University of Birmingham
School of Mathematics

1RA Integration Autumn 2022

Problem Sheet 4

Model Solutions

You have approximately 10 working days from the release of this problem sheet to complete and submit your answers to the $\boxed{\text{SUM}}$ questions (**Q2** and **Q5**) via the Assignments tab on the 1RA Canvas page. You are strongly encouraged to attempt all of the remaining formative questions, and as many of the extra questions as you can, to prepare for the final exam. But only your solutions to the $\boxed{\text{SUM}}$ questions should be submitted to Canvas.

<table>
<tr><td colspan="2">Assignment available from: 18 November
Submission due: 1700 on Wednesday 30 November 2022</td></tr>
<tr><td>Pre-submission</td><td>Post-submission</td></tr>
<tr><td>• Your Guided Study Support Class in Weeks 9-10.
• Tutor meetings in Weeks 8-9.
• PASS from Week 8
• Library MSC from Week 8
• Office Hours (Watson 208):
Friday 1000-1130.</td><td>• Written feedback on your submission (8 December).
• Generic feedback (8 December).
• Model solutions (8 December).
• Tutor meetings in Weeks 10-11.
• Office Hours (Watson 208):
Friday 1000-1130.</td></tr>
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Instructions:

The **deadline** for submission of the two $\boxed{\text{SUM}}$ questions (**Q2** and **Q5**) is as follows:

- **By 1700 on Wednesday 30 November 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

Important:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

Q1. (a) Sketch the graph of $f:[0,4]\to\mathbb{R}$, $f(x)=\sqrt{x}$ and highlight the area covered by the difference $U(f,P)-L(f,P)$ for the partition $P=\{0,1,2,3,4\}$.

(b) Use Riemann's Criterion to prove each of the functions below are integrable:

(i) $f:[0,3]\to[0,\infty)$, $f(x)=x^2$

(ii) $f:[2,4]\to[0,100]$, $f(x)=\begin{cases}5, & x<3;\\ 100, & x=3;\\ 3, & x>3.\end{cases}$

You may use the results stated in **Q8**(b) on Problem Sheet 3 to answer (i).

Solution. (a) Figure 1 shows the graph of $f:[0,4]\to\mathbb{R}$, $f(x)=\sqrt{x}$ with the area covered by the difference $U(f,P)-L(f,P)$ for the partition $P=\{0,1,2,3,4\}$ highlighted in dark blue. This was created using the Riemann–Darboux Sums Calculator.

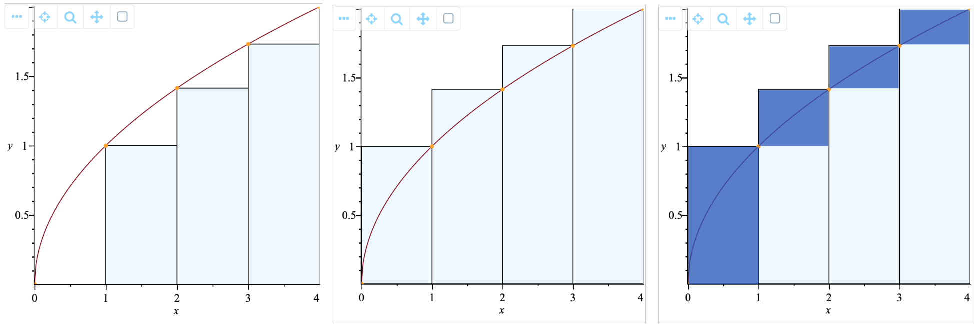

FIGURE 1. Three graphs of $f:[0,4]\to\mathbb{R}$, $f(x)=\sqrt{x}$ showing the areas covered by $L(f,P)$ in light blue, $U(f,P)$ in light blue, and $U(f,P)-L(f,P)$ in dark blue.

(b)(i) $f:[0,3]\to[0,\infty)$, $f(x)=x^2$: Let $\epsilon>0$, and for each $n\in\mathbb{N}$, consider the partition P_n of $[0,3]$ into n subintervals of equal width. Using the values for $L(f,P_n)$ and $U(f,P_n)$ in **Q4**(b), we have

$$\begin{aligned}U(f,P_n)-L(f,P_n)&=\frac{3^3}{6n^3}n(n+1)(2n+1)-\frac{3^3}{6n^3}(n-1)n(2n-1)\\&=\frac{9}{2n^2}[(n+1)(2n+1)-(n-1)(2n-1)]\\&=\frac{9}{2n^2}[(2n^2+3n+1)-(2n^2-3n+1)]\\&=\frac{9}{2n^2}6n\\&=\frac{27}{n}.\end{aligned}$$

To prove Riemann's Criterion, we need to have $\frac{27}{n}<\epsilon$ or equivalently $n>\frac{27}{\epsilon}$. Therefore, we now choose $N\in\mathbb{N}$ with $N>\frac{27}{\epsilon}$, so the above calculations give

$$U(f,P_N)-L(f,P_N)=\tfrac{27}{N}<\epsilon.$$

This proves that f is integrable by Riemann's Criterion.

