University of Birmingham
School of Mathematics
Vectors, Geometry and Linear Algebra
VGLA

Problem Sheet 4

Model Solutions

Remember that there are practise questions under the materials section for each week.

$\boxed{\text{SUM}}$ **Q1**. (i) Let $V = \mathbb{R}^n$ with $n \geq 3$ be a real vector space. Which of the following subsets of V are subspaces of V? In each case prove your assertion.

(a) $A = \{(x_1, x_2, x_3, \ldots, x_n) \mid \alpha x_1 + \beta x_2 + \gamma x_3 = 0\}$ where α, β, γ are fixed elements of $\mathbb{R}$;
(b) $B = \{(x_1, x_2, x_3, \ldots, x_n) \mid 3x_n + 4x_{n-1} + x_{n-2} = 1\}$;
(c) $C = \{(x_1, x_2, x_3, \ldots, x_n) \mid \sum_{i=1}^{n} (i^i)x_i = 0\}$;
(d) $D = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_n - x_{n-1} = x_{n-1} - x_{n-2}\}$;
(e) $E = \{(x_1, x_2, x_3, \ldots, x_n) \mid \prod_{i=1}^{n} ix_i = 0\}$.

(ii) Suppose that $V = \mathbb{C}^3$. Determine whether

$$W = \{(z_1, z_2, z_3) \in V \mid \sum_{i=1}^{3} \mathrm{Im}(z_i) = 0\}$$

is a subspace of V.

(iii) Suppose that A, B, C are subspaces of a vector space V. Set

$$W = (A \cap (B + C)) \cap (B \cap (A + C)) \cap (C \cap (B + A)).$$

Show that W is a subspace of V. Is $W = A \cap B \cap C$? Either give a counterexample which shows that they are not equal, or prove that they are equal.

Solution. (i) Remember that to prove a subset W of V is a subspace, we need to prove that it is non-empty and that for all $\mathbf{v}, \mathbf{w} \in V$ and all $\lambda \in \mathbb{R}$, $\mathbf{v} + \lambda\mathbf{w} \in W$. To prove that a non-empty subset W is not a subspace of V, if suffices to show that there exist $\mathbf{v}, \mathbf{w} \in W$ and $\lambda \in \mathbb{R}$ such that $\mathbf{v} + \lambda\mathbf{w} \notin W$. Remember also that a subspace must contain the zero vector $\mathbf{0}$.

(a) We have A is a subspace. Obviously $\mathbf{0} = (0, 0, 0, \ldots, 0) \in A$ and so A is not empty. Suppose that $\mathbf{v}, \mathbf{w} \in A$ and $\lambda \in \mathbb{R}$. Then we can write $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ where $\alpha v_1 + \beta v_2 + \gamma v_3 = 0$, $\mathbf{w} = (w_1, w_2, w_3, \ldots, w_n)$ where $\alpha w_1 + \beta w_2 + \gamma w_3 = 0$ where $v_i, w_i \in \mathbb{R}$ for $1 \leq i \leq n$. Now

$$\mathbf{v} + \lambda\mathbf{w} = (v_1 + \lambda w_1, v_2 + \lambda w_2, v_3 + \lambda w_3, \ldots, v_n + \lambda w_n).$$

Now we check that $\alpha(v_1 + \lambda w_1) + \beta(v_2 + \lambda w_2) + \gamma(v_3 + \lambda w_3) = 0$. We have

$$\alpha(v_1 + \lambda w_1) + \beta(v_2 + \lambda w_2) + \gamma(v_3 + \lambda w_3) = \alpha v_1 + \beta v_2 + \gamma v_1 + \lambda(\alpha w_1 + \beta w_2 + \gamma w_1).$$

Hence, as $\alpha v_1 + \beta v_2 + \gamma v_3 = 0$ and $\alpha w_1 + \beta w_2 + \gamma w_3 = 0$, we have

$$\alpha(v_1 + \lambda w_1) + \beta(v_2 + \lambda w_2) + \gamma(v_3 + \lambda w_3) = 0$$

and this is true for all $\mathbf{v}, \mathbf{w} \in A$ and $\lambda \in \mathbb{R}$. Hence A is a subspace of V.

