Examples sheet 1 – Solutions – Linear Algebra

VECTOR SPACES.

1. (a) $(\mathbb{N},+,\cdot)$ is not a field, as it fails to have an additive identity (FA2) and as a result FA3 does not make sense. Axiom FM3 also fails, as integers do not have multiplicative inverses.

(b) $(\mathbb{Z},+,\cdot)$ is not a field, as integers do not have multiplicative inverses, so FM3 fails.

(c) $(\mathbb{R},+,\cdot)$ is a field. In particular, $o=0, e=1$ and for any $a\in\mathbb{R}$, $a^-=-a$ and $a^*=1/a$.

(d) $(\mathbb{C},+,\cdot)$ is a field. In particular, $o=0, e=1$ and for any $z\in\mathbb{C}$, $z^-=-z$, $z^*=\overline{z}/\,|z|^2$ $(z\neq 0)$.

(e) $(\mathbb{R},+,-)$ is not a field since the 'multiplication' operation fails associativity (FM1), commutativity (FM4) and distributivity (FD). Note that to satisfy FM2, we need $e=o$ and consequently to satisfy FM3 we need $a^*=a$.

2. (a) $(\mathbb{R},+,\cdot,\mathbb{R})$ is a vector space.

(b) $(\mathbb{C},+,\cdot,\mathbb{C})$ is a vector space.

(c) $(\mathbb{C},+,\cdot,\mathbb{R})$ is a vector space.

(d) $(\emptyset,+,\cdot,\mathbb{R})$ is not a vector space since there are no elements to satisfy the axioms. In particular, the existence axioms VA2 and VA3 fail.

(e) $(\mathbb{E}^3,\times,\cdot,\mathbb{R})$ is not a vector space. Here is the verification of the vector space axioms.
VA0: $\mathbf{v}\times\mathbf{w}\in\mathbb{E}^3$, by definition of cross-product.
VA1: Associativity fails to hold: using the vector triple product formula, we obtain different expressions:

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\bullet\mathbf{w})\mathbf{v}-(\mathbf{u}\bullet\mathbf{v})\mathbf{w},$$

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=-\mathbf{w}\times(\mathbf{u}\times\mathbf{v})=-(\mathbf{v}\bullet\mathbf{w})\mathbf{u}+(\mathbf{u}\bullet\mathbf{w})\mathbf{v}.$$

VA2: There is no vector additive identity, since, working with an orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we find

$$\mathbf{v}\times\mathbf{z}=\mathbf{v}\Longleftrightarrow(v_2z_3-z_2v_3)\mathbf{e}_1+(v_3z_1-v_1z_3)\mathbf{e}_2+(v_1z_2-v_2z_1)\mathbf{e}_3=v_1\mathbf{e}_1+v_2\mathbf{e}_2+v_3\mathbf{e}_3$$

$$\Longleftrightarrow\begin{bmatrix}0&-v_3&v_2\\v_3&0&-v_1\\-v_2&v_1&0\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

which has no solution, in general, since the determinant of the matrix is zero.
VA3: This axiom does not make sense since there is no vector additive identity.
VA4: The vector cross-product is anti-commutative: $\mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u}$.
VM0, VM1, VM4 hold, since they only involve scalar-vector multiplications and $(\mathbb{E}^3,+,\cdot,\mathbb{R})$ is a vector space.
VM2: Distributivity of scalar-vector multiplication fails since $a(\mathbf{u}\times\mathbf{v})\neq(a\mathbf{u})\times(a\mathbf{v})$.
VM3: Distributivity of scalar addition fails since $(a+b)\mathbf{v}\neq(a\mathbf{v})\times(b\mathbf{v})=\mathbf{0}$, in general.

3. Assume that $a\bullet\mathbf{u}\boldsymbol{+}b\bullet\mathbf{v}\in V$ for all $\mathbf{u},\mathbf{v}\in V$ and for all $a,b\in\mathbb{F}$. Then, we can derive axioms VA0, VM0:

- choosing $a=b=e$, we deduce that $\mathbf{u}\boldsymbol{+}\mathbf{v}\in V$;
- choosing $b=o$, we deduce that $a\bullet u\in V$ (as $o\bullet\mathbf{v}=\mathbf{z}$, by the Elementary Properties set 1).

4. We prove elementary properties 3 and 4 included in Proposition 1.1.

3. We show that for all $\mathbf{u}\in V$, $e^-\bullet\mathbf{u}=\mathbf{u}^-$:

$$\mathbf{u}+\mathbf{u}^-=\mathbf{0}=o\bullet\mathbf{u}=(e+e^-)\bullet\mathbf{u}=\mathbf{u}+e^-\bullet\mathbf{u}.$$

The result follows by cancellation in sums.

4. We show that for all $a \in \mathbb{F}$, $a \bullet \mathbf{z} = \mathbf{z}$:

$$a \bullet \mathbf{v} + \mathbf{z} = a \bullet (\mathbf{v} + \mathbf{z}) = a \bullet \mathbf{v} + a \bullet \mathbf{z}.$$

The result follows by cancellation in sums.

5. (a) Assume that there are two additive inverses $\mathbf{v}^-, \tilde{\mathbf{v}}^-$ satisfying

$$\mathbf{v} \boldsymbol{+} \mathbf{v}^- = \mathbf{z} = \mathbf{v} \boldsymbol{+} \tilde{\mathbf{v}}^-.$$

By cancellation in sums, $\mathbf{v}^- = \tilde{\mathbf{v}}^-$. Let $\mathbf{u} = \mathbf{v}^-$. Then $\mathbf{z} = \mathbf{u} \boldsymbol{+} \mathbf{u}^- = \mathbf{v}^- \boldsymbol{+} (\mathbf{v}^-)^-$. On the other hand, $\mathbf{z} = \mathbf{v}^- \boldsymbol{+} \mathbf{v}$. By cancellation in sums, $\mathbf{v} = (\mathbf{v}^-)^-$.

(b) We have, by elementary property 3 in Proposition 1.1

$$(\mathbf{v} \boldsymbol{+} \mathbf{w})^- = \mathbf{e}^- \bullet (\mathbf{v} \boldsymbol{+} \mathbf{w}) = \mathbf{e}^- \bullet \mathbf{v} \boldsymbol{+} \mathbf{e}^- \bullet \mathbf{w} = \mathbf{v}^- \boldsymbol{+} \mathbf{w}^-.$$

(c) Using part **(b)**, we find

$$(\mathbf{v} \boldsymbol{+} \mathbf{w}) + (\mathbf{w} \boldsymbol{+} \mathbf{v})^- = (\mathbf{v} \boldsymbol{+} \mathbf{w}) \boldsymbol{+} \mathbf{w}^- \boldsymbol{+} \mathbf{v}^- = \mathbf{v} \boldsymbol{+} (\mathbf{w} \boldsymbol{+} \mathbf{w}^-) \boldsymbol{+} \mathbf{v}^- = \mathbf{z}.$$

Hence $((\mathbf{w} \boldsymbol{+} \mathbf{v})^-)^- = \mathbf{v} \boldsymbol{+} \mathbf{w}$, and by part **(a)** $\mathbf{w} \boldsymbol{+} \mathbf{v} = \mathbf{v} \boldsymbol{+} \mathbf{w}$ and commutativity holds for all $\mathbf{v}, \mathbf{w} \in V$.

