Examples sheet 3 – Solutions – Linear Algebra

LINEAR MAPS.

1. (a) Let $a, b \in \mathbb{R}$. We have

$$f(ax+by) = \alpha(ax+by)+\beta = (a\alpha x + a\beta) + (b\alpha y + b\beta) + \beta(1-a-b) = af(x)+bf(y),$$

provided $\beta = 0$. Hence, the map is linear if it has the form $f(x) = \alpha x$.

(b) Let $a, b \in \mathbb{R}$. We have

$$f(ax+by) = \begin{bmatrix} \alpha \\ (ax+by)^\beta \end{bmatrix} = \begin{bmatrix} a\alpha \\ ax^\beta \end{bmatrix} + \begin{bmatrix} b\alpha \\ bx^\beta \end{bmatrix} = af(x)+bf(y),$$

provided $\alpha = 0$ and $\beta = 1$. Hence, the map is linear if it has the form $f(x) = \begin{bmatrix} 0 \\ x \end{bmatrix}$.

2. (a) To check linearity, let $p, q \in \mathcal{P}_3(\mathbb{R})$ and $a, b \in \mathbb{R}$. Then

$$f(ap+bq) = \begin{bmatrix} (ap+bq)(-1) \\ 2(ap+bq)(1) \end{bmatrix} = \begin{bmatrix} ap(-1)+bq(-1) \\ 2(ap(1)+bq(1)) \end{bmatrix} = a\begin{bmatrix} p(-1) \\ 2p(1) \end{bmatrix} + b\begin{bmatrix} q(-1) \\ 2q(1) \end{bmatrix} = af(p)+bf(q).$$

Hence f is linear.

(b) The kernel is non-trivial provided it contains non-zero polynomials p satisfying $f(p) = \mathbf{0}_2$. We find

$$f(p) = \begin{bmatrix} p(-1) \\ 2p(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} p(x) = & (x+1)q(x), \\ p(x) = & (x-1)r(x), \end{cases}$$

where $q, r \in \mathcal{P}_2(\mathbb{R})$. Hence, p has factors $(x-1)$ and $(x+1)$ and therefore the form

$$p(x) = (x-1)(x+1)(ax+b) = ax(x-1)(x+1) + b(x-1)(x+1),$$

for some $a, b \in \mathbb{R}$. Therefore

$$p \in \text{span}\left\{x^2-1, x(x^2-1)\right\} =: \text{span}\,\{q_1, q_2\}.$$

We conclude that the kernel is non-trivial. Since $\{q_1, q_2\}$ is a linearly independent set, it is also a basis for the kernel of f, so that $\dim\ker f = 2$. Therefore, f is not injective.

(c) By the rank-nullity formula,

$$\dim\ \text{im}\, f = \text{rank}\ f = \dim \mathcal{P}_4(\mathbb{R}) - \dim\ \ker f = 4-2 = 2 = \dim\ \mathbb{R}^2.$$

On the other hand, $\text{im}\, f \subseteq \mathbb{R}^2$. Hence, by Proposition 4.6, $\text{im}\, f = \mathbb{R}^2$ and the map is surjective.We can confirm this by the following calculation. Let $S = \{x-1, x+1\}$. Then

$$f(x-1) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad f(x+1) = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \Longrightarrow f(S) = \left\{\begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}\right\} \Longrightarrow \text{span}\ f(S) = \mathbb{R}^2$$

since the two vectors in $f(S)$ form a basis for $\mathbb{R}^2$. Hence,

$$\mathbb{R}^2 = \text{span}\ f(S) = f(\text{span}\ S) \subseteq \text{im}\, f.$$

Since $\text{im}\, f \subseteq \mathbb{R}^2$, we conclude that $\text{im}\, f = \mathbb{R}^2$.

3. The aim of this question is to highlight that the concept of linearity involves distinct vector space operations in the domain and in the codomain.

