University of Birmingham
School of Mathematics

1RA Differentiation Autumn 2022

Problem Sheet 1

Model Solutions

You have approximately 10 working days to complete and submit the $\boxed{\text{SUM}}$ questions (**Q4** and **Q11**) and you may begin working on it immediately.

<table>
<tr><td colspan="2">Assignment available from: 28 September
Submission due: 12 October</td></tr>
<tr><td>Pre-submission</td><td>Post-submission</td></tr>
<tr><td>
• Your Guided Study Support Class in Weeks 1-2.
• Tutor meetings in Weeks 2-3.
• PASS from Week 3
• Library MSC from Week 3
• Office Hours: Wednesday 1300-1430 and Friday 1000-1130.
</td><td>
• Written feedback on your submission.
• Generic feedback (20 October).
• Model solutions (20 October).
• Tutor meetings in Week 5.
• Office Hours: Wednesday 1300-1430 and Friday 1000-1130
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Instructions:

You will spend the next two weeks (including your Guided Study Support Class in weeks 2 and 3) working on the $\boxed{\text{SUM}}$ questions (**Q4** and **Q11**).

The **deadline** for submission is as follows:

- **By 17:00 on Wednesday 12 October 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

Important:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

Questions

Q1. For each of the following functions, determine whether it is injective, surjective, or bijective; moreover, in case it is bijective, find its inverse.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$.
(ii) $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^4$.
(iii) $f : (-\infty, 0] \to [0, \infty)$ defined by $f(x) = x^4$.
(iv) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 1$.

Solution. (i). f is not injective, since -1 and 1 belong to the domain $\mathbb{R}$ and $f(-1) = f(1) = 1$. Moreover f is not surjective: indeed -1 belongs to the codomain $\mathbb{R}$ but not to the image of f, since $f(x) = x^4 \geq 0$ for all $x \in \mathbb{R}$, and therefore we cannot have $f(x) = -1$ for any $x \in \mathbb{R}$. Consequently f is not bijective.

(ii). For the same reason as in part (ii), f is not injective. On the other hand, in this case f is surjective, that is, the image $f(\mathbb{R})$ is equal to the codomain $[0, \infty)$: indeed, as in part (ii) we see that $f(x) \geq 0$ for all $x \in \mathbb{R}$, that is, $f(\mathbb{R}) \subseteq [0, \infty)$; to show the opposite inclusion $[0, \infty) \subseteq f(\mathbb{R})$, we observe that, if $y \in [0, \infty)$, then we can take $x = \sqrt[4]{y}$ to have that $x \in \mathbb{R}$ and $f(x) = x^4 = (\sqrt[4]{x})^4 = y$. In any case, since f is not injective, we conclude that f is not bijective.

(iii). We claim that f is invertible (hence bijective, injective and surjective) and its inverse is the function $g : [0, \infty) \to (-\infty, 0]$ given by $g(y) = -\sqrt[4]{y}$. To prove this, we check that both $g \circ f$ and $f \circ g$ are identity functions. Indeed, for all $x \in (-\infty, 0]$,

$$g(f(x)) = g(x^4) = -\sqrt[4]{x^4} = -|x| = x$$

because $x \leq 0$ in this case; moreover, for all $y \in [0, \infty)$,

$$f(g(y)) = f(-\sqrt[4]{y}) = (-\sqrt[4]{y})^4 = y.$$

This proves that g is the inverse of f.

(iv). We claim that f is invertible (hence bijective, injective and surjective) and its inverse is the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = \sqrt[3]{y-1}$. To prove this, as before we check that both $g \circ f$ and $f \circ g$ are identity functions. Indeed, for all $x \in \mathbb{R}$,

$$g(f(x)) = g(x^3 + 1) = \sqrt[3]{(x^3 + 1) - 1} = \sqrt[3]{x^3} = x,$$

and, for all $y \in \mathbb{R}$,

$$f(g(y)) = f(\sqrt[3]{y-1}) = (\sqrt[3]{y-1})^3 + 1 = (y - 1) + 1 = y.$$

□

Q2. For each of the following pairs of sets, determine whether any of the relations $=$, $\subseteq$, $\supseteq$ holds between them. Justify your answers.

(i) $\mathbb{N}$ and $A = \{x \in \mathbb{R} : 1/x \in \mathbb{Q}\}$.
(ii) $B = \{x \in \mathbb{R} : 3x \in \mathbb{Q}\}$ and $\mathbb{Q}$.
(iii) $C = \{x \in \mathbb{R} : 4x \in \mathbb{Z}\}$ and $D = \{x \in \mathbb{R} : x - 3 \in \mathbb{Z}\}$.
(iv) $\mathbb{Q}$ and $E = \{x \in \mathbb{R} : 1/(2 - x) \in \mathbb{Q}\}$.

Solution. (i). $\mathbb{N} \subseteq A$, because every $x \in \mathbb{N}$ is a positive integer, so clearly $x \in \mathbb{R}$, and moreover $1/x$ is a rational number (a fraction of integers), that is, $1/x \in \mathbb{Q}$. On the other hand, $A \nsubseteq \mathbb{N}$: indeed, for example, $1/2 \in A$ (because $1/2 \in \mathbb{R}$ and $1/(1/2) = 2 \in \mathbb{Q}$) but $1/2 \notin \mathbb{N}$. In conclusion, $\mathbb{N} \neq A$.

