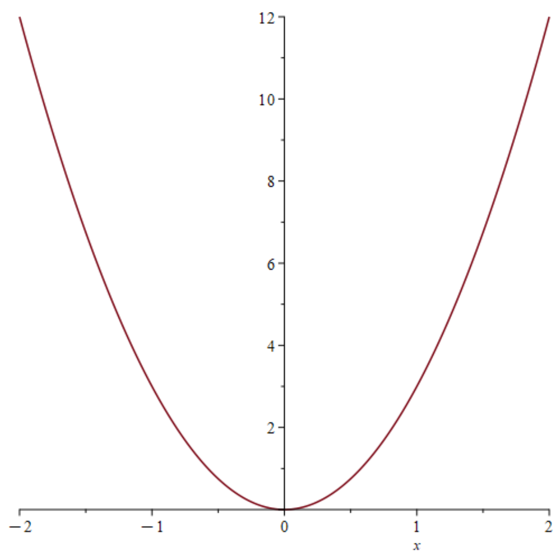
Example sheet 0 - formative

1. sketch the graphs of the following functions:

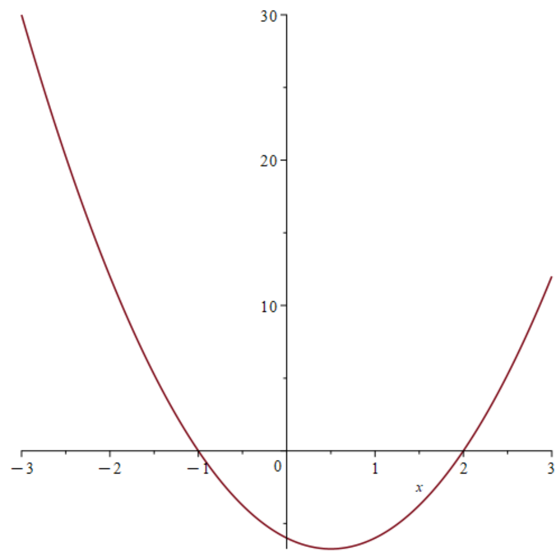
   (a) $f(x) = 3x^2$;
   (b) $y = f(x) = 3x^2 - 3x - 6$;
   (c) $y = -3x^2 + 3x - 6$;
   (d) $y = 3x^2 + 3x + 6$;
   (e) $y = 3x^3 - 6x^2 - 3x + 6$;
   (f) $y = -2x^3 + 22x^2 - 80x + 96$;
   (g) $y = -2x^3 + 4x^2 + 2x + 2$;
   (h) $y = 3x^3 - 6x^2 - 3x + 16$;
   (i) $y = 3x^4 - 15x^2 + 12$;
   (j) $y = 3x^4 - 15x^2 + 22$.

Solution:

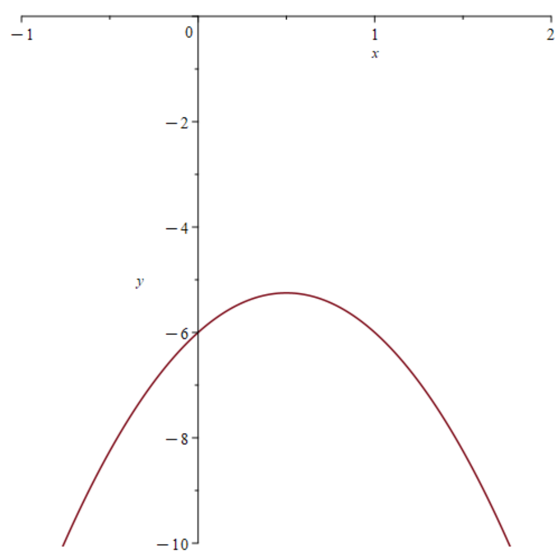
(a) Note that $f(0) = 0$, that there is a minimum at $x = 0, y = 0$ as $f'(0) = 0$ and $f''(0) = 6 > 0$. The sketch is given below:



