

University of Birmingham
School of Mathematics

2RCA/2RCA3 Real and Complex Analysis

Part A: Real Analysis

Semester 2

Summative Problem Sheet 2
Marking Scheme (marked out of **30**)

SUM **Question 5.** Let $(f_n)_{n=4}^\infty$ be the sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} \frac{n^4}{16} \left(x - \frac{4}{n^2}\right)^2, & \text{if } x \in \left[0, \frac{4}{n^2}\right], \\ 0, & \text{if } x \in \left(\frac{4}{n^2}, \frac{9}{n^2}\right), \\ \frac{n^2}{n^2-9} \left(x - \frac{9}{n^2}\right), & \text{if } x \in \left[\frac{9}{n^2}, 1\right]. \end{cases}$$

- (i) Find the pointwise limit of the sequence $(f_n)_{n=4}^\infty$.
(ii) Does the sequence $(f_n)_{n=4}^\infty$ converge uniformly on $[0, 1]$ to its pointwise limit?
Justify your answers. (In particular, state any result you use in the justification of your answer and justify its application.)

Solution. (i) We distinguish the following two cases:

- If for $x = 0$, by definition of f_n we have that $f_n(0) = 1$ for all n .
Therefore

$$f_n(0) = 1 \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

- If $x \in (0, 1]$, since $\frac{9}{n^2}$ tends to 0 as $n \rightarrow \infty$, we have that there exists $N \in \mathbb{N}$ such that

$$\frac{9}{n^2} < x, \quad \text{whenever } n \geq N.$$

Therefore, from the definition of f_n , it follows that

$$f_n(x) = \frac{n^2}{n^2-9} \left(x - \frac{9}{n^2}\right) \quad \text{whenever } n \geq N,$$

and we have that

$$f_n(x) = \frac{n^2}{n^2-9} \left(x - \frac{9}{n^2}\right) \rightarrow x \quad \text{as } n \rightarrow \infty.$$

The above argument shows that the sequence (f_n) converges pointwise to the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$(1) \quad f(x) = \begin{cases} 1, & \text{if } x = 0 \\ x, & \text{if } x \in (0, 1] \end{cases}$$

- (ii) No, the sequence (f_n) does not converge uniformly to f (its pointwise limit) on $[0, 1]$. Indeed, arguing by contradiction, assume that (f_n) converges uniformly to f on $[0, 1]$, where the function f is the function given in (1).

Notice that, since polynomials are continuous functions, for each $n \in \mathbb{N}$ f_n is continuous on $[0, \frac{4}{n^2}) \cup (\frac{4}{n^2}, \frac{9}{n^2}) \cup (\frac{9}{n^2}, 1]$. Moreover, using the continuity of polynomials, we have that

$$\lim_{x \rightarrow \frac{4}{n^2}^-} f_n(x) = \lim_{x \rightarrow \frac{4}{n^2}^-} \frac{n^4}{16} \left(x - \frac{4}{n^2} \right)^2 = 0,$$

$$\lim_{x \rightarrow \frac{4}{n^2}^+} f_n(x) = \lim_{x \rightarrow \frac{4}{n^2}^+} 0 = 0,$$

and $f_n(4/n^2) = \frac{n^4}{16} \left(\frac{4}{n^2} - \frac{4}{n^2} \right)^2 = 0$. Since,

$$\lim_{x \rightarrow \frac{4}{n^2}^-} f_n(x) = \lim_{x \rightarrow \frac{4}{n^2}^+} f_n(x) = f_n\left(\frac{4}{n^2}\right),$$

we have that f_n is continuous at $4/n^2$. A similar argument shows that f_n is continuous at $9/n^2$ (we omit the details here). Consequently, we have that f_n is continuous on $[0, 1]$.

Since the sequence (f_n) is a sequence of continuous functions and we are assuming that f_n converges uniformly to f , then we know that f is a continuous function (result from lectures). However, the function f in (1) is not continuous on $[0, 1]$ (indeed $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$ which is different from $f(0) = 1$). This gives the contradiction. Therefore the sequence (f_n) does not converge uniformly on $[0, 1]$ to its pointwise limit.