(b)(ii) $f:[2,4]\to[0,100]$, $f(x)=\begin{cases}5, & x<3\\ 100, & x=3\\ 3, & x>3\end{cases}$

Let $\epsilon>0$, and for each $\delta\in(0,\frac{1}{10})$, consider the partition $P_\delta:=\{2,3-\delta,3+\delta,4\}$. In the notation $m_i:=\inf\{f(x):x\in[x_{i-1},x_i]\}$ and $M_i:=\inf\{f(x):x\in[x_{i-1},x_i]\}$, where

$\{x_0, x_1, x_2, x_3\} := \{2, 3-\delta, 3+\delta, 4\}$, we find $m_1 = 5$, $m_2 = 3$, $m_3 = 3$, whilst $M_1 = 5$, $M_2 = 100$, $M_3 = 3$, so we have

$$\begin{aligned} L(f, P_\delta) &= \sum_{i=1}^{3} m_i(x_i - x_{i-1}) \\ &= 5(1-\delta) + 3(2\delta) + 3(1-\delta) = 8 - 2\delta \end{aligned}$$

and

$$\begin{aligned} U(f, P_\delta) &= \sum_{i=1}^{3} M_i(x_i - x_{i-1}) \\ &= 5(1-\delta) + 100(2\delta) + 3(1-\delta) = 8 + 192\delta. \end{aligned}$$

This gives

$$U(f, P_\delta) - L(f, P_\delta) = (8 + 192\delta) - (8 - 2\delta) = 194\delta.$$

To prove Riemann's Criterion, we need to have $194\delta < \epsilon$ or equivalently $\delta < \frac{\epsilon}{194}$. Therefore, we now choose $\delta_0 \in (0, \frac{1}{10})$ with $\delta_0 < \frac{\epsilon}{194}$, so the above calculations give

$$U(f, P_{\delta_0}) - L(f, P_{\delta_0}) = 194\delta_0 < \epsilon.$$

This proves that f is integrable by Riemann's Criterion. $\square$

$\boxed{\text{SUM}}$ **Q2.** (a) State Riemann's Criterion for integrability.

(b) For each function defined below, use Riemann's Criterion to prove that it is integrable:

(i) $f : [0, \frac{\pi}{2}] \to \mathbb{R}$, $f(x) = 4\cos^2(x)$

(ii) $g : [1, 10] \to \mathbb{R}$, $g(x) = \begin{cases} 7, & x \in [1, 2); \\ 5, & x \in [2, 10) \\ 3, & x = 10. \end{cases}$

(c) For each function in part (b), use properties of the function and results from Lectures to prove that it is integrable (without using Riemann's Criterion directly).

Solution. (a) It is enough here to *either* quote the following theorem, which states the equivalence of Riemann's Criterion with integrability, *or* to state property (2) from the theorem, which is Riemann's Criterion specifically.

Theorem (Riemann's Criterion)**.** The following properties are equivalent for any bounded function $f : [a, b] \to \mathbb{R}$:

(1) The function f is integrable.

(2) For each $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

(b)(i) Let $\epsilon > 0$. For each $n \in \mathbb{N}$, consider the partition

$$P_n := \{x_0, x_1, \ldots, x_n\} = \{\tfrac{\pi}{2n} i : i = 0, 1, \ldots, n\}$$

of $[0, \frac{\pi}{2}]$ into n subintervals of equal width. For each $i \in \{1, \ldots, n\}$, since f is monotonic decreasing, we have

$$\begin{aligned} m_i &:= \inf\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i), \\ M_i &:= \sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1}). \end{aligned}$$

Together, we have

$$U(f,P_n)-L(f,P_n)=\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})$$
$$=\sum_{i=1}^{n}(f(x_{i-1})-f(x_i))\tfrac{\pi}{2n}$$
$$=[(f(x_0)-f(x_1))+(f(x_1)-f(x_2))+\ldots+(f(x_{n-1})-f(x_n))]\tfrac{\pi}{2n}$$
$$=[f(x_0)+(f(x_1)-f(x_1))+\ldots+(f(x_{n-1})-f(x_{n-1}))+-f(x_n)]\tfrac{\pi}{2n}$$
$$=(f(x_0)-f(x_n))\tfrac{\pi}{2n}$$
$$=(f(0)-f(\tfrac{\pi}{2}))\tfrac{\pi}{2n}$$
$$=4\tfrac{\pi}{2n}=\tfrac{2\pi}{n},$$

where in the fourth line we have simply reordered the terms in the sum to show how they can be grouped into pairs that cancel (the telescoping sum can also be simplified using summation notation as in part (b)(ii) below).

To prove Riemann's Criterion, we need to have $\frac{2\pi}{n}<\epsilon$ or equivalently $n>\frac{2\pi}{\epsilon}$. Therefore, we now choose $N\in\mathbb{N}$ with $N>\frac{2\pi}{\epsilon}$, so the above calculations give

$$U(f,P_N)-L(f,P_N)=\frac{2\pi}{N}<\epsilon.$$

Hence f is integrable by Riemann's Criterion.

(b)(ii) [Method 1] Let $\epsilon>0$. For each $n\in\mathbb{N}$, consider the partition

$$P_n:=\{x_0,x_1,\ldots,x_n\}=\{1+\tfrac{9}{n}i : i=0,1,\ldots,n\}$$

of $[1,10]$ into n subintervals of equal width. For each $i\in\{1,\ldots,n\}$, since f is monotonic decreasing, we have

$$m_i:=\inf\{g(x):x\in[x_{i-1},x_i]\}=g(x_i),$$
$$M_i:=\sup\{g(x):x\in[x_{i-1},x_i]\}=g(x_{i-1}).$$

Together, we have

$$U(g,P_n)-L(g,P_n)=\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})$$
$$=\sum_{i=1}^{n}(g(x_{i-1})-g(x_i))\tfrac{9}{n}$$
$$=\left[\sum_{i=1}^{n}g(x_{i-1})-\sum_{i=1}^{n}g(x_i)\right]\frac{9}{n}$$
$$=\left[\sum_{j=0}^{n-1}g(x_j)-\sum_{i=1}^{n}g(x_i)\right]\frac{9}{n}$$
$$=(g(x_0)-g(x_n))\tfrac{9}{n}$$
$$=(7-1)\tfrac{9}{n}$$
$$=\tfrac{54}{n},$$

where in the fourth line we have made the substitution $j:=i-1$ (the telescoping sum can also be simplified by writing out the terms as in part (b)(i) above).

