2RCA Problem Sheet 3 Solutions

1)

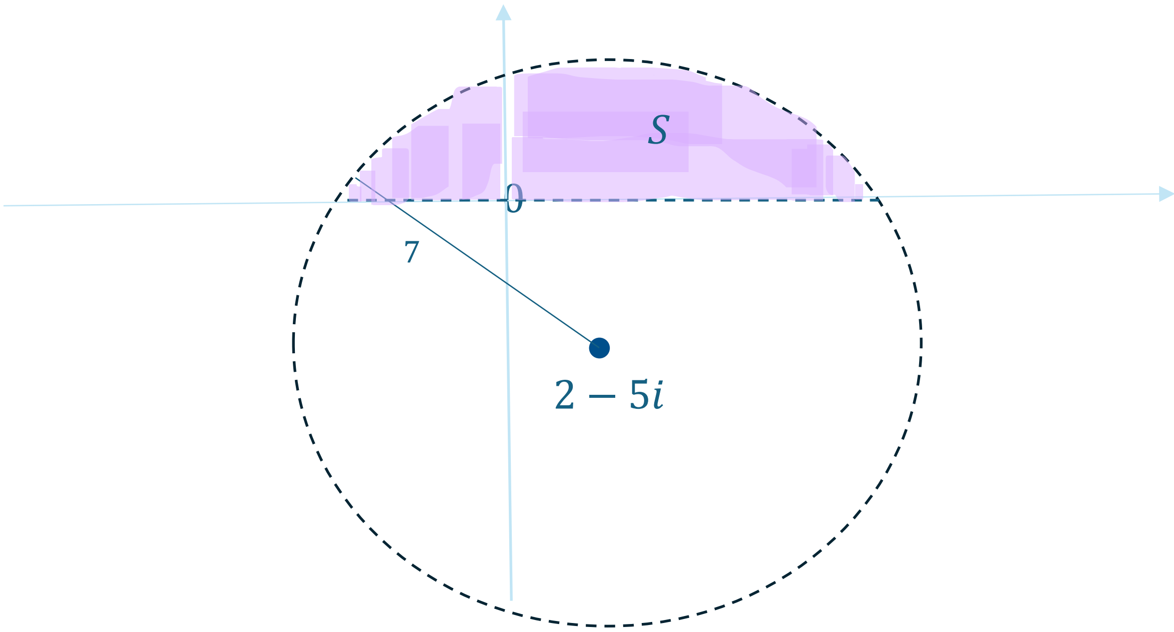
a) Let I denote the set of interior points of S. We check that I is open. Given $z_0 \in I$ our task is to show that z_0 is an interior point of I. Put differently, we need to find $r > 0$ so that $B(z_0, r) \subseteq I$. Since z_0 is an interior point of S we know that there exists $r > 0$ so that $B(z_0, r) \subseteq S$. Moreover, in the lectures we have shown that open balls, in particular, the ball $B(z_0, r)$, are open. So, for each $w \in B(z_0, r)$ we have that w is an interior point of $B(z_0, r)$. Therefore, there exists $s > 0$ so that $B(w, s) \subseteq B(z_0, r) \subseteq S$. This shows that each $w \in B(z_0, r)$ is an interior point of S, hence $B(z_0, r) \subseteq I$.

b) Let ∂S denote the set of boundary points of S. (This is the standard notation for the boundary of a set S). We need to show that $\mathbb{C} \setminus \partial S$ is open. Given $z_0 \in \mathbb{C} \setminus \partial S$, we need to show that z_0 is an interior point of $\mathbb{C} \setminus \partial S$. Put differently, we need to find $r > 0$ so that $B(z_0, r) \subseteq \mathbb{C} \setminus \partial S$. Since z_0 is not a boundary point of S there exists $r > 0$ so that $B(z_0, r)$ does not intersect one of the sets S or $\mathbb{C} \setminus S$. We assume that $B(z_0, r) \cap S = \emptyset$; the other case may be treated similarly. Then $B(z_0, r) \subseteq \mathbb{C} \setminus S$. Moreover, by the argument used in the solution to part a), for each $w \in B(z_0, r)$ we may find $s > 0$ so that $B(w, s) \subseteq B(z_0, r) \subseteq \mathbb{C} \setminus S$, implying $B(w, s) \cap S = \emptyset$. Therefore, each $w \in B(z_0, r)$ is not a boundary point of S and we have shown that $B(z_0, r) \subseteq \mathbb{C} \setminus \partial S$.

2) If S is open then every point of S is an interior point of S. So, given $z_0 \in S$ we may find $r > 0$ such that $B(z_0, r) \subseteq S$, implying $B(z_0, r) \cap (\mathbb{C} \setminus S) = \emptyset$, which means that z_0 is not a boundary point of S. Since this applies to an arbitrary $z_0 \in S$, we have shown that no point of S is a boundary point of S.

