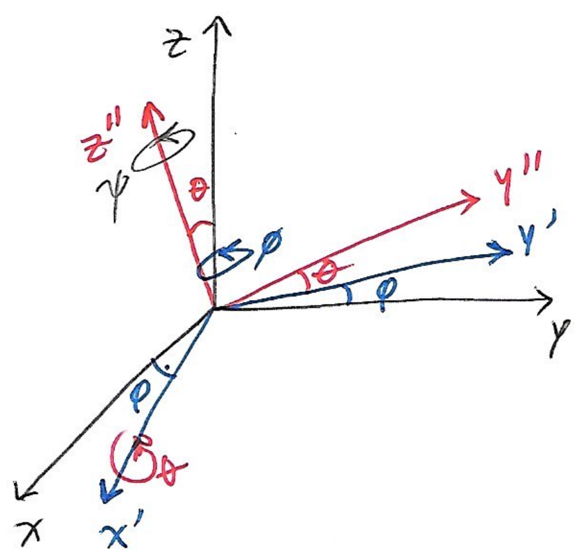
Euler Angles and Euler's theorem 10/14/21

We can carry out the transformation from a given Cartesian system to another by means of <u>three successive rotations performed in a specific sequence</u>.

Remember that the transformation matrices satisfy ~~the~~

- Orthogonality ~~condition~~ $a_{ij} a_{ik} = \delta_{jk}$
- Identity $a_{ki} a_{ij} = \delta_{kj}$
- Proper $|\tilde{A}| = 1$

Consider the following specific sequence.



$\tilde{D}$ is a rotation about z

$$\tilde{D} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\tilde{C}$ is a rotation about x'

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$\tilde{B}$ is a rotation about z''

$$\tilde{B} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

φ, θ, ψ

are the three needed generalized coordinates

independent!

$$\tilde{C}\tilde{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\tilde{B}} \overbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}}^{\tilde{C}\tilde{D}}$$

$$\tilde{A} = \tilde{B}(\tilde{C}\tilde{D}) = \cancel{\cos\psi \sin\psi \cos\phi}$$

$$\cancel{\cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi}$$

Eq. 4.46

$$\begin{bmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} = \tilde{A}$$

Notice that not much is special about this sequence. The only rule is that two adjacent rotations can't be about the same axis. The coordinates then would not be 3+independent. So we have $3 \cdot 2 \cdot 2 = 12$ possible sequences.

Just like by convention the right side or up is positive, or counterclockwise is positive, we can pick a convention for the order of the Euler angles, but there is less standarization.

The book uses the "x-convention": common in celestial mechanics, solidstate

Also known as 313

z-axis $R(\varphi)$ precesion
x'-axis $R(\theta)$ nutation
z''-axis $R(\psi)$ spin

In quantum mechanics, the "y-convention" is common

z-axis $R(\varphi)$ precesion
y'-axis $R(\theta)$ nutation
z''-axis $R(\psi)$ spin

Also known as 323

precesion is the change in orientation of a rotating body

nutation is a rocking motion in the axis of rotation
In celestial mechanics it can be caused by oceans, other planets

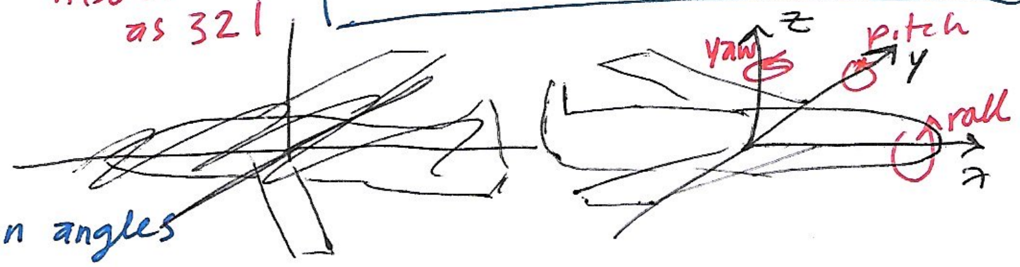
In aircraft and satellites "xyz-convention"

