2DE/2DE3 Example sheet 3: Power series solutions of ODEs

1. Find the intervals of convergence of

   (a)
   $$\sum_{n=0}^{\infty} \frac{3^n}{n!} x^n,$$

   (b)
   $$\sum_{n=0}^{\infty} \frac{n^2}{2^n} (x+2)^n.$$

2. Let
   $$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} 2^{-n} x^{n-1}.$$

   Find $f(x) + g(x)$ as a power series with a single sum.

3. Find $f'(x)$ and $g'(x)$ given

   (a)
   $$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n,$$

   (b)
   $$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

4. In the above example, the series given by $f(x)$ and $g(x)$ converge to $f(x) = (1+x)^{-1}$ and $g(x) = \sin(x)$ (you can check this by deriving the Taylor series of these functions and showing that they are the same as the series above). Verify therefore that your answers for $f'(x)$ and $g'(x)$ are correct by differentiating $f(x) = (1+x)^{-1}$ and $g(x) = \sin(x)$ directly and calculating the first few terms of the Taylor series of $f'(x)$ and $g'(x)$ about $x = 0$.

5. Rewrite the following as sums from $n = 0$ and simplify the resulting series wherever possible:

   (a)
   $$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

   (b)
   $$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

6. Find the Taylor series of $f(x)$ around $x = x_0$ for the following functions and determine the interval of convergence in each case.

(a)

$$f(x) = e^x, \quad x_0 = 0,$$

(b)

$$f(x) = \ln(x), \quad x_0 = 1.$$

7. Determine the a_n (in terms of a_0) such that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

is satisfied.

8. Show that the power series solution about $x = 0$ to

$$2y'' + (x+1)y' + y = 0$$

is given by

$$y = a_0 \left(1 - \frac{x^2}{4} + \frac{x^3}{24} + ...\right) + a_1 \left(x - \frac{x^2}{4} - \frac{x^3}{8} + ...\right),$$

where a_0 and a_1 are constants.

9. Show that the power series solution about $x = 1$ to

$$(x^2 - 2x)y'' + 2y = 0$$

is given by

$$y = a_0 \left(1 + (x-1)^2 + \frac{(x-1)^4}{3} + ...\right) + a_1 \left((x-1) + \frac{(x-1)^3}{3} + \frac{2(x-1)^5}{15} + ...\right),$$

where a_0 and a_1 are constants. [*Hint: express* $x^2 - 2x$ *in terms of* $x - 1$]. Use the Ratio Test on the recurrence relation to show that the series converges in an interval around $x = 1$.

10. Find the solution of the previous question if $y(1) = 2$ and $y'(1) = 4$.

11. Find the general solution of an inhomogeneous version of the above equation (with general initial conditions):

$$(x^2 - 2x)y'' + 2y = x^2 - 2x + 1$$

[*Hint: the only thing that changes is the RHS so you can use most of your working from before. The particular solution will also be a series solution*].

12. Show that the power series solution about $x = 0$ to

$$y'' + xy' + e^x y = 0, \quad y(0) = 0, \quad y'(0) = -1$$

is given by

$$y = -\left(x - \frac{x^3}{3} - \frac{x^4}{12} + ...\right).$$

[*Hint: can you express* e^x *as a power series?*]

13. Show that the power series solution about $x = 0$ to

$$(x+2)x^2y'' - xy' + (1+x)y = 0, \quad x > 0,$$

is given by

$$y = \alpha_1\left(x - \frac{x^2}{3} + \frac{x^3}{10} - \frac{x^4}{30} + ...\right) + \alpha_2\left(x^{\frac{1}{2}} - \frac{3x^{\frac{3}{2}}}{4} + \frac{7x^{\frac{5}{2}}}{32} - \frac{133x^{\frac{7}{2}}}{1920} + ...\right),$$

where α_1 and α_2 are constants.

14. Show that the power series solution about $x = 0$ to

$$3x^2y'' + 2xy' + x^2y = 0, \quad x > 0,$$

is given by

$$y = \alpha_1 x^{\frac{1}{3}}\left(1 - \frac{x^2}{14} + \frac{x^4}{728} + ...\right) + \alpha_2\left(1 - \frac{x^2}{10} + \frac{x^4}{440} + ...\right),$$

where α_1 and α_2 are constants.