Example sheet 1 solutions: Existence and Uniqueness and the Wronskian

1. *A 1st order nonlinear ordinary differential equation initial value problem is given by*

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

(a) *If $f(x, y) = \dfrac{x - y}{2x + 5y}$, establish for which values of (x, y) the functions $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous (hint: in this case f and $\dfrac{\partial f}{\partial y}$ are not continuous on a line; find the equation of this line).*

(b) *Use the appropriate Existence and Uniqueness Theorem to establish the region of the x-y plane in which a unique solution may exist to this differential equation for the given $f(x, y)$.*

(a) The above is a 1st order nonlinear ODE in the form $\dfrac{dy}{dx} = f(x, y)$. We know that a unique solution can only exist where $f(x, y)$ and $\dfrac{\partial f}{\partial y}(x, y)$ are continuous.

$f(x, y) = \dfrac{x - y}{2x + 5y}$ is continuous as long as $2x + 5y \neq 0 \implies y \neq -\dfrac{2}{5}x$.

$$\frac{\partial f}{\partial y}(x, y) = \frac{-1}{(2x + 5y)} - \frac{5(x - y)}{(2x + 5y)^2} = \frac{-(2x + 5y) - 5(x - y)}{(2x + 5y)^2} = \frac{-7x}{(2x + 5y)^2}.$$

Therefore, $\dfrac{\partial f}{\partial y}(x, y)$ is also continuous as long as $2x + 5y \neq 0$.

(b) Since f and $\dfrac{\partial f}{\partial y}$ are continuous on $2x + 5y < 0$ and $2x + 5y > 0$, unique solutions may exist on each of these regions (depending upon the initial condition), i.e. in the shaded regions of Figure 1. The Existence and Uniqueness Theorem for nonlinear 1st order ODEs tells us that a unique solution exists in some interval around x_0 that does not cross the line $2x + 5y = 0$, i.e. in some interval $x_0 - h < x < x_0 + h$ that does not cross this line.

2. *Explain why a unique solution to the differential equation*

$$\frac{dy}{dx} = (x^2 + y^2)^{\frac{3}{2}}$$

may exist throughout the x-y plane (state which Existence-Uniqueness Theorem you are applying).

We use the Existence-Uniqueness Theorem for 1st order nonlinear ODEs. Here,

$$f(x, y) = (x^2 + y^2)^{\frac{3}{2}},$$
$$\frac{\partial f}{\partial y}(x, y) = 2y \times \frac{3}{2}(x^2 + y^2)^{\frac{1}{2}} = 3y(x^2 + y^2)^{\frac{1}{2}}$$

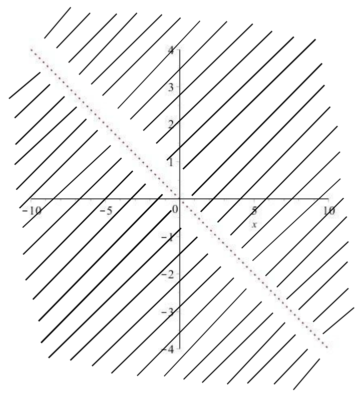


Figure 1: Question 1. Unique solutions may exist either side of the dotted red line $2x+5y=0$.

Since f and $\dfrac{\partial f}{\partial y}$ are continuous everywhere, a unique solution may exist throughout the x-y plane (depending upon the initial condition).

3.

(a) Solve the initial value problem

$$\frac{dy}{dx} = -\frac{4x}{y}, \quad y(0) = y_0.$$

(b) Determine how the interval in which the solution exists depends on the initial value y_0 (i.e. determine for which values of x the solution is real in terms of y_0).

