Example sheet 7 – formative

1. Consider the dynamical system

$$\dot{x} = P(x,y),$$
$$\dot{y} = Q(x,y),$$

where $(x,y) \in \mathbb{R}^2$ and $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are differentiable.

(a) Prove that a necessary condition for the system to be Hamiltonian is that

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0.$$

(b) Determine which of the following two dynamical systems is Hamiltonian and obtain the corresponding Hamiltonian function $H(x,y)$:

(a) $\dot{x} = 1 - xy^2$, $\dot{y} = xy^2 - y$.

(b) $\dot{x} = x + y - x^2$, $\dot{y} = 2xy - y$.

(c) Sketch the global phase portrait of the system from part b which is Hamiltonian, and determine the set of initial conditions $(x_0, y_0) \in \mathbb{R}^2$ for which the solution to this system is periodic.

Solution:

(a) Suppose that the system is Hamiltonian. Then there exists a function $H : \mathbb{R}^2 \to \mathbb{R}$ which is twice continuously differentiable and has $P = H_y$ and $Q = -H_x$. Therefore, $P_x = H_{yx}$ and $Q_y = -H_{xy}$. Second partial derivatives are continuous. so that $H_{xy} = H_{yx}$. Hence,

$$P_x + Q_y = H_{yx} - H_{yx} = 0 \quad \text{for all} \quad (x,y) \in \mathbb{R}^2$$

as required.

(b)   (a) $P = 1 - xy^2$, $Q = xy^2 - y$, and $P_x = -y^2$, $Q_y = 2xy - 1$ with $P_x + Q_y = 2xy - 1 - y^2 \neq 0$. Therefore, this system is not Hamiltonian.

     (b) $P = x + y - x^2$, $Q = 2xy - y$, and $P_x = 1 - 2x$, $Q_y = 2x - 1$ with $P_x + Q_y = 0$. Therefore, the system **may** be Hamiltonian. Suppose it is Hamiltonian with Hamiltonian function $H(x,y)$, then,

$$\frac{\partial H}{\partial y} = x + y - x^2, \tag{1}$$

$$\frac{\partial H}{\partial x} = y - 2xy. \tag{2}$$

Equation (1) gives

$$H(x,y) = xy + \frac{1}{2}y^2 - x^2y + g(x),$$

for some function $g(x)$. On Substituting into (2) we obtain

$$g'(x) = 0 \quad \Rightarrow \quad g = \text{constant}.$$

Without any loss of generality we can set this constant to zero. Therefore,

$$H(x,y) = xy + \frac{1}{2}y^2 - x^2y, \quad (x,y) \in \mathbb{R}^2$$

and this system is Hamiltonian.

(c) We have shown in part b that the system

$$\dot{x} = x + y - x^2,$$
$$\dot{y} = 2xy - y, \tag{3}$$

is Hamiltonian with Hamiltonian function

$$H(x,y) = xy + \frac{1}{2}y^2 - x^2y, \quad (x,y) \in \mathbb{R}^2.$$

System (3) has three equilibrium points

$$(0,0), \quad (1,0), \quad \left(\frac{1}{2}, -\frac{1}{4}\right).$$

Now

$$H_{xx} = -2y, \ H_{yy} = 1, \ H_{xy} = 1 - 2x.$$

Considering each equilibrium point in turn we find that:

1. $H_{xy}^2 - H_{xx}H_{yy}\big|_{(0,0)} = 1 > 0$, therefore $(0,0)$ is a saddle point.

2. $H_{xy}^2 - H_{xx}H_{yy}\big|_{(1,0)} = 1 > 0$, therefore $(1,0)$ is a saddle point.

