University of Birmingham
School of Mathematics

2RCA/2RCA3 Real and Complex Analysis

Part B: Complex Analysis Semester 2

Problem Sheet 5 (formative)

Classification of singularities, Cauchy's residue theorem

Q1. For each of the following functions f, find and classify all the isolated singularities of f. Moreover, find the residue of f at each such singularity.

$$\text{(a)} \quad f(z) = \frac{e^z}{z(1+z)^2}, \qquad \text{(b)} \quad f(z) = \frac{1-\cos(z)}{z^5}, \qquad \text{(c)} \ f(z) = \frac{z}{e^z - 1},$$

$$\text{(d)} \quad f(z) = \sin\left(\frac{1}{(z-2)^2}\right) \cos(z-2).$$

Q2. Evaluate the following contour integrals

(a)
$$\int_C \frac{\sin(3z)}{z + \frac{\pi}{2}} dz,$$
where C is the circle of centre 0 and radius 5 traversed in the anticlockwise direction.

(b)
$$\int_C \frac{e^z}{z(z-7)} dz,$$
where C is the circle of centre 0 and radius 2 traversed in the anticlockwise direction.

(c)
$$\int_C \frac{z^2}{z^2+4} dz,$$
where C is the rectangle with vertices $-2, 2, -2+4i, 2+4i$ traversed in the anticlockwise direction.

(d)
$$\int_C \frac{\sinh(z)}{(z-i\pi)^4} dz,$$
where $C : |z - 2i| = 3$ traversed in the anticlockwise direction.

(e)
$$\int_C \frac{e^z}{z^2 - 2z} dz,$$
where $C : |z| = 4$ traversed in the anticlockwise direction.

(f)
$$\int_C \frac{z+1}{z^2(z-1)} dz,$$
where $C : |z - 2| = \sqrt{2}$ traversed in the anticlockwise direction.

(g)
$$\int_\Gamma \frac{\cos(z)}{(z+1)^2(z+10)}\, dz,$$
where the contour Γ is parametrised by $\gamma : [-\pi, \pi] \to \mathbb{C}$ given by $\gamma(\theta) = 3e^{i\theta} + 1$.

Q3. Use the residue theorem to evaluate the following integrals:

(a)
$$\int_\Gamma \frac{e^{\pi z}}{z^2(z^2+2z+2)}\,dz,$$
where Γ is the circle of centre 0 and radius 3, traversed in the anticlockwise direction.

(b)
$$\int_\Gamma \frac{\sin z}{z^6}\,dz,$$
where Γ is the circle of centre 0 and radius 1, traversed in the anticlockwise direction.

(c)
$$\int_\Gamma z e^{1/z}\,dz,$$
where Γ is the circle of centre 0 and radius 1, traversed in the anticlockwise direction.

(d)
$$\int_\Gamma \frac{z+1}{z(z^2+4)^2}\,dz,$$
where Γ is the circle of centre 0 and radius 5, traversed in the anticlockwise direction.

Q4. Using residue theory, evaluate the integrals

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+9)}\,dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+4}\,dx$

Q5. Using residue theory, evaluate the integrals

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos(\theta)}$,

(b) $\displaystyle\int_0^{2\pi} \frac{d\theta}{3-2\cos(\theta)+\sin(\theta)} d\theta$.