9/23/21 (54)

Nice! We reduced the whole problem to

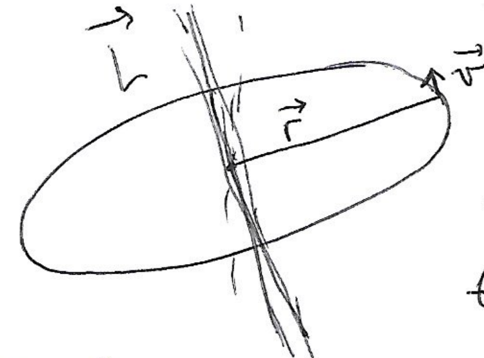
$$\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}^2 - V(r)$$

Let's look at the angular momentum $\vec{L}$

$\boxed{\text{☆ Will it be conserved? Think about planets, accretion disks, galaxies, etc.}}$

Because the potential depends on r only

The system is spherically symmetric, so ~~the~~ angular momentum is conserved. Put $\vec{r}$ and $\dot{\vec{r}}$ on the same plane.



$$\mathcal{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

Since angular momentum is conserved, θ is a cyclic coordinate $\frac{\partial \mathcal{L}}{\partial \theta} = 0$. (p_θ)

$\cancel{\frac{\partial \mathcal{L}}{\partial \theta} = 0 \quad p_\theta}$ $\quad \vec{L} = \vec{r} \times \vec{p}$

We will now focus on the 1-body problem.

$$\mathcal{L} = T - V = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2\right) - V(r)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}mr^2\dot{\theta}^2\right] = \frac{1}{\cancel{2}}mr^2\cancel{2}\dot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 \qquad \text{Lagrange Equation: } \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

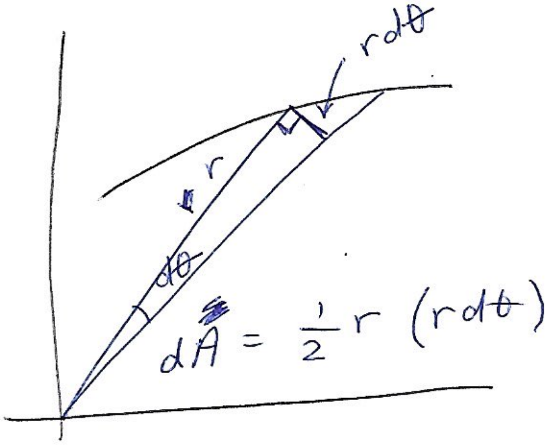
So ~~then~~ $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \Rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$

$$\int d(mr^2\dot{\theta}) = 0\int dt$$

$$mr^2\dot{\theta} + c = 0$$

Eq. 3.8 $\boxed{mr^2\dot{\theta} = \ell}$ constant

Kepler's second law: the radius vector sweeps out equal areas in equal times, so



$$\dot{A} = \frac{dA}{dt} = \text{constant}$$

$$\frac{d}{dt}\left(\frac{1}{2}r^2 d\theta\right) = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\frac{\ell}{m}$$

= constant!

The other equation of motion:

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\cancel{1}}{\cancel{2}}m\cdot \cancel{2}\dot{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\cancel{1}}{\cancel{2}}m\dot{\theta}^2\cdot \cancel{2}r - \frac{\partial V(r)}{\partial r}$$

Force is the gradient of the potential

from angular momentum constant

so $\frac{d}{dt}(m\dot{r}) - m r\dot{\theta}^2 + \frac{\partial V(r)}{\partial r} = 0$

Let $f(r) = -\frac{\partial V(r)}{\partial r}$, then

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

$$\dot{\theta}^2 = \left(\frac{\ell}{mr^2}\right)^2 = \frac{\ell^2}{m^2r^4}$$

$$m\ddot{r} - \frac{\cancel{m}\cancel{r}\,\ell^2}{m^{\cancel{2}}r^{\cancel{4}\,3}} = f(r)$$

Eq. 3.12 $\boxed{m\ddot{r} - \frac{\ell^2}{mr^3} = f(r)}$ Second-order differential equation

There is another conserved quantity:

$$h = \sum_j \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L} \qquad r, \dot{r}, \theta, \dot{\theta}$$

$$h = \dot{r} \frac{\partial \mathcal{L}}{\partial \dot{r}} + \dot{\theta} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L}$$

$$h = \dot{r}\,(m\dot{r}) + \dot{\theta}\,(mr^2\dot{\theta}) - \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

$$h = m\dot{r}^2 + mr^2\dot{\theta}^2 - \frac{1}{2} m\dot{r}^2 - \frac{1}{2} mr^2\dot{\theta}^2 + V(r)$$

