University of Birmingham
School of Mathematics

2RCA/2RCA3 Real and Complex Analysis

Part A: Real Analysis Semester 2

Problem Sheet 1
Model Solutions

The questions indicated with $\boxed{\text{SUM}}$ below constitute the first of the summative assessments of this module, and will contribute to the overall mark of the module. Please submit your answers to the questions indicated with a $\boxed{\text{SUM}}$ below by the deadline of **17:00, Thursday 1 February 2024**, as a single pdf file into 2RCA/2RCA3 Assignment 1 in the Canvas page of the module.

Please note that it is the student's responsibility to make sure that their submission has been uploaded correctly into Canvas and that the uploaded file contains the submission of their assessment (eg, the uploaded file is not corrupted and contains all the pages of their answers).

Please be also aware that where assessments are submitted late without an extension being granted that has been confirmed by the Wellbeing Officer, the standard University penalty of a 5% will be imposed for each working day that the assignment is late. Any work submitted after five working days passed the deadline of submission, with no extension granted by the Wellbeing Officer/s, will be awarded a 0% mark.

In addition to the $\boxed{\text{SUM}}$-questions, Problem Sheet 1 also contains exercises that will not contribute to your module mark. You are strongly encouraged to attempt these before the relevant Guide Study sessions and/or during the course of the semester. Solutions to all exercises will be provided.

The examples/feedback classes (Guided Study) and the lecturer's office hours should be used to ask about the problem sheets.

Important note: Please, be aware that you are not allowed to use L'Hopital's rule to justify the evaluation of the limits in this problem sheet. We will revisit L'Hopital's rule later in the year.

Q1. For each of the following sets, decide whether the set is open, closed and bounded. Justify your answer.

(i) $\{1\}^c$ (that is, the complement of $\{1\}$ on $\mathbb{R}$)
(ii) $\{\pi, 2\pi\}$
(iii) $\mathbb{Q}$
(iv) $(-\infty, 1) \cup \{4\}$

Solution. (i) $\{1\}^c$ is open, it is not closed and it is not bounded. Indeed, first notice that $\{1\}^c = (-\infty, 1) \cup (1, \infty)$, and since $(-\infty, 1)$ and $(1, \infty)$ are open sets (see in the lectures), then the union of these two intervals is open, hence

$\{1\}^c$ is an open set. Notice that the complement of $\{1\}^c$ is the set $\{1\}$ and this set is not an open set (any open interval centered at 1 has points that do not lie in the set $\{1\}$), thus $\{1\}^c$ is not a closed set. Finally, notice that no matter how big or small an open interval centred at 1 is, there will be points of $\{1\}^c$ outside the interval (that is, for all $R > 0$ $\{1\}^c = (-\infty, 1) \cup (1, \infty) \not\subseteq (-R, R)$), hence $\{1\}^c$ is not a bounded set of real numbers.

(ii) $\{\pi, 2\pi\}$ is not open, it is closed, and it is bounded. Indeed, notice that any open interval centred at π (or 2π) has real numbers that do not belong to the set $\{\pi, 2\pi\}$, thus $\{\pi, 2\pi\}$ is not an open set. Second, notice that $\{\pi, 2\pi\}^c = (-\infty, \pi) \cup (\pi, 2\pi) \cup (2\pi, \infty)$ which is an open set since it is the union of four the open sets given by $(-\infty, \pi)$, $(\pi, 2\pi)$, and $(2\pi, \infty)$, thus $\{\pi, 2\pi\}$ is a closed set. Finally notice that there exists $K = 3\pi$ (say) such that

$$|x| \leq 3\pi, \qquad \text{for all} \qquad x \in \{\pi, 2\pi\},$$

hence $\{\pi, 2\pi\}$ is a bounded set.

