Mathematical Induction Practice Questions

The following questions relate to Mathematical Induction. Questions are ranked in difficulty from A (basic) to C (challenging).

(A) Question 1. Prove by mathematical induction that for all $n \in \mathbb{N}$:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

Solution: Let $P(n)$ for $n \in \mathbb{N}$ be the statement

$$\sum_{i=1}^{n} (2i - 1) = n^2. \tag{1}$$

For $n = 1$, we have,

$$\sum_{i=1}^{1} (2i - 1) = 1 = 1^2 \iff P(1) \text{ is true.} \tag{2}$$

Now, assume that for some $k \in \mathbb{N}$ that $P(k)$ is true, i.e.

$$\sum_{i=1}^{k} (2i - 1) = k^2. \tag{3}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{k+1} (2i - 1) &= \sum_{i=1}^{k} (2i - 1) + (2(k + 1) - 1) \\ &= k^2 + 2k + 1 \qquad \text{(via (3))} \\ &= (k + 1)^2. \end{aligned} \tag{4}$$

It follows from (2) and (3) that for $k \in \mathbb{N}$,

$$P(k) \text{ is true} \Longrightarrow P(k + 1) \text{ is true} \tag{5}$$

and via (2), (5) and the PMI, we conclude that $P(n)$ given by (1) is true for all $n \in \mathbb{N}$, as required. ∎

(B) Question 2. Let $P(n)$ be a statement for all $n \in \mathbb{Z}$ such that $n \leq m$. State the base step and induction step, in terms of $P(n)$, that could potentially prove that $P(n)$ is true for all $n \in \mathbb{Z}$ with $n \leq m$.

Solution: Consider the statement $Q(n)$ for $n \in \mathbb{N}$ with $Q(n) = P(m - n + 1)$. It follows from Theorem 2.1 that if $Q(1)$ is true, and for $k \in \mathbb{N}$, we have $Q(k)$ is true implies $Q(k + 1)$ is true, then $Q(n)$ is true for all $n \in \mathbb{N}$ (and hence $P(n)$ is true for all $n \in \mathbb{Z}$ with $n \leq m$).

So, consider the *base step* "$P(m)$ is true" $\iff$ "$Q(1)$ is true". Also, consider the *induction step*, "for $k_1 \in \mathbb{Z}$ with $k_1 \leq m$, we have $P(k_1)$ is true implies $P(k_1 - 1)$ is true" $\iff$ "for $k_2 \in \mathbb{N}$, we have $Q(k_2)$ is true implies $Q(k_2 + 1)$ is true". From the discussion above we conclude that these steps are sufficient. ∎

(B) Question 3. Prove by mathematical induction that:

(a) $n^2 - n$ is even, for all $n \in \mathbb{N}$;

Solution: Consider the statement $P(n)$ for $n \in \mathbb{N}$ given by:

$$n^2 - n \text{ is even.} \tag{6}$$

For $n = 1$, we have

$$1^2 - 1 = 0 \text{ is even } \iff P(1) \text{ is true.} \tag{7}$$

Now, for some $k \in \mathbb{N}$, assume $P(k)$ is true, i.e. that

$$k^2 - k \text{ is even } \iff k^2 - k = 2c(k) \text{ for some } c(k) \in \mathbb{Z}. \tag{8}$$

Therefore,

$$\begin{aligned}(k+1)^2 - (k+1) &= k^2 + 2k + 1 - k - 1 \\ &= k^2 + k \\ &= (k^2 - k) + 2k \\ &= 2(c(k) + k) \quad \text{(via (8))}\end{aligned} \tag{9}$$

and since $(c(k) + k) \in \mathbb{Z}$, we conclude from (8) and (9) that $(k+1)^2 - (k+1)$ is even, i.e.

$$P(k) \text{ is true} \Longrightarrow P(k+1) \text{ is true.} \tag{10}$$

Finally, via (7), (10) and the PMI, we conclude that $P(n)$ given by (6) is true for all $n \in \mathbb{N}$ as required. ■

