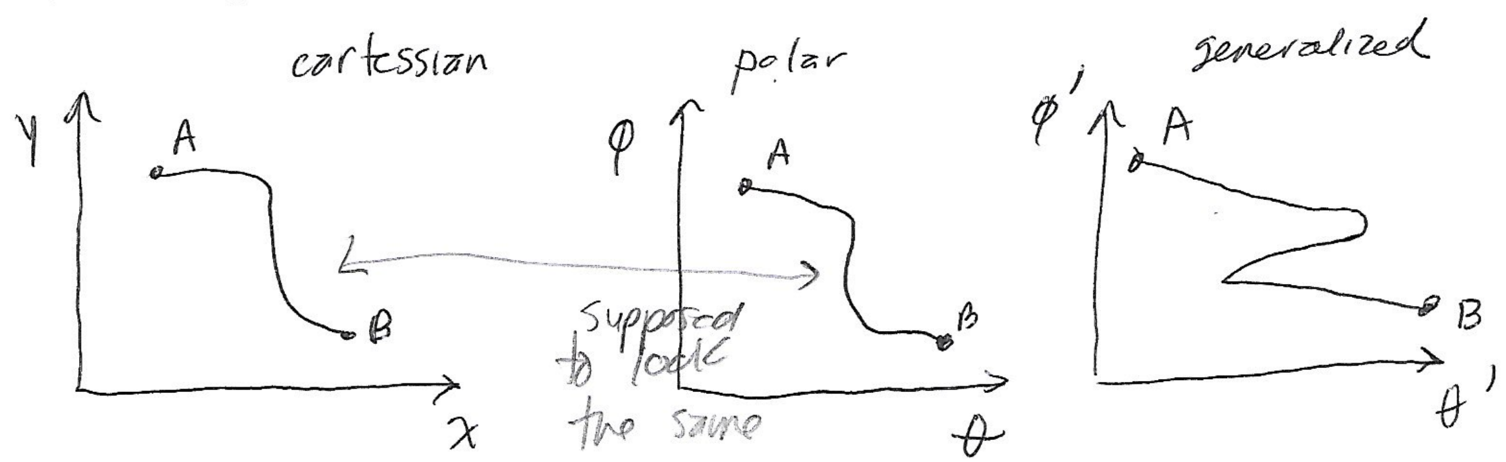
Analytical Mechanics is purely mathematical, so there are no vectors, so we need a way to map one-to-one, correspondence between points in real space and scalars. This mapping is called a <u>coordinate transformation</u>

A well-know transformation is from rectangular to polar coordinates.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

but pretty much anything (that is consistent with the physics) is allowed. Consider a path in cartessian, polar, and "generalized" coordinates.



when we do a coordinate transformation some things will change almost for sure. These are <u>metric properties of the space</u>

- straight lines migth not be straight anymore
- angles and distances will change

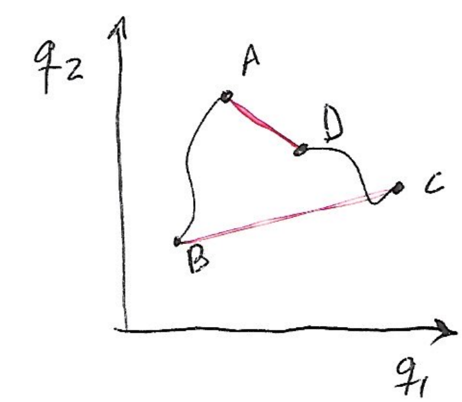
Other things will <u>not</u> change:

- A point remains a point
- The neighborhood of a point remains the neighborhood of that point

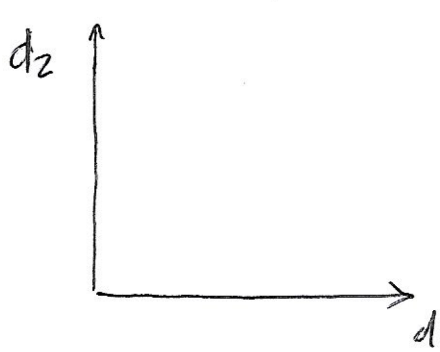
These are topological properties

- A curve remains a curve.
- Adjacent curves remain adjacent curves.
- Continuous and differentiable curves remain continuous and differentiable curves

Consider now two particles moving in two generalized coordinates q_1 and q_2. In order to describe motion we need 4 equations, but if instead describe the separation between particles, we need only 2. This is an example of <u>configuration space</u>, based on the relative positions between particles rather than absolute.



The state at t of A whole system can always be represented as a point for example, the system above would be 1-D



if we add a third particle, we can represent the system as a point in 2-D space (perhaps with extra constraints)

This is a powerful construction. Every system can be represented by a point in sufficiently high-dimensional space. Even human personality:

Big 5 personality traits

- Extraversion — outgoing ———— reserved
- Agreeableness — friendly ———— critical
- Openness to experience — curious ———— consistent
- Conscientiousness — organized ———— careless
- Neuroticism — sensitive ———— resilient

☆ Is there a mapping between the arrangement of the particular neurons in your brain and the above?

delta is used instead of d

1 ← now we know

A virtual displacement is a change in the configuration of the system as the result of any arbitrary infinitesimal change in the coordinates $\delta \vec{r}_i$ consistent 2 with the forces and constraints imposed on the system at the given instant t.