To prove Riemann's Criterion, we need to have $\frac{54}{n}<\epsilon$ or equivalently $n>\frac{54}{\epsilon}$. Therefore, we now choose $N\in\mathbb{N}$ with $N>\frac{54}{\epsilon}$, so the above calculations give

$$U(g,P_N)-L(g,P_N)=\tfrac{54}{N}<\epsilon.$$

Hence f is integrable by Riemann's Criterion.

(b)(ii) [Method 2] Let $\epsilon > 0$, and for each $\delta \in (0, \frac{1}{10})$, consider the partition

$$P_\delta := \{1, 2-\delta, 2+\delta, 10-\delta, 10\} = \{x_0, x_1, x_3, x_4\}.$$

Observe that

$$\begin{aligned} U(f,P_\delta) - L(f,P_\delta) &= \sum_{i=1}^{4}(M_i - m_i)(x_i - x_{i-1}) \\ &= (M_1 - m_1)(1-\delta) + (M_2 - m_2)(2\delta) + (M_3 - m_3)(8-2\delta) + (M_4 - m_4)(\delta) \\ &= (7-7)(1-\delta) + (7-5)(2\delta) + (5-5)(8-2\delta) + (5-3)(\delta) \\ &= 6\delta. \end{aligned}$$

To prove Riemann's Criterion, we need to have $6\delta < \epsilon$ or equivalently $\delta < \frac{\epsilon}{6}$. Therefore, we now choose $\delta_0 \in (0, \frac{1}{10})$ with $\delta_0 < \frac{\epsilon}{6}$, so the above calculations give

$$U(f, P_{\delta_0}) - L(f, P_{\delta_0}) = 6\delta_0 < \epsilon.$$

This proves that f is integrable by Riemann's Criterion.

(c)(i) The function f is monotonic decreasing, since $f'(x) = -8\cos(x)\sin(x) \leq 0$ for all $x \in [0, \frac{\pi}{2}]$, so the integrability of monotonic functions in Theorem 8.2.3 of the Lecture Notes implies that f is integrable.

Alternatively, observe that f is continuous, since it is the composition of a trigonometric function and a polynomial, so Theorem 8.3.5 of the Lecture Notes implies that f is integrable.

Either of these approaches is an acceptable solution.

(c)(ii) The function g is monotonic decreasing by inspection, so the integrability of monotonic functions in Theorem 8.2.3 of the Lecture Notes implies that g is integrable. □

Q3. Suppose that $f : [a, b] \to [0, \infty)$ is bounded and integrable, where $-\infty < a < b < \infty$:

(a) Prove that if $\alpha \geq 0$ and P is a partition of $[a, b]$, then

$$U(\alpha f, P) = \alpha U(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha L(f, P).$$

(b) Use Riemann's Criterion and (a) to prove that αf is integrable.

(c) Explain, without working hard, why the integrability of αf implies that

$$\int_a^b (\alpha f) = \sup\{L(\alpha f, P) : P \text{ is a partition of } [a, b]\}.$$

(d) Use (a) and (c) to conclude that $\int_a^b (\alpha f) = \alpha(\int_a^b f)$.

Solution. (a) Let $P = \{x_0, \ldots, x_n\}$ denote a partition of $[a, b]$ and define

$$\begin{aligned} M_i &:= \sup\{f(x) : x \in [x_{i-1}, x_i]\} \\ M_{i,\alpha} &:= \sup\{(\alpha f)(x) : x \in [x_{i-1}, x_i]\} \\ m_i &:= \inf\{f(x) : x \in [x_{i-1}, x_i]\} \\ m_{i,\alpha} &:= \inf\{(\alpha f)(x) : x \in [x_{i-1}, x_i]\} \end{aligned}$$

for each $i \in \{1, \ldots, n\}$. Observe that $M_{i,\alpha} = \alpha M_i$ and $m_{i,\alpha} = \alpha m_i$ for each $i \in \{1, \ldots, n\}$, hence

$$\begin{aligned} U(\alpha f, P) &= \sum_{i=1}^{n} M_{i,\alpha}(x_i - x_{i-1}) \\ &= \alpha \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \alpha U(f, P) \end{aligned}$$

and

$$\begin{aligned} L(\alpha f, P) &= \sum_{i=1}^{n} m_{i,\alpha}(x_i - x_{i-1}) \\ &= \alpha \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \alpha L(f, P), \end{aligned}$$

as required.

(b) The function f is integrable, so Riemann's Criterion implies that for each $\delta > 0$, there exists a partition P_δ of $[a, b]$ such that $U(f, P_\delta) - L(f, P_\delta) < \delta$. Moreover, by (a), we then have

$$U(\alpha f, P_\delta) - L(\alpha f, P_\delta) = \alpha U(f, P_\delta) - \alpha L(f, P_\delta) < \alpha\delta.$$

We can now use Riemann's Criterion to prove that αf is integrable. Let $\epsilon > 0$ and choose $\delta_0 > 0$ such that $\alpha\delta_0 = \epsilon$ (i.e. $\delta_0 := \epsilon/\alpha$) so that

$$U(\alpha f, P_{\delta_0}) - L(\alpha f, P_{\delta_0}) < \alpha\delta_0 = \epsilon.$$

This proves that αf is integrable by Riemann's Criterion.

(c) The function αf is integrable, which means $\underline{\int_a^b}(\alpha f) = \overline{\int_a^b}(\alpha f)$ by *definition*. If we write out the definition of the lower and upper integrals, then this becomes

$$\begin{aligned} &\sup\{L(\alpha f, P) : P \text{ is a partition of } [a, b]\} \\ &\qquad = \inf\{U(\alpha f, P) : P \text{ is a partition of } [a, b]\} \end{aligned}$$

and the integral $\int_a^b(\alpha f)$ is *defined* to equal this common value.