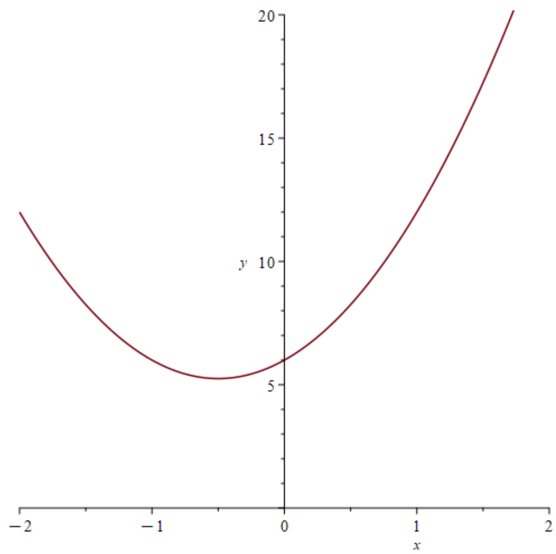
(b) Note that $f(-1) = 0, f(2) = 0$, that there is a minimum at $x = \frac{1}{2}, y = -\frac{27}{4}$ as $f'\left(\frac{1}{2}\right) = 0$ and $f''\left(\frac{1}{2}\right) = 6 > 0$. The sketch is given below:



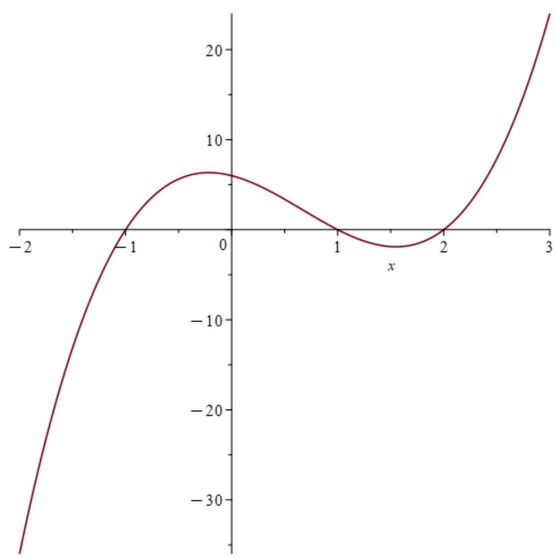
(c) Note that the coefficient of the quadratic term is negative, that there are no zero's and that there is a maximum at $x = \frac{1}{2}, y = -\frac{21}{4}$ as $f'\left(\frac{1}{2}\right) = 0$ and $f''\left(\frac{1}{2}\right) = -6 < 0$. The sketch is given below:



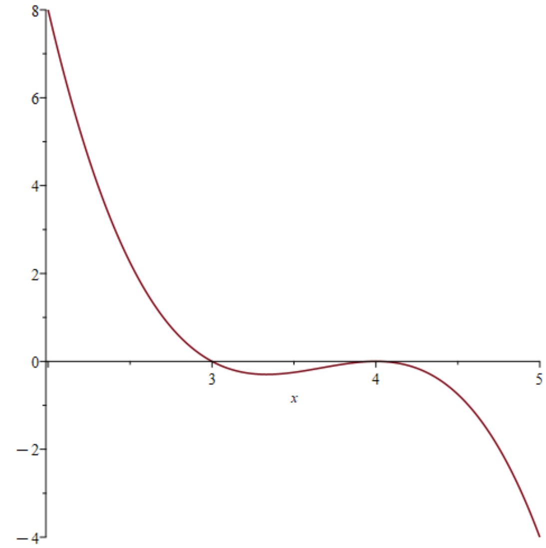
(d) Note that the coefficient of the quadratic term is positive, that there are no zero's and that there is a minimum at $x = -\frac{1}{2}, y = \frac{21}{4}$ as $f'\left(\frac{1}{2}\right) = 0$ and $f''\left(\frac{1}{2}\right) = 6 > 0$. The sketch is given below:



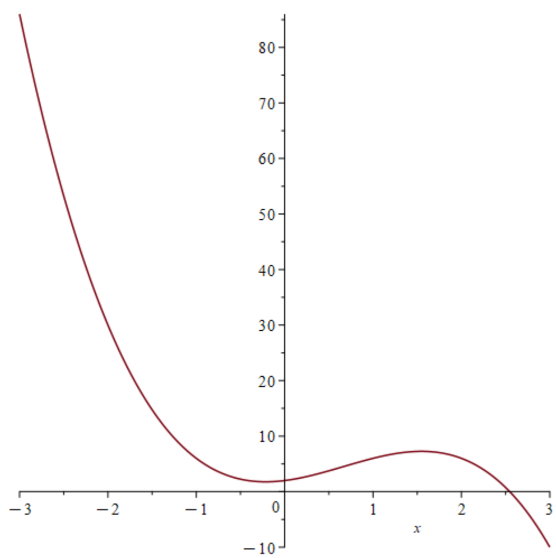
(e) Note that the coefficient of the cubic term is positive, that there are zero's at $x = -1, 1, 2$ and that there is a maximum at $x = \frac{2-\sqrt{7}}{2}$ and a minimum at $x = \frac{2+\sqrt{7}}{2}$. The values of y at the extrema are a bit more complex and are not needed for a rough sketch. However, where more than one curve needs to be plotted, it may be necessary to calculate such values to understand the mutual positioning. The sketch is given below:



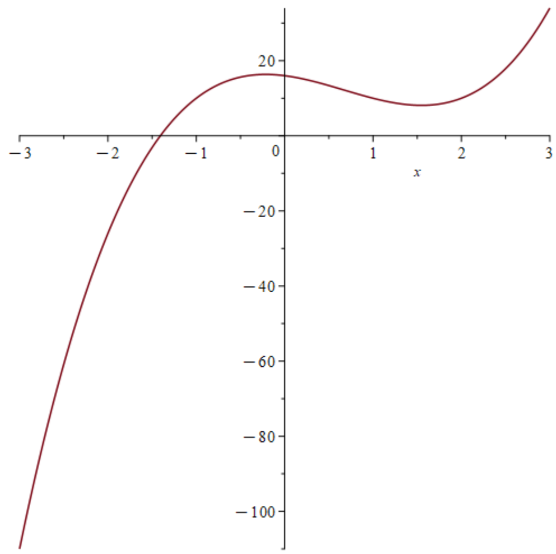
(f) Note that the coefficient of the cubic term is negative, that there are zero's at $x = 3$ and $x = 4$ (twice) and that there is a minimum at $x = \frac{10}{3}$ and a maximum at $x = 4$. The sketch is given below:



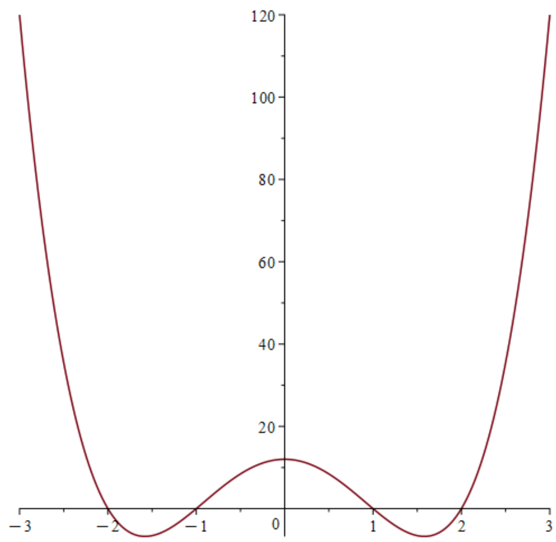
(g) Note that the coefficient of the cubic term is negative, that there is a minimum at $x = \frac{2-\sqrt{7}}{3}$ and a maximum at $x = \frac{2+\sqrt{7}}{3}$. There hence is a zero as well in the interval $\left(\frac{2+\sqrt{7}}{3}, \infty\right)$. Checking the sign of the function, one can narrow this down easily to the interval $(2, 3)$. The sketch is given below:



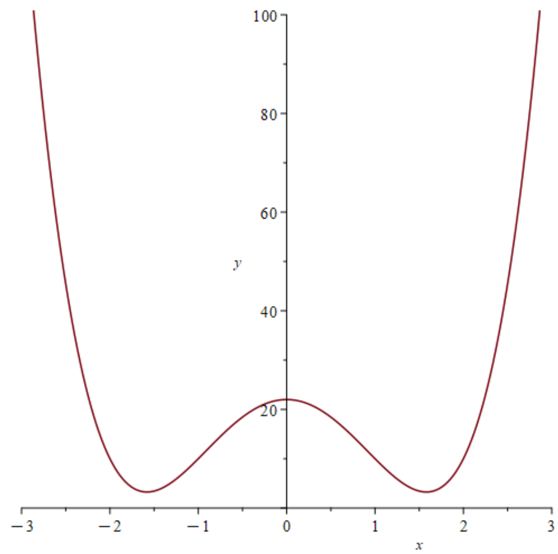
(h) Note that the coefficient of the cubic term is positive, that there is a maximum at $x = \frac{2-\sqrt{7}}{3}$ and a minimum at $x = \frac{2+\sqrt{7}}{3}$. There hence is a zero as well in the interval $\left(-\infty, \frac{2-\sqrt{7}}{3}\right)$. Checking the sign of the function, one can narrow this down easily to the interval $(-2, -1)$. The sketch is given below:



(i) Note that the coefficient of the fourth power is positive, that there are zero's at $x = -2, -1, 1, 2$ and a maximum at $x = 0$ and a minima at $x = -\frac{\sqrt{10}}{2}$ and $x = \frac{\sqrt{10}}{2}$. The sketch is given below:



(j) Note this function only differs from the previous one in the constant term. This means it is shifted vertically, in this case upwards by 10 units. Since the minima in the previous function happen at $\left(-\frac{\sqrt{10}}{2}, -\frac{27}{4}\right)$ and $\left(\frac{\sqrt{10}}{2}, -\frac{27}{4}\right)$, and, $10 > \frac{27}{4}$, this function has no zero's but keeps the same shape. The sketch is given below:



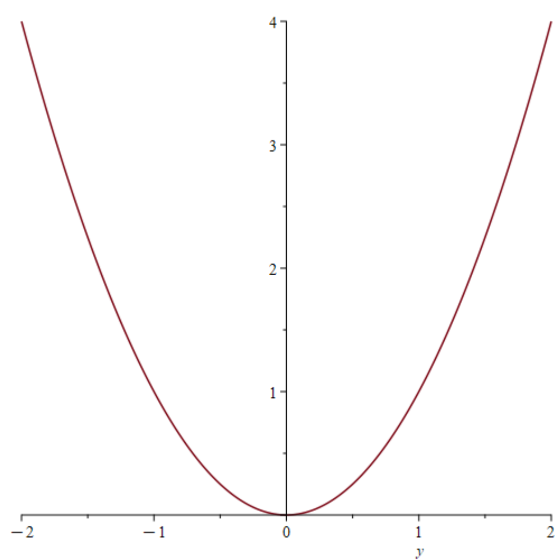
2. Sketch the following curves first on a $x - y$ plane (y on the horizontal axis) and then on a $y - x$ plane (x on the horizontal axis):

(a) $x = y^2$;

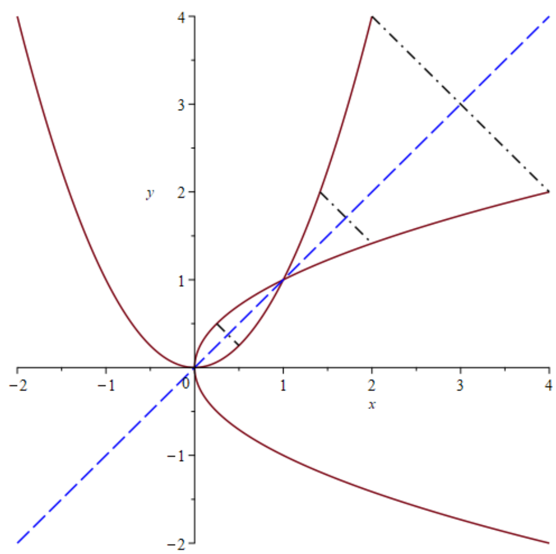
(b) $x = 3y^2 - 3y - 6$.

Solution:

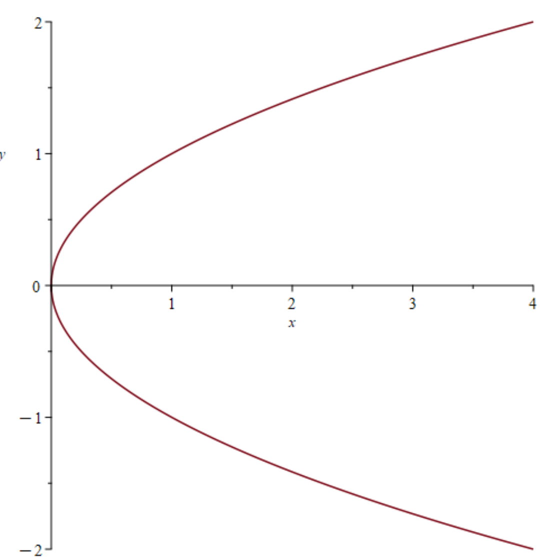
(a) The curve is easily drawn on a $x - y$ plane as given below.