(a) Consider first operations in V_n: we evaluate $a \bullet \mathbf{v} \boldsymbol{+} b \bullet \mathbf{w}$:

$$a \bullet \mathbf{v} = a\mathbf{v} + (a-1)\mathbf{1}_n, \quad b \bullet \mathbf{w} = b\mathbf{w} + (b-1)\mathbf{1}_n$$

so that

$$a \bullet \mathbf{v} \boldsymbol{+} b \bullet \mathbf{w} = a\mathbf{v} + (a-1)\mathbf{1}_n + b\mathbf{w} + (b-1)\mathbf{1}_n + \mathbf{1}_n = a\mathbf{v} + b\mathbf{w} + (a+b-1)\mathbf{1}_n.$$

Hence,

$$f(a \bullet \mathbf{v} \boldsymbol{+} b \bullet \mathbf{w}) = A\left[a\mathbf{v} + b\mathbf{w} + (a+b-1)\mathbf{1}_n\right] + \mathbf{b} + \mathbf{1}_m.$$

We now evaluate $a \bullet f(\mathbf{v}) \boldsymbol{+} b \bullet f(\mathbf{w})$ in V_m:

$$f(\mathbf{v}) = A\mathbf{v} + \mathbf{b} + \mathbf{1}_m, \qquad f(\mathbf{w}) = A\mathbf{w} + \mathbf{b} + \mathbf{1}_m,$$

so that

$$a \bullet f(\mathbf{v}) = a\left(A\mathbf{v} + \mathbf{b} + \mathbf{1}_m\right) + (a-1)\mathbf{1}_m = aA\mathbf{v} + a\mathbf{b} + (2a-1)\mathbf{1}_m,$$
$$b \bullet f(\mathbf{w}) = b\left(A\mathbf{w} + \mathbf{b} + \mathbf{1}_m\right) + (b-1)\mathbf{1}_m = bA\mathbf{w} + b\mathbf{b} + (2b-1)\mathbf{1}_m.$$

Hence,

$$a \bullet f(\mathbf{v}) \boldsymbol{+} b \bullet f(\mathbf{w}) = A\left[a\mathbf{v} + b\mathbf{w}\right] + (a+b)\mathbf{b} + 2(a+b-1)\mathbf{1}_m + \mathbf{1}_m.$$

Therefore, f is a linear map provided

$$(a+b-1)A\mathbf{1}_n + \mathbf{b} = (a+b)\mathbf{b} + 2(a+b-1)\mathbf{1}_m \iff A\mathbf{1}_n = \mathbf{b} + 2 \cdot \mathbf{1}_m.$$

Hence any pair $A, \mathbf{b}$ satisfying the above relation will yield a linear map. For example, taking $\mathbf{b} = -\mathbf{1}_m$ (which is the additive inverse in V_m), yields $A\mathbf{1}_n = \mathbf{1}_m$, i.e., each row of A sums to 1.
Note: One can also obtain this relation by using the property that linear transformations map zero to zero. In this case, $\mathbf{0}_{V_n} = -\mathbf{1}_n$, so we require

$$f(-\mathbf{1}_n) = -\mathbf{1}_m \iff A(-\mathbf{1}_n) + \mathbf{b} + \mathbf{1}_m = -\mathbf{1}_m,$$

which is the same relation we obtained for A and $\mathbf{b}$.

(b) We have

$$f(\mathbf{v}) = \mathbf{0}_{V_m} \iff A\mathbf{v} + \mathbf{b} + \mathbf{1}_m = -\mathbf{1}_m \iff A\mathbf{v} = \mathbf{0}_m,$$

provided $\mathbf{b} = -2\mathbf{1}_m$.

4. (a) The map f is injective provided its kernel is trivial. This is equivalent to requiring the kernel of A to be trivial:

$$A\mathbf{x} = \mathbf{0}_m \iff \sum_{i=1}^{n} x_i \mathbf{c}_i(A) = \mathbf{0}_m,$$

which is equivalent to requiring that the columns of A form a linearly independent set. This is not possible if $n > m$, since a maximal linearly independent set of $\mathbb{R}^m$ (a basis) has m elements, so any additional vectors would render the set linearly dependent. Hence,

- $n > m$: f is not injective;
- $n \leq m$: f may be injective (provided the columns of A are linearly independent).