(ii). $B \subseteq \mathbb{Q}$: indeed, if $x \in B$, then $x \in \mathbb{R}$ and $3x \in \mathbb{Q}$, so $x = \frac{1}{3} \cdot (3x) \in \mathbb{Q}$ (note that $\frac{1}{3} \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under multiplication). Conversely, $\mathbb{Q} \subseteq B$: indeed, if $x \in \mathbb{Q}$, then clearly $x \in \mathbb{R}$, and moreover $3x \in \mathbb{Q}$ as well (again, because $3 \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under multiplication). As a consequence, $B = \mathbb{Q}$.

(iii). $C \nsubseteq D$: indeed $1/4 \in C$ (because $1/4 \in \mathbb{R}$ and $4 \cdot (1/4) = 1 \in \mathbb{Z}$), but $1/4 \notin D$ (because $1/4 - 3 = -11/3$ is not an integer, so $1/4 - 3 \notin \mathbb{Z}$). On the other hand, $D \subseteq C$: indeed, if $x \in D$, then $x \in \mathbb{R}$ and $x - 3 \in \mathbb{Z}$, so $x = (x-3) + 3 \in \mathbb{Z}$ too (note that $3 \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under addition) and as a consequence $4x \in \mathbb{Z}$ as well (because $4 \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under multiplication). Consequently $C \neq D$.

(iv). $\mathbb{Q} \nsubseteq D$: indeed $2 \in \mathbb{Q}$, but $1/(2-2) = 1/0$ is undefined and therefore is not in $\mathbb{R}$, so $2 \notin E$. On the other hand, $D \subseteq \mathbb{Q}$: indeed, if $x \in \mathbb{R}$ and $1/(2-x) \in \mathbb{Q}$, then $2 - x \neq 0$ (because $1/(2-x)$ is defined as an element of $\mathbb{Q}$), so $1/(2-x) \neq 0$ too; consequently $2-x = 1/(1/(2-x)) \in \mathbb{Q}$ (this is because $1/(2-x) \in \mathbb{Q}$ and $\mathbb{Q}$ contains the reciprocal of any of its nonzero elements), and finally $x = 2 - (2-x) \in \mathbb{Q}$ too (because $2 \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under addition and subtraction). □

Q3. Write each of the following subsets of $\mathbb{R}$ as a union of one or more intervals. Use as few intervals as possible in each case. (Here a singleton $\{a\}$ where $a \in \mathbb{R}$ is considered to be an interval.)

(i) $[2,5] \cap (3,9]$.
(ii) $(-\infty,5) \cup (3,7]$.
(iii) $(-4,1) \cup (-1,2) \cup \{\pi\}$.
(iv) $\{x \in \mathbb{R} : x^2 > 144\} \cup \{12\}$.

Solution. (i). $[2,5] \cap (3,9] = (3,5]$.
(ii). $(-\infty,5) \cup (3,7] = (-\infty,7]$.
(iii). $(-4,1) \cup (-1,2) \cup \{\pi\} = (-4,2) \cup \{\pi\}$.
(iv). $\{x \in \mathbb{R} : x^2 > 144\} \cup \{12\} = (-\infty,-12) \cup [12,\infty)$. □

(SUM) **Q4**. Let $f(x) = \frac{1}{1+x}$ and $g(x) = e^{-x}$. Find the following expressions:

(i) $f(2+x)$.
(ii) $f(2x)$.
(iii) $f(x^2)$.
(iv) $f \circ f(x)$.
(v) $f(\frac{1}{f(x)})$.
(vi) $f \circ g(x)$.

Solution. (i). $f(2+x) = \frac{1}{3+x}$.
(ii). $f(2x) = \frac{1}{1+2x}$.
(iii). $f(x^2) = \frac{1}{1+x^2}$.
(iv). $f \circ f(x) = f(f(x)) = \frac{1}{1+\frac{1}{1+x}} = \frac{1+x}{2+x}$.
(v). $f(\frac{1}{f(x)}) = \frac{1}{1+(1+x)} = \frac{1}{2+x}$.
(vi). $f \circ g(x) = f(g(x)) = \frac{1}{1+g(x)} = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$.

Marking scheme: 5 marks for each sub-question. □

Q5. Use sign analysis to solve the inequality

$$\frac{x^3(x-3)^2(x+4)}{x^2-1} \geq 0$$

and write the set of its solutions by using interval notation.

Solution. Observe that

$$x^2 - 1 = (x+1)(x-1),$$

so the denominator of the fraction in the left-hand side of the inequality vanishes for $x = \pm 1$, and for those two values of x the fraction is undefined. We then analyse the sign of the polynomials

$$x, \qquad x-3, \qquad x+4, \qquad x-1 \qquad x+1,$$

which are the first-degree factors of the numerator and the denominator of the fraction. The sign analysis is summarised in the following diagram.