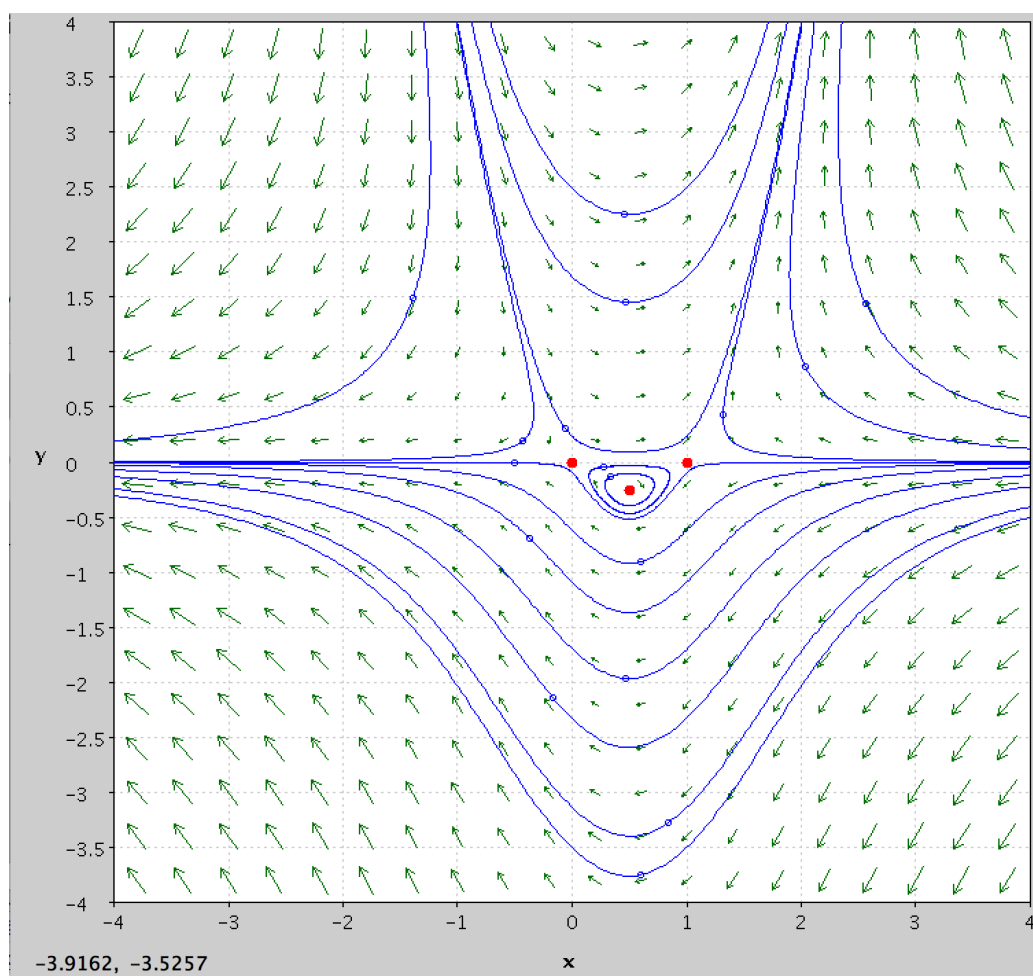3. $H_{xy}^2 - H_{xx}H_{yy}\big|_{(1/2,-1/4)} = -\frac{1}{2} < 0$, therefore $(1/2, -1/4)$ is a centre.

The phase paths through $(0,0)$ and $(1,0)$ have $H \equiv 0$. So, they are given by,

$$y\left(x + \frac{1}{2}y - x^2\right) = 0 \Rightarrow y = 0, \ y = 2x(x - 1).$$

We now sketch the phase portrait of system (3) in he Figure below.

Periodic behaviour for initial conditions in

$$PB = \left\{(x_0, y_0) \in \mathbb{R}^2 : -2x_0(1 - x_0) < y_0 < 0, 0 < x_0 < 1\right\}.$$

-3.9162, -3.5257

2. Consider the dynamical system

$$\dot{x} = 4y\left(x^2 + y^2 + 2\right),$$
$$\dot{y} = -4x\left(x^2 + y^2 - 2\right),$$

where $(x, y) \in \mathbb{R}^2$.

(a) Verify that this dynamical system is Hamiltonian and obtain the corresponding Hamiltonian function $H(x, y)$.

(b) Sketch the global phase portrait of this dynamical system, giving all analytic information (equilibria, eigenvalues and eigenvectors, horizontal and vertical isoclines, direction vectors in each segment, ...) and draw the qualitiatively different trajectories.

(c) Justify your choices of trajectories using symmetry arguments.

(d) Are there any heteroclinic or homoclinic orbits or cycles?

(e) Use the Hamiltonian function to analyse the different types of trajectories in function of the level constant C. Indicate where we have periodic orbits, and justify this by analysing the intersection of the solution curves with $y = mx$. Provide as much detail as possible.