6. Axioms VA0, VA1, VA4 and VM0–VM4 can be verified as in the case of $\mathbb{F}^n(\mathbb{F})$ (and follow from the corresponding entrywise properties of $\mathbb{F}(\mathbb{F})$). Axiom VA2 holds with $\mathbf{z} = (0, 0, \ldots, 0, \ldots)$, while VA3 holds with additive inverse given by $\mathbf{v}^- = (-v_1, -v_2, \ldots, -v_k, \ldots)$.

7. In parts **(a-c)** below, we only need to check the axioms VM0-VM4, as the additive axioms hold for standard addition.

(a) VM0 holds since $a \bullet \mathbf{v} \in \mathbb{F}^2$, as $av_1 + av_2 \in \mathbb{F}$.
VM1 holds since

$$a \bullet (b \bullet \mathbf{v}) = a \bullet \begin{bmatrix} 0 \\ bv_1 + bv_2 \end{bmatrix} = \begin{bmatrix} 0 \\ abv_1 + abv_2 \end{bmatrix} = \begin{bmatrix} 0 \\ (a \cdot b)(v_1 + v_2) \end{bmatrix} = (a \cdot b) \bullet \mathbf{v}.$$

VM2 holds since

$$a \bullet (\mathbf{v} + \mathbf{w}) = \begin{bmatrix} 0 \\ a(v_1 + w_1 + v_2 + w_2) \end{bmatrix} = \begin{bmatrix} 0 \\ a(v_1 + v_2) \end{bmatrix} + \begin{bmatrix} 0 \\ a(w_1 + w_2) \end{bmatrix} = a \bullet \mathbf{v} + a \bullet \mathbf{w}.$$

VM3 holds since

$$(a + b) \bullet \mathbf{v} = \begin{bmatrix} 0 \\ (a + b)(v_1 + v_2) \end{bmatrix} = \begin{bmatrix} 0 \\ a(v_1 + v_2) \end{bmatrix} + \begin{bmatrix} 0 \\ b(v_1 + v_2) \end{bmatrix} = a \bullet \mathbf{v} + b \bullet \mathbf{v}.$$

VM4 fails, since

$$e \bullet \mathbf{v} = \begin{bmatrix} 0 \\ e(v_1 + v_2) \end{bmatrix} \neq \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

(b) VM0 is satisfied since $a \bullet \mathbf{v} = \begin{bmatrix} v_1^a \\ av_2 \end{bmatrix} \in \mathbb{F}^2$.
VM1 holds since

$$a \bullet (b \bullet \mathbf{v}) = a \bullet \begin{bmatrix} v_1^b \\ bv_2 \end{bmatrix} = \begin{bmatrix} v_1^{ab} \\ abv_2 \end{bmatrix} = (a \cdot b) \bullet \mathbf{v}.$$

VM2 fails to hold since

$$a \bullet (\mathbf{v} + \mathbf{w}) = \begin{bmatrix} (v_1 + w_1)^a \\ a(v_2 + w_2) \end{bmatrix} \neq \begin{bmatrix} v_1^a \\ av_2 \end{bmatrix} + \begin{bmatrix} w_1^a \\ aw_2 \end{bmatrix} = a \bullet \mathbf{v} + a \bullet \mathbf{w}.$$

VM3 fails since

$$(a + b) \bullet \mathbf{v} = \begin{bmatrix} v_1^{a+b} \\ (a + b)v_2 \end{bmatrix} \neq \begin{bmatrix} v_1^a \\ av_2 \end{bmatrix} + \begin{bmatrix} v_1^b \\ bv_2 \end{bmatrix} = a \bullet \mathbf{v} + b \bullet \mathbf{v}.$$

VM4 holds if $e = 1$ in $\mathbb{F}$ since

$$e \bullet \mathbf{v} = \begin{bmatrix} v_1^e \\ v_2 \end{bmatrix} = \mathbf{v}.$$

(c) VM0 is satisfied since $a \bullet \mathbf{v} = \begin{bmatrix} \overline{av_1} \\ \overline{av_2} \end{bmatrix} \in \mathbb{F}^2$.

VM1 does not hold since

$$a \bullet (b \bullet \mathbf{v}) = a \bullet \begin{bmatrix} \overline{bv_1} \\ \overline{bv_2} \end{bmatrix} = \begin{bmatrix} \bar{a}bv_1 \\ \bar{a}bv_2 \end{bmatrix} = (a \cdot \bar{b}) \bullet \mathbf{v} \neq (a \cdot b) \bullet \mathbf{v}.$$

VM2 holds since

$$a \bullet (\mathbf{v} + \mathbf{w}) = \begin{bmatrix} \overline{a(v_1 + w_1)} \\ \overline{a(v_2 + w_2)} \end{bmatrix} = \begin{bmatrix} \overline{av_1} \\ \overline{av_2} \end{bmatrix} + \begin{bmatrix} \overline{aw_1} \\ \overline{aw_2} \end{bmatrix} = a \bullet \mathbf{v} + a \bullet \mathbf{w}.$$

VM3 holds since

$$(a + b) \bullet \mathbf{v} = \begin{bmatrix} \overline{a(v_1 + w_1)} \\ \overline{a(v_2 + w_2)} \end{bmatrix} = \begin{bmatrix} \overline{av_1} \\ \overline{av_2} \end{bmatrix} + \begin{bmatrix} \overline{aw_1} \\ \overline{aw_2} \end{bmatrix} = a \bullet \mathbf{v} + a \bullet \mathbf{w}.$$

VM4 fails to hold since

$$e \bullet \mathbf{v} = \begin{bmatrix} \overline{ev_1} \\ \overline{ev_2} \end{bmatrix} = \begin{bmatrix} \overline{v_1} \\ \overline{v_2} \end{bmatrix} \neq \mathbf{v}.$$

(d) Let the standard addition operation be replaced with

$$\mathbf{v} \boldsymbol{+} \mathbf{u} = \begin{bmatrix} v_1 u_1 \\ v_2 + u_2 \end{bmatrix}.$$

We check again the vector space axioms; this time, we need to check all of VA0–VA4, but only VM2 and VM3, which are affected by this change in the addition operation.

