2DE/2DE3 Example Sheet 4: Separation of Variables and Fourier series

1. *Use the method of Separation of Variables to replace the given PDE by two ODEs.*

In each of these examples, we let $u = X(x)T(t)$. Then

$$\begin{aligned}\frac{\partial u}{\partial t} &= X(x)T'(t),\\ \frac{\partial^2 u}{\partial x^2} &= X''(x)T(t),\\ \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) &= X'(x)T'(t).\end{aligned}$$

(a) $x\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial t} = 0;$

Substituting the above into this equation gives:

$$\begin{aligned}xX''(x)T(t) + X(x)T'(t) = 0,&\\ xX''(x)T(t) &= -X(x)T'(t),\\ \frac{xX''(x)}{X(x)} &= -\frac{T'(t)}{T(t)}.\end{aligned}$$

$$\begin{aligned}\implies \frac{xX''(x)}{X(x)} &= k \qquad\text{and}\qquad & -\frac{T'(t)}{T(t)} &= k,\\ xX''(x) - kX(x) &= 0 & T'(t) + kT(t) &= 0,\end{aligned}$$

where k is a constant (remember that it is also okay to use $-k$ on the right hand side to arrive at a slightly different pair of ODEs).

(b) $u_{xx} + u_{xt} + u_t = 0;$

This time, we have

$$\begin{aligned}X''(x)T(t) + X'(x)T'(t) + X(x)T'(t) = 0,&\\ X''(x)T(t) + T'(t)(X'(x) + X(x)) = 0,&\\ X''(x)T(t) &= -T'(t)(X'(x) + X(x)),\\ \frac{X''(x)}{X'(x) + X(x)} &= -\frac{T'(t)}{T(t)}.\end{aligned}$$

$$\begin{aligned}\implies \frac{X''(x)}{X'(x) + X(x)} &= k \qquad\text{and}\qquad & -\frac{T'(t)}{T(t)} &= k,\\ X''(x) - kX'(x) - kX(x) &= 0 & T'(t) + kT(t) &= 0,\end{aligned}$$

where k is a constant (remember that it is also okay to use $-k$ on the right hand side to arrive at a slightly different pair of ODEs).

(c) $[a(x)u_x]_x - b(x)u_{tt} = 0$.

We have

$$\begin{aligned}
a'(x)u_x + a(x)u_{xx} - b(x)u_{tt} &= 0,\\
a'(x)X'(x)T(t) + a(x)X''(x)T(t) - b(x)X(x)T''(t) &= 0,\\
T(t)[a'(x)X'(x) + a(x)X''(x)] - b(x)X(x)T''(t) &= 0,\\
T(t)[a(x)X'(x)]_x - b(x)X(x)T''(t) &= 0,\\
T(t)[a(x)X'(x)]_x &= b(x)X(x)T''(t),\\
\frac{[a(x)X'(x)]_x}{b(x)X(x)} &= \frac{T''(t)}{T(t)}.
\end{aligned}$$

$$\begin{aligned}
\implies [a(x)X'(x)]_x &= kb(x)X(x),\\
a(x)X''(x) + a'(x)X'(x) - kb(x)X(x) &= 0,
\end{aligned}$$

and

$$T''(t) - kT(t) = 0,$$

where k is a constant (remember that it is also okay to use $-k$ on the right hand side to arrive at a slightly different pair of ODEs).

2. *The heat equation in two space dimensions is*

$$\alpha^2(u_{xx} + u_{yy}) = u_t.$$

By looking for a solution in the form

$$u(x, y, t) = X(x)Y(y)T(t),$$

find three ODEs that can be solved to obtain $u(x, y, t)$.(Hint: you will need to introduce two new constants). You do not need to solve the ODEs.

