LECTURE 9

Linear transformations (2)

9.1 Subspaces

It is natural to ask what happens when we try to map a subspace U of V: is $f(U)$ also a subspace of $f(V)$? Are there some obvious and/or some special cases? We consider these matters below. We first consider the image and the kernel of a linear map.

Proposition 9.1 Let $f : V \to W$ be a linear map. Then
i. $\operatorname{im} f$ is a subspace of W: $\operatorname{im} f \leq W$.
ii. $\ker f$ is a subspace of V: $\ker f \leq V$.

Proof. First, note that since f is a linear map, we have $f(\mathbf{0}_{\mathrm{V}}) = \mathbf{0}_{\mathrm{W}}$. Therefore, $\operatorname{im} f$ and $\ker f$ are non-empty subsets of W and V, respectively. This allows us to apply the subspace criterion 2 in both cases.
i. Let $\mathbf{w}_1, \mathbf{w}_2 \in \operatorname{im} f$; then there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{w}_1 = f(\mathbf{v}_1), \mathbf{w}_2 = f(\mathbf{v}_2)$. Then, for any scalars $a_1, a_2 \in \mathbb{F}$, we have

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 = a_1 f(\mathbf{v}_1) + a_2 f(\mathbf{v}_2) = f(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) =: f(\mathbf{v}) \in \operatorname{im} f.$$

Hence, $\operatorname{im} f$ is a subspace of W: $\operatorname{im} f \leq W$.
ii. Let $\mathbf{v}_1, \mathbf{v}_2 \in \ker f$. Then, for any scalars $a_1, a_2 \in \mathbb{F}$, the vector $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ is in $\ker f$ since

$$f(\mathbf{v}) = f(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) = a_1 f(\mathbf{v}_1) + a_2 f(\mathbf{v}_2) = a_1\mathbf{0}_{\mathrm{W}} + a_2\mathbf{0}_{\mathrm{W}} = \mathbf{0}_{\mathrm{W}}.$$

Hence, $\ker f$ is a subspace of V: $\ker f \leq V$. ∎

Proposition 9.2 Let $U \leq V$. Then $f(U) \leq W$.

Proof. The proof is left as an exercise. ∎

9.2 Spans, bases, dimension

Given a spanning set S for a vector space V, by definition, $\operatorname{im} f = f(V) = f(\operatorname{span} S)$. The following result confirms that spanning sets are sufficient to describe the image of a map in the following sense.

Proposition 9.3 Let $f: V \to W$ be a linear map and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in V. Then $f(\text{span}S) = \text{span}f(S)$.

Proof. Let $W \ni \mathbf{w}_i = f(\mathbf{v}_i), i = 1, 2, \ldots, k$. With this notation, the result follows from the identity

$$f\left(\sum_{i=1}^{k} a_i \mathbf{v}_i\right) = \sum_{i=1}^{k} a_i f(\mathbf{v}_i) = \sum_{i=1}^{k} a_i \mathbf{w}_i.$$

■

R This result holds, in particular, for the case where S is a basis for V.

We can establish similar or related results for linearly independent sets.

Proposition 9.4 Let $f: V \to W$ be a linear map with trivial kernel. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \in V$. Then S is a linearly independent set in V if and only if $f(S)$ is a linearly independent set in W.

Proof. We have

$$\mathbf{0}_{\mathrm{V}} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \iff \mathbf{0}_{\mathrm{W}} = a_1 f(\mathbf{v}_1) + a_2 f(\mathbf{v}_2) + \cdots + a_k f(\mathbf{v}_k).$$

Note that the reverse implication always holds as $f(\mathbf{0}_{\mathrm{V}}) = \mathbf{0}_{\mathrm{W}}$ by linearity of f, while the direct implication holds since the kernel of f is trivial, i.e., $\mathbf{0}_{\mathrm{W}} = f(\mathbf{v})$ only if $\mathbf{v} = \mathbf{0}_{\mathrm{V}}$. The result then follows from the above equivalence: a linear combination in V is trivial if and only if it is trivial in W. ■

Note that without the assumption on the kernel of f, we can only show the following.

Proposition 9.5 Let $f: V \to W$ be a linear map and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. If $f(S)$ is a linearly independent set in W, then S is a linearly independent set in V.

Corollary 9.6 Let $f: V \to W$ be a linear map with trivial kernel. Let S be a linearly independent set in V. Then $\dim f(\text{span}S) = |S|$.