(b) $n^3 - n$ is divisible by 3, for all $n \in \mathbb{N}$; and

Solution: Let $P(n)$ for $n \in \mathbb{N}$ with $n \in \mathbb{N}$ be the statement

$$3 \mid n^3 - n. \tag{11}$$

For $n = 1$,

$$3 \mid (1^3 - 1) \iff 3 \mid 0 \iff 3 \times 0 = 0 \Longrightarrow P(1) \text{ is true.} \tag{12}$$

Now assume for $k \in \mathbb{N}$ that $P(k)$ is true, i.e.

$$3 \mid (k^3 - k). \tag{13}$$

Then,

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3(k^2 + k) \\ &= 3(m + k^2 + k) \qquad \text{(via (13))}\end{aligned} \tag{14}$$

for some $m \in \mathbb{Z}$, i.e. $3 \mid (k+1)^3 - (k+1)$. Hence, via (13) and (14), it follows that

$$P(k) \text{ is true } \Longrightarrow P(k+1) \text{ is true.} \tag{15}$$

Finally, via (12), (15) and the PMI, we conclude that $P(n)$ given by (11) is true for all $n \in \mathbb{N}$, as required. ■

(c) $(n^2 + n)^2$ is divisible by 4, for all $n \in \mathbb{N}$.

Solution: Let $P(n)$ for $n \in \mathbb{N}$ with $n \in \mathbb{N}$ be the statement

$$4 \mid (n^2 + n)^2. \tag{16}$$

For $n = 1$,

$$4 \mid (1^2 + 1)^2. \iff 4 \mid 4 \iff 4 \times 1 = 4 \Longrightarrow P(1) \text{ is true.} \tag{17}$$

Now assume for $k \in \mathbb{N}$ that $P(k)$ is true, i.e.

$$4 \mid (k^2 + k)^2. \tag{18}$$

Then,

$$\begin{aligned}((k+1)^2 + (k+1))^2 &= ((k^2 + k) + 2(k+1))^2 \\ &= (k^2 + k)^2 + 4((k^2 + k)(k+1) + (k+1)^2)\end{aligned}$$

$$
\begin{aligned}
&= 4(m + ((k^2+k)(k+1) + (k+1)^2)) \qquad \text{(via (18))} \\
&= 4(m + ((k^2+k)(k+1) + (k+1)^2)) \qquad (19)
\end{aligned}
$$

for some $m \in \mathbb{Z}$, i.e. $4 \mid ((k+1)^2 + (k+1))^2$. Hence, via (18) and (19), it follows that

$$P(k) \text{ is true } \implies P(k+1) \text{ is true.} \tag{20}$$

Finally, via (17), (20) and the PMI, we conclude that $P(n)$ given by (16) is true for all $n \in \mathbb{N}$, as required. ■

(B) Question 4. Prove by mathematical induction that:

(a) $(n+1)! > 2^{n+3}$, for all $n \geq 5$; and

Solution: Let $P(n)$ for $n \in \mathbb{N}$ with $n \geq 5$ be the statement

$$(n+1)! > 2^{n+3}. \tag{21}$$

For $n = 5$, we have

$$(5+1)! = 6! = 720 > 256 = 2^8 = 2^{5+3} \implies P(5) \text{ is true.} \tag{22}$$

Now assume for $k \in \mathbb{N}$ with $k \geq 5$ that $P(k)$ is true, i.e.

$$(k+1)! > 2^{k+3}. \tag{23}$$

Then,

$$
\begin{aligned}
((k+1)+1)! &= (k+2)(k+1)! \\
&> 2 \times 2^{k+3} \qquad \text{(via (23))} \\
&= 2^{(k+1)+3}. \qquad (24)
\end{aligned}
$$

Hence, via (23) and (24), it follows that

$$P(k) \text{ is true } \implies P(k+1) \text{ is true.} \tag{25}$$

Finally, via (22), (25) and the PMI, we conclude that $P(n)$ given by (21) is true for all $n \in \mathbb{N}$ with $n \geq 5$, as required. ■

(b) $n! > 2^n$ for all $n \geq 4$.

