Problem Sheet 4 (Combinatorics part)

Assignment available: Friday 10 November 2023 (Week 7).
Submission deadline: 1700 on Wednesday 22 November 2023 (Week 9).
Required content: All necessary content will be covered by the first lecture of Week 8.

About problem sheet questions: The comments made about the first problem sheet apply equally to this one. In particular, please don't be reluctant to seek help if you are unsure how to proceed towards a solution, or how to express your ideas, as (unlike exam questions) the questions are set on the basis that you have access to this support.

Question 1 (SUM). (a) Prove that for every integer $n \geq 0$ we have

$$3^n = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^n\binom{n}{n}.$$

(b) Prove that for all integers $a, b \geq n \geq 0$ we have

$$\binom{a+b}{n} = \binom{a}{n} + \binom{a}{n-1}\binom{b}{1} + \binom{a}{n-2}\binom{b}{2} + \cdots + \binom{a}{1}\binom{b}{n-1} + \binom{b}{n}.$$

Question 2 (SUM). (a) Prove that $\mathbb{Q}$ is countably infinite.

(b) Prove that $\mathbb{N} \times \mathbb{Q}$ is countably infinite.

Question 3. Let $\Upsilon := \{(a_1, a_2, a_3, \dots) : a_i \in \{0, 1\}$ for every $i \in \mathbb{N}\}$, so in other words Υ is the set of all infinite sequences of zeroes and ones indexed by the natural numbers. Prove that Υ is uncountable.

Hint: Mimic the proof from lectures that $\mathbb{R}$ is uncountable, by considering an arbitrary function $f : \mathbb{N} \to \Upsilon$ and proving that f is not surjective by constructing an element of Υ which is not in the image of f.

Question 4. Let x_1, x_2, x_3, x_4 and x_5 each be chosen uniformly at random in turn from the set $\{1, 2, \dots, 20\}$, independently of each other (in other words, each is chosen uniformly at random regardless of the values of previously-chosen x_i). Find the probabilities of each of the following events.

(a) That there exist $i, j \in \{0, 1, 2, 3, 4\}$ with $i \neq j$ such that three of the integers x_r are congruent to i modulo 5 and the other two integers x_r are congruent to j modulo 5.

(b) That exactly three of the integers x_1, x_2, x_3, x_4 and x_5 are divisible by 2 and exactly three are divisible by 3.