Chapter 2 – Methods for finding closed-form analytical solutions to ODEs

(including the methods of Reduction of Order and Variation of Parameters)

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While general methods for solving 1st order linear equations are well-established, there are no such methods for solving 2nd or higher order linear ODEs (unless the coefficients are constant). In §1 we will consider ways of obtaining additional linearly independent solutions to a linear homogeneous ODE if one solution is already known. We also briefly consider a related method for nonlinear ODEs when the independent variable does not appear explicitly in the ODE. In §2 we will see how to find a solution for an inhomogeneous linear ODE if the general solution to the corresponding homogeneous equation is already known.

1 Reduction of Order method

(for finding a new solution to a homogeneous linear ODE from a known solution, with a note about nonlinear ODEs)

In Chapter 1 we saw how to express the general solution to a general nth order ODE (homogeneous or inhomogeneous) in terms of known solutions $(u_1(x), ..., u_n(x))$ to the homogeneous version of the ODE and a particular solution of the inhomogeneous ODE (y_p), where applicable. We now start to look at ways of obtaining the $u_1(x), ..., u_n(x)$.

1.1 2nd order linear homogeneous ODEs

If we have the general 2nd order linear homogeneous ODE

$$y'' + a(x)y' + b(x)y = 0 \tag{1}$$

then the general solution of (1) takes the form

$$y = \alpha_1 u_1(x) + \alpha_2 u_2(x)$$

where α_1, α_2 are constants and $u_1(x), u_2(x)$ are linearly independent solutions. Unfortunately, unlike for 1st order linear ODEs where we can use integrating factors, there are no general methods for obtaining u_1 or u_2 for all linear 2nd order ODEs analytically (we can always use numerical methods however – this is a very important area of mathematics and is covered in different modules).

The method of Reduction of Order allows us to obtain a second solution, $u_2(x)$, to (1) if a first solution, $u_1(x)$, is already known (note that there is no general method for finding $u_1(x)$ in the first place, however).

We look for solutions of (1) in the form $y = v(x)u_1(x)$ where $v(x)$ is an unknown function.

Substituting $y = vu_1$ into (1) gives

$$
\begin{aligned}
(vu_1)'' + a(x)(vu_1)' + b(x)(vu_1) &= 0,\\
(v''u_1 + 2v'u_1' + vu_1'') + a(x)(v'u_1 + vu_1') + b(x)vu_1 &= 0,\\
v''u_1 + v'(2u_1' + a(x)u_1) + v\cancelto{0}{(u_1'' + a(x)u_1' + b(x)u_1)} &= 0,
\end{aligned}
$$

(because u_1 is a solution to (1))

$$\implies \quad u_1(x)v'' + (2u_1'(x) + a(x)u_1(x))v' = 0,$$

i.e. we have a 2nd order linear ODE in v. If we let $w = v'$ then $w' = v''$ and

$$u_1(x)w' + (2u_1'(x) + a(x)u_1(x))w = 0$$

which is a 1st order linear separable homogeneous ODE in w that can be solved by separation or by integrating factors to obtain $w(x)$. Integrating $w(x)$ will give us $v(x)$, from which we can write down the general solution to (1) as $y(x) = v(x)u_1(x)$.

It is usually easier to follow this method with an example.

Example: Verify that $u_1(x) = x$ is a solution of

$$x^2y'' + 3xy' - 3y = 0, \quad x > 0$$

and use the Reduction of Order method to find a second solution. Give the general solution.

Answer:

Example: Imagine some chemical reaction involving two reactants A and B. A produces B, B produces A and B also naturally degrades over time, with the rate of degradation increasing over time. One way to represent this might be with the following system of ODEs:

$$\frac{dA}{dt} = B, \tag{2}$$

$$\frac{dB}{dt} = A - tB, \quad t > 0. \tag{3}$$

We can transform (2)-(3) into one 2nd order ODE:
(2) also tells us that

$$\frac{d^2A}{dt^2} = \frac{dB}{dt}. \tag{4}$$

Subbing (2) and (4) into (3) gives

$$\frac{d^2A}{dt^2} = A - t\frac{dA}{dt},$$

or

$$\frac{d^2A}{dt^2} + t\frac{dA}{dt} - A = 0. \tag{5}$$

Say we know one solution to (5) already: $u_1(t) = t$. Find the second solution.