VA0 holds, since

$$\mathbf{v} \boldsymbol{+} \mathbf{u} = \begin{bmatrix} v_1 u_1 \\ v_2 + u_2 \end{bmatrix} \in \mathbb{F}^2.$$

VA1 holds, since

$$\begin{aligned} \mathbf{u} \boldsymbol{+} (\mathbf{v} \boldsymbol{+} \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \boldsymbol{+} \begin{bmatrix} v_1 w_1 \\ v_2 + w_2 \end{bmatrix} = \begin{bmatrix} u_1(v_1 w_1) \\ u_2 + (v_2 + w_2) \end{bmatrix} \\ &= \begin{bmatrix} (u_1 v_1) w_1 \\ (u_2 + v_2) + w_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ u_2 + v_2 \end{bmatrix} \boldsymbol{+} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (\mathbf{u} \boldsymbol{+} \mathbf{v}) \boldsymbol{+} \mathbf{w}. \end{aligned}$$

VA2 holds, but with modified additive identity:

$$\mathbf{v} \boldsymbol{+} \mathbf{z} = \begin{bmatrix} v_1 z_1 \\ v_2 + z_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Longleftrightarrow \mathbf{z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

VA3 fails to hold, since

$$\mathbf{v} + \mathbf{v}^- = \mathbf{z} \Longleftrightarrow \begin{bmatrix} v_1 v_1^- \\ v_2 + v_2^- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longleftrightarrow \mathbf{v}^- = \begin{bmatrix} 1/v_1 \\ -v_2 \end{bmatrix},$$

so that any vector $\mathbf{v} \in \mathbb{F}^2$ with $v_1 = 0$ does not have an additive inverse.

VA4 holds, since

$$\mathbf{v} \boldsymbol{+} \mathbf{w} = \begin{bmatrix} v_1 w_1 \\ v_2 + w_2 \end{bmatrix} = \begin{bmatrix} w_1 v_1 \\ w_2 + v_2 \end{bmatrix} = \mathbf{w} \boldsymbol{+} \mathbf{v}.$$

VM2 now holds, since

$$a \bullet (\mathbf{v} \boldsymbol{+} \mathbf{w}) = a \bullet \begin{bmatrix} v_1 w_1 \\ v_2 + w_2 \end{bmatrix} = \begin{bmatrix} (v_1 w_1)^a \\ a(v_2 + w_2) \end{bmatrix} = \begin{bmatrix} v_1^a w_1^a \\ av_2 + aw_2 \end{bmatrix} = \begin{bmatrix} v_1^a \\ av_2 \end{bmatrix} \boldsymbol{+} \begin{bmatrix} v_1^a \\ av_2 \end{bmatrix} = (a \bullet \mathbf{v}) \boldsymbol{+} (a \bullet \mathbf{w}).$$

VM3 also holds now since

$$(a + b) \bullet \mathbf{v} = \begin{bmatrix} v_1^{a+b} \\ (a + b) v_2 \end{bmatrix} = \begin{bmatrix} v_1^a v_1^b \\ av_2 + bv_2 \end{bmatrix} = \begin{bmatrix} v_1^a \\ av_2 \end{bmatrix} \boldsymbol{+} \begin{bmatrix} v_1^b \\ bv_2 \end{bmatrix} = (a \bullet \mathbf{v}) \boldsymbol{+} (b \bullet \mathbf{v}).$$

Hence, $(\mathbb{F}^2, \#, \bullet, \mathbb{F})$ fails to be a vector space, as it fails to satisfy a single axiom: VA3.

SUBSPACES.

8. Assume that U is a non-empty subset of $V(\mathbb{F})$. We show that closure with respect to vector addition and scalar-vector multiplication holds if and only if $a\mathbf{u}+b\mathbf{v}\in U$ for any $\mathbf{u},\mathbf{v}\in U$. The statement will then follow by the Subspace Criterion 1.
$\boxed{\Longrightarrow}$ Assume that U satisfies the closure axioms VA0 and VM0. By VM0, $a\cdot\mathbf{u}\in U$ and $b\cdot\mathbf{v}\in U$ for all $a,b\in\mathbb{F}$ and for all $\mathbf{u},\mathbf{v}\in U$. By VA0, $a\cdot\mathbf{u}+b\cdot\mathbf{v}\in U$.
$\boxed{\Longleftarrow}$ Assume that $a\cdot\mathbf{u}+b\cdot\mathbf{v}\in U$ for all $a,b\in\mathbb{F}$ and for all $\mathbf{u},\mathbf{v}\in U$. Choosing $a=e, b=e$, we find that $\mathbf{u}+\mathbf{v}\in U$, for all $\mathbf{u},\mathbf{v}\in U$, which is VA0. Choosing $b=o$, we find $a\cdot\mathbf{u}\in U$ for all $\mathbf{u}\in U$ and for all $a\in\mathbb{F}$, which is VM0. The above proof and Subspace Criterion 1 show that the following statements are equivalent

- U is a subspace of V
- U satisfies VA0 and VM0 (criterion 1).
- $a\mathbf{u}+b\mathbf{v}\in U$ for all $\mathbf{u},\mathbf{v}\in U$ and all $a,b\in\mathbb{F}$ (criterion 2).

9. **(a)** Let $U_+=\{\mathbf{v}\in\mathbb{R}^n : v_i\geq 0, i=1,2,\ldots,n\}$. The $\mathbf{v}_1+\mathbf{v}_2\in V$ for any $\mathbf{v}_1,\mathbf{v}_2$, but $a\cdot\mathbf{v}\notin V$ for all $a<0$ and for all $\mathbf{v}\in V\setminus\{\mathbf{0}\}$.

(b) Let $U_-=\{\mathbf{v}\in\mathbb{R}^n : v_i\leq 0, i=1,2,\ldots,n\}$ and consider $U_\pm=U_+\cup U_-$. Then $a\cdot\mathbf{v}\in U_\pm$ for all $a\in\mathbb{R}$ and all $\mathbf{v}\in U_\pm$. On the other hand, $\mathbf{e}_1+(-\mathbf{e}_2)\notin U_\pm$, although $\mathbf{e}_1,(-\mathbf{e}_2)\in U_\pm$.

10. False: if $W=V\setminus U$, then $\mathbf{0}\notin W$, so W is not a subspace as FA2 does not hold.

11. False: a counter-example is given by $U+U=U+Z$, where Z is the trivial subspace. More generally, any two distinct subspaces of U as choices for V and W provide couter-examples.

