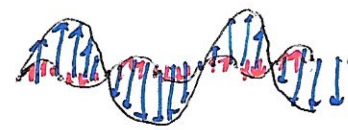
Principal axes transformation for oscillations

This can get a bit confusing. Let's put something physical to use. I will use statements from the book.



"Recall that λ stands for ω^2, so that positive λ corresponds to real frequencies of oscillation."

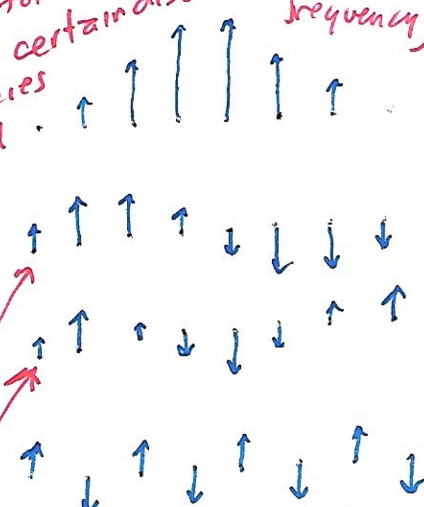
we showed that λ is real as a consequence of $\tilde{T}$ and $\tilde{V}$ being Hermitian, and we built $\tilde{T}$ and $\tilde{V}$ by hand, so we know they truly are Hermitian.

"Neither numerator $(\vec{a}_k^{\,*} \tilde{V} \vec{a}_k)$ nor denominator $(\vec{a}_k^{\,*} \tilde{T} \vec{a}_k)$ can be negative, and the denominator cannot be zero."

total number allowed is "n"

only certain discrete frequencies allowed

(lowest frequency)



(highest frequency)

this one would not survive, actually WHY?

$\vec{a}_k$ is the eigenvector of eigenvalue λ_k and eigenfrequency ω_k^2. The characteristic equation is of n^{th} degree, where "n" is the number of degrees of freedom, so there are "n" eigenvectors, each one with "n" elements.

this is the vector of forces, can be (each row) positive or negative

$$\tilde{V}\vec{a}_k = \begin{bmatrix} V_{11}a_1 + V_{12}a_2 + \ldots + V_{1n}a_n \\ V_{21}a_1 + V_{22}a_2 + \ldots + V_{2n}a_n \\ \vdots \\ V_{n1}a_1 + V_{n2}a_2 + \ldots + V_{nn}a_n \end{bmatrix}$$

there is an amplitude for each degree of freedom. In the diagram above, each particle has 1 degree of freedom

$$\vec{a}_k^{\,*}\,\tilde{V}\,\vec{a}_k = a_1^*(V_{11}a_1 + V_{12}a_2 + \ldots + a_2^*(V_{21}a_1 + V_{22}a_2 + \ldots + \ldots$$

How can we break it?

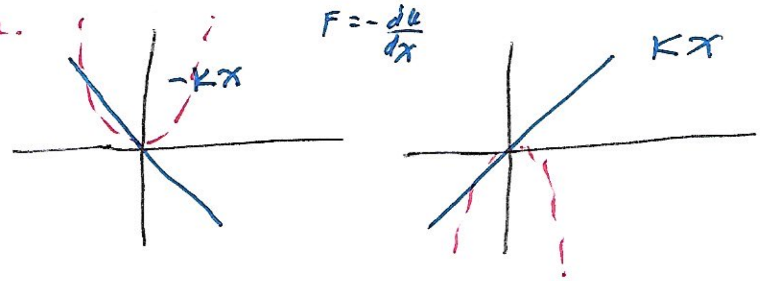
we would need enough elements of $\tilde{V}$ to be negative. These correspond to negative force constants.

Orthogonal directions!

The squares produce positives...

☆ what about $a_1^* V_{12} a_2$? Diagonalization.

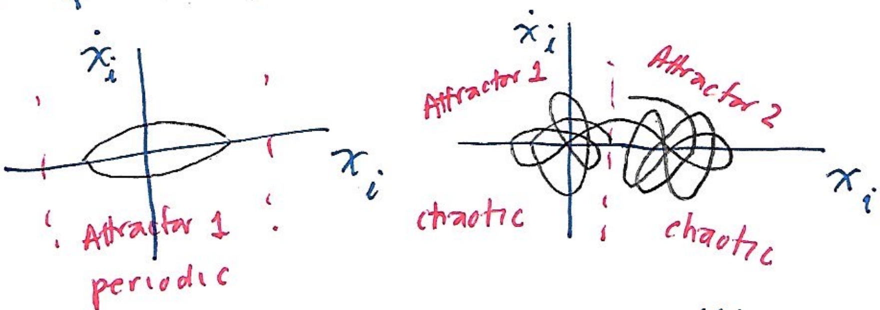
A negative force constant produces a force in the direction of displacement rather in the opposite direction, so it is an unstable system.



