Example sheet 5 – formative

1. Consider the nonlinear dynamical system

$$\dot{x} = x - 2y,$$
$$\dot{y} = 4x - x^3.$$

(a) Determine the equilibrium points of the system.

(b) By constructing the Jacobian matrix, classify the equilibrium points found in part (a).

(c) Sketch the phase portrait of the system. Note the location of the isoclines as broken red lines on the phase portrait.

Solution:

(a) The equilibrium points are given by $\dot{x} = \dot{y} = 0$. We have that

$$\dot{y} = 0 \Rightarrow x = 0, \text{ or } 4 - x^2 = 0.$$

We then have that the equilibrium points are given by $(x, y) = (0, 0)$, $(x, y) = (2, 1)$, $(x, y) = (-2, -1)$.

(b) The Jacobian matrix is given by

$$\mathbf{J} = \begin{pmatrix} 1 & -2, \\ 4 - 3x^2 & 0 \end{pmatrix}.$$

We now calculate the eigenvalues of the Jacobian matrix at each of the equilibrium points.

$$\begin{aligned} \mathbf{J}(0,0) &= \begin{pmatrix} 1 & -2 \\ 4 & 0 \end{pmatrix}, \\ \Rightarrow \begin{vmatrix} 1 - \lambda & -2, \\ 4 & -\lambda \end{vmatrix} &= 0, \\ \Rightarrow \lambda^2 - \lambda + 8 &= 0, \\ \Rightarrow \lambda &= \frac{1}{2} \pm \frac{1}{2} i\sqrt{31}. \end{aligned}$$

The eigenvalues are complex conjugate pairs with positive real part, so the equilibrium point $(0, 0)$ is an unstable spiral.

$$\mathbf{J}(2,1) = \begin{pmatrix} 1 & -2 \\ -8 & 0 \end{pmatrix},$$
$$\Rightarrow \lambda^2 - \lambda - 16 = 0,$$
$$\Rightarrow \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{65}.$$

The eigenvalues are real, distinct, and have opposite sign, so the equilibrium point $(2,1)$ is a saddle point.

$$\mathbf{J}(-2,-1) = \begin{pmatrix} 1 & -2 \\ -8 & 0 \end{pmatrix},$$
$$\Rightarrow \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{65}.$$

The eigenvalues are real, distinct, and have opposite sign, so the equilibrium point $(-2,-1)$ is a saddle point.

(c) At the saddle nodes, the eigenvectors corresponding to the eigenvalues are given by,

$$\begin{pmatrix} 1-\lambda_1 & -2 \\ -8 & -\lambda_1 \end{pmatrix} \mathbf{v}_1 = \mathbf{0},$$
$$\Rightarrow \begin{pmatrix} \frac{1}{2} - \frac{1}{2}\sqrt{65} & -2 \\ -8 & -\frac{1}{2} - \frac{1}{2}\sqrt{65} \end{pmatrix} \mathbf{v_1} = \mathbf{0}.$$

Thus,

$$\mathbf{v}_1 = \left(1, \frac{1}{4} - \frac{1}{4}\sqrt{65}\right)^{\mathsf{T}}.$$

Similarly,

$$\begin{pmatrix} 1-\lambda_2 & -2 \\ -8 & -\lambda_2 \end{pmatrix} \mathbf{v}_2 = \mathbf{0},$$
$$\Rightarrow \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\sqrt{65} & -2 \\ -8 & -\frac{1}{2} + \frac{1}{2}\sqrt{65} \end{pmatrix} \mathbf{v_2} = \mathbf{0}.$$

Thus,

$$\mathbf{v}_2 = \left(1, \frac{1}{4} + \frac{1}{4}\sqrt{65}\right)^{\mathsf{T}}.$$

