Example sheet 1 - formative

1. Turn the following ODEs into autonomous dynamical systems:
   (a) $\ddot{x} - x^2 + x\,\dot{x} = 0$
   (b) $\dddot{x} - \dot{x}\,(1-x)\,(2-x)\,\ddot{x} = 0$
   (c) $\ddot{x} = \cos x - t\,x$

Solution:

(a) This is an autonomous ODE of order 2, so we expect an autonomous dynamical system with 2 variables. We put $x_1 = x$ and $x_2 = \dot{x}$, then

$$\begin{array}{l} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = x^2 - x\,\dot{x} = x_1^2 - x_1 x_2 \end{array} \Rightarrow \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1^2 - x_1 x_2 \end{array}$$

(b) This is an autonomous ODE of order 3, so we expect an autonomous dynamical system with 3 variables. We put $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = \ddot{x}$ then

$$\begin{array}{l} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = x_3 \\ \dot{x}_3 = \dddot{x} = \dot{x}\,(1-x)\,(2-x)\,\ddot{x} = x_2\,(1-x_1)\,(2-x_1)\,x_3 \end{array} \Rightarrow$$

$$\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_2\,(1-x_1)\,(2-x_1)\,x_3 \end{array}$$

(c) This is a non–autonomous ODE of order 2, so we expect a non–autonomous dynamical system with 2 variables or an autonomous dynamical system with three variables. We put $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = t$, then

$$\begin{array}{l} \dot{x}_1 = \dot{x} = x_2 \\ \dot{x}_2 = \ddot{x} = \cos x - t\,x = \cos x_1 - t x_3\,x_1 \\ \dot{x}_3 = 1 \end{array} \Rightarrow \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \cos x_1 - x_3\,x_1 \\ \dot{x}_3 = 1 \end{array}$$

2. For the following autonomous systems find the equilibrium points and sketch the phase line. Then use a graphical argument to classify the stability of the equilibrium points.
   (a) $\dot{x} = x^2 - 1$
   (b) $\dot{x} = x\,(1-x)\,(2-x)$
   (c) $\dot{x} = \cos x$

Solution:

(a) $\dot{x} = x^2 - 1$

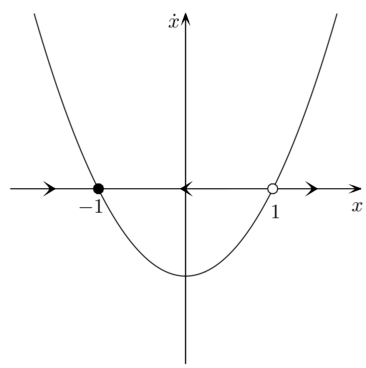
The equilibrium points are given by

$$x^2 - 1 = 0 \quad \Rightarrow \quad x^*_{1,2} = \pm 1.$$

We then have that

$$\dot{x} \begin{cases} > 0, & \text{for } x < -1, \\ < 0, & \text{for } -1 < x < 1, \\ > 0, & \text{for } x > 1. \end{cases}$$

This allows us to draw the phase line, given by



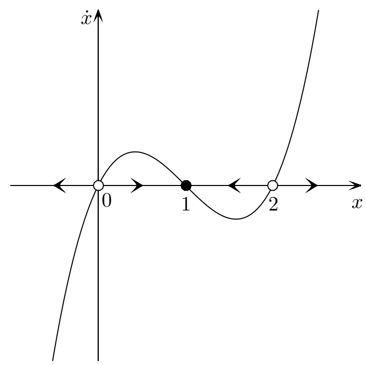
The equilibrium point at $x^*_1 = -1$ is stable, the equilibrium point at $x^*_2 = 1$ is unstable.

(b) $\dot{x} = x\,(1 - x)\,(2 - x)$

The equilibrium points are given by $x^*_1 = 0$, $x^*_2 = 1$, and $x^*_3 = 2$. We then have that

$$\dot{x} \begin{cases} < 0, & \text{for } x < 0, \\ > 0, & \text{for } 0 < x < 1, \\ < 0, & \text{for } 1 < x < 2, \\ > 0, & \text{for } x > 2. \end{cases}$$

We can now draw the phase line, given by



The equilibrium points at $x_1^* = 0$ and $x_3^* = 2$ are unstable, the equilibrium point at $x_2^* = 1$ is stable.