Note: Alternatively, one can show that the sequence $(f_n)_{n=4}^\infty$ does not converge uniformly to its pointwise limit using the criterion for non-uniform convergence. To this end, consider for $k \in \mathbb{N}$

$$n_k = k \quad \text{and} \quad x_k = \frac{1}{k^2}.$$

Using the definition of f_n and the pointwise limit f , observe that for $k \geq 2$ we have

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= \left| f_k\left(\frac{1}{k^2}\right) - f\left(\frac{1}{k^2}\right) \right| \leq \left| \frac{k^4}{16} \left(\frac{1}{k^2} - \frac{4}{k^2} \right)^2 - \frac{1}{k^2} \right| \\ &= \left| \frac{9}{16} - \frac{1}{k^2} \right| = \frac{9}{16} - \frac{1}{k^2} \geq \frac{9}{16} - \frac{1}{4} = \frac{5}{16} > 0. \end{aligned}$$

Consequently the sequence (f_n) does not converge uniformly to f on $[0, 1]$. \square

Marking scheme. [Total of 9 marks]

Part (i). 4 marks:

- **2 marks** for the correct formula of the pointwise limit. In particular, award 1 mark for a valid distinction of cases depending on x even if the student does not find the correct formula for the pointwise limit.
- **2 marks** for a valid justification.

Part (ii). 5 marks:

- **1 mark** for stating that the sequence (f_n) does not converge uniformly;

- **4 marks** for a valid justification of why the sequence does not converge uniformly to its pointwise limit. In particular, if a student uses the result that states that the uniform limit of a sequence of continuous functions is a continuous functions in their justification, then award **2** marks for the justification of the fact that f_n is a continuous function, **1** mark for justification of why the pointwise limit is not a continuous function, and **1** mark for the statement/reference to lecture notes of this result.

If a student justify the non-uniform converge using the criterion for non uniform convergence, please award a total of 4 marks for a valid justification.

SUM **Question 16.** Using the Intermediate Value Theorem and Rolle's Theorem, show that the equation

$$2x - 1 - \sin(x) = 0$$

has exactly one real solution.

Hint: Use the Intermediate Value Theorem to prove that the equation has at least one real solution. Using proof by contradiction and Rolle's theorem show that the equation has exactly one real solution.

Solution. Let $f(x) = 2x - 1 - \sin(x)$. f is continuous and differentiable on the real line (since $\sin(x)$ and $2x - 1$ are differentiable on the real line, and the algebra of continuous and differentiable functions). Notice that

$$f(0) = -1 < 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = \pi - 2 > 0.$$

Thus, by applying the Intermediate Value Theorem to the function f on the interval $[0, \frac{\pi}{2}]$ (since in particular f is continuous on $[0, \pi/2]$), we obtain that there exists $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$, i.e., the equation $f(x) = 0$ has at least a real solution (on the interval $(0, \frac{\pi}{2})$).

We will show that the equation $f(x) = 0$ has exactly one real solution by using Rolle's Theorem. To this end, since $\cos(x) \in [-1, 1]$, we have that

$$f'(x) = 2 - \cos(x) > 0, \quad \text{for all} \quad x \in \mathbb{R}.$$

We will argue by contradiction. Assume that $f(x) = 0$ has two or more real solutions, and let a and b with $a < b$ be two solutions of the equation

$$f(x) = 2x - 1 - \sin(x) = 0.$$

Since a and b are two solutions of the above equation, $f(a) = f(b) = 0$. Then, by applying Rolle's theorem to the function f (notice that in particular f is continuous on the interval $[a, b]$ and differentiable on (a, b)), there exists $\tilde{c} \in (a, b)$ such that

$$f'(\tilde{c}) = 0$$

On the other hand, we have observed that $f'(x) > 0$ for all $x \in \mathbb{R}$, and in particular $f'(\tilde{c}) > 0$. This gives a contradiction. Therefore the equation has a unique real solution.

□

Marking scheme. [Total of 8 marks]

4 marks for the use of the Intermediate Value Theorem to show that the equation has at least one real solution. Precisely,

- **1 mark** for defining an appropriate function and observing/justifying that the function is continuous (on an appropriate interval or on \mathbb{R}).
- **2 marks** for finding a suitable interval of the form $[a, b]$ such that $f(a) > 0$ and $f(b) < 0$.
- **1 mark** for applying the Intermediate Value Theorem to conclude the existence of c such that $f(c) = 0$. Penalise with 1 mark if a student does not explicitly mention the use of this result.

4 marks for the application of Rolle's Theorem to show that the equation has a unique real solution. Precisely,

- **1 mark** for assuming (arguing by contradiction) that the equation has two real solutions a and b and considering the function on the interval determined by a and b .
- **1 mark** for observing/justifying that f is continuous on $[a, b]$ and differentiable on (a, b) .
- **1 mark** for deducing using Rolle's Theorem, the existence of $c \in (a, b)$ such that $f'(c) = 0$.
- **1 mark** for the analysis of f' leading to the observation that $f'(x) \neq 0$, and hence reaching a contradiction.