Answer:

Summary: Reduction of Order can be used to find a fundamental set of two solutions of the homogeneous equation

$$y'' + a(x)y' + b(x)y = 0,$$

given one solution $u_1(x)$.

(a) Let $y = vu_1$ and substitute this into the ODE to obtain a 2nd order linear ODE in v with only v'' and v' terms.

(b) Let $w = v'$ to obtain a 1st order linear separable ODE in w.

(c) Find a nonzero solution for w by separation of variables or using integrating factors.

(d) Use w to obtain v to obtain the general solution.

Note: you will probably find it easier (and it is certainly more useful) to learn the above method, rather than trying to memorise a related formula.

Drawbacks:

- we must know one solution already to find a second;
- we cannot always solve the resulting integrals in terms of elementary functions.

1.2 nth order linear homogeneous ODEs

The Reduction of Order method also applies to nth order linear ODEs. If $u_1(x)$ is one solution of the equation

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + ... + a_0(x)y(x) = 0$$

then the substitution $y = v(x)u_1(x)$ will give a linear differential equation of order $(n-1)$ for v'. However, if $n \geq 3$ the reduced equation is itself of at least 2nd order, meaning that it will rarely be significantly simpler to solve than the original equation. In general, therefore, the Reduction of Order method is not often useful for equations of higher than 2nd order.

1.3 2nd order *nonlinear* ODEs

We have already seen in the example in §1.1 that if the **dependent** variable does not appear explicitly in the ODE, then we can make a substitution to reduce the order of the equation. For example, if $y'' = f(x, y')$ (and not $y'' = g(x, y, y')$) then we can let $w = y'$, which gives $w' = y''$. This allows us to derive the first order ODE $w' = f(x, w)$. If this ODE can be solved for w, then its solution can be integrated to give $y(x)$.

If the **independent** variable does not appear explicitly in an ODE, i.e. $y'' = f(y, y')$ then we can again make the substitution $w = y'$ to this time obtain $y'' = f(y, w)$, but we now must think of y as our independent variable in the new equation. We can then use the chain rule to transform the 2nd order ODE into a 1st order ODE. To begin, if $y' = \dfrac{dy}{dx} = w$ then $\dfrac{dw}{dx} = y''$. However, using the chain rule, we know that $\dfrac{dw}{dx} = \dfrac{dw}{dy}\dfrac{dy}{dx} = w\dfrac{dw}{dy}$. Hence,

$$\begin{aligned} y'' &= f(y, w), \\ \frac{dw}{dx} &= f(y, w), \\ w\frac{dw}{dy} &= f(y, w), \end{aligned}$$

which is a 1st order ODE in $w = w(y)$. It is often easier to understand this by following an example.

Example: Use the Reduction of Order method for nonlinear ODEs to obtain a solution to

$$yy'' + (y')^2 = 0.$$

Answer:

2 Variation of Parameters method

(for finding a particular solution to an inhomogeneous linear ODE given a fundamental set of solutions to the corresponding homogeneous ODE)

In the previous section we saw how to use the Reduction of Order method to find a second solution to a 2nd order homogeneous linear equation if a first solution is known. We now look at using the Variation of Parameters method to obtain the particular solution to a 2nd order inhomogeneous linear equation if a fundamental set of solutions (i.e. linearly independent solutions) to the corresponding homogeneous equation is already known. Combined, this will give us the general solution to the inhomogeneous equation. The method works by manipulating a combination of the two parts of the solution to the homogeneous equation to our advantage.

2.1 2nd order linear inhomogeneous equations

Let $\{u_1(x), u_2(x)\}$ form a fundamental set of solutions to the homogeneous equation

$$y'' + a(x)y' + b(x)y = 0 \tag{6}$$

(note that unless one of $\{u_1, u_2\}$ is already known, in which case we can use the Reduction of Order method to find the other, there are no general methods for attaining $\{u_1, u_2\}$ if a and b are not constants).

