VGLA: Sets and Notation Practice Questions

The following revision type questions relate to Appendix 1, Sets and Notation. Questions are ranked in difficulty from A (basic) to C (challenging).

(A) Question 1. Let $A = \{1, 2, \text{apple}\}, B = \{2, \text{apple}, 1, \text{apple}\}, C = \{2, 1\}, D = \{1, 2, \text{pear}\}$.

(a) Is $1 \in A$?

(b) Is $10 \in A$?

(c) Is $A = B$?

(d) Is $A = C$?

(e) Is $C \subseteq A$?

(f) Is $C \subset A$?

(g) Is $A \subseteq C$?

(h) Is $D \subseteq B$?

Solution:

(a) Yes, $1 \in A$.

(b) No, $10 \notin A$.

(c) Yes since $A \subseteq B$ and $B \subseteq A$.

(d) No since $\text{apple} \in A$ but $\text{apple} \notin C$.

(e) Yes since $C = \{1, 2\} \subseteq \{1, 2, \text{apple}\} = A$.

(f) Yes since $C \subseteq A$ and $\text{apple} \in A \setminus C$.

(g) No since $\text{apple} \in A$ but $\text{apple} \notin C$.

(h) No since $\text{pear} \in D$ but $\text{pear} \notin B$.

$\blacksquare$

(A) Question 2. Write each of the following sets by listing its elements:

(a) $\{x \in \mathbb{Z} \,:\, -1 \leq x < 6\}$,

(b) $\{x \in \mathbb{Z} \,:\, x^2 < 25\}$,

(c) $\{x \in \mathbb{N} \,:\, x^2 < 25\}$,

(d) $\left\{\dfrac{p}{q} \,:\, p \in \mathbb{N} \text{ and } q = 2p+1\right\}$,

(e) $\{x \,:\, x$ is the name of a month containing the letter “R"$\}$,

(f) $\{x \,:\, x$ is letter in “Mississippi”$\}$,

(g) $\{x \,:\, x$ is a month with 33 days$\}$.

Solution:

(a) $\{x \in \mathbb{Z} \,:\, -1 \leq x < 6\} = \{-1, 0, 1, 2, 3, 4, 5\}$.

(b) $\{x \in \mathbb{Z} \,:\, x^2 < 25\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

(c) $\{x \in \mathbb{N} \,:\, x^2 < 25\} = \{1, 2, 3, 4\}$. *Note that the definition of* $\mathbb{N}$ *differs in J1VGLA and J1RAC.*

(d) $\left\{\dfrac{p}{q} \,:\, p \in \mathbb{N} \text{ and } q = 2p+1\right\} = \left\{\dfrac{1}{3}, \dfrac{2}{5}, \dfrac{3}{7}, \dfrac{4}{9}, \dots\right\}$.

(e) $\{x : x$ is the name of a month containing “R”$\} = \{$ JANUARY, FEBRUARY, MARCH, APRIL, SEPTEMBER, OCTOBER, NOVEMBER, DECEMBER$\}$. *The answer to this question depends how the months are written i.e. JANUARY/January and whether or not the letters “R" and “r" are considered different or the same.*

(f) $\{x \,:\, x$ is letter in “Mississippi”$\} = \{$M, i, s, p$\}$.

(g) $\{x \,:\, x$ is a month with 33 days$\} = \emptyset$.

■

(A) Question 3. Let P be the set of prime natural numbers, E the set of even natural numbers and O the set of odd natural numbers. Describe the sets $E \cup O, E \cap O, E \cap P$ and $O \cap P$.

Solution: Since elements in $\mathbb{N}$ are either odd or even (this is an exclusive "or"), then $E \cup O = \mathbb{N}$ and $E \cap O = \emptyset$. Moreover, since every prime number, except $\{2\}$ is an odd natural number it follows that $E \cap P = \{2\}$ and $O \cap P = P \setminus \{2\}$. ■

(A) Question 4. Determine $|A|$ when
(a) $A = \{1, 2, 3\}$, (b) $A = \{\{1, 2, 3\}\}$, (c) $A = \{1, \{2, 3\}\}$.

Solution:

(a) $|\{1, 2, 3\}| = 3$ since the set contains only the 3 elements: 1, 2 and 3.