(d) For the horizontal isoclines we have

$$\dot{y} = 0 \Rightarrow x = 0, \text{ or } 4 - x^2 = 0.$$

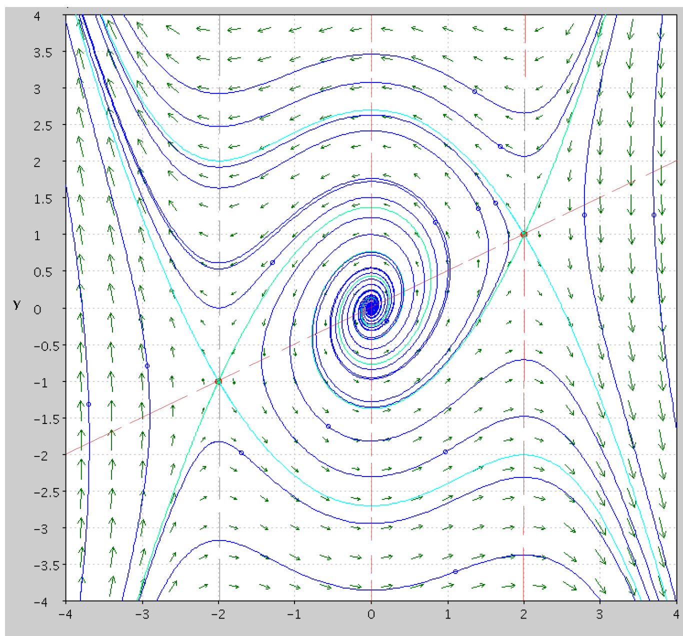
So we have horizontal isoclines at $x = -2$, $x = 0$ and $x = 2$.

1. For $x = -2$ we have that $\dot{x} = -2(1 + y)$ so flow is to the right for $y < -1$ and to the left for $y > -1$.
2. For $x = 0$ we have that $\dot{x} = -2y$ so flow is to the right for $y < 0$ and to the left for $y > 0$.
3. For $x = 2$ we have that $\dot{x} = 2(1 - y)$ so flow is to the right for $y < 1$ and to the left for $y > 1$.

For the vertical isocline, $\dot{x} = m0 \Rightarrow y = \frac{x}{2}$, and $\dot{y} = x(4 - x^2)$, so

1. For $x < -2$ flow is upwards;
2. For $-2 < x < 0$ flow is downwards;
3. For $0 < x < 2$ flow is upwards;
4. For $2 < x$ flow is downwards.

We can now plot the phase portrait for the system, as



2. Locate the equilibrium points of the nonlinear dynamical system,

$$\begin{aligned} \dot{x} &= x(x - y - 3), \\ \dot{y} &= y(x - 5). \end{aligned} \tag{1}$$

Classify the equilibrium points and sketch the phase portrait of (1).

Solution: The nonlinear system (1) has three equilibrium points at $(0,0)$, $(3,0)$ and $(5,2)$. We consider each equilibrium in turn:

Consider $(0,0)$. The associated linear system is given by

$$\dot{x} = -3x,$$
$$\dot{y} = -5y,$$

where $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 0 & -5 \end{pmatrix}$ with eigenvalues $\lambda_{\pm} = -3, -5$ indicating that $(0,0)$ is a stable node. The eigenvectors associated with the eigenvalues $\lambda = -3$ and $\lambda = -5$ are given by $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. Therefore, by the linearization theorem $(0,0)$ is a stable node for the nonlinear system.

Consider $(5,2)$. On transforming to the origin via

$$X = x - 5, \quad Y = y - 2,$$

we obtain the associated linear system as

$$\dot{X} = 5X - 5Y,$$
$$\dot{Y} = 2X,$$

where $\mathbf{A} = \begin{pmatrix} 5 & -5 \\ 2 & 0 \end{pmatrix}$ with eigenvalues $\lambda_{\pm} = \frac{5}{2} \pm \frac{1}{2}\sqrt{15}i$ indicating that $(0,0)$ is an unstable spiral. Therefore, by the linearization theorem $(5,2)$ is an unstable spiral for the nonlinear system.