(iii) $\mathbb{Q}$ is not open, it is not closed and it is not bounded. Indeed, notice that any open interval has real numbers that do not belong to $\mathbb{Q}$ (since every interval has rational and irrational numbers), thus $\mathbb{Q}$ is not an open set. For the same reason $\mathbb{Q}$ is not a closed set. Finally, notice that no matter how big or small an open interval centred at 0 is, there will be some rational numbers points of outside the interval, hence $\mathbb{Q}$ is not a bounded set.

(iv) $A = (-\infty, 1) \cup \{4\}$ is not open, not closed and not bounded. Notice that there exists $4 \in A$ such that for all $r > 0$ the interval $(4 - r, 4 + r)$ contains points outside the set A, thus A is not open. Also notice that $A^c = [1, 4) \cup (4, \infty)$ is not open, indeed there exists $1 \in A^c$ such that for all $r > 0$ the interval $(1 - r, 1 + r)$ contains points outside the set A^c. Therefore, A is not a closed set. Finally, notice that no matter how big or small an open interval centred at 0 is, there will be some real numbers in the set A that lie outside the interval, hence A is not a bounded set.

□

Q2. Using the definition of the limit of a function at a point, prove that

$$\lim_{x \to 2} (2x - 1) = 3.$$

Solution. Let $\varepsilon > 0$ arbitrary. We seek a number $\delta > 0$ such that

$$|(2x - 1) - 3| < \varepsilon, \qquad \text{whenever} \qquad 0 < |x - 2| < \delta.$$

Notice that

$$|(2x - 1) - 3| = |2x - 4| = 2|x - 2|.$$

Then, we can choose $\delta = \varepsilon/2$ so that

$$|(2x - 1) - 3| = 2|x - 2| < 2\delta = \varepsilon, \qquad \text{whenever} \qquad 0 < |x - 2| < \delta.$$

Hence

$$\lim_{x \to 2} (2x - 1) = 3.$$

□

Q3. Using the definition of the limit of a function at a point, prove that

$$\lim_{x \to 2} x^3 = 8.$$

Solution. Let $\varepsilon > 0$. Choose δ to be given by

$$\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}.$$

Note that $\delta > 0$ since $\varepsilon > 0$.

Let x be such that $0 < |x-2| < \delta$. Note first that $\delta \leq 1$ implies that

$$|x| = |(x-2)+2| \leq |x-2| + 2 < \delta + 2 \leq 3$$

and therefore

$$|x^2 + 2x + 4| \leq |x|^2 + 2|x| + 4 \leq 19.$$

Secondly, we have

$$x^3 - 8 = (x-2)(x^2 + 2x + 4)$$

so

$$|x^3 - 8| = |x-2||x^2 + 2x + 4| < \delta|x^2 + 2x + 4| \leq 19\delta \leq \varepsilon.$$

At the last step, we used that $\delta \leq \frac{\varepsilon}{19}$. Hence we have proved that whenever $0 < |x-2| < \delta$ we have $|x^3 - 8| < \varepsilon$. By definition, this means $\lim_{x\to 2} x^3 = 8$. $\square$

Q4. Determine the limits of the following functions f as $x \to a$, if they exist. Justify any assertions that you make.

(i) $f(x) = \dfrac{2-x}{4-x^2}$, where $a = 2$.

(ii) $f(x) = x^3\left(\sin(x) + \dfrac{1}{x}\sin\left(\dfrac{1}{x^2}\right)\right)$, where $a = 0$.

(iii)
$$f(x) = \begin{cases} 1-x & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}, \qquad \text{where } a = 0.$$

Here $\mathbb{Q}$ is the set of rational numbers.

(iv) $f(x) = \dfrac{1-x}{1-x^k}$, where $a = 1$. Here $k \in \{1, 2, 3, \ldots\}$ is fixed.

(v) $f(x) = \dfrac{(x^2 - 3x + 2)^4}{(x-2)^4}$, where $a = 2$.

(vi)
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \notin \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{Q} \end{cases}, \qquad \text{where } \quad a = 0.$$

Here $\mathbb{Q}$ is the set of rational numbers.

