Inertia ellipsoid and Euler Equations of motion

The equation of a circle is $x^2+y^2=1$
it is centered at the origin and the radius is 1

We can elongate one side to form an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
also centered at the origin with the foci at $(\pm c, 0)$ with $c=\sqrt{a^2+b^2}$
The width is $2a$, the height is $2b$, $a>b$.

In 3-D, a sphere is $x^2+y^2+z^2=1$ and for an ellipsoid, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

The moment of inertia is $I=\hat{n}\cdot\tilde{I}\cdot\hat{n}$. Let $\hat{n}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$
where α, β, γ are direction cosines. From the definition of contraction, $I=n_i I_{ij} n_j$

$\tilde{I}$ is symmetric, so $I_{ij}=I_{ji}$

$$\hat{n}\cdot\tilde{I}\cdot\hat{n}=I_{11}\alpha^2+I_{22}\beta^2+I_{33}\gamma^2$$
$$+2\alpha\beta I_{12}+2\alpha\gamma I_{13}+2\beta\gamma I_{32}=I$$

Let $\sqrt{I}\vec{\rho}=\hat{n}$
then we get an extra I on the r.h.s. which we can move to the l.h.s. ~~to~~ to recover

$i=1$: $n_i^2 I_{11}$, $n_1n_2 I_{12}$, $n_1n_3 I_{13}$

$i=2$: $n_2n_1 I_{21}$, $n_2^2 I_{22}$, $n_2n_3 I_{23}$

$i=3$: $n_3n_1 I_{31}$, $n_3n_2 I_{32}$, $n_3^2 I_{33}$

$$1 = I_{11}\rho_1^2 + I_{22}\rho_2^2 + I_{33}\rho_3^2 + 2I_{12}\rho_1\rho_2 + 2I_{13}\rho_1\rho_3 + 2I_{23}\rho_2\rho_3$$

Eq. 5.34 inertial ellipsoid

Notice that we have 3 variables ρ_1, ρ_2, ρ_3

If $\rho_1\rho_2 = \rho_2\rho_3 = \rho_1\rho_3 = 0$ we get the equation of an ellipsoid in which the principal moments of ~~ellipsoid~~ inertia. determine the ~~length~~ length of the axes of the ellipsoid.

If you rotate the ~~axes~~ coordinate system so that

$\rho_1\rho_2 \neq 0$ or $\rho_2\rho_3 \neq 0$ or $\rho_1\rho_3 \neq 0$ you recover the longer Eq.

The shape of the ellipsoid is unchanged, what changes is how you express it.

In terms of R_0, $\vec{\rho} = \hat{n}/R_0\sqrt{M}$

The moment of inertia is $I = m_i r_i^2$. If all the masses are equal, then $I = m\sum_i^n r_i^2$ with $\frac{M}{n} = m$, so

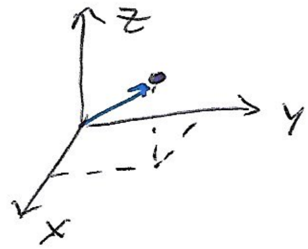
$$I = MR_0^2 = M\left(\sqrt{\frac{1}{n}(r_1^2 + r_2^2 + \ldots + r_n)}\right)^2$$

R_0 is the root mean square of the perpendicular distances from ~~the~~ each particle to the axis of rotation, it is called the radius of gyration. It tells you, if you had to put all the mass of a rigid body in a single particle, how far away that point would have to be from the axis of rotation to have the same moment of inertia as the rigid body.

Euler equations of motion

for a rotating rigid body about a point

Consider the rigid body with the center of mass located a distance R to the origin of the coordinate system



The time derivative of the angular momentum is the torque $\left(\frac{d\vec{L}}{dt}\right) = \vec{N}$

Body coordinates

Using operator 4.86, $\left(\frac{d\vec{L}}{dt}\right)_s = \boxed{\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = \vec{N}}$

In body coordinates, the i^{th} component is

$$\frac{dL_i}{dt} + \epsilon_{ijk}\,\omega_j L_k = N_i$$

ϵ_{ijk} is the Levi-Civita symbol, cross-product in Einstein notation

$\epsilon_{ij\ldots k} = (-1)^p \epsilon_{12\ldots n}$

where n is the number of indices and p the number of pairwise interchanges to get from $ij\ldots k$ to $1,2\ldots n$

If the body axes are aligned with the principal axes of rotation, ~~Since~~ $L_i = I_i \omega_i$, $\tilde{I}$ is diagonal

$$\frac{d(I_i\omega_i)}{dt} = I_i \frac{d\omega_i}{dt} = I_i \dot{\omega}_i$$

principal moments are time independent.

