Angular momentum and the moment of inertia operator

10/26/21

Before we start with Ch. 5, we need to rescue a critical member of our team from Ch. 4: the time derivative of a vector. The rescue operation will be swift. ~~we start with~~ ~~$d\vec{r} = \vec{r} \times d\vec{\Omega}$~~

Right handed system $\boxed{d\vec{r} = d\vec{\Omega} \times \vec{r}}$

$\vec{r} \times \hat{n}$ for right-handed

The Rodrigues' Equation for a right-handed system is

$$\vec{r}\,' = \vec{r}\cancel{\cos\Phi}^{\,1} + \hat{n}(\hat{n}\cdot\vec{r})(\cancel{1-\cos\Phi}^{\,0}) + (\hat{n}\times\vec{r})\cancel{\sin\Phi}^{\,d\Phi}$$

The angle ~~bet~~ of an infinitesimal rotation between $\vec{r}\,'$ and $\vec{r}$ is approximately <u>zero</u>.

$\cos 0 = 1$
$\sin \theta = d\Phi$

so Rodrigues predicts

$$\vec{r}\,' = \vec{r} + (\hat{n}\times\vec{r})\,d\Phi \Rightarrow \vec{r}\,' - \vec{r} = d\vec{r} = (\hat{n}\times\vec{r})\,d\Phi$$

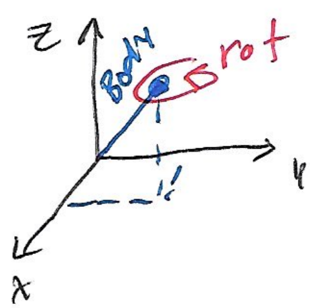
$$d\vec{r} = \hat{n}\,d\Phi \times \vec{r}$$

By inspection, we notice that $d\vec{\Omega} = \hat{n}\,d\Phi$

Many vectors change as <u>time</u> goes by, but the change often depends on the coordinate system being used.

Consider me standing up, I am not moving with respect to the earth nor with respect to you. There is a vector ~~ta~~ that represents my position with respect to the earth, and that vector is still. Nevertheless, to someone observing from space, the earth itself is moving, so the vector representing my position is not zero with respect to galactic coordinates.

Consider an arbitrary vector $\vec{G}$. In one instance, $\vec{G}$ is fixed with respect to the "body" system, ~~so $(d\vec{G})_{body} = 0$~~. We can see from the drawing to the left that the vector that describes the position of the body with respect to space combined with ~~and~~ the rotational state completely describe, as we discussed earlier. So



$$(d\vec{G})_{space} = (d\vec{G})_{body} + (d\vec{G})_{rot} = (d\vec{G})_{body} + d\vec{\Omega} \times \vec{G}$$

Divide everything by dt

$$\underbrace{\left(\frac{d\vec{G}}{dt}\right)_{space} = \left(\frac{d\vec{G}}{dt}\right)_{body} + \vec{\omega} \times \vec{G}}$$

This is who we wanted to save!

Remember that ~~the~~ $d\vec{\Omega}$ is a vector that characterizes rotation, so $\vec{\omega}$ is the angular velocity

We recovered our friend, the time derivative of a vector, from hellish Ch. 4. But she is wearing a costume. Her real identity is

$$\boxed{\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \vec{\omega} \times}$$

Also known as 4.86

$\vec{G}$ is an arbitrary vector. Eq. 4-86 should be seen as an operator, the transformation of the time derivative between two coordinate systems.

☆ Time derivatives are relative to the chosen coordinate system. Solve within vectors before mixing

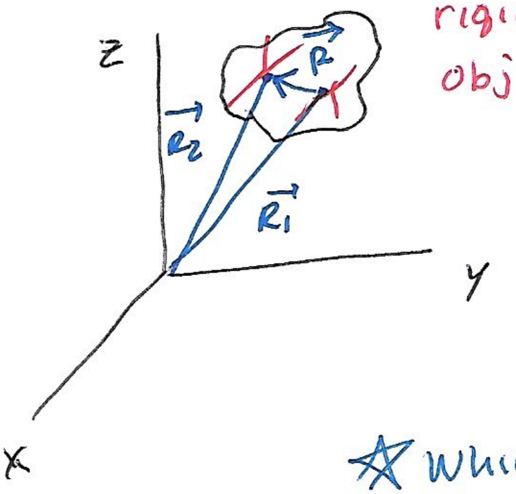
Ch. 5 The rigid body equations of motion

5.1 Angular momentum and kinetic energy of motion about a point

Chasles' theorem, which is actually a corollary to Euler's theorem, states that: "the most general displacement of a rigid body is a translation plus a rotation."

Cartesian coordinates are used to describe translational motion, Euler angles can be used for rotational motion, and we have made such separations before. The total kinetic energy will be of the form $T = \frac{1}{2}Mv^2 + T'(\varphi, \theta, \psi)$

Obligatory drawing of rigid object



Remember that quantities like the torque and the ~~moment of inert~~ angular momentum are not inherently defined; you also need a reference. with respect to which vector?

★ which $\vec{r}$ is best to use?