Solution: Let $P(n)$ for $n \in \mathbb{N}$ with $n \geq 4$ be the statement

$$n! > 2^n. \tag{26}$$

For $n = 4$, we have

$$4! = 24 > 16 = 2^4 \implies P(4) \text{ is true.} \tag{27}$$

Now assume for $k \in \mathbb{N}$ with $k \geq 4$ that $P(k)$ is true, i.e.

$$k! > 2^k. \tag{28}$$

Then,

$$
\begin{aligned}
(k+1)! &= (k+1)k! \\
&> (k+1)2^k \qquad \text{(via (28))} \\
&> 2^{k+1} \qquad \text{(since } k \geq 4\text{).} \qquad (29)
\end{aligned}
$$

Hence, via (28) and (29), it follows that

$$P(k) \text{ is true } \implies P(k+1) \text{ is true.} \tag{30}$$

Finally, via (27), (30) and the PMI, we conclude that $P(n)$ given by (26) is true for all $n \in \mathbb{N}$ with $n \geq 4$, as required. ■

(B) Question 5. Use the Principle of Mathematical Induction to show:

(a) $\left(\bigcup_{k=1}^{n} A_k\right)' = \bigcap_{k=1}^{n} A_k', \quad \forall n \in \mathbb{N},$

Solution: Let $P(n)$ for $n \in \mathbb{N}$, be the statement given by

$$\left(\bigcup_{k=1}^{n} A_k\right)' = \bigcap_{k=1}^{n} A_k'. \tag{31}$$

For $n = 1$, we have,

$$\left(\bigcup_{k=1}^{1} A_k\right)' = (A_1)' = \bigcap_{k=1}^{1} A_1' \iff P(1) \text{ is true.} \tag{32}$$

Now, for some $j \in \mathbb{N}$, assume that $P(j)$ is true, i.e. for any sets $A_k \subseteq U$ (for $k = 1, \ldots, j$), we have

$$\left(\bigcup_{k=1}^{j} A_k\right)' = \bigcap_{k=1}^{j} A_k'. \tag{33}$$

Then,

$$\begin{aligned}
\left(\bigcup_{k=1}^{j+1} A_k\right)' &= \left(\left(\bigcup_{k=1}^{j} A_k\right) \cup A_{j+1}\right)' \\
&= \left(\bigcup_{k=1}^{j} A_k\right)' \cap A_{j+1}' \qquad \text{(via Theorem 1.11 (i))} \\
&= \left(\bigcap_{k=1}^{j} A_k'\right) \cap A_{j+1}' \qquad \text{(via (33))} \\
&= \bigcap_{k=1}^{j+1} A_k'.
\end{aligned} \tag{34}$$

Therefore, via (33) and (34),

$$P(j) \text{ is true} \Longrightarrow P(j+1) \text{ is true.} \tag{35}$$

Finally, via (32), (35) and the PMI, we conclude that $P(n)$ given by (31) is true for all $n \in \mathbb{N}$, as required. *Note that we can use any index, say j for the induction step ... we don't have to use the index k. Additionally note that this result is subsumed by that in (C) Question 21 on the Sets and Notation Practice Questions i.e. this proof is an exercise to practice proof by induction.* ■

(b) $2^n > n, \quad \forall n \in \mathbb{N},$

Solution: Let $P(n)$ for $n \in \mathbb{N}$ be the statement

$$2^n > n. \tag{36}$$

For $n = 1$, we have

$$2^1 = 2 > 1 \Longrightarrow P(1) \text{ is true.} \tag{37}$$

Now assume for $k \in \mathbb{N}$ that $P(k)$ is true, i.e.

$$2^k > k. \tag{38}$$

Then,

$$\begin{aligned}
2^{k+1} &= 2(2^k) \\
&> 2k \qquad \text{(via (38))} \\
&\geq k + 1.
\end{aligned} \tag{39}$$