Substituting $u(x, y, t) = X(x)Y(y)T(t)$ into the equation gives

$$\begin{aligned}
\alpha^2(X''(x)Y(y)T(t) + X(x)Y''(y)T(t)) &= X(x)Y(y)T'(t),\\
\frac{\alpha^2(X''(x)Y(y) + X(x)Y''(y))}{X(x)Y(y)} &= \frac{T'(t)}{T(t)}.
\end{aligned}$$

$$\implies T'(t) - k_1T(t) = 0,$$

(where k_1 is a constant) and

$$\begin{aligned}
\alpha^2(X''(x)Y(y) + X(x)Y''(y)) &= k_1X(x)Y(y),\\
\alpha^2X''(x)Y(y) &= k_1X(x)Y(y) - \alpha^2X(x)Y''(y),\\
&= X(x)(k_1Y(y) - \alpha^2Y''(y)),\\
\frac{\alpha^2X''(x)}{X(x)} &= \frac{k_1Y(y) - \alpha^2Y''(y)}{Y(y)}.
\end{aligned}$$

$$\begin{aligned} &\implies \alpha^2 X''(x) - k_2 X(x) = 0, \\ \text{and} \quad &\alpha^2 Y''(y) + (k_2 - k_1) Y(y) = 0, \end{aligned}$$

(where k_2 is a constant). As above, remember that you may arrive at slightly different ODEs if you've used a negative constant.

3. *Show that, for $m, n \in \mathbb{N}$,*

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \, dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases}$$

Let

$$I_1 = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \, dx,$$

and $n = m$. Using

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta), \\ &= \cos^2(\theta) - (1 - \cos^2(\theta)), \\ &= 2\cos^2(\theta) - 1, \\ \implies \cos^2(\theta) &= \frac{1}{2}(\cos(2\theta) + 1), \end{aligned}$$

we have

$$\begin{aligned} I_1 &= \int_{-L}^{L} \frac{1}{2} \left(\cos\left(\frac{2n\pi x}{L}\right) + 1 \right) \, dx, \\ &= \frac{1}{2} \left[\frac{L}{2n\pi} \sin\left(\frac{2n\pi x}{L}\right) + x \right]_{-L}^{L}, \\ &= \frac{1}{2} \left[\frac{L}{2n\pi} \sin(2n\pi) + L - \frac{L}{2n\pi} \sin(-2n\pi) + L \right], \\ &= L, \quad (\text{if } n = m). \end{aligned}$$

If $n \neq m$, then

$$I_2 = \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \, dx, .$$

Using

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta),$$

we can arrive at

$$\frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta)) = \cos(\alpha)\cos(\beta)$$

We then have that

$$
\begin{aligned}
I_2 &= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right)\, dx, \\
&= \frac{1}{2}\left[\frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right) + \frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)\right]_{-L}^{L}, \\
&= \frac{L}{2\pi}\left[\frac{\sin((n+m)\pi)}{n+m} + \frac{\sin((n-m)\pi)}{n-m} - \frac{\sin(-(n+m)\pi)}{n+m} - \frac{\sin((m-n)\pi)}{n-m}\right], \\
&= 0 \quad (n \neq m, \quad \text{and } \sin(k\pi) = 0 \text{ if } k \in \mathbb{Z}).
\end{aligned}
$$

4. *Show that*

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = 0 \quad \forall m, n \in \mathbb{N}.$$

Let

$$I = \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx.$$

This time we can use

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta), \implies \sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta)).$$

This gives

$$
\begin{aligned}
I &= \frac{1}{2}\int_{-L}^{L} \sin\left(\frac{(n+m)\pi x}{L}\right) + \sin\left(\frac{(n-m)\pi x}{L}\right) dx, \\
&= 0,
\end{aligned}
$$

because $\sin(x)$ is an odd function (you could jump straight to this conclusion by observing $\sin(n\pi x/L)\cos(m\pi x/L)$ is also an odd function).

5. (a) *Find the Fourier series of*

$$f(x) = x \quad -\pi < x < \pi.$$

First notice that $f(x) = x$ is an odd function so its Fourier series will be a Fourier sine series, i.e. $a_n = 0$, $n = 0, 1, 2, \ldots$ and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

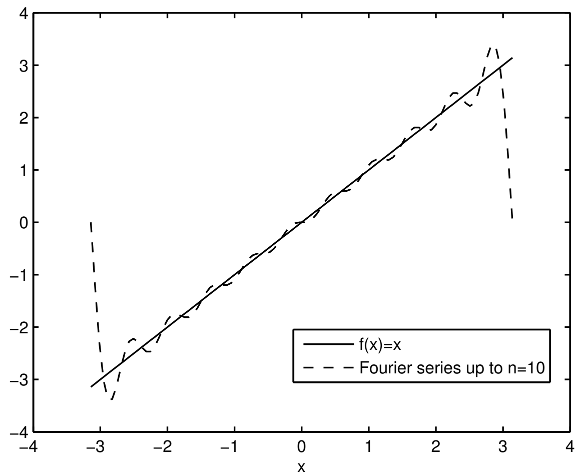

Figure 1: The solid line is $f(x)$ and the dashed line its Fourier series up to $n = 10$ (Question 5a) on $-\pi < x < \pi$. The dashed line oscillates around the solid line, diverging at the end points (as is expected from the Fourier Convergence Theorem).

