Example sheet 7 – formative

1. Consider the dynamical system

$$\dot{x} = \mu x - y + x^2,$$
$$\dot{y} = x - \sigma y + y^2,$$

where $(x, y) \in \mathbb{R}^2$ and μ, σ are constants.

(a) Determine the nature of the equilibrium point $(0, 0)$ for each $\mu, \sigma \geqslant 0$.

(b) Sketch the (σ, μ)-plane and indiate what the equilibrium is in each defined area.

Solution: This system has an equilibrium point at $(0, 0)$, with the associated linear system being given by

$$\dot{x} = \mu x - y,$$
$$\dot{y} = x - \sigma y,$$

where $\mathbf{A} = \begin{pmatrix} \mu & -1 \\ 1 & -\sigma \end{pmatrix}$. The eigenvalues of $\mathbf{A}$ are given by

$$\lambda_{\pm} = -\frac{(\sigma - \mu)}{2} \pm \frac{1}{2}\sqrt{(\sigma - \mu)^2 - 4(1 - \sigma\mu)}.$$

We must now determine the nature of the eigenvalues for $\sigma, \mu \geqslant 0$:

(i) Eigenvalues change from real and distinct to complex conjugate pair when

$$(\sigma - \mu)^2 - 4(1 - \sigma\mu) = 0; \quad \sigma, \mu \geqslant 0,$$

which on rearrangement gives that

$$(\sigma + \mu) = \pm 2.$$

Only the positive option is relevant to our quadrant where $\sigma, \mu \geqslant 0$.

(ii) Eigenvalues are:

Complex conjugate when $0 \leqslant \sigma + \mu < 2$,
Real and distinct when $\sigma + \mu > 2$,
Real and equal when $\sigma + \mu = 2$.

(iii) Eigenvalues are purely imaginary when

$$\sigma = \mu, \quad 0 \leqslant \sigma + \mu < 2.$$

(iv) Zero eigenvalue when

$$\sigma\mu = 1; \quad \sigma, \mu \geqslant 0.$$

(v) For $\sigma\mu < 1$, the two real eigenvalues will have the same sign, positive for $\sigma < \mu$ and negative for $\sigma > \mu$. For $\sigma\mu > 1$ the two real roots will have opposite signs.

The figure below illustrates the σ, μ parameter plane which summarizes all the information obtain above about the nature of the eigenvalues. To conclude we have that:

1. Region A: Eigenvalues real, but of different sign. Equilibrium point $(0, 0)$ is a saddle point.

2. On $\sigma = 2 - \mu$ (for $0 \leqslant \mu \leqslant 2$), eigenvalues real and equal. Equilibrium point $(0, 0)$ is an unstable degenerate node when $1 < \mu \leqslant 2$, while $(0, 0)$ is a stable degenerate node when $0 \leqslant \mu < 1$. Note when $\mu = 1$ $(\sigma = 1)$ the eigenvalues are both zero and the linearization theorem fails to classify the equilibrium point.

