University of Birmingham
School of Mathematics
Vectors, Geometry and Linear Algebra
VGLA

Problem Sheet 5

Remember that there are practise questions under the materials section for each week. Note the question 2 in on page 2.

$\boxed{\text{SUM}}$ **Q1**. (i) Explain why a system of equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ of m equations in n unknowns has a non-zero solution if and only if n is greater than the rank of $\mathbf{A}$.

(ii) Using (i), determine if the following system of equations has a non-zero solution.

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 2x_4 &= 0. \end{aligned}$$

(iii) Using (i), determine if the following system of equations has a non-zero solution.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0. \end{aligned}$$

(iv) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

where $x_1, x_2, x_3 \in \mathbb{R}$. Calculate the determinant of $\mathbf{A}$ (remember Vandermonde) and calculate the rank of A for all possible x_1, x_2 and x_3.

SUM **Q2**. Suppose that V and W are vector spaces over $\mathbb{R}$ with $\dim V = 4$ and $\dim W = 3$. Assume that $T : V \to W$ is a linear transformation.

(i) Explain why $\ker(T) \neq \{\mathbf{0}\}$.
(ii) List all possibilities for the dimension on the image of T.
(iii) Suppose that

$$B_V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

is a basis for V and

$$B_W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

is a basis for W. Assume that

$$\begin{aligned} T(\mathbf{v}_1) &= \mathbf{w}_1 - \mathbf{w}_2 \\ T(\mathbf{v}_2) &= \mathbf{w}_2 - \mathbf{w}_3 \\ T(\mathbf{v}_3) &= \mathbf{w}_3 - \mathbf{w}_1 \\ T(\mathbf{v}_4) &= \mathbf{w}_1 + \mathbf{w}_2 - 2\mathbf{w}_3 \end{aligned}$$

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(a) Suppose that $\mathbf{v} \in V$ has coordinate vector $(1, 2, 3, 4)$ with respect to B_V of V. Calculate $T(\mathbf{v})$ and write down its coordinate vector with respect to the basis B_W.
(b) Write down the matrix representing T with respect to B_V and B_W.
(c) Calculate the dimension of the image of T and find a basis for the image.
(d) Calculate $\ker(T)$.