12. Let $X=X_g:=\{\mathbf{v}=(x_1,x_2,\ldots,x_n,\ldots)\in V=\mathbb{R}^{\mathbb{N}} : x_{n+1}=g(x_n)\}$ Let $\mathbf{v},\mathbf{w}\in V$ with general terms x_n, y_n, respectively. A linear combination $a\mathbf{v}+b\mathbf{w}$ has general term ax_n+by_n. To check that $a\mathbf{v}+b\mathbf{w}$ is in V, we need to check that its general term satisfies the recurrence relation.

(a) Let $g(x)=g_1(x)=\alpha x+\beta$. We have

$$g(ax_n+by_n)=\alpha(ax_n+by_n)+\beta=a(\alpha x_n+\beta)+b(\alpha y_n+\beta)+\beta(1-a-b)=ax_{n+1}+by_{n+1}+\beta-\beta(a+b).$$

Hence, $X=X_{g_1}$ is a subspace of V provided $\beta=0$.

(b) Let $g(x)=g_2(x)=g_1(g_1(x))$. One can view this map as generating a subsequence (x_{2n}) of the sequence (x_n) generated by $g_1(x)$. Now, $g_2(x)=\alpha(\alpha x+\beta)+\beta=\alpha^2x+(\alpha+1)\beta$. By part **(a)**, we need the constant term to be zero, so that either $\alpha=-1,\beta\in\mathbb{R}$ or $\beta=0,\alpha\in\mathbb{R}$. This indicates that the map $g_1(x)=-x+\beta$ with $\beta\neq 0$ yields a sequence (x_n) in $X_{g_1}\not<\mathbb{R}^{\mathbb{N}}$, which contains a subsequence $(x_{2n})\in X_{g_2}<\mathbb{R}^{\mathbb{N}}$.

(c) Let $g(x)=g_3(x)=x(\alpha x+\beta)$. Since this map is nonlinear, the statement in the Subspace Criterion 2 will not hold, so that $X_{g_3}\not<\mathbb{R}^{\mathbb{N}}$.

13. **(a)** Let $V=\mathbb{Z}_2^3$, $U=\{(v_1,v_2,v_3) : v_2=0, v_1,v_3\in\mathbb{Z}_2\}$ and note that $U\subset V$. We check Subspace Criterion 2. Let $a,b\in\mathbb{Z}_2$, $\mathbf{v},\mathbf{w}\in U$ and consider

$$a\mathbf{v}+b\mathbf{w}=a(v_1,0,v_3)+b(w_1,0,w_3)=(av_1+bw_1,0,av_3+bw_3)\in U,$$

since $av_i+bw_i\in\mathbb{Z}_2$ $(i=1,3)$, by field closure. Hence, $U<V$.

(b) Let $V=(\mathbb{Z}_3^2,\oplus_3,\odot_3,\mathbb{Z}_3), U=(\mathbb{Z}_2^2,\oplus_3,\odot_3,\mathbb{Z}_3)$, where $\oplus_3,\odot_3$ denote the modulo operations in $\mathbb{Z}_3$. Note first that $U\subset V$. We check Subspace Criterion 2. Let $a,b\in\mathbb{Z}_3$, $\mathbf{v},\mathbf{w}\in U$ and consider

$$a\mathbf{v}+b\mathbf{w}=a(v_1,v_2)+b(w_1,w_2)=(av_1+bw_1,av_2+bw_2)\notin U,$$

since $av_i+bw_i\notin\mathbb{Z}_2$ $(i=1,2)$, in general (e.g., take $a=b=v_1=w_1=1$). Hence, $U\not<V$.

SPANNING SETS.

14. Let $\mathbf{u}, \mathbf{w} \in U := \text{span}\,\{\mathbf{v}\}$. Let $a, b \in \mathbb{F}$. Then, $\mathbf{u} = r\mathbf{v}, \mathbf{w} = s\mathbf{v}$, for some $r\,s \in \mathbb{F}$ and therefore

$$a\mathbf{u} + b\mathbf{w} = ar\mathbf{v} + bs\mathbf{v} = (ar + bs)\mathbf{v} \in U,$$

since $ar + bs \in \mathbb{F}$, by field closure. Hence, by Subspace Criterion 2, U is a subspace of V.

15. Since $\mathbf{v} \in \text{span}\, S$, with $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we can write it as a linear combination of the column vectors $\mathbf{v}_i$:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \begin{bmatrix} c_1[\mathbf{v}_1]_1 + c_2[\mathbf{v}_2]_1 + c_3[\mathbf{v}_3]_1 \\ c_1[\mathbf{v}_1]_2 + c_2[\mathbf{v}_2]_2 + c_3[\mathbf{v}_3]_2 \\ \vdots \\ c_1[\mathbf{v}_1]_n + c_2[\mathbf{v}_2]_n + c_3[\mathbf{v}_3]_n \end{bmatrix} = \begin{bmatrix} [\mathbf{v}_1]_1 & [\mathbf{v}_2]_1 & [\mathbf{v}_3]_1 \\ [\mathbf{v}_1]_2 & [\mathbf{v}_2]_2 & [\mathbf{v}_3]_2 \\ \vdots & \vdots & \vdots \\ [\mathbf{v}_1]_n & [\mathbf{v}_2]_n & [\mathbf{v}_3]_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A\mathbf{c},$$

where $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ and $[\mathbf{c}]_i = c_i, i = 1, 2, 3$. Note that we used the definition of matrix-vector product in this question: this topic will be discussed later when we study linear maps.

16. First, note that we must have $n \geq 2$, since $1+x+x^2$ cannot be a factor of a polynomial of lower degree. In other words, U is empty for $n < 2$. We will therefore assume throughout that $n \geq 2$; this will allow us to express U as indicated below. Since $U \ni p \in \mathcal{P}_n(\mathbb{R})$, we must have $p(x) = q(x)(x^2 + x + 1)$, for some $q(x) \in \mathcal{P}_{n-2}(\mathbb{R})$. Thus,

$$U = \left\{p \in \mathcal{P}_n(\mathbb{R}) : p(x) = q(x)(x^2 + x + 1), q \in \mathcal{P}_{n-2}(\mathbb{R})\right\}.$$

(a) To check that $U \leq \mathcal{P}_n(\mathbb{R})$, we use Subspace Criterion 2: let $p_1, p_2 \in U$ and let $a, b \in \mathbb{R}$. Then

$$ap_1 + bp_2 = aq_1(x)(x^2 + x + 1) + bq_2(x)(x^2 + x + 1) = (aq_1(x) + bq_2(x))(x^2 + x + 1) \in U,$$

since $aq_1(x) + bq_2(x) \in \mathcal{P}_{n-2}(\mathbb{R})$ by closure. So indeed, U is a subspace of $\mathcal{P}_n(\mathbb{R})$.

