University of Birmingham
School of Mathematics

2RCA/2RCA3 Real and Complex Analysis

Part A: Real Analysis Semester 2

Problem Sheet 1

The questions indicated with $\boxed{\text{SUM}}$ below constitute the first of the summative assessments of this module, and will contribute to the overall mark of the module. Please submit your answers to the questions indicated with a $\boxed{\text{SUM}}$ below by the deadline of **17:00, Thursday 1 February 2024**, as a single pdf file into 2RCA/2RCA3 Assignment 1 in the Canvas page of the module.

Please note that it is the student's responsibility to make sure that their submission has been uploaded correctly into Canvas and that the uploaded file contains the submission of their assessment (eg, the uploaded file is not corrupted and contains all the pages of their answers).

Please be also aware that where assessments are submitted late without an extension being granted that has been confirmed by the Wellbeing Officer, the standard University penalty of a 5% will be imposed for each working day that the assignment is late. Any work submitted after five working days passed the deadline of submission, with no extension granted by the Wellbeing Officer/s, will be awarded a 0% mark.

In addition to the $\boxed{\text{SUM}}$-questions, Problem Sheet 1 also contain exercises that will not contribute to your module mark. You are strongly encouraged to attempt these before the relevant Guide Study sessions and/or during the course of the semester. Solutions to all exercises will be provided.

The examples/feedback classes (Guided Study) and the lecturer's office hours should be used to ask about the problem sheets.

Important note: Please, be aware that you are not allowed to use L'Hopital's rule to justify the evaluation of the limits in this problem sheet. We will revisit L'Hopital's rule later in the year.

Q1. For each of the following sets, decide whether the set is open, closed and bounded. Justify your answer.

(i) $\{1\}^c$ (that is, the complement of $\{1\}$ on $\mathbb{R}$)
(ii) $\{\pi, 2\pi\}$
(iii) $\mathbb{Q}$
(iv) $(-\infty, 1) \cup \{4\}$

Q2. Using the definition of the limit of a function at a point, prove that

$$\lim_{x\to 2}(2x-1) = 3.$$

Q3. Using the definition of the limit of a function at a point, prove that

$$\lim_{x\to 2} x^3 = 8.$$

Q4. Determine the limits of the following functions f as $x \to a$, if they exist. Justify any assertions that you make.

(i) $f(x) = \dfrac{2-x}{4-x^2}$, where $a = 2$.

(ii) $f(x) = x^3\left(\sin(x) + \dfrac{1}{x}\sin\left(\dfrac{1}{x^2}\right)\right)$, where $a = 0$.

(iii)

$$f(x) = \begin{cases} 1-x & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}, \qquad \text{where } a = 0.$$

Here $\mathbb{Q}$ is the set of rational numbers.

(iv) $f(x) = \dfrac{1-x}{1-x^k}$, where $a = 1$. Here $k \in \{1, 2, 3, \ldots\}$ is fixed.

(v) $f(x) = \dfrac{(x^2-3x+2)^4}{(x-2)^4}$, where $a = 2$.

(vi)

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \notin \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{Q} \end{cases}, \qquad \text{where} \quad a = 0.$$

Here $\mathbb{Q}$ is the set of rational numbers.

(vii)

$$f(x) = \begin{cases} x^3\sin\left(\frac{1}{x^2}\right), & \text{if } x \notin \mathbb{Q} \\ 1, & \text{if } x = 0 \\ x, & \text{if } x \in \mathbb{Q} - \{0\} \end{cases}, \qquad \text{where} \quad a = 0.$$

Here $\mathbb{Q}$ is the set of rational numbers.

Q5. Determine the limits of the following functions f as $x \to 0$, if they exist. Justify any assertions that you make.

(i) $f(x) = \dfrac{3x + |x|}{7x - 5|x|}$

(ii) $f(x) = \dfrac{\sin|x|}{|x|}$

Q6. Using the fact that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$, determine the following limits, if they exist. Justify any assertions that you make.

(i) $\displaystyle\lim_{x\to 0}\frac{\sin(\pi x)}{\sin(3x)}$.

(ii) $\displaystyle\lim_{x\to 0}\frac{x\sin x}{1-\cos x}$.

(iii) $\displaystyle\lim_{x\to 0}\frac{x^4+x^2}{1-\cos x}$.

(iv) $\displaystyle\lim_{x\to 0}\frac{\tan^3 x - x}{x+x^2}$.