Conversely, suppose that S is not open. Then not every point of S is an interior point of S. In other words, there exists $z_0 \in S$ such that for every $r > 0$ the ball $B(z_0, r)$ fails to be contained in S. Put differently, for every $r > 0$ we have $B(z_0, r) \cap (\mathbb{C} \setminus S) \neq \emptyset$. Since $z_0 \in B(z_0, r) \cap S$ for every $r > 0$, we also have $B(z_0, r) \cap S \neq \emptyset$ for every $r > 0$. Hence, $z_0 \in S$ is a boundary point of S and we have shown that S contains some of its boundary points.

3)



S is open: Given $z_0 \in S$ we verify that z_0 is an interior point of S. In other words, we find $r > 0$ such that $B(z_0, r) \subseteq S$. Since $z_0 \in S$ we have $|z_0 - 2 + 5i| < 7$ and $Im(z_0) > 0$. Let $r > 0$ be a number that we will specify later. Then for $z \in B(z_0, r)$ we have

$$|z-2+5i| = |(z-z_0)+(z_0-2+5i)| \leq |z-z_0| + |z_0-2+5i| < r + |z_0-2+5i|.$$

and

$$Im(z) = Im\big(z_0-(z_0-z)\big) = Im(z_0) - Im(z_0-z) \geq Im(z_0) - |Im(z_0-z)| \geq Im(z_0) - |z-z_0| > Im(z_0) - r.$$

Therefore, taking

$$r = \frac{1}{2}\min\{7-|z_0-2+5i|, Im(z_0)\} > 0,$$

We have that $|z-2+5i| < r + |z_0-2+5i| < 7$ and $Im(z) > Im(z_0) - r > 0$ for every $z \in B(z_0, r)$. Hence, $B(z_0, r) \subseteq S$.
S is connected because, from the sketch, it is clear that every two points in S may be connected by a path in S. Since S is both open and connected, it is a domain. From the sketch it is also clear that any closed simple loop in S may be continuously shrunk to a point inside of S. Hence, S is simply connected. Since $S \subseteq B(2-5i, 7) \subseteq B(0, |2-5i|+7) \subseteq B(0,14)$ we have that S is bounded.
We have already shown that every point in S is an interior point of S. We claim that the set ∂S of boundary points of S is given by
$\partial S = \{z \in \mathbb{C}: Im(z) = 0, |z-5+2i| < 7\} \cup \{z \in \mathbb{C}: Im(z) > 0, |z-5+2i| = 7\}$.
Label the first set in this union T_1 and the second set T_2. Let $z_0 \in T_1$. We verify that z_0 is a boundary point of S. Given $r > 0$ we need to show that $B(z_0, r) \cap S \neq \emptyset$ and $B(z_0, r) \cap (\mathbb{C} \setminus S) \neq \emptyset$. Observe that the point

$$w = z_0 + \frac{1}{2}\min\{r, 7 - |z_0 - 5 + 2i|\}\, i$$

satisfies $|w - z_0| = \frac{1}{2}\min\{r, 7 - |z_0 - 2 + 5i|\} < r$,

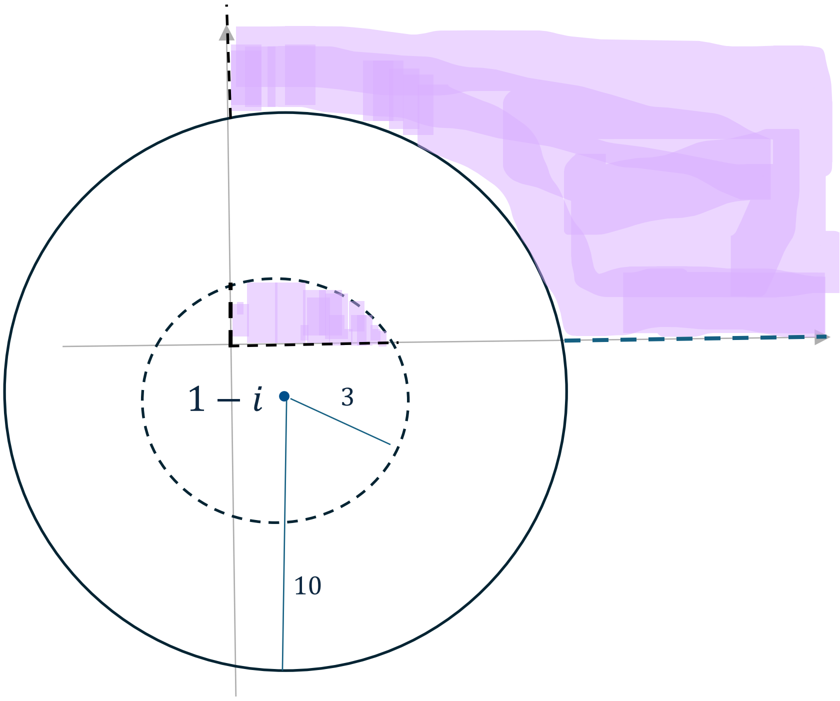
$$Im(w) = Im(z_0) + \frac{1}{2}min\{r, 7 - |z_0 - 2 + 5i|\} = 0 + \frac{1}{2}min\{r, 7 - |z_0 - 2 + 5i|\} > 0,$$

and

$$|w - 2 + 5i| \leq |w - z_0\Big| + |z_0 - 5 + 2i| = \frac{1}{2}min\{r, 7 - |z_0 - 5 + 2i|\Big\} + |z_0 - 5 + 2i| < 7.$$

Hence $w \in B(0, r) \cap S \neq \emptyset$. On the other hand the point $u = z_0 - \frac{r}{2}i$ satisfies $|u - z_0| = \frac{r}{2} < r$ and $Im(u) = Im(z_0) - \frac{r}{2} = -\frac{r}{2} < 0$. Hence, $u \in B(z_0, r) \cap (\mathbb{C} \setminus S) \neq \emptyset$.

