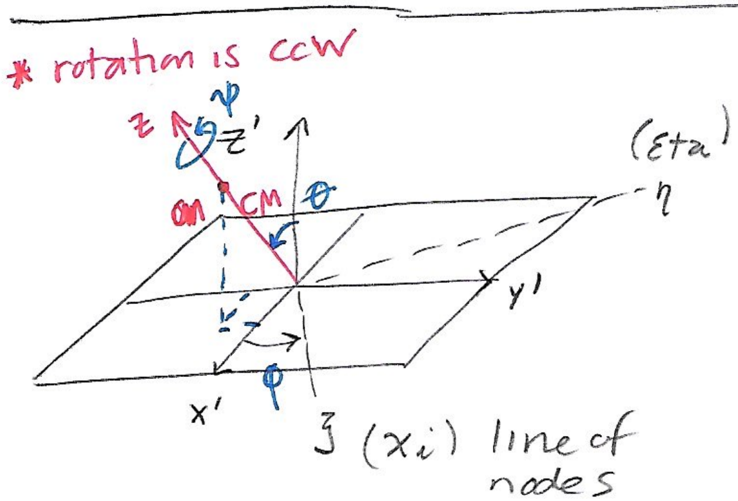
Forced heavy symmetric top



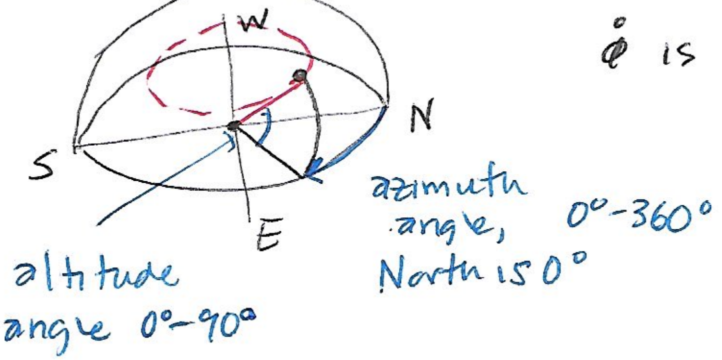
The primed coordinate system is the space system, but we only need the body coordinate system to describe the symmetric top.

- We will use the 313 rotation convention, from this rotation we get the Euler angles ψ, θ, φ.
- The second rotation aligns the z-axis with the vector normal to the plane of rotation (the axis of rotation, and for the symmetric top with $I_x = I_y \neq I_z$, also one of the principal moments of inertia), so the first rotation needs to align the yz-plane with the center of mass.
- The ξ-axis is called the line of nodes, it is the intersection of the system and body planes
- The third rotation is about the z-axis, it aligns the x-axis (not shown) with the inital angle ψ_0, which could be zero. This one is often arbitrary.

Hence:

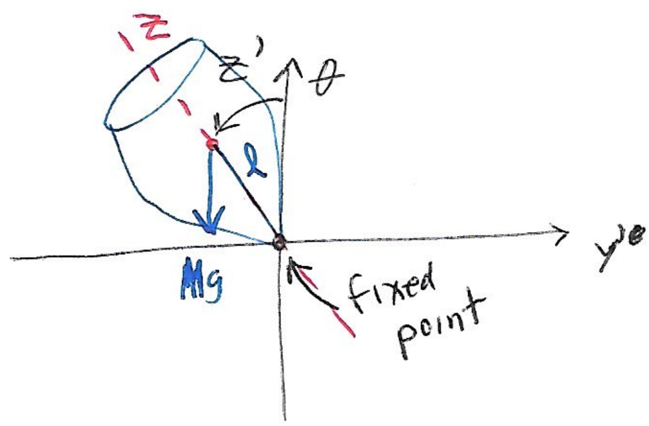
ψ is the rotation angle of the top about its own z-axis (axis of rotation) and $\dot{\psi}$ is how fast it is spinning.

ϕ is the azimuth of the top about the vertical, so $\dot{\phi}$ is how fast the precession is



θ is the inclination of the z-axis from the vertical (the z'-axis). If the top is spinning very fast, θ will not change much with time, but when the spin decreases the top will start to wobble, to rock. ~~θ~~ This is called nutation, so $\dot{\theta}$ is how fast the nutation is occurring.