Solution:

(a) For the system to be Hamiltonian, we need that $P_x + Q_y = 0$ for all x and y. Now

$$P_x + Q_y = 8xy - 8xy = 0,$$

hence the system is Hamiltonian. To find the Hamiltonian function, we observe that

$$H_y = 4y\left(x^2 + y^2 + 2\right),$$

so that

$$H(x, y) = 2x^2y^2 + y^4 + 4y^2 + f(x),$$

by integration. As

$$H_x = -\left(-4x\left(x^2 + y^2 - 2\right)\right),$$

we obtain that

$$4xy^2 + f'(x) = 4x\left(x^2 + y^2 - 2\right),$$

so that

$$f'(x) = 4x^3 - 8x,$$

or

$$f(x) = x^4 - 4x^2 + C.$$

Combining these results yields

$$H(x, y) = x^4 + 2x^2y^2 + y^4 + 4\left(y^2 - x^2\right) + C.$$

(b) (i) First find the equilibrium points. The obvious one is $(0,0)$. $\dot{y}$ can also be zero when $(x^2 + y^2) = 2$, which when substituted into $\dot{x}$ gives $\dot{x} = 16y$, so that $y = 0$ and hence $x^2 = 2$ or $x = \pm\sqrt{2}$. So there are three equilibrium points: $(0,0)$, $\left(\sqrt{2},0\right)$ and $\left(-\sqrt{2},0\right)$.

(ii) The Jacobian is given by

$$\mathbf{J} = \begin{pmatrix} 8xy & 4\left(x^2 + 3y^2 + 2\right) \\ -4\left(3x^2 + y^2 - 2\right) & -8xy \end{pmatrix}$$

For $(0,0)$, we have that

$$\mathbf{J}_{(0,0)} = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}$$

with eigenvalues $\lambda = \pm 8$ and eigenvectors $\mathbf{v} = (1,1)^\mathsf{T}$ for $\lambda = 8$ and $\mathbf{w} = (1,-1)^\mathsf{T}$ for $\lambda = -8$. Because of the theorem on linearisation, this will be a saddle point in the non-linear system.

For $\left(\sqrt{2},0\right)$ and $\left(-\sqrt{2},0\right)$, we get

$$\mathbf{J}_{\left(\pm\sqrt{2},0\right)} = \begin{pmatrix} 0 & 16 \\ -16 & 0 \end{pmatrix}$$

with eigenvalues $\lambda = \pm 16i$. Hence this indicates both are centres in the linear approximation, which may not be the case in the non-linear system.

(iii) The horizontal isocline is found by solving $-4x\left(x^2 + y^2 - 2\right) = 0$, or $x = 0$ or $x^2 + y^2 = 2$.

For $x = 0$, the direction is given by $\dot{x} = 4y^3 + 8y = 4y\left(y^2 + 2\right)$ so that x increases with t when $y > 0$ and x decreases with t when < 0.

For $x^2 + y^2 = 2$, $\dot{x} = 8y$, so again, x increases with t when $y > 0$ and x decreases with t when < 0.

(iv) The vertical isocline is given by $4y\left(x^2 + y^2 + 2\right) = 0$ which can only be satisfied by $y = 0$. In this case, $\dot{y} = -4x\left(x^2 - 2\right) = -4x\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$. Therefore, y increases with t when $0 < x < \sqrt{2}$ or $x < -\sqrt{2}$. y decreases with t when $-\sqrt{2} < x < 0$ or $x > \sqrt{2}$.

(v) We notice that the isoclines divide the plane in 8 segments in which the general direction (sign of $\dot{x}$ and $\dot{y}$) is the same within each segment. We can determine this general direction by analysing the direction vectors along $y = x$ and $y = -x$.

(vi) Along $y = x$, we have that

$$\dot{x} = 8x\left(x^2 + 1\right),$$
$$\dot{y} = -8x\left(x^2 - 1\right),$$

so that x increases for x positive and decreases for x negative. But y increases for $x < -1$ or $0 < x < 1$, while y decreases for $-1 < x < 0$ or $x > 1$. Also notice that

$$\left| \frac{dy}{dx} \right| = \left| \frac{-(x^2 - 1)}{(x^2 + 1)} \right| < 1.$$

