University of Birmingham
School of Mathematics

1RA Differentiation Autumn 2022

Problem Sheet 1

issued Week 1

You have approximately 10 working days to complete and submit the $\boxed{\text{SUM}}$ questions (**Q4** and **Q11**) and you may begin working on it immediately.

<table>
<tr><td colspan="2">Assignment available from: 28 September
Submission due: 12 October</td></tr>
<tr><td>Pre-submission</td><td>Post-submission</td></tr>
<tr><td>• Your Guided Study Support Class in Weeks 1-2.
• Tutor meetings in Weeks 2-3.
• PASS from Week 3
• Library MSC from Week 3
• Office Hours: Wednesday 1300-1430 and Friday 1000-1130.</td><td>• Written feedback on your submission.
• Generic feedback (20 October).
• Model solutions (20 October).
• Tutor meetings in Week 5.
• Office Hours: Wednesday 1300-1430 and Friday 1000-1130</td></tr>
</table>

Instructions:

You will spend the next two weeks (including your Guided Study Support Class in weeks 2 and 3) working on the $\boxed{\text{SUM}}$ questions (**Q4** and **Q11**).

The **deadline** for submission is as follows:

- **By 17:00 on Wednesday 12 October 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

Important:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

Questions

Q1. For each of the following functions, determine whether it is injective, surjective, or bijective; moreover, in case it is bijective, find its inverse.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$.
(ii) $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^4$.
(iii) $f : (-\infty, 0] \to [0, \infty)$ defined by $f(x) = x^4$.
(iv) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 1$.

Q2. For each of the following pairs of sets, determine whether any of the relations $=$, $\subseteq$, $\supseteq$ holds between them. Justify your answers.

(i) $\mathbb{N}$ and $A = \{x \in \mathbb{R} : 1/x \in \mathbb{Q}\}$.
(ii) $B = \{x \in \mathbb{R} : 3x \in \mathbb{Q}\}$ and $\mathbb{Q}$.
(iii) $C = \{x \in \mathbb{R} : 4x \in \mathbb{Z}\}$ and $D = \{x \in \mathbb{R} : x - 3 \in \mathbb{Z}\}$.
(iv) $\mathbb{Q}$ and $E = \{x \in \mathbb{R} : 1/(2 - x) \in \mathbb{Q}\}$.

Q3. Write each of the following subsets of $\mathbb{R}$ as a union of one or more intervals. Use as few intervals as possible in each case. (Here a singleton $\{a\}$ where $a \in \mathbb{R}$ is considered to be an interval.)

(i) $[2, 5] \cap (3, 9]$.
(ii) $(-\infty, 5) \cup (3, 7]$.
(iii) $(-4, 1) \cup (-1, 2) \cup \{\pi\}$.
(iv) $\{x \in \mathbb{R} : x^2 > 144\} \cup \{12\}$.

(SUM) **Q4**. Let $f(x) = \frac{1}{1+x}$ and $g(x) = e^{-x}$. Find the following expressions:

(i) $f(2 + x)$.
(ii) $f(2x)$.
(iii) $f(x^2)$.
(iv) $f \circ f(x)$.
(v) $f(\frac{1}{f(x)})$.
(vi) $f \circ g(x)$.

Q5. Use sign analysis to solve the inequality

$$\frac{x^3(x-3)^2(x+4)}{x^2-1} \geq 0$$

and write the set of its solutions by using interval notation.

Q6. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the rule $f(x) = x^2 - x - 1$. Evaluate f at:

$$-1,\ (1+\sqrt{5})/2,\ \pi,\ x+1,\ 3t+y.$$

Give your answers exactly, not as decimals (so you might involve the symbol π in your answer, for example).

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ is defined by the rule $f : x \mapsto 9^x$, for all $x \in \mathbb{R}$.

(a) Find the outputs of f corresponding to the inputs:

$$-3,\ -5/2,\ -1/2,\ 1/2,\ 3/2.$$

(b) What input has 81 as its output?

(c) Are 0 and 1 outputs of f?

(d) What is the image of f?

(iii) Determine whether the rule

"associate to the number x the number y such that $y^2 = x + 1$"

defines a function:

(a) from $\mathbb{R}$ to $\mathbb{R}$;

(b) from $[-1, \infty)$ to $\mathbb{R}$;

(c) from $[-1, \infty)$ to $[0, \infty)$.

Q7. Use the Domain Convention to determine the domain and the range of the real valued functions of a real variable defined by the following rules:

(a) $f(x) = 2x^2 + 1$;

(b) $g(x) = (2 - x)/(3 + x)$.

Is either of these functions one-to-one? If so, find an appropriate real-valued inverse function.