(d) It follows from (a) that $\sup_P L(\alpha f, P) = \alpha \sup_P L(f, P)$ whilst (c) shows that $\int_a^b(\alpha f) = \sup_P L(\alpha f, P)$ and $\int_a^b f = \sup_P L(f, P)$, hence

$$\int_a^b(\alpha f) = \sup_P L(\alpha f, P) = \alpha \sup_P L(f, P) = \alpha \int_a^b f,$$

as required. □

Q4. Suppose that $f : [a, b] \to [0, \infty)$ is bounded and integrable, where $-\infty < a < b < \infty$, and that $0 \leq f(x) \leq M$ for all $x \in [a, b]$ and some $M > 0$:

(a) Prove that $0 \leq \underline{\int_a^b} f$ and that $\overline{\int_a^b} f \leq M(b - a)$.

(b) Use (a) and the definition of the integral to prove that $0 \leq \int_a^b f \leq M(b - a)$.

(c) Use (b) to prove that $\lim_{h\to 0^+}\left(\int_a^{a+h} f\right) = 0$ and $\lim_{h\to 0^+}\left(\int_{b-h}^{b} f\right) = 0$.

Solution. (a) Let $P = \{x_0, x_1, \ldots, x_n\}$ denote a partition of $[a, b]$. Observe that $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} \geq 0$ for each $i \in \{0, 1, \ldots, n\}$, hence

$$L(f, P) := \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \geq 0.$$

We combine this with the definition of the lower integral to obtain

$$\underline{\int_a^b} f := \sup_P L(f, P) \geq 0.$$

Next, observe that $M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\} \leq M$ for each $i \in \{0, 1, \ldots, n\}$, hence

$$U(f, P) := \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \leq M \sum_{i=1}^{n}(x_i - x_{i-1}) = M(b - a).$$

We combine this with the definition of the upper integral to obtain

$$\overline{\int_a^b} f := \inf_P U(f,P) \leq M(b-a).$$

(b) The function f is integrable, so (a) implies that

$$0 \leq \underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f \leq M(b-a),$$

as required

(c) If $h \in (0, b-a)$, then (b) implies that

$$0 = 0((a+h)-a) \leq \int_a^{a+h} f \leq M((a+h)-a) = Mh$$

and

$$0 = 0(b-(b-h)) \leq \int_{b-h}^b f \leq M(b-(b-h)) = Mh.$$

Now $\lim_{h\to 0^+}\left(\int_a^{a+h} f\right) = 0$ and $\lim_{h\to 0^+}\left(\int_{b-h}^b f\right) = 0$ follow from the Sandwich Theorem, since $\lim_{h\to 0^+} Mh = 0$. □

$\boxed{\text{SUM}}$ **Q5**. (a) State the First Fundamental Theorem of Calculus.

(b) Use the First Fundamental Theorem of Calculus to prove that each of the following functions is differentiable and find a formula for its derivative function that does not contain the integral symbol:

(i) $F : [2,4] \to \mathbb{R},\ F(x) := \displaystyle\int_2^x \frac{1}{\log(t)}\ \mathrm{d}t,\ x \in [2,4]$

(ii) $G : [-1,1] \to \mathbb{R},\ G(x) := \displaystyle\int_{-5}^{5x^4+3x^2+1} e^{-t^2}\ \mathrm{d}t,\ x \in [-1,1]$

(iii) $H : [\pi, 2\pi] \to \mathbb{R},\ H(x) := \displaystyle\int_x^{2\pi} \frac{\sin(t)}{t}\ \mathrm{d}t,\ x \in [\pi, 2\pi]$

Solution. (a) The First Fundamental Theorem of Calculus is stated below. This is Theorem 8.3.5 in the Lecture Notes. It is good practice, although not essential, to use a theorem environment to express the result.

Theorem. Let $f : [a,b] \to \mathbb{R}$ denote a bounded integrable function and suppose that $F : [a,b] \to \mathbb{R}$ is given by $F(x) := \int_a^x f$ for all $x \in [a,b]$. If f is continuous at c, where $c \in [a,b]$, then F is differentiable at c with derivative $F'(c) = f(c)$.

(b)(i) Let $f : [2,4] \to \mathbb{R}$ be given by $f(x) := 1/\log(x)$ for all $x \in [2,4]$. This is a continuous function because the Logarithm function is continuous and $\log(x) > 0$ for all $x \in [2,4]$. The First Fundamental Theorem of Calculus applies at each point c in the interval $[2,4]$, because f is continuous at each such point, so F is differentiable with $F'(x) = f(x) = 1/\log(x)$ for all $x \in [2,4]$.

(b)(ii) Let $f : [-5,9] \to \mathbb{R}$ be given by $f(x) := e^{-x^2}$ for all $x \in [-5,9]$. The First Fundamental Theorem of Calculus does not apply directly here because $5x^4 + 3x^2 + 1$ appears in upper endpoint of the integral. Instead, let $F : [1,9] \to \mathbb{R}$ be given by

$$F(x) := \int_{-5}^x e^{-t^2}\ \mathrm{d}t = \int_{-5}^x f(t)\ \mathrm{d}t$$

for all $x \in [1,9]$. The function f is continuous, since it is a composition of the exponential functional and a polynomial, so the First Fundamental Theorem of Calculus implies that F is differentiable with $F'(x) = f(x) = e^{-x^2}$ for all $x \in [-5,9]$.