(b) B is not a subspace as $\mathbf{0} = (0, 0, 0, \ldots, 0) \notin B$.

(c) C is a subspace. Obviously $\mathbf{0} = (0, 0, 0, \ldots, 0) \in C$ and so C is not empty.
Suppose that $\mathbf{v}, \mathbf{w} \in C$ and $\lambda \in \mathbb{R}$. Then we can write $\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$ where $\sum_{i=1}^{n}(i^i)v_i = 0$, and $\mathbf{w} = (w_1, w_2, w_3, \ldots, w_n)$ where $\sum_{i=1}^{n}(i^i)w_i = 0$, where $v_i, w_i \in \mathbb{R}$ for $1 \leq i \leq n$. Then consider $\mathbf{v} + \lambda\mathbf{w} = (z_1, \ldots, z_n)$ where $z_k = k^k v_k + \lambda k^k w_k$ for $1 \leq k \leq n$. Hence

$$\sum_{i=1}^{n} z_i = \sum_{i=1}^{n}(i^i v_i + \lambda i^i w_i) = (\sum_{i=1}^{n} i^i v_i) + \lambda(\sum_{i=1}^{n} i^i w_i) = 0 + \lambda 0 = 0.$$

(d) D is a subspace. First note that $\mathbf{0} \in D$, so D is non-empty. Suppose that $\mathbf{v}, \mathbf{w} \in D$ and $\lambda \in \mathbb{R}$. Then we can write

$$\mathbf{v} = (v_1, v_2, v_3, \ldots, v_n)$$

with $v_n - 2v_{n-1} + v_{n-2} = 0$ and

$$\mathbf{w} = (w_1, w_2, w_3, \ldots, w_n)$$

with $w_n - 2w_{n-1} + w_{n-2} = 0$ where $v_i, w_i \in \mathbb{R}$ for $1 \leq i \leq n$.

$$\mathbf{v} + \lambda\mathbf{w} = (v_1 + \lambda w_1, v_2 + \lambda w_2, v_3 + \lambda w_3, \ldots, v_n + \lambda w_n).$$

Now

$$v_n + \lambda w_n - 2(v_{n-1} + \lambda w_{n-1}) + v_{n-2} + \lambda w_{n-2} = v_{n-1} - 2v_{n-1} + v_{n-2} + \lambda(w_n - 2w_{n-2} + w_{n-2}) = 0.$$

Hence $\mathbf{v} + \lambda\mathbf{w} \in D$ for all $\mathbf{v}, \mathbf{w} \in D$ and $\lambda \in \mathbb{R}$. Therefore D is a subspace.

(e) This is not a subspace. Take $\mathbf{v} = (1, 1, \ldots, 1, 0)$ and $\mathbf{w} = (0, \ldots, 0, 1)$. Then $\mathbf{v}$ and $\mathbf{w} \in V$ but $\mathbf{v} + \mathbf{w} = (1, 1, \ldots, 1) \notin V$.

(ii) We have $(1, 0, 0) \in W$ and $i(1, 0, 0) \notin W$. Hence W is not a subspace.

(iii) It suffice to show that $W = A \cap B \cap C$ as the intersection of subspaces is a subspace. Since $(A \cap (B + C)) \subseteq A$, $(B \cap (A + C)) \subseteq B$ and $(C \cap (B + A)) \subseteq C$, we have $W \subseteq A \cap B \cap C$.

Conversely, we have $B + C \supseteq B \cap C$. Hence $A \cap (B + C) \supseteq A \cap (B \cap C) = A \cap B \cap C$. Similarly, $(B \cap (A + C)) \supseteq A \cap B \cap C$ and $(C \cap (B + A)) \supseteq A \cap B \cap C$. Hence $W \supseteq A \cap B \cap C$. Therefore $A \cap B \cap C \supseteq W \supseteq A \cap B \cap C$ and we conclude that $W = A \cap B \cap C$.

□

SUM **Q2**. (i) Determine the quadratic equation satisfied by the points $z = x + iy$ on the Argand diagram which satisfy the following equation

$$||z - 2i| - |z + 2i|| = 2.$$

(ii) Consider the ellipse given by the equation

$$5x^2 + 5y^2 + 6xy = 8.$$

This is obtained from the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $0 < b < a$ by rotating through some angle α. Find

(a) the angle of rotation α;
(b) the coordinates of the foci of the rotated ellipse;
(c) the length of the major and minor axes.