(c) $\dot{x} = \cos x$

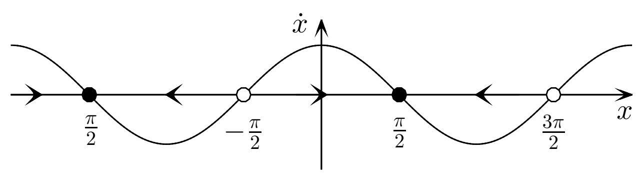
The equilibrium points are given by

$$x_n^* = \frac{1}{2}\pi + n\pi, \quad \text{for } n \in \mathbb{Z}.$$

We then have that

$$\dot{x} \begin{cases} > 0, & \text{for } -\frac{1}{2}\pi + n\pi < x < \frac{1}{2}\pi + n\pi, \\ < 0, & \text{for } \frac{1}{2}\pi + n\pi < x < \frac{3}{2}\pi + n\pi, \end{cases}$$

for $n \in \mathbb{Z}$. We can now draw the phase line, given by



The equilibrium points at x_n^* with n even are stable, those for n odd are unstable.

3. Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails (because $f'(x^*) = 0$, then use a graphical argument.

   (a) $\dot{x} = x(2 - x)$

(b) $\dot{x} = 1 - e^{-x}$

(c) $\dot{x} = \log x$

(d) $\dot{x} = x^4$

Solution:

(a) $\dot{x} = x(2 - x)$

The equilibrium points are $x_1^* = 0$ and $x_2^* = 2$. As shown in the lectures, linear stability analysis tells us that we need to look at the gradient of $\dot{x}$ at the equilibrium point to establish stability. Writing $\dot{x} = f(x)$, we have that

$$f'(x) = 2 - 2x \quad \begin{cases} > 0, & \text{at } x_1^* (= 0), \\ < 0, & \text{at } x_2^* (= 2). \end{cases}$$

Thus, via linear stability analysis, x_1^* is an unstable equilibrium point, while x_2^* is a stable equilibrium point.

(b) $\dot{x} = 1 - e^{-x}$

There is a single equilibrium point of this system, at $x^* = 0$. Writing $\dot{x} = f(x)$ we have that

$$f'(x) = e^{-x}.$$

Thus $f'(x^*) = 1 > 0$, and, via linear stability analysis, $x^* = 0$ is an unstable equilibrium point.

(c) $\dot{x} = \log x$

There is a single equilibrium point of this system, at $x^* = 1$ (with no solutions when $x < 0$). Writing $\dot{x} = f(x)$ we have that

$$f'(x) = \frac{1}{x}.$$

Thus $f'(x^*) = 1 > 0$, and, via linear stability analysis, $x^* = 1$ is an unstable equilibrium point.

(d) $\dot{x} = x^4$

There is a single equilibrium point of this system, at $x^* = 0$. Writing $\dot{x} = f(x)$ we have that $f'(0) = 0$ and so linear stability analysis fails in this case. Examining the system, we see that $\dot{x} > 0$ for $x < 0$ and $x > 0$, and so the equilibrium point $x^* = 0$ is half-stable.

