Problem Sheet 5 — Model Solutions and Feedback

Question 1 (SUM). Let G be a graph with $n \geq 3$ vertices and $\delta(G) \geq n/2$.

(a) Prove that G is connected.

(b) Using (a), or otherwise, prove that for each $k < n$, if G contains a copy of C_k then G contains a copy of P_k.

(c) Prove that for each k, if G contains a copy of P_k then G contains either a copy of P_{k+1} or a copy of C_{k+1}.

(d) Using (b) and (c), or otherwise, prove that G contains a copy of C_n (i.e. a cycle including every vertex of G).

The purpose of this question is to lead you step-by-step through the proof of one of the founding results of extremal graph theory, namely Dirac's theorem (1952), which states that every graph G on $n \geq 3$ vertices with $\delta(G) \geq n/2$ contains a Hamilton cycle, that is, a cycle of length n (in other words, a cycle that includes every vertex of G. Please note that each part of the problem uses a different argument, so if you are struggling with an earlier part, please do go on to consider later parts of the question despite this.

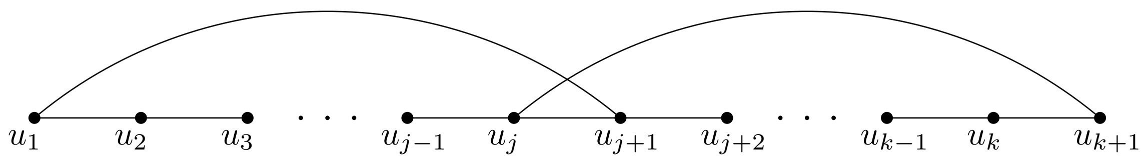
There will be a Canvas announcement early in Week 10 to provide further support as to how you might try to begin each part of the question. I have kept this separate from the problem sheet as I would strongly encourage you first to attempt the questions before you look at this additional content, and to complete as much as you can without it.

The definition of a connected graph will be presented in the lecture on the Monday of week 10; all the remaining graph theory concepts you need were covered in week 9.

Solution. For (a), let u and v be arbitrary vertices of G. If uv is an edge, then (u, v) is a walk from u to v in G. On the other hand, if uv is not an edge then $N(u), N(v) \subseteq V(G) \setminus \{u, v\}$, so $|N(u) \cup N(v)| \leq n-2$. Since $|N(u)|, |N(v)| \geq \delta(G) \geq n/2$, by inclusion-exclusion we have $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq n/2 + n/2 - (n-2) > 0$, so we may choose a vertex $w \in N(u) \cap N(v)$, meaning that w is adjacent to both u and v, so (u, w, v) is a walk from u to v in G. So in both cases we have found such a walk; since u and v were arbitrary it follows that G is connected.

For (b), let C be a copy of C_k in G, and let $v_1, \dots, v_k$ be the vertices of C, numbered in order as they appear around the cycle. Observe that since $k < n$ we may choose a vertex $x \in V(G)$ which is not a vertex of C. Since G is connected there is a walk in G from v_1 to x, and at some step this walk must leave C, meaning that there exist a vertex v_i of C and a vertex $y \in V(G) \setminus V(C)$ for which $v_i y$ is an edge. The vertices $v_{i+1}, v_{i+1}, \dots, v_k, v_1, v_2, \dots, v_i, y$, in that order, then form a path in G with $k+1$ vertices, that is, a copy of P_{k+1}.

For (c), let P be a copy of P_k in G, and let $u_1, \dots, u_{k+1}$ be the vertices of P, numbered in order as they appear on the path. Observe that if u_1 (respectively u_{k+1}) has a neighbour $x \notin V(P)$, then the vertices $x, u_1, \dots, u_{k+1}$ (respectively $u_1, \dots, u_{k+1}, x$), in that order, form a copy of P_{k+1} in G. So we may assume that $N(u_1) \subseteq P$ and $N(u_{k+1}) \subseteq P$. Now let $S = \{j : u_1 u_{j+1} \in E(G)\}$ and $T = \{j : u_j u_{k+1} \in E(G)\}$. Then our previous remark implies that $|S| \geq \deg(u_1) \geq n/2$ and $|T| \geq \deg(u_{k+1}) \geq n/2$, and by definition we have $S, T \subseteq \{1, 2, \dots, k\}$, so $|S \cup T| \leq k \leq n-1$ (the final condition holds because we have a copy of P_k in G; since P_k has $k+1$ vertices, and G has n vertices, we must have $k+1 \leq n$). By inclusion-exclusion we therefore have $|S \cap T| \geq |S| + |T| - |S \cup T| \geq 1$, and so we may choose $j \in S \cap T$. From the definition of S and T it follows that $u_1 u_{j+1}$ and $u_j u_{k+1}$ are both edges of G, and so the vertices $u_1, u_2, \dots u_j, u_{k+1}, u_k, \dots, u_{j+1}, u_1$, in that order, form a copy of C_{k+1} in G (see the diagram below).