(vii)
$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x^2}\right), & \text{if } x \notin \mathbb{Q} \\ 1, & \text{if } x = 0 \\ x, & \text{if } x \in \mathbb{Q} - \{0\} \end{cases}, \qquad \text{where } \quad a = 0.$$

Here $\mathbb{Q}$ is the set of rational numbers.

Solution. (i) For each x with $0 < |x-2| < 1$ we have

$$\frac{2-x}{4-x^2} = \frac{2-x}{(2-x)(2+x)} = \frac{1}{2+x}.$$

Since $\lim_{x\to 2} \frac{1}{2+x} = \frac{1}{4}$, it follows that $\lim_{x\to 2} f(x) = \frac{1}{4}$.

(ii) We have

$$f(x) = x^3 \sin x + x^2 \sin(\tfrac{1}{x^2}).$$

Now $x^3 \to 0$ and $\sin x \to 0$ as $x \to 0$ so $x^3 \sin x \to 0$ as $x \to 0$ by the Algebra of Limits.

For the other term we cannot use the Algebra of Limits since $\lim_{x\to 0} \sin(\frac{1}{x^2})$ does not exist. However, for each $x \neq 0$ we have

$$-x^2 \leq x^2 \sin(\tfrac{1}{x^2}) \leq x^2$$

and since $\pm x^2 \to 0$ as $x \to 0$, it follows from the Sandwich Theorem that $\lim_{x\to 0} x^2 \sin(\frac{1}{x^2}) = 0$. Hence, by the Algebra of Limits, $\lim_{x\to 0} f(x) = 0$.

(iii) We claim that $\lim_{x\to 0} f(x)$ does not exist. To prove this, define two sequences (a_n) and (b_n) as follows:

$$a_n = \frac{1}{n} \qquad \text{and} \qquad b_n = \frac{\pi}{n}.$$

Then $a_n \neq 0$ and $a_n \in \mathbb{Q}$ for each $n \in \mathbb{N}$. This means $f(a_n) = 1 - \frac{1}{n}$ and therefore $f(a_n) \to 1$ as $n \to \infty$. Also, $b_n \neq 0$ and $b_n \notin \mathbb{Q}$ for each $n \in \mathbb{N}$, which means $f(b_n) = \frac{\pi}{n}$ and therefore $f(b_n) \to 0$ as $n \to \infty$. It follows that $\lim_{x\to 0} f(x)$ does not exist.

(iv) We use the identity

$$(1-x)(1+x+x^2+\cdots+x^{k-1}) = 1-x^k$$

(or, equivalently, using the formula for the sum of a geometric series) to get that

$$\frac{1-x}{1-x^k} = \frac{1}{1+x+x^2+\cdots+x^{k-1}}$$

for $x \neq 1$. Therefore,

$$\lim_{x\to 1} \frac{1-x}{1-x^k} = \frac{1}{k}$$

by the Algebra of Limits.

(v) Since $x^2 - 3x + 2 = (x-1)(x-2)$ we have that

$$\frac{(x^2-3x+2)^4}{(x-2)^4} = (x-1)^4$$

for $x \neq 2$. Therefore,

$$\lim_{x\to 2} \frac{(x^2-3x+2)^4}{(x-2)^4} = 1$$

by the Algebra of Limits.

(vi) The will show that $\lim_{x\to 0} f(x)$ does not exists. Indeed, consider the sequences (a_n) and (b_n) defined by

$$a_n = \frac{1}{2\pi n} \qquad \text{and} \qquad b_n = \frac{1}{n} \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Notice that a_n and $b_n \neq 0$ for all $n \in \mathbb{N}$, and

$$a_n = \frac{1}{2\pi n} \longrightarrow 0, \qquad \text{and} \qquad b_n = \frac{1}{n} \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty.$$

Moreover, from the definition of f it follows that

$$f(a_n) = f\left(\frac{1}{2\pi n}\right) = \sin(2\pi n) = 0 \longrightarrow 0,$$

and

$$f(b_n) = f\left(\frac{1}{n}\right) = 1 \longrightarrow 1 \qquad \text{as} \qquad n \to \infty.$$