This implies, e.g., that the trajectory along the eigenvector $(1, 1)^{\mathsf{T}}$ leaving the equilibrium at $(0, 0)$ will remain underneath the line $y = x$.
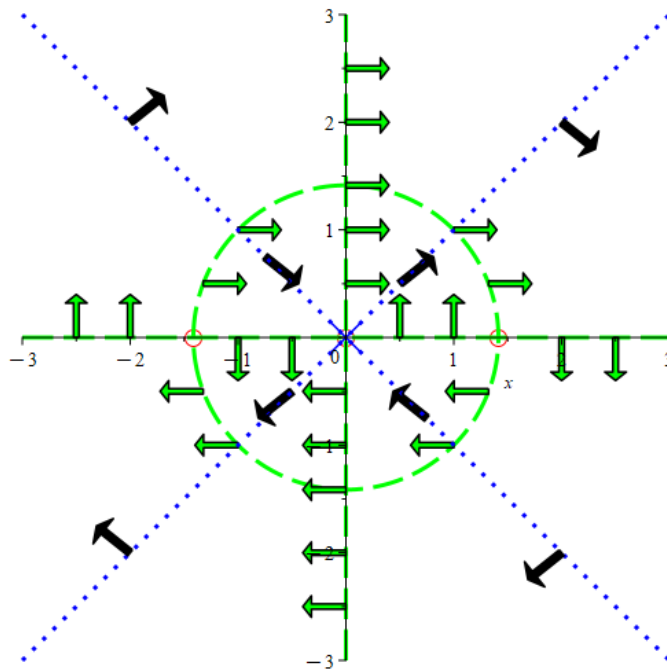
(vii) Along $y = -x$, we have that

$$\dot{x} = -8x \left(x^2 + 1 \right),$$
$$\dot{y} = -8x \left(x^2 - 1 \right),$$

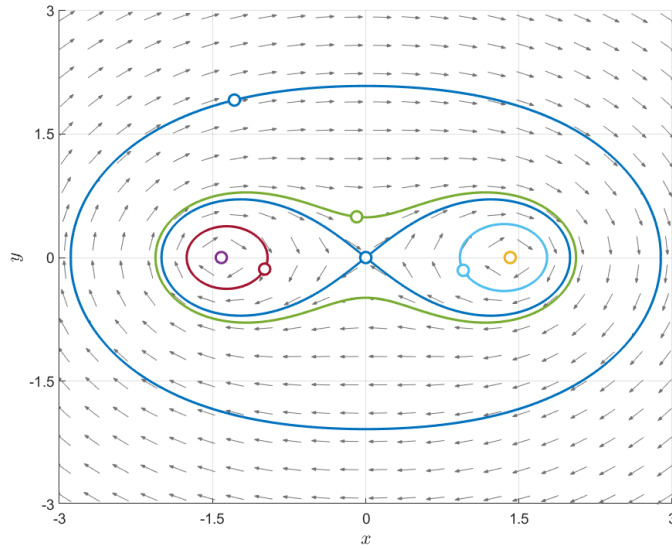so that x increases for x negative and decreases for x positive. But y increases for $x < -1$ or $0 < x < 1$, while y decreases for $-1 < x < 0$ or $x > 1$. Also notice that

$$\left| \frac{dy}{dx} \right| = \left| \frac{(x^2 - 1)}{(x^2 + 1)} \right| < 1.$$

This yields the following information for the phase portrait:

The qualitatively different trajectories are given as

(c) Note the symmetry in these equations. If for (x_o, y_o) the direction vector is given by $(\dot{x}, \dot{y}) = (\alpha, \beta)$, then the direction vector for $(-x_o, y_o)$ is given by $(\alpha, -\beta)$, the direction vector for $(x_o, -y_o)$ is given by $(-\alpha, \beta)$, and, the direction vector for $(-x_o, -y_o)$ is given by $(-\alpha, -\beta)$. This means that if we have traced the trajectory in the first quadrant of the plane $(x > 0$ and $y > 0)$, we can then fold the graph along the y-axis and trace the same trajectory in the fourth quadrant $(x < 0$ and $y > 0)$. Having thus established the trajectory in the upper half plane (and note that this trajectory is continuous across the y-axis), we can then fold the plane along the x-axis to trace the trajectory in the lower half plane. This then implies that all trajectories are closed curves.

(d) There are two homoclinic orbits at the equilibirum $(0,0)$ and hence a homoclinic cycle.

