Example sheet 3 – formative

1. Consider the second order linear differential equation

$$\ddot{x} + 2\dot{x} - 3x = 0.$$

   (a) Recast the problem as a system of first order differential equations

   (b) Write the system in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Show that the eigenvalues of $\mathbf{A}$ are given by $\lambda_1 = 1$, $\lambda_2 = -3$, and find the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$

   (c) Use your answer to part (b) to classify the equilibrium point $(0, 0)$.

2. Consider the second order differential equation

$$\ddot{x} - \dot{x} - 2x = 0, \quad x \in \mathbb{R}.$$

   (a) Write the above ODE as a system of first order differential equations.

   (b) Determine the location and nature of the equilibrium points of the resulting system.

   (c) Determine the horizontal and vertical isoclines and the direction of the flow along them.

   (d) Hence sketch the phase plane of the system, indicating all qualitatively different solutions.

3. Sketch the phase portraits of the following linear dynamical systems. This should include details of the equilibria, their type, eigenvalues and eigenvectors and horizontal and vertical isoclines. In each case state how all qualitatively different solutions $(x(t), y(t))$ behave as $t \to \infty$ for varying initial conditions (x_0, y_0).

   (a)

$$\begin{aligned} \dot{x} &= -x - 3y, \\ \dot{y} &= -x + y. \end{aligned}$$

   (b)

$$\begin{aligned} \dot{x} &= 2x - y, \\ \dot{y} &= -x + 2y. \end{aligned}$$

   (c)

$$\begin{aligned} \dot{x} &= -x, \\ \dot{y} &= -3y. \end{aligned}$$

   (d)

$$\begin{aligned} \dot{x} &= -3x, \\ \dot{y} &= -3y. \end{aligned}$$