

Examples sheet 2 – Linear Algebra

The exercises below correspond to material from Lectures 5–7. Selected exercises will be covered in the Examples class scheduled in week 5. Solutions will be available on Canvas.

INNER PRODUCTS. NORMS.

1. Establish if the functions $\mathcal{B}(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ given below satisfy the inner product properties over the vector spaces V indicated.

(a) $V = \mathcal{P}_n(\mathbb{R})$,

$$\mathcal{B}(p, q) := \int_{-1}^1 p'(x)q'(x)dx + p(0)q(0).$$

(b) $V = \mathbb{Z}_2^3(\mathbb{Z}_2)$,

$$\mathcal{B}(\mathbf{v}, \mathbf{w}) := (v_1w_1 + v_2w_2 + v_3w_3) \pmod{2}.$$

(c) $V = \mathbb{R}^2$,

$$\mathcal{B}(\mathbf{v}, \mathbf{w}) := v_1a_{11}w_1 + v_1a_{12}w_2 + v_2a_{21}w_1 + v_2a_{22}w_2, \quad [a_{ij}] := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =: A.$$

(d) $V = \mathbb{R}^2$,

$$\mathcal{B}(\mathbf{v}, \mathbf{w}) := v_1w_2 + v_2w_1.$$

2. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Show that $\langle \mathbf{0}, \mathbf{v} \rangle = 0$ for any $\mathbf{v} \in V$.

3. Let $V(\mathbb{R})$ be a vector space. Let $\langle \cdot, \cdot \rangle$ be an inner product defined on $V \times V$. Consider the function $n : V \rightarrow \mathbb{R}_+ \cup \{0\}$ defined via

$$n(\mathbf{v}) = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Check that $n(\cdot)$ satisfies the norm properties.

4. Let $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ and let $\|\cdot\|$ denote the corresponding induced norm. Show that

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2),$$

for any $\mathbf{v}, \mathbf{w} \in V$.

5. Check that the following functions defined on \mathbb{R}^n satisfy the norm properties.

(a) $n_1(\mathbf{v}) := \|\mathbf{v}\|_1 := |v_1| + |v_2| + \cdots + |v_n|$.

(b) $n_\infty(\mathbf{v}) := \|\mathbf{v}\|_\infty := \max_{1 \leq j \leq n} |v_j|$.

6. Consider the norms defined in **Q5** for the case $n = 2$. Show that they do not satisfy the property in **Q4**. Deduce that they cannot be induced norms.

7. Consider the inner-product over $V = \mathcal{P}_n(\mathbb{R})$ given by

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx.$$

- (a) Find the length of $p(x) = 1$.
- (b) Find the distance between $p(x) = 1$ and $q(x) = x - 1$.
- (c) Find the angle between p and q .

ORTHOGONALITY

8. Let $\mathbf{u}, \mathbf{v} \in V$, where $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Define

$$\mathbf{w} := \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v}.$$

- (a) Show that $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.
 - (b) Use this orthogonality together with Pythagoras' theorem to derive the Cauchy-Schwarz inequality.
9. Let $(V(\mathbb{R}), \langle \cdot, \cdot \rangle)$ be an inner product space and let $\|\cdot\|$ denote the induced norm. Show that $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ if and only if $\|\mathbf{v}\| \leq \|\mathbf{v} + a\mathbf{w}\|$ for any $a \in \mathbb{R}$.
10. Let $(V, \langle \cdot, \cdot \rangle)$ be an n -dimensional inner product space. Let S be an orthogonal set of vectors in V . Show that if $|S| = n + 1$, then $\mathbf{0} \in S$.
11. Let $(V, \langle \cdot, \cdot, \cdot \rangle)$ be an inner product space. Let $U = \text{span } \{\mathbf{u}\}$. Given any $\mathbf{v} \in V$, show that

$$\mathbf{v}_U^\perp = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}.$$

12. Let $(V, \langle \cdot, \cdot, \cdot \rangle)$ be an inner product space. Let $U \leq V$. Show that for any $\mathbf{v} \in V$

$$\left\| \mathbf{v} - \mathbf{v}_U^\parallel \right\| \leq \|\mathbf{v} - \mathbf{u}\|, \quad \text{for all } \mathbf{u} \in U.$$

13. Let \mathbb{R}^4 be equipped with the Euclidean inner product and let $U = \text{span } \{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Find $\mathbf{u} \in U$ such that $\|\mathbf{u} - \mathbf{1}\|$ is as small as possible, where $\|\cdot\|$ is the induced norm (i.e., the Euclidean norm) and $\mathbf{1} \in \mathbb{R}^4$ is the vector of ones.