4)



S is not open. We show this by demonstrating that the point $1 + 9i$ belongs to S but is not an interior point of S. Observe that $|(1 + 9i) - 1 + i| = |10i| = 10$, $Re(1 + 9i) = 1 > 0$ and $Im(1 + 9i) = 9 > 0$. Therefore, $1 + 9i \in S$. Now let $r > 0$. We show that the ball $B(1 + 9i, r)$ contains points of $\mathbb{C} \setminus S$, so it is not contained in S. We may assume that $r < 1$. Consider the point $w = 1 + \left(9 - \frac{r}{2}\right)i$. Then $|w - (1 + 9i)| = |\frac{r}{2}i| = \frac{r}{2} < r$. So $w \in B(z, r)$. On the other hand we have $|w - 1 + i| = |\left(9 - \frac{r}{2} + 1\right)i| = 10 - \frac{r}{2} < 10$ and $|w - 1 + i| = 10 - \frac{r}{2} > 9 > 3$. Therefore $w \in \mathbb{C} \setminus S$.

S is not closed. We show this by demonstrating that the point 0 belongs to $\mathbb{C} \setminus S$ but it is not an interior point of $\mathbb{C} \setminus S$. Since $Re(0) = 0$ we have that $0 \in \mathbb{C} \setminus S$. However, given any $r > 0$ we may take $\delta = \frac{1}{2}\min(r, 1)$ and show that the point $w = \delta(1 + i)$ belongs to $B(0, r) \cap S$. Indeed, observe

$Re(w) = \delta > 0, Im(w) = \delta > 0$ and $|w - 1 + i|^2 = (\delta - 1)^2 + (\delta + 1)^2 = 2\delta^2 + 2 < 3^2$. Hence, $w \in S$. Moreover, we have $|w - 0| = \delta\sqrt{2} < r$, so $w \in B(0, r)$. We have shown that open ball with centre 0 is not contained in $\mathbb{C} \setminus S$. Hence, 0 is not an interior point of $\mathbb{C} \setminus S$.

S is not connected because, for example the points $1 + i$ and $1 + 9i$ both belong to S but they cannot be connected by a path in S (see the picture). Since S is not connected it is also not simply connected.

Since S is not connected, it is not a domain.

Given any $R > 10$ we may observe that the point $1 + i(2R - 1)$ belongs to S and has absolute value greater than R. Hence, S is unbounded.

The set of interior points of S is

$$I = \{z \in \mathbb{C} : (|z - 1 + i| < 3 \ or \ |z - 1 + i| > 10), Re(z) > 0, Im(z) > 0\}.$$

Let $z \in I$. We verify that z is an interior point of S. If $|z - 1 + i| < 3$, we set $\delta = \frac{1}{2}\min(3 - |z - 1 + i|, Re(z), Im(z))$ and observe for all $w \in B(z, \delta)$ that

$$|w - 1 + i| \leq |w - z| + |z - 1 + i| < \delta + |z - 1 + i| < 3,$$
$$Re(w) = Re(z) - Re(z - w) \geq Re(z) - |Re(z - w)|$$
$$\geq Re(z) - |z - w| > Re(z) - \delta > 0$$

and similarly $Im(w) > 0$. Hence $w \in S$ and we have shown that $B(z, \delta) \subseteq S$.

In the remaining case we have $|z - 1 + i| > 10$ and take $\delta = \frac{1}{2}\min(10 - |z - 1 + i|, Re(z), Im(z))$. Then, for any $w \in B(z, \delta)$ we have $|w - 1 + i| \geq |z - 1 + i| - |w - z| > |z - 1 + i| - \delta > 10, Re(w) > 0$ and $Im(w) > 0$ (the latter two inequalities are shown similarly to in the first case). Hence $w \in S$ and we have shown that $B(z, \delta) \subseteq S$.

