LECTURE 3

Linear independence

3.1 Linear independence

The three examples of spans of column vectors in the previous lecture indicate that

- different sets of vectors can yield the same span;
- the same subspace can be spanned by sets with different cardinality.

The latter observation suggests that there may exist in general a representation of a subspace that employs the least number of vectors. This is indeed the case: sets with 'unnecessary' vectors in this context are known as linearly dependent sets. To make this more precise, consider the following result.

Proposition 3.1 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and let $\mathbf{u} \in \text{span}S$. Then

$$\text{span}S = \text{span}S \cup \{\mathbf{u}\}\,.$$

If $\mathbf{u} \in \text{span}S$, there holds

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \iff a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k + (-1)\mathbf{u} = \mathbf{0}.$$

In other words, $\mathbf{u}$ can be expressed in terms of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, and is thus dependent on these vectors, provided the expression on the right holds. This observation justifies the following definition.

Definition 3.1 — Linearly dependent set. Let $V(\mathbb{F})$ be a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ denote a set of vectors in V. We say S is linearly dependent provided there exist scalars $a_1, a_2, \ldots, a_k \in \mathbb{F}$ not all zero such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}. \tag{3.1}$$

R Another interpretation of (3.1) is that the zero vector can be written as a non-trivial linear combination of the vectors in S.

Definition 3.2 — Linearly independent set. A set of vectors that is not linearly dependent is called linearly independent.

A less intuitive example employing this definition is contained in the following result.

Proposition 3.2 The empty set is a linearly independent set.

Proof. If $S = \emptyset$, there are no vectors that can be employed to construct a linear combination equal to the zero vector. Hence the empty set is not linearly dependent. ∎

When a set is non-empty, one can rephrase the definition of linear independence as the logical complement of Definition 3.1 (see also the remark following it).

Definition 3.3 — Linearly independent set. Let $V(\mathbb{F})$ be a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ denote a non-empty set of vectors in V. We say S is linearly independent provided the zero vector can only be written as the trivial linear combination of vectors in S. In other words,

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0} \Longrightarrow a_1 = a_2 = \cdots = a_k = 0. \tag{3.2}$$

The following results concern some special cases.

Proposition 3.3 Any set S containing the zero vector is linearly dependent.

Proof. Let $\mathbf{v} := \mathbf{0} \in S$. Then $1\mathbf{v} = \mathbf{0}$, so that $\mathbf{0}$ can be written as a non-trivial linear combination of vectors in S. ∎

Corollary 3.4 Let $V(\mathbb{F})$ be a vector space. Then V is a linearly dependent set.

Proof. Since $V(\mathbb{F})$ is a subspace, there holds $\mathbf{0} \in V$. By Proposition 3.3, the set V is linearly dependent. ∎

Proposition 3.5 A set S containing a single non-zero vector is linearly independent.

Proof. Let $\mathbf{0} \neq \mathbf{v} \in S$. By Proposition 1.2, $a\mathbf{v} = \mathbf{0} \Longrightarrow a = 0$, so that S is linearly independent. ∎

Let us examine some examples of linear dependence and independence.

Example 3.1 Consider again the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

We note that

$$\mathbf{v}_3 = \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2,$$

so that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. On the other hand, and two vectors of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. For example,

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{0} \Longrightarrow \begin{bmatrix} a_1 \\ 2a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{cases} a_1 + a_2 & = 0 \\ \quad 2a_1 & = 0 \end{cases} \Longrightarrow a_1 = a_2 = 0.$$

Example 3.2 Let $\mathbf{v}_1 = (1,2), \mathbf{v}_2 = (2,1)$ and let $S = \{\mathbf{v}_1\mathbf{v}_2\} \subset \mathbb{Z}_3^2$. Then S is a linearly dependent set in $V = Z_3^2$ since the zero vector can be written as a non-trivial linear combination:

$$\mathbf{v}_1 \oplus \mathbf{v}_2 = (1,2) \oplus (2,1) = (3,3) \bmod 3 = (0,0).$$

Example 3.3 Let $S = \{a_0 + a_1 x : a_0, a_1 \in \mathbb{R}\}$. Then $S(\mathbb{R})$ is a vector space, as it is the set of polynomials of degree at most one: $S(\mathbb{R}) = \mathcal{P}_1(\mathbb{R})$. By Corollary 3.4, the set S is linearly dependent. On the other hand, the set $S = \{1, x\}$ is linearly independent, since

$$a_1 1 + a_2 x = \mathbf{0} \Longrightarrow a_1 + a_2 x = \mathbf{0} \Longrightarrow p = \mathbf{0} \Longrightarrow a_1 = a_2 = 0,$$

where we defined p to be the polynomial $p(x) = a_1 + a_2 x$. Note that while we tried to use the formalism from Lecture 2, in general, one could also view the statement on the left as an identity that has to be satisfied for any real value assigned to x, e.g., for $x = 1$ and $x = 2$ we obtain a linear system in a_1, a_2:

$$a_1 + a_2 x = 0 \Longrightarrow \left[\begin{array}{ll} a_1 + a_2 & = 0 \\ a_1 + 2a_2 & = 0 \end{array} \right. \Longrightarrow a_1 = a_2 = 0.$$