We are looking for the general solution to

$$y'' + a(x)y' + b(x)y = c(x) \tag{7}$$

given by

$$y = y_p(x) + \alpha_1 u_1(x) + \alpha_2 u_2(x)$$

where $y_p(x)$ is a particular solution to (7). Unlike when $a(x)$ and $b(x)$ are constants, there is no set procedure to find this general solution. However, if the general solution to the corresponding homogeneous equation is known (i.e. $u_1(x), u_2(x)$), then we can find $y_p(x)$.

We look for functions $v_1(x), v_2(x)$ such that

$$y = v_1(x)u_1(x) + v_2(x)u_2(x) \tag{8}$$

is a solution of (7). The method is called Variation of Parameters because v_1 and v_2 are allowed to vary with x. However, we choose v_1 and v_2 such that the first derivative of y is the same as if v_1 and v_2 were constants.

From (8), we have that

$$y' = v_1'u_1 + v_1u_1' + v_2'u_2 + v_2u_2' \tag{9}$$

but we want v_1, v_2 such that

$$y' = v_1u_1' + v_2u_2', \tag{10}$$

for this method. Equating (9) and (10) means we must choose v_1, v_2 such that

$$v_1'u_1 + v_2'u_2 = 0. \tag{11}$$

(11) is our first condition on v_1, v_2.

We also need that $y = v_1u_1 + v_2u_2$ satisfies the differential equation (7) for it to be a solution. From (10) we can calculate the second derivative of y:

$$y'' = v_1'u_1' + v_1u_1'' + v_2'u_2' + v_2u_2''. \tag{12}$$

Substituting (8), (10) and (12) into our ODE (7) gives

$$\begin{aligned}
v_1'u_1' + v_1u_1'' + v_2'u_2' + v_2u_2'' + a(x)(v_1u_1' + v_2u_2') + b(x)(v_1u_1 + v_2u_2) &= c(x),\\
v_1(\cancelto{0}{u_1'' + a(x)u_1' + b(x)u_1}) + v_2(\cancelto{0}{u_2'' + a(x)u_2' + b(x)u_2}) + v_1'u_1' + v_2'u_2' &= c(x),\\
v_1'u_1' + v_2'u_2' &= c(x). \qquad (13)
\end{aligned}$$

(13) is our second condition on v_1, v_2, i.e. if v_1' and v_2' satisfy the linear algebraic equations:

$$v_1'u_1 + v_2'u_2 = 0, \tag{14}$$
$$v_1'u_1' + v_2'u_2' = c(x) \tag{15}$$

then $y = v_1u_1 + v_2u_2$ is the general solution of the inhomogeneous equation (7).

We can solve (14) and (15) by elimination (remember u_1, u_2, u_1', u_2' are known functions, v_1', v_2' are unknown): ($u_2'\times$(14))-($u_2\times$(15)) to give

$$\begin{aligned}
v_1'u_1u_2' + v_2'u_2u_2' - v_1'u_1'u_2 - v_2'u_2u_2' &= -c(x)u_2,\\
v_1'u_1u_2' - v_1'u_1'u_2 &= -c(x)u_2,\\
v_1'(u_1u_2' - u_1'u_2) &= -c(x)u_2.
\end{aligned}$$

Notice that $u_1u_2' - u_1'u_2$ is the Wronskian, $W(u_1, u_2)$, of u_1 and u_2. Since we know that u_1 and u_2 form a fundamental set of solutions to (6), we know that $W \neq 0$ and so we can divide through by W to give

$$v_1' = -\frac{c(x)u_2(x)}{W(u_1, u_2)}. \tag{16}$$

Similarly, we can obtain

$$v_2' = \frac{c(x)u_1(x)}{W(u_1, u_2)}. \tag{17}$$

[Exercise: check this.] Integrating (16) and (17) will give us $v_1(x)$ and $v_2(x)$ and substituting these into (8) will yield the general solution (and therefore also the particular solution) to (7).

