Example sheet 2 – formative

1. Find and classify all equilibrium points for the following dynamical systems. Hence, determine the bifurcation point(s) and sketch the bifurcation diagram.
   (a) $\dot{y} = cy + 4y^3$
   (b) $\dot{x} = c + x - \dfrac{x}{1+x}$, for $c < 4$, $x \neq -1$
   (c) $\dot{x} = cx - \log(1+x)$, for $x \geqslant -1$

Solution:

(a) $\dot{y} = cy + 4y^3$

The equilibrium points are given by

$$0 = cy + 4y^3 = y\left(c + 4y^2\right),$$

so $y = 0$ is an equilibrium point for all values of c. We then have that the solutions of $c + 4y^2 = 0$ are given by

$$y^* = \pm\frac{\sqrt{-16c}}{8} = \pm\frac{1}{2}\sqrt{-c}.$$

We then have the following cases:

(i) $c < 0$: The system has three equilibrium points, $y_1^* = 0$, $y_{2,3}^* = \pm\frac{1}{2}\sqrt{-c}$
(ii) $c \geqslant 0$: The system has one equilibrium point $y_1^* = 0$

Using linear stability analysis, writing $\dot{y} = f(y)$, we have $f'(y) = c + 12y^2$. Thus the equilibrium point $y_1^* = 0$ is stable if $c < 0$ and unstable if $c > 0$.

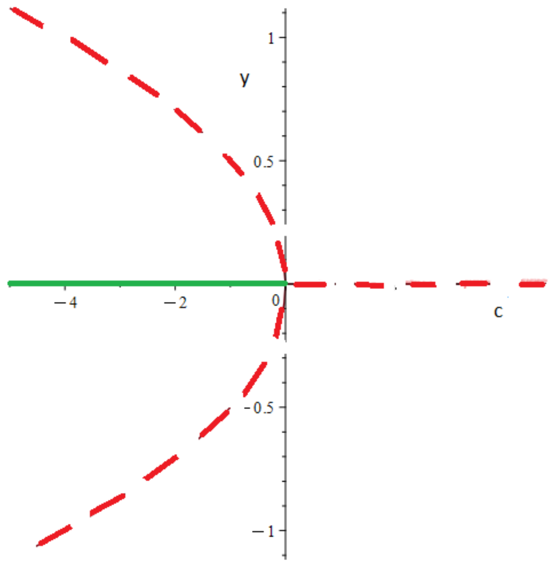
At $y_{2,3}^* = \pm\frac{1}{2}\sqrt{-c}$,

$$f'\left(y_{2,3}^*\right) = c + 12\left(\pm\frac{1}{2}\sqrt{-c}\right)^2 = c + 3(-c) = -2c.$$

This is positive for $c < 0$ so both equilibrium points are unstable.

Thus the point $c = 0$ is a subcritical pitchfork bifurcation.

For the bifurcation diagram, noticce that $c = -4y^2$. The bifurcation diagram is given below.



(b) $\dot{x} = c + x - \dfrac{x}{1+x}$, for $x \geqslant 0$

The equilibrium points of the system are solutions of

$$x^2 + cx + c = 0,$$

which are given by

$$x^* = \frac{-c \pm \sqrt{c^2 - 4c}}{2}.$$

Now, since $c^2 - 4c < 0$ for $0 < c < 4$, there are no equilibrium points for these values of c. We then have the following cases:

(i) $4 > c > 0$: The system has no equilibrium points.

(ii) $c = 0$: The system has one equilibrium point at $x^* = 0$.

(iii) $c < 0$: The system has two equilibrium points at $x^*_{1,2} = \frac{-c \pm \sqrt{c^2-4c}}{2}$.

Using linear stability analysis, we have that

$$f'(x) = 1 - \frac{1}{1+x} + \frac{x}{(1+x)^2} = 1 - \frac{1}{(1+x)^2}.$$

Hence, at $x^* = 0$, $f'(0) = 0$, so we need to use a graphical approach. This would also be helpful for determining the stability of the equilibria for $c < 0$ as the algebra becomes quite involved. So let us look at the function

$$c + x - \frac{x}{1+x}.$$

The constant c merely means a shift up or down, so let us look in detail at the case $c = 0$ as the other cases can be easily derived from that. We can readily see that there is a zero at $x = 0$ and a vertical asymptotte at $x = -1$. Also

$$\lim_{x\to\infty} \frac{x^2}{1+x} = \infty, \quad \lim_{x\to-\infty} \frac{x^2}{1+x} = -\infty.$$