Mathematically, the matrix product can be negative, but it does not describe a very interesting system, there are no oscillations strong enough to return the system back to its original position or close.

I will not get to chaos, so...

We will see that in addition to generalized coordinates, we can describe a system using its phase space representation, position momenta space. A very simple one, one particle with one degree of freedom, in harmonic motion. What if each oscillation is a bit different and eventually moves back and forth between "centroids"? The regions where the system gets "trapped" are described by a

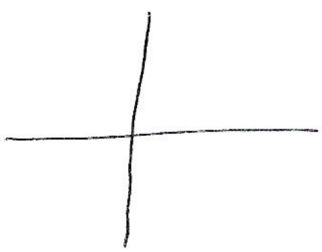


region of space, so ~~reccgenerate~~ it could be a manifold, this region is called an "attractor." The more interesting ones have fractal structure and produce chaotic motion. They are called "strange attractors."

Applications are many and varied.

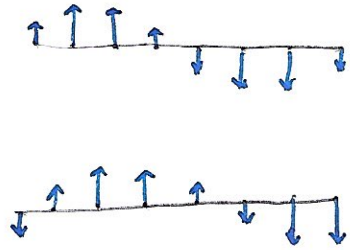
The eigenvectors are interesting. When the matrix $\tilde{A}$ ~~$\tilde{V}$ "operates" on $\vec{a}_k$~~ is used to diagonalize $\tilde{V}$ it produces a potential energy. The same matrix $\tilde{A}$ used to diagonalize $\tilde{T}$ produces ~~the~~ a kinetic energy.

$$\vec{a}_k^{\,*}\, \tilde{T}\, \vec{a}_k = a_1^*\left(m_{11}a_1 + m_{12}\, a_2 + \ldots\right.$$
$$+ a_2^*\left(m_{21}\, a_1 + m_{22}\, a_2 + \ldots\right.$$
$$+ \ldots$$

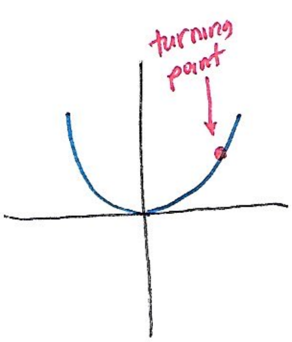


Can we break it? Well... if $\tilde{A}$ diagonalizes $\tilde{T}$, all the off-diagonal elements are gone. We only have squares of the velocities, which produce positive numbers or zero. The masses are also positive or zero, can't be negative like the force constants. If all the masses are zero, then the matrix product is zero, but this does not describe anything physical. Perhaps we can make all the amplitudes (velocities) equal to zero?

"For each of these values of ω^2, Eqs. (6.12) may be solved for the amplitudes of a_i, or more precisely, for $n-1$ of the amplitudes in terms of the remaining a_i."



The solutions are equivalent, but you have to "select" one of the amplitudes to begin with.



If you start the system at a turning point so that its kinetic energy is zero, it will have a potential energy maximum, which the system will minimize by increasing the kinetic energy. Can you carefully craft a configuration that produces zero kinetic energy? No, the more particles you displace the more energy you put in the system!

Going back to Eq. 6.17:

Eq. 6.17

$$0 = (\lambda_k - \bar{\lambda}_\ell)\, \vec{a}_\ell^{\,*}\, \tilde{T}\, \vec{a}_k$$

This holds if $\lambda_k - \bar{\lambda}_\ell = 0$ or $\vec{a}^{\,*}\, \tilde{T}\, \vec{a}_k$

The former case implies that the system has degeneracies, distinct ways of accomodating the same energy. ~~The latter implies that~~ $\tilde{A}^* \tilde{T} \tilde{A} = \tilde{1}$

we remove the degeneracies with 5 since now there is one value for each row, the system is determined.

Before we saw the similarity transformation $\tilde{C}' = \tilde{B}\tilde{C}\tilde{B}^{-1}$

Now we have $\tilde{C}' = \tilde{A}^* \tilde{C} \tilde{A}$ which is called the congruence transformation. They are the same if $\tilde{A}$ is orthogonal.

If we introduce a diagonal matrix with the eigen values ~~of the~~,

$$\tilde{\lambda} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

Eq. 6.15 becomes $\tilde{V}\tilde{A} = \tilde{T}\tilde{A}\tilde{\lambda}$

$$\tilde{A}^*\tilde{V}\tilde{A} = \tilde{A}^*\tilde{T}\tilde{A}\tilde{\lambda}$$

$$\tilde{A}^*\tilde{V}\tilde{A} = \tilde{1}\tilde{\lambda} = \tilde{\lambda}$$

so the solutions are $|\tilde{V} - \tilde{\lambda}\tilde{1}| = 0$ for $\tilde{V}$ diagonal.

$$\Rightarrow \tilde{A}^*\tilde{A} = \tilde{1} \quad \text{and} \quad \tilde{A}^*\tilde{V}\tilde{A} = \tilde{V}_{diag}$$