(b) The map f is surjective provided rank f = rank $A = m$, or equivalently col $A = \mathbb{R}^m$. Since the rank of a matrix is equal to the number of linearly independent columns (i.e., the dimension of its column space), we find that it is possible to have rank $A = m$, provided the number of columns n is at least equal to the dimension of the codomain $\mathbb{R}^m$, i.e., $n \geq m$. Otherwise a number $n < m$ of columns cannot span a space of dimension m. Hence,

- $n < m$: f is not surjective;
- $n \geq m$: f may be surjective (provided we have m columns of A that are linearly independent).

5. **(a)** We have

VA0 Additive closure holds, since $f+g$ is a linear map:

$$(f+g)(a\mathbf{v}+b\mathbf{w}) = f(a\mathbf{v}+b\mathbf{w})+g(a\mathbf{v}+b\mathbf{w}) = af(\mathbf{v})+bf(\mathbf{w})+ag(\mathbf{v})+bg(\mathbf{w}) = a(f+g)(\mathbf{v})+b(f+g)(\mathbf{w}).$$

VA1 Associativity holds for linear maps, since it holds for function addition.

VA2 The additive identity is the zero function $o : V \to W$, which is linear.

VA3 The additive inverse corresponding to $f \in \mathcal{L}(V, W)$ is $-f$, which is also linear.

VA4 Commutativity holds for linear maps, as it holds for functions in general.

VM0 Multiplicative closure holds, since $\alpha \cdot f$ is a linear map for any $\alpha \in \mathbb{F}$:

$$(\alpha \cdot f)(a\mathbf{v} + b\mathbf{w}) = \alpha \cdot f(a\mathbf{v} + b\mathbf{w}) = \alpha \cdot (af(\mathbf{v}) + bf(\mathbf{w})) = a(\alpha \cdot f)(\mathbf{v}) + b(\alpha \cdot f)(\mathbf{v})$$

VM1 Associativity of scalar-vector multiplication holds for linear maps as it holds for functions in general.

VM2 Distributivity of scalar-vector multiplication holds for linear maps as it holds for functions in general.

VM3 Distributivity of scalar addition holds for linear maps as it holds for functions in general.

VM4 The multiplicative identity property holds, as it holds for functions in general.

(b) $\mathcal{L}_0(V, W) \not\leq \mathcal{L}(V, W)$ since the additive identity is not in $\mathcal{L}_0(V, W)$, as its kernel is not trivial: $\ker o = V$.

6. **(a)** Let $\mathbf{x}$ be the vector of coordinates of a generic vector $\mathbf{v} \in V$ in the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. We have

$$\varphi_B(a\mathbf{v} + b\mathbf{v}') = a\mathbf{x} + b\mathbf{x}' = a\varphi_B(\mathbf{v}) + b\varphi_B(\mathbf{v}'),$$

so that φ_B is linear.

(b) The coordinate map is bijective since it is

- injective:

$$\varphi_B(\mathbf{v}) = \varphi_B(\mathbf{v}') \Longrightarrow \sum_{i=1}^{n} x_i\mathbf{v}_i = \sum_{i=1}^{n} x_i'\mathbf{v}_i \Longrightarrow x_i = x_i' \Longrightarrow \mathbf{v} = \mathbf{v}'.$$

- surjective: for any $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{v} \in V$ given by $\mathbf{v} = \sum_{i=1}^{n} x_i\mathbf{v}_i$, such that $\varphi_B(\mathbf{v}) = \mathbf{x}$.

7. By subspace criterion 2, $a\mathbf{u}_1 + b\mathbf{u}_2 \in U$ for any $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $a, b \in \mathbb{F}$. By linearity of f,

$$f(a\mathbf{u}_1 + b\mathbf{u}_2) = af(\mathbf{u}_1) + bf(\mathbf{u}_2).$$

Since $f(\mathbf{u}_1), f(\mathbf{u}_2), f(a\mathbf{u}_1 + b\mathbf{u}_2) \in f(U)$ and $f(U) \subseteq W$, by subspace criterion 2 we conclude that $f(U) \leq W$.