Eq. 3.13 "Energy function" Hamiltonian, total energy

$$h = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

$$\frac{dh}{dt} = -\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow h = T + V = \text{constant} = E$$

Remember that $\dot{\theta}^2 = \frac{l^2}{m^2 r^4}$, so

Eq. 3.15

$$E = \frac{1}{2} m \left(\dot{r}^2 + \frac{\cancel{r^2}\, l^2}{m^2 \cancel{r^4}\, r^2}\right) + V(r) = \boxed{\frac{1}{2} m\dot{r}^2 + \frac{l^2}{2mr^2} + V = \text{constant}}$$

This equation can also be derived from Eq. 3.12, see book for details

This system has ~~to~~ two variables, r and θ, so to solve it we need 4 integrations. With conservation laws for the angular momentum and total energy, we have performed 2 integrations, so two remain.

$$\frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V = E$$

$$\Rightarrow \sqrt{\cancel{\frac{2}{m}}\cancel{\frac{1}{2}}\cancel{m}\dot{r}^2} = \sqrt{\left(E - V - \frac{\ell^2}{2mr^2}\right)\frac{2}{m}} = \frac{dr}{dt}$$

$$\Rightarrow \int_0^{t^*} dt = \int_{r_0}^{r^*} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}}$$ Eq. 3.18

This is the general solution. Notice that this gives the solution as $t(r) = F(r, r_0, E, \ell)$. In principle (at least), we can solve for $r(t, r_0, E, \ell)$

Once we have r, since $mr^2\dot{\theta} = \ell \Rightarrow \frac{d\theta}{dt} = \frac{\ell}{mr^2}$

$$\int d\theta = \int \frac{\ell}{mr^2}\, dt$$ if the initial value is θ_0,

$$\theta = \ell \int_0^{t^*} \frac{dt}{mr^2(t)} + \theta_0$$

$\boxed{\text{Although we have solved the equivalent one-body problem formally, practically speaking, the integrals are usually quite unmanageable and is often more convenient to perform the integration in some other fashion.}}$

We can still say a few things about the motion in the general case. Remember that

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r) \Rightarrow m\ddot{r} = f(r) + \frac{l^2}{mr^3}$$

Let $\boxed{f'(r) = f(r) + \frac{l^2}{mr^3}}$ Effective force

then $m\ddot{r} = f'(r)$ This is an effective one-dimensional situation, depends only on r

Notice that $\frac{l^2}{mr^3} = mr\dot{\theta}^2 = \frac{mr^2\dot{\theta}^2}{r} = \frac{mv_\theta^2}{r}$ is the centrifugal force

★ is this centrifugal force real?

$$f = -\frac{dV}{dx}$$

$$-\int f\,dx = \int dV$$

$$V = -\int f\,dx$$

$$\cancel{f'(r) = -\frac{\partial}{\partial r}\left[f(r) + \frac{l^2 r^{-3}}{m}\right]}$$

$$\cancel{f'(r) = -\frac{\partial f(r)}{\partial r} - \frac{l^2}{m}\frac{\partial}{\partial r} r^{-3}}$$

$$\cancel{f(r) =}$$

$$V'(r) = -\int f(r)\,dr - \frac{l^2}{m}\int r^{-3}\,dr = V(r) - \frac{l^2}{m}\frac{r^{-2}}{(-2)}$$

Eq. 3.22

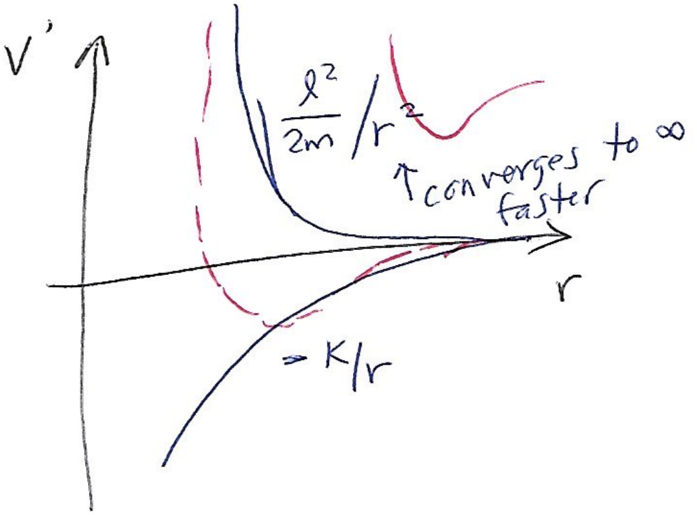
$$\boxed{V'(r) = V(r) + \frac{l^2}{2mr^2}}$$

The effective potential is the original potential plus a term that depends on the square angular momentum and proportional to the inverse of the squared of the separation.

