Problem Sheet 1 (Combinatorics part)

Assignment available: Friday 29 September 2023 (Week 1).
Submission deadline: 1700 on Wednesday 11 October 2023 (Week 3).
Required content: All necessary content will be covered in the Week 1 lectures, except for Question 4(a) you need the fact that a set of size n has 2^n subsets, which we will see in Week 2.

About problem sheet questions: The problems for the assessed problem sheets are carefully chosen to help develop your understanding of the course material through the process of solving the problem (with assistance if necessary) and presenting your answer. Many problems are challenging, and you are very much encouraged to discuss the problems with friends, and to seek assistance where you feel it is necessary (both for how to solve the problem, and for how to present an answer clearly and effectively).

I would like to emphasise strongly that **problem sheet questions are different from exam questions**. They are set in the knowledge that you have a lengthy assessment window with many forms of assistance and feedback available to you, and in the expectation that you will use these. Engaging with these problems in the right way will develop your subject understanding and communication skills far more than simply drilling yourself in exam-type questions, which is the point. But do not panic: you will not find your exam paper full of similarly-challenging questions to be solved with limited time and no assistance.

We will discuss exam-type questions later in the course. For the time being the focus is on developing your understanding of this area of mathematics, and your ability to present mathematical arguments, as much as possible.

Question 1. (SUM) What is the greatest number of bishops which can be placed on a chessboard so that no bishop is attacking another? (In other words, what is the greatest number of squares you can choose on an 8×8 grid without having two squares on the same diagonal.) In your answer you should show how this number can be achieved and also prove that a greater number is not possible.

Question 2. (SUM) Using a similar argument to the proof of Theorem 1.2, or otherwise, prove the following useful variant of the Pigeonhole Principle:

> Let $k, \ell_1, \ldots, \ell_k \in \mathbb{N}$. If at least $\ell_1 + \cdots + \ell_k + 1$ pigeons are placed in k pigeonholes numbered $1, \ldots, k$, then there exists some $i \leq k$ for which there are at least $\ell_i + 1$ pigeons in pigeonhole i.

Question 3. Prove that some member of the following sequence is divisible by 2023.

$$7, 77, 777, 7777, 77777, 777777, 7777777, 77777777, \ldots$$

Hint: consider the differences $x_i - x_j$ for elements x_i and x_j in the sequence above. It may help to consider the example from the first lecture in which the Pigeonhole Principle was applied to show that in any sequence of r integers there is a consecutive subsequence whose sum is divisible by r.

Question 4. A set X has five elements, each of which is a natural number, and the sum of the elements of X is at most 30.

(a) How many non-empty proper subsets of X are there?

(b) Show that the sum of the members of any non-empty proper subset $A \subsetneq X$ must be between 1 and 29.

(c) Show that there are two non-empty proper subsets of X whose sums are the same.

(d) Show that there are two *disjoint* non-empty proper subsets of X whose sums are the same.

(e) Adapt your arguments to show that the same conclusion holds if X instead has 60 elements, each of which is an natural number with at most 16 digits.

Remark: the final part verifies the claim I made at the very start of the first lecture, since if we colour one of the subsets red and the other blue, then the sum of the red numbers and the sum of the blue numbers are equal.