1Mech — Mechanics

Mechanics exercises 5 (weeks 9 and 10)

This sheet's assessed question is #5.

1. A smooth sphere of radius a has its centre at the origin. If (ρ, θ, z) give cylindrical polar coordinates, with the z axis pointing vertically upwards, the surface of the sphere is given by $\rho^2 + z^2 = a^2$. A particle of mass m moves on the interior surface of the sphere under the action of gravity, with position vector $\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$, where $\mathbf{e}_\rho$, $\mathbf{e}_z$ are basis vectors pointing in the $\rho(t)$ and $z(t)$ direction respectively, with normal reaction $\mathbf{R}$ acting purely perpendicular to the surface.

   (a) Show that the particle's velocity satisfies $|\dot{\mathbf{r}}|^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$.

   (b) Hence show that

   $$\begin{aligned} \rho^2\dot{\theta} &= h, \\ \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) + mgz &= E, \end{aligned}$$

   where h and E are constants. What do these expressions represent physically?

   (c) Show that

   $$2\rho\dot{\rho} + 2z\dot{z} = 0,$$

   and hence

   $$\dot{\rho}^2 = \frac{z^2\dot{z}^2}{a^2 - z^2}.$$

   (d) If the particle is initially located at $\rho = a/\sqrt{2}$, $z = -a/\sqrt{2}$, moving with speed V in the direction of the horizontal tangent of the surface, find the values of h and E.

   (e) Hence show that the motion satisfies

   $$a^2\dot{z}^2 = \left(z + a/\sqrt{2}\right)\left(2g\left(z^2 - a^2\right) - V^2\left(z - a/\sqrt{2}\right)\right).$$

   (f) By differentiating to find an expression for $\ddot{z}$, show that the particle will initially rise if $V^2 > ag/\sqrt{2}$.

   (g) [**Optional extension**] Suppose instead that the particle is initially located at $\rho = a$, $z = 0$, and is projected vertically with speed V. Find the smallest value of V that will allow the particle to reach $z = a$.

2. A smooth surface of revolution has equation $z = a^2/\rho$ where $a > 0$ is a constant, (ρ, θ, z) are cylindrical polar coordinates, with z pointing downwards. A small particle of mass m slides on the interior of the surface.

   (a) Briefly explain why

   $$\begin{aligned} \rho^2\dot{\theta} &= h, \\ \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz &= E, \end{aligned}$$

   are both constants.

(b) If the particle initially moves horizontally with velocity $\rho\dot{\theta} = a\omega > 0$ at depth $z = a$ below the origin, show that

$$\rho^2\dot{\theta} = a^2\omega, \tag{1}$$

$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz = \frac{1}{2}m\omega^2 a^2 - mga. \tag{2}$$

(c) By rewriting every term in (2) in terms of z and/or $\dot{z}$, show that the particle moves between $z = a$ and $z = 2g/\omega^2 - a$. Hence show that the particle moves in a circle if $g = a\omega^2$.

3. A rocket of mass m launches from rest and expels exhaust gases at speed u relative to the rocket's motion. The rocket burns fuel such that the total mass of the rocket over time is given by $m = m_0 e^{-bt}$ where $b > g/u$ is a constant, and moves under the action of gravity g and air resistance assumed to be of the form kmv^2 where v is the velocity of the rocket.

   (a) Find the velocity of the rocket as a function of time.

   (b) Hence find the limiting velocity of the rocket as time tends to infinity.

4. **[Challenging; optional]** Two railway workers, each of mass m, are standing on a frictionless railway cart of mass M which is on a horizontal track and is initially stationary. The railway workers run from the front of the cart and jump off of the rear of the cart with a speed u relative to the cart.

   (a) What is the final speed of the cart if both workers jump simultaneously?

   (b) What is the final speed of the cart if the workers do not jump at the same time (i.e. the second worker only jumps after the first worker has already jumped off the cart).

   (c) If instead there are N workers, what is the final speed if they all jump sequentially?

5. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

   A particle of mass m moves without friction on the inner surface of a bowl, whose shape is given in cylindrical polar coordinates (ρ, θ, z) by

$$z = \frac{\rho^2}{a},$$

   where $a > 0$ is a constant and the z-axis points vertically upwards. This shape is known as a *paraboloid* since it is obtained by revolving a parabola around its axis. The particle is released from a point where $z = z_0$ for some $z_0 > 0$, with initial velocity $v\mathbf{e}_\theta$ for some $v \geq 0$.

   You may assume the following expression for the particle's velocity:

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z.$$

(a) Using the conservation of angular momentum and of energy, show that the particle's z-coordinate obeys the following equation (which is valid for $z > 0$):

$$\dot{z}^2 \left(1 + \frac{a}{4z}\right) = -2gz + 2gz_0 + v^2 - \frac{v^2 z_0}{z}. \tag{3}$$

You may assume any generic forms of the conservation equations, as long as you state them clearly.

(b) By writing the right-hand side of (3) as

$$-\frac{2g}{z}(z - z_1)(z - z_2),$$

for some constants z_1 and z_2 which you should determine, show that the particle moves in a circle if $v = \sqrt{2gz_0}$ and find the period of the circular motion.

(c) By differentiating (3) with respect to t, show that:

i. If $v > \sqrt{2gz_0}$, then the particle rises initially, and it will begin to fall again if and when it reaches the height $z = v^2/(2g)$.

ii. If $v < \sqrt{2gz_0}$, then the particle falls initially, and it will begin to rise again if and when it reaches the height $z = v^2/(2g)$.

(d) In the special case $v = 0$, describe the motion of the particle. Be as specific as you can, but you do not need to justify your answer.