Since $0 \neq 1$, we conclude that $\lim_{x\to 0} f(x)$ does not exists

(vii) We will show that $\lim_{x\to 0} f(x) = 0$. To this end notice that, for $x \notin \mathbb{Q}$ and such that $0 < |x| < 1$, we have that

(1) $$|f(x)| = \left|x^3 \sin\left(\frac{1}{x^2}\right)\right| \leq |x|^3 < |x|.$$

Also, for $x \in \mathbb{Q}$ with $x \neq 0$,

(2) $$|f(x)| = |x|.$$

Therefore, from (1) and (2), we have that

$$|f(x)| \leq |x| \qquad \text{when} \qquad 0 < |x| < 1,$$

or equivalently

$$-|x| \leq f(x) \leq |x| \qquad \text{when} \qquad 0 < |x| < 1.$$

Since $\lim_{x\to 0} \pm|x| = 0$, from the above inequality and the Sandwich theorem we have that

$$\lim_{x\to 0} f(x) = 0.$$

□

Q5. Determine the limits of the following functions f as $x \to 0$, if they exist. Justify any assertions that you make.

(i) $f(x) = \dfrac{3x + |x|}{7x - 5|x|}$

(ii) $f(x) = \dfrac{\sin|x|}{|x|}$

Solution. (i) $\lim_{x\to 0} f(x)$ does not exists. Indeed, notice that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{3x + |x|}{7x - 5|x|} = \lim_{x\to 0^+} \frac{3x + x}{7x - 5x} = \lim_{x\to 0^+} \frac{4}{2} = 2$$

but

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{3x + |x|}{7x - 5|x|} = \lim_{x\to 0^-} \frac{3x - x}{7x + 5x} = \lim_{x\to 0} \frac{2}{12} = \frac{1}{6}.$$

Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, we conclude that $\lim_{x\to 0} f(x)$ does not exists.

(ii) We will show that $\lim_{x\to 0} \frac{\sin|x|}{|x|} = 1$. Indeed, notice that

$$\lim_{x\to 0^+} \frac{\sin|x|}{|x|} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1,$$

since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Also

$$\lim_{x\to 0^-} \frac{\sin|x|}{|x|} = \lim_{x\to 0^-} \frac{\sin(-x)}{(-x)} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1,$$

since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$, we conclude that $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} f(x) = 1$.

□

Q6. Using the fact that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, determine the following limits, if they exist. Justify any assertions that you make.

(i) $\lim_{x\to 0} \dfrac{\sin(\pi x)}{\sin(3x)}$.

(ii) $\lim_{x\to 0} \dfrac{x\sin x}{1-\cos x}$.

(iii) $\lim_{x\to 0} \dfrac{x^4+x^2}{1-\cos x}$.

(iv) $\lim_{x\to 0} \dfrac{\tan^3 x - x}{x+x^2}$.

Solution. (i) For $0 < |x| < \frac{1}{2}$ we have

$$\frac{\sin(\pi x)}{\sin(3x)} = \frac{\pi x}{3x} \times \frac{\sin(\pi x)}{\pi x} \times \frac{3x}{\sin(3x)} = \frac{\pi}{3} \times \frac{\sin(\pi x)}{\pi x} \times \frac{3x}{\sin(3x)}.$$

Since $\lim_{y\to 0} \frac{\sin y}{y} = 1$ it follows that $\frac{\sin(\pi x)}{\pi x} \to 1$ as $x \to 0$ and $\frac{3x}{\sin(3x)} \to 1$ as $x \to 0$. Hence, by the Algebra of Limits, $\lim_{x\to 0} f(x) = \frac{\pi}{3}$.

(ii) For $0 < |x| < \frac{\pi}{2}$ we have

$$\frac{x\sin x}{1-\cos x} = \frac{x\sin x(1+\cos x)}{1-\cos^2 x} = \frac{x\sin x(1+\cos x)}{\sin^2 x} = \frac{x}{\sin x}(1+\cos x).$$

Since $\cos x \to 1$ and $\frac{x}{\sin x} \to 1$ as $x \to 0$ we get

$$\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = 2$$

by the Algebra of Limits.

