LH: Homework sheet 1 – Linear Algebra

Instructions. Please upload your solutions to the Canvas assignment page by 5pm, 5 November.

Plagiarism check: Your submission should be your own work and should not be identical or substantially similar to other submissions. A check for plagiarism will be performed on all submissions.

Assessment: This sheet is assessed, with a maximum contribution to your final mark of 5%.

Note. You should make sure that you state clearly any results you use in your proofs or derivations.

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1. Let $(V, \langle\cdot,\cdot\rangle)$ denote a real inner product space and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ denote a basis for V.

Consider the Gram-Schmidt procedure applied to B: set $\mathbf{u}_1 = \mathbf{v}_1$ and compute

$$\mathbf{u}_{k+1} = \mathbf{v}_{k+1} - \sum_{j=1}^{k} \frac{\langle \mathbf{v}_{k+1}, \mathbf{u}_j\rangle}{\langle \mathbf{u}_j, \mathbf{u}_j\rangle}\mathbf{u}_j, \qquad k = 1, 2, \dots, n-1.$$

(a) Show that $\mathbf{v}_{k+1} \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k+1}\}$.

(b) Show by induction on k that

$$\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\},$$

for $k = 1, 2, \dots, n$.

(c) Show that $\langle \mathbf{u}_{k+1}, \mathbf{u}_{k+1}\rangle = \langle \mathbf{u}_{k+1}, \mathbf{v}_{k+1}\rangle$. Deduce that $\|\mathbf{u}_k\| \le \|\mathbf{v}_k\|$ for $k = 1, \dots, n$.

2. Let $\mathcal{P}_n(\mathbb{R})$ denote the space of polynomials p of degree no greater than n and with real coefficients. When viewed as functions of a single real variable, say $p : [a,b] \to \mathbb{R}$, the polynomials in $\mathcal{P}_n(\mathbb{R})$ induce a vector space denoted by $\mathcal{P}_n([a,b])$:

$$\mathcal{P}_n([a,b]) := \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid p : [a,b] \to \mathbb{R}, a_0, \dots, a_n \in \mathbb{R}\}.$$

For $a = -1, b = 1$, this is an inner product space when equipped with the inner product

$$\langle p, q\rangle := \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1-x^2}}\mathrm{d}x.$$

In the following, we let $U := \mathcal{P}_2([-1,1])$. We also denote by $\|\cdot\|$ the norm induced by $\langle\cdot,\cdot\rangle$ on U.

(a) Check that the set $S = \{1, x\}$ is an orthogonal set in U with respect to $\langle\cdot,\cdot\rangle$. Find another *monic* polynomial[1] in U that is orthogonal to S.

(b) Let $q(x) = x^3$. Use part **(a)** to find $p^* \in U$ such that $\|q - p^*\| \le \|q - p\|$ for all $p \in U$. Please state clearly any results you use in your derivation.

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[1] A polynomial of degree n is said to be monic if $a_n = 1$.