		-4		-1		0		1		3	
x	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
x^3	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x-3)^2$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	0	$+$
$x+4$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x^3(x-3)^2(x+4)$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
x^2-1	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$
$\frac{x^3(x-3)^2(x+4)}{x^2-1}$	$+$	0	$-$	n.d.	$+$	0	$-$	n.d.	$+$	0	$+$

(Here "n.d." stands for "not defined".) According to this diagram, the initial inequality is verified if and only if $x \le -4$ or $-1 < x \le 0$ or $x > 1$. In other words, the solution set is $(-\infty, 4] \cup (-1, 0] \cup (1, \infty)$. $\square$

Q6. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the rule $f(x) = x^2 - x - 1$. Evaluate f at:

$$-1, \; (1+\sqrt{5})/2, \; \pi, \; x+1, \; 3t+y.$$

Give your answers exactly, not as decimals (so you might involve the symbol π in your answer, for example).

(ii) The function $f : \mathbb{R} \to \mathbb{R}$ is defined by the rule $f : x \mapsto 9^x$, for all $x \in \mathbb{R}$.

(a) Find the outputs of f corresponding to the inputs:

$$-3, \; -5/2, \; -1/2, \; 1/2, \; 3/2.$$

(b) What input has 81 as its output?

(c) Are 0 and 1 outputs of f?

(d) What is the image of f?

(iii) Determine whether the rule

"associate to the number x the number y such that $y^2 = x + 1$"

defines a function:

(a) from $\mathbb{R}$ to $\mathbb{R}$;

(b) from $[-1, \infty)$ to $\mathbb{R}$;

(c) from $[-1, \infty)$ to $[0, \infty)$.

Solution. (i). If $f(x) = x^2 - x - 1$ for all $x \in \mathbb{R}$, then

$$f(-1) = 1, \qquad f\left(\frac{1+\sqrt{5}}{2}\right) = 0, \qquad f(\pi) = \pi^2 - \pi - 1,$$

$$f(x+1) = x^2 + x - 1, \qquad f(3t+y) = 9t^2 + 6ty + y^2 - 3t - y - 1.$$

(ii). (a). If $f(x) = 9^x$ for all $x \in \mathbb{R}$, then

$$f(-3) = \frac{1}{729}, \qquad f\left(-\frac{5}{2}\right) = \frac{1}{243}, \qquad f\left(-\frac{1}{2}\right) = \frac{1}{3},$$
$$f\left(\frac{1}{2}\right) = 3, \qquad f\left(\frac{3}{2}\right) = 27.$$

(b). If $81 = f(x) = 9^x$, then $x = \log_9 81 = 2$.

(c). 0 is not an output of f, since $f(x) = 9^x > 0$ for all $x \in \mathbb{R}$. On the other hand, 1 is an output of f, since $f(0) = 9^0 = 1$.

(d). The image of f is the open half-line $(0, \infty)$, that is, the set of all positive real numbers: indeed $f(x) = 9^x > 0$ for all $x \in \mathbb{R}$, and moreover, for all $y > 0$, if we take $x = \log_9 x$, then $f(x) = 9^x = y$.

(iii). (a). No, the rule does not define a function from $\mathbb{R}$ to $\mathbb{R}$. For example, if we take $x = -2$, then $x \in \mathbb{R}$, but $x + 1 = -1 < 0$, so there is no real number y such that $y^2 = x + 1$ in this case.

(b). No, the rule does not define a function from $[-1, \infty)$ to $\mathbb{R}$. In this case, for all $x \in [-1, \infty)$, we have $x + 1 \geq 0$, so there exist real numbers y such that $y^2 = x + 1$; the problem is that there may be more than one $y \in \mathbb{R}$ satisfying the condition (for example, in the case $x = 0$, we have $x + 1 = 1$, so both $y = -1$ and $y = 1$ satisfy the condition $y^2 = x+1$), which is in contradiction with the definition of function (the rule defining a function associates to every element of the domain exactly one element of the codomain).

(c). Yes, the rule does define a function from $[-1, \infty)$ to $[0, \infty)$. Indeed, in this case, $y = \sqrt{1 + x}$ is a well-defined element of $[0, \infty)$ for all $x \in [-1, \infty)$. $\square$

Q7. Use the Domain Convention to determine the domain and the range of the real valued functions of a real variable defined by the following rules:

(a) $f(x) = 2x^2 + 1$;
(b) $g(x) = (2 - x)/(3 + x)$.

Is either of these functions one-to-one? If so, find an appropriate real-valued inverse function.

Solution. (a). The domain of f is $\mathbb{R}$ (the expression $2x^2 + 1$ makes sense for all $x \in \mathbb{R}$). The range of f is $f(\mathbb{R}) = [1, \infty)$. Indeed $2x^2+1 \geq 1$ for all $x \in \mathbb{R}$; moreover, every element $y \in [1, \infty)$ can be written in the form $2x^2 + 1$ for some $x \in \mathbb{R}$ (just take $x = \sqrt{(y-1)/2}$). The function f is not injective, since $f(1) = f(-1) = 3$.

(b). The domain of g is $\mathbb{R} \setminus \{-3\}$ (the denominator vanishes at $x = -3$, so -3 must be removed from the domain). The range is $g(\mathbb{R}) = \mathbb{R} \setminus \{-1\}$. Indeed,

$$y = (2 - x)/(3 + x) \quad \Longleftrightarrow \quad (3 + x)y = 2 - x \quad \Longleftrightarrow \quad (y + 1)x = 2 - 3y;$$

the latter equality is impossible if $y = -1$ (thus -1 is not in the range of g); assuming $y \neq -1$, we deduce that

$$y = (2 - x)/(3 + x) \quad \Longleftrightarrow \quad x = (2 - 3y)/(y + 1),$$

whence any real number $y \neq 1$ is the image of some x in the domain of g. The same argument shows that every $y \in \mathbb{R} \setminus \{-1\}$ has exactly one preimage x in the domain of g, thus g is injective, and its real-valued inverse is the function $h : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ given by

$$h(y) = (2 - 3y)/(y + 1).$$

$\square$

Q8. Let f and g be both elementary functions. Show the following functions

$$M(x) = \max\{f(x), g(x)\};$$
$$m(x) = \min\{f(x), g(x)\},$$

are also elementary functions.