(iii) For $0 < |x| < \pi$ we have

$$\frac{x^4+x^2}{1-\cos x} = \frac{(x^4+x^2)(1+\cos x)}{1-\cos^2 x} = \frac{(x^4+x^2)(1+\cos x)}{\sin^2 x} = \left(\frac{x}{\sin x}\right)^2 (x^2+1)(1+\cos x).$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ it follows that

$$\lim_{x\to 0} \frac{x^4+x^2}{1-\cos x} = 2$$

by the Algebra of Limits.

(iv) For $x \neq 0$, we may write

$$\frac{\tan^3 x - x}{x+x^2} = \frac{\frac{\sin^3 x}{\cos^3 x} - x}{x+x^2} = \frac{(\sin x/x)^3 x^3 - x\cos^3 x}{\cos^3 x(x+x^2)} = \frac{(\sin x/x)^3 x^2 - \cos^3 x}{\cos^3 x(1+x)}$$

and so

$$\lim_{x\to 0} \frac{\tan^3 x - x}{x+x^2} = -1$$

since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \cos x = 1,$$

and using the Algebra of Limits.

□

Q7. For each of the following functions f and points a, determine whether $\lim_{x\to a} f(x)$ exists, and compute the limit if it exists. In each case, justify your answer (that is, if the limit does not exist you need to justify why it does not exists; if the limit exists you need to state the results you have used in the evaluation of the limit).

(i) $f(x) = \dfrac{(x^2 - x - 6)^2}{(x+2)^2}$, where $a = -2$.

(ii) $f(x) = \dfrac{(\sin|x|)^2}{x}$, $a = 0$

(iii) $f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x^2}\right), & \text{if } x \notin \mathbb{Q} \\ x, & \text{if } x \in \mathbb{Q} \end{cases}$, $a = 0$

(iv) $f(x) = \sin\left(\dfrac{1}{(2-x)^3}\right)$ $a = 2$

Solution. (i) Factorising the numerator, we obtain that if $x \neq -2$

$$\frac{(x^2 - x - 6)^2}{x+2} = \frac{\left((x-3)(x+2)\right)^2}{(x+2)^2} = (x-3)^2.$$

Thus, using the Algebra of the Limits,

$$\lim_{x \to -2} \frac{(x^2 - x - 6)^2}{x+2} = \lim_{x \to -2} (x-3)^2 = \left(\lim_{x \to -2} (x-3)\right)^2 = (-2-3)^2 = 25$$

(ii) Notice that, since we know that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and $\lim_{x \to 0} \sin(x) = 0$, using the algebra of limits it follows that

$$\lim_{x \to 0^+} \frac{(\sin|x|)^2}{x} = \lim_{x \to 0^+} \frac{(\sin(x))^2}{x} = \lim_{x \to 0^+} \left(\frac{\sin(x)}{x}\right)(\sin(x)) = 1 \times 0 = 0$$

and

$$\begin{aligned} \lim_{x \to 0^-} \frac{(\sin|x|)^2}{x} &= \lim_{x \to 0^-} \frac{(\sin(-x))^2}{x} = \lim_{x \to 0^-} \frac{(-\sin(x))^2}{x} \\ &= \lim_{x \to 0^+} \left(\frac{\sin(x)}{x}\right)(\sin(x)) = 1 \times 0 = 0. \end{aligned}$$

Since $\lim_{x \to 0^\pm} \frac{(\sin|x|)^2}{x}$ both exist and equal to 0, we conclude that $\lim_{x \to 0} \frac{(\sin|x|)^2}{x}$ exists and

$$\lim_{x \to 0} \frac{(\sin|x|)^2}{x} = 0.$$

(iii) Notice that, if $x \notin \mathbb{Q}$, since $|\cos(x)| \leq 1$ for all $x \in \mathbb{R}$,

$$|f(x)| = \left| x^2 \cos\left(\frac{1}{x^2}\right) \right| \leq |x|^2 \leq |x|, \qquad \text{for all } |x| \leq 1 \qquad \text{and } x \notin \mathbb{Q}.$$