Hence, via (38) and (39), it follows that

$$P(k) \text{ is true } \implies P(k+1) \text{ is true.} \tag{40}$$

Finally, via (37), (40) and the PMI, we conclude that $P(n)$ given by (36) is true for all $n \in \mathbb{N}$, as required. ■

(c) $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{1}{2}n(n+1)\right]^2, \quad \forall n \in \mathbb{N}.$

Solution: Let $P(n)$ for $n \in \mathbb{N}$, be given by

$$\sum_{k=1}^{n} k^3 = \left[\frac{1}{2}n(n+1)\right]^2. \tag{41}$$

For $n = 1$, we have,

$$\sum_{k=1}^{1} k^3 = 1^3 = 1 = \left[\frac{1}{2}(1)(1+1)\right]^2 \iff P(1) \text{ is true.} \tag{42}$$

Now, for some $j \in \mathbb{N}$, assume that $P(j)$ is true, i.e.

$$\sum_{k=1}^{j} k^3 = \left[\frac{1}{2}j(j+1)\right]^2. \tag{43}$$

Then,

$$\begin{aligned} \sum_{k=1}^{j+1} k^3 &= \sum_{k=1}^{j} k^3 + (j+1)^3 \\ &= \left[\frac{1}{2}j(j+1)\right]^2 + (j+1)^3 \quad \text{(via (43))} \\ &= \left[\frac{1}{2}(j+1)\right]^2 (j^2 + 4j + 4) \\ &= \left[\frac{1}{2}(j+1)(j+2)\right]^2. \end{aligned} \tag{44}$$

Therefore, via (43) and (44),

$$P(j) \text{ is true} \implies P(j+1) \text{ is true.} \tag{45}$$

Finally, via (42), (45) and the PMI, we conclude that $P(n)$ given by (41) is true for all $n \in \mathbb{N}$, as required. ■

(B) Question 6. Prove the following:
Theorem: Let $P(n)$ be a statement for each $n \in \mathbb{N}$ with $n \geq m$. Additionally, suppose that both of the following statements are satisfied:

(i) $P(m)$ is true; and

(ii) for each $k \in \mathbb{N}$, we have

$$P(n) \text{ is true } \forall m \leq n \leq k \implies P(k+1) \text{ is true.}$$

Then $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq m$. *Hint - Consider Theorem 2.2.*

Solution: Consider the statement $Q(n)$ for $n \in \mathbb{Z}$ with $n \geq m$, given by

$$P(l) \text{ for all } m \leq l \leq n.$$

Then $Q(m)$ is true via (i). Also, via (ii), $Q(k)$ is true $\implies Q(k+1)$ is true. Therefore, via the PMI (Theorem 2.2), it follows that $Q(n)$ is true for all $n \in \mathbb{Z}$ with $n \geq m$. By the definition of $Q(n)$ it follows that $P(n)$ is also true for all $n \in \mathbb{Z}$ with $n \geq m$, as required. *This type of induction is known as total/complete induction.* ■

(B) Question 7. For all $n \in \mathbb{N}$, using the PMI, prove that $5^n - 1$ is divisible by 4. *Recall that $n \in \mathbb{N}$ is divisible by a $m \in \mathbb{N}$ if there exists $k \in \mathbb{N}$ such that $n = mk$.*

Solution: For each $n \in \mathbb{N}$, let $P(n)$ be the statement

$$5^n - 1 \text{ is divisible by 4.} \tag{46}$$

Since

$$5^1 - 1 = 4 \iff P(1) \text{ is true.} \tag{47}$$

Now, assume that $P(k)$ is true for some $k \in \mathbb{N}$ i.e. that there exists $l \in \mathbb{N}$ such that

$$5^k - 1 = 4l. \tag{48}$$

Then, we have

$$5^{k+1} - 1 = 5^k.5 - 1 = 5(5^k - 1) + (5 - 1) = 5.l.4 + 4 = 4(5l + 1)$$

for some $l \in \mathbb{N}$, and hence,

$$P(k) \text{ is true } \implies P(k+1) \text{ is true for } k \in \mathbb{N}. \tag{49}$$