Solution. We note that

$$M(x) = \max\{f(x), g(x)\} = \frac{1}{2}\big[f(x) + g(x) + \sqrt{(f(x) - g(x))^2}\big];$$
$$m(x) = \min\{f(x), g(x)\} = \frac{1}{2}\big[f(x) + g(x) - \sqrt{(f(x) - g(x))^2}.$$

□

Q9. For each of the following functions, determine its infimum and supremum and whether it has a (global) maximum and/or a minimum.

(a) $f(x) = 3^x$;
(b) $g(x) = 1/(1 + x^2)$;
(c) $h(x) = \sin(2x)$.

Solution. (a). By the Domain Convention, the domain of f is $\mathbb{R}$. Consequently the image of f is the open half-line $(0, \infty)$. Indeed $3^x > 0$ for all $x \in \mathbb{R}$, and every $y > 0$ can be written in the form 3^x for some $x \in \mathbb{R}$ (just take $x = \log_3 y$). Then $\inf f = \inf(0, \infty) = 0$ and $\sup f = \sup(0, \infty) = \infty$. The set $(0, \infty)$ has neither maximum nor minimum, so f has neither maximum nor minimum.

(b). By the Domain Convention, the domain of g is $\mathbb{R}$. We claim that the image of g is the interval $(0, 1]$. Indeed, clearly $1 + x^2 \geq 1$ for all $x \in \mathbb{R}$, whence $0 < 1/(1 + x^2) \leq 1$ for all $x \in \mathbb{R}$, which shows that the image of g is contained in $(0, 1]$. Conversely, if $y \in (0, 1]$, then $1/y \geq 1$, so we can find $x \in \mathbb{R}$ such that $1 + x^2 = 1/y$ (just take $x = \sqrt{1/y - 1}$) and therefore $y = 1/(1 + x^2)$; this shows that $(0, 1]$ is contained in the image of g, and in conclusion that $g(\mathbb{R}) = (0, 1]$. From this we deduce that $\inf g = \inf(0, 1] = 0$ and $\sup g = \sup(0, 1] = 1$. The set $(0, 1]$ has no minimum, so g has no minimum; on the other hand, $\max g = \max(0, 1] = 1$.

(c). By the Domain Convention, the domain of h is $\mathbb{R}$, and consequently its image is the interval $[-1, 1]$; indeed, clearly $-1 \leq \sin(2x) \leq 1$ for all $x \in \mathbb{R}$, and moreover, if $y \in [-1, 1]$, then we can write $y = \sin(2x)$ by choosing, e.g., $x = \arcsin(y)/2$. Thus $\inf h = \min h = \min[-1, 1] = -1$ and $\sup h = \max h = \max[-1, 1] = 1$. □

Q10. Prove the following statements by using the definition of limit.

(i) $\displaystyle\lim_{x\to\infty} \frac{1}{x^2 + x} = 0.$
(ii) $\displaystyle\lim_{x\to 0} \frac{1}{x^4} = \infty.$

Solution. (i). Note first that $x^2 + x = x(x + 1)$, so (according to the Domain Convention) the domain of $x \mapsto \frac{1}{x^2+x}$ is $A = \{x \in \mathbb{R} : x \neq 0, -1\} = (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$, which is unbounded above. According to the definition of limit, we must prove that

$$\forall \epsilon > 0 : \exists N \in \mathbb{R} : \forall x \in A : \left(x > N \Rightarrow \left|\frac{1}{x^2 + x} - 0\right| < \epsilon\right). \tag{1}$$

Let $\epsilon > 0$ be given. Observe that, if $x > 0$, then $x^2 + x > x > 0$, and therefore

$$\left|\frac{1}{x^2 + x} - 0\right| = \left|\frac{1}{x^2 + x}\right| = \frac{1}{x^2 + x} < \frac{1}{x}$$

in this case; moreover clearly

$$\frac{1}{x} < \epsilon \quad \iff \quad x > \frac{1}{\epsilon}.$$

Consequently, if we take $N = 1/\epsilon$, then $N > 0$, and therefore, for all $x > N$, we have that $x > 0$ and by the previous inequalities we conclude that

$$\left| \frac{1}{x^2 + x} - 0 \right| < \frac{1}{x} < \frac{1}{N} = \epsilon,$$

as desired. Since $\epsilon > 0$ was arbitrary, this proves (1), that is, $\lim\limits_{x\to\infty} \dfrac{1}{x^2 + x} = 0$.