3. Region B: Eigenvalues are real, distinct and negative. Equilibrium point $(0, 0)$ is a stable node.

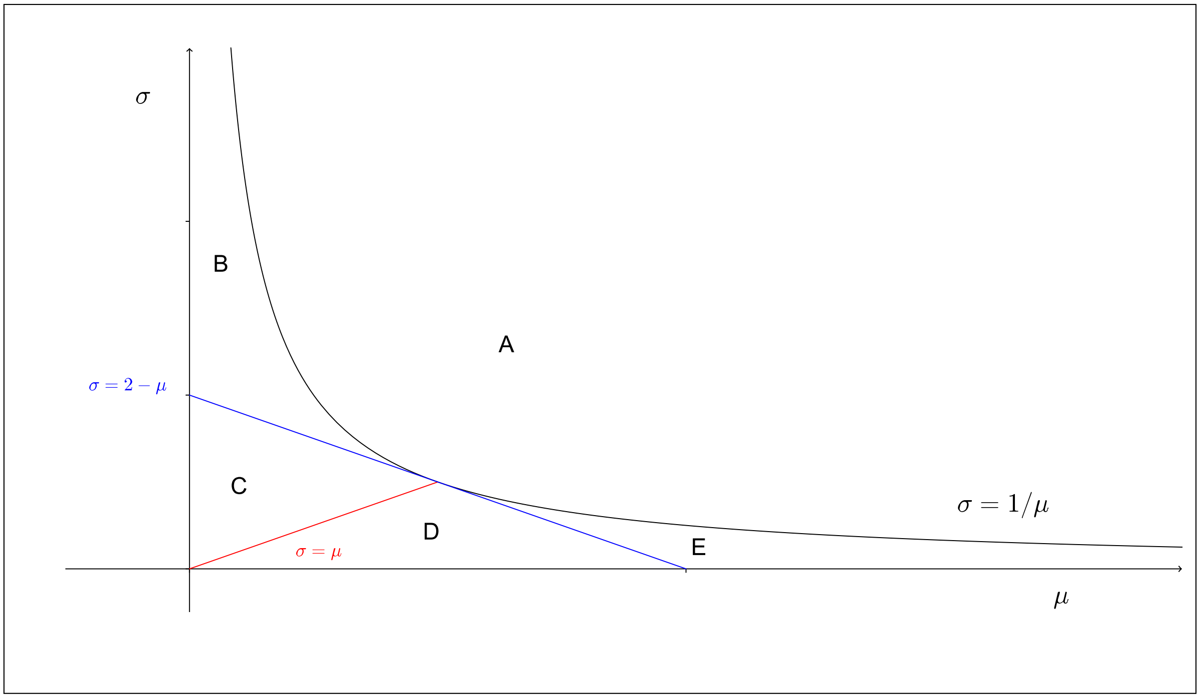
4. Region E: Eigenvalues are real, distinct and positive. Equilibrium point $(0, 0)$ is an unstable node.

5. On $\sigma = \mu$ (for $0 \leqslant \mu \leqslant 1$), eigenvalues are purely imaginary and the linearization theorem fails to classify the equilibrium point.

6. Region C: Eigenvalues are a complex conjugate pair with negative real part. Equilibrium point $(0, 0)$ is a stable spiral.

7. Region D: Eigenvalues are a complex conjugate pair with positive real part. Equilibrium point $(0, 0)$ is an unstable spiral.

8. On $\sigma = 1/\mu$ (for $\mu \geqslant 0$), we have a zero eigenvalue and the linearization theorem fails to classify the equilibrium point.



2. Consider the 2-dimensional dynamical system

$$\begin{aligned} \dot{x} &= (1 - x - y)x, \\ \dot{y} &= (4 - 7x - 3y)y, \end{aligned} \tag{1}$$

where $x, y \geqslant 0$.

(a) Find the equilibrium points of (1).

(b) Determine the horizontal and vertical isoclines for dynamical system (1). and find the direction of the flow on them.

(c) Consider the region $D = \left\{(x, y) : 0 < x < 1, 0 < y < \frac{3}{2}\right\}$ with boundary $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where:

$$\begin{aligned} C_1 &= \{(x, y) : y = 0, x \in [0, 1]\} \\ C_2 &= \left\{(x, y) : x = 1, y \in \left[0, \frac{3}{2}\right]\right\} \\ C_3 &= \left\{(x, y) : y = \frac{3}{2}, x \in [0, 1]\right\} \\ C_3 &= \left\{(x, y) : x = 0, y \in \left[0, \frac{3}{2}\right]\right\} \end{aligned}$$

and establish that $D \cup C$ is a positively invariant set for dynamical system (1).

(d) Using parts b and c locate any positively invariant sets for (1) within $D \cup C$.