(b) When $n = 2$, $q \in \mathcal{P}_0(\mathbb{R})$, so that $p(x) = a_0(1 + x + x^2) =: a_0p_0(x)$ for some $a_0 \in \mathbb{R}$. Therefore, a spanning set for U is

$$S = \{p_0\} := \left\{1 + x + x^2\right\}.$$

When $n = 3$, $q \in \mathcal{P}_1(\mathbb{R})$, so that $p(x) = (a_0 + a_1x)(1 + x + x^2) =: a_0p_0(x) + a_1p_1(x)$, for some $a_0, a_1 \in \mathbb{R}$. Therefore, a spanning set for U is

$$S = \{p_0, p_1\} := \left\{1 + x + x^2, x(1 + x + x^2)\right\}.$$

The spans are minimal, as removing one element removes the spanning property of the set.

LINEAR INDEPENDENCE

17. To establish linear independence, we need to consider the representation of $\mathbf{0}$ as a linear combination of the vectors provided. This usually requires the solution of a linear system $M\mathbf{a} = \mathbf{0}$. Thus, the set if linearly independent if and only if $\det M \neq 0$.

(a) Consider

$$\mathbf{0} = a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} =: M\mathbf{a}.$$

Since $\det M = 0$, there are non-trivial solutions $\mathbf{a}$, e.g., $a_1 = -1, a_2 = 1, a_3 = 1$, which provide a non-trivial representation of the zero vector, so that S is linearly dependent. [Note that you do not need to specify a non-trivial solution to answer this question].

(b) Consider

$$0 = a_1(1 + x - x^2) + a_2(2x^2 - 1) + a_3(x + 3) \iff \begin{cases} a_1 - a_2 + 3a_3 & = 0 \\ a_1 + a_3 & = 0 \\ -a_1 + 2a_2 & = 0 \end{cases} \iff \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{0}.$$

Since $\det M = 5 \neq 0$, we have a unique solution, which is the trivial solution: $a_1 = a_2 = a_3 = 0$. Hence, the set S is linearly independent.

18. (a) True: since S is linearly dependent, we have

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0},$$

with some a_i non-zero. In particular, we must have a_1 non-zero, since otherwise $\{\mathbf{v}_2, \mathbf{v}_3\} \subset S'$ would be linearly dependent, while S' is given as linearly independent: this contradicts the statement of **Q7(b)**. Hence,

$$\mathbf{v}_1 = -\frac{a_2}{a_1}\mathbf{v}_2 - \frac{a_3}{a_1}\mathbf{v}_3 = b\mathbf{v}_2 + \mathbf{c}\mathbf{v}_3,$$

so that $\mathbf{v}_1 \in \text{span}\,\{\mathbf{v}_2, \mathbf{v}_3\} \subset \text{span}\,\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{span}\,S'$.

(b) False: if $\mathbf{v}_4 \in \text{span}\,S$, then

$$\mathbf{v}_4 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1(b\mathbf{v}_2 + \mathbf{c}\mathbf{v}_3) + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \in \text{span}\,\{\mathbf{v}_2, \mathbf{v}_3\},,$$

which implies that S' is linearly dependent, a contradiction.

19. (a) We run the usual test for linear independence:

$$\mathbf{0} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \iff a_1(c\mathbf{i} - s\mathbf{j}) + a_2(s\mathbf{i} + c\mathbf{j}) = (a_1c + a_2s)\mathbf{i} + (a_2c - a_1s)\mathbf{j} \iff \begin{cases} a_1c + a_2s & = 0, \\ a_2c - a_1s & = 0, \end{cases}$$

which is a linear system

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} =: M_1\mathbf{a} = \mathbf{0}.$$

The solution is trivial, as $\det M_1 = c^2 + s^2 \neq 0$. Hence, $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a linearly independent set.

(b) The set $\{\mathbf{e}_1, \mathbf{e}_2\}$ is linearly independent in this case also, as

$$\det M_2 := \det \begin{bmatrix} c & s \\ c & -s \end{bmatrix} = -2cs \neq 0.$$

(c) In each case, we require $c^2 + s^2$. In the first case, the vectors are orthgonal for any $s, c \in \mathbb{R} \setminus \{0\}$. In the second case, we require $c^2 - s^2$, so only values satisfying $c = \pm s$ will ensure orthogonality.

20. (a) Let us label the elements in U, V such that

$$V = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_m\} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\} \cup \{\mathbf{v}_{k+1}, \dots, \mathbf{v}_m\} =: U \cup \{\mathbf{v}_{k+1}, \dots, \mathbf{v}_m\}.$$

Since U is linearly dependent, there exists a non-trivial linear combination of vectors in U representing the zero vector:

$$\mathbf{0} = a_1\mathbf{v}_1 + \cdots a_k\mathbf{v}_k.$$

Hence, the following is also a non-trivial linear combination of vectors in V representing the zero vector:

$$\mathbf{0} = a_1\mathbf{v}_1 + \cdots a_k\mathbf{v}_k + 0 \cdot \mathbf{v}_{k+1} + \cdots + 0 \cdot \mathbf{v}_m.$$

Therefore, V is linearly dependent.

(b) If we assume U is linearly dependent, by part **(a)** V is linearly dependent – a contradiction.

21. Recall that a set is linearly dependent if $\mathbf{0}$ can be written as a non-trivial linear combination of its elements.

(a) Let $S_1 = \{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{k-1} - \mathbf{v}_k, \mathbf{v}_k - \mathbf{v}_1\} =: \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$. Then

$$\mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_k = (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_2 - \mathbf{v}_3) + \cdots (\mathbf{v}_{k-1} - \mathbf{v}_k) + (\mathbf{v}_k - \mathbf{v}_1) = \mathbf{0}.$$

Hence, $\mathbf{0}$ can be written as a non-trivial linear combination of the elements in S_1, so this is a linearly dependent set.

(b) Let $S_2 = \left\{\mathbf{v}_1 - \mathbf{v}_{\pi(1)}, \mathbf{v}_2 - \mathbf{v}_{\pi(2)}, \ldots, \mathbf{v}_{k-1} - \mathbf{v}_{\pi(k-1)}, \mathbf{v}_k - \mathbf{v}_{\pi(k)}\right\} =: \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$. Then

$$\mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_k = \sum_{j=1}^{k} \mathbf{v}_j - \sum_{j=1}^{k} \mathbf{v}_{\pi(j)} = \mathbf{0}.$$

Hence, S_2 is linearly dependent as well.