Next, let $g : [-1,1] \to \mathbb{R}$ be given by $g(x) := 5x^4 + 3x^2 + 1$ for all $x \in [-1,1]$. The polynomial g is differentiable with derivative $g'(x) = 20x^3 + 6x$ for all $x \in [-1,1]$. Moreover, we have $G = F \circ g$, where F is also differentiable, so the Chain Rule implies that G is differentiable with derivative

$$\begin{aligned} G'(x) &= (F \circ g)'(x) = F'(g(x))g'(x) \\ &= e^{-(g(x))^2}(20x^3 + 6x) \\ &= e^{-(5x^4+3x^2+1)^2}(20x^3 + 6x) \end{aligned}$$

for all $x \in [-1,1]$.

(b)(iii) Let $f : [\pi, 2\pi] \to \mathbb{R}$ be given by $f(x) := \sin(x)/x$ for all $x \in [\pi, 2\pi]$. The First Fundamental Theorem of Calculus does not apply directly here because the variable x appears in the lower endpoint, not the upper endpoint, of the integral. Instead, let $F : [\pi, 2\pi] \to \mathbb{R}$ be given by

$$F(x) := \int_\pi^x \frac{\sin(t)}{t}\ \mathrm{d}t = \int_\pi^x f(t)\ \mathrm{d}t$$

for all $x \in [\pi, 2\pi]$. The function f is continuous, since the Sine function is continuous and $x > 0$ for all $x \in [\pi, 2\pi]$, so the First Fundamental Theorem of Calculus implies that F is differentiable with $F'(x) = f(x) = \sin(x)/x$ for all $x \in [\pi, 2\pi]$.

Next, we use the restriction and extension properties of the integral in Theorem 8.4.1 of the Lecture Notes to write

$$\begin{aligned} H(x) &= \int_x^{2\pi} \frac{\sin(t)}{t}\ \mathrm{d}t = \int_\pi^{2\pi} \frac{\sin(t)}{t}\ \mathrm{d}t - \int_\pi^x \frac{\sin(t)}{t}\ \mathrm{d}t \\ &= \int_\pi^{2\pi} \frac{\sin(t)}{t}\ \mathrm{d}t - F(x) \end{aligned}$$

for all $x \in [\pi, 2\pi]$. The first term on the right-hand side of the equality above is a real number, which does not depend on x, and constant functions are differentiable with derivatives identically equal to zero, so the linearity of derivatives implies that H is differentiable with $H'(x) = 0 - F'(x) = -\sin(x)/x$ for all $x \in [\pi, 2\pi]$. □

Q6. (a) State the Second Fundamental Theorem of Calculus.

(b) Suppose that $-\infty < a < b < \infty$. Use the Second Fundamental Theorem of Calculus to prove that

$$\int_a^b |x|\ \mathrm{d}x = \begin{cases} \frac{1}{2}(b^2 - a^2), & a \geq 0; \\ \frac{1}{2}(a^2 + b^2), & a < 0 \leq b; \\ \frac{1}{2}(a^2 - b^2), & b < 0. \end{cases}$$

You must verify all hypotheses required to apply the Second Fundamental Theorem of Calculus. In particular, if you use the fact that a certain function is an antiderivative of the absolute value function, then you must prove this fact (be careful proving differentiability at the origin).

(c) Let $f : \mathbb{R} \setminus [-3,-2] \to \mathbb{R}$ be defined by $f(x) := |x|$ for all $x \in \mathbb{R} \setminus [-3,-2]$. Find two antiderivatives F_1 and F_2 of f such that

$$F_1(x) - F_2(x) = \begin{cases} 9, & x > -2; \\ 3, & x < -3. \end{cases}$$

You must prove that your choices for F_1 and F_2 are antiderivatives of f.

Solution. (a) The Second Fundamental Theorem of Calculus is stated below. It is good practice, although not essential, to use a theorem environment to express the result.

Theorem. If $f : [a,b] \to \mathbb{R}$ is a bounded integrable function, and there exists a differentiable function $g : [a,b] \to \mathbb{R}$ such that $f = g'$, then $\int_a^b f = g(b) - g(a)$.

(b) The function $f : [a, b] \to \mathbb{R}$ given by $f(x) := |x|$ for all $x \in [a, b]$ is continuous, hence it is integrable by Theorem 8.3.5 in the Lecture Notes. An antiderivative of f is any differentiable function g such that $f = g'$. We claim that $g : [a, b] \to \mathbb{R}$ given by

$$g(x) := \begin{cases} \frac{1}{2}x^2, & \text{if } x \geq 0; \\ -\frac{1}{2}x^2, & \text{if } x < 0, \end{cases}$$

is differentiable with $g'(x) = |x| = f(x)$ for all $x \in [a, b]$. To prove this, we consider the following three cases:

• If $a \geq 0$, then $g(x) = \frac{1}{2}x^2$ for all $x \in [a, b]$, so g is a polynomial around each point in its domain, and which is differentiable with derivative $g'(x) = x = |x|$ for all $x \in [a, b]$.

• If $b < 0$, then $g(x) = -\frac{1}{2}x^2$ for all $x \in [a, b]$, so g is a polynomial around each point in its domain, and which is differentiable with derivative $g'(x) = -x = |x|$ for all $x \in [a, b]$.

• If $a < 0 \leq b$, then g is differentiable on $[a, 0)$ and on $(0, b]$ as in the above two cases. We must check if it is differentiable at the origin, however, because in this case g is *not* given by a *single* polynomial expression in any open interval containing the origin. To do this, we calculate

$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{\frac{1}{2}h^2 - 0}{h} = \lim_{h \to 0^+} \tfrac{1}{2}h = 0$$

and

$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{-\frac{1}{2}h^2 - 0}{h} = \lim_{h \to 0^-} -\tfrac{1}{2}h = 0.$$

The equality of these limits proves that g is differentiable at 0 with

$$g'(0) := \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = 0 = f(0).$$

It is important to recognise here that differentiability away from the origin is immediate from the differentiability of polynomials on $\mathbb{R}$. Around the origin, however, the function g is not given by a *single* polynomial, so we must instead argue directly using the definition of differentiability.