8. By the rank-nullity formula, $\text{nullity}A = 7 - \text{rank } A$ and since $0 \leq \text{rank } A \leq 3$, we find $4 \leq \text{nullity}(A) \leq 7$. The upper bound is achieved when $\text{rank } A = 0$, i.e., for the zero matrix $O_{3,7}$. The lower bound is achieved for any matrix with three linearly independent columns.

9. **(a)** Since $f(\mathbf{x}) = A\mathbf{x}$, where $A \in \mathbb{R}^{m\times n}$, it follows that $f : \mathbb{R}^n \to \mathbb{R}^m$; hence, $\text{im} f \subset \mathbb{R}^m$, so that $\dim imf \leq m$. By the rank-nullity formula,

$$\text{nullity}(f) = n - \text{rank } f = n - \dim \text{im} f > n - m > 0.$$

Hence, $\dim \ker f > 0$, so that $\ker f$ is not trivial.

(b) Assume f is injective. Then its kernel is trivial and, by the rank-nullity formula,

$$\text{rank } f = n - \text{nullity}(f) = n.$$

Since the dimension of the codomain is the same as that of the image, the two are the same: $\text{im } f = \mathbb{R}^n$ (see Proposition 4.6). Hence f is surjective (by Proposition 8.2ii.). Assume now that f is surjective. By Proposition 8.2ii., $\text{rank } f = n$, so that $\text{nullity}(f) = 0$ and the kernel of f is trivial. By Proposition 8.2i., f is injective.

MATRIX REPRESENTATIONS. KERNEL AND RANK.

10. Let

$$B_V = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\} =: \{\mathbf{v}_1, \mathbf{v}_2\}, \quad B_W = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}.$$

We first represent the given vectors in the two bases:

$$\begin{bmatrix}0\\2\end{bmatrix} = 2(\mathbf{v}_1 - \mathbf{v}_2), \quad \begin{bmatrix}2\\1\end{bmatrix} = \mathbf{v}_1 + \mathbf{v}_2$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = 3\mathbf{w}_3 - \mathbf{w}_2 - \mathbf{w}_1, \quad \begin{bmatrix}2\\1\\0\end{bmatrix} = \mathbf{w}_1 + \mathbf{w}_2.$$

We find, using the linearity of f,

$$\begin{aligned} f(2(\mathbf{v}_1 - \mathbf{v}_2)) &= 2f(\mathbf{v}_1) - 2f(\mathbf{v}_2) = 3\mathbf{w}_3 - \mathbf{w}_2 - \mathbf{w}_1 \\ f(\mathbf{v}_1 + \mathbf{v}_2) &= f(\mathbf{v}_1) + f(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2 \end{aligned}$$

and solving for $f(\mathbf{v}_1)$ and $f(\mathbf{v}_2)$ we find the matrix representation with respect to the given bases:

$$\begin{cases} f(\mathbf{v}_1) = & \frac{1}{4}\mathbf{w}_1 + \frac{1}{4}\mathbf{w}_2 + \frac{1}{4}\mathbf{w}_3 \\ f(\mathbf{v}_2) = & \frac{3}{4}\mathbf{w}_1 + \frac{3}{4}\mathbf{w}_2 - \frac{3}{4}\mathbf{w}_3 \end{cases} \Longrightarrow A = \frac{1}{4}\begin{bmatrix}1 & 3\\1 & 3\\3 & -3\end{bmatrix}.$$

11. (a) We have

$$f(\mathbf{e}_1) = \begin{bmatrix}3\\-1\\2\end{bmatrix} = 3\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$$

and similarly,

$$f(\mathbf{e}_2) = \mathbf{e}_1 - 2\mathbf{e}_2, \quad f(\mathbf{e}_3) = -\mathbf{e}_1 + 2\mathbf{e}_2,$$

so that the matrix representation is

$$A = \begin{bmatrix}3 & 1 & -1\\-1 & -2 & 2\\2 & 0 & 0\end{bmatrix}.$$

(b) Since $f(\mathbf{x}) = A\mathbf{x}$, the image of f is $\operatorname{col} A$. The last two columns of A are linearly dependent, so

$$\operatorname{col} A = \operatorname{span}\left\{ \begin{bmatrix}3\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix} \right\},$$

so that $\operatorname{rank} f = 2$.