(b) $|\{\{1, 2, 3\}\}| = 1$ since the only element in the set is $\{1, 2, 3\}$.

(c) $|\{1, \{2, 3\}\}| = 2$ since the set contains only the 2 elements: 1 and $\{2, 3\}$.

■

(A) Question 5. Put the correct sign $\Rightarrow, \Leftarrow$ or $\Leftrightarrow$ between the following pairs of conditions on a real number x. Briefly explain your answers.

(a) $x \in \mathbb{N}$; $1/x \in \mathbb{Q}$.

(b) $3x \in \mathbb{Q}$; $x \in \mathbb{Q}$.

(c) $3x \in \mathbb{Z}$; $x + 3 \in \mathbb{Z}$.

(d) $x \in \mathbb{Q}$; $1/(1 - x) \in \mathbb{Q}$.

Solution:

(a) If $x \in \mathbb{N}$ then $x \in \mathbb{Z}$ and $x > 0$. Thus $\frac{1}{x} \in \mathbb{Q}$. Therefore,

$$x \in \mathbb{N} \Longrightarrow \frac{1}{x} \in \mathbb{Q}.$$

Since $x = \frac{1}{2}$ satisfies $\frac{1}{x} = 2 \in \mathbb{Q}$ but $x \notin \mathbb{N}$, it follows that

$$\frac{1}{x} \in \mathbb{Q} \not\Longrightarrow x \in \mathbb{N}.$$

Note that to explain if a statement is true we need to provide a proof. To explain why a statement is false, we need a counterexample.

(b) If $3x \in \mathbb{Q}$ then $3x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q > 0$. Thus $x = \frac{p}{3q} \in \mathbb{Q}$ since $3q \in \mathbb{Z}$ ($\mathbb{Z}$ is closed under multiplication) and $3q > 0$. Therefore

$$3x \in \mathbb{Q} \Longrightarrow x \in \mathbb{Q}.$$

If $x \in \mathbb{Q}$, then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $q > 0$. Thus, $3x = \frac{3p}{q} \in \mathbb{Q}$ since $3p \in \mathbb{Z}$ and $q > 0$. Therefore

$$x \in \mathbb{Q} \Longrightarrow 3x \in \mathbb{Q}.$$

We conclude that

$$x \in \mathbb{Q} \iff 3x \in \mathbb{Q}.$$

(c) If $x = \frac{1}{3}$ then $3x = 1 \in \mathbb{Z}$ and $x + 3 = \frac{10}{3} \notin \mathbb{Z}$. Therefore

$$3x \in \mathbb{Z} \not\Longrightarrow x + 3 \in \mathbb{Z}.$$

If $x + 3 \in \mathbb{Z}$, then $(x + 3) - 3 = x \in \mathbb{Z}$ and hence $3x \in \mathbb{Z}$ (since $\mathbb{Z}$ is closed under addition and multiplication). Therefore

$$x + 3 \in \mathbb{Z} \Longrightarrow 3x \in \mathbb{Z}.$$

(d) Since $x = 1 \in \mathbb{Q}$ but $\frac{1}{1-x} = \frac{1}{0} \notin \mathbb{Q}$ it follows that

$$x \in \mathbb{Q} \not\Longrightarrow \tfrac{1}{1-x} \in \mathbb{Q}.$$

If $\frac{1}{1-x} = \frac{p}{q} \in \mathbb{Q}$ then $p, q \in \mathbb{Z}$ with $q > 0$ and $p \neq 0$. Therefore

$$\frac{1}{1-x} = \frac{p}{q} \Longrightarrow -1 = \frac{p(x-1)}{q} \Longrightarrow x = \frac{(p-q)(-1)^n}{p(-1)^n}$$

with $(-1)^n p > 0$ (for suitable choice of either odd or even n). It follows that

$$\frac{1}{1-x} \in \mathbb{Q} \Longrightarrow x \in \mathbb{Q}.$$

$\blacksquare$

(A) Question 6. Considering the answers to the previous exercise, give the correct relationship between the following pairs of sets. Use only the symbols $\subset, \subseteq$ or $=$. Explain your answers.