SUM **Question 23.** Using the Mean Value Theorem show that

$$\arccos(x) - \frac{\pi}{2} \leq -x \quad \text{for all } x \in [0, 1].$$

Here, $\arccos : [-1, 1] \rightarrow [0, \pi]$ is the inverse of the cosine function.

Solution. Notice that, since $\arccos(0) = \frac{\pi}{2}$ and $\arccos(1) = 0$, then the inequality in the statement of the question is true when $x = 0$ or $x = 1$. We need to show that the inequality is also true for all $x \in (0, 1)$.

Now, let $x \in (0, 1)$. Consider the function $f(t) = \arccos(t)$ on the interval $[0, x]$. Notice that, since the arccosine function is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, and $[0, x] \subseteq [-1, 1]$ for any $x \in (0, 1)$, then f is differentiable on $(0, x)$ and continuous on $[0, x]$, and applying the Mean Value theorem we obtain that there exists $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c),$$

or equivalently

$$(2) \quad \frac{\arccos(x) - \frac{\pi}{2}}{x} = -\frac{1}{\sqrt{1-c^2}}.$$

Finally notice that since $c \in (0, x)$ and $x \in (0, 1)$, we have that $0 < c < x < 1$, and in particular $0 < 1 - c^2 < 1$. Hence

$$-\frac{1}{\sqrt{1-c^2}} < -1.$$

From the above inequality and (2), we have that

$$(3) \quad \frac{\arccos(x) - \frac{\pi}{2}}{x} = -\frac{1}{\sqrt{1-c^2}} < -1,$$

and, using that $x > 0$, we have that

$$\arccos(x) - \frac{\pi}{2} < -x,$$

as desired. \square

Marking scheme. [Total of 7 marks]

- **1 mark** for observing that the identity is true for $x = 0$ and $x = 1$.
- **2 marks** for observing/justifying that $f(t) = \arccos(t)$ on $[0, x]$ for $0 < x < 1$ satisfies the hypothesis of the Mean Value Theorem,
- **2 marks** for applying the Mean Value Theorem to show the existence of $c \in (0, x)$ such that the identity (2) holds.
- **2 marks** for a correct control of $f'(c)$ to get the desired inequality. Do not penalise if a student forgets to mention that they use the fact that $x > 0$ in the last step.

Note: Observe that it is not necessary to distinguish the case $x = 1$, though I have done in the model solution. The argument works when $x \in (0, 1]$, instead of the range $x \in (0, 1)$ considered in the solution.

SUM **Question 25.** Using L'Hôpital's Rule, show that the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{x^2} - \frac{1}{(\tan(x))^2}, & \text{if } x \neq 0, \\ \frac{2}{3}, & \text{if } x = 0 \end{cases}$$

is differentiable at $x_0 = 0$ and give the value of $f'(0)$.

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Solution. By the definition of differentiability of a function at a point, f is differentiable at 0 if the following limit exists

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}.$$

Now, for $h \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $h \neq 0$ we have

$$\begin{aligned} \frac{f(h) - f(0)}{h} &= \frac{1}{h^3} - \frac{1}{h(\tan(h))^2} - \frac{2}{3h} = \frac{1}{h^3} - \frac{(\cos(h))^2}{h(\sin(h))^2} - \frac{2}{3h} \\ &= \frac{3(\sin(h))^2 - 3h^2(\cos(h))^2 - 2h^2(\sin(h))^2}{3h^3(\sin(h))^2} \end{aligned}$$

so we let $g_1(h) = 3(\sin(h))^2 - 3h^2(\cos(h))^2 - 2h^2(\sin(h))^2$ and $g_2(h) = 3h^3(\sin(h))^2$. Observe that g_1 and g_2 are differentiable functions of any order on the real line (since sine, cosine and polynomial functions are and the algebra of differentiable functions).