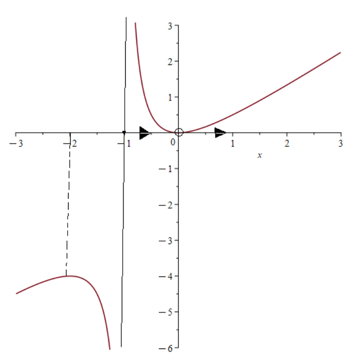
Near the vertical asymptote we have

$$\lim_{x\to-1^-} \frac{x^2}{1+x} = -\infty, \quad \lim_{x\to-1^+} \frac{x^2}{1+x} = \infty.$$

The maxima and minima can be found by looking for the zero's of the derivative:

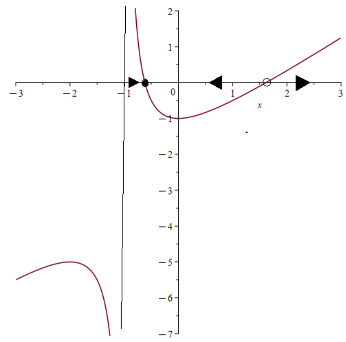
$$\left(\frac{x^2}{1+x}\right)' = \frac{2x}{1+x} - \frac{x^2}{(1+x)^2} = \frac{x\,(2+x)}{(1+x)^2}.$$

So we have zero's at $x = -2$ (local maximum) and at $x = 0$ (local minimum). We can use the second derivative to establish the nature of these extrema. Hence, for $c = 0$, the phaseline looks like



From the figure above we see that the equilibrium at $x^* = 0$ for $c = 0$ is halfstable.

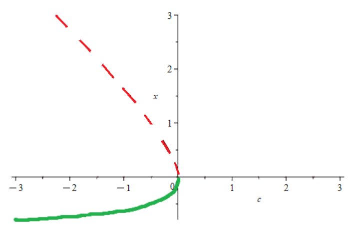
As $c < 4$, any vertical shift upwards ($c > 0$) will not reach the point where the maximum touches the x-axis. However, any shift downwards ($c < 0$) will produce the two equilibria consistent with our analysis above. The phaseline for $c = -1$ is given below and is qualitatively identical to all other cases for $c < 0$. Hence the equilibrium at x_1^* is stable and that at x_2^* is unstable, where $x_1^* < x_2^*$.



Thus the bifurcation point is $c = 0$ and is a saddle-node bifurcation as the two equilibria for $c < 0$ annihilate each other at $c = 0$ so there is no equilibrium point for $0 < c < 4$. As for the equilibria,

$$c = -\frac{x^2}{1+x},$$

we can obtain the bifurcation diagram by first rotating the grpah above (for $c = 0$) with respect to the x-axis and then flipping the axis over. This yields the bifurcation diagram below.

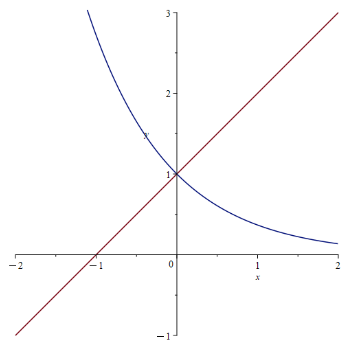


(c) $\dot{x} = cx - \log(1+x)$, for $x \geqslant 0$

The equilibrium points are solutions of

$$\log(1+x) = cx \Leftrightarrow 1 + x = e^{cx}.$$

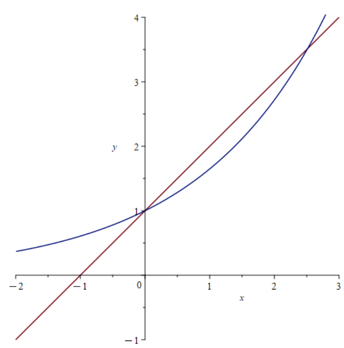
Now, clearly there is an equilibrium point at $x^* = 0$ for all values of c. We can ploit both left hand side and right hand side for difference values of c to understand how many times both will intersect. For $c < 0$, it is clear from the typical plot below that there is only the one point of intersection, i.e. $x^* = 0$:



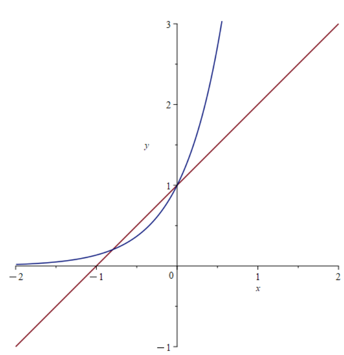
For $c = 0$, $x^* = 0$ is also the only equilibrium. For $c > 0$, we have three different scenario's, depending on the slope of e^{cx} at $x = 0$, i.e.

