Some formal properties of $\tilde{A}$ the transformation matrix

Consider two succesive transformations. The first one is denoted by $\tilde{B}$ $x_k' = b_{kj} x_j$

$\vec{r}' = \tilde{B}\vec{r}$

the second one by $\tilde{A}$ $x_i'' = a_{ik} x_k'$

$\vec{r}'' = \tilde{A}\vec{r}'$

then $x_i'' = a_{ik} b_{kj} x_j$. with $c_{ij} = a_{ik} b_{kj}$, $x_i'' = c_{ij} x_j$

we can see that two succesive linear transformations is equivalent to a third transformation applied independently.

$$\tilde{C} = \tilde{A}\tilde{B}$$

Notice, though, that matrix multiplication is <u>not commutative</u>, so in general $\tilde{A}\tilde{B} \neq \tilde{B}\tilde{A}$. If $\tilde{D} = \tilde{B}\tilde{A}$, $d_{ij} = b_{ik} a_{kj} \neq c_{ij}$ in general

Matrix multiplication is <u>associative</u>, so

not commutative but associative

$$(\tilde{A}\tilde{B})\tilde{C} = \tilde{A}(\tilde{B}\tilde{C})$$

There is an operation that changes $\vec{r}'$ back to $\vec{r}$, it is the inverse of $\tilde{A}$ and it is denoted $\tilde{A}^{-1}$. In this case, rather than

$x_i' = a_{ij} x_j$ we have $x_i = a_{ij}' x_j'$

$x_k' = a_{ki} x_i$ ← must be consistent

use k for consistency

So $\boxed{x_k' = a_{ki}\, a_{ij}'\, x_j'}$ Eq. 4.29

$\boxed{\vec{x} = 1\vec{x}}$

$\cancel{\tilde{A}\tilde{A}^{-1} = 1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

a_{ki} columns a_{ij}'

rows

$$\begin{array}{cc} & \begin{array}{ccc} i=1 & i=2 & i=3 \end{array} \\ \begin{array}{c} k=1 \\ k=2 \\ k=3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \begin{array}{cc} \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} & \\ \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} & \begin{array}{c} i=1 \\ i=2 \\ i=3 \end{array} \end{array}$$

columns in a_{ki} become rows in a_{ij}.

This product is also a matrix multiplication

$$\cancel{a_{ki}a_{ij}' = a_{11}a_{11}' + a_{12}a_{21}' + a_{13}a_{31}'}$$

$x_k' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$ if $j=k$, then $x_j' = x_k'$ and $a_{ki}a_{ij}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We need:

$$a_{11}a_{11}' + a_{12}a_{21}' + a_{13}a_{31}' = 1$$
$$a_{21}a_{12}' + a_{22}a_{22}' + a_{23}a_{32}' = 1$$
$$a_{31}a_{13}' + a_{32}a_{23}' + a_{33}a_{33}' = 1$$

Einstein notation: if the indeces are repeated, sum over all indices

$$a_{ki}a_{ik}' = a_{k1}a_{1k}' + a_{k2}a_{2k}' + a_{k3}a_{3k}' = 1$$

We also need:

$$a_{11}a_{12}' + a_{12}a_{22}' + a_{13}a_{32}' = 0 \quad k=1\ j=2$$
$$a_{11}a_{13}' + a_{12}a_{23}' + a_{13}a_{33}' = 0 \quad k=1\ j=3$$
$$a_{21}a_{11}' + a_{22}a_{21}' + a_{23}a_{31}' = 0 \quad k=2\ j=1$$
$$a_{21}a_{13}' + a_{22}a_{23}' + a_{23}a_{33}' = 0 \quad k=2\ j=3$$
$$a_{31}a_{11}' + a_{32}a_{21}' + a_{33}a_{31}' = 0 \quad k=3\ j=1$$
$$a_{31}a_{12}' + a_{32}a_{22}' + a_{33}a_{32}' = 0 \quad k=3\ j=2$$

Thus Eq. 430

$$\boxed{a_{ki}a_{ij}' = \delta_{kj}}$$

Just like $x \cdot \frac{1}{x} = 1$, with $\tilde{1} = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$ identity matrix

inverse of x

if $a_{ki} a'_{ij} = \delta_{kj}$, then a'_{ij} is the inverse of a_{ki}.

Eq. 4.32 $\boxed{\tilde{A}\tilde{A}^{-1} = \tilde{1}}$ In this particular case, the multiplication commutes

It is called the identity transformation, $\vec{x} = \tilde{1}\vec{x}$.