Solution:

(a) The equilibrium points of (1) are found by solving the following equations

$$\begin{aligned} (1-x-y)x &= 0 \qquad && \Rightarrow x = 0 \text{ or } y = 1-x, \\ (4-7x-3y)y &= 0 \qquad && \Rightarrow y = 0 \text{ or } y = \frac{4}{3} - \frac{7}{3}x, \end{aligned}$$

giving four possible equilibrium points

$$(0,0), \quad \left(0, \frac{4}{3}\right), \quad (1,0), \quad \left(\frac{1}{4}, \frac{3}{4}\right).$$

(b) Considering

$$\frac{dy}{dx} = \frac{(4-7x-3y)y}{(1-x-y)x} \quad \begin{cases} 0, & y = 0 \text{ or } 3y = 4-7x, \\ \infty, & x = 0 \text{ or } y = 1-x. \end{cases}$$

The horizontal isoclines are given by $y = 0$ and $3y = 4-7x$, while the vertical isoclines are given by $x = 0$ and $y = 1-x$.

The direction of flow on the horizontal and vertical isoclines is calculated as follows:

For $y = 0$, $\dot{x} = x(1-x)$, so x is increasing with increasing t when $0 < x < 1$ (arrow pointing to the right) and decreasing for $x > 1$ (arrows pointing to the left).

For $3y = 4-7x$, $\dot{x} = \frac{1}{3}(4x-1)x$, so x is increasing with increasing t when $x > \frac{1}{4}$ (arrow pointing to the right) and decreasing for $0 < x < \frac{1}{4}$ (arrows pointing to the left).

For $x = 0$, $\dot{y} = (4-3y)y$, so y is increasing with increasing t when $0 < y < \frac{4}{3}$ (arrow pointing upwards) and decreasing for $y > \frac{4}{3}$ (arrows pointing downwards).

For $y = 1-x$, $\dot{y} = (1-4x)(1-x)$, so y is increasing with increasing t when $x < \frac{1}{4}$ or $x > 1$ (arrow pointing upwards) and decreasing for $\frac{1}{4} < x < 1$ (arrows pointing downwards).

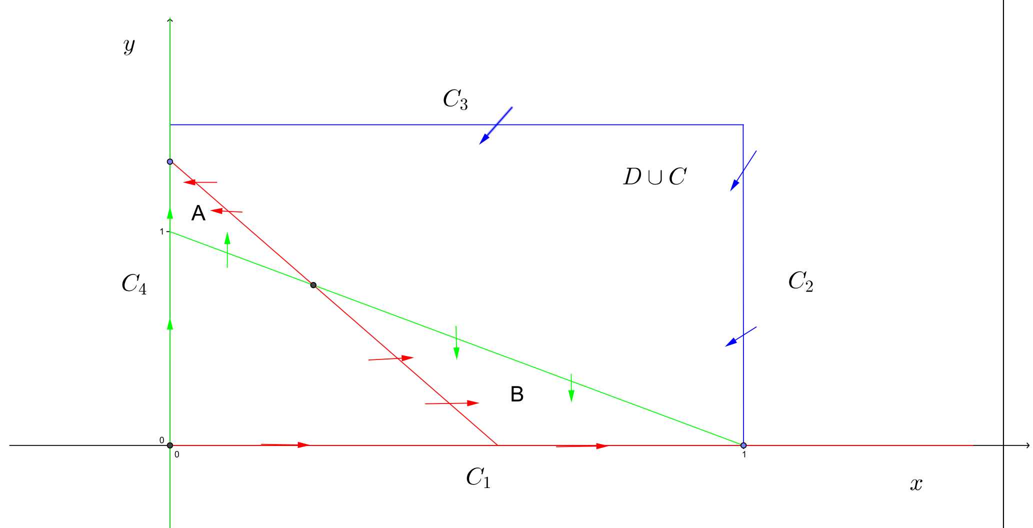
(c) The vector field $(P(x,y), Q(x,y)) = ((1-x-y)x, (4-7x-3y)y)$ and the outward normals to the boundaries C_i $(i = 1,2,3,4)$ are given by $\mathbf{n}_1 = (0,-1)$, $\mathbf{n}_2 =$