Here $L = \pi$, so

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx, \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx, \\
&= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \int \frac{1}{n}\cos(nx)\,dx\right]_0^{\pi}, \\
&= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^{\pi}, \\
&= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - 0 - \frac{1}{n^2}\sin(0)\right], \\
&= -\frac{2}{n}(-1)^n, \\
&= \frac{2(-1)^{n+1}}{n}.
\end{aligned}$$

Hence the Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx),$$

see Figure 1.

(b) *If $f(x)$ is defined to be periodic with period 2π, by the Fourier Convergence Theorem, to what function does this series converge?*

$f(x)$ and $f'(x)$ are piecewise continuous on $(-\pi, \pi)$ and $f(x)$ is continuous on the whole interval. Thus by the FCT, its Fourier series converges to a 2π periodic function $g(x)$ where

$$g(x) = \begin{cases} x, & -\pi < x < \pi, \\ \frac{1}{2}(f(-\pi^+) + f(-\pi^-)) = \frac{1}{2}(-\pi + \pi) = 0, & x = -\pi, \\ \frac{1}{2}(f(\pi^+) + f(\pi^-)) = \frac{1}{2}(-\pi + \pi) = 0, & x = \pi. \end{cases}$$

6. (a) *Find the Fourier series of*

$$f(x) = x^2 \quad -\pi < x < \pi.$$

Notice that $f(x) = x^2$ is an even function, so we are looking for a Fourier cosine series (i.e. $b_n = 0, \quad \forall n$). Here $L = \pi$, hence

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ &= \frac{2}{\pi}\int_0^{\pi} x^2\,dx, \\ &= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}, \\ &= \frac{2}{\pi}\frac{\pi^3}{3}, \\ &= \frac{2\pi^2}{3}, \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx, \\ &= \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx, \\ &= \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) - \int \frac{2x}{n}\sin(nx)\,dx\right]_0^{\pi}, \\ &= \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) - \frac{2}{n}\left(-\frac{x}{n}\cos(nx) + \int \frac{1}{n}\cos(nx)\,dx\right)\right]_0^{\pi}, \\ &= \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n}\left(\frac{1}{n^2}\sin(nx)\right)\right]_0^{\pi}, \\ &= \frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_0^{\pi}, \\ &= \frac{2}{\pi}\left(\frac{2\pi(-1)^n}{n^2}\right), \\ &= \frac{4(-1)^n}{n^2}. \end{aligned}$$

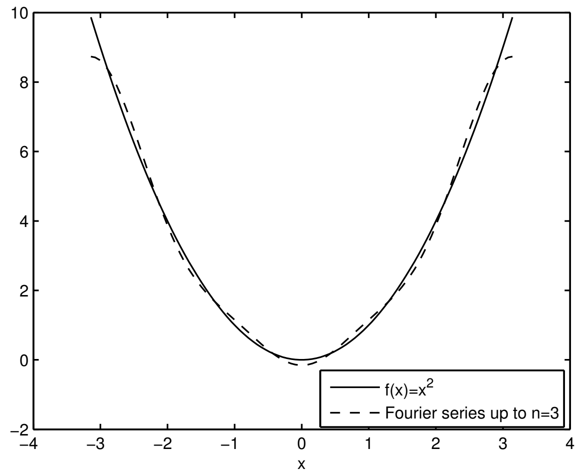

Figure 2: Question 6a: The Fourier series (dashed line) of $f(x) = x^2$ (solid line) up to $n = 3$ on $-\pi < x < \pi$. The dashed line tracks the solid line very closely.

$$\implies f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx).$$

See Figure 2.