(c) To find the kernel, we consider $A\mathbf{x} = \mathbf{0}$:

$$A\mathbf{x} = \mathbf{0} \iff \begin{bmatrix}3 & 1 & -1\\-1 & -2 & 2\\2 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \mathbf{0} \iff \begin{cases} x = 0 \\ y = z \end{cases} \Longrightarrow \ker A = \operatorname{span}\left\{ \begin{bmatrix}0\\1\\1\end{bmatrix} \right\},$$

so that $\dim\ker f = 1$.

12. **(a)** Let $A \in \mathbb{R}^{3\times 5}$ have orthogonal columns. Then 2 columns must be zero and we find rank $A = 3$, since the other 3 columns are linearly independent. The dimension of the kernel is then $\dim\ker A = 5 - 3 = 2$. We can find a basis for $\ker A$, if we assume A takes a certain form. Let's permute the columns of A so that it has the structure

$$A = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\ \mathbf{0}\ \mathbf{0}].$$

Then

$$\ker A = \operatorname{span}\{\mathbf{e}_4, \mathbf{e}_5\} = \operatorname{span}\left\{\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}.$$

(b) Let $B \in \mathbb{R}^{5\times 3}$ have orthogonal (and therefore linearly independent) columns. Since $n = 3 < 5 = m$, by **Q4(a)**, f is injective and therfore $\ker B = \{\mathbf{0}_3\}$, so that $\dim\ker B = 0$ and rank $B = 3$.

13. **(a)** Let $S = \varphi_V(\ker f) := \{\mathbf{x} \in \mathbb{R}^n : f(\varphi_V^{-1}(\mathbf{x})) = \mathbf{0}_W\}$. We show $S \subseteq \ker A$ and $\ker A \subseteq S$. Let $\mathbf{x} \in S$. Then $f(\varphi_V^{-1}(\mathbf{x})) = \mathbf{0}_W$. Define $\mathbf{v} := \varphi_V^{-1}(\mathbf{x})$. Then $\mathbf{x} = \varphi_V(\mathbf{v})$ and therefore we can apply Proposition 11.1:

$$f(\mathbf{v}) = \mathbf{0}_W \implies \mathbf{v} \in \ker f \overset{\text{Prop 11.1}}{\implies} \mathbf{x} \in \ker A \implies S \subseteq \ker A.$$

Let $\mathbf{x} \in \ker A$. By Proposition 11.1, the vector $\mathbf{v} \in V$ with coordinates x_i relative to the basis B_V satisfies $\mathbf{v} \in \ker f$. Hence, since $\mathbf{v} = \varphi_V^{-1}(\mathbf{x})$, we find

$$f(\mathbf{v}) = \mathbf{0}_W \implies f(\varphi_V^{-1}(\mathbf{x})) = \mathbf{0}_W \implies \mathbf{x} \in S \implies \ker A \subseteq S.$$

(b) Let $S = \varphi_W(\operatorname{im} f) := \{\mathbf{y} \in \mathbb{R}^m : \varphi_W^{-1}(\mathbf{y}) = f(\mathbf{v}),\ \text{for some } \mathbf{v} \in V\}$. We show $S \subseteq \operatorname{col} A$, $\operatorname{col} A \subseteq S$. Let $\mathbf{y} \in S$. Then $\varphi_W^{-1}(\mathbf{y}) = f(\mathbf{v})$ for some $\mathbf{v} \in V$. Define $\mathbf{w} := \varphi_W^{-1}(\mathbf{y}) = f(\mathbf{v})$. Then $\mathbf{y} = \varphi_W(\mathbf{w})$ and therefore we can apply Proposition 11.4:

$$\mathbf{w} = f(\mathbf{v}) \implies \mathbf{w} \in \operatorname{im} f \overset{\text{Prop 11.4}}{\implies} \mathbf{y} \in \operatorname{col} A \implies S \subseteq \operatorname{col} A.$$

Let $\mathbf{y} \in \operatorname{col} A$. By Proposition 11.4, the vector $\mathbf{w} \in W$ with coordinates $\mathbf{y}$ relative to the basis B_W satisfies $\mathbf{w} \in \operatorname{im} f$. Hence, since $\mathbf{w} = \varphi_W^{-1}(\mathbf{y})$, we find

$$\mathbf{w} \in \operatorname{im} f \implies \varphi_W^{-1}(\mathbf{y}) = \mathbf{w} = f(\mathbf{v}),\ \text{for some } \mathbf{v} \in V \implies \mathbf{y} \in S \implies \operatorname{im} f \subseteq S.$$

14. Let $\mathbf{e}_i, \mathbf{e}'_j$ denote the canonical basis vectors in $\mathbb{R}^5$ and $\mathbb{R}^3$, respectively.

