9/21/21



The momentum conservation theorems are closely connected to the symmetry properties of the system.

Often, the inspection of the symmetries of even complicated systems results in knowledge about the constants of motion.

* If a system is invariant to translation along a certain direction, the linear momentum along that direction is conserved
* " " " " " rotation about a certain axes, the angular momentum about that axis is conserved.

WHAT ABOUT CONSERVATION OF ENERGY?

Again consider only conservative forces, so $V(q_j)$.

The Lagrangian is $\mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$

Hence, its total time derivative is

chain rule

$$\frac{d\mathcal{L}}{dt} = \sum_j \frac{\partial \mathcal{L}}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial \mathcal{L}}{\partial t}$$

Lagrange's equations: $\frac{\partial \mathcal{L}}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)$

so

$$\frac{d\mathcal{L}}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)\frac{dq_j}{dt} + \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial \mathcal{L}}{\partial t}$$

$\dot{q}_j$

Product rule $d(u \cdot v) = u\,dv + v\,du$

Let $u = \dot{q}_j$ $\quad du = \frac{d\dot{q}_j}{dt}$ $\quad$ so $\frac{d\mathcal{L}}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) + \frac{\partial \mathcal{L}}{\partial t}$

$dv = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)$ $\quad v = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$

Moving $\frac{d\mathcal{L}}{dt}$ to the other side,

$$0 = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{d\mathcal{L}}{dt} + \frac{\partial \mathcal{L}}{\partial t}$$

we distribute $\frac{d\mathcal{L}}{dt}$ to get

$$\sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}\right) + \frac{\partial \mathcal{L}}{\partial t} = 0$$

or big $H$

Let $\sum_j \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L} = h(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)$

be the "energy function"

This is identical to the Hamiltonian

Then $\frac{dh}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$ Eq. 2.54

☆ If the Lagrangian is independent of time, $\frac{dh}{dt} = 0$, so the energy function $h$ is conserved. This integral of motion is sometimes called Jacobi's integral

Recall that the kinetic energy $T$ could be written as

$T = T_0 + T_1 + T_2$. $T_0$ is a function of generalized coordinates only
$T_1$ is linear in the generalized velocities
$T_2$ is quadratic in the generalized velocities

The Lagrangian can be similarly decomposed, althoug we require homogenuity

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$

$L_0$ is a fn. of gen. coords only, $0^{th}$ degree homogeneous
$L_1$ is linear in gen. velocities, $1^{st}$ degree homogeneous
$L_2$ is quadratic in gen velocities, $2^{nd}$ degree homogeneous

~~For~~ ~~A~~ A function of two variables $f(x,y)$ is homogeneous of degree $k$ if $f(rx, ry) = r^k f(x,y)$

$\mathcal{L}_0(q,t)$

$\mathcal{L}_1(q,\dot{q},t) \Rightarrow \mathcal{L}_1(q, r\dot{q}, t) = r^1 \mathcal{L}_1(q,\dot{q},t)$

$\mathcal{L}_2(q,\dot{q},t) \Rightarrow \mathcal{L}_2(q, r\dot{q}, t) = r^2 \mathcal{L}_2(q,\dot{q},t)$

Euler's homogeneous function theorem states that if $f$ is a homogeneous function of degree $r$ in the variables $x_i$, then $\sum_i x_i \frac{\partial f}{\partial x_i} = rf$

c.f. the Energy Function $h = \sum_j \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}$

$$h = \sum_j \dot{q}_j \frac{\partial}{\partial \dot{q}_j}\left(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2\right) - \mathcal{L}$$

Using the theorem, $\mathcal{L}_0 \rightarrow 0$ ; $\mathcal{L}_1 \rightarrow \mathcal{L}_1$ ; $\mathcal{L}_2 \rightarrow 2\mathcal{L}_2$ so

$$h = \mathcal{L}_1 + 2\mathcal{L}_2 - \mathcal{L} = (-\mathcal{L}_0) + (\mathcal{L}_1 - \mathcal{L}_1) + (2\mathcal{L}_2 - \mathcal{L}_2) = \mathcal{L}_2 - \mathcal{L}_0$$