(a)

$$\begin{aligned} \frac{dy}{dx} &= -\frac{4x}{y}, \\ \int y\,dy &= -4\int x\,dx, \\ \frac{y^2}{2} &= -2x^2 + c, \end{aligned}$$

but $y(0) = y_0 \implies \dfrac{{y_0}^2}{2} = c.$

$$\begin{aligned} \implies \frac{y^2}{2} &= -2x^2 + \frac{{y_0}^2}{2}, && (1) \\ y^2 &= -4x^2 + y_0^2, \\ y &= \pm\sqrt{y_0^2 - 4x^2}. && (2) \end{aligned}$$

But $y(0) = y_0 \implies y = \sqrt{y_0^2 - 4x^2}.$

(b) From (1) we can see that $y_0 = 0$ will only give a zero solution in the real plane. From (2), for a nonzero solution we require

$$\begin{aligned} y_0^2 - 4x^2 &> 0, \\ 4x^2 &< {y_0}^2, \\ x^2 &< \frac{{y_0}^2}{4}, \\ |x| &< \frac{|y_0|}{2}. \end{aligned}$$

4. *Solve the initial value problem*

$$\frac{dy}{dx} = 2xy^2, \quad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value y_0 *(consider the cases* $y_0 = 0, y_0 > 0$ *and* $y_0 < 0$*).*

$$\begin{aligned} \frac{dy}{dx} &= 2xy^2, \\ \int y^{-2}\, dy &= \int 2x\, dx, \\ -\frac{1}{y} &= x^2 + c, \\ y &= \frac{-1}{x^2 + c}, \end{aligned}$$

Imposing the initial condition gives $y_0 = -\frac{1}{c} \implies c = -\frac{1}{y_0}$ if $y_0 \neq 0$.

If $y_0 = 0$ then $y(x) \equiv 0$ is a solution for all x.

If $y_0 \neq 0$, this gives $y = \frac{-1}{x^2 - \frac{1}{y_0}} = \frac{y_0}{1 - x^2 y_0}$ so that y blows up as $x^2 \to \frac{1}{y_0}$.

However, x^2 will never tend towards $1/y_0$ if $y_0 < 0$.

Therefore,

- if $y_0 \leq 0$ a solution exists on $-\infty < x < \infty$,
- if $y_0 > 0$ a solution exists on $|x| < \frac{1}{\sqrt{y_0}}$.

5. (a) *Verify that both* $y_1(x) = 1 - x$ *and* $y_2(x) = -\frac{x^2}{4}$ *are solutions of the initial value problem*

$$\frac{dy}{dx} = \frac{-x + (x^2 + 4y)^{\frac{1}{2}}}{2}, \quad y(2) = -1$$

and determine where these solutions are valid.

$$\text{LHS} = \frac{dy_1}{dx} = -1$$

and

$$\begin{aligned}\text{RHS} &= \frac{-x + \sqrt{x^2 + 4 - 4x}}{2}, \\ &= \frac{-x + \sqrt{(x-2)^2}}{2}, \\ &= \frac{-x + |x-2|}{2}, \\ &= \begin{cases} -1 & \text{if } x \geq 2, \\ 1 - x & \text{if } x < 2. \end{cases}\end{aligned}$$

Check the initial condition

$$y_1(2) = -1,$$

so y_1 is a solution on $x \geq 2$.

$$\begin{aligned}\text{LHS} &= \frac{dy_2}{dx}, \\ &= -\frac{2x}{4}, \\ &= -\frac{x}{2}\end{aligned}$$

and

$$\begin{aligned}\text{RHS} &= \frac{-x + (x^2 - x^2)^{\frac{1}{2}}}{2}, \\ &= -\frac{x}{2},\end{aligned}$$

and $y_2(2) = -1$, so y_2 is also a solution.
This solution is valid on all x.

(b) *Explain why the existence of multiple solutions does not invalidate the Existence-Uniqueness (Picard's) theorem for nonlinear 1st order ODEs.*

f is continuous for all x but $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}\dfrac{(x^2+4y)^{-\frac{1}{2}}}{2} \times 4 = \dfrac{1}{\sqrt{x^2+4y}}$ is not continuous when $x^2 + 4y = 0$, e.g. at (2,-1).
Therefore, the Existence-Uniqueness theorem cannot apply here.