Solution. (i) Let $z = x + iy$. Then $|z - 2i| - |z + 2i| = 2$ if and only if

$$|z - 2i| = 2 + |z + 2i|$$

gives by squaring

$$x^2 + (y - 2)^2 = 4 + 4|z + 2i| + x^2 + (y + 2)^2$$

if and only if

$$-8y - 4 = 4|z + 2i|$$

if and only if

$$-2y - 1 = |z + 2i|.$$

Square both sides to obtain

$$(2y + 1)^2 = |z + 2i|^2 = (x^2 + (y + 2)^2)$$

which is if and only if

$$3y^2 - x^2 - 3 = .0$$

Hence the locus is a hyperbola which in standard form is

$$y^2 - \frac{1}{3}x^2 = 1.$$

As an alternative approach, the described set of points is those with difference of the distances from $2i$ and $-2i$ equal to 2. This is the definition of a hyperbola. This hyperbola is in standard form as the centre is 0 and its major axis in on the imaginary axis. Here the difference of the distances is $2a = 2$ and so $a = 1$ and $c = 2$. Thus $b^2 = c^2 - a^2 = 4 - 1 = 3$ so that $b = \sqrt{3}$. Now the hyperbola is in standard form and so we obtain

$$y^2 - \frac{x^2}{3} = 1.$$

(ii) We are told that the ellipse has been rotated. If the major axis has angle α to the major axis of the ellipse before it was rotated, then we know that it has equation where we may assume that $b < a$.

$$\left(\frac{\cos^2(\alpha)}{a^2} + \frac{\sin^2(\alpha)}{b^2}\right)x^2 + \left(\frac{\sin^2(\alpha)}{a^2} + \frac{\cos^2(\alpha)}{b^2}\right)y^2 + 2\cos(\alpha)\sin(\alpha)\left(\frac{1}{a^2} - \frac{1}{b^2}\right)xy = 1.$$

Hence equating coefficients we have the three equations

$$\left(\frac{\cos^2(\alpha)}{a^2} + \frac{\sin^2(\alpha)}{b^2}\right) = \tfrac{5}{8} \tag{1}$$

$$\left(\frac{\sin^2(\alpha)}{a^2} + \frac{\cos^2(\alpha)}{b^2}\right) = \tfrac{5}{8} \tag{2}$$

and

$$2\cos(\alpha)\sin(\alpha)(\frac{1}{a^2} - \frac{1}{b^2}) = \tfrac{6}{8} \tag{3}$$

Subtracting equation (2) from (1) we get

$$\cos(2\alpha)(\frac{1}{a^2} - \frac{1}{b^2}) = 0. \tag{4}$$

Hence as equation (3) shows that $(\frac{1}{a^2} - \frac{1}{b^2}) \neq 0$ we see that $\cos(2\alpha) = 0$ which means that $\alpha = \pi/4$ or $3\pi/4$ as we may suppose the rotation is between 0 and π.

Now we know that $\sin(2\alpha) = 2\cos(\alpha)\sin(\alpha) = \pm 1$ as so $b < a$, we must have $\sin(2\alpha) = 1$ so that $\alpha = 3\pi/4$. Hence equation (3) gives

$$\frac{1}{a^2} - \frac{1}{b^2} = -\frac{6}{8} = -\frac{3}{4}.$$

whereas equation (1) gives

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{5}{4}.$$

It follows that

$$\frac{1}{a^2} = \frac{1}{4}$$

so that $a = 2$ and then $b = 1$. Hence

$$5x^2 + 5y^2 + 6xy = 8$$

is the ellipse described by the rotation of the ellipse

$$\frac{1}{4}x^2 + y^2 = 1$$

through angle $3\pi/4$. Hence
(a) the angle of rotation is $3\pi/4$;
(b) the foci of the unrotated ellipse are at $(\pm\sqrt{3}, 0)$. Hence the foci of the the ellipse in question are at $(\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}})$ and $(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}})$.
(c) The major axis has length 4 and the minor axis has length 2.