Consider the figure above: $\vec{R}_2 = \vec{R}_1 + \vec{R}$, so

$$\left(\frac{d\vec{R}_2}{dt}\right)_s = \left(\frac{d\vec{R}_1}{dt}\right)_s + \left(\frac{d\vec{R}}{dt}\right)_s$$

The difference between R_1 and R_2 is constant, but rotational system does not move

$$\left(\frac{d\vec{R}}{dt}\right)_s = \cancel{\left(\frac{d\vec{R}}{dt}\right)_r} + \vec{\omega}_1 \times \vec{R}$$

Hence $$\left(\frac{d\vec{R}_2}{dt}\right)_s = \left(\frac{d\vec{R}_1}{dt}\right)_s + \vec{\omega}_1 \times \vec{R}$$

similarly for $\vec{R}_1 = \vec{R}_2 - \vec{R}$

$$\left(\frac{d\vec{R}_1}{dt}\right)_s = \left(\frac{dR_2}{dt}\right)_s - \vec{\omega}_2 \times \vec{R}$$

$$(\vec{\omega}_1 \times \vec{R}) + (-\vec{\omega}_2 \times \vec{R}) = \left(\frac{d\vec{R}_2}{dt}\right)_s - \left(\frac{d\vec{R}_1}{dt}\right)_s + \left(\frac{d\vec{R}_1}{dt}\right)_s - \left(\frac{dR_2}{dt}\right)_s$$

$$(\vec{\omega}_1 - \vec{\omega}_2) \times \vec{R} = 0$$

So either $\vec{\omega}_1 - \vec{\omega}_2 = 0$

or angle between $\vec{\omega}_1 - \vec{\omega}_2$ and $\vec{R}$ is zero, so they lie on $\vec{R}$

Nevertheless, $\vec{R}_1$ and $\vec{R}_2$ are arbitrary, so this is not a viable solution.

$$\boxed{\vec{\omega}_1 = \vec{\omega}_2}$$

There is only 1 angular velocity for all coordinate system fixed in the rigid body.

This is of course a corollary to the definition of rigid body. By definition, distances between particles does not change, so angular velocity can't be different

When a rigid body moves with one point stationary, ~~at a radius r~~ the total angular momentum is $\vec{L} = \sum_i \vec{r}_i \times (m_i \vec{v}_i)$ or, in Einstein notation

$$\vec{L} = m_i (\vec{r}_i \times \vec{v}_i) \quad \text{Eq. 5.1}$$

$\vec{r}_i$ is the distance (radius) between the i^{th} particle and the rotation point. $\vec{v}_i$ is the change in $\vec{r}_i$ w.r.t. time, which comes exclusively from the rotational motion since radial motion is impossible.

Applying 4.86, $\underbrace{\left(\frac{d\vec{r}_i}{dt}\right)_s}_{\vec{v}_i} = \cancel{\left(\frac{d\vec{r}_i}{dt}\right)_r}^{\,0} + \vec{\omega} \times \vec{r}_i = \vec{v}_i$

Hence, $\vec{L} = m_i \left[\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\right]$

since $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})$ 106

$$= \vec{\omega}\,(\vec{r_i}\cdot\vec{r_i}) - \vec{r_i}\,(\vec{r_i}\cdot\vec{\omega}),$$

Eq. 5.3

$$\vec{L} = m_i\left[r_i^2\,\vec{\omega} - \vec{r_i}\,(\vec{r_i}\cdot\vec{\omega})\right]$$

Looking at the components,

$$L_x = m_i\left[\omega_x r_i^2 - x_i\,(x_i\omega_x + y_i\omega_y + z_i\omega_z)\right]$$

Eq 5.4
$$L_x = \omega_x m_i(r_i^2 - x_i^2) - \omega_y m_i x_i y_i - \omega_z m_i x_i z_i$$

$L_y = \dots \qquad \omega_y m_i(r_i^2 - y_i^2)$

$L_z = \dots \quad$ similar $\qquad \omega_z m_i(r_i^2 - z_i^2)$

☆ Each component L_x, L_y, L_z is a linear function of all the components of the angular velocity.

We can emphasize the similarity with Eq. 4.12

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

The 9 coefficients I_{xx}, I_{xy}, etc. are the 9 elements of the transformation matrix. We can write the equation as $\boxed{\vec{L} = \tilde{I}\,\vec{\omega}}$ Eq. 5.9

☆ Notice that so far no coordinate system was specified. Although $x, y, z, \omega_x, \omega_y, \omega_z$ are used, they represent arbitrary orthogonal vectors. It is agnostic.

☆ The diagonal elements of the Moment of Inertia tensor (or matrix) are called the moment of inertia coefficients and have the following forms

$$I_{xx} = m_i \left(r_i^2 - x_i^2\right)$$

or

$$I_{xx} = \int_V \rho(\vec{r})\left(r^2 - x^2\right) dV$$

☆ The off-diagonal elements are called products of inertia and have the following forms

$$I_{xy} = - m_i x_i y_i$$

or

$$I_{jk} = \int_V \rho(\vec{r})\left(r^2 \delta_{jk} - x_j x_k\right) dV$$

☆ In this case, the interpretation of the operator is clear, it is operating on the angular velocity vector and not on the cartesian system.

☆ Unlike the rotation operators, $\tilde{I}$ has units $(kg\, m^2)$ and it is not subject to orthogonality conditions.