To conclude, since f is integrable, and we have found an antiderivative g (i.e. $f = g'$), we can apply the Second Fundamental Theorem of Calculus (Theorem 9.2.1 in the Lecture Notes) to obtain

$$\begin{aligned} \int_a^b |x| \, \mathrm{d}x &= \int_a^b f = \int_a^b g' \\ &= g(b) - g(a) = \begin{cases} \frac{1}{2}(b^2 - a^2), & a \geq 0; \\ \frac{1}{2}(a^2 + b^2), & a < 0 \leq b; \\ \frac{1}{2}(a^2 - b^2), & b < 0, \end{cases} \end{aligned}$$

as required.

(c) Now suppose that $f : \mathbb{R} \setminus [-3, -2] \to \mathbb{R}$ is defined by $f(x) := |x|$ for all $x \in \mathbb{R} \setminus [-3, -2]$. The domain of f is the set $\mathbb{R} \setminus [-3, -2] = (-\infty, -3) \cup (-2, \infty)$, which consists of two *disconnected* intervals $(-\infty, -3)$ and $(-2, \infty)$. It is this property which allows us to obtain antiderivatives, F_1 and F_2, which do *not* differ by an additive constant. For example, the functions $F_1 : \mathbb{R} \setminus [-3, -2] \to \mathbb{R}$ defined by

$$F_1(x) := \begin{cases} \frac{1}{2}x^2 + 9, & \text{if } x \in [0, \infty); \\ -\frac{1}{2}x^2 + 9, & \text{if } x \in (-2, 0); \\ -\frac{1}{2}x^2 + 3, & \text{if } x \in (-\infty, -3), \end{cases}$$

and $F_2 : \mathbb{R} \setminus [-3, -2] \to \mathbb{R}$ defined by

$$F_2(x) := \begin{cases} \frac{1}{2}x^2, & \text{if } x \in [0, \infty); \\ -\frac{1}{2}x^2, & \text{if } x \in (-2, 0); \\ -\frac{1}{2}x^2, & \text{if } x \in (-\infty, -3), \end{cases}$$

are both antiderivatives of f with

$$F_1(x) - F_2(x) = \begin{cases} 9, & x > -2; \\ 3, & x < -3, \end{cases}$$

as required. In particular, the functions F_1 and F_2 are both differentiable with $F_1' = F_2' = f$, as in part (b). It is enough to reference the analogous discussion and computations in part (b) for justification here, but the details are included below for completeness.

The functions F_1 and F_2 are each given by a single polynomial around each point in their domain $(-\infty, -3) \cup (-2, \infty)$, except at the origin. The differentiability of F_1 at the origin is proved by computing

$$\lim_{h \to 0^+} \frac{F_1(0+h) - F_1(0)}{h} = \lim_{h \to 0^+} \frac{(\frac{1}{2}h^2 + 9) - 9}{h} = \lim_{h \to 0^+} \tfrac{1}{2}h = 0$$

and

$$\lim_{h \to 0^-} \frac{F_1(0+h) - F_1(0)}{h} = \lim_{h \to 0^-} \frac{(-\frac{1}{2}h^2 + 9) - 9}{h} = \lim_{h \to 0^-} -\tfrac{1}{2}h = 0,$$

to deduce that $F_1'(0) = 0 = f(0)$. The differentiability of F_2 at the origin is proved likewise by computing

$$\lim_{h \to 0^+} \frac{F_2(0+h) - F_2(0)}{h} = \lim_{h \to 0^+} \frac{\frac{1}{2}h^2 - 0}{h} = \lim_{h \to 0^+} \tfrac{1}{2}h = 0$$

and

$$\lim_{h \to 0^-} \frac{F_1(0+h) - F_1(0)}{h} = \lim_{h \to 0^-} \frac{-\frac{1}{2}h^2 - 0}{h} = \lim_{h \to 0^-} -\tfrac{1}{2}h = 0,$$

to deduce that $F_2'(0) = 0 = f(0)$.

More generally, we could choose

$$F_1(x) := \begin{cases} \frac{1}{2}x^2 + c_1, & \text{if } x \in [0, \infty); \\ -\frac{1}{2}x^2 + c_1, & \text{if } x \in (-2, 0); \\ -\frac{1}{2}x^2 + c_2, & \text{if } x \in (-\infty, -3), \end{cases}$$

and

$$F_2(x) := \begin{cases} \frac{1}{2}x^2 + c_3, & \text{if } x \in [0, \infty); \\ -\frac{1}{2}x^2 + c_3, & \text{if } x \in (-2, 0); \\ -\frac{1}{2}x^2 + c_4, & \text{if } x \in (-\infty, -3), \end{cases}$$

for any real numbers c_1, c_2, c_3 and c_4, such that $c_1 - c_3 = 9$ and $c_2 - c_4 = 3$. □

Q7. Find the following antiderivatives and integrals (henceforth $\log(x) := \log_e(x)$):

(a) $\displaystyle\int_1^e x^2 \log(x)\, \mathrm{d}x$

(b) $\displaystyle\int (\log(x))^2\, \mathrm{d}x$

(c) $\displaystyle\int_0^\pi e^x \cos(x)\, \mathrm{d}x$

Solution. (a) We apply Integration by Parts with

$$\begin{cases} u(x) = \log(x) \\ v(x) = \frac{1}{3}x^3 \end{cases} \quad \text{and} \quad \begin{cases} u'(x) = x^{-1} \\ v'(x) = x^2 \end{cases}$$

to obtain

$$\begin{aligned}\int_1^e x^2\log(x)\ \mathrm{d}x &= \left[\tfrac{1}{3}x^3\log(x)\right]_1^e - \int_1^e x^{-1}(\tfrac{1}{3}x^3)\ \mathrm{d}x\\ &= \tfrac{1}{3}e^3 - 0 - \left[\tfrac{1}{9}x^3\right]_1^e\\ &= \tfrac{1}{3}e^3 - \tfrac{1}{9}e^3 + \tfrac{1}{9}\\ &= \tfrac{2}{9}e^3 + \tfrac{1}{9}.\end{aligned}$$