For (d), let P be a longest path in G, and suppose that P has length k (so P is a copy of P_k in G). By (c) G contains either a copy of P_{k+1} or a copy of C_{k+1}, and the former cannot hold since P was a longest path in G. So G contains a copy of C_{k+1}. If $k \leq n-2$ then by (b) we would have that G contains a copy of P_{k+1}, again contradicting the fact that P is a longest path in G. Since G has only n vertices, we must therefore have $k = n-1$, meaning that our copy of C_{k+1} is a copy of C_n in G. □

Feedback. The key points I would look for in a good answer are as follows:

- For (a), the overall structure of the argument should be to consider some arbitrary $u, v \in V(G)$ and show that there is a walk in G from u to v, from which you draw the conclusion that G is connected. Within this outline, you should explain clearly why there is such a walk; in particular, in each case of your answer you should be clear about what this walk is (in the model solution this is (u, v) in one case and (u, w, v) in the other case).

- For (b), you should start by indicating that you are considering a copy of C_k in G. Then you need to explain why connectedness implies that there is an edge with one vertex in this copy and the other not in this copy, and then explain why this gives a copy of P_k. To do the latter clearly I think it's necessary to give names to the vertices and spell out exactly which order of vertices gives the path.

- For (c), you should again start by indicating that you are considering a copy of P_k in G. Then somehow it needs to be clear how your argument splits into the case where you get a copy of P_{k+1} and the case where you get a copy of C_{k+1}. In each case it's again important to be clear about how these copies are formed, for which I think it's necessary to label the vertices and spell out exactly which order of vertices gives the path/cycle. The other key part of the argument is to explain why there has to be some j such that u_1u_{j+1} and u_ju_{k+1} are both edges of G, as done by considering the sets S and T in the model solution, though alternatives are possible. It's really helpful to include a diagram to illustrate how you get a cycle in each case.

- For (d) you can proceed by considering a longest path in G, or alternatively you can prove by induction on k that G contains P_k for each $k < n$ (the base case $k = 1$ is trivial since you just need to find an edge, whilst the inductive step holds by applying (c) to find P_{k+1} or C_{k+1}, and then in the latter case applying (b) to find P_{k+1}). Either way, you need to consider paths and not cycles for the minimality/induction argument, since G does not necessarily contain cycles of all lengths (e.g. G could be bipartite, in which case it would not contain C_3, C_5, C_7...). However, once you have found a copy of P_{n-1}, as in the proof you can apply (c) to get a copy of C_n.

Over the course of this answer you have proved that if G is a graph on $n \geq 3$ vertices with $\delta(G) \geq n/2$ then G contains a *Hamilton cycle*, that is, a cycle which contains every vertex of G. This celebrated result was proved by Dirac in 1952, and is another foundational result of extremal graph theory. Indeed, a great deal of current active research is concerned with finding generalisations of Dirac's theorem for other subgraphs and other types of degree condition. The argument used in (c), to find either a longer path or a cycle from a given path, is known as *Pósa's rotation-extension technique*, and is a versatile method which can be applied to a wide range of problems about paths and cycles in graphs.

Note that the theorem does not hold for $n = 2$, as shown by taking G to be the graph with two vertices and one edge. Where does the model solution fail in this instance? You may feel that an additional comment should have been added at the appropriate point to reflect this. Observe also that this theorem is best-possible in the sense that the theorem would not hold for any weaker minimum degree condition. Indeed, if we let A and B be sets each of size $\lceil \frac{n}{2} \rceil$ with $|A \cup B| = n$ (meaning that $|A \cap B|$ is zero if n is even and one if n is odd), then the graph with vertex set $A \cup B$ with edge set $\{aa' : a, a' \in A\} \cup \{bb' : b, b' \in A\}$ has n vertices and $\delta(G) \geq \lceil \frac{n}{2} \rceil - 1$ but does not contain a Hamilton cycle (can you explain why not?).