We have that

$$\begin{aligned}
g_1'(h) &= -6h \cos^2(h) + 2(3 + h^2) \cos(h) \sin(h) - 4h \sin^2(h) \\
g_1''(h) &= 2h^2 \cos^2(h) + 8h \cos(h) \sin(h) - 2(5 + h^2) \sin^2(h) \\
g_1'''(h) &= -4(-3h \cos^2(h) + (3 + 2h^2) \cos(h) \sin(h) + 3h \sin^2(h)) \\
g_1^{(iv)}(h) &= -8h(h \cos^2(h) + 8 \cos(h) \sin(h) - h \sin^2(h)) \\
g_1^{(v)}(h) &= 16(-5h \cos^2(h) + 2(-2 + h^2) \cos(h) \sin(h) + 5h \sin^2(h)) \\
&= 16((-9 + 2h^2) \cos^2(h) + 24h \cos(h) \sin(h) + (9 - 2h^2) \sin^2(h))
\end{aligned}$$

and

$$\begin{aligned}
g_2'(h) &= 3h^2 \sin(h)(2h \cos(h) + 3 \sin(h)) \\
g_2''(h) &= 6h(h^2 \cos^2(h) + 6h \cos(h) \sin(h) - (-3 + h^2) \sin^2(h)) \\
g_2'''(h) &= -6(-9h^2 \cos^2(h) + 2h(-9 + 2h^2) \cos(h) \sin(h) + 3(-1 + 3h^2) \sin^2(h)) \\
g_2^{(iv)}(h) &= -24(h(-9 + h^2) \cos^2(h) + 6(-1 + 2h^2) \cos(h) \sin(h) - h(-9 + h^2) \sin^2(h)) \\
g_2^{(v)}(h) &= 24(-15(-1 + h^2) \cos^2(h) + 4h(-15 + h^2) \cos(h) \sin(h) + 15(-1 + h^2) \sin^2(h))
\end{aligned}$$

So $g_1(0) = g_1'(0) = g_1''(0) = g_1^{(iii)}(0) = g_1^{(iv)}(0) = g_1^{(v)}(0) = 0$ and $g_2(0) = g_2'(0) = g_2''(0) = g_2'''(0) = g_2^{(iv)}(0) = 0$, and $g_2^{(v)}(0) = 360 \neq 0$.

Since g_2 is differentiable to any order on \mathbb{R} (using a lemma from the lectures) there exists (a, b) such that $g_2^{(j)}(x) \neq 0$ for all $x \in (a, b) \setminus \{0\}$ for $j = 1, 2, 3, 4$. Hence, by the 0/0 form of L'Hôpital's Rule (repeatedly) and the Algebra of Limits, it follows that

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{g_1(h)}{g_2(h)} = \lim_{h \rightarrow 0} \frac{g_1'(h)}{g_2'(h)} = \lim_{h \rightarrow 0} \frac{g_1''(h)}{g_2''(h)} \\
&= \lim_{h \rightarrow 0} \frac{g_1'''(h)}{g_2'''(h)} = \lim_{h \rightarrow 0} \frac{g_1^{(iv)}(h)}{g_2^{(iv)}(h)} = \lim_{h \rightarrow 0} \frac{g_1^{(v)}(h)}{g_2^{(v)}(h)} = \frac{0}{360} = 0.
\end{aligned}$$

Therefore f is differentiable at 0 with $f'(0) = 0$. Here we have used the continuity of the functions $g_1^{(v)}$ and $g_2^{(v)}$ at 0 in the evaluation of the last limit. \square

Marking scheme. [Total of 6 marks]

- **1 mark** for the definition of differentiability of a function at 0.
- **5 marks** for using L'Hôpital's Theorem to show that the function is differentiable at 0, and $f'(0) = 0$. Precisely,
 - **1 mark** for stating/justifying the differentiability properties of the functions they consider in their application of L'Hôpital's Theorem.
 - **1 mark** for the correct calculation of the derivatives of g_1 and g_2 and their evaluation at the point 0. Students have been told that they do not need to include their working of the evaluation of the derivatives.
 - **1 mark** for stating that since g_2 is differentiable to any order (or at least of order 4), and $g_2(0) = g_2'(0) = g_2''(0) = g_2'''(0) = 0$ and $g_2^{(iv)}(0) \neq 0$, then there exists (a, b) such that $g_2^{(j)}(x) \neq 0$ for all $x \in (a, b) \setminus \{0\}$ for $j = 1, 2, 3, 4$.- As seen in lectures.
 - **1 mark** for a justification of the evaluation of the last limit.
 - **1 mark** for the numerical answer: $f'(0) = 0$.

Note: There may be some students who *incorrectly* study the continuity of the function at 0 instead of the differentiability. If this is the case, award a maximum of 4 for a valid application of L'Hôpital's Theorem to show that f is continuous at 0.