Example sheet 4 – formative

1. Sketch the phase portraits of the following linear dynamical systems. This should include details of the equilibria, their type, eigenvalues and eigenvectors and horizontal and vertical isoclines. In each case state how all qualitatively different solutions $(x(t), y(t))$ behave as $t \to \infty$ for varying initial conditions (x_0, y_0).

   (a)
$$\dot{x} = -x - y,$$
$$\dot{y} = 2x - 3y.$$

   (b)
$$\dot{x} = 2x - 5y,$$
$$\dot{y} = x - 2y.$$

   (c)
$$\dot{x} = 3x + 6y,$$
$$\dot{y} = x + 2y.$$

   (d)
$$\dot{x} = 2x - 4y,$$
$$\dot{y} = x - 2y.$$

   (e)
$$\dot{x} = 2x + y - 3,$$
$$\dot{y} = 6x - 3y - 15.$$

Solution:

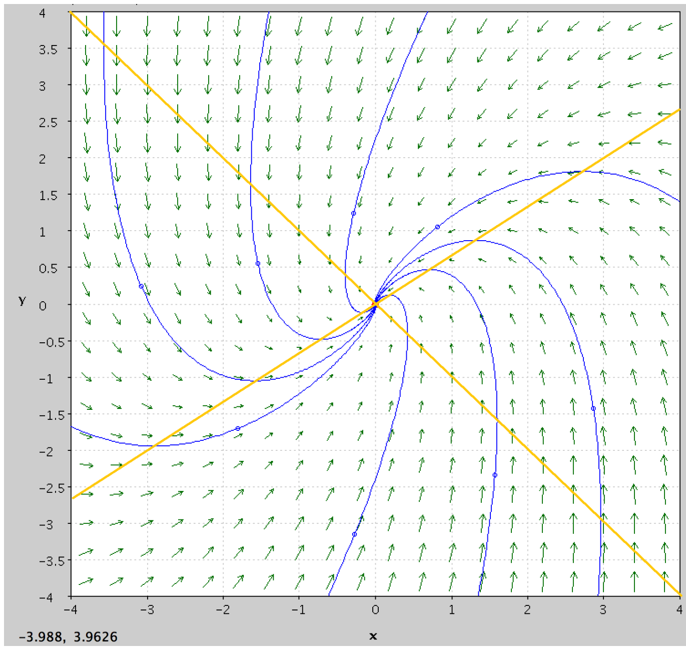
(a) Equilibrium point is $(0,0)$, eigenvalues are $\lambda_1 = -2+i, \lambda_2 = -2-i$. Eigenvalues are complex conjugate pairs with negative real part, so equilibrium point is a **stable spiral**.

The horizontal isocline is given by $y = \frac{2x}{3}$ and along the horizontal isocline

$$\dot{x} = -\frac{5}{3}x \quad \begin{cases} < 0, & x > 0, \\ > 0, & x < 0. \end{cases}$$

The vertical isocline is given by $y = -x$ and along the vertical isocline,

$$\dot{y} = 5x \quad \begin{cases} > 0, & x > 0, \\ < 0, & x < 0. \end{cases}$$



All trajectories $(x(t), y(t)) \to (0,0)$ as $t \to \infty$.

(b) Equilibrium point is $(0,0)$, eigenvalues are $\lambda_1 = i$, $\lambda_2 = -i$. Eigenvalues are complex conjugate pairs with zero real part, so equilibrium point is a **centre**.

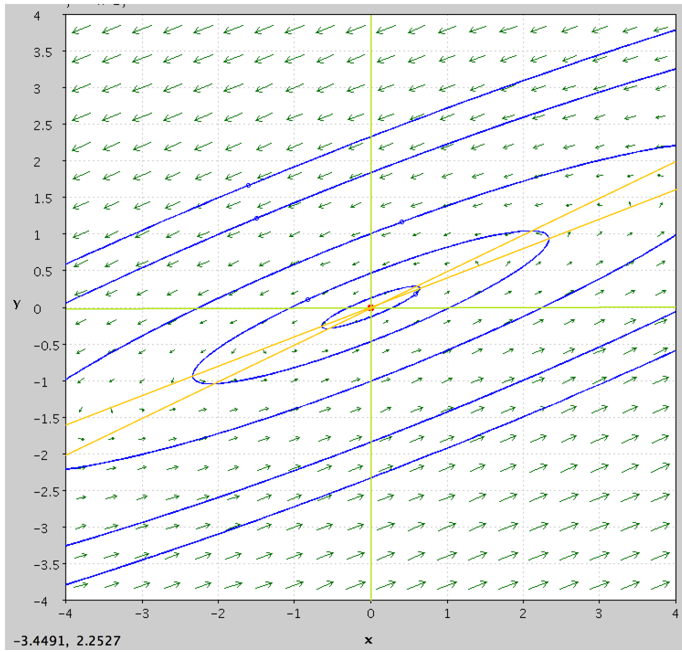
The horizontal isocline is given by $y = \frac{x}{2}$ and along the horizontal isocline

$$\dot{x} = -\frac{1}{2}x \quad \begin{cases} < 0, & x > 0, \\ > 0, & x < 0. \end{cases}$$