Question 2. Let G be a bipartite graph with vertex classes A and B. Prove that

$$\sum_{v \in A} \deg(v) = |E(G)|.$$

Solution. Since G is bipartite with vertex classes A and B, every edge of G includes one vertex of A and one vertex of B. Place a pebble on each edge at the end which is a vertex of A. Then one pebble is placed on each edge, so the total number of pebbles placed is $|E(G)|$. On the other hand, for each vertex $v \in A$ the number of pebbles placed at v is precisely the number of edges incident to v, that is, $\deg(v)$, so the total number of pebbles placed is $\sum_{v \in A} \deg(v)$. We conclude that $\sum_{v \in A} \deg(v) = |E(G)|$. □

Feedback. This statement can be seen as a version of the handshaking lemma for bipartite graphs, and the proof is very similar to that of the handshaking lemma. The key thing is to explain clearly why counting the number of incidences between edges and vertices of A in each way (by the edges, or by the vertex degrees) gives the RHS and LHS of the equation, and that these are therefore equal. I find that the clearest way to do this is through the idea of placing an object (in this case a pebble) at each incidence, but alternatives are fine as well.

Question 3. Prove that any tree with at least two vertices has at least two leaves.

Solution. Let T be a tree on $n \geq 2$ vertices. Then T has $n-1$ edges by Corollary 5.8, and so the Handshaking Lemma implies that $\sum_{v \in V} d(v) = 2n-2$. We know from Lemma 5.9 that T has a leaf, say v. Suppose for a contradiction that v is the only leaf of T. Then each of the $n-1$ remaining vertices of T has degree at least two (zero is not possible since T is a tree and therefore connected, and one is not possible as then the vertex would be a leaf). So we would have $\sum_{v \in V} d(V) \geq 1 + 2(n-1) = 2n-1$, a contradiction. We conclude that v is not the only leaf of T, so T has at least two leaves. □

Feedback. An alternative proof for this question proceeds by considering a longest path in T, and observing that the endvertices of this path must both be leaves (can you see why?). A related problem, shown on the Extra Exercises sheet, is to show that a tree T with maximum degree $\Delta(T) = \Delta$ has at least Δ leaves. Proceeding by the model solution, the key thing to look for in an answer is a clear calculation of why you can't have just one leaf, with the source of each term explained.

Question 4.

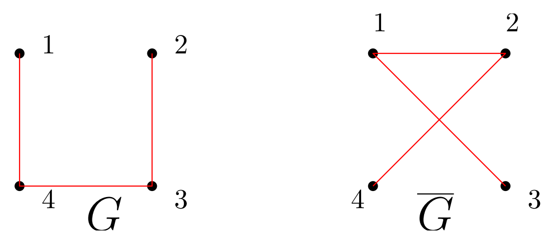
(a) For which $n \in \mathbb{N}$ does there exist a graph G with n vertices such that $|E(G)| = |E(\overline{G})|$?

(b) Find a graph G with at least two vertices for which G and $\overline{G}$ are isomorphic.

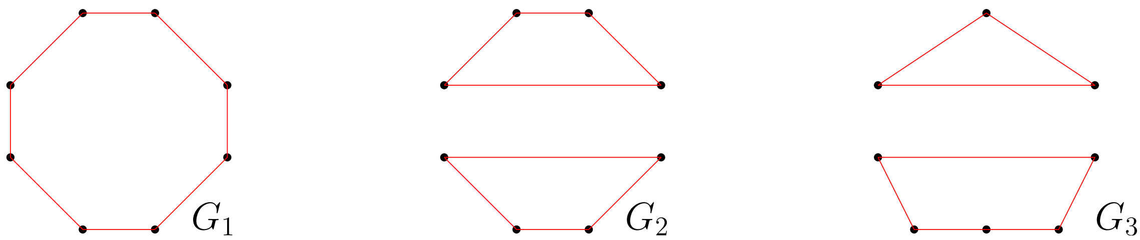
(c) Up to isomorphism, how many 2-regular graphs are there with 8 vertices?