Via (47), (49) and the PMI, it follows that $P(n)$ is true for all $n \in \mathbb{N}$, as required. ■

(C) Question 8. Consider the following argument:
"**Theorem:** $n^2 + n$ is odd for all $n \in \mathbb{N}$.
Proof: Let $P(n)$ be the statement $n^2 + n$ is odd for each $n \in \mathbb{N}$. Certainly it is true that $n = 1$ is odd. Now, suppose that $P(k)$ is true, i.e. $k^2 + k$ is odd. Then,

$$\begin{aligned}(k+1)^2 + (k+1) &= (k^2 + 2k + 1) + (k+1)\\ &= k^2 + k + (2k+2)\\ &= \text{odd } + \text{ even}\\ &= \text{odd.}\end{aligned}$$

Therefore, by the principle of mathematical induction,

$$n^2 + n \text{ is odd for all } n \in \mathbb{N}.''$$

State whether or not the above proof is correct, and justify your answer.

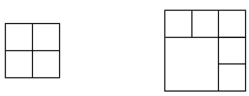
Solution: The proof is not correct. To see this, let n be any odd natural number. Then since n^2 is odd and n is odd, it follows that $n^2 + n$ is even. The mistake in the proof is in the *base step*, specifically, the statement "Certainly it is true that $n = 1$ is odd" although true, is not a suitable base step. ■

(C) Question 9. Let $P(n)$ for $n \geq 3$ be the statement, 'The sum of the internal angles of a convex n-sided polygon is $(n-2)\pi$ radians". Explain in one sentence why $P(3)$ is true, and then use mathematical induction to establish that $P(n)$ is true for all $n \geq 3$.

Solution: For $n = 3$ the result follows from the known result for triangles, namely 'the sum of the internal angles of a triangle is π radians'. Now suppose the statement is true for $n = k$ for some $k \in \mathbb{N}$ with $k \geq 3$. Then, for any convex $(k+1)$-sided polygon, for any 3 adjacent vertices, say v_1, v_2 and v_3, we can remove the triangle $\triangle v_1 v_2 v_3$ and define a k-sided convex polygon from all vertices of the (k+1)-sided polygon, except v_2. Since the sum of the internal angles of the original $(k+1)$-sided polygon is equal to the sum of the internal angles of the triangle $\triangle v_1 v_2 v_3$ (which is π) and the sum of the internal angles of the remaining k-sided polygon (which via the induction assumption is $(k-2)\pi$) it follows that the sum of the internal angles of the $(k+1)$-sided polygon is $((k+1)-2)\pi$, and hence $P(k)$ is true implies $P(k+1)$ is true. From the PMI, it follows that $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq 3$ as required.

■

(C) Question 10. A natural number $n \in \mathbb{N}$ is *fulfilling* if it is possible to fill a square with n sub-squares (not necessarily of the same size). For instance, 4 and 6 are fulfilling since:



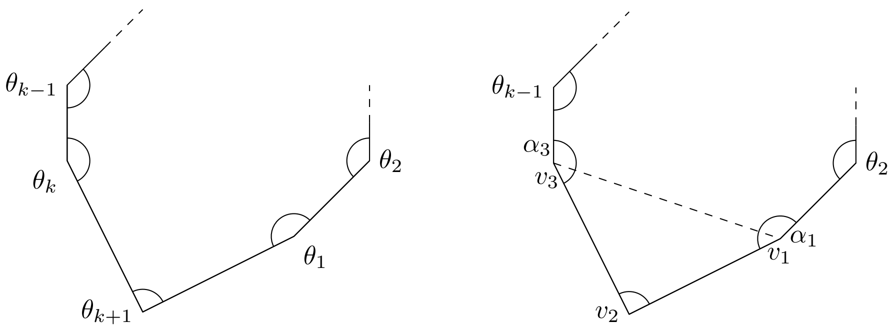


Figure 1: The idea of the inductive step of the proof in question 9 is highlighted in the diagrams above. Namely, we split the $k+1$ sided polygon into 2 polygons with 3 and k sides respectively. Moreover, since $\alpha_1 + v_1 = \theta_1$, $v_2 = \theta_{k+1} =$ and $\alpha_3 + v_3 = \theta_k$, it follows that we can use $P(3)$ is true on the triangle and the induction assumption $P(k)$ is true on the remaining polygon.

