Chapter 5 – Separable solutions of partial differential equations

(using the heat equation, wave equation and Laplace's equation as examples)

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In the previous chapter, we saw how to apply Separation of Variables to convert certain types of PDEs into ODEs and how Fourier series may be useful in finding solutions to these equations. We recap this approach here before considering specific examples. We will look at the heat, wave and Laplace equations as examples, but the theory is applicable to many more PDEs (generally linear systems with a pair of homogeneous boundary conditions).

(a) Look for solutions to the PDE for $u(x,t)$ in the form $u(x,t) = X(x)T(t)$ and reduce the PDE into two ODEs in $X(x)$ and $T(t)$ (Separation of Variables). The resulting ODEs will be dependent on an unknown separation constant.

(b) Convert the pair of homogeneous boundary conditions into the appropriate new variable (avoiding the trivial solution).

(c) Employ the new boundary conditions to identify possible values for the separation constant and solve the ODEs.

(d) Combine $X(x)$ and $T(t)$ to obtain $u(x,t)$ and apply the superposition principle to obtain a general solution for $u(x,t)$.

(e) Impose the remaining condition(s).

Note that the exact approach (e.g. which conditions to use when) will depend on the nature of the PDE system under consideration. The above method can be generalised to PDE systems in more than two independent variables. As usual, it is generally easier to understand this method by working through examples.

1 The heat equation

1.1 Fixed temperature at the ends of the rod

Recall from Chapter 4 §1 that a model for the distribution of heat across a rod with fixed temperature at the ends of the rod (subject to certain assumptions) is given by:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \quad t > 0,$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \qquad t > 0,$$

$$u(x,0) = f(x), \qquad 0 < x < L,$$

Example: Use Separation of Variables and Fourier series to find the separable solution of the heat equation with fixed temperature at the ends of the rod

Answer: In Chapter 4 §1 we derived the solution to the above system:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right),$$

where the b_n must be chosen to satisfy the initial condition $u(x,0) = f(x)$, i.e.

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

i.e. the b_n are the coefficients of the Fourier sine series for $f(x)$ on $-L < x < L$ if $f(x)$ is odd on $-L < x < L$ (a Fourier sine series can only converge to an odd function).

$f(x)$ is defined on $0 < x < L$ and undefined elsewhere (so we can effectively choose how it is defined elsewhere in a way that will be helpful to us). If we extend $f(x)$ to be an odd function on $-L < x < L$, then finding the b_n is equivalent to finding the coefficients of the Fourier (sine) series of $f(x)$ where

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \end{aligned}$$

Example: Solve the heat equation, as given above, with $L = \pi$ and $u(x,0) = f(x) = 1$ on $0 < x < \pi$.

Answer:

See Canvas for an animation of this solution.

1.2 Insulated ends of the rod

The heat equation and its associated boundary and initial conditions are adaptable to different circumstances. So far we have only considered boundary conditions that reflect the ends of the rod/wire being held at a fixed temperature of 0°C ($u(0,t) = u(L,t) = 0$). What happens if we change this to consider a rod/wire with insulated ends so that no heat can flow in or out?

The boundary conditions become

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \quad t > 0.$$

We now need to find the solution to the heat equation given by

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \quad t > 0, \tag{1}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \qquad t > 0, \tag{2}$$

$$u(x,0) = f(x), \qquad 0 < x < L. \tag{3}$$

Example: Find the separable solution to the above system using Separation of Variables and Fourier series.

Answer: We know from §1 of Chapter 4 that, by looking for a solution in the form $u(x,t) = X(x)T(t)$ (i.e. using Separation of Variables), we can obtain the following ODEs in x and t:

$$X''(x) + \sigma X(x) = 0, \tag{4}$$
$$T'(t) + \alpha^2 \sigma T(t) = 0. \tag{5}$$

We have that

$$\frac{\partial u}{\partial x}(x,t) = X'(x)T(t),$$

so that the boundary conditions (2) become

$$X'(0)T(t) = X'(L)T(t) = 0.$$

This means that either $T(t) \equiv 0$ (i.e. $u(x,t) \equiv 0$) or $X'(0) = X'(L) = 0$. Since we are not interested in the trivial solution we must have

$$X'(0) = X'(L) = 0. \tag{6}$$

As in §1 of Chapter 4, we consider three regimes of σ to solve (4) subject to (6).

$\underline{\sigma = 0}$

$\underline{\sigma < 0}$

$\underline{\sigma > 0}$

and $X(x) = k_1 \cos\left(\frac{n\pi x}{L}\right)$.