Let's consider the motion of a symmetrical body in a uniform gravitational field when one point on the symmetry axis is fixed in space, such as a top. The y'z'-plane looks like this:



$$\vec{L} = \tilde{I}\vec{\omega} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k}$$

Assume that the top is spinning very fast, so most of its angular momentum is along the $\hat{k}$ direction

$$\vec{L} \approx I_3\omega_3\hat{k} = L_3\hat{k}$$

In this case there <u>is</u> a torque produced by gravity as it acts on the center of mass, so $\frac{d\vec{L}}{dt} = \vec{N}$

and we know $\vec{N} = \vec{R}_{cm} \times \vec{F} = \vec{R}_{cm} \times (-Mg)\hat{k}'$

Cross product properties

$\vec{N} = Mg\,\hat{k}' \times \ell\hat{k} = Mg\ell\,(\hat{k}' \times \hat{k})$

Since $\vec{L} \approx L_3\hat{k}$, $\hat{k} \approx \frac{\vec{L}}{L_z}$, so $\vec{N} \approx \frac{Mg\ell}{L_z}(\hat{k}' \times \vec{L})$

system axes

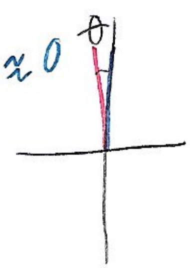
$\cancel{\Omega = \omega_3\left(\frac{I_3 - I_1}{I_1}\right) = \frac{\omega_z I_z - \omega_z I_x}{I_x}}$

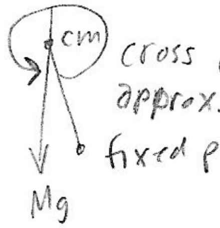
$\cancel{I_x\,\Omega}$ From operator 4.86, $\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = \vec{N}$

This torque does not affect the spin too much if the top is spinning fast and θ is not too large

in this scenario:

$\left(\frac{d\vec{L}}{dt}\right)_b \approx 0$





cross product approx. zero

$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_s \approx \vec{\Omega} \times \vec{L}$

Comparing with the equation above,

$\frac{Mg\ell}{L_z}\hat{k}' = \vec{\Omega} = \frac{Mg\ell}{I_z\omega_z}$

We can be a bit more formal. Previously, we derived the following system of equations

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_1) = N_1$$

$$I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) = N_2$$

$$I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = N_3$$

And looket at the case $\vec{N} = \vec{0}$ with $I_1 = I_2 \neq I_3$.

Now consider the next more complicated case, so $I_1 = I_2 \neq I_3$, $\omega_3 \neq 0$, $\omega_1 = \omega_2 = 0$, but $\vec{N} = N_1 \hat{i}$. Initially we have

$$\cancel{I_1 \dot{\omega}_1 = N_1}$$

$$\cancel{I_2 \dot{\omega}_2 = 0}$$

$$I_3 \dot{\omega}_3 = 0$$

$$I_1 \dot{\omega}_1 + \boxed{\cancel{\boxed{\omega_2}\,\omega_3 (I_3 - I_1)}} = N_1$$ ← $N_1 \neq 0$, so $\dot{\omega}_1 \neq 0$

$$I_2 \dot{\omega}_2 + \boxed{\cancel{\omega_3 \boxed{\omega_1} (I_1 - I_3)}} = 0$$ ← this makes $\omega_1 \neq 0$

$$I_3 \dot{\omega}_3 + \boxed{\cancel{\boxed{\omega_1}\boxed{\omega_2} (I_2 - I_1)}} = 0$$

Next instant:

$$I_1 \dot{\omega}_1 + \boxed{\cancel{\boxed{\omega_2}\,\omega_3 (I_3 - I_1)}} = N_1$$

$$I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) = 0$$ ← Since the N_2 component of the torque is zero, $\dot{\omega}_2 \neq 0$

$$I_3 \dot{\omega}_3 + \boxed{\cancel{\omega_1 \boxed{\omega_2} (I_2 - I_1)}} = 0$$

we recover the original system of equations

Let's attempt the Lagrangian procedure

The kinetic energy is $T = \frac{1}{2} I_j \omega_j^2$. If $I_1 = I_2 \neq I_3$,

$$T = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$, but we need the angular velocities in terms of the Euler angles

The infinitesimal rotation associated with $\vec{\omega}$ consists of 3 successive infinitesimal rotations with $\omega_\varphi = \dot{\varphi}$, $\omega_\theta = \dot{\theta}$, $\omega_\psi = \dot{\psi}$. They are not orthogonal, rather they are: $\vec{\omega}_\varphi$ along the ~~space~~ z^*-axis
$\vec{\omega}_\theta$ along the line of nodes
$\vec{\omega}_\psi$ along the z-axis

To bring $\vec{\omega}_\varphi$ from z^* to z', we need to apply all the rotations, Remember $\tilde{A} = \tilde{B}\tilde{C}\tilde{D}$ Eq. 4.46