The set of boundary S may be written as a union of three sets T_1,T_2,T_3, where

$$T_1 = \{z \in \mathbb{C} : (|z - 1 + i| = 3 \ or \ |z - 1 + i| = 10), Re(z) \geq 0, Im(z) \geq 0\},$$
$$T_2 = \{z \in \mathbb{C} : (|z - 1 + i| < 3 \ or \ |z - 1 + i| > 10), Re(z) = 0, Im(z) \geq 0)\},$$
$$T_3 = \{z \in \mathbb{C} : (|z - i + 1| < 3 \ or \ |z - 1 + i| > 10), Re(z) \geq 0, Im(z) = 0\}$$

We first verify that no point outside of $T_1 \cup T_2 \cup T_3$ is a boundary point of S. First we may observe that

$$\mathbb{C} \setminus (T_1 \cup T_2 \cup T_3) = I \cup \{z \in \mathbb{C} : Re(z) < 0\} \cup \{z \in \mathbb{C}: Im(z) < 0\}.$$

We have already proved that every point of I is an interior point of S and, by definition, any interior point of S is not a boundary point of S. Now suppose that $z \in \mathbb{C}$ with $Re(z) < 0$. Then, setting $\delta = \frac{1}{2}|Re(z)|$ we see that $B(z,\delta) \subseteq \{z \in \mathbb{C} : Re(z) < 0\}$ and therefore $B(z,\delta) \cap S = \emptyset$ and z is not a boundary point of S. Similarly, we may show that any point $z \in \mathbb{C}$ with $Im(z) < 0$ is not a boundary point of S.

It remains to check that each point of $T_1 \cup T_2 \cup T_3$ is a boundary point of S. Suppose $z = x + iy \in T_1$. Then $|z - 1 + i| = 3$ or 10 and we have $y \geq 0$. We will assume that $|z - 1 + i| = 3$; a similar proof works for the other case $|z - 1 + i| = 10$. Assuming $|z - 1 + i| = 3$, we set $w(\delta) = z + \delta i$ and observe that $|w(\delta) - z| = \delta$ and

$$|w(\delta) - 1 + i|^2 = (x-1)^2 + (y + \delta + 1)^2 > (x-1)^2 + (y+1)^2$$
$$= |z - 1 + i|^2 = 3^2,$$

and $|w(\delta) - 1 + i| \leq |w(\delta) - z| + |z - 1 + i| = 3 + \delta$. So, given $r > 0$ we conclude for $\delta = \frac{1}{2}\min(r, 1)$, that the point $w(\delta)$ belongs to $B(z,r) \cap (\mathbb{C} \setminus S)$. On the other hand if we study for $\delta \in (0,1)$ the point $u(\delta) = z - \delta i$ we observe that $|u(\delta) - 1 + i|^2 = (x-1)^2 + (y - \delta + 1)^2 < (x-1)^2 + (y+1)^2 = 3^2$, $Re\big(u(\delta)\big) > Re(z) - \delta$, $Im\big(u(\delta)\big) > Im(z) - \delta$ and $|u(\delta) - z| = \delta$. Therefore, given $r > 0$ we have for $\delta = \frac{1}{2}\min(r, 1, Re(z), Im(z))$ that $u(\delta) \in B(z,r) \cap S$.

Finally we check that every point of $T_2 \cup T_3$ is a boundary point of S. Since T_2 and T_3 are similar, we only treat T_2. Let $z \in T_2$. Then $z = iy$ where $y \geq 0\}$. We assume that $|z - 1 + i| < 3$; the case $|z - 1 + i| > 10$ is dealt with similarly. Given $r > 0$ we note that the point $z - \frac{r}{2}$ satisfies $Re\left(z - \frac{r}{2}\right) = -\frac{r}{2} < 0$ and $|\left(z - \frac{r}{2}\right) - z| = \frac{r}{2} < r$. Hence, $z - \frac{r}{2} \in B(z,r) \cap (\mathbb{C} \setminus S)$. Now, for $\delta > 0$ we study the point $w(\delta) = z + \delta(1+i) = \delta + i(y + \delta)$ and observe that $|w(\delta) - z| = \delta\sqrt{2}$ and

$|w(\delta) - 1 + i| \leq |w(\delta) - z| + |z - 1 + i| = \delta\sqrt{2} + |z - 1 + i|$.

Moreover, we have $Re\big(w(\delta)\big) = \delta > 0$ and $Im(w(\delta) = y + \delta > 0$. Given $r > 0$, we may take $\delta = \frac{1}{2}\min(r, 3 - |z - 1 + i|)$ and then the above derived inequalities express that $w(\delta) \in B(z,r) \cap S$. We have shown that for every $r > 0$ we have $B(z,r) \cap S \neq \emptyset$ and $B(z,r) \cap (\mathbb{C} \setminus S) \neq \emptyset$.