Consider $(3,0)$. On transforming to the origin via

$$X = x - 3, \quad Y = y,$$

we obtain the associated linear system as

$$\dot{X} = 3X - 3Y,$$
$$\dot{Y} = -2Y,$$

where $\mathbf{A} = \begin{pmatrix} 3 & -3 \\ 0 & -2 \end{pmatrix}$ with eigenvalues $\lambda_{\pm} = 3, -2$ indicating that $(0,0)$ is a saddle point. The eigenvectors associated with the eigenvalues $\lambda = 3$ and $\lambda = -2$ are given by $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ respectively. Therefore, the straight line path corresponding to $\lambda = -2$ is given by $Y = \frac{5}{3}X$, which when written in terms of the original variables x, y becomes $y = \frac{3}{5}(x-3)$. Note that this straight line path is only valid 'very' close to the equilibrium point. Therefore, by the linearization theorem $(3,0)$ is a saddle point for the nonlinear system.

Alternatively, the Jacobian

$$\mathbf{J} = \begin{pmatrix} 2x-y-3 & -x, \\ y & x-5 \end{pmatrix}.$$

can be used to obtain the three matrices above.

On returning to the full nonlinear system (1) we examine to location of the horizontal and vertical isoclines, given by

$$\frac{dy}{dx} = \frac{y(x-5)}{x(x-y-3} \quad \begin{cases} 0, & y=0 \text{ or } x=5, \\ \infty, & x=0 \text{ or } y=x-3. \end{cases}$$

We now consider the direction of flow on the horizontal and vertical isoclines. When $x = 5$ the $\dot{x}$ equation becomes

$$\dot{x} = 5(2-y)\begin{cases} >0, & y<2, \\ <0, & y>2. \end{cases}$$

Therefore, when $y > 2$, $\dot{x} < 0$ and x decreases as t increases, while when $y < 2$, $\dot{x} > 0$ and x increases as t increases. Similar analysis can be carried out on the remaining isoclines and the coordinate axes.

When $y = 0$ the $\dot{x}$ equation becomes

$$\dot{x} = x(x-3)\begin{cases} >0, & x<0 \text{ or } x>3, \\ <0, & 0<x<3. \end{cases}$$

Therefore, x decreases as t increases when $0 < x < 3$, and x increases as t increases elsewqhere. Notice that teh line $y = 0$ is a trajectory, as it is horizontal and all flow is horizontal along it.

When $x = 0$ the $\dot{y}$ equation becomes

$$\dot{y} = -5y\begin{cases} >0, & y<0, \\ <0, & y>0. \end{cases}$$

Therefore, the flow is upwards ($\dot{y} > 0$) when $y < 0$ and downwards for $y > 0$. Note that the line $x = 0$ is also a trajectory as it is a vertical line along which the flow is vertical.

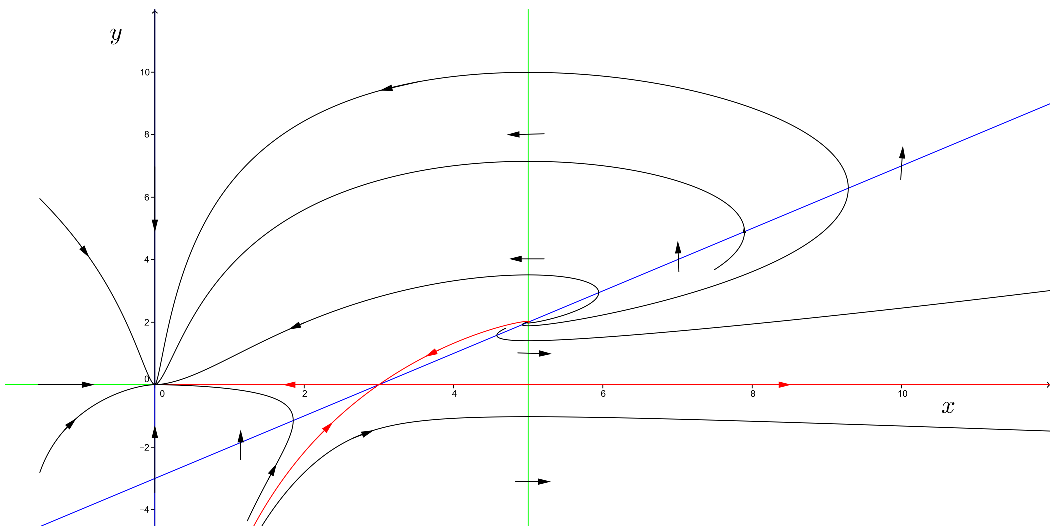
When $y = x - 3$ the $\dot{y}$ equation becomes

