In the moment of inertia tensor, the diagonal elements are of the form $I_{xx} = m_i (r_i^2 - x_i^2)$ and the off-diagonal are of the form $I_{xy} = -m_i x_i y_i$.

It is easy to see that $I_{xy} = I_{yx}$, so out of the 9 coefficients, only 6 are independent. All the ~~coe~~ coefficients are real, so if we take the transpose and then the conjugate,

$$\tilde{A} \qquad \tilde{A}^T \qquad \overline{\tilde{A}^T}$$

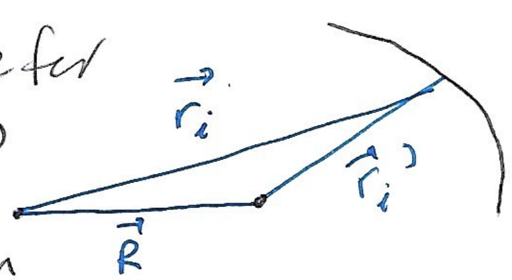
$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \qquad \begin{bmatrix} I_{xx}^* & I_{xy}^* & I_{xz}^* \\ I_{xy}^* & I_{yy}^* & I_{yz}^* \\ I_{xz}^* & I_{zy}^* & I_{zz}^* \end{bmatrix} \qquad \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

we end up with the same matrix. Hence, the moment of inertia tensor is a Hermitian operator.

Previously we considered the case in which a rigid body rotates about a single point, and that point was located at the origin.

☆ What if the origin is located somewhere else?

\# ignore for now

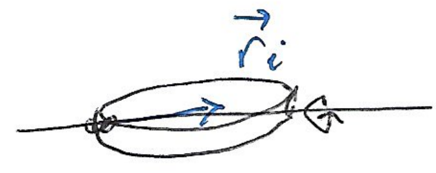


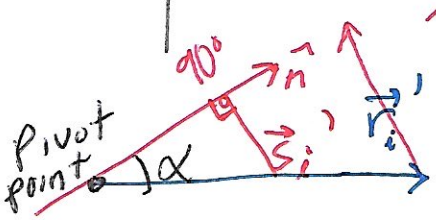
Now $\vec{r}_i$ is given by $\vec{R} + \vec{r}_i{}'$, where $\vec{R}$ is the distance from the origin to the axis of rotation and $\vec{r}_i{}'$ is the distance from particle i which is part of the rigid body, to the axis of rotation.

In intro mechanics you learned that the moment of inertia is $I = \sum_i m_i \cdot \vec{s}_i{}^2$ or $m_i \vec{s}_i{}^2$ in Einstein notation

where $\vec{s}_i$ is the distance from particle i to the axis of rotation. (the pivot point)

*Which one has higher moment of inertia?

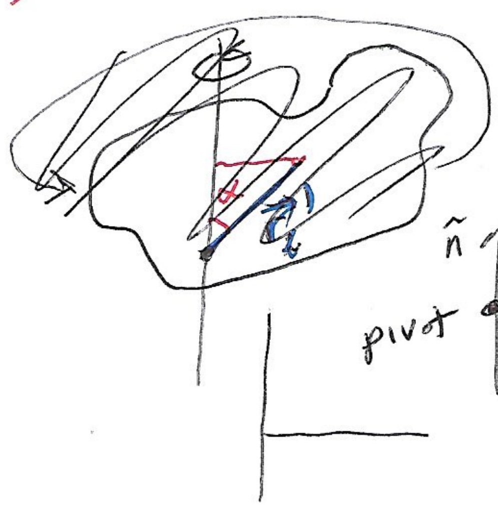




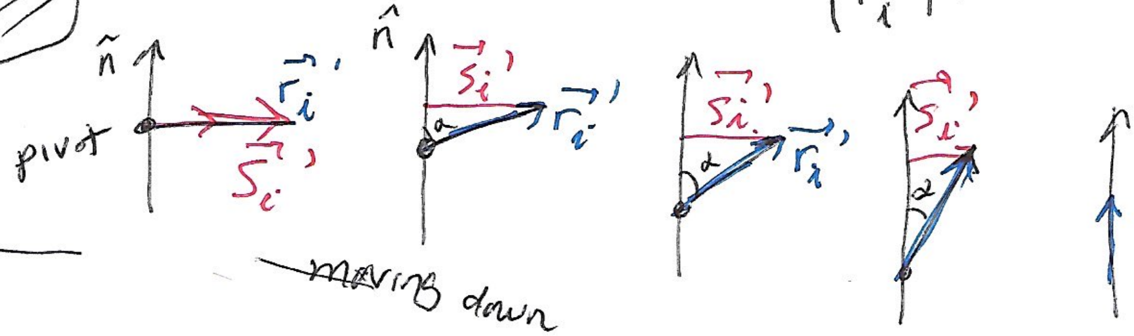
90° between $\vec{r}_i{}'$ and $\hat{n}$

0° between $\vec{r}_i{}'$ and $\hat{n}$

Distance to the axis is now 0



The perpendicular distance to the axes of rotation is $|\vec{r}_i{}'| \sin\alpha$



We can express s_i as the magnitude of the cross product between $\vec{r}_i'$ and $\hat{n}$, then

$$\vec{s}_i'^{\,2} = \vec{s}_i' \cdot \vec{s}_i' = (\vec{r}_i' \times \hat{n}) \cdot (\vec{r}_i' \times \hat{n})$$

We defined $\omega \hat{n} = \vec{\omega}$, so $\hat{n} = \dfrac{\vec{\omega}}{\omega}$, so