[Hint: use the result of Q12.]

14. Let $(V, \langle \cdot, \cdot \rangle)$ be an n -dimensional inner product space. Let $Z = \{\mathbf{0}\}$ denote the corresponding trivial vector space. Show the following properties.

- (a) $V^\perp = Z$.
- (b) $Z^\perp = V$.
- (c) $U \cap U^T \subseteq Z$ for any $U \subseteq V$.

15. Let $V = \mathbb{R}^3$ be equipped with the Euclidean inner product. Define the subset $U \subset V$ via

$$U := \text{span} \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y - z = 0 \right\}.$$

- (a) Show that $U < V$
- (b) Find a spanning set for U^\perp .

ORTHOGONAL SETS. ORTHOGONALISATION.

16. Use the Gram-Schmidt procedure to orthogonalise the following basis set for \mathbb{R}^3 :

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

17. Find an orthogonal basis for the following subspace of \mathbb{R}^4 :

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : \begin{cases} x + y + z + w = 0 \\ y + z = 0 \end{cases} \right\}.$$

18. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^4$. Let us view the first step in the Gram-Schmidt orthonormalisation procedure as modifying the first vector \mathbf{v}_1 , while leaving the others unchanged. We write this in matrix form as follows:

$$\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} = \begin{bmatrix} + & \times & \times \\ + & \times & \times \\ + & \times & \times \\ + & \times & \times \end{bmatrix} \begin{bmatrix} \oplus & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Note that the first vector \mathbf{v}_1 , indicated in bold \times is replaced with the normalised vector $\mathbf{v}_1 / \|\mathbf{v}_1\|$, which is the first column on the right indicated by $+$, while the entry \oplus in the second matrix is $\|\mathbf{v}_1\|$. The unit diagonal entries indicate that the vectors $\mathbf{v}_2, \mathbf{v}_3$ are kept unchanged. Write the next two steps of the procedure in matrix form in a similar way.

19. Let $n \in \mathbb{N}$. Let $\mu(x) > 0$ for all $x \in [a, b]$ and let $\langle \cdot, \cdot \rangle_\mu$ be an inner product defined via

$$\langle p, q \rangle_\mu = \int_a^b \mu(x)p(x)q(x)dx.$$

Consider the polynomials defined by the three-term recurrence

$$q_{i+1}(x) = (x - \alpha_{i+1})q_i(x) - \beta_i q_{i-1}(x), \quad (i = 0, \dots, n-1),$$

where $q_{-1}(x) = 0$, $q_0(x) = 1$ and

$$\begin{aligned} \alpha_{i+1} &= \frac{\langle xq_i, q_i \rangle_\mu}{\langle q_i, q_i \rangle_\mu} \quad (i = 0, 1, \dots), \\ \beta_i &= \frac{\langle q_i, q_i \rangle_\mu}{\langle q_{i-1}, q_{i-1} \rangle_\mu} \quad (i = 1, 2, \dots). \end{aligned}$$

Check that $\{q_0, q_1, q_2\}$ are orthogonal for the following choices:

- (a) $\mu(x) = 1$, $a = -1$, $b = 1$.
- (b) $\mu(x) = e^{-x}$, $a = 0$, $b = \infty$.

20. (a) Let $C_0(x) = 1$, $C_1(x) = x$. Use the three-term recurrence

$$C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x),$$

to find $C_2(x)$ and $C_3(x)$.

- (b) The roots of $C_n(x)$ are

$$\rho_{k,n} = \cos \frac{(2k+1)\pi}{2n}, \quad k = 0, \dots, n-1.$$

Check this for the case $n = 3$.

- (c) Let $n = 3$. Check that $\langle C_i, C_j \rangle = 0$ for $i \neq j = 0, \dots, n-1$, where

$$\langle C_i, C_j \rangle = \sum_{k=0}^{n-1} C_i(\rho_{k,n}) C_j(\rho_{k,n}).$$