$\square$

Q3. Suppose that V is a vector space over $\mathbb{R}$ of finite dimension $n \geq 1$. Assume that $U_1, \ldots, U_k$ is a finite collection of subspaces of V with $\dim U_j \leq n-1$ for $1 \leq j \leq k$. Show that

$$\bigcup_{i=1}^{k} U_i \neq V.$$

Sketch:

(i) Use induction on $\dim V$. What is the inductive hypothesis?
(ii) Why is the result true when $n = 1$?
(iii) Assume that $n \geq 2$. Show that there are an infinite number of subspaces of V of dimension $n-1$. You could do this by fixing a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and defining subspaces

$$V_\lambda = \begin{cases} \langle \lambda\mathbf{v}_1 + \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle & n \geq 3 \\ \langle \lambda\mathbf{v}_1 + \mathbf{v}_2 \rangle & n = 2. \end{cases}$$

Show that for $\lambda_1, \lambda_2 \in \mathbb{R}$, $V_{\lambda_1} = V_{\lambda_2}$ if and only if $\lambda_1 = \lambda_2$.
(iv) Using (iii), let W be a subspace of dimension $n-1$ with $W \notin \{U_1, \ldots U_k\}$.
(v) Show that for each $1 \leq j \leq k$, $W \cap U_j$ is a subspace of W of dimension at most $n-2$.
(vi) Suppose that $V = \bigcup_{i=1}^{k} U_i$. Show that $W = \bigcup_{i=1}^{k} (W \cap U_i)$, apply the inductive hypothesis and conclude the proof.

Is the same true if the vector space is over a finite field and finite dimensional? Either prove it, or explain why the result is not true.

Solution. Let $P(m)$ be the statement that "an m-dimensional real vector space is not the union of a finite number of subspaces of dimension less than m." If V has dimension 1, then all the subspaces of dimension less than 1 have dimension 0. This means that they are all the subspace $\{0\}$. Hence, if $U_1, \ldots, U_k$ are such subspaces, then $\bigcup_{i=1}^{k} U_i = \{0\} \neq V$. Hence $P(1)$ is true.

Assume that $P(n-1)$ is true. Let V be a vector space over $\mathbb{R}$ of dimension $n \geq 2$. Assume that $U_1, \ldots, U_k$ is a finite collection of subspaces of V with $\dim U_j \leq n-1$ for $1 \leq j \leq k$.

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V.

Let's prove the claim in (c). Suppose that $W = U_\mu = U_\lambda$ with $\lambda \neq 0$. Then $\lambda_1\mathbf{v}_1 + \mathbf{v}_2 - (\mu\mathbf{v}_1 + \mathbf{v}_2) = (\lambda - \mu)\mathbf{v}_1 \in W$. Multiplying this by $(\lambda - \mu)^{-1}$ then gives $\mathbf{v}_1 \in W$. Hence W contains $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ which is a linearly independent set of size n. This contradicts $\dim W = n-1$. Hence $U_\lambda \neq U_\mu$. It follows that there are an infinite number of distinct subspaces of V of dimension $n-1$.

Since $\{U_1, \ldots, U_k\}$ is finite, we can select $\tau \in R$ such that $W = U_\tau \notin \{U_1, \ldots, U_k\}$.

Since the intersection of subspaces of V is again a subspace, we have

$$\{U_i \cap W \mid 1 \leq i \leq k\}$$

is a finite set of subspaces of W.

Assume that for some i with $1 \leq i \leq k$, $\dim(W \cap U_i) \geq n-1$. Then, as $U_i \cap W$ is a subspace of W and a subspace of U_i both of which have dimension at most $n-1$, we have $W = W \cap U_i = U_i$. This contradicts the choice of W. Hence $\dim(W \cap U_i) \leq n-2$.

Assume that $V = \bigcup_{i=1}^{k} U_i$. Then

$$W = W \cap V = W \cap \bigcup_{i=1}^{k} U_i = \bigcup_{i=1}^{k} (W \cap U_i).$$

Since $P(n-1)$ is true, this is impossible. We conclude that $V \neq \bigcup_{i=1}^{k} U_i$. Hence $P(n)$ hold. But then $P(m)$ is true for all natural number m. $\square$