Consider the double sum $a_{k\ell} a_{ki} a'_{ij}$

associative, so OK

with $c_{\ell i} = a_{k\ell} a_{ki}$, we get $c_{\ell i} a'_{ij}$

with $d_{kj} = a_{ki} a'_{ij}$, we get $a_{k\ell} d_{kj}$

Eq. 4.15

The orthogonality condition states that $a_{ij} a_{ik} = \delta_{jk}$

$c_{\ell i} = a_{k\ell} a_{ki} = \delta_{\ell i}$

so $a_{k\ell} a_{ki} a'_{ij} = \delta_{\ell i} a'_{ij}$, which is ~~only~~ not zero only if $\ell = i$, so $= a'_{\ell j}$

Eq. 4.30

The identity transformation states that $a_{ki} a'_{ij} = \delta_{kj}$

so $a_{k\ell} a_{ki} a'_{ij} = a_{k\ell} \delta_{kj}$ which is not zero only if $k = j$, so $a_{j\ell}$

Combining results, $a'_{\ell j} = a_{j\ell}$ Interchanges rows and columns! This is know as the transpose

In explicit matrix form, $\boxed{\tilde{A}^{-1} = \tilde{A}^{T}}$ Important!

Eq. 4.35

I use the tilde to denote a matrix. "Any" matrix.

But the book uses the tilde to denote the reciprocal matrix, and Bold font to denote matrices. I use the symbol T to denote the transpose. There won't be many issues like this in the course, but this is one case.

This result is very important because in general it is difficult to get the inverse, but for orthogonal matrices, the reciprocal matrix is the transposed matrix, which is exceedingly easy to compute.

~~Finally, without proof, the determinant of an orthogonal matrix can be +1 or -1, only.~~

~~The Euler Angles~~

We know that the 9 directional cosines can't be used as generalized coordinates because they are not independent. The orthogonality conditions reduce the number of independent elements to 3, but there is one more condition needed.

The determinant of a matrix is a scalar value that is a function of the elements of a square matrix. The determinant is non-zero iff the matrix is invertible. The determinant of a product of matrices is the product of its determinants.

$|\tilde{A}\tilde{B}| = |\tilde{A}| \cdot |\tilde{B}|$ Vertical bars denote the determinant.

Let $\tilde{A} = \tilde{B}$, then $|\tilde{A}\tilde{A}| = |\tilde{A}|\cdot|\tilde{A}| = |\tilde{A}|^2$

Now $\tilde{A}^{-1} = \tilde{A}^{T}$, so $\tilde{A}\tilde{A}^{-1} = \tilde{A}\tilde{A}^{T} = \tilde{1}$

$$|\tilde{A}\tilde{A}^{T}| = |\tilde{1}| = 1 = |\tilde{A}|\cdot|\tilde{A}^{T}|$$

Interchanging rows and columns does not change the determinant, so $|\tilde{A}^{T}| = |\tilde{A}|$, which implies that $|\tilde{A}|\cdot|\tilde{A}| = |\tilde{A}|^2 = 1$ Eq. 4.42

The determinant of an orthogonal matrix can only be +1 or −1.

To quickly remind you of the determinant,

$$|\tilde{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

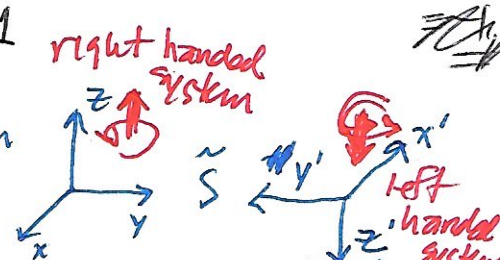
$$|\tilde{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$ and so on.

Consider the simplest 3x3 matrix with determinant −1

$$\tilde{S} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -\tilde{1} \quad ; \quad |\tilde{S}| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1\left[0-(-1)\right] + 0(\sim\sim) + 0(\sim\sim)$$
$$= -1$$

$$(\vec{r}\,') = \tilde{S}\vec{r}$$

↙ operates on unprimed system, transforms it into the primed system



This has the effect of changing the sign of each of the coordinate axes

It is called an <u>inversion</u>.

One way to achieve ~~this~~ inversion is by performing a rotation by 180° and then a reflection in the direction of the axis of rotation.

$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

But it is impossible to ~~get~~ reproduce $\tilde{S}$ with just rotations

Determinant of the rotation

$-1(-1) = +1$

Determinant of the reflexion

$1(-1) = -1$

Any matrix with determinant of -1 includes inversion, which can't be physically achieved with only rotations. Orthogonal transformations with determinant $\underline{\underline{+1}}$ are called <u>proper</u>, those with determinant $\underline{\underline{-1}}$ are called <u>improper</u>.

Only proper orthogonal transformations are physical and thus can be used ~~an~~ with the Lagrangian formulation. One set of parameters that satisfy orthogonality and produce proper transformations is the set of Euler angles.