Also, if $x \in \mathbb{Q}$, we have that

$$|f(x)| = |x| \qquad \text{if} \qquad x \in \mathbb{Q}.$$

The above two inequalities gives

$$|f(x)| \leq |x| \qquad \text{for all} \qquad x \in \mathbb{R} \quad \text{such that} \quad |x| \leq 1,$$

or equivalently

$$-|x| \leq f(x) \leq |x| \qquad \text{for all} \qquad x \in \mathbb{R} \quad \text{such that} \quad |x| \leq 1. \tag{3}$$

Since we know that $\lim_{x \to 0} |x| = 0$ and $\lim_{x \to 0^-} (-|x|) = 0$, using the Sandwich Theorem from the inequality (3) we conclude that

$$\lim_{x \to 0} f(x) = 0.$$

(iv) Consider the sequences (a_n) and (b_n) defined as follows

$$a_n = 2 - \left(\frac{1}{\pi n}\right)^{\frac{1}{3}} \quad \text{and} \quad b_n = 2 - \left(\frac{1}{\frac{\pi}{2} + 2\pi n}\right)^{\frac{1}{3}}$$

Notice that a_n, $b_n \neq 2$ for all $n \in \mathbb{N}$. Also

$$a_n = 2 - \left(\frac{1}{\pi n}\right)^{\frac{1}{3}} \longrightarrow 2, \quad \text{and} \quad b_n = 2 - \left(\frac{1}{\frac{\pi}{2} + 2\pi n}\right)^{\frac{1}{3}} \longrightarrow 2 \quad \text{as} \qquad n \to \infty.$$

However

$$f(a_n) = \sin(n\pi) = 0 \longrightarrow 0, \quad \text{and} \qquad f(b_n) = \sin(\frac{\pi}{2} + 2n\pi) = 1 \longrightarrow 1 \quad \text{as} \quad n \to \infty.$$

Since $0 \neq 1$, we conclude that the limit

$$\lim_{x \to 2} \sin\left(\frac{1}{(2-x)^3}\right)$$

<u>does not</u> exist.

$\square$

Q8. Suppose that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at x_0. Show that there exists $M > 0$ and $\delta > 0$ such that

$$|f(x)| \leq M \qquad \text{whenever} \qquad |x - x_0| < \delta.$$

Solution. Since by hypothesis f is continuous at x_0, given $\varepsilon = 1$ (say), there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < 1 \qquad \text{whenever} \qquad |x - x_0| < \delta.$$

From the above inequality, and using the triangle inequality for real numbers, it follows that

$$|f(x)| = |(f(x) - f(x_0)) + f(x_0)| \leq |f(x) - f(x_0)| + |f(x_0)| < 1 + |f(x_0)|$$

whenever $|x - x_0| < \delta$. Thus there exists $M = 1 + |f(x_0)| > 0$ such that

$$|f(x)| < M \qquad \text{whenever} \qquad |x - x_0| < \delta,$$

as required.

$\square$

$\boxed{\text{SUM}}$ **Q9**. Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a real function. Suppose that there exists $M > 0$ and $\delta > 0$ such that

$$|f(x)| \leq M \qquad \text{whenever} \quad 0 < |x| < \delta.$$

Define the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} x^2 f(x), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Show that g is continuous at 0.