Q7. For each of the following functions f and points a, determine whether $\lim_{x\to a} f(x)$ exists, and compute the limit if it exists. In each case, justify your answer (that is, if the limit does not exist you need to justify why it does not exists; if the limit exists you need to state the results you have used in the evaluation of the limit).

(i) $f(x) = \dfrac{(x^2-x-6)^2}{(x+2)^2}$, where $a = -2$.

(ii) $f(x) = \dfrac{(\sin|x|)^2}{x}$, $a = 0$

(iii) $f(x) = \begin{cases} x^2\cos\left(\dfrac{1}{x^2}\right), & \text{if } x \notin \mathbb{Q} \\ x, & \text{if } x \in \mathbb{Q} \end{cases}$, $a = 0$

(iv) $f(x) = \sin\left(\dfrac{1}{(2-x)^3}\right)$ $a = 2$

Q8. Suppose that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at x_0. Show that there exists $M > 0$ and $\delta > 0$ such that

$$|f(x)| \le M \qquad \text{whenever} \qquad |x - x_0| < \delta.$$

SUM **Q9**. Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a real function. Suppose that there exists $M > 0$ and $\delta > 0$ such that

$$|f(x)| \le M \qquad \text{whenever} \quad 0 < |x| < \delta.$$

Define the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} x^2 f(x), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Show that g is continuous at 0.

Q10. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has the property that, for some $C, \alpha > 0$, we have

$$|f(x) - f(y)| \le C|x-y|^{\alpha} \qquad \text{for all } x, y \in \mathbb{R}.$$

Using the $\varepsilon - \delta$ definition of continuity, prove that f is continuous on $\mathbb{R}$.

Note: You must not use the Sandwich Theorem here.

Q11. Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which fails to be continuous at any point $x \in \mathbb{R}$.

Q12. Let $\alpha \in \mathbb{R}$ and $f_\alpha : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ be the function defined by

$$f_\alpha(x) = \begin{cases} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{\sin(x)}, & \text{if} \quad x \in \left(-\frac{\pi}{2}, 0\right), \\ \cos(\alpha x^2) - \alpha, & \text{if} \quad x \in \left[0, \frac{\pi}{2}\right). \end{cases}$$

Study the continuity of f_α on $(-\pi/2, \pi/2)$ depending on the values of α.

$\boxed{\text{SUM}}$ **Q13**. Let $\alpha \in \mathbb{N} \cup \{0\}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} |x|^\alpha \cos\left(\frac{1}{x}\right), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

(i) Show that f is a continuous at 0 for all $\alpha \in \mathbb{N}$.
(ii) Show that f is not continuous at 0 when $\alpha = 0$.
(iii) Is f a continuous at $x \neq 0$ for all $\alpha \in \mathbb{N} \cup \{0\}$? Justify your answer.

Q14. (i) Show that if $a, b \in \mathbb{R}$, then $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$.
(ii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Using part (i), prove/deduce that the function $h : \mathbb{R} \to \mathbb{R}$, given by

$$h(x) = \max\{f(x), g(x)\}$$

for each $x \in \mathbb{R}$, is continuous. Justify your answer.

$\boxed{\text{SUM}}$ **Q15**. Suppose that the function $f : [-1, 1] \to [-1, 1]$ is continuous. Use the Intermediate Value Theorem to prove that there exists $c \in [-1, 1]$ such that

$$f(c) = c^5.$$

Note: You should carefully justify each of the hypotheses of the theorem.

Q16. Use the Intermediate Value Theorem to show that the equation

$$-\pi^2 \sin^2(x) + \sin(x) = -1$$

has at least two distinct real solutions on $(0, \infty)$. Justify your answer.

$\boxed{\text{SUM}}$ **Q17**. Use the Intermediate Value Theorem to prove that the equation

$$x^2 + 2\sin(x) - \cos(x) = 1$$

has at least two real solutions.

Note: You should carefully justify each of the hypotheses of the theorem.

Hint: You need to carefully choose your own intervals to apply the Intermediate Value Theorem.

Q18. Use the Intermediate Value Theorem to show that if the function $f : [0, 1] \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x \in [0, 1]$, then *either* $f(x) > 0$ for all $x \in [0, 1]$, *or* $f(x) < 0$ for all $x \in [0, 1]$.

Hint: You may want to argue by contradiction.

Q19. Use the Boundedness Theorem to show that if the function $f : [0, 1] \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x \in [0, 1]$, then there exists $\delta > 0$ such that $|f(x)| > \delta$ for all $x \in [0, 1]$.