Example 3.4 Let $S = \{\sin x, \cos x\}$. This is a linearly independent set since

$$a_1 \sin x + a_2 \cos x = \mathbf{0} \Longrightarrow f = \mathbf{0},$$

where $f(x) := a_1 \sin x + a_2 \cos x$ is the zero function, i.e., $f(x) = 0$ for all x. Choosing $x = 0$ and $x = \pi/2$ as arguments of f, we get $a_2 = 0$ and $a_1 = 0$, so that S is indeed linearly independent.

We end this section with a result that confirms the importance of linear independence.

Proposition 3.6 — Uniqueness of representation. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ denote a set of vectors. Let $\mathbf{v} \in \operatorname{span} S$. Then $\mathbf{v}$ has a unique representation as a linear combination of vectors in S if and only if S is a linearly independent set.

Proof. Let

$$\mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots a_k \mathbf{v}_k.$$

$\boxed{\Longrightarrow}$: we are given that the above representation is unique, i.e., there are no other coefficients that can represent $\mathbf{v}$ as a linear combination of vectors in S. Assume now that S is not a linearly independent set. We show that this assumption yields a contradiction. By definition, linear dependence implies that the zero vector can be written as a non-trivial linear combination:

$$\mathbf{0} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots b_k \mathbf{v}_k,$$

for some scalars $b_1, b_2, \ldots, b_k$, not all zero. Taking the sum of the above two relations we obtain

$$\mathbf{v} = (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \cdots + (a_k + b_k)\mathbf{v}_k =: c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k,$$

where $c_i := a_i + b_i, i = 1, \ldots, k$ satisfy $c_i - a_i = b_i$ are not all zero. Hence, $c_i \neq a_i$ for some i. Thus, the representation of $\mathbf{v}$ is not unique – a contradiction. Hence, S must be a linearly independent set.
$\boxed{\Longleftarrow}$: we are given that S is a linearly independent set. Assume that the coefficients $a_1, a_2, \ldots, a_k$ are not unique in the representation of $\mathbf{v}$. Then

$$\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots b_k \mathbf{v}_k,$$

for some other coefficients $b_i \neq a_i$ for at least one i. Taking the difference, we find

$$\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_k - b_k)\mathbf{v}_k.$$

This implies that $\mathbf{0}$ can be written as a non-trivial linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, which contradicts their assumed linear independence. Hence, we must have $a_i = b_i$ for all $i = 1, \ldots, k$ and the representation of $\mathbf{v}$ as a linear combination of $\mathbf{v}_i$ is unique. ■

The above result is an important characterisation of uniqueness of representation of a vector which highlights the importance of linear independence of sets. The following two sections provide a further insight into the concept of linear independence.

3.2 Removing vectors. Minimal spanning sets.

Lemma 3.7 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. Then S is a linearly dependent set if and only if there exists $j \in \{1, 2, \dots, k\}$ such that $\text{span}S = \text{span}S \setminus \{\mathbf{v}_j\}$.

Proof. $\boxed{\Longrightarrow}$: we are given that S is a linearly dependent set; we need to show that there exists an index j such that $\text{span}S = \text{span}S \setminus \{\mathbf{v}_j\}$. To prove this set equivalence, we show that double inclusion holds. First, for any $j = 1, 2, \dots, k$,

$$S \setminus \{\mathbf{v}_j\} \subseteq S \Longrightarrow \text{span}S \setminus \{\mathbf{v}_j\} \subseteq \text{span}S.$$

To show the second inclusion, let $\mathbf{v} \in \text{span}S$ have the representation

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k.$$

Since S is a linearly dependent set, the zero vector can be written as a non-trivial linear combination of vectors in S:

$$\mathbf{0} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k.$$

This means that there exists an index j such that $b_j \neq 0$. Define $c = a_j/b_j$. Then

$$\text{span}S \ni \mathbf{v} = \mathbf{v} - c\mathbf{0} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k) - c\,(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k) =$$

$$= \sum_{i=1}^{k}\left(a_i - \frac{a_j}{b_j}b_i\right)\mathbf{v}_i = \sum_{i\neq j}^{k}\left(a_i - \frac{a_j}{b_j}b_i\right)\mathbf{v}_i \in \text{span}S \setminus \{\mathbf{v}_j\}.$$

