Simple Applications of the Lagrangian formulation

We used:
- generalized coordinates
- D'Alembert's principle (Principle of virtual work)
- Newton's second law

To derive Lagrange's Equations $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$

where $\mathcal{L} \doteq T - V$ is the Lagrangian.

You need 2 things

☆ To work with the Lagrangian formulation of mechanics you need the kinetic energy T and the potential energy V in generalized coordinates. To go from Cartesian to generalized coordinates, use the transformation equations

$$\vec{r}_1 = \vec{r}_1(q_1, q_2, \ldots, q_{3N-K}, t)$$

$$\vdots$$

$$\vec{r}_N = \vec{r}_N(q_1, q_2, \ldots, q_{3N-K}, t)$$

The kinetic energy in generalized coordinates is given by

Using Eq. 1.46

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right)^2$$

Notice that after all the terms are multiplied,

there will be 1 term $\left(\frac{\partial \vec{r}_i}{\partial t}\right)^2$ ← independent of velocity

linear in velocity

as many $\left(\frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right)\left(\frac{\partial \vec{r}_i}{\partial t}\right)$ as there are degrees of freedom

quadratic in velocity

and DOF squared $\left(\frac{\partial \vec{r}_i}{\partial \dot{q}_j}\dot{q}_j\right)\left(\frac{\partial \vec{r}_i}{\partial \dot{q}_k}\dot{q}_k\right)$

so $$\boxed{T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2}\sum_j \sum_k M_{jk}\dot{q}_j \dot{q}_k}$$ Eq. 1.71

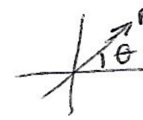
with $M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2$ $\qquad M_{jk} = \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_k}$

$M_j = \sum_i m_i \frac{\partial \vec{r}_i}{\partial t}\cdot\frac{\partial \vec{r}_i}{\partial q_j}$

Notice also that if the system is not explicitely time-dependent, then the $\partial \vec{r}_i / \partial t$ terms will be zero, so the M_0 and M_j terms will vanish. The M_{jk} terms will be fine.

Example: 1 particle in plane with polar coordinates

Degrees of freedom: 2



Generalized coordinates: $q_1 = r \quad q_2 = \theta$

Not explicitely time-dependent, so $M_0 = M_j = 0$

The kinetic energy must be given in terms of r and θ

The transformation equations are: $x = r\cos\theta$

$y = r\sin\theta$

Using Eq. 1.71, we would get 4 jk combinations:

$$\left|\ \frac{\partial \vec{r}}{\partial r}\cdot\frac{\partial \vec{r}}{\partial r}\ \right|\ \frac{\partial \vec{r}}{\partial r}\cdot\frac{\partial \vec{r}}{\partial \theta}\ \left|\ \frac{\partial \vec{r}}{\partial \theta}\ \frac{\partial \vec{r}}{\partial r}\ \right|\ \frac{\partial \vec{r}}{\partial \theta}\ \frac{\partial \vec{r}}{\partial \theta}\ \Big|$$

In this particular case, it is easier to notice that

udv + vdu

$$\frac{d}{dt}x = \frac{d}{dt}(\overset{u}{r}\overset{v}{\cos\theta}) = \frac{1}{dt}\left[-r\sin\theta\, d\theta + \cos\theta\, dr\right] = \overset{\dot{x}}{\dot{r}\cos\theta - r\dot{\theta}\sin\theta}$$

$$\frac{d}{dt}y = \frac{d}{dt}(\overset{u}{r}\overset{v}{\sin\theta}) = \frac{1}{dt}\left[r\cos\theta\, d\theta + \sin\theta\, dr\right] = \overset{\dot{y}}{\dot{r}\sin\theta + r\dot{\theta}\cos\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left[(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2\cos^2\theta - \cancel{2r\dot{r}\dot{\theta}\sin\theta\cos\theta} + r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\sin^2\theta + \cancel{2r\dot{r}\dot{\theta}\sin\theta\cos\theta} + r^2\dot{\theta}^2\cos^2\theta\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2(\overset{1}{\cancel{\cos^2\theta + \sin^2\theta}}) + r^2\dot{\theta}^2(\overset{1}{\cancel{\sin^2\theta + \cos^2\theta}})\right]$$

$$T = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2\right]$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0$$