(ii). Note first that, by the Domain Convention, the domain of $x \mapsto \frac{1}{x^4}$ is $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$, of which 0 is an accumulation point. According to the definition of limit, we must prove that

$$\text{(2)} \qquad \forall M > 0 : \exists \delta > 0 : \forall x \in \mathbb{R} \setminus \{0\} : \left(0 < |x - 0| < \delta \Rightarrow \frac{1}{x^4} > M \right).$$

Let $M > 0$. Note that, for all $x \in \mathbb{R} \setminus \{0\}$, $x^4 > 0$ and therefore

$$\frac{1}{x^4} > M \iff x^4 < \frac{1}{M} \iff |x| < \frac{1}{\sqrt[4]{M}}$$

Hence, if we take $\delta = \frac{1}{\sqrt[4]{M}}$, then $\delta > 0$ and, for all $x \in \mathbb{R} \setminus \{0\}$, if $0 < |x - 0| < \delta$, we have $|x| < \delta = \frac{1}{\sqrt[4]{M}}$ and therefore $\frac{1}{x^4} > M$. Since $M > 0$ was arbitrary, this proves (2), that is, $\lim\limits_{x\to 0} \dfrac{1}{x^4} = \infty$. $\square$

(SUM) **Q11**. (i) Determine whether the following functions are even or odd (or neither):

(a) $g(x) = x \cdot \frac{2^x - 1}{2^x + 1}$,
(b) $h(x) = x + \sin x$,
(c) $k(x) = x^3 + \cos(\pi x)$.

(ii) Let $a, b \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a \sin x + b \cos x$. What can you say about a and b if f is an odd function? What can you say if f is an even function? Is it possible for f to be both even and odd?

Solution. (i). The domain of these functions is $\mathbb{R}$, which is symmetric with respect to the origin. [*5 marks*]

(a). g is even [*5 marks*]; indeed

$$\begin{aligned} g(-x) &= (-x) \cdot \frac{2^{-x} - 1}{2^{-x} + 1} \\ &= -x \cdot \frac{2^{-x}(1 - 2^x)}{2^{-x}(1 + 2^x)} \\ &= -x \cdot \frac{1 - 2^x}{1 + 2^x} = g(x) \end{aligned}$$

for all x in the domain of g [*5 marks* for the computation].

(b). h is odd [*5 marks*]; indeed $h(-x) = (-x) + \sin(-x) = -x - \sin x = -h(x)$, for all $x \in \mathbb{R}$. [*5 marks* for the computation]

(c). k neither even nor odd [*5 marks*]; indeed $k(1) = 1 + \cos(\pi) = 1 - 1 = 0$, while $k(-1) = (-1)^3 + \cos(-\pi) = -1 - 1 = -2$. Therefore, $k(-1) \neq \pm k(1)$. [*5 marks* for the computation] (ii). We also note that the domain of f is $\mathbb{R}$, which is symmetric with respect to the origin. Thus f is an odd function if and only if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. On the other hand,

$$f(-x) = -f(x) \quad \iff \quad -a\sin x + b\cos x = -(a\sin x + b\cos x).$$

[*5 marks*] and the last equality holds for all $x \in \mathbb{R}$ if and only if $b = 0$ [*5 marks*]. So, f is an odd function if and only if $b = 0$ [*5 marks*]. Similarly, f is even if and only if $f(-x) = f(x)$ for all $x \in \mathbb{R}$; on the other hand,

$$f(-x) = f(x) \quad \iff \quad -a\sin x + b\cos x = a\sin x + b\cos x,$$

[*5 marks*] and the last equality holds for all $x \in \mathbb{R}$ if and only if $a = 0$ [*5 marks*]. This shows that f is even if and only if $a = 0$ [*5 marks*].

Consequently, f can be simultaneously even and odd, but this happens if and only if $a = b = 0$, that is, $f(x) = 0$ for all $x \in \mathbb{R}$. [*5 marks*] □

Q12. Prove the following limit by using the definition of limit.

(i) $\lim_{x\to\infty} x^{\frac{3}{2}} = \infty$.

(ii) $\lim_{x\to 1} \sqrt[3]{x} = 1$.

Solution. (i) According to the Domain Convention, the domain of the function $x \mapsto x^{\frac{3}{2}} = (\sqrt{x})^3$ is $[0,\infty)$, which is unbounded above. So, according to the definition of limit, we must prove that

$$\text{(3)} \qquad \forall M > 0 : \exists N \in \mathbb{R} : \text{ such that } \forall x \in [0,\infty) \text{ we have } \Big(x > N \Rightarrow x^{\frac{3}{2}} > M\Big).$$

Let $M > 0$. Note that, for all $x \in [0,\infty)$,

$$x^{\frac{3}{2}} > M \quad \iff \quad x > M^{\frac{2}{3}}$$

(here we are using that both $x, M \geq 0$). Consequently, if we choose $N = M^{\frac{2}{3}}$, then $N > 0$ and, for all $x \in [0,\infty)$, if $x > N$, then $x^{\frac{3}{2}} > N^{\frac{3}{2}} = (M^{\frac{2}{3}})^{\frac{3}{2}} = M$. Since $M > 0$ was arbitrary, this proves (3), that is, $\lim_{x\to\infty} x^{\frac{3}{2}} = \infty$.