(a) We evaluate

$$f(\mathbf{e}_1) = \mathbf{e}'_1, \quad f(\mathbf{e}_2) = \mathbf{0}_3, \quad f(\mathbf{e}_3) = \mathbf{e}'_2, \quad f(\mathbf{e}_4) = \mathbf{0}_3, \quad f(\mathbf{e}_5) = \mathbf{e}'_3,$$

so that the matrix representation is

$$A = \begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\end{bmatrix}.$$

(b) We similarly evaluate

$$g(\mathbf{e}'_1) = \begin{bmatrix}1\\1/2\\0\\0\\0\end{bmatrix} = 1\cdot\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2, \quad g(\mathbf{e}'_2) = \begin{bmatrix}0\\1/2\\1\\1/2\\0\end{bmatrix} = \frac{1}{2}\mathbf{e}_2 + 1\cdot\mathbf{e}_3 + \frac{1}{2}\mathbf{e}_4, \quad g(\mathbf{e}'_3) = \begin{bmatrix}0\\0\\0\\1/2\\1\end{bmatrix} = \frac{1}{2}\mathbf{e}_4 + 1\cdot\mathbf{e}_5.$$

Hence, the matrix representation is

$$B = \begin{bmatrix}1&0&0\\1/2&1/2&0\\0&1&0\\0&1/2&1/2\\0&0&1\end{bmatrix}.$$

(c) The matrix representation of $h = f \circ g$ is

$$C = AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, we can view A as a left inverse for B and B as a right inverse for A. These are examples of generalised inverses (or pseudo-inverses). One thing to note is that, unlike inverses of square matrices, generalised inverses are not unique: for example you can replace rows 2 and 4 of B with any values, while preserving the right inverse property.

(d) Consider the restriction of the identity map on $\mathbb{R}^5$ to the subspace

$$U = \left\{ \mathbf{v} \in \mathbb{R}^5 : v(2) = v(4) = 0 \right\},$$

given by $\widetilde{id}(\mathbf{u}) = id(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u} \in U$. This is also called the (natural) **inclusion map**. Noting that U is isomorphic to $\mathbb{R}^3$, we conclude that the linear map f is induced by this inclusion map. Such maps are generally referred to as inclusion maps. The map g can be viewed as an **interpolation map**, as its action interlaces the entries in a vector $\mathbf{v} \in \mathbb{R}^3$ with neighbour averages. Finally, h is the identity, which in this case can be viewed as the composition of interpolation followed by inclusion/restriction.

15. **(a)** Let $\mathbf{b} \in \operatorname{col} A$. Then we can represent it as a linear combination of the columns of A:

$$\mathbf{b} = \sum_{i=1}^{n} x_i \mathbf{c}_i(A) = A\mathbf{x}.$$

Hence, the solution of the linear system is given by the coefficients of $\mathbf{b}$ in the linear combination. Since the number of columns n is larger than the dimension m of the space $\mathbf{b}$ and the columns are in, the columns form a linearly dependent set in general (since the size of a maximal linearly independent set is m). Hence, the representation of $\mathbf{b}$ as a linear combination is non-unique. This means that we may have more than one solution to the linear system.

(b) Assume now that $\ker A$ is non-trivial (this is the case anyway when $n > m$: see **Q4(a)**). So if we have a solution $\mathbf{x}$ to the linear system, then we have infinitely-many, since $\mathbf{x}+\mathbf{z}$ is also a solution for any $\mathbf{z} \in \ker A$.