Let $\mathcal{L} = T - V$ with $-V$ just an arbitrary potential
-or zero

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2\right]\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\cancel{\frac{1}{2}}mr^2\,\cancel{2}\dot{\theta}\right) - 0 = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

This is conservation of angular momentum
if potential is not zero, we would have the torque exerted by the force field

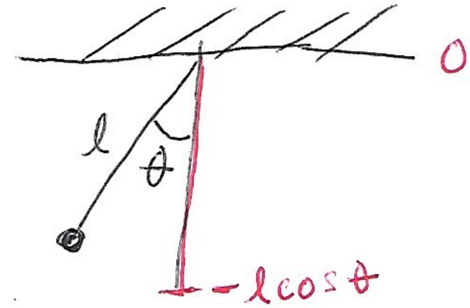
$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}}\left[\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2\right]\right) - \frac{\partial}{\partial r}\mathcal{L} = 0$$

$$\frac{d}{dt}\left(\cancel{\frac{1}{2}}m\,\cancel{2}\dot{r}\right) - 0 = 0$$

$$\frac{d}{dt}(m\dot{r}) = 0$$

conservation of linear momentum
if potential is not zero, then force of the force field is change in momentum

Example : Pendulum



Degrees of freedom: 1

Generalized coordinate: $q_1 = \theta$

Not explicetely time-dependent, so $M_0 = M_j = 0$

$M_{jk} = M_{\theta\theta}$

The velocity is a radial velocity times a distance, $v_\theta = l\dot\theta$ $\quad \vec{r}_\theta = l\theta$

$$T = \frac{1}{2}\sum_j^1 \sum_k^1 M_{jk}\dot q_j \dot q_k = \frac{1}{2} M_{\theta\theta}\dot\theta\dot\theta$$

$$M_{\theta\theta} = \sum_i^1 m_i \frac{\partial \vec{r}_i}{\partial\theta}\cdot\frac{\partial \vec{r}_i}{\partial\theta}$$

(There is only 1 particle and position vector changes linearly with angle θ)

$= m l^2$

$\therefore T = \frac{1}{2} m l^2 \dot\theta^2$

The potential is just due to gravity, so $\sim mgh$

$$V = -mgl\cos\theta$$

$$\mathcal{L} = \frac{1}{2} m l^2 \dot\theta^2 - (-mgl\cos\theta)$$

$$\mathcal{L} = \frac{1}{2} m l^2 \dot\theta^2 + mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

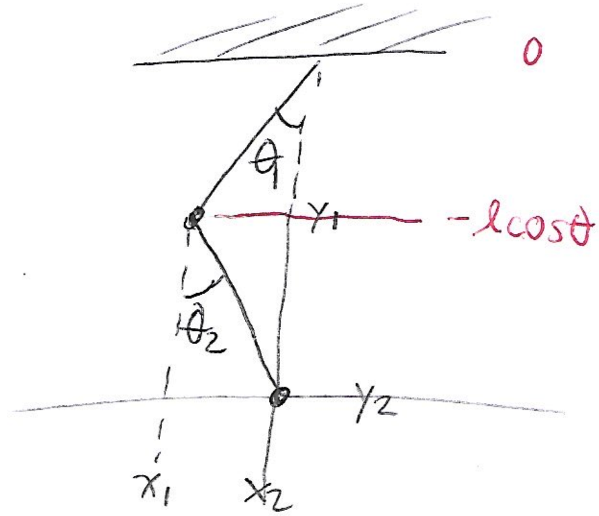
$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = \cancel{\tfrac{1}{2}} m l^2 \cdot \cancel{2}\dot\theta$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = -\sin\theta\cdot mgl$$

