# Rotations as vectors

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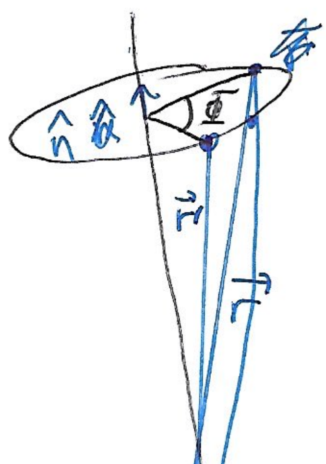
Equation 4.62 is called by the book the "rotation formula." and it is given by

Eq. 4.62

$$\vec{r}\,' = \vec{r}\cos\Phi + \hat{n}\,(\hat{n}\cdot\vec{r})(1-\cos\Phi) + (\vec{r}\times\hat{n})\sin\Phi$$

It is more commonly know as the Rodrigues' rotation formula in honor of Olinde Rodrigues, a French mathematician and banker who first derived it. Eq. 4.62 is in his Ph.D thesis, published in 1815. and titled "Mouvement de rotation d'un corps de révolucion pesant."

$\vec{r}$ is the orignal vector and $\vec{r}\,'$ the displaced vector, this applies the active use of the rotation operator. The pasive use rotates the axes and we want that rotation to be right-handed so as to no ~~have~~ cause any issues.




This means that the rotation is $\boxed{\text{left-handed.}}$ The angle $\Phi$ is the rotation angle and it goes in the clock-wise direction. Typically $\Phi$ is given by the Euler angles $\varphi, \theta, \psi$

Finally, $\hat{n}$ is the unit vector along the axis of rotation. In Eq. 4.62,

- The first term on the right $\boxed{\vec{r}\cos\Phi}$ In general has components along all orthogonal directions. but not along directions perpendicular to itself
- The second term on the right $\boxed{\hat{n}(\hat{n}\cdot\vec{r})(1-\cos\Phi)}$ only has non-zero components for the $\hat{n}$ direction, parallel to the rotation axis.
- The third term on the right $\boxed{(\vec{r}\times\hat{n})\sin\Phi}$ ~~only~~ has ~~non~~ zero components for the $\hat{n}$ direction by definition and its non-zero components are perpendicular to the rotation axis.

(*) Explain the following after explaining #

Remember the definition of the cross product

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{vmatrix} = \hat{i}\,|c_y d_z - d_y c_z| - \hat{j}\,|c_x d_z - d_x c_z| + \hat{k}\,|c_x d_y - d_x c_y|$$

$$\text{So } \vec{c}\times\vec{d} = \begin{bmatrix} (\vec{c}\times\vec{d})_x \\ (\vec{c}\times\vec{d})_y \\ (\vec{c}\times\vec{d})_z \end{bmatrix} = \begin{bmatrix} c_y d_z - d_y c_z \\ c_x d_z - d_x c_z \\ c_x d_y - d_x c_y \end{bmatrix}$$

The third term on the right can be written as

notice sign

$$\cancel{\cos\Phi(\hat{n}} \qquad -\sin\Phi\,(\hat{n}\times\vec{r})$$

Can we find a matrix $\tilde{C}$ such that we can write the ~~dot~~ cross product above as $\tilde{C}\vec{r} = \hat{n}\times\vec{r}$?

Sure we can, it is called the cross-product-matrix

$$\tilde{C} = \begin{bmatrix} 0 & -C_z & C_y \\ C_z & 0 & -C_x \\ -C_y & C_x & 0 \end{bmatrix} \qquad \text{so}$$

Now go to &

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~~#~~ In general, a vector ~~is equal to~~ can be decomposed into components that are parallel to a given axis and components that are perpendicular, so $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$. The parallel side will be given in this case by $\vec{v}_{\parallel} = \hat{n}(\hat{n}\cdot\vec{r})$ ~~times any scalar~~ and the perpendicular one by $v_{\perp} = \vec{v} - v_{\parallel} = \vec{r} - \hat{n}(\hat{n}\cdot\vec{r})$ times any scalar

The vector triple product is $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

with $\vec{A} = \hat{n}$
$\vec{B} = \hat{n}$
$\vec{C} = \vec{r}$

$$\hat{n}\times(\hat{n}\times\vec{r}) = (\hat{n}\cdot\vec{r})\hat{n} - \cancel{(\hat{n}\cdot\hat{n})}\,\vec{r}$$

Notice that $\vec{v}_{\parallel}$ is aligned with the $\hat{n}$ axis, so it is invariant, (magnitude and direction) upon rotation.

Notice that $\vec{v}_{\perp}$ can be given by cylindrical coordinates because we have the original radius $\vec{r}$, with parameterization $x = r\cos\Phi$ ; $y = r\sin\Phi$ as usual. The vector substraction ensures there is nothing orthogonal to x and y (above the xy plane)

The magnitude of $\vec{v}_{\perp}$ does not change upon rotation, but its direction does. Usually the coordinates on the $z=0$ plane are given by a radial and tangential components, ~~which~~ call them $\vec{v}_{\perp,1}$, $\vec{v}_{\perp,2}$

$\cancel{\vec{r}' = \vec{r}_{\parallel} + \cos\theta\, \vec{r}_{\perp,\parallel}}$

$$\vec{r}' = \vec{r}_{\parallel} + \cos\Phi\, \vec{r}_{\perp,1} + \sin\Phi\, \vec{r}_{\perp,2}$$

Using the properties of the cross product and triple product, as well as $v_{\perp} = \vec{v} - v_{\parallel}$, you recover Eq. 4.62

8 We know that $\tilde{C}\vec{r} = \hat{n} \times \vec{r}$

but due to the commutation of the matrix multiplication operation, ~~so~~ $\tilde{C}^2\vec{r} = \hat{n} \times (\hat{n} \times \vec{r})$.

We can rearrange the terms of Rodrigues rotation formula so that it starts with the least number of $\tilde{C}$'s applied to it and ends with the most

left hand system

$$\vec{r}\,' = \vec{r}\cos\theta + \cancel{\hat{\phi}}(\hat{n} \times \vec{r})\sin\Phi + \hat{n}(\hat{n}\cdot\hat{r})(1-\cos\Phi)$$

$$= (\cancel{\cos\theta} + 1 - \cancel{\cos\theta})\vec{r} + (\hat{n} \times \hat{r})\sin\Phi + \hat{n} \times (\hat{n} \times \hat{r})(1-\cos\theta)$$

in matrix form $\vec{r}\,' = \vec{r} + \sin\Phi\,\tilde{C}\vec{r} + (1-\cos\Phi)\tilde{C}^2\vec{r}$

which we can rewrite as $\vec{r}\,' = \tilde{A}\vec{r}$ which of course makes it explicit that this is a rotation matrix, $\tilde{A} = \tilde{1} + \sin\theta\,\tilde{C} + (1-\cos\theta)\,\tilde{C}^2$

Kill the 1

This is very important

$\tilde{A}$ is an element of the rotation group $SO(3)$

Lie algebra