spin

x-axis $R(\varphi)$ roll
y'-axis $R(\theta)$ pitch
z''-axis $R(\psi)$ yaw

Also known as 321

In this case: Tait-Bryan angles

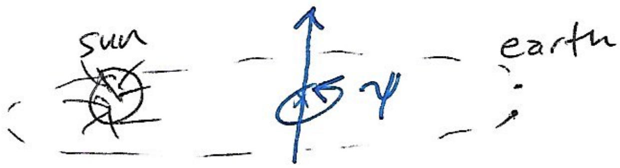


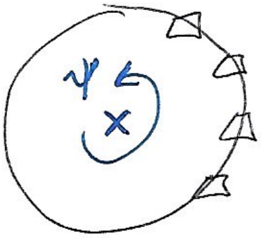
Euler's theorem.

We have seen that the orientation of a body can be specified by an orthogonal transformation. In general, the orientation will change with time. If the body axes are aligned with the space axes at $t=0$, then $\tilde{A}(0)=\tilde{1}$.

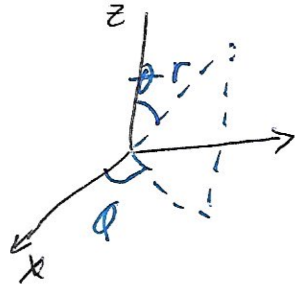
Since motion is continuous, then $\tilde{A}(t)$ is a continuos function of time.

Euler's theorem for rigid bodies: the general displacements of a rigid body with one point fixed is a rotation about some axis.

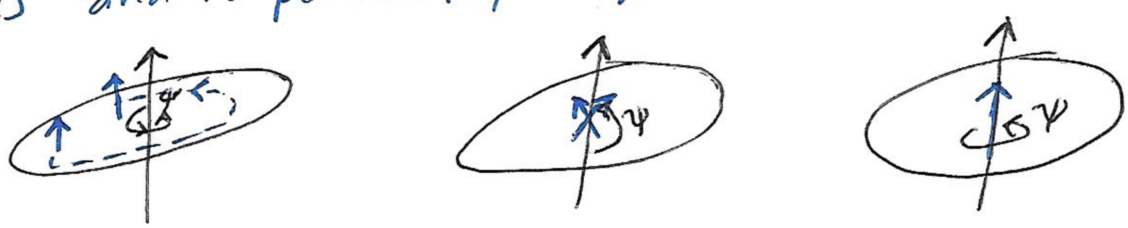




It is always possible to find an axis through the fixed point with orientation in polar angles θ, ϕ such that a rotation by ψ duplicates the general rotation (described also by Euler angles).



This is a single rotation. ~~The axis~~ Notice that every vector would be rotated, except for the one that passes through this axes and is perfectly aligned.



There is only one vector that has the same components in the original and rotated systems. For this vector,

$\vec{R}' = \tilde{A}\vec{R} = \vec{R}$, or more generally $\vec{R}' = \tilde{A}\vec{R} = \lambda\vec{R}$

Eq. 4.48 Eq. 4.49

λ is a (potentially complex) constant, so it affects the magnitude but not the direction of $\vec{R}$. ~~The vector will have the same magnitude in the rotated system due to the orthogonality conditions.~~ Nevertheless, λ must be $+1$ in order for Eq. 4.48 to hold. The rotation matrices are real, so $\tilde{A}$ must be real. Because there is only one vector that satisfies Eq. 4.48, we can restate Euler's theorem as:

The real orthogonal matrix specifying the physical motion of a rigid body with one point fixed always has the eigenvalue $\underline{\underline{+1}}$ $\Rightarrow (\tilde{A} - \lambda \tilde{1})\vec{R} = 0$ Eigenvalue Equations

Expanding, we can see that the ratios of the components are specified, but not their absolute value. The number of solutions is $\underline{\text{infinite}}$, so the matrix is $\underline{\text{singular}}$, so its determinant is zero.

$$(a_{11}-\lambda)X + a_{12}Y + a_{13}Z = 0$$
$$a_{21}X + (a_{22}-\lambda)Y + a_{23}Z = 0$$
$$a_{21}X + a_{32}Y + (a_{33}-\lambda)Z = 0$$

$$|\tilde{A} - \lambda\tilde{1}| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