(ii) According to the Domain Convention, the domain of the function $x \mapsto \sqrt[3]{x}$ is $\mathbb{R}$, of which 1 is an accumulation point. According to the definition of limit, we must prove

$$\text{(4)} \quad \forall \epsilon > 0 : \exists \delta > 0 : \text{ such that } \forall x \in \mathbb{R} : \text{ we have } 0 < |x-1| < \delta \Rightarrow \big|\sqrt[3]{x} - 1\big| < \epsilon.$$

Let $\epsilon > 0$. Note that, for all $x \in \mathbb{R}$,

$$\begin{aligned} |x^3 - 1| < \epsilon &\iff -\epsilon < \sqrt[3]{x} - 1 < \epsilon \\ &\iff 1 - \epsilon < \sqrt[3]{x} < 1 + \epsilon \\ &\iff (1-\epsilon)^3 < x < (1+\epsilon)^3, \end{aligned}$$

that is, the set of the $x \in \mathbb{R}$ such that $|\sqrt[3]{x} - 1| < \epsilon$ is the interval $((1-\epsilon)^3, (1+\epsilon)^3)$. Similarly, for any $\delta > 0$, the set of the $x \in \mathbb{R}$ such that $0 < |x-1| < \delta$ is the set $(1-\delta, 1+\delta) \setminus \{1\}$. Hence, in order for the implication in (4) to hold for all $x \in \mathbb{R}$, we must find $\delta > 0$ so that

$$\text{(5)} \qquad (1-\delta, 1+\delta) \setminus \{1\} \subseteq ((1-\epsilon)^3, (1+\epsilon)^3).$$

This inclusion does hold provided $1 - \delta \geq (1-\epsilon)^3$ and $1 + \delta \leq (1+\epsilon)^3$, that is, $\delta \leq 1 - (1-\epsilon)^3$ and $\delta \leq (1+\epsilon)^3 - 1$. In particular, (5) holds if we take

$$\delta = \min\{1 - (1-\epsilon)^3, (1+\epsilon)^3 - 1\}.$$

Note that, if δ is defined by this formula, then $\delta > 0$; indeed, since $\epsilon > 0$, clearly $(1-\epsilon)^3 < 1$ and $(1+\epsilon)^3 > 1$, so both $1 - (1-\epsilon)^3$ and $(1+\epsilon)^3 - 1$ are positive. In other words, for all $\epsilon > 0$, we have found a $\delta > 0$ such that the inclusion (5) holds. In view of the above discussion, this proves (4), that is, $\lim_{x\to 1} \sqrt[3]{x} = 1$. $\square$

Extra Questions

EQ1. If S is a finite set, we write $|S|$ for the number of elements of S. The nonnegative integer $|S|$ is also called the *cardinality* of S.

(i) Compute $|\{200, 2, \sqrt{2}\}|$, $|\{\text{fish}, \text{pear}\}|$, and $|\{200, 2, \sqrt{2}, 200\}|$.
(ii) Let A and B be finite *disjoint* sets, i.e. sets such that $A \cap B = \emptyset$. Express $|A \cup B|$ in terms of $|A|$ and $|B|$.
(iii) Let $A = \{0, 1, 2, 3\}$ and $B = \{2, 5, 6, 7, 8, 9\}$. What is $|A \cup B|$? Compare it with $|A| + |B|$.
(iv) Let A and B be finite sets. Find a formula for $|A \cup B|$ in terms of $|A|$, $|B|$ and $|A \cap B|$. Explain why your formula works.

Solution. (i). $|\{200, 2, \sqrt{2}\}| = 3$. $|\{\text{fish}, \text{pear}\}| = 2$. $|\{200, 2, \sqrt{2}, 200\}| = 3$. (Note that $\{200, 2, \sqrt{2}, 200\} = \{200, 2, \sqrt{2}\}$, and this set has three distinct elements.)

(ii). If A and B are disjoint, then $|A \cup B| = |A| + |B|$.

(iii). If $A = \{0, 1, 2, 3\}$ and $B = \{2, 5, 6, 7, 8, 9\}$, then $A \cup B = \{0, 1, 2, 3, 5, 6, 7, 8, 9\}$, so $|A \cup B| = 9$. On the other hand, $|A| = 4$, $|B| = 6$, so $|A| + |B| = 10$. So $|A| + |B|$ and $|A \cup B|$ differ by 1, which corresponds to the fact that A and B have one element in common: $A \cap B = \{2\}$ and $|A \cap B| = 1$.

(iv). The formula is $|A \cup B| = |A| + |B| - |A \cap B|$. This is because, when counting the elements of the union $A \cup B$, if we first count the elements of A and then the elements of B, the common elements (that is, the elements of the intersection $A \cap B$) are counted twice. So we need to subtract $|A \cap B|$ from $|A| + |B|$ in order to obtain the correct number of elements of $A \cup B$. $\square$

EQ2. Give examples of:

(i) a function whose domain is not equal to the codomain;
(ii) a function whose domain is not equal to the image;
(iii) a function whose codomain is not equal to the image;
(iv) a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ f = f$;
(v) two functions $f, g : [0, \infty) \to [0, \infty)$ such that $f \circ g$ and $g \circ f$ are not equal;
(vi) two different functions $f, g : \mathbb{R} \to \mathbb{R}$ whose restrictions to $[-1, 0]$ are equal.

Solution. [Note: there are many possible correct solutions to this question, the following are just examples.]

(i), (ii), (iii). The function $f : \{0\} \to \{0, 1\}$ defined by $f(0) = 1$ has domain $\{0\}$, codomain $\{0, 1\}$ and image $\{1\}$, which are three different sets.