5)

a) For $(x,y) \in \mathbb{R}^2$ we have

$$f(x + iy) = (x + iy)^2 + (x - iy) = (x^2 - y^2 + i2xy) + (x - iy)$$
$$= (x^2 - y^2 + x) + i(2xy - y).$$

So $u(x,y) = x^2 - y^2 + x$ and $v(x,y) = 2xy - y$.

b) For $(x,y) \in \mathbb{R}^2 \setminus \{(0,1)\}$ we have

$$f(x+iy)=\frac{x+iy}{x-iy+i}=\frac{(x+iy)(x-i(1-y))}{(x+i(1-y))(x-i(1-y))}$$
$$=\frac{x^2+y(1-y)+i(xy-x(1-y))}{x^2+(1-y)^2}$$
$$=\frac{x^2+y(1-y)}{x^2+(1-y)^2}+i\frac{2xy-x}{x^2+(1-y)^2}.$$

So $u(x,y)=\frac{x^2+y(1-y)}{x^2+(1-y)^2}$ and $v(x,y)=\frac{2xy-x}{x^2+(1-y)^2}$.

c) For $(x,y)\in\mathbb{R}^2$ we have
$$f(x+iy)=e^{x+iy}=e^xe^{iy}=e^x(\cos y+i\sin y)=e^x\cos y+i\,e^x\sin y.$$
So $u(x,y)=e^x\cos y$ and $v(x,y)=e^x\sin y$.

6) We may take for example $S=\{z\in\mathbb{C} : Re(z)\in\mathbb{Q}\}$. Let $z\in\mathbb{C}$. We verify that z is a boundary point of S. Let $r>0$. We know that the interval $(0,r)\subseteq\mathbb{R}$ contains both rational and irrational numbers. Let $q\in(0,r)\cap\mathbb{Q}$ and $p\in(0,r)\cap(\mathbb{R}\setminus\mathbb{Q})$. Then $|(z+p)-z|=|p|=p<r$ and $|(z+q)-z|=|q|=q<r$ and so we have $z+p,z+q\in B(z,r)$. Moreover, $Re(z+p)=Re(z)+Re(p)\in\mathbb{R}\setminus\mathbb{Q}$, because the sum of a rational number and an irrational number is always irrational. Similarly, $Re(z+q)=Re(z)+Re(q)\in\mathbb{Q}$, because the sum of two rational numbers is always rational. Hence $p\in B(z,r)\cap(\mathbb{C}\setminus S)\neq\emptyset$ and $q\in B(z,r)\cap S\neq\emptyset$. This demonstrates that z is a boundary point of S.

7)

a) Let $A>0$ be arbitrary and $\delta>0$ be a number that we will specify later. For $z\in\mathbb{C}$ with $0<|z+i|<\delta$ we have $|\frac{1}{(z+i)^5}|=\frac{1}{|z+i|^5}>\frac{1}{\delta^5}$. Therefore, if we choose $\delta=(\frac{1}{A})^{\frac{1}{5}}$ we have $|\frac{1}{(z+i)^5}|>\frac{1}{\delta^5}=A$, whenever $z\in\mathbb{C}$ with $0<|z+i|<\delta$.

b) Fix $\varepsilon>0$ and let $\delta>0$ be a number that we will specify later. For $z\in\mathbb{C}$ with $0<|z-3i|<\delta$ we have
$$|\frac{z^2-2iz+3}{z-3i}-4i|=|\frac{(z-3i)(z+i)}{z-3i}-4i|=|z-3i|<\delta$$
We now specify that $\delta=\varepsilon$. Then, the above shows that $|\frac{z^2-2iz+3}{z-3i}-4i|<\delta=\varepsilon$ whenever $z\in\mathbb{C}$ and $0<|z-3i|<\delta$.

8)

a) For every $z\in\mathbb{C}\setminus\{i\}$ we have
$$\frac{z^3+iz^2-z+3i}{z-i}=\frac{(z-i)(z^2+2iz-3)}{z-i}=z^2+2iz-3.$$
Therefore
$$\lim_{z\to i}\frac{z^3+iz^2-z+3i}{z-i}=\lim_{z\to i}(z^2+2iz-3)=i^2+2i(i)-3=-6.$$

For the second last equation we used the fact that polynomials are continuous.

b) Using the triangle inequality we have for every $z \in \mathbb{C} \setminus B(0,3)$

$$|\frac{z^2}{iz^3+3z-1}| = \frac{|z|^2}{|iz^3+3z-1|} \leq \frac{|z|^2}{|i||z|^3-3|z|-1} = \frac{|z|^2}{|z|^3-3|z|-1} \leq \frac{2|z|^2}{|z|^3}.$$

In the last inequality we used that $R \geq 3$ implies $R^3 \geq 8R \geq 6R+2$ and so $R^3-3R-1 \geq \frac{1}{2}R^3$. Therefore, given $\varepsilon > 0$ we may choose $M = \frac{3}{\varepsilon}$ and observe for all $z \in \mathbb{C}$ with $|z| \geq M$ that

$$|\frac{z^2}{iz^3+3z-1}| \leq \frac{2}{|z|} < \varepsilon.$$

This shows that

$$\lim_{z\to\infty} \frac{z^2}{iz^3+3z-1} = 0.$$

c) For each $z \in \mathbb{C} \setminus \{2i\}$ we have

$$\frac{1}{(z^2+4)} = \frac{1}{(z+2i)(z-2i)}.$$

We observe that as $z \to 2i$, the denominator tends to zero, so the absolute value of the quotient tends to ∞. Thus, we will verify that the limit is ∞. Fix A>0 and observe that if $\delta > 0$ and $z \in \mathbb{C}$ with $0 < |z-2i| < \delta$, then

$$|\frac{1}{z^2+4}| = |\frac{1}{(z-2i)(z+2i)}| = \frac{1}{|z-2i|}\frac{1}{|z+2i|}$$
$$\geq \frac{1}{\delta}\frac{1}{|2i+2i|-|2i-z|} \geq \frac{1}{\delta}\frac{1}{4-\delta}.$$