$$\dot{y} = (x-3)(x-5) \begin{cases} > 0, & x < 3 \text{ or } x > 5, \\ < 0, & 3 < x < 5. \end{cases}$$

Therefore, the flow is upwards ($\dot{y} > 0$) when $x < 3$ or $x > 5$ and downwards for $3 < x < 5$.

The horizontal and vertical isoclines have been plotted in the phase portrait depicted below. Note that the line coloured blue corresponds to the vertical isocline $y = x - 3$.

Combining all the information we are now in a position to be able to sketch the phase portrait.



3. The damped pendulum equation

$$I\ddot{\theta} + \mu\dot{\theta} + mgl\sin\theta = 0,$$

can be written in the form

$$\ddot{x} + \varepsilon\dot{x} + k^2 \sin x = 0,$$

where $\varepsilon > 0$.

(a) Write the damped pendulum equation as a system of first order differential equations, and establish the location of the equilibrium points.

(b) Calculate the Jacobian matrix for the nonlinear system.

(c) Use the Jacobian matrix to classify the equilibrium points.

(d) In the case $\varepsilon = k = 1$, sketch the phase portrait of the system for $x \in (-4, 4)$. Note the location of the isoclines as dotted lines on the phase portrait.

Solution:

(a) Writing the damped pendulum equation as a system of first order ODEs we get

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -\varepsilon y - k^2 \sin x.\end{aligned}$$

The equilibrium points are given by $y = 0$, with $x = n\pi$ $(n \in \mathbb{Z})$.

(b) The Jacobian matrix is given by

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -k^2 \cos x & -\varepsilon \end{pmatrix}.$$

(c) We have that

$$\begin{aligned}\mathbf{J}(n\pi, 0) &= \begin{pmatrix} 0 & 1 \\ -k^2 \cos(n\pi) & -\varepsilon \end{pmatrix},\\ &= \begin{pmatrix} 0 & 1 \\ (-1)^{n+1} k^2 & -\varepsilon \end{pmatrix}.\end{aligned}$$

The eigenvalues of $\mathbf{J}$ are given by

$$\begin{aligned}&\begin{vmatrix} -\lambda & 1 \\ (-1)^{n+1} k^2 & -\varepsilon - \lambda \end{vmatrix} = 0,\\ &\Rightarrow \lambda^2 + \varepsilon\lambda + (-1)^n k^2 = 0,\\ &\qquad \Rightarrow \lambda = \frac{1}{2}\left(-1 \pm \sqrt{\varepsilon^2 - 4k^2 (-1)^n}\right).\end{aligned}$$

If n is odd, the the eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(-\varepsilon \pm \sqrt{\varepsilon^2 + 4k^2}\right).$$

Thus $\lambda_1 < 0 < \lambda_2$, and the equilibrium point is a saddle node.