(b) After recalling that $\int \log(x)\ \mathrm{d}x = x(\log(x) - 1)$ by Example 9.3.2 in the Lecture Notes, we apply Integration by Parts with

$$\begin{cases} u(x) = \log(x) \\ v(x) = x(\log(x) - 1) \end{cases} \qquad \text{and} \qquad \begin{cases} u'(x) = x^{-1} \\ v'(x) = \log(x) \end{cases}$$

to obtain

$$\begin{aligned}&\int (\log(x))^2\ \mathrm{d}x\\ &= (\log(x))x(\log(x) - 1) - \int x^{-1}x(\log(x) - 1)\\ &= (\log(x))x(\log(x) - 1) - \int (\log(x) - 1)\\ &= (\log(x))x(\log(x) - 1) - x(\log(x) - 1) + x\\ &= (\log(x) - 1)x(\log(x) - 1) + x\\ &= x(\log(x) - 1)^2 + x\\ &= x(\log(x))^2 - 2x\log(x) + 2x.\end{aligned}$$

(c) We apply Integration by Parts twice. First with $u(x) = e^x$ and $v(x) = \sin(x)$, so $u'(x) = e^x$ and $v'(x) = \cos(x)$, to obtain

$$\int_0^\pi e^x\cos(x)\ \mathrm{d}x = [e^x\sin(x)]_0^\pi - \int_0^\pi e^x\sin(x)\ \mathrm{d}x.$$

Next, with $u(x) = e^x$ and $v(x) = -\cos(x)$, so $u'(x) = e^x$ and $v(x)' = \sin(x)$, to obtain

$$\begin{aligned}\int_0^\pi e^x\sin(x)\ \mathrm{d}x &= [-e^x\cos(x)]_0^\pi - \int_0^\pi e^x(-\cos(x))\ \mathrm{d}x\\ &= [-e^x\cos(x)]_0^\pi + \int_0^\pi e^x\cos(x)\ \mathrm{d}x.\end{aligned}$$

We combine the previous two identities to obtain

$$\int_0^\pi e^x\cos(x)\ \mathrm{d}x = [e^x\sin(x)]_0^\pi - \left([-e^x\cos(x)]_0^\pi + \int_0^\pi e^x\cos(x)\ \mathrm{d}x\right),$$

hence

$$\int_0^\pi e^x\cos(x)\ \mathrm{d}x = \tfrac{1}{2}\left[e^x(\sin(x) + \cos(x))\right]_0^\pi = -\tfrac{1}{2}(e^\pi + 1).$$

Q8. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{x-4}{x^2 - 5x + 6}\ \mathrm{d}x$

(b) $\displaystyle\int_1^2 \frac{x^5 + x - 1}{x^3 + 1}\ \mathrm{d}x$

(c) $\displaystyle\int \frac{x^2 + 2x - 1}{x^3 - x}\ \mathrm{d}x$

Solution. (a) The integrand is a rational function so we use the method of partial fractions. The denominator $x^2 - 5x + 6 = (x-3)(x-2)$ has two linear factors, so the appropriate form of the partial fraction expansion is

$$\frac{x-4}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}.$$

After finding a common denominator, we obtain

$$x - 4 = A(x-2) + B(x-3) = (A+B)x + (-2A - 3B).$$

We equate the coefficients of both x and x^0 to obtain the simultaneous equations $A+B=1$ and $(-2A-3B) = -4$ with the solution $A = -1$ and $B = 2$, hence

$$\begin{aligned} \int \frac{x-4}{x^2-5x+6}\,\mathrm{d}x &= \int \frac{-1}{x-3}\,\mathrm{d}x + \int \frac{2}{x-2}\,\mathrm{d}x \\ &= -\log|x-3| + 2\log|x-2|. \end{aligned}$$

(b) The integrand is a rational function so we use the method of partial fractions. The degree of the numerator is larger than degree of the denominator, so we use polynomial long division to obtain

$$\frac{x^5+x-1}{x^3+1} = x^2 + \frac{-x^2+x-1}{x^3+1}.$$

The denominator $x^3+1 = (x+1)(x^2-x+1)$ has a linear factor and an irreducible quadratic factor, so the appropriate form of the partial fraction expansion is

$$\frac{-x^2+x-1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

After finding a common denominator, we obtain

$$\begin{aligned} -x^2 + x - 1 &= A(x^2-x+1) + (Bx+C)(x+1) \\ &= (A+B)x^2 + (-A+B+C)x + (A+C). \end{aligned}$$

We equate the coefficients of x^2, x and x^0 to obtain the simultaneous equations

$$\begin{aligned} A + B &= -1 \\ -A + B + C &= 1 \\ A + C &= -1 \end{aligned}$$

with the solution $A = -1$, $B = 0$ and $C = 0$, hence

$$\begin{aligned} \int_1^2 \frac{x^5+x-1}{x^3+1}\,\mathrm{d}x &= \int_1^2 x^2\,\mathrm{d}x + \int_1^2 \frac{-1}{x+1}\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 - \log|x+1|\right]_1^2 \\ &= \tfrac{7}{3} + \log(\tfrac{2}{3}). \end{aligned}$$