If, additionally, we have $\delta < 1$ then the last quantity is at least $\frac{1}{3\delta}$. Hence, if we choose $\delta = \min(\frac{1}{2}, \frac{1}{3A})$, then

$$|\frac{1}{z^2+4}| \geq \frac{1}{3\delta} > A \text{ whenever } z \in \mathbb{C} \text{ and } 0 < |z-2i| < \delta.$$

Hence, $\lim_{z\to 2i} \frac{1}{z^2+4} = \infty$.

9) Fix $z_0 \in \mathbb{C} \setminus \{p\}$. We verify, via the definition of the derivative, that

$$f'(z_0) = -\frac{1}{(z_0-p)^2}.$$

For $z \in \mathbb{C} \setminus \{p, z_0\}$ we observe that

$$\begin{aligned}\frac{f(z)-f(z_0)}{z-z_0} &= \frac{1}{z-z_0}\left(\frac{1}{z-p}-\frac{1}{z_0-p}\right)\\ &= \frac{1}{z-z_0}\left(\frac{(z_0-p)-(z-p)}{(z-p)(z_0-p)}\right)\\ &= \frac{1}{z-z_0}\frac{z_0-z}{(z-p)(z_0-p)} = -\frac{1}{(z-p)(z_0-p)}.\end{aligned}$$

Now we may use that the denominator of the last expression is a continuous function of z and the Algebra of Limits to conclude that

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} -\frac{1}{(z-p)(z_0-p)} = -\frac{1}{(z_0-p)^2}.$$

10)

a) We have $f(x+iy) = u(x,y) + iv(x,y)$, with $u(x,y) = xy^3$ and $v(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$. We compute the partial derivatives of u and v as $\frac{\partial u}{\partial x} = y^3$, $\frac{\partial u}{\partial y} = 3xy^2$, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$. Therefore, the Cauchy Riemann equations hold for u and v at $(x,y) \in \mathbb{R}^2$ if and only if

$$y^3 = 0 \text{ and } 3xy^2 = 0.$$

This pair of equations is satisfied if and only if $y = 0$, so the Cauchy Riemann equations hold at every point of the form $(x,0)$ with $x \in \mathbb{R}$. By Theorem 6.1, we conclude that f is not differentiable at any point $x+iy$ with $y \neq 0$. To decide whether f is differentiable at $x_0 \in \mathbb{R}$ we investigate whether the limit defining $f'(x_0)$ exists. So, for $z = x+iy \in \mathbb{C}\setminus\{0\}$ we observe

$$\frac{f(z)-f(x_0)}{z-x_0} = \frac{xy^3}{z-x_0} = \frac{(x-x_0)y^3 + x_0y^3}{z-x_0}.$$

We note that the numerator of the latter expression has absolute value at most

$$|x-x_0||y|^3 + |x_0||y|^3.$$

Since $z - x_0 = (x-x_0) + iy$ we also have $|x-x_0| \leq |z-x_0|$ and $|y| \leq |z|$. (Here we are using the inequalities $|Re(w)| \leq |w|$ and $|Im(w)| \leq |w|$ which hold for all complex numbers w). Therefore the absolute value of the numerator may be bounded above by

$$|z-x_0|^4 + |x_0||z-x_0|^3.$$

Hence the absolute value of the numerator approaches zero much quicker than that of the denominator as z tends to zero. We therefore aim to verify precisely that $\lim_{z\to 0}\frac{f(z)-f(x_0)}{z-x_0} = 0$. Fix $\varepsilon > 0$ and observe, for $0 < \delta < 1$ and $z = x+iy \in \mathbb{C}$ with $0 < |z| < \delta$ that

$$|\frac{f(z)-f(x_0)}{z-x_0}| \le \frac{|z-x_0|^4 + |x_0||z-x_0|^3}{|z-x_0|}$$
$$= |z-x_0|^3 + |x_0||z-x_0|^2 \le (1+|x_0|)\delta.$$

Therefore, if we choose $\delta = \frac{\varepsilon}{1+|x_0|}$, we get

$$|\frac{f(z)-f(0)}{z-0}| < (1+|x_0|)\delta = \varepsilon$$

whenever $z \in \mathbb{C}$ and $0 < |z| < \delta$.