(a) $\mathbb{N}$ and $\{x \in \mathbb{R} : 1/x \in \mathbb{Q}\}$.

(b) $\{x \in \mathbb{R} : 3x \in \mathbb{Q}\}$ and $\mathbb{Q}$.

(c) $\{x \in \mathbb{R} : 3x \in \mathbb{Z}\}$ and $\{x \in \mathbb{R} : x + 3 \in \mathbb{Z}\}$.

(d) $\mathbb{Q}$ and $\{x \in \mathbb{R} : 1/(1 - x) \in \mathbb{Q}\}$.

Solution: *We use results from the question A5 to justify the following conclusions.*

(a) Since $y \in \mathbb{N}$ implies $y \in \{x \in \mathbb{R} \ : \ \frac{1}{x} \in \mathbb{Q}\}$, it follows that

$$\mathbb{N} \subseteq \left\{x \in \mathbb{R} \ : \ \frac{1}{x} \in \mathbb{Q}\right\}.$$

Since $\frac{1}{2} \in \{x \in \mathbb{R} \ : \ \frac{1}{x} \in \mathbb{Q}\}$ but $\frac{1}{2} \notin \mathbb{N}$, we conclude that

$$\mathbb{N} \subset \left\{x \in \mathbb{R} \ : \ \frac{1}{x} \in \mathbb{Q}\right\}.$$

(b) Since $y \in \{x \in \mathbb{R} \ : \ 3x \in \mathbb{Q}\} \iff y \in \mathbb{Q}$, we conclude that

$$\mathbb{Q} = \{x \in \mathbb{R} \ : \ 3x \in \mathbb{Q}\}.$$

(c) Since $y \in \{x \in \mathbb{R} \ : \ x + 3 \in \mathbb{Z}\} \Longrightarrow y \in \{x \in \mathbb{R} \ : \ 3x \in \mathbb{Z}\}$, it follows that

$$\{x \in \mathbb{R} \ : \ x + 3 \in \mathbb{Z}\} \subseteq \{x \in \mathbb{R} \ : \ 3x \in \mathbb{Z}\}.$$

Additionally, since $\frac{1}{3} \in \{x \in \mathbb{R} \ : \ 3x \in \mathbb{Z}\}$ but $\frac{1}{3} \notin \{x \in \mathbb{R} \ : \ x+3 \in \mathbb{Z}\}$, we conclude that

$$\{x \in \mathbb{R} \ : \ x + 3 \in \mathbb{Z}\} \subset \{x \in \mathbb{R} \ : \ 3x \in \mathbb{Z}\}.$$

(d) Since $y \in \left\{x \in \mathbb{R} \ : \ \frac{1}{1-x} \in \mathbb{Q}\right\} \Longrightarrow y \in \mathbb{Q}$, it follows that

$$\left\{x \in \mathbb{R} \ : \ \frac{1}{1 - x} \in \mathbb{Q}\right\} \subseteq \mathbb{Q}.$$

Since $1 \in \mathbb{Q}$ but $1 \notin \left\{x \in \mathbb{R} \ : \ \frac{1}{1-x} \in \mathbb{Q}\right\}$, we conclude that

$$\left\{x \in \mathbb{R} \ : \ \frac{1}{1 - x} \in \mathbb{Q}\right\} \subset \mathbb{Q}.$$

■

(A) Question 7. In this question, the universal set is

$$\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, \text{fish}, \text{fowl}\}.$$

Let $A = \{1, 2, 3\}$, $B = \{2, 3, \text{fish}\}$ and $C = \{2, \text{fowl}, 7, 8\}$. Work out each of the following sets using only the definition of union, intersection and complement.

(a) $A \cap (B \cup C)$.

(b) $(A \cap B) \cup (A \cap C)$.

(c) $A \cup (B \cap C)$.

(d) $(A \cup B) \cap (A \cup C)$.

(e) C'.

(f) $(C')'$.

(g) $(A \cap C)'$.

(h) $A' \cup C'$.

(i) $(A \cup B)'$.

(j) $A' \cap B'$.

Solution:

(a) $A \cap (B \cup C) = \{1, 2, 3\} \cap \{2, 3, \text{fish}, \text{fowl}, 7, 8\} = \{2, 3\}$.