Example: Given that $\{x, x^3\}$ is a fundamental set of solutions for $x^2y'' - 3xy' + 3y = 0$, find the general solution of

$$x^2y'' - 3xy' + 3y = 4x^7, \quad x > 0.$$

Answer:

Summary: Variation of Parameters is a method for calculating a particular solution of $y'' + a(x)y' + b(x)y = c(x)$ given a fundamental set of solutions $\{u_1, u_2\}$ to the corresponding homogeneous equation $y'' + a(x)y' + b(x)y = 0$.

(a) Substitute $y = v_1u_1 + v_2u_2$ into the ODE and by assuming the first derivative of y is the same as if v_1 and v_2 were constants, arrive at the following linear algebraic equations to solve for v_1' and v_2':

$$\begin{aligned} v_1'u_1 + v_2'u_2 &= 0, \\ v_1'u_1' + v_2'u_2' &= c(x). \end{aligned}$$

(b) Use v_1' and v_2' to obtain v_1 and v_2 to give the general solution to $y'' + a(x)y' + b(x)y = c(x)$.

The general solution is therefore given by

$$y = \alpha_1 u_1(x) + \alpha_2 u_2(x) - u_1(x) \int \frac{c(x)u_2(x)}{W(u_1, u_2)}\, dx + u_2(x) \int \frac{c(x)u_1(x)}{W(u_1, u_2)}\, dx.$$

In the above example, we have employed the formula for the general solution directly. However, you must be able to derive this formula so you should learn the whole method and not just any associated formulae.

Drawbacks:

- we must know a fundamental set of solutions to the corresponding homogeneous equation (this is not trivial unless the coefficients are constant);
- evaluating the integrals to obtain v_1, v_2 depends entirely on the nature of $u_1(x), u_2(x)$ and $c(x)$ and will often be tricky or impossible to express in terms of elementary functions;
- the original ODE must be arranged into the appropriate form if formulae are to be employed.

2.2 nth order linear inhomogeneous equations

We can extend the theory behind Variation of Parameters to ODEs of higher order, but we again need a fundamental set of solutions to the corresponding homogeneous equation.

Suppose that $\{u_1, ..., u_n\}$ is a fundamental set of solutions of the corresponding homogeneous equation for

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0 y = b(x).$$

We look for a solution of the form

$$y = v_1 u_1 + ... + v_n u_n. \tag{18}$$

Differentiating (18) gives

$$\begin{aligned} y' &= v_1' u_1 + v_1 u_1' + ... + v_n' u_n + v_n u_n', \\ &= (v_1' u_1 + ... + v_n' u_n) + (v_1 u_1' + ... + v_n u_n'). \end{aligned}$$

As before, we look for functions $v_1, ..., v_n$ such that the first derivative of y is the same as if the $v_1, ..., v_n$ were constants, i.e. we set

$$v_1' u_1 + ... + v_n' u_n = 0. \tag{19}$$

This provides our first condition on the $v_1, ..., v_n$. Since we have n unknowns, we require $(n-1)$ more conditions to find the $v_1, ..., v_n$. Differentiating (18) twice with the condition (19) gives

$$y'' = (v_1' u_1' + ... + v_n' u_n') + (v_1 u_1'' + ... + v_n u_n'').$$

Similar to above, we set
$$v_1'u_1' + ... + v_n'u_n' = 0.$$
Continuing this process until you have $(n-1)$ conditions, and subbing the $y, y', ..., y^{(n)}$ into the ODE gives the following linear algebraic system for $v_1', ..., v_n'$:

$$\begin{aligned}
v_1'u_1 + ... + v_n'u_n &= 0, \\
v_1'u_1' + ... + v_n'u_n' &= 0, \\
&\vdots \\
v_1'u_1^{(n-2)} + ... + v_n'u_n^{(n-2)} &= 0, \\
v_1'u_1^{(n-1)} + ... + v_n'u_n^{(n-1)} &= \frac{b(x)}{a_n(x)}.
\end{aligned}$$

Since the $u_1, ..., u_n$ form a fundamental set of solutions for the homogeneous equation, their Wronskian, $W(u_1, ..., u_n) \neq 0$ and the above system has a solution for the $v_1', ..., v_n'$. Solving this system and using the result to obtain $v_1, ..., v_n$ yields the general solution for the inhomogeneous equation.