The virial theorem (widely applicable!) 9/28/21

Consider a system of massive particles with position vectors $\vec{r}_i$ and applied forces $\vec{F}_i$, then $\dot{\vec{p}}_i = \vec{F}_i$ are the EoM.

$u \quad v$

We are interested in the quantity $G = \sum_i \vec{p}_i \cdot \vec{r}_i$, mostly

$d(uv) = u\,dv + v\,du$

because $$\frac{dG}{dt} = \sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i + \sum_i \vec{r}_i \cdot m_i \ddot{\vec{r}}_i$$

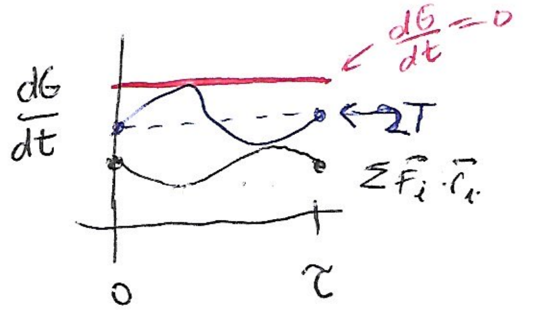
$$\frac{dG}{dt} = \sum_i m_i \dot{\vec{r}}_i^{\,2} + \sum_i \vec{r}_i \cdot \vec{F}_i = \underbrace{2T}_{\text{kinetic}} + \underbrace{\sum_i F_i \cdot r_i}_{\text{potential}}$$

In general, the kinetic energy and the potential energy will be different at each instant, even if the sum is constant. Nevertheless, if the motion is periodic, we can get a time average by integrating over the period and dividing by the period.

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{\cancel{dt}}\cancel{dt} \equiv \overline{\frac{dG}{dt}} = \overline{2T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} = \frac{1}{\tau}\left[G(\tau) - G(0)\right]$$

$= 0$ by definition if periodic

≈ 0 if not periodic, can find a time τ such that close to zero



with $d\vec{F}_i = -P\hat{n}\,dA$

$$-\frac{P}{2}\int \hat{n}\cdot r\,dA = -\frac{P}{2}\int \nabla\cdot\vec{r}\,dV = -\frac{P}{2}(3V)$$

so $\frac{3}{2}Nk_B T = \frac{3}{2}PV$

In any case $$\overline{T} = -\frac{1}{2}\overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$ Eq. 3.26 Virial theorem

We previously derived the Equation $\cancel{\left(\frac{dr}{dt}\right)^2}$ $\dot{r}^2$ ~~= 2E~~

$$\frac{dr}{dt} = \left[\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)\right]^{1/2}$$ and showed that, at least in principle, it is separable

$$\int dt = \pm \int \frac{dr}{\sqrt{\frac{2}{m}(E - V - \ell^2/2mr^2)}}$$ This implies we can get $\boxed{r(t) \text{ and } \theta(t)}$ we have the orbit as a fn. of time

Eq. 3.18

Sometimes this is called "reduce to quadrature," this essentially means that you got the problem to integrals that can, at least in principle, be evaluated analytically or numerically.

We have an orbit in terms of time. Consider the "Kepler Problem," the force follows the inverse-squre law. In this case $V(r) = -\frac{K}{r}$. We want the orbit in space, so $\underline{r(\theta)}$.

$$m\ddot{r} - \frac{\ell^2}{mr^3} = \overset{-\frac{dV(r)}{dr}}{f(r)} \Rightarrow m\ddot{r} = \frac{\ell^2}{mr^3} - \frac{K}{r^2}$$

Remember $\ell = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt} \Rightarrow d\theta = \frac{\ell}{mr^2}dt \Rightarrow \frac{d}{dt} = \frac{\ell}{mr^2}\frac{d}{d\theta}$

$$\Rightarrow \boxed{\frac{dr}{dt} = \frac{\ell}{mr^2}\frac{dr}{d\theta}}$$

Let $\boxed{u = 1/r}$, then

chain rule

$$\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{r(\theta)}\right) = \frac{d(1/r)}{dr}\frac{dr}{d\theta} = \boxed{-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{du}{d\theta}}$$

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{K}{r^2}$$

$\frac{d}{dt}$ $\frac{du}{dt}$

$$m \cdot \frac{d}{dt}\left(\frac{dr}{dt}\right) = \cancel{m}\cdot\frac{l}{\cancel{m}r^2}\frac{d}{d\theta}\left(\frac{l}{\cancel{m}r^2}\frac{dr}{d\theta}\right) = \frac{l^2}{\cancel{m}r^3} - \frac{Km}{r^2}$$

$$\Rightarrow l u^{\cancel{2}} \frac{d}{d\theta}\left(-l\frac{du}{d\theta}\right) = l^2 u^{\cancel{3}} - K u^{\cancel{2}} m$$ divide both sides by u^2

$$\Rightarrow -\frac{\cancel{l^2}}{\cancel{l^2}}\frac{d^2u}{d\theta^2} = \frac{\cancel{l^2}u}{\cancel{l^2}} - \frac{Km}{l^2}$$ divide both sides by l^2

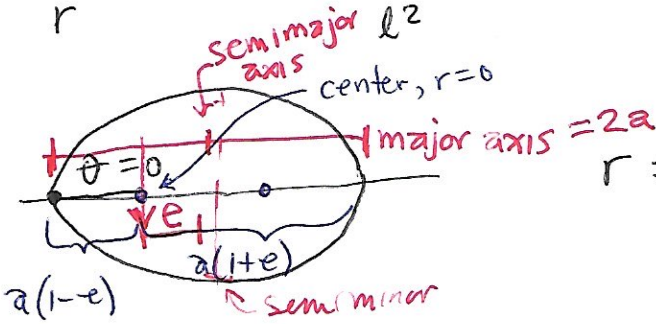
Compare to Eq. 3.34

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{Km}{l^2}}$$

$$\Rightarrow +\frac{d^2u}{d\theta^2} + u = +\frac{Km}{l^2}$$

This is a famous diff. Eq. its solution is $u = A\cos\theta + \frac{Km}{l^2}$

$$u = \frac{1}{r} = A\cos\theta + \frac{mK}{l^2} \Rightarrow r = \frac{1}{A\cos\theta + mK/l^2} \quad ①$$



$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad ②$$ The equation of an ellipse is this

semimajor axis, eccentricity

Consider the case $\theta = 0$, then $r(\theta) = r(0) = \frac{a(1-e^2)}{1+e} = \frac{a\cancel{(1+e)}(1-e)}{\cancel{(1+e)}}$

By comparing ① and ② we can get some intuition about the parameters

$a(1-e^2) = 1$ and $\frac{mK}{l^2} = 1$, so

$$a(1-e^2) = \frac{mK}{l^2} \Rightarrow l = \sqrt{\frac{mK}{a(1-e^2)}}$$

Eq. 3.63

$\frac{\ell^2}{mk} = 1$ as well, so $a(1-e^2) = \frac{\ell^2}{mk}$

$\Rightarrow \ell = \sqrt{amk(1-e^2)}$