(b) $(A \cap B) \cup (A \cap C) = \{2, 3\} \cup \{2\} = \{2, 3\}$.

(c) $A \cup (B \cap C) = \{1, 2, 3\} \cup \{2\} = \{1, 2, 3\}$.

(d) $(A \cup B) \cap (A \cup C) = \{1, 2, 3, \text{fish}\} \cap \{1, 2, 3, \text{fowl}, 7, 8\} = \{1, 2, 3\}$.

(e) $C' = U \setminus C = \{1, 3, 4, 5, 6, \text{fish}\}$.

(f) $(C')' = C = \{2, \text{fowl}, 7, 8\}$.

(g) $(A \cap C)' = \{2\}' = \{1, 3, 4, 5, 6, 7, 8, \text{fish}, \text{fowl}\}$.

(h) $A' \cup C' = \{4, 5, 6, 7, 8, \text{fish}, \text{fowl}\} \cup \{1, 3, 4, 5, 6, \text{fish}\} = \{1, 3, 4, 5, 6, 7, 8, \text{fish}, \text{fowl}\}$.

(i) $(A \cup B)' = \{1, 2, 3, \text{fish}\}' = \{4, 5, 6, 7, 8, \text{fowl}\}$.

(j) $A' \cap B' = \{4, 5, 6, 7, 8, \text{fish}, \text{fowl}\} \cap \{1, 4, 5, 6, 7, 8, \text{fowl}\} = \{4, 5, 6, 7, 8, \text{fowl}\}$.

Note that the sets in: (a) and (b); (c) and (d); (g) and (h); and (i) and (j), respectively, are equal. For other sets A, B and C one can construct Venn diagrams to formally see that these equalities hold. ■

(A) Question 8. Express each of the following sets as a single interval.

(a) $(1,3) \cup (2,15)$. (b) $[1,8] \cap [4,16]$. (c) $[66,76] - [72,100]$
(d) $[0,\infty) \cup (-10,10)$ (e) $[27,29] - (26,28)$

Solution:

(a) $(1,3) \cup (2,15) = (1,15)$.

(b) $[1,8] \cap [4,16] = [4,8]$.

(c) $[66,76] - [72,100] = [66,72)$.

(d) $[0,\infty) \cup (-10,10) = (-10,\infty)$.

(e) $[27,29] - (26,28) = [28,29]$

$\blacksquare$

(A) Question 9. If X, Y are sets containing m, n elements respectively, how many elements are there in the Cartesian product $X \times Y$?

Solution: $|X \times Y| = |X||Y| = mn$ i.e. there are mn elements in $X \times Y$. $\blacksquare$

(B) Question 10. (a) Find, if possible, infinite sets A and B such that $A \cap B = \{0\}$ and $A \cup B = \mathbb{Z}$.

(b) Find, if possible, sets C and D such that $C \cup D = \{b, i, g\}$ and $C \cap D = \{s, m, a, l, l\}$.

Solution:

(a) *There are infinitely many valid answers to this question ... here is one of them.* Let E be the set with all even integers as elements and O be the set with all odd integers as elements and 0. By setting $A = E$ and $B = O$, it follows (as in (A) Question 3) that

- A and B have infinitely many elements;
- $A \cap B = \{0\}$; and
- $A \cup B = \mathbb{Z}$, as required.

(b) Assume sets C and D exist. Then since $s \in C \cap D$ it follows that $s \in C$. Since $s \in C$, it follows that $s \in C \cup D$, which is a contradiction (since $s \notin C \cup D = \{b, i, g\}$). Therefore, such sets do not exist.

$\blacksquare$

(B) Question 11. Recall that, if S is a finite set we let

$$|S| \quad = \quad \text{the number of elements in } \; S.$$

$|S|$ is called the **cardinality** of S; it is always a non-negative integer.

(a) Find $|\{200, 2, \sqrt{2}\}|$, $|\{\text{fish,pear}\}|$ and $|\{200, 2, \sqrt{2}, 200\}|$.

(b) Let A and B be finite disjoint sets. Express $|A \cup B|$ in terms of $|A|$ and $|B|$.