In contrast to actual displa3cement of the system in time interval dt in which forces and constraints might change.

1 Configuration

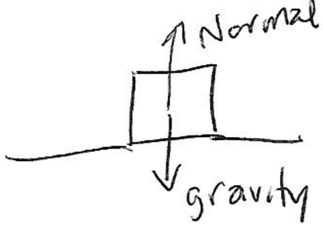
Both particles move d in the same direction

Move d towards each other

Not virtual displacement

Change in configuration

② Consistent with forces and constraints instantaneously



Can't go "into the ground" or move left or right

③ Frozen in time

Force might be different infinitesimaly later, we are just exploring

If the system is in equilibrium, the virtual work (the dot product of the force) is zero.

$$\vec{F}_i = F_i^{(applied)} + f_i \leftarrow \text{constraint}$$

$$\sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r} + \sum_i f_i \cdot \delta \vec{r}_i = 0$$

We will restrict ourselves to situations in which $\sum_i f_i \cdot \delta \vec{r}_i = 0$

☆ In groups, come up with a case in which this holds and another case in which it doesn't.

So now we only have $\sum_i \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = 0$ ~~D'Alembert's Principle~~

If the system is not in equilibrium, then the correct equation is $\vec{F}_i = \vec{p}_i \Rightarrow \vec{F}_i - \vec{p}_i = 0$

$$\sum_i (\vec{F}_i - \vec{p}_i) \cdot \delta \vec{r}_i = 0$$

The virtual work ~~of~~ the forces of constraint is zero

$$\delta W = \sum_i f_i \cdot \delta \vec{r}_i = 0$$

This is D'Alembert's principle. It is an assumption but will only consider cases in which it is fullfilled.

Let's get the Equations of Motion, but getting rid of the forces of constraint (we can make them go away) with D'Alembert

$$\vec{F}_i = \dot{\vec{p}}_i$$

$$\vec{F}_i = \dot{\vec{p}}_i$$

$$\sum_i \vec{F}_i = \sum_i \dot{\vec{p}}_i$$

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i$$

$$① \quad \sum_i \vec{F}_i^{(a)} \cdot \delta\vec{r}_i + \underbrace{\cancel{\sum_i \vec{f}_i \cdot \delta r_i}}_{\text{D'Alembert}} = \sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i$$

Now move to generalized coordinates, each $r_i(q_1, q_2, \ldots, q_k, t)$

using the chain rule $\vec{v}_i \equiv \dfrac{d\vec{r}_i}{dt}$

holonomic constraint

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial \vec{r}_i}{\partial q_2}\frac{\partial q_2}{\partial t} + \ldots + \cancel{\frac{\partial \vec{r}_i}{\partial t}}\frac{\partial t}{\partial t}$$

Eq. 1.46

$$\vec{v}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$$

$$\delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_1}\,\delta q_1 + \frac{\partial \vec{r}_i}{\partial q_2}\,\delta q_2 + \dots$$

1.47

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

(1) $$\sum_i \vec{F}_i \cdot \delta r_i = \sum_i m_i \dot{\vec{v}}_i \cdot \delta \vec{r}_i$$

On the LHS $$\sum_i \sum_j \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

Let $$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial r_i}{\partial q_j}$$

Q is the <u>generalized force</u>, Q_j is the j^{th} component of the generalized force

Then LHS $$\sum_j Q_j\,\delta q_j$$

RHS $$\sum_i m_i \dot{\vec{v}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i$$

$$= \sum_i \sum_j m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

Let's focus on only 1 j, using product rule

$$u\,dv + v\,du = d(u \cdot v) \Rightarrow u\,dv = d(u \cdot v) - v\,du$$

Let $$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

$$\underbrace{m_i \ddot{\vec{r}}_i}_{dv} \cdot \underbrace{\frac{\partial \vec{r}_i}{\partial q_j}}_{u} = d(uv) - v\,du$$

remove one dot due to integration

$$= \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

① ②

independent of time

① $$m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{dt}{dt} = m_i \dot{\vec{r}}_i \frac{\frac{\partial \vec{r}_i}{dt}}{\frac{\partial q_j}{dt}} = m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

② $$m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

The kinetic energy $T = \sum_i \frac{1}{2} m \dot{\vec{r}}^2$; $\partial T = \frac{2}{2} m \dot{\vec{r}}\, \partial \dot{\vec{r}}$

all components

so ① $$m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

② $$\frac{\partial T}{\partial q_j} = m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

whole thing

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}$$

RHS = LHS

Rewritting our equation

LHS = RHS

$$\cancel{\sum_j} Q_j \, \cancel{\delta q_j} = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] \cancel{\delta q_j}$$

Must hold for each j q_j

Like any force, the generalized force can be expressed, e.g

$$F = -\frac{du}{dx} \sim Q = -\frac{dV}{dq}$$

$$Q_j = -\frac{\partial V}{\partial q_j}$$

so

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

since V does not depend on the velocity, we can write

$$\frac{d}{dt} \frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0 \quad \text{Let } \mathcal{L} = T - V$$

$$\frac{d}{dt} \frac{\mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

Lagrange's equations

1.57