(b) *If $f(x)$ is defined to be periodic with period 2π, by the Fourier Convergence Theorem, to what function does this series converge?*

$f(x)$ and $f'(x)$ are piecewise continuous on $(-\pi, \pi)$ and $f(x)$ is actually continuous throughout $(-\pi, \pi)$ and since $g(\pi) = g(-\pi)$, the Fourier series will converge to the 2π periodic function given by

$$g(x) = x^2, \quad -\pi \le x \le \pi.$$

7. (a) *Find the Fourier series of*

$$f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$$

This time we have $L = 1$. Notice that $f(x)$ is odd, so all $a_n = 0$ and we are

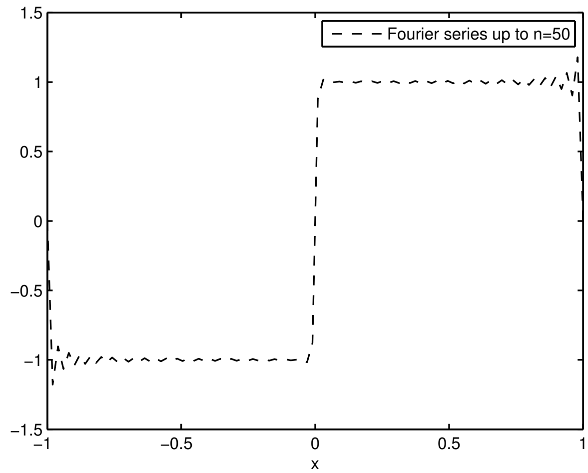

Figure 3: The Fourier series of $f(x)$ given in Question 7a up to $n = 50$ on $-1 < x < 1$. The Fourier series tracks $f(x)$ closely wherever the function is continuous.

looking for a Fourier sine series.

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \\ &= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx, \\ &= 2\int_{0}^{1}\sin(n\pi x)\,dx, \\ &= 2\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_0^1, \\ &= 2\left(-\frac{1}{n\pi}\cos(n\pi)+\frac{1}{n\pi}\cos(0)\right), \\ &= 2\left(\frac{(-1)^{n+1}}{n\pi}+\frac{1}{n\pi}\right), \\ &= \frac{2}{n\pi}(1+(-1)^{n+1}). \end{aligned}$$

$$\implies f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}(1+(-1)^{n+1})\sin(n\pi x).$$

See Figure 3.

(b) *If $f(x)$ is defined to be periodic with period 2, by the Fourier Convergence Theorem, to what function does this series converge?*

$f(x)$ and $f'(x)$ are piecewise continuous on $(-1, 1)$, so the Fourier series will converge to the periodic function (period 2) given by

$$g(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1, \\ 0, & x = -1, 0, 1. \end{cases}$$

8. (a) *Find the Fourier series of*

$$f(x) = e^x \quad -\pi < x < \pi.$$

The function $f(x)$ is neither even nor odd, so we cannot simplify our calculations in this example. We have $L = \pi$. Hence

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx, \\ &= \frac{1}{\pi}\Big[e^x\Big]_{-\pi}^{\pi}, \\ &= \frac{1}{\pi}(e^{\pi} - e^{-\pi}), \end{aligned}$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)\,dx.$$

To evaluate the a_n we can use integration by parts twice and rearrange the subsequent equation to obtain the correct integral:

$$\begin{aligned} \int e^x\cos(nx)\,dx &= \frac{1}{n}e^x\sin(nx) - \frac{1}{n}\int e^x\sin(nx)\,dx, \\ &= \frac{1}{n}e^x\sin(nx) - \frac{1}{n}\left(-\frac{e^x}{n}\cos(nx) + \int \frac{1}{n}e^x\cos(nx)\,dx\right), \\ &= \frac{1}{n}e^x\sin(nx) + \frac{1}{n^2}e^x\cos(nx) - \frac{1}{n^2}\int e^x\cos(nx)\,dx, \\ \left(\frac{n^2+1}{n^2}\right)\int e^x\cos(nx)\,dx &= e^x\left(\frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right), \\ \int e^x\cos(nx)\,dx &= \frac{e^x(n\sin(nx) + \cos(nx))}{n^2+1}. \end{aligned}$$