CHANGE OF BASIS

16. **(a)** Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $B' = \{\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3\} =: \{\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3\}$. Then

$$\mathbf{v}_1 = \mathbf{v}'_2, \mathbf{v}_2 = \mathbf{v}'_1, \mathbf{v}_3 = \mathbf{v}'_3 \Longrightarrow M = M_{BB'} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) The matrix M is a permutation matrix. In fact, this is an elementary permutation since $P^2 = I_3$. Hence, $P = P^{-1}$. Multiplying a matrix from the left will swap rows 1 and 2. Multiplying a matrix from the right will swap columns 1 and 2. Consider now the matrix representations of a linear map f under bases B and B'. Given A_{BB}, the matrix under the change of basis is

$$A_{B'B'} = M_{BB'} A_{BB} M_{BB'}.$$

Hence, the new matrix will have rows 1 and 2 and also columns 1 and 2 swapped over, through multiplication by the transition matrix (a permutation in this case) from the left and from the right.

17. **(a)** Let $B = \{2x, 4x - 2\} =: \{q_1, q_1\}$, $B' = \{x - 1, x + 1\} =: \{q'_1, q'_2\}$. To find the transition matrix, we express q_1, q_2 in terms of q'_1, q'_2:

$$\begin{cases} q_1 & = q'_1 + q'_2 \\ q_2 & = 3q'_1 + q'_2 \end{cases} \Longrightarrow M_{BB'} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}.$$

(b) Let us denote the B-coordinates as $\mathbf{x}'$. We find

$$\mathbf{x}' = M_{BB'}\mathbf{x} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

(c) We find

$$p(x) = 3q_1(x) + q_2(x) = 10x - 2 = 6q_1(x)' + 4q_2(x)'.$$

18. Consider the linear map $f : V \to V$ defined by $f \mapsto f + f'$, where f' is the derivative of f. Let

$$B = \{2x, 4x - 2\} =: \{q_1, q_1\}, \quad B' = \{x - 1, x + 1\} =: \{q_1', q_2'\}.$$

(a) We find

$$f(q_1') = 1 + x - 1 = x = \frac{1}{2}\left(q_1' + q_2'\right), \quad f(q_2') = 1 + x + 1 = x + 2 = -\frac{1}{2}q_1' + \frac{3}{2}q_2'.$$

Hence

$$A_{B'B'} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

(b) The matrix representation with respect to basis B is

$$A_{BB} = M_{BB'}^{-1} A_{B'B'} M_{BB'} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}.$$

We can check this result:

$$f(q_1) = 2x + 2 = 3q_1 - q_2, \quad f(q_2) = 4x - 2 + 4 = 4x - 2 = 4q_1 - q_2,$$

so that indeed

$$A_{BB} = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}.$$

19. By definition, matrices $A, B \in \mathbb{F}^{m\times n}$ are equivalent, if there exist invertible matrices $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$ such that $B = M^{-1}AN$. We write $A \sim B$. To show matrix equivalence is an equivalence relation, we need to check the three properties of an equivalence.

i. reflexivity: $A \sim A$. Let $M = I_m, N = I_n$. Then $A = M^{-1}AN$, with M, N invertible, so that $A \sim A$.

ii. symmetry: $A \sim B \Longleftrightarrow B \sim A$. We have

$$A \sim B \Longleftrightarrow B = M^{-1}AN \Longleftrightarrow MBN^{-1} = A \Longleftrightarrow B \sim A,$$

since M, N^{-1} are invertible matrices of suitable sizes.

iii. transitivity: $A \sim B, B \sim C \Longrightarrow A \sim C$. Let M, N and Q, P be the invertible matrices arising in the equivalences $A \sim B$, $B \sim C$, respectively. We find

$$\begin{cases} A \sim B & \Longleftrightarrow B = M^{-1}AN \\ B \sim C & \Longleftrightarrow C = Q^{-1}BP \end{cases} \Longrightarrow C = Q^{-1}M^{-1}ANP \Longrightarrow A \sim C,$$

since the matrices $Q^{-1}M^{-1} \in \mathbb{F}^{m\times m}$ and $NP \in \mathbb{F}^{n\times n}$ are invertible, as products of invertible matrices.