(c) Following the hint, we let $k = 3$, so that $S_3 = \{\mathbf{v}_1 - 2\mathbf{v}_2, \mathbf{v}_2 - 2\mathbf{v}_3, \mathbf{v}_3 - 2\mathbf{v}_1\} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. We investigate the following representation of $\mathbf{0}$:

$$\mathbf{0} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1(\mathbf{v}_1 - 2\mathbf{v}_2) + a_2(\mathbf{v}_2 - 2\mathbf{v}_3) + a_3(\mathbf{v}_3 - 2\mathbf{v}_1) = (a_1 - 2a_3)\mathbf{v}_1 + (a_2 - 2a_1)\mathbf{v}_2 + (a_3 - 2a_2)\mathbf{v}_3.$$

Since S is linearly independent, the expression on the right is a trivial linear combination, so that

$$\begin{cases} a_1 - 2a_3 &= 0 \\ a_2 - 2a_1 &= 0 \\ a_3 - 2a_2 &= 0 \end{cases} \iff a_3 = 2a_2 = 2(2a_1) = 2(2(2a_3)) = 2^3 a_3 \iff a_1 = a_2 = a_3 = 0.$$

A similar argument for the general case results in $a_1 = a_2 = \cdots = a_k = 0$, so that S_3 is linearly independent.

(d) Let $S_4 = \{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \ldots, \mathbf{v}_{k-1} - \mathbf{v}_k, \mathbf{v}_k\}$. We proceed as in part **(c)**: we investigate the following representation of $\mathbf{0}$:

$$\mathbf{0} = a_1(\mathbf{v}_1 - \mathbf{v}_2) + a_2(\mathbf{v}_2 - \mathbf{v}_3) + \cdots + a_{k-1}(\mathbf{v}_{k-1} - \mathbf{v}_k) + a_k\mathbf{v}_k = a_1\mathbf{v}_1 + (a_2 - a_1)\mathbf{v}_2 + (a_3 - a_2)\mathbf{v}_3 + \cdots + (a_k - a_{k-1})\mathbf{v}_k.$$

As before, due to the linear independence of S we find

$$\begin{cases} a_1 &= 0 \\ a_2 - a_1 &= 0 \\ \cdots & \\ a_k - a_{k-1} &= 0 \end{cases} \iff a_1 = a_2 = \cdots = a_k = 0.$$

Hence, S_4 is linearly independent.

Remark. All of the above examples yield a linear system for the coefficients a_j: $M\mathbf{a} = \mathbf{0}$. The solution is trivial (i.e., $\mathbf{a} = \mathbf{0}$) if and only if M is non-singular; this can be established in several way, e.g., by checking that $\det M \neq 0$, or that M is singular because all the rows (or columns) sum to zero. For example, while in part **(a)** we spotted a non-trivial linear combination, we could have established instead that the corresponding matrix $M = M_1$ is singular:

$$\det M_1 = \det \begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix} = 1 \cdot \begin{vmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{vmatrix} + (-1)(-1)^{k+1} \begin{vmatrix} -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{vmatrix} = 1 + (-1)^{2k-1} = 0.$$

22. **(a)** To construct a maximal linearly independent set, we need to find only one more vector that is not in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$. This is because a maximal linearly independent set is a basis and a linearly independent set with cardinality matching the dimension of the vector space (in this case 3), is a basis. For linear independence we require

$$\mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \begin{bmatrix} 1 & 3 & x \\ 2 & 0 & y \\ 0 & 2 & z \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Consider the following determinant evaluations:

$$\det \begin{bmatrix} 1 & 3 & x \\ 2 & 0 & y \\ 0 & 2 & z \end{bmatrix} \overset{y=z=0}{=} \det \begin{bmatrix} 1 & 3 & x \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \overset{x=1}{=} 4 \neq 0 \Longrightarrow \mathbf{v}_3 = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) In this case, we need to append 2 vectors, $\mathbf{v}_3$ and $\mathbf{v}_4$ so that the resulting set is linearly independent. We can base this construction on part **(a)**: since the first 3 entries in $\mathbf{v}_1$ and $\mathbf{v}_2$ match those from part **(a)**, we can construct $\mathbf{v}_3$ in a similar way, by appending a zero. Finally, we note that we can take $\mathbf{v}_4 = \mathbf{e}_4$, since this vector is orthogonal to the other three, hence cannot be dependent on any of them. Therefore a maximal linearly indepedent set is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

23. **(a)** Let $\ell_1(x) = a_0 + a_1 x + a_2 x^2$. Using $\ell_1(x_j) = \delta_{1j}$, we find

$$\begin{cases} a_0 + (-1)a_1 + (-1)^2 a_2 & = 1 \\ a_0 + 0 \cdot a_1 + 0^2 a_2 & = 0 \\ a_0 + 1 \cdot a_1 + 1^2 a_2 & = 0 \end{cases} \Longrightarrow \begin{cases} a_0 & = 0 \\ a_1 & = -1/2 \\ a_2 & = 1/2 \end{cases} \Longrightarrow \ell_1(x) = \frac{1}{2}(x^2 - x).$$

Working similarly, we find

$$\ell_2(x) = 1 - x^2, \quad \ell_3(x) = \frac{1}{2}(x^2 + x).$$

(b) To show that S is a linearly independent set, we consider a representation of the zero vector in $\mathcal{P}_2$ as a linear combination of ℓ_i:

$$0 = c_1\ell_1(x) + c_2\ell_2(x) + c_3\ell(x) = c_1\frac{1}{2}(x^2 - x) + c_2(1 - x^2) + c_3\frac{1}{2}(x^2 + x) = c_2 + \frac{1}{2}(c_3 - c_1)x + \frac{1}{2}(c_3 + c_1 - c_2)x^2,$$

so that $c_1 = c_2 = c_3 = 0$ and the set S is linearly independent, indeed.

(c) By definition, S is a maximal subset of V if adding to S any vector from $V \setminus \{\mathbf{0}\}$ results in a linearly dependent set. Therefore, consider the test of linear independence for $S \cup \{p\}$, where $0 \neq p(x) = a + bx + cx^2$:

$$0 = c_1\ell_1(x) + c_2\ell_2(x) + c_3\ell_3(x) + c_4(a + bx + cx^2) = (c_2 + ac_4) + \frac{1}{2}(c_3 - c_1 + 2bc_4)x + \frac{1}{2}(c_3 + c_1 - c_2 + 2cc_4)x^2$$

which yields the following system of 3 equations in 4 variables:

$$\begin{cases} c_2 + ac_4 & = 0, \\ c_3 - c_1 + 2bc_4 & = 0, \\ c_3 + c_1 - c_2 + 2cc_4 & = 0. \end{cases}$$

This is an under-determined system of equations, which in general has infinitely-many solutions. For our purposes, it suffices to exhibit one non-zero solution. One can produce such a non-trivial solution by setting a value for one of the unkowns and solving for the remaining ones. For example, setting $c_4 = 1$, we find $c_2 = -a$ and solve for c_1 and c_3:

$$\begin{cases} c_3 - c_1 = & -2b \\ c_3 + c_1 = & -a - 2c \end{cases} \Longrightarrow \begin{cases} c_3 = & -\frac{1}{2}(a + 2b + 2c), \\ c_1 = & -\frac{1}{2}(a - 2b + 2c). \end{cases}$$

Hence, a non-trivial linear combination of the elements of $S \cup \{p\}$ for the zero vector in V uses coefficients

$$c_1 = -\frac{1}{2}(a - 2b + 2c), \quad c_2 = -a, \quad c_3 = -\frac{1}{2}(a + 2b + 2c), \quad c_4 = 1.$$

Therefore, $S \cup \{p\}$ is a linearly dependent set for any $p \in \mathcal{P}_2$; by definition, S is a maximal linearly independent subset of V.