Solution. To show that g is continuous at 0, we will show that

$$\lim_{x \to 0} g(x) = g(0).$$

We will show that the above limit exists and equals $g(0) = 0$ using the Sandwich Theorem. Indeed, using the hypothesis on f we observe that

$$|g(x)| = |x^2 f(x)| \leq x^2 M \qquad \text{whenever} \qquad 0 < |x| < \delta,$$

and since $|g(0)| = 0$, we note the above inequality also holds for $x = 0$. Hence, we have

$$|g(x)| \leq x^2 M \qquad \text{whenever} \qquad |x| < \delta,$$

or equivalently,

$$-Mx^2 \leq g(x) \leq Mx^2, \qquad \text{whenever} \qquad |x| < \delta. \tag{4}$$

Since $\lim_{x\to 0} \pm Mx^2 = 0$ (since x^2 is continuous at 0), and (4) holds, using the Sandwich Theorem we conclude that

$$\lim_{x\to 0} g(x) = 0 = g(0).$$

Therefore, g is continuous at 0.

Remark: Alternatively, the converge of the above limit to zero can be shown using the definition the $\varepsilon - \delta$ definition of limit of a function at a point.

□

Q10. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has the property that, for some $C, \alpha > 0$, we have

$$|f(x) - f(y)| \leq C|x - y|^\alpha \qquad \text{for all } x, y \in \mathbb{R}.$$

Using the $\varepsilon - \delta$ definition of continuity, prove that f is continuous on $\mathbb{R}$.

Note: You must not use the Sandwich Theorem here.

Solution. Fix $x_0 \in \mathbb{R}$.

Let $\varepsilon \in \mathbb{R}^+$. Choose $\delta = (\varepsilon/C)^{\frac{1}{\alpha}}$. We note that this choice of δ is well-defined and $\delta > 0$ since $C, \varepsilon > 0$.

Now, whenever $|x - x_0| < \delta$ we have

$$|f(x) - f(x_0)| \leq C|x - x_0|^\alpha < C\delta^\alpha = \varepsilon.$$

Hence, f is continuous at x_0. Since x_0 was chosen arbitrarily, it follows that f is continuous on $\mathbb{R}$.

□

Q11. Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which fails to be continuous at any point $x \in \mathbb{R}$.

Solution. There are loads of examples of such functions. One example is

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(Here $\mathbb{Q}$ denotes the set of rational numbers.)

□

Q12. Let $\alpha \in \mathbb{R}$ and $f_\alpha : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ be the function defined by

$$f_\alpha(x) = \begin{cases} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{\sin(x)}, & \text{if} \quad x \in \left(-\frac{\pi}{2}, 0\right), \\ \cos(\alpha x^2) - \alpha, & \text{if} \quad x \in \left[0, \frac{\pi}{2}\right). \end{cases}$$

Study the continuity of f_α on $(-\pi/2, \pi/2)$ depending on the values of α.

Solution. Let $\alpha \in \mathbb{R}$.

First observe that sine, cosine and polynomial functions are continuous on $\mathbb{R}$, $1/x$ is continuous on $\mathbb{R} \setminus \{0\}$, and $\sin(x) \neq 0$ on $(-\pi/2, \pi/2) \setminus \{0\}$. Thus, using the algebra of continuous functions, we have that f_α is continuous on $(-\pi/2, 0)$ and on $(0, \pi/2)$ for all values of the parameter $\alpha \in \mathbb{R}$.

Therefore, f_α is continuous on $(-\pi/2, \pi/2)$ if and only if f_α is continuous at $x_0 = 0$, or equivalently iff

$$\lim_{x\to 0^+} f_\alpha(x) = \lim_{x\to 0^-} f_\alpha(x) = f_\alpha(0). \tag{5}$$

Now, using the definition of f_α, we have that

$$f_\alpha(0) = 1 - \alpha. \tag{6}$$

Also

$$\lim_{x\to 0^+} f_\alpha(x) = \lim_{x\to 0^+} \cos(\alpha x^2) - \alpha = 1 - \alpha. \tag{7}$$

Here, we have used the continuity of the function $\cos(\alpha x^2) - \alpha$ in evaluating the above limit. In order to evaluate $\lim_{x\to 0^-} f_\alpha(x)$, we observe that, for all $x \in (-\pi/2, 0)$, the function f_α can be rewritten as follows

$$f_\alpha(x) = \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin(x)} = \frac{1}{\frac{\sin(x)}{x}} \left(x \sin\left(\frac{1}{x}\right)\right), \qquad \text{for all} \qquad x \in (-\pi/2, 0). \tag{8}$$