4. For the dynamical systems in question two, sketch the direction fields on the following domains,

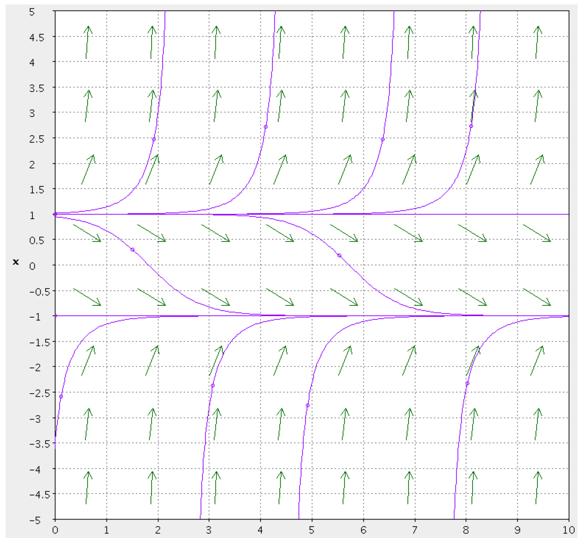
(a) $(x,t) \in (-5,5) \times (0,10)$

(b) $(x,t) \in (-1,3) \times (0,10)$

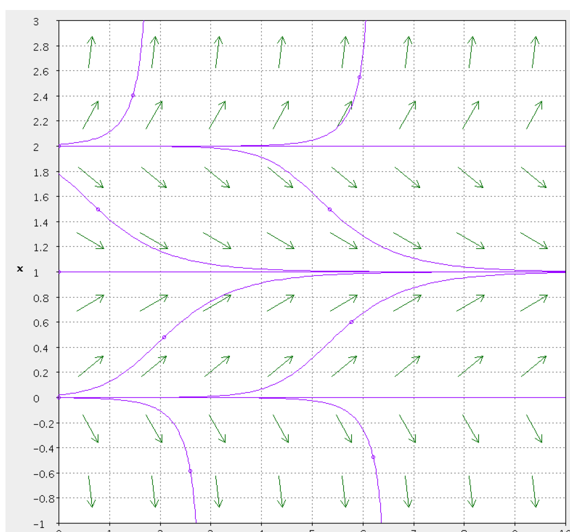
(c) $(x,t) \in (-2\pi, 2\pi) \times (0,10)$

Solution:

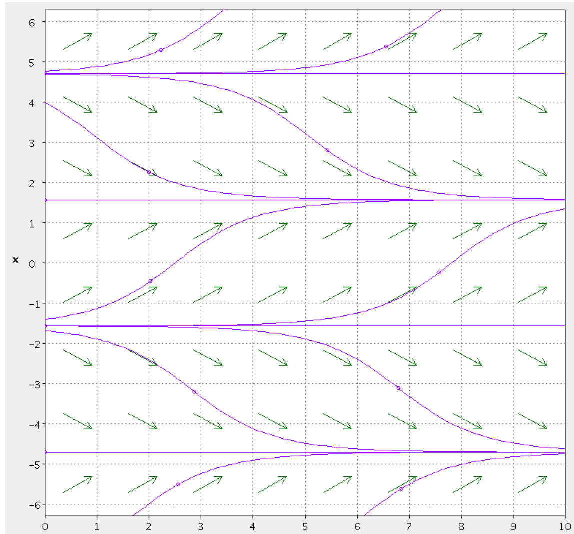
(a) $\dot{x} = x^2 - 1$



(b) $\dot{x} = x(1-x)(2-x)$



(c) $\dot{x} = \cos x$



5. The Logistic equation,

$$\frac{dn}{dt} = n\left(a - bn\right), \quad n\left(t\right) \geqslant 0, b \neq 0$$

for some constants a and b, was first introduced as a way of describing population growth by Verhulst in 1838, where $n\left(t\right)$ is the size of the population at time t.

(a) Find the equilibrium points of the logistic equation. Then, using linear stability analysis, establish the stability of the equilibrium points of the logistic equation for all non-zero values of a and b.

(b) For the case $a = b = 1$, by considering $\dot{n}\left(t\right)$ at various points, sketch the direction field for the Logistic equation for $0 \leqslant n \leqslant 2$, $t > 0$. Then use this sketch to draw the solution curves for the system.

(c) Finally, using part (b), describe what happens to the size of a population which at time $t = 0$ is given by $n\left(0\right) = 0.1$.

Solution:

(a) The equilibrium points of the Logistic equation are given by

$$\dot{n} = 0 \Rightarrow n = 0 \quad \text{or} \quad n = \frac{a}{b}.$$

Writing $\dot{n} = f\left(n\right)$, we then have

$$f'\left(n\right) = a - 2bn.$$

Linear stability analysis tells us that the sign of f' at the equilibrium point dictates the stability. We have

$$f'(0) = a \quad \Rightarrow \begin{cases} f'(0) > 0, & a > 0, \\ f'(0) < 0, & a < 0. \end{cases}$$

Thus, the equilibrium point $n = 0$ is stable if $a < 0$, and unstable if $a > 0$.