(c) Let $A = \{0, 1, 2, 3, 4\}$ and $B = \{2, 5, 6, 7, 8\}$. What is $|A \cup B|$? Compare it with $|A| + |B|$.

(d) Let A and B be finite sets. Find a formula for $|A \cup B|$ in terms of $|A|$, $|B|$ and $|A \cap B|$. Explain why your formula works.

Solution:

(a) $|\{200, 2, \sqrt{2}\}| = 3$, $|\{\text{fish,pear}\}| = 2$ and $|\{200, 2, \sqrt{2}, 200\}| = 3$.

(b) Since A and B are disjoint, it follows that $A \cap B = \emptyset$. Since A and B are finite, then $|A|$ and $|B|$ exist. Moreover, every element in $A \cup B$ is either in A or B (exculsive or), and hence

$$|A \cup B| = |A| + |B|.$$

(c) Since $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ it follows that $|A \cup B| = 9$. Additionally observe that $|A| = 5$ and $|B| = 5$ so $|A| + |B| = 10 > |A \cup B|$ (2 is counted twice in $|A| + |B|$). *Note that for finite sets A and B: $|A \cup B| < |A| + |B|$ if and only if $A \cap B \neq \emptyset$.*

(d) For $n \in \mathbb{N}$, if sets C_i for $i = 1 \ldots n$ are finite and disjoint, then

$$\left| \bigcup_{i=1}^{n} C_i \right| = \sum_{i=1}^{n} |C_i|. \tag{1}$$

Also, for finite sets D_1 and D_2,

$$D_1 = (D_1 \setminus D_2) \cup (D_1 \cap D_2), \tag{2}$$

where $(D_1 \setminus D_2)$ and $(D_1 \cap D_2)$ are disjoint. Moreover, for finite sets D_1 and D_2,

$$D_1 \cup D_2 = (D_1 \setminus D_2) \cup (D_1 \cap D_2) \cup (D_2 \setminus D_1), \tag{3}$$

where $(D_1 \setminus D_2)$, $(D_1 \cap D_2)$ and $D_1 \setminus D_1$ are disjoint. Therefore,

$$\begin{aligned}
|A| + |B| &= ||(A \setminus B) \cup (A \cap B)| + |(B \setminus A) \cup (B \cap A)| && \text{(via (2))}\\
&= (|(A \setminus B)| + |(A \cap B)| + |(B \setminus A)|) + |(B \cap A)| && \text{(via (1))}\\
&= |(A \setminus B) \cup (A \cap B) \cup (B \setminus A)| + |(B \cap A)| && \text{(via (1) and (3))}\\
&= |(A \cup B)| + |(B \cap A)| && \text{(via (3))}.
\end{aligned}$$

Re-arranging the equation above gives

$$|A| + |B| - |A \cap B| = |A \cup B|.$$

More detail is given in this solution than required to see this result, but this illustration is helpful for seeing the bigger picture ... see the second last question on this sheet and it's generalisation.

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(B) Question 12. Let l and m be integers with $l < m$. Write down formulae for the cardinalities of the following sets in terms of l and m.
(a) $\{x \in \mathbb{Z} \,:\, \ell \le x \le m\}$, (b) $\{x \in \mathbb{Z} \,:\, \ell < x \le m\}$,
(c) $\{x \in \mathbb{Z} \,:\, \ell \le x \le 2\ell\}$, (d) $\{x \in \mathbb{Z} \,:\, 2^\ell \le x < 2^{\ell+1}\}$.

Solution:

(a) $|\{x \in \mathbb{Z} \,:\, \ell \le x \le m\}| = m - l + 1$.

(b) $|\{x \in \mathbb{Z} \,:\, \ell < x \le m\}| = m - l$.

(c) $|\{x \in \mathbb{Z} \,:\, \ell \le x \le 2\ell\}| = 2l - l + 1 = l + 1$.

(d) $|\{x \in \mathbb{Z} \,:\, 2^\ell \le x < 2^{\ell+1}\}| = 2^{l+1} - 2^l = 2^l(2-1) = 2^l$.

■

(B) Question 13. Let A and B be finite sets. Suppose that $A \subseteq B$ and that $|A| = |B|$. Explain why $A = B$.