6. *For the equation*

$$x^2 y'' - 2y = 0, \quad x > 0,$$

(a) *verify that $u_1 = x^2$ and $u_2 = x^{-1}$ are linearly independent solutions;*

$$\begin{aligned} u_1' &= 2x, & u_2' &= -x^{-2}, \\ u_1'' &= 2, & u_2'' &= 2x^{-3}. \end{aligned}$$

Subbing these into the equation gives

$$\begin{aligned} x^2u_1'' - 2u_1 &= 2x^2 - 2x^2 = 0, \\ x^2u_2'' - 2u_2 &= 2x^2x^{-3} - 2x^{-1} = 0, \end{aligned}$$

so that both u_1 and u_2 are indeed solutions of the differential equation.
To check if they are linearly independent, we must calculate the Wronskian:

$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -1 - 2 = -3 \neq 0.$$

Therefore, u_1 and u_2 are linearly independent.

(b) *write down the general solution;*

$$y = \alpha_1 x^2 + \alpha_2 x^{-1}$$

where α_1, α_2 are arbitrary constants.

(c) *find the solution that satisfies the initial conditions $y(1) = -2$ and $y'(1) = -7$.*

$$y' = 2\alpha_1 x - \alpha_2 x^{-2}$$

Sub in the initial conditions:

$$\begin{aligned} y(1) &= \alpha_1 + \alpha_2 = -2, \\ y'(1) &= 2\alpha_1 - \alpha_2 = -7. \end{aligned}$$

Solving these with elimination gives $\alpha_1 = -3$ and $\alpha_2 = 1$.
The unique solution is therefore

$$y = -3x^2 + \frac{1}{x}.$$

7. *For the equation*

$$xy'' - (x+2)y' + 2y = 0, \quad x > 0,$$

(a) *verify that $u_1 = e^x$ and $u_2 = x^2 + 2x + 2$ are linearly independent solutions;*

$$\begin{aligned} u_1' &= e^x, & u_2' &= 2x + 2, \\ u_1'' &= e^x, & u_2'' &= 2. \end{aligned}$$

Subbing these into the equation gives

$$\begin{aligned}
xu_1'' - (x+2)u_1' + 2u_1 &= xe^x - (x+2)e^x + 2e^x = 0,\\
xu_2'' - (x+2)u_2' + 2u_2 &= 2x - (x+2)(2x+2) + 2x^2 + 4x + 4 = 0,
\end{aligned}$$

so that both u_1 and u_2 are indeed solutions of the differential equation.
To check if they are linearly independent, we must calculate the Wronskian:

$$\begin{aligned}
W &= \begin{vmatrix} e^x & x^2+2x+2 \\ e^x & 2x+2 \end{vmatrix},\\
&= e^x(2x+2) - e^x(x^2+2x+2),\\
&= e^x(2x+2-x^2-2x-2),\\
&= -x^2e^x,\\
&\neq 0,
\end{aligned}$$

since $x > 0$. Therefore, u_1 and u_2 are linearly independent.

(b) *write down the general solution;*

$$y = \alpha_1 e^x + \alpha_2(x^2+2x+2)$$

where α_1, α_2 are arbitrary constants.

(c) *find the solution that satisfies the initial conditions $y(1) = 0$ and $y'(1) = 1$.*

$$y' = \alpha_1 e^x + \alpha_2(2x+2)$$

Sub in the initial conditions:

$$\begin{aligned}
y(1) &= \alpha_1 e + \alpha_2(1+2+2) = \alpha_1 e + 5\alpha_2 = 0,\\
y'(1) &= \alpha_1 e + 4\alpha_2 = 1.
\end{aligned}$$

Solving these with elimination gives $\alpha_1 = 5e^{-1}$ and $\alpha_2 = -1$.
The unique solution is therefore

$$y = 5e^{x-1} - x^2 - 2x - 2.$$

8. *This question is designed to give you some insight into methods used in the proof of the Existence and Uniqueness theorem. You can use software such as Wolfram Alpha, Maple or Matlab to help with plotting the functions.*

(a) *Use the method of successive approximations to solve the initial value problem*

$$\frac{dy}{dx} = 2(y+1), \quad y(0) = 0$$

with $u_0(x) = 0$, by obtaining a general $u_n(x)$.