Hence,

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[\frac{e^x(n\sin(nx)+\cos(nx))}{n^2+1}\right]_{-\pi}^{\pi}, \\ &= \frac{1}{\pi}\left(\frac{e^{\pi}(-1)^n}{n^2+1} - \frac{e^{-\pi}(-1)^n}{n^2+1}\right), \\ &= \frac{(e^{\pi}-e^{-\pi})(-1)^n}{\pi(n^2+1)}. \end{aligned}$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\,dx,$$

but

$$\begin{aligned}
\int e^x \sin(nx)\,dx &= -\frac{e^x}{n}\cos(nx) + \frac{1}{n}\int e^x \cos(nx)\,dx,\\
&= -\frac{e^x}{n}\cos(nx) + \frac{1}{n}\left(\frac{e^x}{n}\sin(nx) - \frac{1}{n}\int e^x \sin(nx)\,dx\right),\\
&= -\frac{e^x}{n}\cos(nx) + \frac{e^x}{n^2}\sin(nx) - \frac{1}{n^2}\int e^x \sin(nx)\,dx,\\
\left(\frac{n^2+1}{n^2}\right)\int e^x \sin(nx)\,dx &= e^x\left(\frac{\sin(nx)}{n^2} - \frac{\cos(nx)}{n}\right),\\
\int e^x \sin(nx)\,dx &= \frac{e^x(\sin(nx) - n\cos(nx))}{n^2+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\left[\frac{e^x(\sin(nx) - n\cos(nx))}{n^2+1}\right]_{-\pi}^{\pi},\\
&= \frac{1}{\pi}\left(\frac{e^\pi n(-1)^{n+1} - e^{-\pi}(-n(-1)^n)}{n^2+1}\right),\\
&= \frac{1}{\pi(n^2+1)}(e^\pi n(-1)^{n+1} + n(-1)^n e^{-\pi}),\\
&= \frac{n}{\pi(n^2+1)}(e^\pi(-1)^{n+1} - (-1)^{n+1}e^{-\pi}),\\
&= \frac{(e^\pi - e^{-\pi})(-1)^{n+1}n}{\pi(n^2+1)},
\end{aligned}$$

and

$$f(x) = \frac{(e^\pi - e^{-\pi})}{2\pi} + \sum_{n=1}^{\infty}\frac{(e^\pi - e^{-\pi})(-1)^n}{\pi(n^2+1)}\cos(nx) + \sum_{n=1}^{\infty}\frac{(e^\pi - e^{-\pi})(-1)^{n+1}n}{\pi(n^2+1)}\sin(nx).$$

See Figure 4.

(b) *If $f(x)$ is defined to be periodic with period 2π, by the Fourier Convergence Theorem, to what function does this series converge?*

$f(x)$ and $f'(x)$ are piecewise continuous on $(-\pi,\pi)$ and $f(x)$ is actually continuous throughout $(-\pi,\pi)$. Hence the Fourier series will converge to the 2π periodic function given by

$$\begin{aligned}
g(x) &= \begin{cases} e^x, & -\pi < x < \pi,\\ \frac{1}{2}(e^\pi + e^{-\pi}), & x = \pm\pi,\end{cases}\\
&= \begin{cases} e^x, & -\pi < x < \pi,\\ \cosh(x), & x = \pm\pi.\end{cases}
\end{aligned}$$

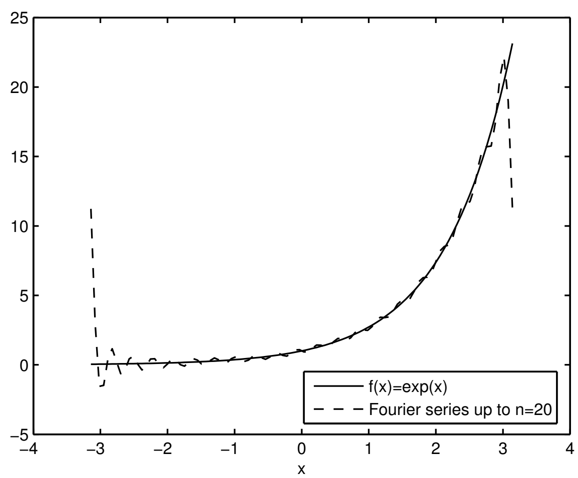

Figure 4: Question 8a: The Fourier series (dashed line) up to $n = 20$ for $f(x) = e^x$ (solid line) for $-\pi < x < \pi$. The Fourier series oscillates around $f(x)$, diverging at the end points (as would be expected by the Fourier Convergence Theorem).