Since $\vec{\omega}_\varphi$ is along the z°-axis, $\vec{\omega}_\varphi = [0, 0, \dot{\varphi}]$ and

$$\tilde{A}\vec{\omega}_\varphi = \begin{bmatrix} \square & \square & \sin\psi\sin\theta \\ \square & \square & \cos\psi\sin\theta \\ \square & \square & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix}$$

$$\tilde{A}\vec{\omega}_\varphi = \begin{bmatrix} \dot{\varphi}\sin\psi\sin\theta \\ \dot{\varphi}\cos\psi\sin\theta \\ \dot{\varphi}\cos\theta \end{bmatrix}$$

To bring $\vec{\omega}_\theta$ from ξ to x' we need to apply the $\tilde{B}$ rotation. Since $\vec{\omega}_\theta$ is along the ξ-axis,

Eq. 4.45

$\vec{\omega}_\theta = [\,\dot{\theta}, 0, 0\,]$ and

$$\tilde{B}\vec{\omega}_\theta = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix}$$

$\vec{\omega}_\psi$ is already along the z'-axis, so $\omega_\psi = [0, 0, \dot{\psi}]$

Adding the components,

$$\omega_x = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_y = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_z = \dot{\phi}\cos\theta + \dot{\psi}$$

Eq. 4.87

In terms of the Euler angles, the kinetic energy is

$$T = \frac{1}{2} I_1 \left[(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 \right] + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2$$

$$T = \frac{1}{2} I_1 \left[\dot{\phi}^2\sin^2\theta\sin^2\psi \cancel{+ 2\dot{\phi}\dot{\theta}\sin\theta\sin\psi\cos\psi} + \dot{\theta}^2\cos^2\psi + \dot{\phi}^2\sin^2\theta\cos^2\psi \cancel{- 2\dot{\phi}\dot{\theta}\sin\theta\cos\psi\sin\psi} + \dot{\theta}^2\sin^2\psi \right]$$

$+ \ldots$

$$T = \frac{1}{2} I_1 \left[\dot{\phi}^2 (\sin^2\theta\sin^2\psi + \sin^2\theta\cos^2\psi) + \dot{\theta}^2 (\underbrace{\cos^2\psi + \sin^2\psi}_{1}) \right] + \ldots$$

$$T = \frac{1}{2} I_1 \left\{ \dot{\varphi}^2 \left[\sin^2\theta \left(\cancel{\sin^2\psi + \cos^2\psi}^{\,1} \right) \right] + \dot{\theta}^2 \right\} + \dots$$

Eq. 5.50

$$T = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\varphi}\cos\theta \right)^2$$

The potential energy $V = m_i \vec{r}_i \cdot \vec{g} = M\vec{R} \cdot \vec{g}$

where $\vec{R} = l\hat{k}$ and $\vec{g} = g\hat{k}'$, so $V = Mgl(\hat{k}' \cdot \hat{k})$

Similar to the cross product before

$$\hat{k}' \cdot \hat{k} = |\hat{k}'||\hat{k}| \cos\alpha_{\hat{k}'\hat{k}} = \cos\theta \qquad V = Mgl\cos\theta$$

$\mathcal{L} = T - V$, so we get Eq. 5.52

☆ What is the next step in solving this problem? Constants.

The following generalized coordinates appear in the Lagrangian:

	$\dot{\theta}$	kinetic energy first term
	θ	all terms
	$\dot{\varphi}$	kinetic energy, first & second terms
cyclic coordinates $\longrightarrow$ constant of motion	φ	None
	$\dot{\psi}$	kinetic energy second term
cyclic coordinates $\longrightarrow$ constant of motion	ψ	None

so p_ψ $\quad \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) = \dfrac{\partial \mathcal{L}}{\partial \psi} = 0 \Rightarrow \dfrac{\partial \mathcal{L}}{\partial \dot{\psi}}$ is constant

$\cancel{\frac{1}{2} I_3 \cdot 2 \left[\dot{\psi}^2 + \dot{\varphi}\cos\theta\right]}$

likewise

p_φ

$$\frac{\partial}{\partial \dot{\psi}} \left[\frac{1}{2} I_3 \left(\dot{\psi}^2 + 2\dot{\psi}\dot{\varphi}\cos\theta + \dot{\varphi}^2 \cos^2\theta \right) \right]$$

$$= \frac{1}{2} I_3 \left(2\dot{\psi} + 2\dot{\varphi}\cos\theta \right) = I_3 \left(\dot{\psi} + \dot{\varphi}\cos\theta \right)$$