Since $\mathbf{v}$ was arbitrary, we conclude that $\text{span}S \subseteq \text{span}S \setminus \{\mathbf{v}_j\}$. Hence, $\text{span}S = \text{span}S \setminus \{\mathbf{v}_j\}$.
$\boxed{\Longleftarrow}$: we are given that $\text{span}S = \text{span}S \setminus \{\mathbf{v}_j\}$ for some j. To show S is linearly dependent, note that $\mathbf{v}_j \in \text{span}S = \text{span}S \setminus \{\mathbf{v}_j\}$. Thus, $\mathbf{v}_j \in \text{span}S \setminus \{\mathbf{v}_j\}$ so that

$$\mathbf{v}_j = \sum_{i\neq j} a_i\mathbf{v}_i \Longleftrightarrow a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + (-1)\mathbf{v}_j + \cdots + a_k\mathbf{v}_k = \mathbf{0},$$

which means that $\mathbf{0}$ is a non-trivial linear combination of the vectors in S (since at least the coefficient of $\mathbf{v}_j$ is nonzero). Hence, S is linearly dependent. ∎

We illustrate this result with an example.

Example 3.5 Consider again the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

The set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, so that there is a vector that we can remove without changing the span. In this particular case, any vector can be removed (i.e., the index j in Lemma 3.7 can be 1,2 or 3). However, one can notice that removing any vector yields a linearly independent set, so we cannot remove a further vector without changing the span.

Lemma 3.7 indicates that we can 'trim down' a linearly dependent set without changing its span: when this is the case, the representation of a vector will use fewer vectors from the span and so it will be more 'economical'. However, when the span changes, then we can identify a spanning set that is linearly independent. By definition, this is a minimal spanning set. The next result indicates that linear independence provides a characterisation of minimal spanning sets.

Theorem 3.8 Let S be a spanning set for a finite-dimensional vector space $V(\mathbb{F})$. Then S is a minimal spanning set for V if and only if it is a linearly independent set in V.

Proof. If $V(\mathbb{F})$ is the trivial vector space, then its minimal spanning set is the empty set, which is a linearly independent spanning set for V (see Definition 3.3). Let us assume now that $V(\mathbb{F})$ is a non-trivial vector space. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a (non-empty) spanning set for V.
$\boxed{\Longrightarrow}$: Let S be a minimal spanning set. We show that S is linearly independent. We do this by contradiction: assume that S is linearly dependent. By Lemma 3.7, $\text{span}S = \text{span}S \setminus \{\mathbf{v}_j\}$ for some j. Hence, $\text{span}S = V = \text{span}S \setminus \{\mathbf{v}_j\}$ and therefore S is not a minimal spanning set, as it does not satisfy Definition 2.5.
$\boxed{\Longleftarrow}$: Let S be a linearly independent spanning set. We show that S is a minimal spanning set for V. We do this by contradiction: assume that S is not a minimal spanning set. Then there exists an index j such that $\text{span}S \setminus \{\mathbf{v}_j\} = V$. But $V = \text{span}S$. Hence, there exists an index j such that $\text{span}S = \text{span}S \setminus \{\mathbf{v}_j\}$ and by Lemma 3.7 this implies that S is a linearly dependent set – a contradiction. ∎

3.3 Adding vectors. Maximal linearly independent sets.

Lemma 3.7 describes the change in the span of a set S under the removal of a vector from S. What happens when we augment S with a given vector? The following result provides an answer.

Lemma 3.9 Let S be linearly independent and let $\mathbf{v}$ be given. Then $S \cup \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \notin \text{span}S$.

Proof. $\boxed{\Longrightarrow}$: let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ and $S \cup \{\mathbf{v}\}$ be linearly independent sets. We show that $\mathbf{v} \notin \text{span}S$. We do this by contradiction: assume $\mathbf{v} \in \text{span}S$. Then

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \Longleftrightarrow a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k + (-1)\mathbf{v} = \mathbf{0}.$$