(iv). The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ satisfies $f \circ f = f$, since $f(f(x)) = f(x) = x$ for all $x \in \mathbb{R}$.

(v). If we define f and g by $f(x) = x + 1$ and $g(x) = 2x$, then

$$f \circ g(x) = f(g(x)) = 2x + 1, \qquad g \circ f(x) = g(f(x)) = 2x + 2,$$

so, for example, $f \circ g(0) = 1 \neq 2 = g \circ f(0)$, and therefore $f \circ g \neq g \circ f$.

(vi). If we define f and g by

$$f(x) = -x, \qquad g(x) = |x|,$$

then, for all $x \in [-1, 0]$, we have $g(x) = -x = f(x)$ (since $x \leq 0$) and therefore $f|_{[-1,0]} = g|_{[-1,0]}$. $\square$

EQ3. (i) By expanding the expression $(a-b)^2$, or otherwise, prove that

$$ab \leq \frac{a^2+b^2}{2}$$

for all real numbers a and b.

(ii) Deduce further that

$$abcd \leq \frac{a^4+b^4+c^4+d^4}{4}$$

for all real numbers a, b, c and d.

(iii) Is it true that

$$abc \leq \frac{a^3+b^3+c^3}{3}$$

for all real numbers a, b and c? Justify your answer.

Solution. (i). Note that $(a-b)^2$ is a square of a real number, hence it is nonnegative. Therefore

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2.$$

If we bring the term $2ab$ to the left-hand side, then we obtain

$$2ab \leq a^2 + b^2,$$

and the desired inequality is obtained by dividing by 2 both sides.

(ii). Set $A = ab$ and $B = cd$. If apply the inequality in part (i) to the numbers A and B, then we obtain

$$AB \leq \frac{A^2+B^2}{2},$$

that is,

$$abcd \leq \frac{(ab)^2+(cd)^2}{2}.$$

Similarly, by applying the inequality in part (i) to the numbers a^2 and b^2, and then to the numbers c^2 and d^2, we also deduce that

$$a^2b^2 \leq \frac{a^4+b^4}{2}, \qquad c^2d^2 \leq \frac{c^4+d^4}{2}.$$

Note that $(ab)^2 = a^2b^2$ and $(cd)^2 = c^2d^2$. By combining the previous inequality we finally obtain that

$$abcd \leq \frac{(ab)^2+(cd)^2}{2} \leq \frac{\frac{a^4+b^4}{2}+\frac{c^4+d^4}{2}}{2} = \frac{a^4+b^4+c^4+d^4}{4},$$

as desired.

(iii). No, the inequality does not hold for all real numbers a, b, c. For example, if we take $a = b = -1$ and $c = 1$, then

$$abc = 1, \qquad \frac{a^3+b^3+c^3}{3} = -\frac{1}{3},$$

so $abc > -\frac{a^3+b^3+c^3}{3}$ in this case. □

EQ4. Recall the *Triangle Inequality* for real numbers: If $a, b \in \mathbb{R}$ then

$$|a+b| \leq |a| + |b|.$$

(i) Using the Triangle Inequality prove that if $a, b \in \mathbb{R}$ then

$$|a| - |b| \leq |a-b|. \tag{6}$$

(ii) Deduce further that if $a, b \in \mathbb{R}$ then

$$||a| - |b|| \leq |a-b|.$$

(iii) For which real numbers a, b does this last inequality hold with equality?

Solution. (i). If we apply the Triangle Inequality to the real numbers $a - b$ and b, we obtain

$$|(a-b)+b| \leq |a-b|+|b|,$$

that is,

$$|a| \leq |a-b|+|b|;$$

if we bring $|b|$ to the left-hand side, the desired inequality follows.

(ii). We distinguish two cases. If $|a|-|b| \geq 0$, then

$$\begin{aligned} ||a|-|b|| &= |a|-|b| \\ &\leq |a-b| \qquad \text{(by the inequality (6))}. \end{aligned}$$

If instead $|a|-|b|<0$, then

$$\begin{aligned} ||a|-|b|| &= -(|a|-|b|) \\ &= |b|-|a| \\ &\leq |b-a| \qquad \text{(by the inequality (6))} \\ &= |a-b|. \end{aligned}$$

In both cases, the desired inequality is verified.

(iii). We must determine all the real numbers a and b such that

$$||a|-|b|| = |a-b|.$$

Since both sides are nonnegative, this equality is equivalent to the one obtained by squaring both sides, that is,

$$(|a|-|b|)^2 = (a-b)^2.$$

By expanding the squares, we see that the above equality holds if and only if

$$a^2+b^2-2|a||b| = a^2+b^2-2ab,$$

that is (by cancelling a^2+b^2 from both sides and dividing by 2) if and only if

$$|ab| = ab.$$

However this is true if and only if

$$ab \geq 0,$$

that is, if and only if

$$\text{both } a, b \geq 0 \qquad \text{or} \qquad \text{both } a, b \leq 0.$$

In other words, the initial equality holds if and only if a and b have the same sign, or one of them is zero. □

EQ5. Use the Domain Convention to determine the domain of the real-valued function of a real variable defined by the rule

$$f(x) = \sqrt{x^2-x-2}.$$

Determine the range of this function. Is the function injective? If so, determine its real-valued inverse. If not, restrict the domain in such a way that it is possible to determine an inverse and find this function. (Recall that, by convention, we always take $\sqrt{x}$ to be nonnegative.)

