Examples sheet 1 – Linear Algebra

The exercises below correspond to material from Lectures 1–4. Selected exercises will be covered in the Examples class scheduled in week 3. Solutions will be available on Canvas.

VECTOR SPACES.

1. Identify which of the structures indicated below is a field.

(a) $(\mathbb{N}, +, \cdot)$;
(b) $(\mathbb{Z}, +, \cdot)$;
(c) $(\mathbb{R}, +, \cdot)$;
(d) $(\mathbb{C}, +, \cdot)$;
(e) $(\mathbb{R}, +, -)$.

2. Identify which of the structures indicated below is a vector space.

(a) $(\mathbb{R}, +, \cdot, \mathbb{R})$;
(b) $(\mathbb{C}, +, \cdot, \mathbb{C})$;
(c) $(\mathbb{C}, +, \cdot, \mathbb{R})$;
(d) $(\emptyset, +, \cdot, \mathbb{R})$;
(e) $(\mathbb{E}^3, \times, \cdot, \mathbb{R})$.

3. Let $(V, \boldsymbol{+}, \bullet, \mathbb{F})$ be a vector space. Show that the vector space axioms VA0 and VM0 can be replaced with the requirement that $a \bullet \mathbf{u} \boldsymbol{+} b \bullet \mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$ and for all $a, b \in \mathbb{F}$.

4. Let $(V, \boldsymbol{+}, \bullet, \mathbb{F})$ be a vector space. Use the vector space axioms to prove the following properties (see statements 3 and 4 in Proposition 1.1, L1 - Lecture Notes).

3. For all $\mathbf{u} \in V$, $e^- \bullet \mathbf{u} = \mathbf{u}^-$.
4. For all $a \in \mathbb{F}$, $a \bullet \mathbf{z} = \mathbf{z}$.

Recall that $e^- \in \mathbb{F}$ is the scalar additive inverse of the multiplicative identity and $\mathbf{z} \in V$ is the vector additive identity.

5. Let $V(\mathbb{F}) = (V, \boldsymbol{+}, \bullet, \mathbb{F})$ be an algebraic structure satisfying all axioms except for VA4 (commutativity). Let $\mathbf{v}, \mathbf{w} \in V$ be arbitrary. Assume that any additive inverse satisfies $\mathbf{v} \boldsymbol{+} \mathbf{v}^- = \mathbf{v}^- \boldsymbol{+} \mathbf{v} = \mathbf{z}$.

(a) Show that the additive inverse $\mathbf{v}^-$ is unique. Deduce that $(\mathbf{v}^-)^- = \mathbf{v}$.
(b) Show that $(\mathbf{v} \boldsymbol{+} \mathbf{w})^- = \mathbf{v}^- \boldsymbol{+} \mathbf{w}^-$.
(c) Show that $(\mathbf{v} \boldsymbol{+} \mathbf{w}) \boldsymbol{+} (\mathbf{w} \boldsymbol{+} \mathbf{v})^- = \mathbf{z}$. Deduce that vector addition is commutative.

6. Let $\mathbb{F}$ denote a field and define the set of sequences of elements from $\mathbb{F}$

$$V := \{\mathbf{v} := (v_1, v_2, \ldots, v_k, \ldots) : v_i \in \mathbb{F}\}.$$

Show that $V(\mathbb{F})$ is a vector space when equipped with the entrywise operations of vector addition and scalar-vector multiplication:

$$\mathbf{v} \boldsymbol{+} \mathbf{w} := (v_1 + w_1, v_2 + w_2, \ldots, v_k + w_k, \ldots), \qquad a \bullet \mathbf{v} = (av_1, av_2, \ldots, av_k, \ldots),$$

for $\mathbf{v}, \mathbf{w} \in V$ and $a \in \mathbb{F}$. [This vector space is denoted by $\mathbb{F}^{\mathbb{N}}$ and is called the **sequence space**.]

7. Let $\mathbb{F}$ be a field and consider the structure $V(\mathbb{F}) := (\mathbb{F}^2, +, \bullet, \mathbb{F})$, where the scalar-vector multiplication operation $\bullet$ is given below for generic scalars $a \in \mathbb{F}$ and vectors $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{F}^2$.

(a) $a \bullet \mathbf{v} := \begin{bmatrix} 0 \\ av_1 + av_2 \end{bmatrix}$;

(b) $a \bullet \mathbf{v} := \begin{bmatrix} v_1^a \\ av_2 \end{bmatrix}$;

(c) $a \bullet \mathbf{v} := \begin{bmatrix} \overline{av_1} \\ \overline{av_2} \end{bmatrix}, \quad \mathbb{F} = \mathbb{C}$.

In each case, check the vector space axioms and hence decide if $V(\mathbb{F})$ is a vector space.

(d) Redo part **(b)**, for the same scalar-vector multiplication, but with the vector addition operation defined via

$$\mathbf{v} \boldsymbol{+} \mathbf{u} = \begin{bmatrix} v_1 u_1 \\ v_2 + u_2 \end{bmatrix}.$$

SUBSPACES.