(d) The spanning property follows from the derivation in part **(c)**:

$$0 = c_1\ell_1(x) + c_2\ell_2(x) + c_3\ell_3(x) + c_4 p(x) \Longleftrightarrow p(x) = -c_1\ell_1(x) - c_2\ell_2(x) - c_3\ell_3(x)$$

so that any $p \in \mathcal{P}_2(\mathbb{R})$ of the form $p(x) = a + bx + cx^2$, can be written as a linear combination of ℓ_j:

$$p(x) = \frac{1}{2}(a - 2b + 2c)\ell_1(x) + a\ell_2(x) + \frac{1}{2}(a + 2b + 2c)\ell_3(x).$$

(e) It is straightforward to find that $\ell_1(x) + \ell_2(x) + \ell_3(x) = 1$. Since S is linearly independent, this representation of $p(x) = 1$ as a linear combination of ℓ_j is unique. To show that S is minimal, we need to check that removing any element from S removes the spanning property of S. Since $\ell_1(x) + \ell_2(x) + \ell_3(x) = 1$ uniquely, we deduce that $p(x) = 1 \notin \operatorname{span}\{\ell_i, \ell_j\}$ for any $i, j \in \{1, 2, 3\}$, so that S is indeed minimal.

(f) Let us evaluate $p(x) = 1 + 2x + 3x^2$ at x_j:

$$p(x) = a_1\ell_1(x) + a_2\ell_2(x) + a_3\ell_3(x) \Longleftrightarrow p(x_j) = \sum_{i=1}^{3} a_i\ell_i(x_j) = \sum_{i=1}^{3} a_i\delta_{ij} = a_j \Longleftrightarrow \begin{cases} a_1 = & p(x_1) = 2, \\ a_2 = & p(x_2 = 1, \\ a_3 = & p(x_3) = 6. \end{cases}$$

(g) A similar approach would require us to choose 4 **distinct** values of x_j. We could add another value, or we could choose them to be equispaced in the interval $[-1, 1]$ or we could simply choose any 4 values in $\mathbb{R}$. This would allow us to construct polynomials ℓ_i of degree 3, which can be shown to satisfy all the properties derived in this question. More generally, one can show that these polynomials exist and are unique for any set of distinct values x_j: they are known as **Lagrange polynomials** and represent an important tool in Approximation Theory (e.g., in the study of interpolation of functions by polynomials).

24. To construct a minimal spanning set for V, we need to remove vectors without changing the span until we obtain a set with cardinality equal to the dimension of V (as one further removal will lose the spanning property of the set).

(a) We first note that the fifth vector is a multiple of the first, so we can remove it without changing the span. The fourth vector is the sum of the first and second, so we can remove this one as well. We are left with the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}.$$

Removing any vector from this list will yield the required minimal spanning set as that will result in a spanning set (as each associated 2×2 matrix is non-singular) with cardinality equal to 2.

(b) Since $\dim V = 3$, we need to remove a single element without changing the span. We can consider this task as being equivalent to finding a non-trivial linear combination as a representation of zero. In this case, we could establish the dependence of one vector on the others - and consequently remove it. We have

$$\mathbf{0} = a_1(1 + x + x^2) + a_2(x - 2) + a_3(x^2 - 1) + a_4(1 - 2x + x^2) \Longleftrightarrow \begin{cases} a_1 - 2a_2 - a_3 + a_4 & = 0, \\ a_1 + a_2 - 2a_4 & = 0, \\ a_1 + a_3 & = 0. \end{cases}$$

To find a non-trivial solution, we can choose one coefficient and solve for the other three: take $a_4 = 1$ and solve for a_1, a_2, a_3. The resulting linear system is

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \implies \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}.$$

Hence, all the coefficients are non-zero and any vector can be written as a linear combination of the other three, which means that we can remove any without changing the span. Since the above linear system has a unique solution, we conclude that the first three elements are linearly independent, so a minimal spanning set can be taken to be

$$S = \left\{1 + x + x^2, x - 2, x^2 - 1\right\}.$$

BASES. COORDINATES

25. **(a)** The task amounts to solving a linear system: for example,

$$\mathbf{v}_4 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \iff \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(b) This task amounts to checking that the determinant of 3×3 matrices with columns drawn from the set S is non-zero. There are four cases to consider. We will derive in part II results that would allow for a different (simpler) approach to this question.

26. **(a)** Let $V = \mathbb{C}^2(\mathbb{R})$. In this case we need to check that we can write any element in $\mathbb{C}^2$ as a linear combination of elements in S, with the coefficients (denoted by x, y, z, w) drawn from $\mathbb{R}$. In the following, the complex entries in the $\mathbb{C}^2$-vectors are denoted by $a + ib$ and $a' + ib'$.Consider S_1:

$$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a + ib \\ a' + ib' \end{bmatrix} \implies \begin{cases} x &= a + ib \\ y &= a' + ib' \end{cases},$$

which cannot hold if $b, b' \neq 0$. Hence, S_1 does not span V and therefore is not a basis for V. Consider S_2:

$$x \begin{bmatrix} 1 + i \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 - i \end{bmatrix} = \begin{bmatrix} a + ib \\ a' + ib' \end{bmatrix} \implies \begin{cases} x + ix &= a + ib \\ y + iy &= a' + ib' \end{cases},$$

which cannot hold if $a \neq b$ and $a' \neq b'$. Hence, S_2 does not span and therefore is not a basis for V. Consider S_3:

$$x \begin{bmatrix} i \\ i \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a + ib \\ a' + ib' \end{bmatrix} \implies \begin{cases} ix + y &= a + ib \\ ix + y &= a' + ib' \end{cases},$$

which cannot hold if $a \neq a'$ and $b \neq b'$. Hence, S_3 does not span and therefore is not a basis for V. Consider S_4:

$$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ i \end{bmatrix} + z \begin{bmatrix} i \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a + ib \\ a' + ib' \end{bmatrix} \implies \begin{cases} x + zi &= a + ib \\ w + iy &= a' + ib' \end{cases},$$

which holds for any a, a', b, b', with coefficients $x = a, y = b', z = b, w = a'$. Hence, S_4 is a spanning set for V and also a basis since it is evident that it is a minimal spanning set. Finally, it is straightforwad to check that S_5 is also a basis for V.Note that $\dim \mathbb{C}^2(\mathbb{R}) = 4$. This may seem counter-intuitive at first, but since we can only work over the reals, we can see that generating a vector in $\mathbb{C}^2$ using only real numbers would require 4 real parameters.