Solution: Suppose $A \subset B$. Then there exists at least 1 element $x \in B \setminus A$. Therefore,

$$\begin{aligned} |B| &= |B \setminus A| + |B \cap A| && \text{(via (1))} \\ &= |B \setminus A| + |A| && \text{(since } A \subset B\text{)} \\ &> |A| && \text{(since } B \setminus A \neq \emptyset\text{)} \end{aligned}$$

which contradicts $|A| = |B|$. We conclude that $A = B$, as required. ■

(B) Question 14. (a) On a Venn diagram, indicate the set $A \cup (B \cap C)$.

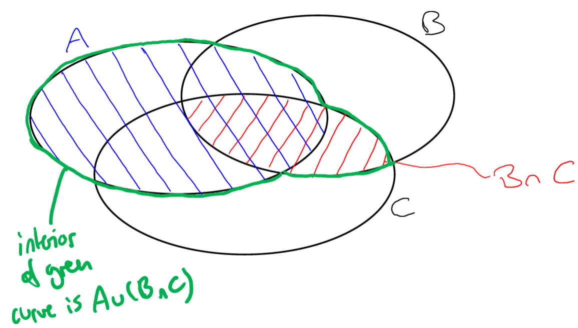
(b) On another Venn diagram, indicate the set $(A \cup B) \cap (A \cup C)$.

(c) Persuade yourself that

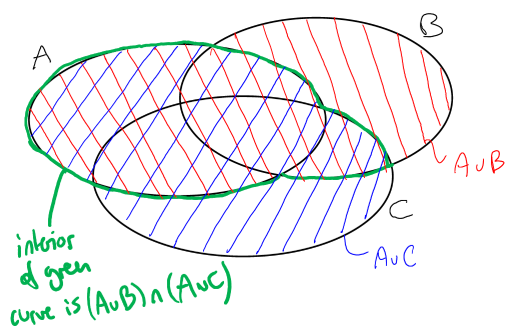
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

[This is the another version of distributive law.]

Solution:



(a)



(b)

(c) See (B) Question 12.

■

(B) Question 15. (a) Let A, B, C and D be sets. Prove that

$$A \cap (B \cup C \cup D) = (A \cap B) \cup (A \cap C) \cup (A \cap D).$$

(b) Let A, B, C and D be sets. State and prove a formula for

$$A \cup (B \cap C \cap D).$$

Solution:

(a) Since

$$\begin{aligned} x \in A \cap (B \cup C \cup D) &\iff (x \in A) \text{ and } ((x \in B) \text{ or } (x \in C) \text{ or } (x \in D)) \\ &\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \text{ or } (x \in A \text{ and } x \in D) \\ &\iff x \in (A \cap B) \cup (A \cap C) \cup (A \cap D), \end{aligned}$$

it follows since all implications above are if and only if that

$$A \cap (B \cup C \cup D) = (A \cap B) \cup (A \cap C) \cup (A \cap D),$$

as required.

(b) Let $E = C \cap D$. Then

$$\begin{aligned} A \cup (B \cap C \cap D) &= A \cup (B \cap E) \\ &= (A \cup B) \cap (A \cup E) && \text{(via (B) Question \textbf{??})} \\ &= (A \cup B) \cap (A \cup (C \cap D)) \\ &= (A \cup B) \cap ((A \cup C) \cap (A \cup D)) && \text{(via (B) Question \textbf{??})} \\ &= (A \cup B) \cap (A \cup C) \cap (A \cup D) \end{aligned}$$

■

(B) Question 16. The **symmetric difference** of two sets A and B, written $A \Delta B$ is defined by

$$A \Delta B \quad = \quad (A \cup B) - (A \cap B).$$

(a) Illustrate this set using a Venn Diagram. On another Venn Diagram, illustrate the set $(A - B) \cup (B - A)$ and deduce that

$$A \Delta B \;=\; (A - B) \cup (B - A).$$

(b) Is Δ commutative?