(c) The integrand is a rational function so we use the method of partial fractions. The denominator $x^3 - x = x(x+1)(x-1)$ has three linear factors, so the correct form of the partial fraction decomposition is

$$\frac{x^2+2x-1}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

After finding a common denominator, we obtain

$$\begin{aligned} x^2 + 2x - 1 &= A(x^2-1) + B(x^2-x) + C(x^2+x) \\ &= (A+B+C)x^2 + (-B+C)x + (-A). \end{aligned}$$

We equate the coefficients of x^2, x and x^0 to obtain the simultaneous equations

$$\begin{aligned} A + B + C &= 1 \\ -B + C &= 2 \\ -A &= -1 \end{aligned}$$

with the solution $A = 1$, $B = -1$ and $C = 1$, hence

$$\begin{aligned} \int \frac{x^2 + 2x - 1}{x^3 - x} \, \mathrm{d}x &= \int \frac{1}{x} \, \mathrm{d}x + \int \frac{-1}{x+1} \, \mathrm{d}x + \int \frac{1}{x-1} \, \mathrm{d}x \\ &= \log|x| - \log|x+1| + \log|x-1|. \end{aligned}$$

Extra Questions

EQ1. Suppose that $f : [a, b] \to [0, \infty)$ is bounded and integrable, where $-\infty < a < b < \infty$. A *tagged partition* (P, T) of $[a, b]$ consists of a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and a collection $T = \{t_1, \ldots, t_n\}$ of *tags* satisfying $t_1 \in [x_0, x_1], \ldots, t_n \in [x_{n-1}, x_n]$. The corresponding *Riemann Sum* is defined by

$$R(f, P, T) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

(a) Prove that $L(f, P) \leq R(f, P, T) \leq U(f, P)$ for any tagged partition (P, T).
(b) Prove that $L(f, P) \leq \int_a^b f \leq U(f, P)$ for any partition P.
(c) Use Riemann's Criterion to prove that for each $\epsilon > 0$, there exists a partition P such that $|R(f, P, T) - \int_a^b f| < \epsilon$ whenever T is a collection of tags for P.

EQ2. (a) For each function defined below, use the properties of the function and results from Lectures/Lectures Notes to prove that it is integrable:
  (i) $f : [0, 10] \to \mathbb{R}$, $f(x) = 3x^2 + 5x + 9$
  (ii) $g : [1, 100] \to \mathbb{R}$, $g(x) = \lfloor x \rfloor := \max\{n \in \mathbb{N} : n \leq x\}$
(b) Use Riemann's Criterion to prove that each function in part (a) is integrable. You may use the results stated in **Q8**(b) on Problem Sheet 3 (but you are not required to do so).

EQ3. (a) Prove that if $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both uniformly continuous, then $f + g$ is uniformly continuous.
(b) Suppose that $f : [a, \infty) \to \mathbb{R}$ is a continuous function, where $-\infty < a < b < \infty$. Prove that if f is uniformly continuous on both $[a, b]$ and $[b, \infty)$, then f is uniformly continuous on $[a, \infty)$.

EQ4. Suppose that $f : [1, 5] \to \mathbb{R}$ and $g : [2, 6] \to [-2, 10]$ are both integrable functions, whilst $10 \leq f(x) \leq 1000$ for all $x \in [2, 4]$. For each of the integrals below, use the properties of integrable functions to prove that the integral exists, and then find an upper bound and a lower bound for the value of the integral:
(a) $\int_2^4 f$
(b) $\int_2^5 (f - g)$
(c) $\int_3^4 6fg$

EQ5. Use the First Fundamental Theorem of Calculus and apply it to prove that each of the following functions are differentiable and find a formula for their derivatives:
(a) $F : [0, 2] \to \mathbb{R}, \ F(x) := \displaystyle\int_0^x \sin(t^2) \, \mathrm{d}t, \ x \in [0, 2]$

(b) $G : [1, 2] \to \mathbb{R},\ G(x) := \displaystyle\int_1^x \sin(t^2)\ \mathrm{d}t,\ x \in [1, 2]$

(c) $H : [0, 1] \to \mathbb{R},\ H(x) := \displaystyle\int_x^1 \sin(t^2)\ \mathrm{d}t,\ x \in [0, 1]$

(d) $I : [0, 1] \to \mathbb{R},\ I(x) := \displaystyle\int_0^{2x^3} \sin(t^2)\ \mathrm{d}t,\ x \in [0, 1]$

(e) $J : [0, 2] \to \mathbb{R},\ J(x) := \left(\displaystyle\int_0^x \sin(t^2)\ \mathrm{d}t\right)^2,\ x \in [0, 2]$

The formula for the derivative $J'(x)$ may contain an integral expression.

EQ6. Suppose that $f : [1, 3] \to \mathbb{R}$ is differentiable and that its derivative f' is continuous:

(a) If $f(1) = 10$ and $\int_1^3 f' = 16$, then calculate $f(3)$.
(b) Explain why the continuity of f' allowed for the application of the Fundamental Theorem of Calculus in part (a).
(c) State a weaker condition on f' that would suffice to apply the Fundamental Theorem of Calculus in part (a).

EQ7. Find the following antiderivatives and integrals:

(a) $\displaystyle\int x \sin(5x)\ \mathrm{d}x$

(b) $\displaystyle\int_1^2 \frac{(\log(x))^2}{x^3}\ \mathrm{d}x$

(c) $\displaystyle\int e^{2x} \sin(3x)\ \mathrm{d}x$

EQ8. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{x^2 + 1}{x + 4}\ \mathrm{d}x$

(b) $\displaystyle\int \frac{10}{(x - 1)(x^2 + 4)}\ \mathrm{d}x$

(c) $\displaystyle\int_3^4 \frac{x^2 + 1}{x^2 - 4x + 4}\ \mathrm{d}x$