(e) Consider the Hamiltonian function of which the level curves form the trajectories. Hence each trajectory can be described as

$$H(x, y) = x^4 + 2x^2 y^2 + y^4 + 4\left(y^2 - x^2\right) + C = 0,$$

for a given value of C. Let us consider the intersection of the trajectories with the line $y = mx$ through the equilibrium point. Then we obtain the equation

$$\left(m^2 + 1\right) x^4 + 4\left(m^2 - 1\right) x^2 + C = 0,$$

for the points of intersection with this line. There could possibly be four x-values that satisfy this equation. First we solve this for x^2:

$$x^2 = -2\frac{m^2 - 1}{\left(m^2 + 1\right)^2} \pm \frac{1}{\left(m^2 + 1\right)^2}\sqrt{4\left(m^2 - 1\right)^2 - \left(m^2 + 1\right)C}.$$

It is useful to carefully examine the behaviour of the discriminant, i.e.

$$D = 4 \left(m^2 - 1\right)^2 - \left(m^2 + 1\right)^2 C,$$

which is itself a quartic function in m which allows us to analyse the area's where D is positive and hence where real solutions for x^2 can be found. The other condition worth checking when $D > 0$ is if

$$D > 4 \left(m^2 - 1\right)^2,$$

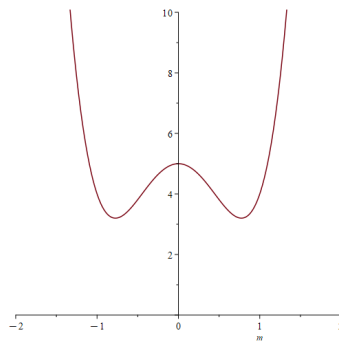in which case there will be a positive and negative solution for x^2. We note that

$$\frac{dD}{dm} = 2m \left((4 - C) m^2 - (4 + C)\right),$$

so the extrema of the quartic D can be found at $m = 0$ and
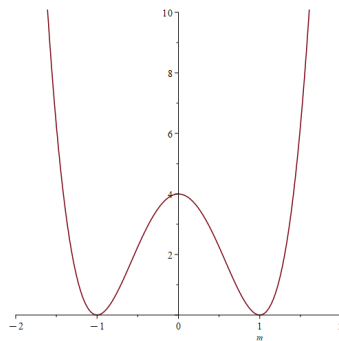
$$m^2 = \frac{4 + C}{4 - C}.$$

Hence for $C \geqslant 4$ or $C \leqslant -4$ the function D has only one extremum at $m = 0$, there are three extrema for $-4 \leqslant C \leqslant 4$. The value of D at $m = 0$ is $4 - C$. If they exist, the value at the other extrema is $\frac{-16C}{4-C} < 4 - C$. The function D is plotted below for a number of values of C.
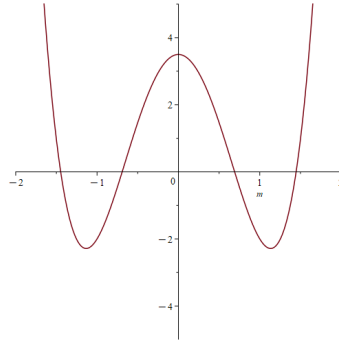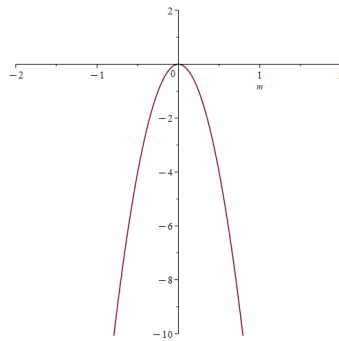
For $-4 < C < 0$:

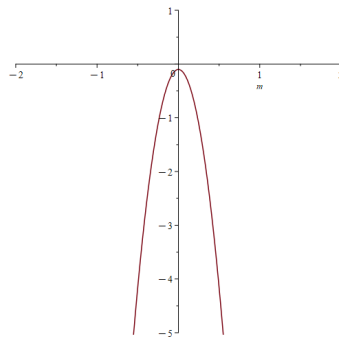
For $C = 0$:

For $0 < C < 4$:

For $C = 4$:

For $C > 4$:

(i) $C < 0$. In this case $D > 0$ and $D > 4 \left(m^2 - 1\right)^2$ for all values of m and there is one positive solution for x^2, i.e. $x^2 = \xi$ (whether $m^2 - 1 > 0$ or $m^2 - 1 < 0$). Hence the line will cross a trajectory at exactly two points, i.e. $x = \pm\sqrt{\xi}$, whatever the value of m is. Therefore, trajectories cannot be spirals but must be closed as they move around the centre of the plane.