Solution. The domain of f is the set $D = \{x \in \mathbb{R} : x^2 - x - 2 \geq 0\}$. More precisely, since $x^2 - x - 2 = (x-2)(x+1)$, it follows that $D = (-\infty, -1] \cup [2, \infty)$. The range of f is $f(D) = [0, \infty)$. Indeed, if $y = \sqrt{x^2 - x - 2}$, then $y \geq 0$, so the image is contained in $[0, \infty)$; on the other hand, if $y \geq 0$,

$$y = \sqrt{x^2 - x - 2} \quad \iff \quad y^2 = x^2 - x - 2 \quad \iff \quad x^2 - x - (2 + y^2) = 0,$$

and the last quadratic equation in x has a solution in $\mathbb{R}$ for every value of y (the discriminant is $1 + 4(2 + y^2) \geq 9 > 0$), so every $y \geq 0$ is in the image. The same argument actually shows that g is not injective: since the discriminant is strictly positive, the quadratic equation

$$x^2 - x - (2 + y^2) = 0$$

in x has two distinct real solutions, that is, every $y \in [0, \infty)$ has two distinct preimages, given by

$$x = \frac{1 \pm \sqrt{1 + 4(2 + y^2)}}{2}.$$

Note that, if we take the $+$ sign in place of $\pm$ in the last expression, then $x \in [2, \infty)$; if instead we take the $-$ sign, then $x \in (-\infty, -1]$. Consequently, if we restrict f to one of the two intervals $(-\infty, 1]$ or $[2, \infty)$, then we only have one preimage in the domain of the restriction, thus the restriction is injective. The real-valued inverse of the restriction $f|_{[2,\infty)}$ is the function $g : [0, \infty) \to \mathbb{R}$ given by

$$g(y) = \frac{1 + \sqrt{1 + 4(2 + y^2)}}{2},$$

while the real-valued inverse of $f|_{(-\infty,-1]}$ is the function $h : [0, \infty) \to \mathbb{R}$ given by

$$h(y) = \frac{1 - \sqrt{1 + 4(2 + y^2)}}{2}.$$

□

* **EQ6**. What can you say about the real numbers x and a if you are told that, for every $\epsilon > 0$, $|x - a| < \epsilon$? What if you are told that, for every positive integer n, $|x - a| < 1/n$?

Solution. If x and a are real numbers, then $|x - a|$ is also a real number and $|x - a| \geq 0$.

Suppose that $|x - a| < \epsilon$ for all $\epsilon > 0$. Then we claim that $|x - a| = 0$. Indeed, it cannot be $|x - a| > 0$, because otherwise we could take $\epsilon = |x - a|$ and obtain a contradiction to our assumption (we cannot have $\epsilon < \epsilon$). In other words, a nonnegative real number which is less than any positive real number must be zero. Finally, from $|x - a| = 0$ we deduce $x = a$.

For a similar reason, if we assume that $|x - a| < 1/n$ for all $n \in \mathbb{N}$, then again $|x - a| = 0$ and consequently $x = a$. Here we are using the following fundamental fact: if r is a nonnegative real number such that $r < 1/n$ for all $n \in \mathbb{N}$, then $r = 0$. (Indeed, if there existed $r > 0$ such that $r < 1/n$ for all $n \in \mathbb{N}$, then we would have $1/r > n$ for all $n \in \mathbb{N}$, that is, $1/r$ would be a real number larger than any natural number, but this would contradict the Archimedean property of $\mathbb{R}$.) □

* **EQ7**. Let $f : (0, \infty) \to \mathbb{R}$.

(i) Prove that, if $\lim_{x\to\infty} f(x) = \infty$, then f is unbounded.

(ii) Suppose instead that $\lim_{x\to\infty} f(x) = \ell$ for some $\ell \in \mathbb{R}$. Is it necessarily true that f is bounded in this case? Justify your answer.

Solution. (i). We need to prove that the image $f((0,\infty))$ is unbounded. More precisely, we will show that $f((0,\infty))$ is unbounded above. For a contradiction, assume instead that $f((0,\infty))$ is bounded above. Then there exists an upper bound for $f((0,\infty))$, that is, an element $M \in \mathbb{R}$ such that

$$f(x) \leq M \qquad \forall x \in (0,\infty). \tag{7}$$

Note that any real number larger than M is an upper bound of $f((0,\infty))$ as well; so, up to replacing M with a larger number if necessary, we may assume that $M > 0$. On the other hand, since $\lim_{x\to\infty} f(x) = \infty$, and $M > 0$, by definition we can find $N \in \mathbb{R}$ such that, for all $x \in (0,\infty)$,

$$x > N \qquad \Longrightarrow \qquad f(x) > M.$$

In particular, if we take $x = \max\{N, 0\} + 1$, then $x \in (0,\infty)$ and $x > N$, and therefore $f(x) > M$. This however contradicts the inequality (7).

(ii). No, f need not be bounded in this case. For example, let $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1/x = 0 \in \mathbb{R}$, as shown in lectures. However f is unbounded: indeed, as discussed in lectures, $f((0,\infty)) = (0,\infty)$, and therefore $\sup f = \infty$, that is, f is unbounded above. $\square$