Taking these results together,

$$X_n(x) = c_n \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, ...$$

are all solutions to (4), subject to the boundary conditions (6) (where the c_n are undetermined coefficients).

We now need to solve (5). Imposing our derived condition on σ gives

$$T'(t) + \left(\frac{\alpha n\pi}{L}\right)^2 T(t)$$

This is the same as we had in §1 of Chapter 4, so we know this will give the solution

$$T_n(t) = \beta_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right), \quad n = 0, 1, 2, ...$$

Thus,

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t), \\ &= c_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right) \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, ... \end{aligned}$$

absorbing the β_n inside the c_n. All $u_n(x,t)$ for $n = 0, 1, 2, ...$ will satisfy (1) and (2) so by the superposition principle, a linear combination of all the u_n also will, i.e.

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t), \\ &= \sum_{n=0}^{\infty} c_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right)\cos\left(\frac{n\pi x}{L}\right), \\ &= c_0 + \sum_{n=1}^{\infty} c_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right)\cos\left(\frac{n\pi x}{L}\right), \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right)\cos\left(\frac{n\pi x}{L}\right), \end{aligned}$$

(where we have replaced c_0 with $c_0 = \dfrac{a_0}{2}$ and the c_n with $c_n = a_n$ for all $n = 1, 2, 3, ...$).

To satisfy the initial condition (3), we must choose the coefficients a_n such that

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x),$$

i.e. obtaining the coefficients a_n is equivalent to finding the coefficients in the Fourier cosine series for $f(x)$.

Note that as $t \to \infty$,

$$u(x,t) \to \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx,$$

i.e. the average initial temperature.

Example: Find the separable solution to the heat equation as given by (1) and (2) subject to the initial condition

$$u(x,0) = x, \quad 0 < x < L.$$

Answer:

See Canvas for an animation of this solution.

2 The wave equation

A string of length L is fixed at both ends to be stretched across a straight horizontal distance. When plucked, its vertical displacement $u(x,t)$ is given by

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < L, \quad t > 0, \tag{7}$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \qquad t > 0, \tag{8}$$

$$u(x,0) = f(x), \qquad 0 < x < L, \tag{9}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < L. \tag{10}$$

We have that a is the wave speed and $a^2 = T/\rho$ where T is the tension in the string and ρ is the mass per unit length of the string. (This equation also applies to studies of acoustic waves, water waves, electromagnetic waves, seismic waves...)

In the above equations we make the following assumptions:

- damping effects such as air resistance are negligible;
- the amplitude of motion is relatively small;
- there is a nonzero initial displacement, i.e. $f(x) \not\equiv 0$;
- initial velocity is zero, i.e. equation (10).

See Figure 1.

Example: Use Separation of Variables and Fourier series to find a separable solution to the above problem.

Answer: As for the heat equation, we look for a solution in the form $u(x,t) = X(x)T(t)$. Substituting this into (7) gives

$$a^2 X''(x)T(t) = X(x)T''(t),$$
$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)}.$$

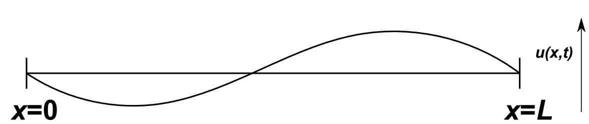

Figure 1: The vertical displacement of the string at any point x and time t is given by $u(x,t)$. The string is fixed at both ends and its vertical displacement here takes on the shape of a wave.

For the above to hold, we must have both sides equal to a constant, $-\sigma$ say. Thus we can reduce (7) to the following two ODEs:

$$X''(x) + \sigma X(x) = 0, \tag{11}$$
$$T''(t) + a^2\sigma T(t) = 0. \tag{12}$$

Imposing the boundary conditions (8) gives

$$u(0,t) = 0 \implies X(0)T(t) = 0 \implies \text{either} \quad X(0) = 0 \quad \text{or} \quad T(t) \equiv 0,$$
$$u(L,t) = 0 \implies X(L)T(t) = 0 \implies \text{either} \quad X(L) = 0 \quad \text{or} \quad T(t) \equiv 0.$$

Therefore we take

$$X(0) = 0 \quad \text{and} \quad X(L) = 0 \tag{13}$$

to be boundary conditions (to avoid the trivial solution $u(x,t) \equiv 0$) of the ODE in x given by (11).