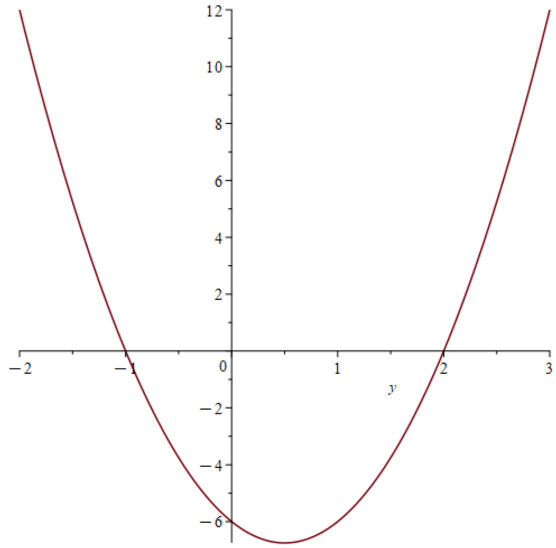
We then need to realise that to draw this curve on a $y-x$ plane, we need to reflect it about the first bisectrix $(y=x)$, as illustrated below:



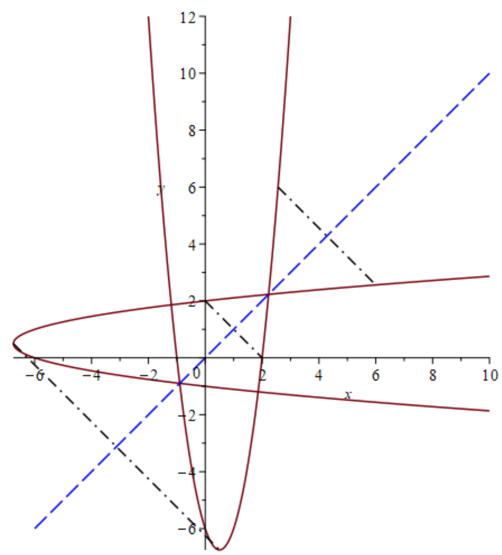
This gives us the graph of the curve in the $y-x$ plane:



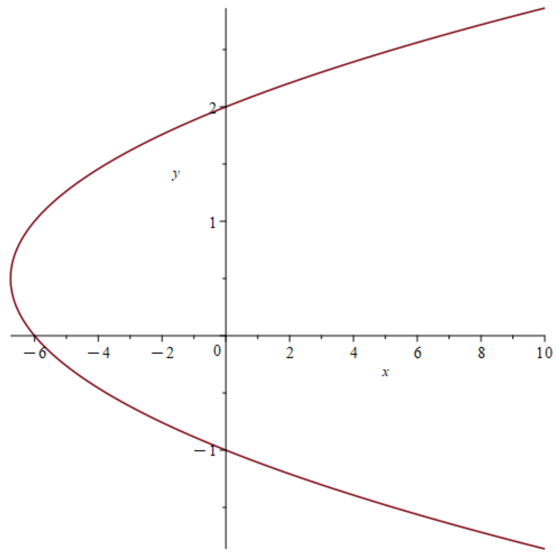
(b) In the $x-y$ plane, this curve has a positive coefficient of the quadratic term (in y) and zero's at $y=-1$ and $y=2$, which gives the following sketch:



We then need to realise that to draw this curve on a $y - x$ plane, we need to reflect it about the first bisectrix $(y = x)$, as illustrated below:



This gives us the graph of the curve in the $y - x$ plane:

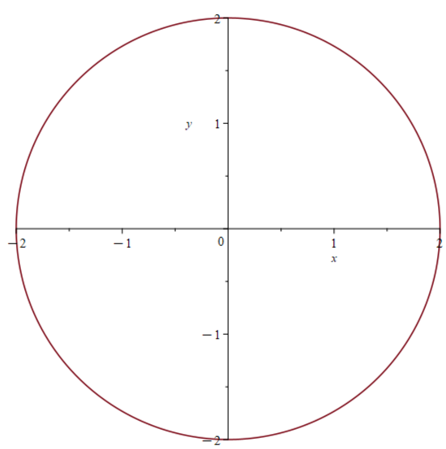


3. Sketch the following curves:

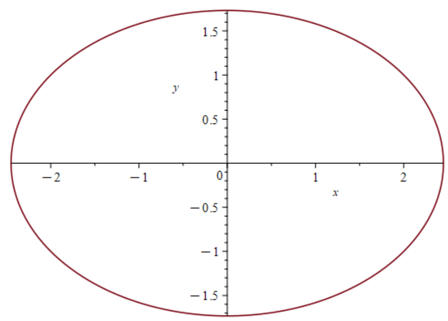
   (a) $3x^2 + 3y^2 = 12$;
   (b) $3x^2 + 6y^2 = 18$;
   (c) $3x^2 - 6y^2 = 3$;
   (d) $3x^2 - 6y^2 = 0$.

Solution:

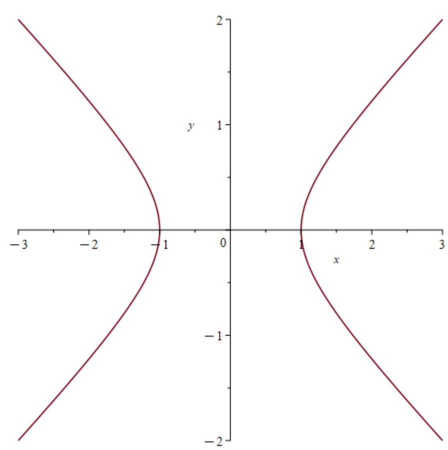
(a) Dividing the equation on both sides by 3, we get the equation $x^2 + y^2 = 4$, which is easily recognised as a circle with center at $(0, 0)$ and radius 2:



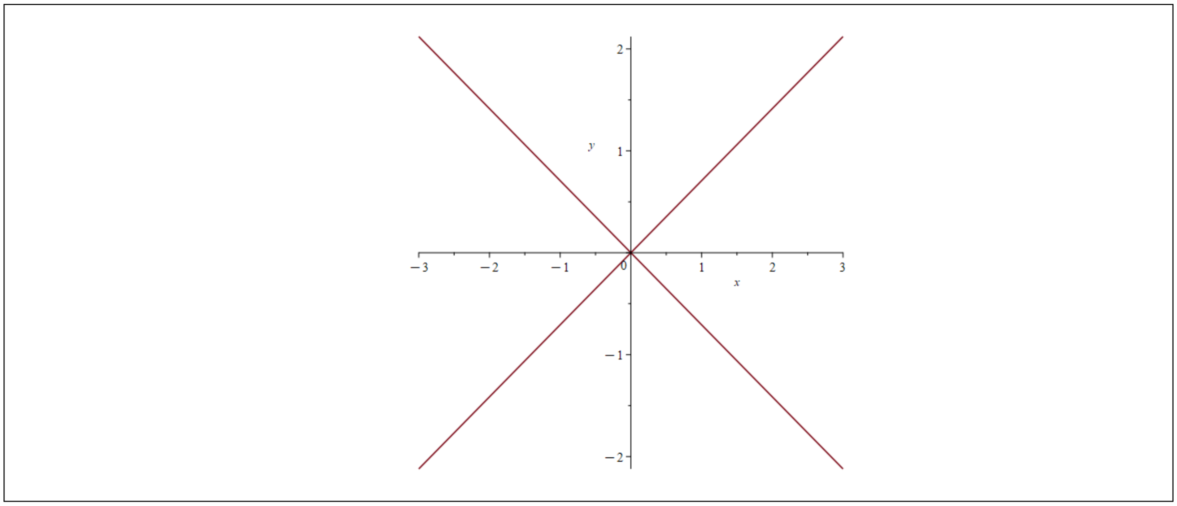
(b) Dividing the equation on both sides by 3, we get the equation $x^2 + 2y^2 = 6$, which is easily recognised as an ellipse with vertices $\left(0, -\sqrt{3}\right)$, $\left(0, \sqrt{3}\right)$, $\left(-\sqrt{6}, 0\right)$, and $\left(\sqrt{6}, 0\right)$:



(c) This is the equation of an hyperbola with asymptotes $y = \frac{x}{\sqrt{2}}$ and $y = -\frac{x}{\sqrt{2}}$, and vertices at $(-1, 0)$ and $(1, 0)$:



(d) Note that $3x^2 - 6y^2 = 3\left(x - \sqrt{2}y\right)\left(x + \sqrt{2}y\right) = 0$ so the curve consists of two straight lines;



y
x