8. Prove Subspace criterion 2:

> Let $V(\mathbb{F})$ be a vector space. A non-empty subset U of V is a subspace of V over $\mathbb{F}$ if and only if for any $\mathbf{u}, \mathbf{v} \in U$ and for any $a, b \in \mathbb{F}$, there holds $a\mathbf{u} + b\mathbf{v} \in U$.

9. Consider the structure $V(\mathbb{R}) := (\mathbb{R}^n, +, \cdot, \mathbb{R})$.

(a) Give an example of a subset of V that is closed under addition but not multiplication.

(b) Give an example of a subset of V that is closed under multiplication but not addition.

[Hint: you may want to consider the case $n = 2$, first.]

10. Let $V(\mathbb{F})$ be a vector space and let $U(\mathbb{F})$ be a subspace: $U(\mathbb{F}) \leq V(\mathbb{F})$. Let W denote the (relative) complement of the set U in V.

True or false: $W(\mathbb{F}) \leq V(\mathbb{F})$.

11. Let U, V, W be subspaces of some vector field and let $U + V = U + W$.

True or false: $V = W$. [If true, prove the statement; if false, give a counter-example].

12. Let $\alpha, \beta \in \mathbb{R}$. Consider the set X of sequences defined recursively via $x_{n+1} = g(x_n)$ for $n \in \mathbb{N}$, where

(a) $g_1(x) = \alpha x + \beta$;

(b) $g_2(x) = (g_1 \circ g_1)(x)$;

(c) $g_3(x) = x(\alpha x + \beta)$.

In each case, check whether X is a subspace of $\mathbb{R}^{\mathbb{N}}$ for some α, β (see definition of $\mathbb{R}^{\mathbb{N}}$ in **Q6**).

13. Consider the field $\mathbb{F} = (\mathbb{Z}_q, \oplus, \odot)$, where $\mathbb{Z}_q = \{1, 2, \ldots, q-1\}$ and $\oplus, \odot$ are the operations of addition and multiplication modulo q, where q is a prime. Let $V(\mathbb{F})$ be the vector space $(\mathbb{Z}_q^n, \boldsymbol{+}, \bullet, \mathbb{F})$, where $\mathbb{Z}_q^n$ is the set of n-tuples with entries in $\mathbb{Z}_q$ and $\boldsymbol{+}, \bullet$ indicate elementwise addition and multiplication by a scalar of vectors in $\mathbb{Z}_q^n$. Let $U \subset V$ be the sets defined below.

(a) $V = \mathbb{Z}_2^3$, $U = \{(v_1, v_2, v_3) : v_2 = 0\}$,

(b) $V = \mathbb{Z}_3^2$, $U = \mathbb{Z}_2^2$.

In each case, establish if $U(\mathbb{F}) < V(\mathbb{F})$.

SPANNING SETS.

14. Let $V(\mathbb{F})$ be a vector space and let $\mathbf{v} \in V$. Show that span $\{\mathbf{v}\}$ is a subspace of V.

15. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of column vectors in $\mathbb{R}^n$. Let $V = \text{span } S$. Show that any $\mathbf{v} \in V$ satisfies

$$\mathbf{v} = A\mathbf{c}, \qquad A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3],$$

where A is a $n \times 3$ matrix with 3 columns given by the vectors in S and with $\mathbf{c} \in \mathbb{R}^3$.

16. Let U be the set of polynomials of degree n divisible by $x^2 + x + 1$.

(a) Is U a subspace of $\mathcal{P}_n(\mathbb{R})$?

(b) Find a spanning set for U when $n = 2$. Do it also for $n = 3$. Are your spans minimal?

LINEAR INDEPENDENCE

17. Establish which of the following sets $S \subset V(\mathbb{F})$ is linearly independent.

(a) $V = \mathbb{R}^n(\mathbb{R})$,

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(b) $V = \mathcal{P}_2(\mathbb{R})$,

$$S = \left\{1 + x - x^2, 2x^2 - 1, x + 3\right\}.$$

18. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set and $S' = \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly independent. True or false:

(a) $\mathbf{v}_1 \in \text{span } S'$.

(b) $\mathbf{v}_4 \in \text{span } S$.

19. Let $c, s \in \mathbb{R} \setminus \{0\}$ and let $\mathbf{i}, \mathbf{j}$ be the usual Cartesian vectors.

(a) Let $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{E}^2$ be given by

$$\begin{cases} \mathbf{e}_1 &= c\mathbf{i} - s\mathbf{j}, \\ \mathbf{e}_2 &= s\mathbf{i} + c\mathbf{j}. \end{cases}$$

Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a linearly independent set.

(b) Redo part **(a)** for the case where $\mathbf{e}_1, \mathbf{e}_2$ are given by

$$\begin{cases} \mathbf{e}_1 &= c\mathbf{i} + s\mathbf{j}, \\ \mathbf{e}_2 &= c\mathbf{i} - s\mathbf{j}. \end{cases}$$

(c) In each case, find a condition on c, s such that $\mathbf{e}_1, \mathbf{e}_2$ are unit vectors. Are they orthogonal?