$(1,0)$, $\mathbf{n}_3 = (0,1)$, $\mathbf{n}_4 = (-1,0)$, respectively. Now computing $(P(x,y), Q(x,y)).\mathbf{n}_i$ on each C_i we have that

$$\begin{aligned}
&\text{On } C_1 : ((1-x)x, 0).(0,-1) = 0, \\
&\text{On } C_2 : (-y, -3y(1+y)).(1,0) = -y \leqslant 0, \\
&\text{On } C_3 : \left(\left(-\frac{1}{2} - x\right)x, \frac{3}{2}\left(-\frac{1}{2} - 7x\right)\right).(0,1) = -\frac{3}{2}\left(\frac{1}{2} + 7x\right) < 0, \\
&\text{On } C_4 : (0, (4-3y)y).(-1,0) = 0,
\end{aligned}$$

Since $(P(x,y), Q(x,y)).\mathbf{n}_i \leqslant 0$ on each C_i we conclude that $D \cup C$ is a positively invariant set for dynamical system (1).

(d) The Figure below gives the phase plane for (1) showing the equilibrium points found in part a, the horizontal and vertical isoclines determined in part b and the positively invariant region $D \cup C$ found in part c. Consideration of this figure then indicates that there are two positively invariant regions contained within $D \cup C$ which we label as A and B.



3. Show that the nonlinear system

$$\dot{x} = -y + x\left(1 - \sqrt{x^2 + y^2}\right),$$
$$\dot{y} = x + y\left(1 - \sqrt{x^2 + y^2}\right),$$

has a limit cycle given by $x^2 + y^2 = 1$.

Solution: We first write the nonlinear system in terms of polar coordinates (via $x = r\cos\theta, y = r\sin\theta$) to obtain

$$\begin{aligned} \dot{r} &= r(1-r), \\ \dot{\theta} &= 1 (\text{counterclockwise rotation}). \end{aligned} \tag{2}$$

We observe that

$$\dot{r} \begin{cases} < 0, & r > 1, \\ = 0, & r = 1, \\ > 0, & 0 < r < 1, \\ = 0, & r = 0. \end{cases}$$

Therefore, we have a stable limit cycle when $r = 1$, that is when $x^2 + y^2 = 1$.

4. Use the Poincaré-Benedixson theorem to establish that the dynamical system

$$\begin{aligned} \dot{x} &= x + y - x^3 + xy^2 - x\left(x^2 + y^2\right)^2, \\ \dot{y} &= y - x - x^2 y + y^3 - y\left(x^2 + y^2\right)^2, \end{aligned}$$

where $(x, y) \in \mathbb{R}^2$ has at least one periodic orbit surrounding the origin. Note that $(0, 0)$ is the only equilibrium point.

Solution: We first write the nonlinear system in terms of polar coordinates (via $x = r\cos\theta, y = r\sin\theta$) to obtain

$$\begin{aligned} \dot{r} &= r - r^3\cos^2\theta + r^3\sin^2\theta - r^5, \\ \dot{\theta} &= -1. \end{aligned} \tag{3}$$

It is clear from (3) that we choose $r = R$ sufficiently large so that

$$\dot{r}\Big|_{r=R} < 0 \quad \text{for all} \quad 0 \leqslant \theta < 2\pi.$$

In addition, we may also choose $r = \varepsilon$ sufficiently small so that

$$\dot{r}\Big|_{r=\varepsilon} > 0 \quad \text{for all} \quad 0 \leqslant \theta < 2\pi.$$

These two conditions together with $(3)_2$ ensure that the region

$$D: \quad \varepsilon \leqslant r \leqslant R, \quad 0 \leqslant \theta < 2\pi$$

is a positively invariant region for the dynamical system. Moreover, region D contains no equilibrium points, and we conclude, via the Poincaré-Bendixson Theorem, that D must contain at least one periodic orbit of the dynamical system.