

2DE/2DE3 Example sheet 5: Partial Differential Equations

1. Find the separable solution to the heat equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad (1)$$

subject to the boundary conditions

$$u(0, t) = 0, \quad u(\pi, t) = 0 \quad (2)$$

and the initial condition

$$u(x, 0) = \sin(x) - 2\sin(2x) + 7\sin(10x) \quad (3)$$

when $\alpha = 2$. By substituting your solution back into the original equation, check that it is indeed a solution and that it satisfies the boundary conditions.

2. Write down the PDE, boundary conditions and initial condition representing the temperature of a straight, thin wire of length 1m, initially at 10°C across the wire, except at one end which is suddenly heated to 17°C. Assume that the wire is insulated across the length of the wire and the temperature at the ends is held fixed. (You do not need to solve the resulting model).
3. What are the PDE, boundary conditions and initial condition representing the temperature of a straight, thin wire of length 1m, with both ends held at 10°C, the mid-point 5°C initially, and the initial distribution of heat across the wire given by a quadratic function? Assume that the wire is insulated across the length of the wire and the temperature at the ends is held fixed.(You do not need to solve the resulting model).
4. Find the separable (non-trivial) solution to

$$\begin{aligned} 8 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, & 0 < x < \pi, \\ u(0, t) &= u(\pi, t) = 0, & t > 0, \\ u(x, 0) &= x^2, & 0 < x < \pi. \end{aligned}$$

5. Find the separable solution to the previous question using the boundary conditions:

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0,$$

instead.

6. Find the separable solution to the following wave problem:

$$\begin{aligned} 2 \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2}, & 0 < x < \pi, \quad t > 0, \\ u(0, t) &= 0, \quad u(\pi, t) = 0, & t > 0, \\ u(x, 0) &= x, & 0 < x < \pi, \\ \frac{\partial u}{\partial t}(x, 0) &= 0, & 0 < x < L. \end{aligned}$$

7. Find the separable solution to the following wave problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2}, & 0 < x < 1, \quad t > 0, \\ u(0, t) &= 0, \quad u(1, t) = 0, & t > 0, \\ u(x, 0) &= x(1 - x), & 0 < x < 1, \\ \frac{\partial u}{\partial t}(x, 0) &= \sin(7\pi x), & 0 < x < 1. \end{aligned}$$

8. Find the separable solution to Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0, y) &= 0, & u_x(a, y) &= f(y), & 0 < y < b, \\ u_y(x, 0) &= 0, & u_y(x, b) &= 0, & 0 < x < a. \end{aligned}$$

Derive expressions for the coefficients and show that

$$\int_0^b f(y) dy = 0$$

is a necessary condition for the problem to be solvable.

9. Find the separable solution to Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

subject to the mixed boundary conditions

$$\begin{aligned} u(0, y) &= 0, & u(a, y) &= 0, & 0 < y < b, \\ u_y(x, 0) &= 0, & u_y(x, b) &= kx(a - x), & 0 < x < a. \end{aligned}$$