Hence, $f'(x_0) = \lim_{z\to x_0} \frac{f(z)-f(x_0)}{z-x_0}$ exists and equals 0. We have shown that f is differentiable at each $x_0 \in \mathbb{R}$ with $f'(x_0) = 0$.

b) We have

$$f(x+iy) = \big((x-iy)+i\big)^2 = \big(x+i(1-y)\big)^2$$
$$= x^2 - (1-y)^2 + i2x(1-y).$$

Hence, $f(x+iy) = u(x,y) + iv(x,y)$ with $u(x,y) = x^2 - (1-y)^2$ and $v(x,y) = 2x(1-y)$ for all $(x,y) \in \mathbb{R}^2$. We compute the partial derivatives of u and v as $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2(1-y)$, $\frac{\partial v}{\partial x} = 2(1-y)$ and $\frac{\partial v}{\partial y} = -2x$. Therefore, the Cauchy Riemann equations are satisfied at (x,y) if and only if $2x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -2x$ and $2(1-y) = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2(1-y)$. The only solution of this pair of equations is $(x,y) = (0,1)$. We conclude, by Theorem 6.1, that f is not differentiable at any point $x+iy \in \mathbb{C}$ with $(x,y) \neq (0,1)$, that is, at any point in $\mathbb{C} \setminus \{i\}$. To determine whether f is differentiable at i we investigate whether the limit defining $f'(i)$ exists. For $z \in \mathbb{C} \setminus \{i\}$ we observe

$$\frac{f(z)-f(i)}{z-i} = \frac{(\overline{z}+i)^2}{z-i} = \frac{\left(\overline{z-i}\right)^2}{z-i}.$$

Note that the absolute value of the numerator is given by $|\overline{z-i}|^2 = |z-i|^2$, and that this quantity tends to zero, much quicker than the absolute value of the denominator. We will therefore aim to show precisely that $\lim_{z\to i} \frac{f(z)-f(i)}{z-i} = 0$. Fix $\varepsilon > 0$ and observe that for any $\delta > 0$ and $z \in \mathbb{C}$ with $0 < |z-i| < \delta$, we have

$$|\frac{\left(\overline{z-i}\right)^2}{z-i}| = \frac{|z-i|^2}{|z-i|} = |z-i| < \delta.$$

Therefore, if we choose $\delta = \varepsilon$, we have

$$|\frac{\big(f(z)-f(i)\big)}{z-i}| < \delta < \varepsilon \text{ whenever } z \in \mathbb{C} \text{ and } |z-i| < \delta.$$

Hence, $f'(i) = \lim_{z\to i} \frac{f(z)-f(i)}{z-i} = 0$, and so f is differentiable at i.

c) We have $f(x+iy) = u(x,y) + iv(x,y)$, where $u(x,y) = e^y \sin x$ and $v(x,y) = e^y \cos x$ for all $(x,y) \in \mathbb{R}^2$. We compute the partial derivatives of u and v as $\frac{\partial u}{\partial x} = e^y \cos x, \frac{\partial u}{\partial y} = e^y \sin x, \frac{\partial v}{\partial x} = -e^y \sin x$ and $\frac{\partial v}{\partial y} = e^y \cos x$. Observe that at every $(x, y) \in \mathbb{R}^2$ we have $\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) = e^y \cos x$ and $\frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y) = e^y \sin x$. Hence, u and v satisfy the Cauchy Riemann equations at every $(x, y) \in \mathbb{R}^2$. Moreover, all of the functions $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous on $\mathbb{R}^2$. We conclude, by Theorem 8.3, that f is differentiable at every point in $\mathbb{C}$.

11) Writing $f(x + iy) = u(x, y) + iv(x, y)$ we have $\overline{f}(x) = \tilde{u}(x) + i\tilde{v}(x, y)$, with $\tilde{u}(x, y) = u(x, y)$ and $\tilde{v}(x, y) = -v(x, y)$. Since both f and $\overline{f}$ are holomorphic, the functions u and v and $\tilde{u}$ and $\tilde{v}$ satisfy the Cauchy Riemann equations at every point in $\mathbb{R}^2$. So we have
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial x} = \frac{\partial \tilde{v}}{\partial y} = -\frac{\partial v}{\partial y},$$
$$\text{and } \frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \tilde{v}}{\partial x} = \frac{\partial v}{\partial x}.$$
Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x}$ hold everywhere in $\mathbb{R}^2$, which implies all the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are constant zero on the whole of $\mathbb{R}^2$. We conclude, via the formula for the derivative of f in Theorem 6.1, that $f'(z) = 0$ for every $z \in \mathbb{C}$. Therefore, by Theorem 7.2, f is constant, which implies that $\overline{f}$ is constant as well.

12) We compute
$$\frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2,$$
$$\frac{\partial u}{\partial y} = bx^2 + 2cxy + 3dy^2,$$
$$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by,$$
$$\frac{\partial^2 u}{\partial y^2} = 2cx + 6dy,$$
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = 2bx + 2cy.$$
Note that all these second order partial derivatives exist at every point $(x, y) \in \mathbb{R}^2$ and that they are continuous on $\mathbb{R}^2$. Therefore, $u: \mathbb{R}^2 \to \mathbb{R}$ is harmonic if and only if
$$(6ax + 2by) + (2cx + 6dy) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
for all $(x, y) \in \mathbb{R}^2$. This holds if and only if $6a + 2c = 0$ and $2b + 6d = 0$. So the set of all 4-tuples for which u is harmonic is given by
$$\{(a, b, c, d) : u \textit{ is harmonic}\} = \{(p, -3q, -3p, q) : p, q \in \mathbb{R}\}.$$

13) We compute

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2},$$
$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2},$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2},$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2},$$
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2+y^2)^2}.$$

We note that all second order derivatives of u exist everywhere in $\mathbb{R}^2 \setminus \{0\}$ and they are all continuous on $\mathbb{R}^2 \setminus \{0\}$. Moreover, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2} = 0.$$

Therefore, u satisfies all the conditions of Definition 8.1 and is harmonic.