Solution. (a) Let G be a graph on n vertices, and observe that there are $\binom{n}{2}$ unordered pairs $\{u, v\}$ of distinct vertices of G. By definition of complement each such pair is an edge of precisely one of G and $\overline{G}$. So $|E(G)| + |E(\overline{G})| = \binom{n}{2}$. If also $|E(G)| = |E(\overline{G})|$, then we have $|E(G)| = \frac{1}{2}\binom{n}{2}$, which has an integer solution if and only if $\binom{n}{2} = \frac{n(n-1)}{2}$ is even, that is, if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$ (and in the case that a solution exists, we can take G to be any graph on n vertices with this many edges).

(b) There are many possible examples; one is shown in the diagram below (to see that these graphs are isomorphic, observe that taking $f(1) = 3$, $f(2) = 4$, $f(3) = 2$ and $f(4) = 1$ gives an isomorphism from G to $\overline{G}$).



(c) Up to isomorphism, there are three 2-regular graphs with 8 vertices, as shown below. To see this, we argue as follows. Let H be a 2-regular graph on n vertices, and arbitrarily choose a vertex x_1 of H. Now choose a neighbour x_2 of x_1 in H, and for each $i \geq 2$ choose a neighbour x_{i+1} of x_i in H other than x_{i-1}; we can do this since x_i has degree two. Stop when we have $x_{i+1} = x_j$ for some $j \leq i-2$, which must occur at some point since we cannot keep continue choosing new vertices forever. We then cannot have $j \geq 2$, since then x_j would have x_{j-1}, x_{j+1} and x_{i+1} as neighbours, so would have degree at least 3. So we must have $j = 1$, that is, that there is an edge from x_i to x_1. Then each vertex in the set $S = \{x_1, x_2, \ldots, x_i\}$ has exactly two neighbours in S, so there are no edges incident to both a vertex of S and a vertex of $V \setminus S$. It follows that if we delete from H all vertices of S and all edges incident to vertices of S, we are left with a 'leftover' graph H' which is 2-regular and has $n - |S|$ vertices. Observe also that we must have $|S| \geq 3$ as we chose x_{i+1} not to be x_{i-1}.



Now let G be a 2-regular graph with eight vertices, and form a set $S \subseteq V(G)$ as described above. Then we must have $|S| \in \{3,4,5,8\}$. Indeed, we cannot have $|S| \in \{6,7\}$ as then the 'leftover' graph would be a 2-regular graph on 1 or 2 vertices, and no such graph exists. If $|S| = 8$ then G is isomorphic to the graph G_1 shown on the left of the diagram above. On the other hand, if $|S| \in \{3,4,5\}$ then the 'leftover graph' G' has 3, 4 or 5 vertices. We can then apply the above argument again (with $H = G'$ to obtain a set $S' \subseteq V(G')$. We must then have $|S'| = |V(G')|$ since otherwise we would obtain a second 'leftover graph G'' which would be a 2-regular graph on 1 or 2 vertices, and no such graph exists. So if $|S| = 3$ or $|S| = 5$ then we obtain the graph G_2 shown on the right of the diagram above, whilst if $|S| = 4$ then we obtain the graph G_3 shown in the middle of the diagram above. $\square$

Feedback. For (a), once you get that $|E(\overline{G})| = \binom{n}{2} - |E(G)|$, hopefully this leads straightforwardly to $|E(G)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$, and we need this to be an integer, which gives the solution. In your answer the key is to be clear about these steps. Also make sure you have actually answered the question – if your answer effectively is saying that if such a G exists then n is 0 or 1 modulo 4, then, whilst that statement is true, you haven't actually asserted that such graphs exist for these values of n.

For (b) you simply need to give an example (drawing it is best) and show that it has the property in the question statement (by giving an isomorphism from the graph to its complement). I think trial and error is the easiest way to do this, using (a) to rule out certain number of vertices (you know from (a) that 4 is the smallest possible order of such a graph, so just look for an example with four vertices and you should find one quickly).

Part (c) is a good example of how the best way to get started with a question can simply be to try to find examples of the graphs in question and understand their behaviour. Hopefully this will bring you to the key idea that these graphs are unions of cycles, which your answer should state clearly, and then you need to prove this. The idea for this of just considering vertex by vertex where you could walk to from a given starting vertex is a common idea which you may find useful for other proofs. Once you have proved that statement it remains to see how 8 vertices can be divided up into cycles, and in particular you should explain how you know there are no more.

The idea in (c) is a special case of a more general useful fact (which can be proved by a similar argument applying induction): in a graph G with maximum degree at most two, each connected component is either a path or a cycle. If the graph is 2-regular we can't have paths (since the endvertices would have degree one) so we just have cycles.