Proof. By Proposition 9.4, $f(S)$ is a linearly independent set in W and therefore a basis for the subspace U of W that it spans, namely $U = \text{span}f(S)$. By Proposition 9.3, $U = \text{span}f(S) = f(\text{span}S)$. Hence $\dim f(\text{span}S) = |f(S)| = |S|$. ■

9.3 Rank and nullity

Definition 9.1 — Rank and nullity. Let $f: V \to W$ be a linear map.
The **rank** of f is the dimension of the image of f: $\text{rank } f = \dim \text{ im } f$.
The **nullity** of f is the dimension of the kernel of f: $\text{nullity } f = \dim \ker f$.

Example 9.1 If $f = o$ (the zero map), then $\text{im } o = \{\mathbf{0}_{\mathrm{W}}\}$, so that $\text{rank } o = 0$. Since for all $\mathbf{v} \in V$, $o(\mathbf{v}) = \mathbf{0}_{\mathrm{W}}$, $\text{nullity } o = \dim V$. On the other hand, if $f = id$, then $\text{im } id = V$, so that $\text{rank } id = \dim W = \dim V$. Finally, since $id(\mathbf{0}_{\mathrm{V}}) = \mathbf{0}_{\mathrm{V}}$, the kernel of id is trivial and hence $\text{nullity } id = 0$.

The following result contains observations based on previous definitions.

Proposition 9.7 Let $f: V \to W$ be a linear map, where V is a finite dimensional vector space. Then

$$0 \le \text{rank } f \le \dim W, \qquad 0 \le \text{nullity } f \le \dim V.$$

We are now ready to prove the following fundamental result.

Theorem 9.8 — Rank-nullity formula. Let V be an $n-$dimensional vector space. Let $f : V \to W$ be a linear map. Then

$$\text{rank } f + \text{nullity } f = n.$$

Proof. Let B denote a basis set for V containing a basis B_1 for $\ker f$ (see Proposition 4.3). Denote by B_2 the complement of B_1 in B; then $B = \{B_1, B_2\}$, where, by construction,

- B_1 and B_2 are disjoint sets;
- B_2 is a linearly independent set.

Define $k := |B_1| = \dim \ker f = \text{nullity } f$ and $r := |B_2|$. With this notation,

$$n = \dim V = |B| = |B_1| + |B_2| = k + r = \text{nullity } f + r.$$

Claim: $r = \text{rank } f$. To see this, consider the linear map $\tilde{f} : \text{span} B_2 \to W$ defined via $\tilde{f}(\mathbf{v}) = f(\mathbf{v})$ for $\mathbf{v} \in \text{span} B_2$. Note that $f(B_2) = \tilde{f}(B_2)$. Since the kernel of $\tilde{f}$ is trivial due to the disjointness of B_1 and B_2, we can use Proposition 9.4 to deduce that $\tilde{f}(B_2)$ is a linearly independent set in W. Moreover, it is a spanning set for im f since

$$\text{im } f = f(V) = f(\text{span} B) = \text{span} f(B) = \text{span}\{f(B_1), f(B_2)\} = \text{span}\{\mathbf{0}_{\mathrm{W}}, f(B_2)\} = \text{span} f(B_2) = \text{span} \tilde{f}(B_2),$$

where we used the result of Proposition 9.3. Hence, $\tilde{f}(B_2)$ is a basis for im f and, by definition,

$$\text{rank } f = \dim \text{span} \tilde{f}(B_2) = \left|\tilde{f}(B_2)\right| = |B_2| = r$$

and the result follows. ∎

We end this lecture with the following results on injectivity and surjectivity.

Proposition 9.9 Let V be an $n-$dimensional vector space. Let $f : V \to W$ be a linear map.
i. If $\dim V > \dim W$, then f is not injective.
ii. If $\dim V < \dim W$, then f is not surjective.

Proof. Both results are consequences of the rank-nullity formula.
i. Let $\dim V > \dim W$. Then f is not injective as the kernel of f is not trivial since

$$\dim \ker f = \text{nullity } f = n - \text{rank } f \geq n - \dim W > 0$$

ii. Let $\dim V < \dim W$. Then f is not surjective as im $f \neq W$ since

$$\dim \text{im } f = \text{rank } f = n - \text{nullity } f \leq \dim V < \dim W.$$

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