We have that $f(x,y) = 2(y+1)$, so the corresponding integral equation is

$$u(x) = \int_0^x 2(u+1)\,ds.$$

$$
\begin{aligned}
u_0 &= 0, \\
u_1 &= \int_0^x 2\,ds = \Big[2s\Big]_0^x = 2x, \\
u_2 &= 2\int_0^x (2s+1)\,ds = 2\Big[s^2+s\Big]_0^x = 2x^2+2x, \\
u_3 &= 2\int_0^x (2s^2+2s+1)\,ds = 2\left[\frac{2s^3}{3}+s^2+s\right]_0^x = \frac{4x^3}{3}+2x^2+2x, \\
&\vdots \\
u_n &= \sum_{k=1}^{n} \frac{2^k x^k}{k!}.
\end{aligned}
$$

(b) *Plot $u_n(x)$ for $n = 1, ..., 4$. Does it look like the u_n are converging to a solution?*
See Figure 2(a).

(c) *Express*

$$\lim_{n\to\infty} u_n(x) = u(x)$$

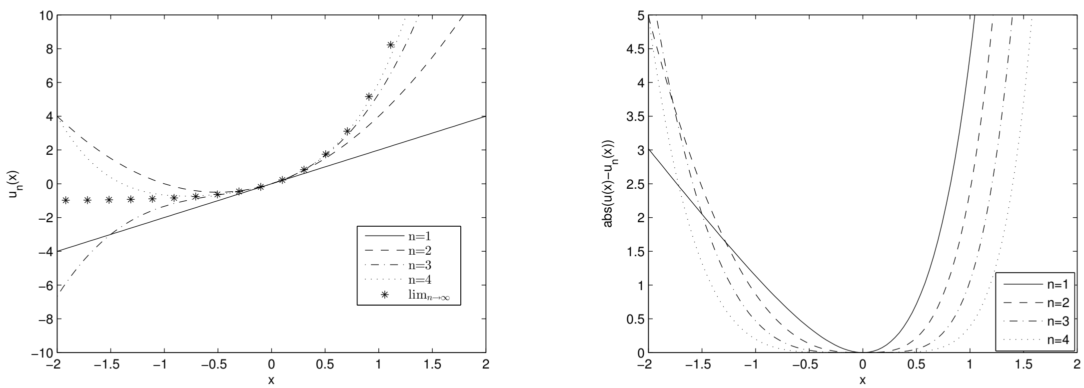
in terms of elementary functions (i.e. solve the initial value problem). Add $u(x)$ to your plot to see how well the $u_i(x)$ approximate the solution.

$$
\begin{aligned}
\frac{dy}{dx} &= 2(y+1), \\
\int \frac{1}{y+1}\,dy &= \int 2\,dx, \\
\ln \| y+1| &= 2x + c, \\
|y+1| &= Ae^{2x}.
\end{aligned}
$$

Imposing the initial condition $y(0) = 0 \implies y = e^{2x} - 1$, i.e.

$$\lim_{n\to\infty} u_n(x) = e^{2x} - 1.$$

(d) *On a new graph, plot $|u(x) - u_n(x)|$ for $n = 1, ..., 4$ to visualise the error between the $u_i(x)$ and the solution. Use this to estimate the interval in which each $u_i(x)$ is a reasonably good approximation to the solution.*
See Figure 2(b).
Observe how each successive $u_i(x)$ approximates the full solution over a wider interval.



(a) (b)

Figure 2: Question 6. (a) $u_n(x)$ (b) $|u(x) - u_n(x)|$. As n increases, the approximation in (a) matches the full solution over a wider range of x around $x = 0$. Corresponding to this, as n tends to infinity, the error between the approximation and the full solution in (b) is relatively small over these wider ranges of x.