Infinitesimal rotations

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We have been ~~ta~~ talking for a while about how matrix multiplication and hence rotation operations don't commute. The order of operations affects the final result.

Infinitesimal transformations

Before, we used $\boxed{a_{ij}}$ to represent the elements of the matrix, but let's use $\boxed{\epsilon_{ij}}$ to make explicit that they are infinitesimally small. Now,

original system

infinitesimal change

$$x_i' = x_i + \epsilon_{ij} x_j$$
$$= (\delta_{ij} + \epsilon_{ij}) x_j$$

In matrix form: $\vec{x}' = (\tilde{1} + \tilde{\epsilon})\vec{x}$

For transformation matrices we had $\vec{r}' = \tilde{A}\vec{r}$, so infinitesimal transformations are analogous, but the elements are quite different.

Do these matrices commute? Let's check

$$\begin{bmatrix} 1+\epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & 1+\epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & 1+\epsilon_{33} \end{bmatrix}\begin{bmatrix} 1+\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & 1+\varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & 1+\varepsilon_{33} \end{bmatrix}$$

upper left 11 site

$$(1+\epsilon_{11})(1+\varepsilon_{11}) + \epsilon_{12}\varepsilon_{12} + \epsilon_{13}\varepsilon_{13} \quad \cancel{(1+\epsilon_{11})\epsilon_{12}}$$

$$1 + \varepsilon_{11} + \epsilon_{11} + \overset{\text{too tiny}}{\cancel{\epsilon_{11}\varepsilon_{11}}}$$

top middle

$$(1+\epsilon_{11})\varepsilon_{12} + \epsilon_{12}(1+\varepsilon_{22}) + \epsilon_{13}\varepsilon_{32}$$

$$\varepsilon_{12} + \cancel{\epsilon_{11}\varepsilon_{12}} + \epsilon_{12} + \cancel{\epsilon_{12}\varepsilon_{22}} + \cancel{\epsilon_{13}\varepsilon_{32}} \text{ too tiny}$$

upper right

$$(1+\epsilon_{11})\varepsilon_{13} + \epsilon_{12}\varepsilon_{23} + \epsilon_{13}(1+\varepsilon_{33})$$

$$\varepsilon_{13} + \underset{\text{too tiny}}{\cancel{\epsilon_{11}\varepsilon_{13}}} + \overset{\text{too tiny}}{\cancel{\epsilon_{12}\varepsilon_{23}}} + \epsilon_{13} + \cancel{\epsilon_{13}\varepsilon_{32}} \text{ too tiny}$$

And we can start to see the pattern:

$$(\tilde{1} + \tilde{\epsilon}_1)(\tilde{1} + \tilde{\varepsilon}_2) = \cancel{1+\epsilon_1+\varepsilon_2}$$

$$\begin{bmatrix} 1+\varepsilon_{11}\epsilon_{11} & \varepsilon_{12}+\epsilon_{12} & \varepsilon_{13}+\epsilon_{13} \\ \varepsilon_{21}+\epsilon_{21} & 1+\varepsilon_{22}+\epsilon_{22} & \varepsilon_{23}+\epsilon_{23} \\ \varepsilon_{31}+\epsilon_{31} & \varepsilon_{32}+\epsilon_{32} & 1+\varepsilon_{33}\epsilon_{32} \end{bmatrix}$$

This is equivalent to adding three matrices,

$$(\tilde{1}+\tilde{\epsilon}_1)(\tilde{1}+\tilde{\varepsilon}_2) = \tilde{1} + \tilde{\epsilon}_1 + \tilde{\varepsilon}_2$$

For the case of $(\tilde{1}+\tilde{\varepsilon}_2)(\tilde{1}+\tilde{\epsilon}_1)$ we get exactly the same result since scalar product and addition are commutative and you have the same values in each of the ~~ve~~ matrix elements

We know the definition of inverse, $\tilde{A}^{-1}\tilde{A} = \tilde{1}$

In this case $\tilde{A} = \tilde{1}+\tilde{\epsilon}$, so $\tilde{A}^{-1} = \tilde{1} - \tilde{\epsilon}$

$$\cancel{\text{so } \tilde{A}^{-1}(\tilde{1}+\tilde{\epsilon}) = \tilde{1}}$$

$$\cancel{\Rightarrow \tilde{A}^{-1} + \tilde{A}^{-1}\tilde{\epsilon} = \tilde{1}}$$

$$\cancel{\Rightarrow \tilde{A}^{-1} = \tilde{1} - \tilde{A}^{-1}\tilde{\epsilon} = \tilde{1} - \tilde{A}^{-1}(\tilde{A} - \tilde{1})}$$

$$\tilde{A}^{-1}\tilde{A} = (\tilde{1}-\tilde{\epsilon})(\tilde{1}+\tilde{\epsilon}) = \tilde{1} - \tilde{\epsilon} + \tilde{\epsilon} = \tilde{1}$$

Using the pattern that we uncovered before, this is

For orthogonal transformations, we have $\tilde{A}^{-1} = \tilde{A}^{T} = \tilde{1} - \tilde{\epsilon}$

but also $\tilde{A}^{T} = (\tilde{1} + \tilde{\epsilon})^{T} = \tilde{1}^{T} + \tilde{\epsilon}^{T} = \underline{1} + \tilde{\epsilon}^{T}$

$$\Rightarrow \underline{\underline{\tilde{\epsilon}^{T} = -\tilde{\epsilon}}}$$

The matrix $\tilde{\epsilon}$ is antisymmetric by definition

$$\tilde{\epsilon} = \begin{bmatrix} 0 & -\epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & 0 & -\epsilon_{23} \\ -\epsilon_{31} & \epsilon_{32} & \end{bmatrix} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$$

Convenient notation

With these infinitesimal rotations, we can do something that we couldn't before: $\vec{r}' - \vec{r} \equiv d\vec{r} = \tilde{\epsilon}\vec{r}$

In expanded form
$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 0 + x_2 d\Omega_3 - x_3 d\Omega_2 \\ -x_1 d\Omega_3 + 0 + x_3 d\Omega_1 \\ x_1 d\Omega_3 - x_2 d\Omega_1 + 0 \end{bmatrix}$$

Using the "convenient notation", we can express

$$d\vec{r} = \vec{r} \times d\vec{\Omega} \qquad \text{Eq. 4.72}$$