Before

$$I = \frac{m_i}{\vec{\omega}^2} (\vec{r}_i' \times \vec{\omega}) \cdot (\vec{r}_i \times \vec{\omega}) = \frac{m_i}{\vec{\omega}^2} \vec{v}_i^{\,2}$$

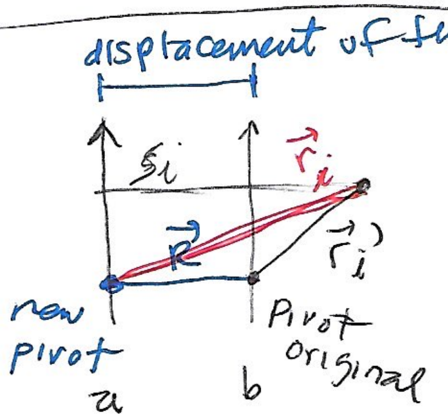
Before

We also derived $\vec{v}_i = \vec{\omega} \times \vec{r}$ and of course Eq. 5.17

$$T = \frac{1}{2} m_i v_i^2 \text{, so } I = \frac{2T}{\omega^2} \Rightarrow \boxed{T = \frac{1}{2} I \omega^2}$$

which is the same we derived using the tenso contraction $T = \dfrac{\vec{\omega}^2}{2} \hat{n} \cdot \mathrm{I} \cdot \hat{n} = \dfrac{1}{2} I \omega^2$

Note that I and $\tilde{\mathrm{I}}$ are different



The moment of inertia about this new axis/pivot a is:

$$I_a = m_i \vec{s}_i^{\,2} = m_i (\vec{r}_i \times \hat{n})^2$$

Since $\vec{r}_i = \vec{R} + \vec{r}_i'$,

$$I_a = m_i \left[(\vec{R} + \vec{r}_i') \times \hat{n} \right]^2 = m_i \left[(\vec{R} \times \hat{n}) + (\vec{r}_i' \times \hat{n}) \right]^2$$

$$I_a = \sum_i m_i (\vec{R} \times \hat{n}) + 2 m_i (\vec{R} \times \hat{n}) \cdot (\vec{r}_i' \times \hat{n}) + m_i (\vec{r}_i' \times \hat{n})^2$$

Since $(s\vec{A})\times\vec{B} = \vec{A}\times(s\vec{B}) = s(\vec{A}\times\vec{B})$

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

$$2m_i(\vec{R}\times\hat{n})\cdot(\vec{r}_i{}'\times\hat{n}) = -2(\vec{R}\times\hat{n})\cdot m_i(\hat{n}\times\vec{r}_i{}')$$

$$= -2(\vec{R}\times\hat{n})\cdot(\hat{n}\times m_i\vec{r}_i{}')$$

The definition of the center of mass is

$$\sum_i m_i(\vec{r}_i - \vec{R}) = 0$$

$\vec{R}$ are the coordinates of the center of mass

Since $\vec{r}_i = \vec{R} + \vec{r}_i{}'$

$$\vec{r}_i - \vec{R} = \vec{r}_i{}'$$

$$\sum_i m_i\vec{r}_i{}' = 0$$

$$I_a = M(\vec{R}\times\hat{n})^2 + m_i(\vec{r}_i{}'\times\hat{n})^2$$

sum over the mass

So the whole term vanishes

what we had before moving the axis. Let's call it I_b

~~$I_a = I_b + M(R\sin\theta)^2$~~

$$I_a = I_b + M(R\sin\theta)^2$$ Eq. 5.21

perpendicular distance between center of mass and new axis

The moment of inertia about any given axis (a) is equal to the moment of inertia about a parallel axis (b) that goes through the center of mass (I_b) <u>plus</u> the moment of inertia of the body taken as a particle located at the center of mass about axis a.

The previous derivations show us, as if we didn't know, that the coefficients of $\tilde{I}$ depend on the location of the origin and the orientation of the rigid body w.r.t. the axes.

This suggest that there is a set of coordinates in which the tensor is diagonal $\tilde{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$

$\vec{L} = \tilde{I}\vec{\omega}$, so $L_1 = I_1 \omega_1$

$L_2 = I_2 \omega_2$

$L_3 = I_3 \omega_3$

Last time we derived $T = \dfrac{\vec{\omega} \cdot \tilde{I} \cdot \vec{\omega}}{2}$

and the definition of contraction $T = \frac{1}{2} \omega_j I_{jk} \omega_k$

so $T = \frac{1}{2} \left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right)$ Eq. 5.26

Remember that you express vectors and tensors using coordinate systems, but they exist independently, in particular, they have a magnitude and direction

And they follow definition Eq. 5.10.

In general, a 3-D vector will be $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$, its magnitude $|\vec{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$, the orientation can be given with polar angles θ and φ

Remember that $\vec{r}\,' = \tilde{A}\vec{r}$, so we can use two Euler angles to align any cartesian system with the vector (the z-axis, for example).

We learned before that to align an operator (a matrix) we need $\tilde{I}_D = \tilde{R}\tilde{I}\tilde{R}^T$. In general you will need all three Euler coordinates, but you can make $\tilde{I}$ diagonal, $\tilde{I}_D = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$

Immediately we see that I_1, I_2, I_3 are the eigenvalues of $\tilde{I}$. Each one is a component of the principal moment of inertia tensor.

The eigenvectors are in the rows and columns of $\tilde{R}$ and $\tilde{R}^T$. The eigenvectors are called the principal axes.

Once you have $\tilde{I}_D$, the inertia tensor relative to any axis that goes through the center of mass is given by $\tilde{I} = \tilde{S}\tilde{I}_D\tilde{S}^T$ where $\tilde{S}$ is the appropriate transformation. In this case, the eigenvalues are obtained by solving the secular equation

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0$$

And you can move the origin using the parallel axis theorem.