For time-independent (so $T = T_2$) systems and conservative forces (so $\mathcal{L}_0 = -V$ and $\mathcal{L}_2 = T = T_2$)

$$\boxed{h = T + V = E}$$

$\frac{dh}{dt} = -\frac{\partial \mathcal{L}}{\partial t} = 0$ for time-independent

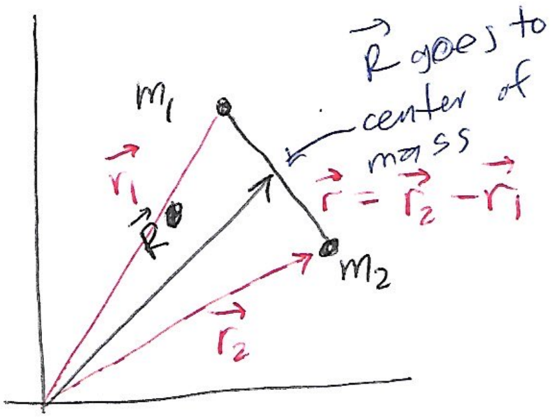
Total energy is conserved!

# The central force problem

From here on, applications, until Chapter 8.

~~Key~~ Historically, the two-body problem concerned mainly people working in cellestial dynamics. This problem can be reduced to a one-body problem. Consider the following figure:

(system is conservative)




Since $\vec{R}$ goes from the origin to the center of mass,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$V = V(\vec{r})$$ ← distance between particles

As always, write down Lagrangian

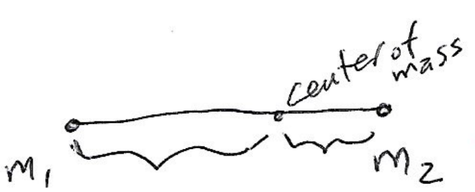
$\cancel{\mathcal{L} = T(\dot{\vec{R}}, \dot{\vec{r}}) - V(\vec{r})}$ ~~Naively~~ $\cancel{\mathcal{L} = T(\dot{\vec{r}}_1, \dot{\vec{r}}_2}$

~~Naively,~~ $\cancel{\mathcal{L}(\dot{\vec{r}}_1, \dot{\vec{r}}_2)}$ $\quad \mathcal{L} = T(\dot{\vec{r}}_1, \dot{\vec{r}}_2) + V(\vec{r})$

But the potential is in terms of $\vec{r}$, so need $\mathcal{L} = T(\dot{\vec{R}}, \dot{\vec{r}}) + V(\vec{r})$

Go from

$$T = \tfrac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\vec{r}}_2^{\,2}$$ to something that depends on $\dot{\vec{R}}, \dot{\vec{r}}$




Eq. 3.2

$\vec{r}_1{}' = \dfrac{m_2}{m_1+m_2}\vec{r} \quad \vec{r}_2{}' = \dfrac{m_1}{m_1+m_2}\vec{r}$

If $m_2 \to \infty$, $\vec{r}_1{}' = \dfrac{m_2}{m_2}\vec{r} = \vec{r}$

$\vec{r}_2{}' = \dfrac{0}{m_2} = 0\,\vec{r}$

center of mass on $m_2$

we can see that $\vec{r}_1 = \vec{R} + \vec{r}_1{}'$

$\vec{r}_2 = R + \vec{r}_2{}'$, so

$$T = \frac{1}{2}m_1\left(\dot{\vec{R}} + \dot{\vec{r}}_1{}'\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} + \dot{\vec{r}}_2{}'\right)^2$$