Hence $\mathbf{0}$ is a non-trivial linear combination of the vectors in the set $S \cup \{\mathbf{v}\}$; therefore, by definition, this set is linearly dependent, which contradicts our initial assumption.
$\boxed{\Longleftarrow}$: let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be linearly independent and assume $\mathbf{v} \notin \text{span}S$. We show that $S \cup \{\mathbf{v}\}$ is linearly independent. We do this again by contradiction: assume that $S \cup \{\mathbf{v}\}$ is linearly dependent. Then

$$\mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k + b\mathbf{v},$$

where not all coefficients are zero. We consider two cases:
(i) $b = 0$: we get

$$\mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k,$$

which implies $a_i = 0$ for all i, since S is linearly independent. Therefore, $\mathbf{0}$ is a trivial linear combination of vectors in $S \cup \{\mathbf{v}\}$, which implies that this set is linearly independent – a contradiction.
(ii) $b \neq 0$: in this case we can write $\mathbf{v}$ as follows:

$$\mathbf{v} = \frac{a_1}{b}\mathbf{v}_1 + \frac{a_2}{b}\mathbf{v}_2 + \cdots + \frac{a_k}{b}\mathbf{v}_k.$$

Hence $\mathbf{v} \in \text{span}S$ – a contradiction.
We conclude that $S \cup \{\mathbf{v}\}$ must be linearly independent. ∎
Let us consider some examples.

Example 3.6 Let $p_j(x) = x^j$, for $j = 0, 1, \dots, k$. Let $S = \{p_0, p_1, \dots, p_k\}$. Then S is linearly independent; one can immediately see that $\text{span}S = \mathcal{P}_k$, the space of polynomials of degree k. We can augment S with p_{k+1}, which is not in $\text{span}S$; this results in the set $S \cup \{p_{k+1}\}$ which spans $\mathcal{P}_{k+1}$. Thus, polynomial spaces can be constructed by augmenting existing sets of monomials with monomials of higher degree.

Example 3.7 Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set and $\mathbf{v} \notin \operatorname{span} S$. By Lemma 3.9, the set $S \cup \{\mathbf{v}\}$ is linearly independent, a fact which we can verify directly.

Lemma 3.9 allows us to construct linearly independent sets of ever larger cardinality, by choosing new vectors in a suitable way. We can therefore augment successively linearly independent sets, while preserving the property of independence. However, this cannot be done indefinitely as otherwise V would have to be an infinite-dimensional vector space, which we excluded from our discussion. This observation suggests the following definition (compare it with Definition 2.5).

Definition 3.4 — Maximal linearly independent set. Let S be a linearly independent set of vectors from a vector space $V(\mathbb{F})$.

i. If $V(\mathbb{F})$ is trivial, the maximal linearly independent set if defined to be the empty set.
ii. If $V(\mathbb{F})$ is non-trivial, S is called a maximal linearly independent set in V if

$$S \cup \{\mathbf{v}\} \text{ is linearly dependent for any } \mathbf{v} \in V \setminus \{\mathbf{0}\}.$$

R The above definition considers separately the case where $V(\mathbb{F})$ is the trivial vector space, as there are no non-zero vectors to augment S with (so the criterion for the non-trivial case ii. does not make sense). Note that the definition 'works' if $V(\mathbb{F})$ is the trivial vector space: by Proposition 3.2, $S = \emptyset$ is linearly independent.

We end with the following characterisation of maximal linearly independent sets.

Theorem 3.10 Let S be a linearly independent set of vectors from a finite-dimensional vector space $V(\mathbb{F})$. Then S is a maximal linearly indepedent set in V if and only it is a spanning set for V.

Proof. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a linearly independent set in V.
$\boxed{\Longrightarrow}$ Let S be maximal. Let $\mathbf{v} \in V \setminus \{\mathbf{0}\}$. By the definition of maximal set, the set $S \cup \{\mathbf{v}\}$ is linearly dependent. Hence, there exist scalars $a_1, a_2, \ldots, a_k$, not all zero, such that

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k.$$

Since $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ is arbitrary, we conclude that $V \setminus \{\mathbf{0}\} \subseteq \operatorname{span} S$. Given that $\mathbf{0} \in \operatorname{span} S$, we conclude that $V \subseteq \operatorname{span} S$. On the other hand, $\operatorname{span} S \subseteq V$ (by closure). Hence, $V = \operatorname{span} S$ and therefore S is a spanning set for V.
$\boxed{\Longleftarrow}$ Let S be a linearly indepedent spanning set. We need to show that S is maximal. By Theorem 3.8, S is a minimal spanning set for V and therefore any $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ can be written as a non-trivial linear combination

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \Longleftrightarrow \mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k - \mathbf{v},$$

which means that the zero vector can be written as a non-trivial linear combination of vectors in the set $S \cup \{\mathbf{v}\}$. Therefore, this set is linearly dependent. Hence, by Definition 3.4, S is a maximal linear independent set in V. ■

The above discussion indicates that minimal spanning sets and maximal linearly independent sets are special and should be further investigated.