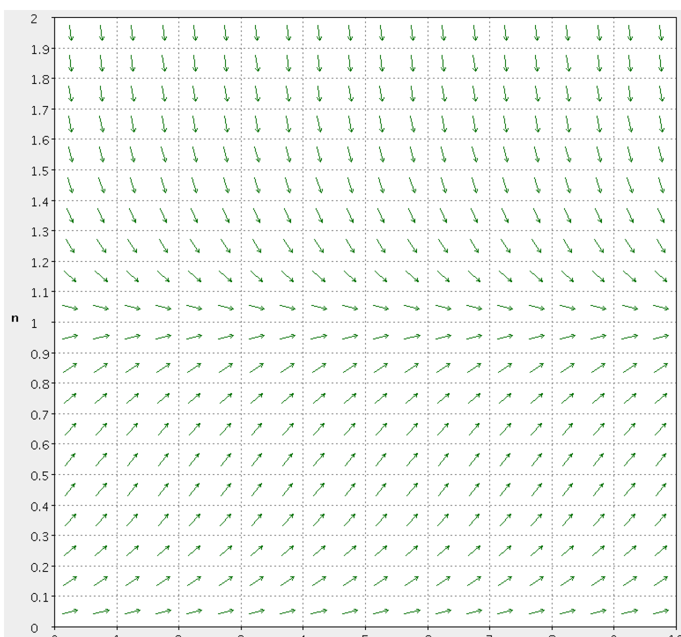
Also,

$$f'\left(\frac{a}{b}\right) = -a \quad \Rightarrow \begin{cases} f'(0) > 0, & a < 0, \\ f'(0) < 0, & a > 0. \end{cases}$$

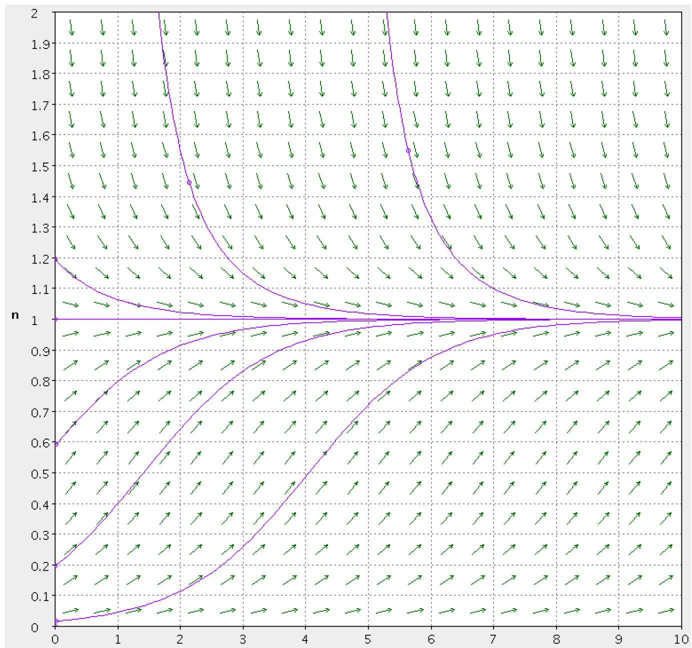
Thus, the equilibrium point $n = \frac{a}{b}$ is stable if $a > 0$, and unstable if $a < 0$.

(b) For $a = b = 1$, we have $\dot{n} = n(1 - n)$. Examining $\dot{n}(t)$, we have

n	0	0.25	0.5	0.75	1	1.25	1.5
$\dot{n}$	0	0.1875	0.25	0.1875	0	−0.3125	−0.75



Using the direction field from part (b), we can sketch the solution curves,



(c) If $n(0) = 0.1$, since $\dot{n}(0.1) = 0.1(0.9) = 0.01 > 0$, the population will grow with $n(t) \to 1$, as $t \to \infty$.