Note that we cannot apply Theorem 8.3 to u as a harmonic function $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ because $\mathbb{R}^2 \setminus \{0\}$ is not simply connected. Therefore, a priori it is not clear whether a harmonic conjugate of u exists. However, we will find one.

Suppose $v: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is a harmonic conjugate of $u: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$. Then u and v satisfy the Cauchy Riemann equations in $\mathbb{R}^2 \setminus \{0\}$. So we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2},$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{2y}{x^2+y^2}.$$

Integrating the first equation with respect to y, we get

$$v(x,y) = 2\arctan\left(\frac{y}{x}\right) + c(x),$$

for some function $c(x)$. Differentiating with respect to x and applying the second Cauchy Riemann equation, we obtain

$$\frac{\partial v}{\partial x} = \frac{2}{1+\frac{y^2}{x^2}} \frac{-y}{x^2} + c'(x) = -\frac{2y}{x^2+y^2}.$$

We conclude that $c'(x) = 0$ for all x, so $c(x) = c$ is a constant. To obtain a candidate of a harmonic conjugate of $u: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ we may choose any value of the constant c that we wish. Hence we take, for example $v(x,y) = 2\arctan\frac{y}{x} + i$ for $(x,y) \in \mathbb{R}^2 \setminus \{0\}$ as our candiate for a harmonic conjugate of $u: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$. By our construction of v we know that u and v satisfy the Cauchy Riemann equations at all points of $\mathbb{R}^2 \setminus \{0\}$. Moreover, we may compute the partial derivatives of v up until the partial derivatives of second order and observe that they are all defined and continuous on $\mathbb{R}^2 \setminus \{0\}$. Hence, v is harmonic and a harmonic conjugate of u.

14) For $z \in T_1 \setminus \{0\}$ we have $\overline{z} = -z$ and so $\frac{z^2}{\overline{z}z} = \frac{z^2}{-z^2} = -1$. Hence

$$\lim_{\substack{z \to 0 \\ z \in T_1}} \frac{z^2}{\overline{z}z} = \lim_{\substack{z \to 0 \\ z \in T_1}} \frac{z^2}{-z^2} = \lim_{\substack{z \to 0 \\ z \in T_2}} -1 = -1.$$

On the other hand, for $z \in T_2 \setminus \{0\}$ we have $\overline{z} = z$ and so $\frac{z^2}{\overline{z}z} = \frac{z^2}{z^2} = 1$. Hence

$$\lim_{\substack{z \to 0 \\ z \in T_2}} \frac{z^2}{\overline{z}z} = \lim_{\substack{z \to 0 \\ z \in T_2}} \frac{z^2}{z^2} = \lim_{\substack{z \to 0 \\ z \in T_2}} 1 = 1.$$

Since we have two sets T_1 and T_2 such that

$$\lim_{\substack{z \to 0 \\ z \in T_1}} \frac{z^2}{\overline{z}z} \neq \lim_{\substack{z \to 0 \\ z \in T_2}} \frac{z^2}{\overline{z}z},$$

we conclude, by Lemma 3.3 in the lecture notes, that $\lim_{z \to 0} \frac{z^2}{\overline{z}z}$ does not exist.

15) We take $T_1 = \{x \in \mathbb{R} : x > 0\}$ and $T_2 = \{x \in \mathbb{R} : x < 0\}$. Note that 0 is a boundary point of $T_i \setminus \{0\}$ for both $i = 1,2$, so we may investigate $\lim_{\substack{z \to 0 \\ z \in T_i}} \frac{z}{|z|}$ for $i = 1,2$. For $z \in T_1$ we have $z = |z|$ and so $\frac{z}{|z|} = 1$. Hence,

$$\lim_{\substack{z \to 0 \\ z \in T_1}} \frac{z}{|z|} = \lim_{\substack{z \to 0 \\ z \in T_1}} 1 = 1.$$

On the other hand, for $z \in T_2$ we have $|z| = -z$ and so

$$\lim_{z \to 0} \frac{z}{|z|} = \lim_{z \to 0} -1 = -1.$$

Since we have two sets T_1 and T_2 such that

$$\lim_{\substack{z \to 0 \\ z \in T_1}} \frac{z^2}{\overline{z}z} \neq \lim_{\substack{z \to 0 \\ z \in T_2}} \frac{z^2}{\overline{z}z},$$

we conclude, by Lemma 3.3 in the lecture notes, that $\lim_{z \to 0} \frac{z^2}{\overline{z}z}$ does not exist.