Eq. 5.39

$$I_i\dot{\omega}_i + \epsilon_{ijk}\,\omega_j\omega_k I_k = N_i$$

No Einstein convention in i since component

$$I_1\dot{\omega}_1 + \epsilon_{123}\,\omega_2\omega_3 I_3 + \epsilon_{132}\,\omega_3\omega_2 I_2 = N_1$$

$$I_1\dot{\omega}_1 + \omega_2\omega_3 I_3 - \omega_2\omega_3 I_2 = N_1$$

$i=1$ $$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = N_1$$

From convention, $\epsilon_{123} = 1$

ϵ_{123} needs zero switches to get to ϵ_{123}

so $\epsilon_{123} = (-1)^0 = 1$

ϵ_{132} needs one switch $(3 \rightleftarrows 2)$, so $(-1)^1 = -1$

123
231
312

$$I_2\dot{\omega}_2 + \epsilon_{231}\omega_3\omega_1 I_1 + \epsilon_{213}\omega_1\omega_3 I_3$$

$(+1)$ $(-1)^2 = 1$

$$I_2\dot{\omega}_2 + \omega_3\omega_1\left(I_1 - I_3\right) = N_2$$

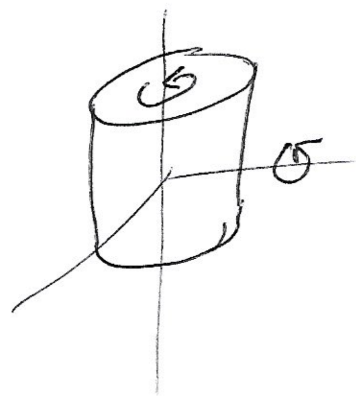
213 231
(-1) 123 132 I
123 ''

Similarly, $I_3\dot{\omega}_3 + \omega_1\omega_2\left(I_2 - I_1\right) = N_3$

In the case $I_1 = I_2 \neq I_3$, for example an ellipsoid of revolution

the second term on the l.h.s. is zero, $I_3\dot{\omega}_3 = N_3$.

A torque with no z component does not affect to rotation along z.



You can change the spin with a torque $\vec{N} = N_1\hat{i} + N_2\hat{j}$, but you can't rotate

Consider the case in which the torque is zero, then

$$I_1\dot{\omega}_1 = -\omega_2\omega_3\left(I_3 - I_2\right) = \omega_2\omega_3\left(I_2 - I_3\right)$$

$$I_2\dot{\omega}_2 = \omega_3\omega_1\left(I_3 - I_1\right)$$

$$I_3\dot{\omega}_3 = \omega_1\omega_2\left(I_1 - I_2\right)$$

Non-linear system of differential equations. Since there is no torque, the kinetic energy and the angular momentum are conserved, so we have 2 integrals of the motion. It is possible to integrate using elliptical functions, but then it becomes more of a math problem.

(what is next?!)

If there is no torque and $I_1 = I_2 \neq I_3$

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3)$$
$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)$$
$$I_3 \dot{\omega}_3 = 0$$

$\Rightarrow$ ω_3 is a constant which can be determined by initial conditions

so

$$\dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3)/I_1 \quad \text{and} \quad \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)/I_2 .$$

with $\Omega = \omega_3 \dfrac{I_3 - I_1}{I_1}$, $\quad \dot{\omega}_1 = -\omega_2 \Omega$, $\quad \dot{\omega}_2 = \omega_1 \Omega$

$$\frac{d}{dt}\dot{\omega}_1 = \ddot{\omega}_1 = -\frac{d}{dt}(\omega_2 \Omega) = -\Omega \dot{\omega}_2 = -\Omega^2 \omega_1$$

$\ddot{\omega}_1 = -\Omega^2 \omega_1$ is the equation of a simple harmonic oscillator

the solution is $\omega_1 = A\cos(\Omega t)$

$$\vec{\omega} = \left[A\cos(\Omega t),\ A\sin(\Omega t),\ \omega_3\right]$$

$d\omega_2 = dt\, \omega_1 \Omega$ $\qquad \tau = \Omega t$, $d\tau = \Omega dt$

$\int d\omega_2 = \omega_2 = \int A\cos \Omega t\, dt \leftarrow \frac{d\tau}{\Omega}$

$A\int \cos\tau\, d\tau$

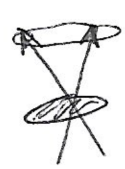
$A\sin\tau + C$

$\omega_2 = A\sin \Omega t$

but we can see that

$$\sqrt{\omega_1^2 + \omega_2^2} = \left[A^2\cos^2(\Omega t) + A^2\sin^2(\Omega t)\right]^{1/2}$$
$$= A$$

so ω_1 and ω_2 is the parametric equation of a circle of radius A.




$\cancel{T = \frac{\vec{L}^2}{2I}}$ $\cancel{T = \frac{1}{2}\tilde{I}\omega = \frac{1}{2} I_1 \omega_1}$

$$T = \frac{1}{2}\,\vec{\omega}\cdot\tilde{I}\cdot\vec{\omega} = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_1 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

$$T = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \omega_3^2$$

Since both A and ω_3 are constant, the kinetic energy is constant

$$\vec{L} = \tilde{I}\vec{\omega} = I_1 \omega_1 \hat{i} + I_1 \omega_2 \hat{j} + I_3 \omega_3 \hat{k}$$

$$L^2 = I_1^2 (\omega_1^2 + \omega_2^2) + I_3^2 \omega_3^2 = I_1^2 A^2 + I_3^2 \omega_3^2$$

the magnitude of the angular momentum also constant.