Equation of motion

$\frac{d}{dt}(ml^2\dot\theta) = ml^2\ddot\theta$, so $\cancel{m}l^{\cancel{2}}\ddot\theta + \cancel{m}g\cancel{l}\sin\theta = 0 \Rightarrow \boxed{\ddot\theta = -\frac{g}{l}\sin\theta}$

Example: Double pendulum



Degrees of freedom: 2

Generalized coordinates: $q_1 = \theta_1$
$q_2 = \theta_2$

Not explicitely time-dependent

so $M_0 = M_j = 0$

The first pendulum is the same as the single pendulum, so $T_1 = \frac{1}{2} m l^2 \dot{\theta}_1^2$. Nevertheless, the second pendulum depends on the first one. $T_2 = \frac{1}{2} m \left(\dot{x}_2^2 + \dot{y}_2^2 \right)$

$\cancel{u}$ v constant

$$x_2 = l \sin\theta_1 + l \sin\theta_2$$
$$y_2 = -l\cos\theta_1 - l\cos\theta_2$$

$udv + vdu$

$$\frac{d x_2}{dt} = \frac{d}{dt}\left[l \overset{\cos}{\cancel{\sin}} \theta_1 \, d\theta_1 + l \overset{\cos}{\cancel{\sin}} \theta_2 \, d\theta_2 \right] = l\cos\theta_1 \, \dot{\theta}_1 + l\cos\theta_2 \, \dot{\theta}_2$$

$$\frac{dy_2}{dt} = l \sin\theta_1 \, \dot{\theta}_1 + l \sin\theta_2 \, \dot{\theta}_2$$

$$\text{so } T_2 = \frac{1}{2} m \Big[l^2 \cos^2\theta_1 \, \dot{\theta}_1^2 + 2 l^2 \cos\theta_1 \cos\theta_2 \, \dot{\theta}_1 \dot{\theta}_2 + l^2 \cos^2\theta_2 \, \dot{\theta}_2^2$$
$$+ l^2 \sin^2\theta_1 \, \dot{\theta}_1^2 + 2 l^2 \sin\theta_1 \cancel{\theta_1} \sin\theta_2 \, \dot{\theta}_1 \dot{\theta}_2 + l^2 \sin^2\theta_2 \, \dot{\theta}_2^2 \Big]$$

$$T_2 = \frac{1}{2} m \Big[(l\dot{\theta}_1)^2 \underbrace{(\cos^2\theta_1 + \sin^2\theta_1)}_{1} + (l\dot{\theta}_2)^2 \underbrace{(\cos^2\theta_2 + \sin^2\theta_2)}_{1}$$
$$+ 2 l^2 \dot{\theta}_1 \dot{\theta}_2 \overbrace{(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)}^{\cos(\theta_1 - \theta_2)} \Big]$$

$$T_2 = \frac{1}{2} m \left[(l\dot{\theta}_1)^2 + (l\dot{\theta}_2)^2 + 2l^2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

Ch. 2 Variational Principles and Lagrange's Equations

Hamilton's Principle: the motion of the system from time t_1 to time t_2 is such that the line integral (called the "action")

★ In configurational space

$$I = \int_{t_1}^{t_2} \mathcal{L}\, dt \qquad \text{Eq. 2.1}$$

where $\mathcal{L} = T - V$, has a stationary value for the actual path of the motion. In other words, the motion is such that the variation of the line integral for fixed t_1 and t_2 is zero:

Eq. 2.2

$$\delta I = \delta \int_{t_1}^{t_2} \mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)\, dt = 0$$

★ Calculus of variations, functions, maxima, minima, 1st derivative
★ Principle of virtual work assumed
★ Is nature really lazy?
★ More general than Newton's laws