$$= \frac{1}{2}m_1\dot{\vec{R}}^2 + \cancel{\tfrac{1}{2}}m_1\cancel{2}\dot{\vec{R}}\cdot\dot{\vec{r}}_1{}' + \frac{1}{2}m_1\dot{\vec{r}}_1{}'^2 + \frac{1}{2}m_2\dot{\vec{R}}^2 + \cancel{\tfrac{1}{2}}m_2\cancel{2}\dot{\vec{R}}\cdot\dot{\vec{r}}_2{}' + \frac{1}{2}m_2\dot{\vec{r}}_2{}'^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}m_1\dot{\vec{r}}_1{}'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2{}'^2 + \underbrace{\dot{\vec{R}}\cdot\left(m_1\dot{\vec{r}}_1{}' + m_2\dot{\vec{r}}_2{}'\right)}$$

$$\vec{R}\cdot\frac{d}{dt}\left(m_1\vec{r}_1{}' + m_2\vec{r}_2{}'\right)$$

using definitions above,

$$\vec{R}\cdot\frac{d}{dt}\left[m_1\left(\vec{r}_1 - \vec{R}\right) + m_2\left(\vec{r}_2 - \vec{R}\right)\right]$$

$\vec{R}(m_1+m_2) = m_1\vec{r}_1 + m_2\vec{r}_2$ using definition of center of mass

$$\vec{R}\cdot\frac{d}{dt}\left[m_1\vec{r}_1 - m_1\vec{R} + m_2\vec{r}_2 - m_2\vec{R}\right]$$

$$\vec{R}\cdot\frac{d}{dt}\left[m_1\vec{r}_1 + m_2\vec{r}_2 - (m_1+m_2)\vec{R}\right]$$

$$\vec{R}\cdot\frac{d}{dt}\left[(m_1+m_2)\vec{R} - (m_1+m_2)\vec{R}\right]$$

the whole term goes to zero

Let $T' = \frac{1}{2}m_1\dot{\vec{r}}_1{}'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2{}'^2$, then we can write

$$T = \underbrace{\frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2}_{\text{kinetic energy of the center of mass}} + \underbrace{T'}_{\text{kinetic energy relative to the center of mass}}$$

$$\vec{r}_1{}' = \frac{m_2}{m_1+m_2}\vec{r} \qquad \vec{r}_2{}' = \frac{m_1}{m_1+m_2}\vec{r}$$

$$(\dot{\vec{r}}_1{}')^2 = \left(\frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 = \frac{m_2^2}{(m_1+m_2)^2}\dot{\vec{r}}^2 \qquad (\dot{\vec{r}}_2{}')^2 = \left(\frac{m_1}{m_1+m_2}\dot{\vec{r}}\right)^2 = \frac{m_1^2}{(m_1+m_2)^2}\dot{\vec{r}}^2$$

$$T' = \frac{1}{2}m_1\frac{m_2^2}{(m_1+m_2)^2}\dot{\vec{r}}^2 + \frac{1}{2}m_2\frac{m_1^2}{(m_1+m_2)^2}\dot{\vec{r}}^2$$

$$2T' = \frac{m_1 m_2^2\cancel{(m_1+m_2)^2} + m_1^2 m_2\cancel{(m_1+m_2)}}{(m_1+m_2)^{\cancel{4}\,2}}\dot{\vec{r}}^2$$

$$2T' = \frac{m_1 m_2\cancel{(m_2+m_1)}}{(m_1+m_2)^{\cancel{2}}}\dot{\vec{r}}^2$$

Eq. 3.4

$$T' = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2 \qquad \mu = \frac{m_1 m_2}{m_1+m_2} \text{ is the reduced mass}$$

conservative

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}) \qquad \text{Eq 3.3}$$

The central force motion of two bodies about their center of mass can always be reduced to an equivalent one-body problem. First we should reduce the dimensionality.

Notice that $\mathcal{L}(\vec{r}, \dot{\vec{r}}, -, \dot{\vec{R}})$, so the coordinate $\vec{R}$ is cyclic using Eq. from page 45 of my notes, $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\vec{R}}\right) = 0$

$\frac{\partial\mathcal{L}}{\partial\vec{R}}$ is constant. This can occur if $\dot{\vec{R}} = 0$, the center of mass is not moving or is moving at constant speed (so it is an inertial system)