The vertical isocline is given by $y = \frac{2x}{5}$ and along the vertical isocline,

$$\dot{y} = \frac{1}{5}x \quad \begin{cases} > 0, & x > 0, \\ < 0, & x < 0. \end{cases}$$

Here, it is also useful to look at the gradients on $x = 0$ and $y = 0$. On $x = 0$, $\dot{x} = -5y$ and $\dot{y} = -2y$. On $y = 0$, $\dot{x} = 2x$ and $\dot{y} = x$.



Trajectories $(x(t), y(t))$ orbit the equilibrium point $(0,0)$ for all t.

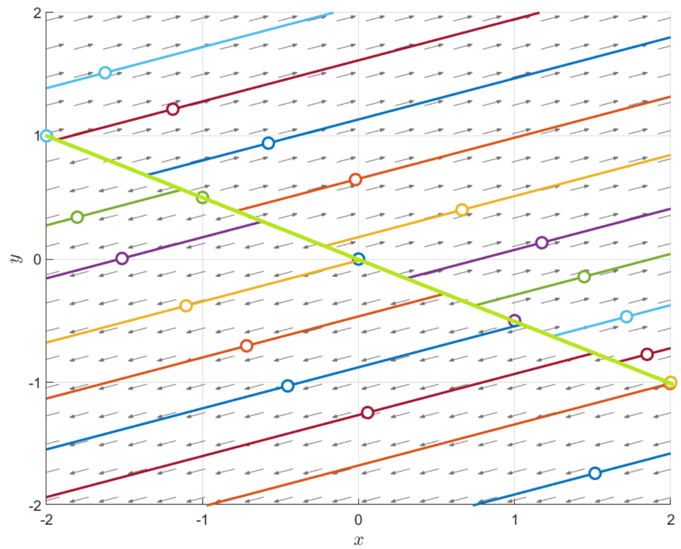
(c) The line $y = -\frac{x}{2}$ consists of equilibrium points. The eigenvalues are $\lambda_1 = 0$ (with eigenvector $\mathbf{v} = (-2, 1)$) and $\lambda_2 = 5$ (with eigenvector $\mathbf{w} = (3, 1)$). Since the non-zero eigenvalue is positive, we have an **unstable comb**.

Looking for horizontal and vertical isoclines will return the line with equilibrium points. The trajectories are straight lines with slope $\frac{1}{3}$ and the movement on all these is away from the line $y = -\frac{x}{2}$, i.e. the line with equilibrium points. This can be checked by, e.g. looking at the sign of $\dot{x}$ and $\dot{y}$ on the vertical line $x = 0$:

$$\dot{x} = 6y \quad \begin{cases} > 0, & y > 0, \\ < 0, & y < 0, \end{cases}$$

and

$$\dot{y} = 2y \quad \begin{cases} > 0, & y > 0, \\ < 0, & y < 0. \end{cases}$$



The line with equilibrium points is rendered in green. Trajectories $(x(t), y(t)) \to (\infty, \infty)$ as $t \to \infty$ when (x_0, y_0) is above the line with equilibrium points, and $(x(t), y(t)) \to (-\infty, -\infty)$ as $t \to \infty$ when (x_0, y_0) is below the line with equilibrium points.

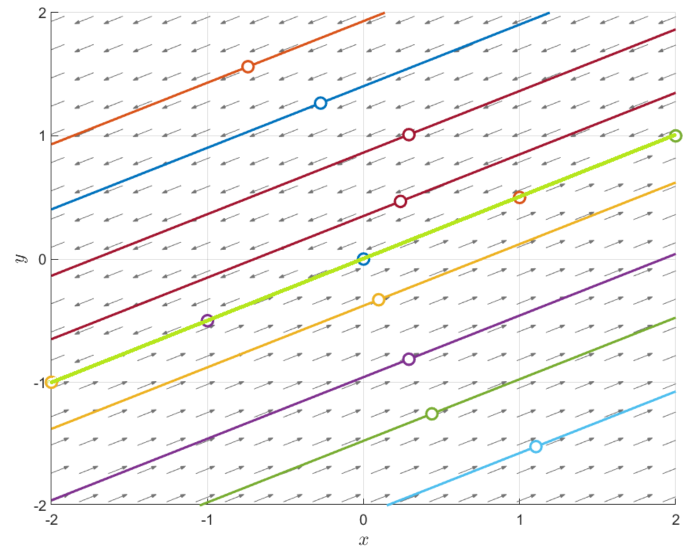
(d) The line $y = \frac{x}{2}$ consists of equilibrium points. The eigenvalue is $\lambda_1 = 0$ (with eigenvector $\mathbf{v} = (2, 1)$) . Since the second eigenvalue is zero as well, and there is only a single independent eigenvector, we have a **shear**.