If n is even, the eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(-\varepsilon \pm \sqrt{\varepsilon^2 - 4k^2}\right).$$

We then have the following sub-cases:

$$\begin{cases} \varepsilon^2 > 4k^2, & \lambda_1 < \lambda_2 < 0 \Rightarrow \text{stable node},\\ \varepsilon^2 = 4k^2, & \lambda_1 = \lambda_2 < 0 \Rightarrow \text{degenerate stable node},\\ \varepsilon^2 < 4k^2, & \lambda_1,\ \lambda_2 \text{ complex conjugate pairs, with } \mathrm{Re}(\lambda) < 0, \Rightarrow \text{stable focus}. \end{cases}$$

(d) For $x \in (-4, 4)$ we have three equilibrium points, $(x, y) = (-\pi, 0)$, $(0, 0)$, and $(\pi, 0)$.

$\underline{(0,0)}$: We have $\lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Stable spiral.

$\underline{(\pm\pi,0)}$: We have $\lambda_{1,2} = \frac{1}{2}\left(-1 \pm \sqrt{5}\right)$. Saddle nodes.

For the saddle nodes, the eigenvector corresponding to $\lambda_1 = \frac{1}{2}\left(\sqrt{5} - 1\right)$ is $\mathbf{v}_1 = \left(\frac{1}{2}\left(1 + \sqrt{5}\right), 1\right)^T$.

The eigenvector corresponding to $\lambda_2 = -\frac{1}{2}\left(1 + \sqrt{5}\right)$ is $\mathbf{v_2} = \left(\frac{1}{2}\left(1 - \sqrt{5}\right), 1\right)^T$.

Isoclines are given by

$$\frac{dy}{dx} = -\frac{y + \sin x}{y} = \begin{cases} 0, & y = -\sin x, \\ \infty, & y = 0. \end{cases}$$

When $y = -\sin x$ the $\dot{x}$ equation becomes

$$\dot{x} = -\sin x \begin{cases} > 0, & -\pi < x < 0 \text{ or } \pi < x < 4, \\ < 0, & -4 < x\_\pi \text{ or } 0 < x < \pi. \end{cases}$$

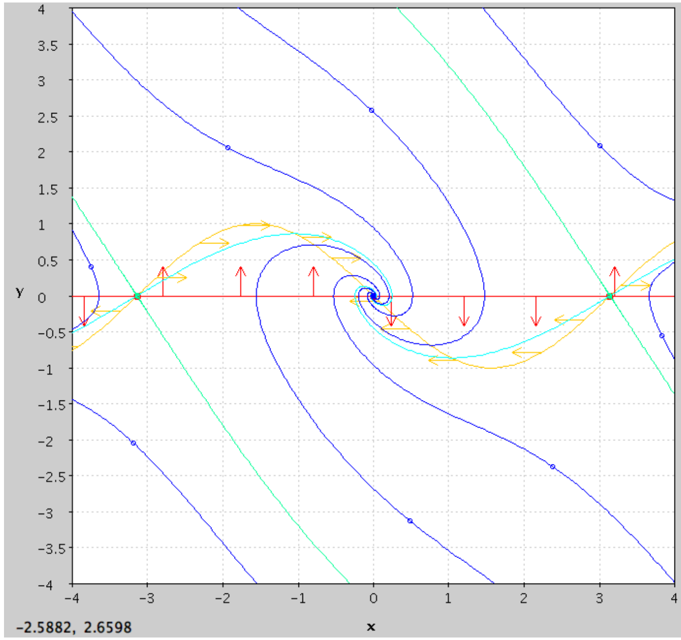
Therefore, the flow is to the left (x decreasing as t increases) when $-4 < x < -\pi$ or $0 < x < \pi$, and to the right elsewhere.

When $y = 0$ the $\dot{y}$ equation becomes

$$\dot{y} = -\sin x \begin{cases} > 0, & -\pi < x < 0 \text{ or } \pi < x < 4, \\ < 0, & -4 < x\_\pi \text{ or } 0 < x < \pi. \end{cases}$$

Therefore, the flow is downwards (y decreasing as t increases) when $-4 < x < -\pi$ or $0 < x < \pi$, and upwards elsewhere.

The phaseportrait can now be drawn:



y
x
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4
-3
-2
-1
0
1
2
3
4
-2.5882, 2.6598