(ii) $C = 0$. Then $D = 4 \left(m^2 - 1\right)^2$ which is always positive and the solutions for x^2 are $x^2 = 0$ or

$$x^2 = \frac{-4 \left(m^2 - 1\right)}{\left(m^2 + 1\right)^2}.$$

So for all values of m, $x = 0$ is a solution, but only for $m^2 - 1 < 0$ do we have other points of intersection, i.e. when the slope of the line satisfies $-1 \leqslant m \leqslant 1$ (where $m = \pm 1$ yields another two $x = 0$ solutions). We also note that for $m = 0$, the additional intersections are at $x = \pm 2$. Hence, for $C = 0$ we recover the homoclinic orbits and we know now they cross the x-axis at $x = \pm 2$. We also find the equilibrium point at $x = 0$.

(iii) $0 < C < 4$. Now we need to find those intervals for m^2 where D is positive, i.e., we need to solve $D = 0$ first. Hence we write

$$D = 4\left((4 - C)\, m^4 - 2\left(()\, 4 + C\right)m^2 + (4 - C)\right),$$

so that

$$m^2 = \frac{4 + C}{4 - C} \pm \frac{1}{4 - C}\sqrt{(4 + C)^2 - (4 - C)^2},$$

or

$$m^2 = \frac{4 + C}{4 - C} \pm \frac{4}{4 - C}\sqrt{C}.$$

One can easily show that $4 + C - 4\sqrt{C} > 0$ for $0 < C < 4$ so that both solutions for m^2 are positive, and thus we find four possible solutions for m which will define the values for m for which D is positive:

$$m_1 = \sqrt{\frac{4 + C + 4\sqrt{C}}{4 - C}}, \quad m_2 = -\sqrt{\frac{4 + C + 4\sqrt{C}}{4 - C}},$$

$$m_3 = \sqrt{\frac{4 + C - 4\sqrt{C}}{4 - C}}, \text{ and, } m_4 = -\sqrt{\frac{4 + C - 4\sqrt{C}}{4 - C}}.$$

From the graph of D we can see that $D > 0$ for $m < m_2$, $m_4 < m < m_3$, and $m > m_1$. One can also show that

$$m_1^2 > 1, \quad m_2^2 > 1, \quad m_3^2 < 1, \text{ and, } m_4^2 < 1.$$

Now consider the points of intersection, given by

$$x^2 = -2\frac{m^2 - 1}{(m^2 + 1)^2} \pm \frac{1}{(m^2 + 1)^2}\sqrt{4\left(m^2 - 1\right)^2 - \left(m^2 + 1\right)C}.$$

When $m^2 \geqslant 1$, we need

$$\sqrt{4\left(m^2 - 1\right)^2 - \left(m^2 + 1\right)C} \geqslant 2\left(m^2 - 1\right),$$

to have any positive solutions for x^2, which requires

$$4\left(m^2 - 1\right)^2 - \left(m^2 + 1\right)C \geqslant 4\left(m^2 - 1\right)^2,$$

which can never be satisfied. Hence we're restricted to slopes that satisfy $m^2 < 1$ or $-1 < m < 1$. In this case, we always have one positive solution for x^2 and we'll have one more if

$$\sqrt{4\left(1-m^2\right)^2 - \left(m^2+1\right)C} \leqslant 2\left(1-m^2\right),$$

which is always the case for $C > 0$. So we find four intersection points, which are symmetric, with two positive and two negative x-values. But they will only exist for slopes between m_4 and m_3. One can also show that

$$\lim_{C \to 0} m_3 = 1, \quad \lim_{C \to 0} m_4 = -1,$$

and that

$$\lim_{C \to 4} m_3 = 0, \quad \lim_{C \to 4} m_4 = 0.$$

So where for C close to 0 we'll have solutions close to the homoclinic orbits, they will become more confined as c increases. One can also show that the largest values for x^2, which we'll achieve for $C \to 0$ and $m = 0$, is 4, so these trajectories lie entirely within the homoclinic orbits. The fact that within each homoclinic orbit we can find at most two points of intersections also means the trajectories in this case are periodic orbits, or, to be more precise, are non-isolated periodic orbits.

(iv) $C = 4$. In this case $D = -16m^2$ and the only m-value for which intersections are possible is $m = 0$. In that case, we get that $x^2 = 2$, yielding the equilibrium points $x = \pm\sqrt{2}$.

(v) Finally, for $C > 4$, $D < 0$ for all m so no intersections do exists, hence no trajectories.