Looking for horizontal and vertical isoclines will return the line with equilibrium points. The trajectories are straight lines with slope $\frac{1}{2}$. The direction of the trajectories can be checked by, e.g. looking at the sign of $\dot{x}$ and $\dot{y}$ on the vertical line $x = 0$:

$$\dot{x} = -4y \quad \begin{cases} < 0, & y > 0, \\ > 0, & y < 0, \end{cases}$$

and

$$\dot{y} = -2y \quad \begin{cases} < 0, & y > 0, \\ > 0, & y < 0. \end{cases}$$



The line with equilibrium points is rendered in green. Trajectories $(x(t), y(t)) \to (-\infty, -\infty)$ as $t \to \infty$ when (x_0, y_0) is above the line with equilibrium points, and $(x(t), y(t)) \to (\infty, \infty)$ as $t \to \infty$ when (x_0, y_0) is below the line with equilibrium points.

(e) The equilibrium point is at $(2, -1)$. So we transform the dynamical system of equations using $X = x - 2, Y = y + 1$ into

$$\dot{x} = 2X + Y,$$
$$\dot{y} = 6X - 3Y.$$

The eigenvalues are $\lambda_1 = 3$ (with eigenvector $\mathbf{v} = (-1, 6)$) and $\lambda_2 = -4$ (with eigenvector $\mathbf{w} = (1, 1)$). Since there are two non-zero real eigenvalues of opposite sign, the equilibrium is a **saddle point**.

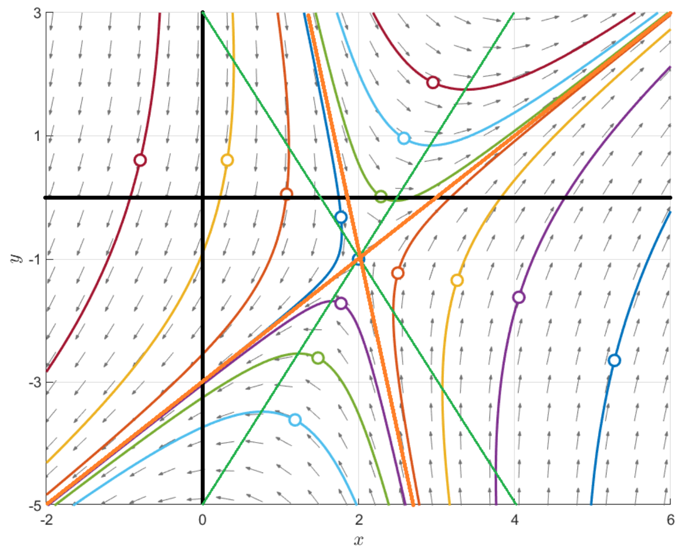
The straight line solution associated with the first eigenvalue λ_1 is given by $(y + 1) = (x - 2)$ or $y = x - 3$. The straight line solution associated with the second eigenvalue λ_2 is given by $(y + 1) = -6(x - 2)$ or $y = -6x + 11$.

The horizontal isocline is given by $y = 2x - 5$ and along the horizontal isocline

$$\dot{x} = 2x + (2x - 5) - 3 = 4(x - 2) \quad \begin{cases} > 0, & x > 2, \\ < 0, & x < 2. \end{cases}$$

The vertical isocline is given by $y = -2x + 3$ and along the vertical isocline

$$\dot{y} = 6x - 3(-2x + 3) - 15 = 12(x - 2) \quad \begin{cases} > 0, & x > 0, \\ < 0, & x < 0. \end{cases}$$



The horizontal and vertical isoclines are rendered in green. The straight line solutions along the eigenvectors are given in orange. Trajectories $(x(t), y(t)) \to (\infty, \infty)$ as $t \to \infty$ when (x_0, y_0) is to the right of the line $y = -6x + 11$, associated with the negative eigenvalue λ_2, and $(x(t), y(t)) \to (-\infty, -\infty)$ as $t \to \infty$ when (x_0, y_0) is to the left of this line.