$$(e^{cx})' = ce^{cx},$$

so at $x = 0$ this slope equals c. When $0 < c < 1$, the slope of the exponential is less then the slope of the line $x + 1$ and there will be a second point of intersection $x_2^* > 0$. This is illustrated in the figure below:



For $c = 1$ the exponential and the line $x + 1$ have the same slope so they touch each other, and there is no second point of intersection. When $c > 1$, the slope of the exponential is steeper than that of the line $x + 1$ and there will be a second point of intersection $-1 < x_1^* < 0$ as illustrated in the typical plot below:



To study the stability, we try linear stability analysis,

$$f'(x) = (cx - \log(1+x))' = c - \frac{1}{1+x}.$$

For $x^* = 0$, $f'(0) = c - 1$, so this equilibrium point is stable for $c < 1$ and unstable for $c > 1$. For $c = 1$ we observe that, as the exponential touches, but does not cross, the line $x + 1$ at $x = 0$, on both the left and the right of $x^* = 0$,

$$e^{cx} > x + 1 \Rightarrow cx > \log(x+1) \Rightarrow cx - \log(x+1) > 0,$$

so the flow is to the right both on the left and on the right and so the equilibrium point $x^* = 0$ is halfstable for $c = 1$. For the additional equilibrium point for $c > 0$, the linear stability analysis fails to provide a clear answer (as we have no exact formula for x_a^*), but a similar analysis than the one used for $x^* = 0$ and $c = 1$ will tell us that for $0 < c < 1$ the additional equilibrium point x_a^* is unstable and for $c > 1$ is stable. In summary:

(i) $c \leqslant 0$: The system has one stableequilibrium point at $x^* = 0$.

(ii) $0 < c < 1$: The system has two equilibrium points, $x^* = 0$ is a stable equilibrium point, $x_1^* > 0$ is an unstable equilibrium point.

(iii) $c = 1$: the system has one equilibrium point $x^* = 0$ which is halfstable.

(iv) $c > 1$: The system has two equilibrium points at $-1 < x_1^* < 0$ which is stable and at $x^* = 0$ which is unstable.

The change at $c = 0$ from one to two equilibria does not match any of the seen bifurcations, but the bifurcation point at $c = 1$ involves the exchange of stability and hence is a transcritical bifurcation. For the bifurcation diagram, we need to plot the function

$$c = \frac{\log(x+1)}{x}.$$

This function has a vertical asymptote at $x = -1$ with

$$\lim_{x\to -1^+} \frac{\log(x+1)}{x} = \infty.$$

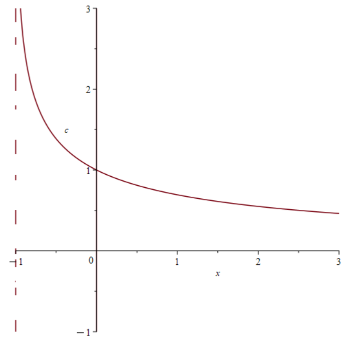
At $x = 0$ the function yields $\frac{0}{0}$ but

$$\lim_{x\to 0} \frac{\log(x+1)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1.$$

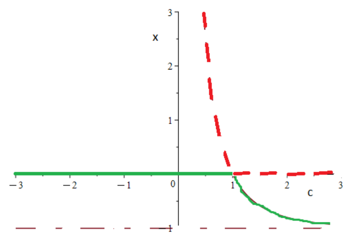
The function is always positive and

$$\lim_{x\to\infty} \frac{\log(x+1)}{x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{1+x} = 0.$$

So we can plot c versus x as below:



Flipping this over to obtain a x versus c plot, we obtain the bifurcation diagram:



2. Given the nonlinear dynamical system

$$\dot{x} = (2 - x - y)\,x,$$
$$\dot{y} = (1 - 3x - 4y)\,y,$$

(a) find all the equilibrium points of the system;

(b) determine the horizontal and vertical isoclines of the system;

(c) establish the direction of flow along the horizontal and vertical isoclines of the system;

(d) establish the direction of the flow in the regions defined by the vertical and horizontal isoclines;

(e) check for any straight lines which may contain trajectories;

(f) sketch all information obtained on a phase portrait.