Notice that (11) and (13) are just as we saw in the solution of the heat equation in Chapter 4. Thus, following the same working, we have that

$$\sigma = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, ...$$

and

$$X_n(x) = k_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, ... \tag{14}$$

We must then solve

$$T''(t) + \left(\frac{an\pi}{L}\right)^2 T(t) = 0$$

to get

$$T_n(t) = b_n \cos\left(\frac{an\pi t}{L}\right) + c_n \sin\left(\frac{an\pi t}{L}\right). \tag{15}$$

Putting (14) and (15) together, we have

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(b_n \cos\left(\frac{an\pi t}{L}\right) + c_n \sin\left(\frac{an\pi t}{L}\right)\right),$$

(where we have absorbed the k_n constants into b_n and c_n). Imposing the first initial condition (9),

$$u(x,0) = f(x) \implies \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x),$$

i.e. the b_n are the coefficients of the Fourier sine series of period $2L$ for $f(x)$:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, .. \tag{16}$$

Imposing the second initial condition (10)

$$\frac{\partial u}{\partial t}(x,0) = 0 \implies \sum_{n=1}^{\infty} \frac{c_n a n\pi}{L}\sin\left(\frac{n\pi x}{L}\right) = 0,$$

i.e. the $c_n a n\pi/L$ must be coefficients of the Fourier sine series of period $2L$ for $g(x) \equiv 0$:

$$\begin{aligned}\frac{c_n a n\pi}{L} &= \frac{2}{L}\int_0^L 0\,dx,\\ &= 0, \implies c_n = 0 \quad \forall \quad n = 1, 2, ...\end{aligned}$$

Thus the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi t}{L}\right)$$

with the b_n given by (16).

Example: Solve the following wave problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < 10, \quad t > 0, \tag{17}$$

$$u(0,t) = 0 \quad \text{and} \quad u(10,t) = 0, \qquad t > 0, \tag{18}$$

$$u(x,0) = x(10 - x), \qquad 0 < x < 10, \tag{19}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 < x < 10. \tag{20}$$

Answer:

See Canvas for an animation of this solution.

What would happen if we do not assume that the initial velocity is zero? i.e. the initial condition (10) is replaced by

$$\frac{\partial u}{\partial t}(x,0) = g(x). \tag{21}$$

Then

$$\begin{aligned} \frac{c_n a n\pi}{L} &= \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \\ c_n &= \frac{2}{an\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \end{aligned} \tag{22}$$

i.e. the $c_n an\pi/L$ must be the coefficients of the Fourier sine series of period $2L$ for general $g(x)$ and the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(b_n \cos\left(\frac{an\pi t}{L}\right) + c_n \sin\left(\frac{an\pi t}{L}\right)\right)$$

with the b_n given by (16) and the c_n given by (22).

3 Laplace's equation

The Laplace equation in two dimensions is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b. \tag{23}$$

It is often also known as the **potential equation** and arises in the study of, amongst many other things, heat conduction (this is the steady state of the heat equation in two dimensions), electric charge, incompressible fluids and elasticity.

We require boundary conditions to solve Laplace's equation (note that there is no time-dependence, so initial conditions are not required). If we specify values of $u(x, y)$ at the boundary, these conditions are called **Dirichlet boundary conditions**. If we specify values of the derivative of $u(x, y)$ at the boundary, these are called **Neumann boundary conditions**. If a combination of the two are given, these are called **mixed boundary conditions**.

3.1 An example using Dirichlet conditions

Example: Find the separable solution to the Laplace equation given the following Dirichlet boundary conditions

$$u(x, 0) = 0 \quad \text{and} \quad u(x, b) = 0, \qquad 0 < x < a, \tag{24}$$

$$u(0, y) = 0 \quad \text{and} \quad u(a, y) = f(y), \qquad 0 < y < b. \tag{25}$$

Answer: We look for solutions in the form $u(x, y) = X(x)Y(y)$ and use Separation of Variables. Substituting this into (23) gives

$$X''(x)Y(y) = -X(x)Y''(y),$$
$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The above can only hold if both sides are equal to a constant, σ, say, i.e.

$$X''(x) - \sigma X(x) = 0, \tag{26}$$
$$Y''(y) + \sigma Y(y) = 0. \tag{27}$$

Imposing the homogeneous boundary conditions from (24) and (25) we have

$$u(x, 0) = 0 \implies X(x)Y(0) = 0 \implies Y(0) = 0, \tag{28}$$
$$u(x, b) = 0 \implies X(x)Y(b) = 0 \implies Y(b) = 0, \tag{29}$$
$$u(0, y) = 0 \implies X(0)Y(y) = 0 \implies X(0) = 0. \tag{30}$$

Solving (27) subject to (28) and (29) is equivalent to solving (4) and (6) in §1 of Chapter 4. Therefore, we know we must have $\sigma = \left(\frac{n\pi}{b}\right)^2$, $n = 1, 2, \ldots$ and

$$Y_n(y) = c_n \sin\left(\frac{n\pi y}{b}\right).$$

We must then solve (26) with these values of σ:

$$X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x) = 0, \implies X(x) = k_1 \cosh\left(\frac{n\pi x}{b}\right) + k_2 \sinh\left(\frac{n\pi x}{b}\right).$$