The characteristic polynomial is

$$(a_{11}-\lambda)\left[(a_{22}-\lambda)(a_{33}-\lambda) - a_{32}a_{23}\right] - a_{12}\left[a_{21}(a_{33}-\lambda) - a_{32}a_{23}\right]$$
$$+ a_{13}\left[a_{21}a_{32} - a_{31}(a_{22}-\lambda)\right] = 0$$

we know that one of the eigenvalues is 1 $\lambda = 1$, so

$$(a_{11}-1)\left[a_{22}a_{33} - a_{22} - a_{33} + 1 - a_{32}a_{23}\right] - a_{12}\left[a_{21}a_{33} - a_{21} - a_{32}a_{23}\right]$$
$$+ a_{13}\left[a_{21}a_{32} - a_{31}a_{22} - a_{31}\right] = 0$$

$$a_{11}a_{22}a_{33} - \cancel{a_{11}a_{22}} - \cancel{a_{11}}a_{33} + \cancel{a_{11}} - a_{11}a_{32}a_{23}$$

$$- \cancel{a_{22}}a_{33} + \cancel{a_{22}} + \cancel{a_{33}} - \cancel{1} + a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{21} + \cancel{a_{12}}a_{32}a_{23}$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} - a_{13}a_{31} = 0 \qquad \begin{bmatrix} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let a_{ij} have the values of the transformation matrix

$$\cancel{(a_{11} - \lambda)[a_{22}a_{33} - a_{22} - a_{33} - a_{32}a_{23}}$$

$$(a_{11} - \lambda)\left[a_{22}a_{33} - \lambda a_{22} - \lambda a_{33} + \lambda^2 - a_{32}a_{23}\right]$$

$$-a_{12}\left[a_{21}a_{33} - \lambda a_{21} - a_{32}a_{23}\right] + a_{13}\left[a_{21}a_{32} - a_{31}a_{22} + \lambda a_{31}\right] = 0$$

$$a_{11}a_{22}a_{33} - \lambda a_{11}a_{22} - \lambda a_{11}a_{33} + \lambda^2 a_{11} - a_{11}a_{23}a_{32}$$
$$- \lambda a_{22}a_{33} + \lambda^2 a_{22} + \lambda^2 a_{33} - \lambda^3 + \lambda a_{23}a_{32}$$
$$- a_{12}a_{21}a_{33} + \lambda a_{12}a_{21} + a_{12}a_{23}a_{32} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} + \lambda a_{13}a_{31} = 0$$

$$A^{-1} = A^T$$

$$-\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda\Big[\overset{a_{22}}{(a_{11}a_{33} - a_{13}a_{31})} + \overset{a_{33}}{(a_{11}a_{22} - a_{12}a_{21})} + \overset{a_{11}}{(a_{22}a_{33} - a_{23}a_{32})}\Big]$$

$$+ \underbrace{a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{32} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}}_{|\tilde{A}| = 1} = 0$$

$$-\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(a_{11} + a_{22} + a_{33}) + 1 = 0$$

A quaternion is a number system that extends the complex numbers. They are generally represented as

$$\overset{\text{scalar angle}}{a} + \overbrace{b\hat{i} + c\hat{j} + d\hat{k}}^{\text{complex vector}} \qquad a, b, c, d \text{ real numbers}$$

$$q = [q_0 \ q_1 \ q_2 \ q_3] \qquad \hat{i}, \hat{j}, \hat{k} \text{ basic quaternions}$$

For 321,

$$q = \begin{bmatrix} \cos(\psi/2) \\ 0 \\ 0 \\ \sin(\psi/2) \end{bmatrix} \begin{bmatrix} \cos(\theta/2) \\ 0 \\ \sin(\theta/2) \\ 0 \end{bmatrix} \begin{bmatrix} \cos(\phi/2) \\ \sin(\phi/2) \\ 0 \\ 0 \end{bmatrix}$$

The rotation matrix can be expressed in terms of quaternions

c.f. Eq. 4.47

$$\tilde{A} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_0 q_2 + q_1 q_3) \\ 2(q_1 q_2 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_3) & 2(q_0 q_1 + q_2 q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$