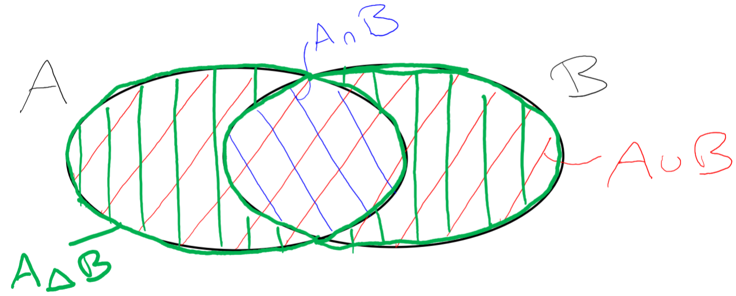
(c) What is $A \Delta A$?

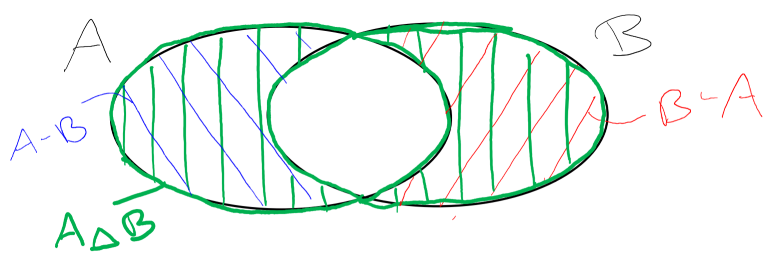
(d) What is $A \Delta \emptyset$?

(e) Suppose $A \subseteq B$, what is $A \Delta B$?

Solution:

(a) The Venn diagrams indicate that the sets $(A \cup B) - (A \cap B)$ and $(A - B) \cup (B - A)$ are equal (although this is not a proof).





Since,

$$\begin{aligned} x \in (A \cup B) - (A \cap B) &\iff x \in (A \cup B) \text{ and } x \notin (A \cap B) \\ &\iff (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A) \end{aligned}$$

$$\begin{aligned}&\iff (x \in (A-B)) \text{ or } (x \in (B-A))\\ &\iff x \in (A-B) \cup (B-A),\end{aligned}$$

as required.

(b) Since $\cup$ and $\cap$ are commutative, it follows that

$$A\Delta B = (A \cup B) - (A \cap B) = (B \cup A) - (B \cap A) = B\Delta A.$$

Therefore, Δ is commutative.

(c) $A\Delta A = (A \cup A) - (A \cap A) = A - A = \emptyset$.

(d) $A\Delta\emptyset = (A \cup \emptyset) - (A \cap \emptyset) = A - \emptyset = A$.

(e) If $A \subseteq B$, then $A\Delta B = (A \cup B) - (A \cap B) = B - A$.

■

(C) Question 17. Let A, B and C be finite sets. Let $D = B \cup C$. By substituting for D in the formula

$$|A \cup D| = |A| + |D| - |A \cap D|$$

and making repeated use of the formula for $|X \cup Y|$, prove that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Solution: Letting $D = B \cup C$, it follows from repeated application of the formula above and Theorem 1.11, that

$$\begin{aligned}|A \cup B \cup C| &= |A \cup D|\\ &= |A| + |D| - |A \cap D|\\ &= |A| + |B \cup C| - |A \cap (B \cup C)|\\ &= |A| + (|B| + |C| - |B \cap C|) - (|(A \cap B) \cup (A \cap C)|)\\ &= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|\\ &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|,\end{aligned}$$

as required. ■

(C) Question 18. Let U be a universal set and $A_i \subset U$ be sets for $i \in \mathbb{Z}$. Prove that

$$\left(\bigcup_{i\in\mathbb{Z}} A_i\right)' = \bigcap_{i\in\mathbb{Z}} A_i'.$$

Hint - Prove the result directly. Avoid mathematical induction.

Solution: For a generic element

$$\begin{aligned} x \in \left(\bigcup_{i\in\mathbb{Z}} A_i\right)' &\iff x \notin \left(\bigcup_{i\in\mathbb{Z}} A_i\right) \\ &\iff x \notin A_i \quad \forall i \in \mathbb{Z} \\ &\iff x \in A_i' \quad \forall i \in \mathbb{Z} \\ &\iff x \in \bigcap_{i\in\mathbb{Z}} A_i', \end{aligned}$$

and hence, the result follows. *Note that this argument holds with* $\mathbb{Z}$ *replaced by any suitable index set e.g. a finite index set.* ■