Imposing the boundary condition (30):

$$X(0) = 0 \implies k_1 = 0,$$

i.e.

$$X_n(x) = k_n \sinh\left(\frac{n\pi x}{b}\right).$$

Thus

$$u_n(x, y) = c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right),$$

(where we have absorbed the k_n into the c_n) satisfies (23) subject to the homogeneous boundary conditions in (24) and (25) for $n = 1, 2, \ldots$ and we can use the superposition principle to show that

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

is also a solution.

Imposing the final boundary condition from (25), we have

$$u(a, y) = f(y) \implies \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = f(y),$$

i.e. the $c_n \sinh(n\pi a/b)$ must be coefficients of the Fourier sine series of period $2b$ for $f(y)$:

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy,$$

$$c_n = \frac{2}{b \sinh(n\pi a/b)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy.$$

3.2 An example using mixed boundary conditions

Example: Find a solution to the Laplace equation given the following mixed boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(a,y) = 0, \qquad 0 < y < b, \tag{31}$$

$$u(x,b) = 0 \quad \text{and} \quad u(x,0) = f(x), \qquad 0 < x < a. \tag{32}$$

Answer: As before, we look for a solution in the form $u(x,y) = X(x)Y(y)$ to get (26) and (27). Imposing the boundary conditions (31) gives

$$X'(0) = X'(a) = 0. \tag{33}$$

Using working from §1.2 we can solve (26) subject to (33) to get

$$\sigma = -\left(\frac{n\pi}{a}\right)^2$$

and

$$X_n(x) = c_n \cos\left(\frac{n\pi x}{a}\right), \quad n = 0, 1, 2, ...$$

We must then solve

$$Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) = 0.$$

If $n = 0$,

$$Y_0 = A_0 + B_0 y,$$

otherwise

$$Y_n = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right).$$

Imposing the first of the boundary conditions in (32) to $u(x,y) = X(x)Y(y)$ gives

$$Y(b) = 0. \tag{34}$$

Imposing the new boundary condition (34) gives, if $n = 0$,

$$\begin{aligned} Y_0(b) = 0 \implies A_0 + B_0 b = 0 &\implies A_0 = -bB_0, \\ &\implies Y_0 = B_0(y - b), \end{aligned}$$

otherwise

$$Y_n(b) = 0 \implies A_n \cosh\left(\frac{n\pi b}{a}\right) + B_n \sinh\left(\frac{n\pi b}{a}\right) = 0 \implies A_n = -B_n \tanh\left(\frac{n\pi b}{a}\right),$$

$$\implies Y_n = B_n\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\right).$$

Thus, from $u_n(x,y) = X_n(x)Y_n(y)$, we have

$$\begin{aligned} u_0(x,y) &= c_0 B_0(y-b), \\ &= r_0(y-b), \end{aligned}$$

(with $r_0 = c_0 B_0$) and

$$\begin{aligned} u_n(x,y) &= c_n \cos\left(\frac{n\pi x}{a}\right) B_n\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\right), \\ &= r_n \cos\left(\frac{n\pi x}{a}\right)\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\right), \end{aligned}$$

(with $r_n = c_n B_n$), $n \geq 1$. Taking the sum of all the possible solutions, therefore by the Superposition Principle,

$$u(x,y) = r_0(y-b) + \sum_{n=1}^{\infty} r_n \cos\left(\frac{n\pi x}{a}\right)\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\right).$$

Finally, imposing the second boundary condition in (32) gives

$$f(x) = -r_0 b + \sum_{n=1}^{\infty} -r_n \tanh\left(\frac{n\pi b}{a}\right)\cos\left(\frac{n\pi x}{a}\right),$$

i.e. we have a Fourier cosine series series for $f(x)$ of period $2a$ where

$$\begin{aligned} -r_0 b &= \frac{1}{a}\int_0^a f(x)\,dx, \\ r_0 &= -\frac{1}{ab}\int_0^a f(x)\,dx, \end{aligned}$$

and

$$\begin{aligned} -r_n \tanh\left(\frac{n\pi b}{a}\right) &= \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)\,dx, \\ r_n &= \frac{-2}{a\tanh\left(n\pi b/a\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)\,dx. \end{aligned}$$

Remember this theory applies to many more PDEs than those we have considered as examples in this chapter. In addition to these, and those on the exercise sheet, try to find some more examples for your own practice.