(a) Show that 7 and 8 are fulfilling.

(b) Show that if k is fulfilling, then so is $k + 3$.

(c) Hence, prove that that all natural numbers greater than 5 are fulfilling.

(d) State whether or not 1, 2, 3, 4 and 5 are fulfilling, and hence describe all fulfilling numbers.

Solution: An abridged solution (in comparison to the previous solutions) is given below:

(a) 7 and 8 are demonstrated to be fulfilling in Figure 2.

(b) If k is fulfilling, then by replacing any sub-square with 4 sub-squares illustrates that $k+3$ is fulfilling.

(c) Consider the statement $P(n)$ given by:

$$n,\ n+1,\ n+2 \text{ are fulfilling.} \tag{50}$$

First, observe via (a) and Figure 2 that $P(6)$ is true. i.e. the base step is true for $m = 6$. Moreover, if we assume that $P(k)$ is true, then via (b), it follows that k, $k+1$, $k+2$ and $k+3$ are fulfilling. Hence $P(k)$ is true implies $P(k+1)$ is true for any $k \in \mathbb{N}$. Therefore, via the PMI, it follows that $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq 6$, as required.

(d) 1 is fulfilling since any square can be filled with 1 sub-square ... itself (we take a sub-square to be a subset of the unit square that is also a square ... but note that an alternative description of a sub-square may exclude this). If there are more than 1 sub-squares, then it is necessary to have at least 4 distinct sub-squares, so that there is a distinct sub-square in each of the corners of the unit square. Therefore, 2 and 3 are not fulfilling. To illustrate that 5 is not fulfilling (I don't feel comfortable calling this a proof), first consider the 4 sub-squares in the corners. Observe that all 4 sub-squares cannot have side length $\frac{1}{2}$ (otherwise there would only be 4 sub-squares and not 5, as required for a counter-example). Moreover, at least one of these corner sub-squares has side length $x > \frac{1}{2}$, otherwise these 4 sub-squares, when removed from the unit square, will leave a non-square region which cannot be filled by a further 1 sub-square. Consequently, the side lengths of the remaining 3 corner squares are less than or equal to $1-x$. Hence, it follows that if we remove the 4 corner sub-squares from the unit square, the remaining region will have edges on at least 2 distinct sides of the unit square since $2(1-x) < 1$ but include none on the corners of the unit square. Again, this remaining region is a not square and hence, cannot be filled in by 1 sub-square. Therefore, 5 is not fulfilling.

■

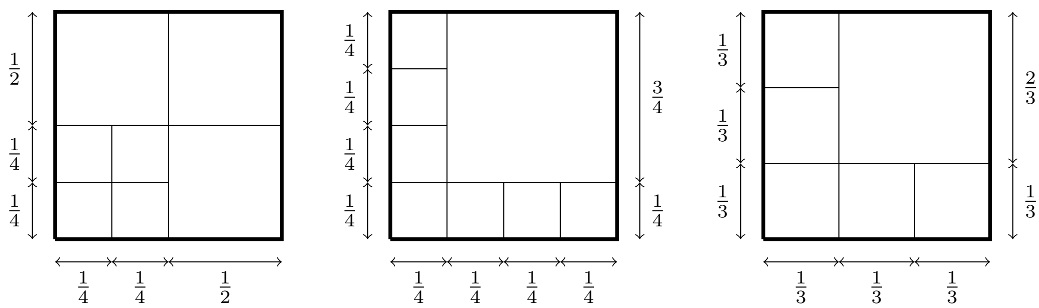


Figure 2: 'Proof by Tikz' that 6, 7 and 8 are fulfilling. To clarify, Tikz is the LaTeX package used to draw these diagrams.