + y\sin\phi)\hat{i} + (-x\sin\phi + y\cos\phi)\hat{j}$