Q8. Let f and g be both elementary functions. Show the following functions

$$M(x) = \max\{f(x), g(x)\};$$
$$m(x) = \min\{f(x), g(x)\},$$

are also elementary functions.

Q9. For each of the following functions, determine its infimum and supremum and whether it has a (global) maximum and/or a minimum.

(a) $f(x) = 3^x$;

(b) $g(x) = 1/(1 + x^2)$;

(c) $h(x) = \sin(2x)$.

Q10. Prove the following statements by using the definition of limit.

(i) $\displaystyle\lim_{x\to\infty} \frac{1}{x^2 + x} = 0.$

(ii) $\displaystyle\lim_{x\to 0} \frac{1}{x^4} = \infty.$

(SUM) **Q11**. (i) Determine whether the following functions are even or odd (or neither):

(a) $g(x) = x \cdot \frac{2^x - 1}{2^x + 1}$,

(b) $h(x) = x + \sin x$,

(c) $k(x) = x^3 + \cos(\pi x)$.

(ii) Let $a, b \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a \sin x + b \cos x$. What can you say about a and b if f is an odd function? What can you say if f is an even function? Is it possible for f to be both even and odd?

Q12. Prove the following limit by using the definition of limit.

(i) $\lim\limits_{x\to\infty} x^{\frac{3}{2}} = \infty$.

(ii) $\lim\limits_{x\to 1} \sqrt[3]{x} = 1$.

Extra Questions

[Questions marked with a * may be more challenging than others.]

EQ1. If S is a finite set, we write $|S|$ for the number of elements of S. The nonnegative integer $|S|$ is also called the *cardinality* of S.

(i) Compute $|\{200, 2, \sqrt{2}\}|$, $|\{\text{fish}, \text{pear}\}|$, and $|\{200, 2, \sqrt{2}, 200\}|$.

(ii) Let A and B be finite *disjoint* sets, i.e. sets such that $A \cap B = \emptyset$. Express $|A \cup B|$ in terms of $|A|$ and $|B|$.

(iii) Let $A = \{0, 1, 2, 3\}$ and $B = \{2, 5, 6, 7, 8, 9\}$. What is $|A \cup B|$? Compare it with $|A| + |B|$.

(iv) Let A and B be finite sets. Find a formula for $|A \cup B|$ in terms of $|A|$, $|B|$ and $|A \cap B|$. Explain why your formula works.

EQ2. Give examples of:

(i) a function whose domain is not equal to the codomain;

(ii) a function whose domain is not equal to the image;

(iii) a function whose codomain is not equal to the image;

(iv) a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = f$;

(v) two functions $f, g : [0, \infty) \to [0, \infty)$ such that $f \circ g$ and $g \circ f$ are not equal;

(vi) two different functions $f, g : \mathbb{R} \to \mathbb{R}$ whose restrictions to $[-1, 0]$ are equal.

EQ3. (i) By expanding the expression $(a - b)^2$, or otherwise, prove that

$$ab \leq \frac{a^2 + b^2}{2}$$

for all real numbers a and b.

(ii) Deduce further that

$$abcd \leq \frac{a^4 + b^4 + c^4 + d^4}{4}$$

for all real numbers a, b, c and d.

(iii) Is it true that

$$abc \leq \frac{a^3 + b^3 + c^3}{3}$$

for all real numbers a, b and c? Justify your answer.

EQ4. Recall the *Triangle Inequality* for real numbers: If $a, b \in \mathbb{R}$ then

$$|a + b| \leq |a| + |b|.$$

(i) Using the Triangle Inequality prove that if $a, b \in \mathbb{R}$ then

$$|a| - |b| \leq |a - b|.$$

(ii) Deduce further that if $a, b \in \mathbb{R}$ then

$$||a| - |b|| \leq |a - b|.$$

(iii) For which real numbers a, b does this last inequality hold with equality?

EQ5. Use the Domain Convention to determine the domain of the real-valued function of a real variable defined by the rule

$$f(x) = \sqrt{x^2 - x - 2}.$$

Determine the range of this function. Is the function injective? If so, determine its real-valued inverse. If not, restrict the domain in such a way that it is possible to determine an inverse and find this function. (Recall that, by convention, we always take $\sqrt{x}$ to be nonnegative.)

* **EQ6**. What can you say about the real numbers x and a if you are told that, for every $\epsilon > 0$, $|x - a| < \epsilon$? What if you are told that, for every positive integer n, $|x - a| < 1/n$?

* **EQ7**. Let $f : (0, \infty) \to \mathbb{R}$.

(i) Prove that, if $\lim_{x\to\infty} f(x) = \infty$, then f is unbounded.
(ii) Suppose instead that $\lim_{x\to\infty} f(x) = \ell$ for some $\ell \in \mathbb{R}$. Is it necessarily true that f is bounded in this case? Justify your answer.