(b) Let $V = \mathbb{C}^2(\mathbb{C})$. In this case we need to check that we can write any element in $\mathbb{C}^2$ as a linear combination of elements in S, with the coefficients (denoted by z_1, z_2, z_3, z_4) drawn from $\mathbb{C}$. As before, the complex entries in the $\mathbb{C}^2$-vectors are denoted by $a + ib$ and $a' + ib'$.Consider S_1:

$$z_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a + ib \\ a' + ib' \end{bmatrix} \implies \begin{cases} z_1 &= a + ib \\ z_2 &= a' + ib' \end{cases},$$

which holds for any a, a', b, b', with coefficients $z_1 = a + ib, z_2 = a' + ib'$. Hence S_1 is a spanning set, which is also minimal, hence a basis for V.We find the same is the case for S_2, while S_3 is linearly dependent in V, since the first vector is a (complex) scalar multiple of the second. Finally, the sets S_4, S_5 are linearly dependent in V, so they are not bases.Note that $\dim \mathbb{C}^2(\mathbb{C}) = 2$. Comparing this with the previous case, it is now clear that to generate a vector in $\mathbb{C}^2$ using complex scalars requires only 2 (complex) parameters. Thus, $\mathbb{C}^2$ is an example of a set where the choice of field yields vector spaces with different dimensions.

27. (a) By the Subspace Criterion 2, $U < \mathbb{R}^2$ since $U \subset \mathbb{R}^2$ and any linear combination of vectors $\mathbf{v}, \mathbf{v}' \in U$ satisfies the constraint:

$$a\mathbf{v} + b\mathbf{v}' = a \begin{bmatrix} x \\ y \\ z \end{bmatrix} + b \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} ax + bx' \\ ay + by' \\ az + bz' \end{bmatrix} =: \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \in U$$

since

$$2x'' - y'' + 3z'' = a(2x - y + 3z) + b(2x' - y' + 3z') = a \cdot 0 + b \cdot 0 = 0.$$

(b) To find a basis, consider writting a general vector in U as a linear combination of certain vectors (to be derived). Since we have one constraint for our 3 entries in a general element of U, we can assign generic values to two variables and write the third in terms of these:

$$2x - y + 3z = 0 \overset{x=a, z=b}{\Longrightarrow} y = 2a + 3b,$$

so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 2a + 3b \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\} =: \text{span } \{\mathbf{u}_1, \mathbf{u}_2\} =: \text{span } S.$$

Hence S is a spanning set for U. It is also linearly independent, since

$$a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow a = b = 0.$$

Hence S is a basis for U.

(c) Any vector $\mathbf{w} \notin \text{span } S$ can be used to construct a basis $B = S \cup \{\mathbf{w}\}$, so that any $\mathbf{v} \in V$ can be written as $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in U, \mathbf{w} \in \text{span } \{\mathbf{w}\} =: W$. Since B is a basis, the set sum will be a direct sum $V = U \oplus W$, since this representation is unique, due to the linear independence of vectors in B. However, one could define another set $W' = \text{span } \{\mathbf{u}_2, \mathbf{w}\}$, so that, assuming $\mathbf{u} = a\mathbf{u}_1 + b\mathbf{u}_2$,

$$\mathbf{v} = \mathbf{u} + \mathbf{w} = a\mathbf{u}_1 + b\mathbf{u}_2 + \mathbf{w} = [a\mathbf{u}_1 + (b - c)\mathbf{u}_2] + [c\mathbf{u}_2 + \mathbf{w}] =: \mathbf{u}' + \mathbf{w}',$$

where $\mathbf{u}' \in U, \mathbf{w}' \in W'$, so that $V = U + W'$.

28. Let $p \in \mathcal{P}_n(\mathbb{R})$ and consider the two constraints indicated in the question:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \Longrightarrow \begin{cases} p(0) & = a_0 = 0, \\ p(1) & = a_1 + a_2 + \cdots + a_{n-1} + a_n = 0. \end{cases}$$

Hence, any polynomial in U can be written as

$$p(x) = a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} - (a_1 + a_2 + \cdots + a_{n-1}) x^n = a_1(x - x^n) + a_2(x^2 - x^n) + \cdots a_{n-1}(x^{n-1} - x^n).$$

Hence,

$$U = \text{span } B := \text{span } \left\{ x - x^n, x^2 - x^n, \ldots, x^{n-1} - x^n \right\}.$$

Note that every element in the basis B satisfies the constraints in the definition of U. Note also that

$$\dim U = |B| = n - 1 = \dim \mathcal{P}_n(\mathbb{R}) - m,$$

where $m = 2$ is the number of constraints.

29. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ denote a basis for V. Then, due to the uniqueness of representation of a vector $\mathbf{v} \in V$ in the basis B, we can write V as a direct sum of subspaces $U_i = \text{span}\,\{\mathbf{v}_i\}$.

30. **(a)** We have $p_1(x) = 1, p_2(x) = 1 - x, p_3(x) = 1 - 2x^2$. This is a linearly independent set since

$$\mathbf{0} = a_1p_1 + a_2p_2 + a_3p_3 \iff 0 = a_1 + a_2(1-x) + a_3(1-2x^2) \implies \begin{cases} a_1 + a_2 + a_3 & = 0 \\ -a_2 & = 0 \\ -2a_3 & = 0 \end{cases} \implies a_1 = a_2 = a_3 = 0.$$

Since $\dim B = 3 = \dim \mathcal{P}_2(\mathbb{R})$, the set B is a basis.

(b) We find

$$p = a_1p_1 + a_2p_2 + a_3p_3 \iff 1 + 2x + 3x^2 = a_1 + a_2(1-x) + a_3(1-2x^2) \implies \begin{cases} a_1 + a_2 + a_3 & = 1 \\ -a_2 & = 2 \\ -2a_3 & = 3 \end{cases},$$

so that $a_2 = -2, a_3 = -3/2, a_1 = 9/2$ and $p = \frac{9}{2}p_1 - 2p_2 - \frac{3}{2}p_3$.