Solution:

The equilibrium points are given by $\dot{x} = \dot{y} = 0$. We have that

$$\dot{x} = 0 \quad \Rightarrow \quad x = 0, \quad \text{or} \quad y = 2 - x.$$

If $x = 0$ then

$$\dot{y} = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = \frac{1}{4}.$$

If $y = 2 - x$, then

$$\dot{y} = 0 \quad \Rightarrow \quad x = 2 \ (y = 0) \quad \text{or} \quad x = 7 \ (y = -5).$$

Thus, the equilibrium points are $(0, 0)$, $\left(0, \frac{1}{4}\right)$, $(2, 0)$, $(7, -5)$.

The **horizontal isocline** is the curve in the phase plane upon which

$$\frac{dy}{dx} = 0,$$

$(Q(x, y) = 0)$. The **vertical isocline** is the curve in the phase plane upon which

$$\frac{dy}{dx} = \infty,$$

$(P(x, y) = 0)$.

We have

$$\frac{dy}{dx} = \frac{(1 - 3x - 4y)\,y}{(2 - x - y)\,x} = \begin{cases} 0, & \text{when } y = 0, \text{ or } y = \frac{1-3x}{4}, \\ \infty, & \text{when } x = 0, \text{ or } y = 2 - x. \end{cases}$$

Thus the horizontal isoclines are given by $y = 0$ and $y = \frac{1 - 3x}{4}$, while the vertical isoclines are given by $x = 0$, and $y = 2 - x$.

Along the horizontal isocline $y = 0$,

$$\dot{x} = (2 - x)\,x = \begin{cases} < 0 & \text{when } x < 0, \text{ or } x > 2, \\ > 0, & \text{when } 0 < x < 2. \end{cases}$$

Note that there are straight line solutions along this isocline.

Along the horizontal isocline $y = \frac{1 - 3x}{4}$,

$$\dot{x} = (7 - x)\,\frac{x}{4} = \begin{cases} < 0 & \text{when } x < 0, \text{ or } x > 7, \\ > 0, & \text{when } 0 < x < 7. \end{cases}$$

Along the vertical isocline $x = 0$,

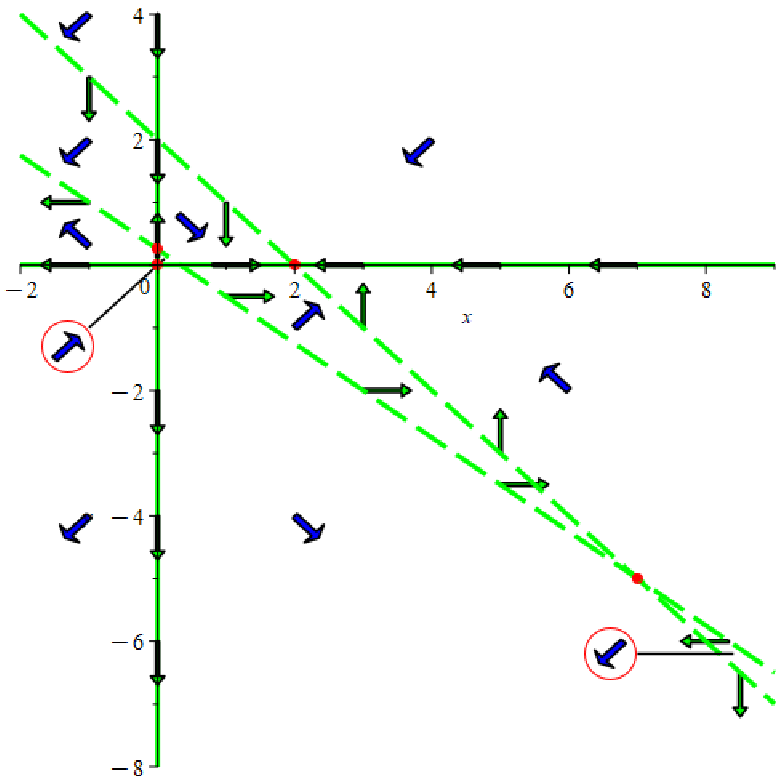
$$\dot{y} = (1 - 4y)\,y = \begin{cases} < 0 & \text{when } y < 0, \text{ or } y > \frac{1}{4}, \\ > 0, & \text{when } 0 < y < \frac{1}{4}. \end{cases}$$

Note that there are straight line solutions on this isocline.