As $r \to 0$, the centrifugal force term blows up. Consider an attractive inverse-square law force, $f = -\frac{k}{r^2} \Rightarrow V = +k\int dr\, r^{-2} = -(1)kr^{-1} = -\frac{k}{r}$

$$V'(r) = -\frac{k}{r} + \frac{l^2}{2mr^2}$$

We can plot the two terms

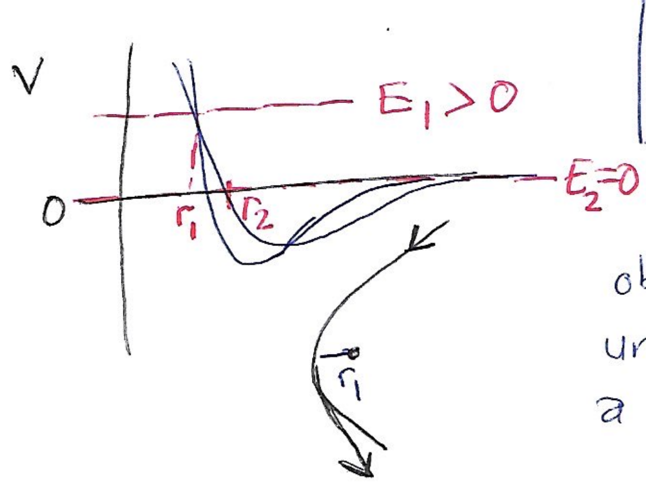


Notice that the effective potential is a Lennard-Jones potential!

$$\lim_{r\to 0} V'(r) = \infty$$

$$\lim_{r\to\infty} V'(r) = 0 \quad \text{(approaches from below)}$$

Consider the case for total energy E_1. Remember that $E = T + V$



$\boxed{\text{★ What kind of motion do you get}}$

$E_1 > 0$ At r_1, all the energy is potential and object can't get any closer, but unbound, r can go to infinity. Produces a hyperbola.

$E_2 = 0$ Similar to previous case, but $r_2 > r_1$. It's kinetic energy, and hence velocity, is zero at $r = \infty$, but it is positive (since potential is negative) at any distance $r > r_2$. Produces a parabola

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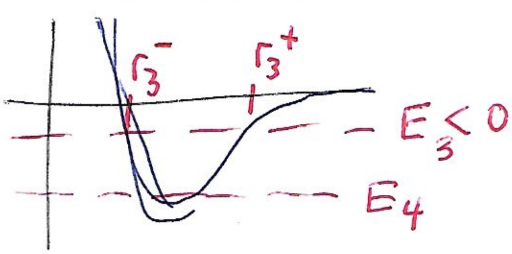
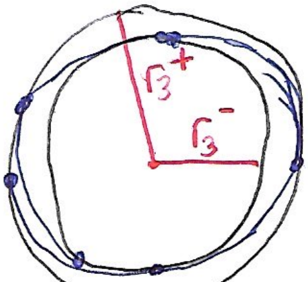
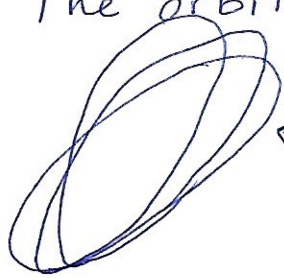


In this case the total energy is negative, which means that the object is bounded oscillating between r_3^- and r_3^+. The one-dimensional potential is rotating about the $r=0$ axis. It is possible that $r = r_3^-$ at $\theta = 0$ always, but in general this will not happen.





The orbit will be eliptical but in general there will be an ← apsidal precesion

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If $E = E_4$, then $r_3^- = r_3^+ = r_4$. The two bounds coincide so the orbit is perfectly circular!

Notice that the eccentricity of an orbit depends on how much "extra kinetic energy" needs to be accomodated. Evidence for gravitational waves were first provided by binary pulsars. The fact that they orbit each other radiates energy, reducing E. At some point they will be in a perfect circular orbit, ~~and~~ before colliding.

In materials, the energy barrier is produced by Coulomb repulsion of the electrons, and the attraction could be due to van der Waals interactions. At zero temperature we have the E_4 case. At finite temperature there is "kinetic energy" that needs to be accomodated. Typically the bottom of the potential is a bit assymmetric,

with $|r_3^+ - r_4| > |r_3^- - r_4|$, so the material expands to accomodate the kinetic energy. E_2 is the phase transition between a solid and a fluid, with E_1 a very hot fluid.