Example sheet 7 – formative

1. Consider the dynamical system

$$\dot{x} = P(x, y),$$
$$\dot{y} = Q(x, y),$$

where $(x, y) \in \mathbb{R}^2$ and $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are differentiable.

(a) Prove that a necessary condition for the system to be Hamiltonian is that

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0.$$

(b) Determine which of the following two dynamical systems is Hamiltonian and obtain the corresponding Hamiltonian function $H(x, y)$:

(a) $\dot{x} = 1 - xy^2$, $\dot{y} = xy^2 - y$.

(b) $\dot{x} = x + y - x^2$, $\dot{y} = 2xy - y$.

(c) Sketch the global phase portrait of the system from part b which is Hamiltonian, and determine the set of initial conditions $(x_0, y_0) \in \mathbb{R}^2$ for which the solution to this system is periodic.

2. Consider the dynamical system

$$\dot{x} = 4y\left(x^2 + y^2 + 2\right),$$
$$\dot{y} = -4x\left(x^2 + y^2 - 2\right),$$

where $(x, y) \in \mathbb{R}^2$.

(a) Verify that this dynamical system is Hamiltonian and obtain the corresponding Hamiltonian function $H(x, y)$.

(b) Sketch the global phase portrait of this dynamical system, giving all analytic information (equilibria, eigenvalues and eigenvectors, horizontal and vertical isoclines, direction vectors in each segment, ...) and draw the qualitiatively different trajectories.

(c) Justify your choices of trajectories using symmetry arguments.

(d) Are there any heteroclinic or homoclinic orbits or cycles?

(e) Use the Hamiltonian function to analyse the different types of trajectories in function of the level constant C. Indicate where we have periodic orbits, and justify this by analysing the intersection of the solution curves with $y = mx$. Provide as much detail as possible.