Along the vertical isocline $y = 2 - x$,

$$\dot{y} = (x-7)(2-x) = \begin{cases} < 0 & \text{when } x < 2, \text{ or } x > 7, \\ > 0, & \text{when } 0 < x < 7. \end{cases}$$

We can than indicate the general direction of the flow of the solutions in each area separated by these isoclines. The picture thus obtained is therefore:



If we attempt to find straight line solutions, we obtain a quadratic in x which is only satisfied for all values of x when

$$\begin{cases} -3m^2 - 2m & = 0, \\ (-7q-1)m - 3q & = 0, \\ -4q^2 + q & = 0. \end{cases}$$

which has only one solution, i.e. $m = 0, q = 0$ which returns the horizontal isocline $y = 0$.

3. Given the nonlinear dynamical system

$$\dot{x} = 2x + xy - 5y,$$
$$\dot{y} = x^2 - 2x.$$

(a) find all the equilibrium points of the system;

(b) determine the horizontal and vertical isoclines of the system;

(c) establish the direction of flow along the horizontal and vertical isoclines of the system;

(d) establish the direction of the flow in the regions defined by the vertical and horizontal isoclines;

(e) check for any straight lines which may contain trajectories;

(f) sketch all information obtained on a phase portrait.

Solution:

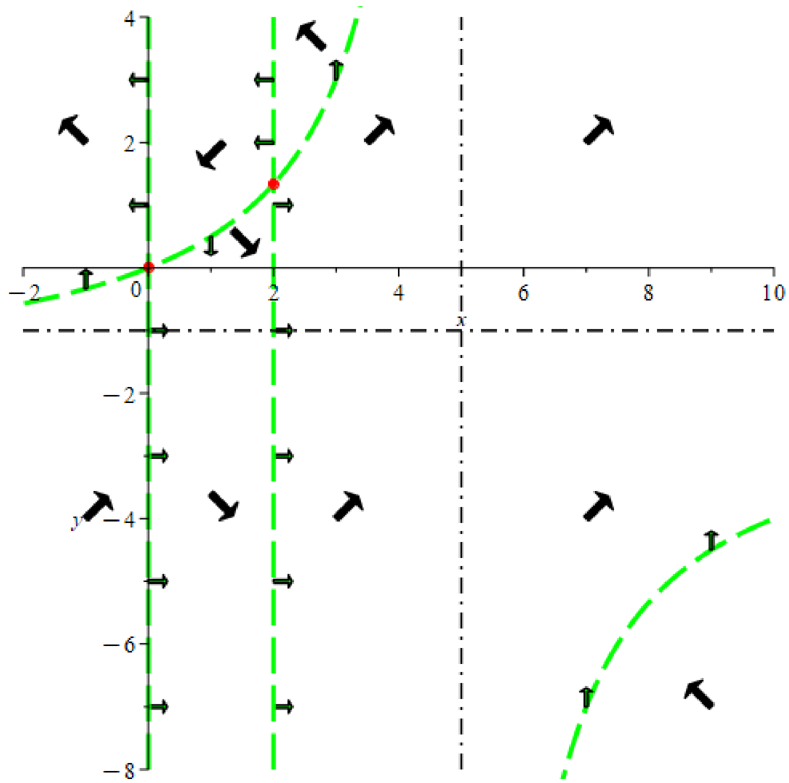
(a) Starting with $\dot{y} = 0$ we find that either $x = 0$ or $x = 2$. For $x = 0$, $\dot{x} = -5y$ so we have an equilibrium point at $(0, 0)$. For $x = 2$, $\dot{x} = 4 - 3y$, so we have an equilibrium point at $\left(2, \dfrac{4}{3}\right)$.

(b) To find the horizontal isoclines we must solve $x^2 - 2x = 0$, that gives $x = 0$ and $x = 2$. To find the vertical ones we must solve $2x + xy - 5y = 0$ that gives a hyperbola with asymptotes $x = 5$ and $y = -2$ (try to plot $y = \frac{2x}{x-5}$).

(c) on the horizontal isocline $x = 0$ we have $\dot{x} = -5y$, positive for y negative and vice-versa. On the horizontal isocline $x = 2$, we have $\dot{x} = 4-3y$ that is positive for $y < 4/3$, negative otherwise.

On the vertical isocline we have $\dot{y} = x^2 - 2x$ that is positive for $x > 2$ and $x < 0$, negative otherwise.

We can than indicate the general direction of the flow of the solutions in each area separated by these isoclines. The picture thus obtained is therefore:



If we attempt to find straight line solutions, we obtain a quadratic in x which is only satisfied for all values of x when

$$\begin{cases} m^2 - 1 & = 0, \\ 2 + (q+2)m - 5m^2 & = 0, \\ 5mq & = 0. \end{cases}$$

which has no slutions for m and q.