20. Let U, V be finite sets with $U \subseteq V$.

(a) Show that if U is linearly dependent, then so is V.

(b) Show that if V is linearly independent, then so is U.

21. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a linearly independent set. Establish which of the following sets is linearly independent.

(a) $S_1 = \{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \dots, \mathbf{v}_{k-1} - \mathbf{v}_k, \mathbf{v}_k - \mathbf{v}_1\}$.

(b) $S_2 = \left\{\mathbf{v}_1 - \mathbf{v}_{\pi(1)}, \mathbf{v}_2 - \mathbf{v}_{\pi(2)}, \dots, \mathbf{v}_{k-1} - \mathbf{v}_{\pi(k-1)}, \mathbf{v}_k - \mathbf{v}_{\pi(k)}\right\}$, where π is a permutation of the index set $\{1, 2, \dots, k\}$.

(c) $S_3 = \{\mathbf{v}_1 - 2\mathbf{v}_2, \mathbf{v}_2 - 2\mathbf{v}_3, \dots, \mathbf{v}_{k-1} - 2\mathbf{v}_k, \mathbf{v}_k - 2\mathbf{v}_1\}$.
[Hint: you may want to consider small values of k, e.g., $k = 3$, before providing a general proof.]

(d) $S_4 = \{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \dots, \mathbf{v}_{k-1} - \mathbf{v}_k, \mathbf{v}_k\}$.

22. Let S be sets of vectors as given below. In each case construct a maximal linearly independent set.

(a) $S = \{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

(b) $S = \{\mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^4$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

23. Let $V = \mathcal{P}_2(\mathbb{R})$ and let $x_1 = -1, x_2 = 0, x_3 = 1$. Consider the set

$$S = \{\ell_i \in V : \ell_i(x_j) = \delta_{ij}, 1 \leq i, j \leq 3\}.$$

(a) Find ℓ_i for $i = 1, 2, 3$.

(b) Show that S is a linearly independent set.

(c) Show that S is a maximal linearly independent set in V.

(d) Show that S is a spanning set for V.

(e) Check that $\ell_1(x) + \ell_2(x) + \ell_3(x) = 1$. Deduce that S is a minimal spanning set for V.

The above results indicate that S is a basis set for $\mathcal{P}_2(\mathbb{R})$.

(f) Find the coordinates of $p(x) = 1 + 2x + 3x^2$ with respect to S.
[Hint: you may want to use the 'delta property' of the basis elements: $\ell_i(x_j) = \delta_{ij}$, for $j = 1, 2, 3$.]

(g) How would you generalise this approach in order to construct a basis set for $\mathcal{P}_3$?

24. Consider the following spanning sets S for the indicated vector spaces V. In each case, construct a minimal spanning set.

(a) $V = \mathbb{R}^2$,

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}.$$

(b) $V = \mathcal{P}_2(\mathbb{R})$,

$$S = \left\{1 + x + x^2, x - 2, x^2 - 1, 1 - 2x + x^2\right\}.$$

BASES. COORDINATES

25. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(a) Choose an element of S and write it as a linear combination of the other three.

(b) Check that any three elements of S form a basis for $\mathbb{R}^3$.

26. Consider the following sets in $\mathbb{C}^2$.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1+i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1-i \end{bmatrix} \right\}, \quad S_3 = \left\{ \begin{bmatrix} i \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},$$

$$S_4 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad S_5 = \left\{ \begin{bmatrix} 1+i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1-i \end{bmatrix}, \begin{bmatrix} 1-i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1+i \end{bmatrix} \right\}.$$

(a) Which of the above sets is a basis for $\mathbb{C}^2(\mathbb{R})$?

(b) Which of the above sets is a basis for $\mathbb{C}^2(\mathbb{C})$?

27. Let $V = \mathbb{R}^3$ and let $U \subset V$ be given below.

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y + 3z = 0 \right\}.$$

(a) Show that $U < \mathbb{R}^3$.

(b) Find a basis for U.

(c) Hence, find another subspace W of V such that $V = U + W$.

28. Let U be the set of polynomials p of degree $n \geq 2$ satisfying $p(0) = p(1) = 0$. Show that U is a subspace of $\mathcal{P}_n(\mathbb{R})$. Find a basis for U.

29. Let $V(\mathbb{F})$ be a vector space with $\dim V = n$. Show that

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n,$$

for some subspaces U_i of V with $\dim U_i = 1$, for $i = 1, 2, \ldots, n$.

30. Let $B = \{p_1, p_2, p_3\}$, where $p_i(x) = 1 - (i-1)x^{i-1}$ for $i = 1, 2, 3$.

(a) Show that B is a basis for $\mathcal{P}_2(\mathbb{R})$.

(b) Find the coordinates of $p(x) = 1 + 2x + 3x^2$ relative to the basis B.