Since we know that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, using the algebra of limits we get that

$$\lim_{x\to 0} \frac{1}{\frac{\sin(x)}{x}} = 1. \tag{9}$$

Next, we continue to show that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Indeed, we have that

$$\left| x \sin\left(\frac{1}{x}\right) \right| \leq |x| \qquad \text{for} \qquad x \neq 0$$

since $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$. Therefore,

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x| \qquad \text{for} \qquad x \neq 0.$$

From the above inequality, and the fact that $\lim_{x\to 0} \pm|x| = 0$, using the Sandwich Theorem we obtain that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0. \tag{10}$$

From (8), (9) and (10), and using the algebra of limits, we have that

(11)

$$\lim_{x\to 0^+} f_\alpha(x) = \lim_{x\to 0^+} \frac{1}{\frac{\sin(x)}{x}} \left(x \sin\left(\frac{1}{x}\right)\right) = \left(\lim_{x\to 0^+} \frac{1}{\frac{\sin(x)}{x}}\right) \left(\lim_{x\to 0^+} x \sin\left(\frac{1}{x}\right)\right) = 1 \times 0 = 0.$$

Identities (5), (6), (7) and (11) show that f_α is continuous at 0 if and only if $1 - \alpha = 0$, i.e. $\alpha = 1$.

The above argument shows that f_α is continuous on $(-\pi/2, \pi/2)$ if and only is $\alpha = 1$.

$\square$

$\boxed{\text{SUM}}$ **Q13**. Let $\alpha \in \mathbb{N} \cup \{0\}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} |x|^\alpha \cos\left(\frac{1}{x}\right), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

(i) Show that f is a continuous at 0 for all $\alpha \in \mathbb{N}$.
(ii) Show that f is not continuous at 0 when $\alpha = 0$.
(iii) Is f a continuous at $x \neq 0$ for all $\alpha \in \mathbb{N} \cup \{0\}$? Justify your answer.

Solution. (i) Fix $\alpha \in \mathbb{N}$. In order to show that f is continuous at 0, we need to show that

$$\lim_{x\to 0} f(x) = f(0),$$

or equivalently

$$\lim_{x\to 0} |x|^\alpha \cos\left(\frac{1}{x}\right) = 0,$$

since $f(0) = 0$. Notice that for $x \neq 0$

$$\left| |x|^\alpha \cos\left(\frac{1}{x}\right) \right| \leq |x|^\alpha$$

and thus

$$-|x|^\alpha \leq |x|^\alpha \cos\left(\frac{1}{x}\right) \leq |x|^\alpha, \qquad x \neq 0$$

Since the absolute function is continuous on $\mathbb{R}$, using the algebra of continuous functions, $\pm|x|^\alpha$ is continuous at 0 for any $\alpha \in \mathbb{N}$, and consequently $\lim_{x\to 0} \pm|x|^\alpha = 0$. From this remark, the above inequality, and using the Sandwich Theorem, we conclude that

$$\lim_{x\to 0} |x|^\alpha \cos\left(\frac{1}{x}\right) = 0,$$

as required.

(ii) We will show that f is not continuos at 0 when $\alpha = 0$ by showing that the following limit

$$\lim_{x\to 0} f(x)$$

does <u>not</u> exist.

Indeed, consider (for example) the following sequences

$$a_n = \frac{1}{2\pi n} \qquad \text{and} \qquad b_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$$

We have that $a_n, b_n \neq 0$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0,$$

however

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \cos(1/a_n) = \lim_{n\to\infty} \cos(2\pi n) = \lim_{n\to\infty} 1 = 1,$$

and

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \cos(1/b_n) = \lim_{n\to\infty} \cos\left(\frac{\pi}{2} + 2\pi n\right) = \lim_{n\to\infty} 0 = 0,$$

since $1 \neq 0$, we conclude that the limit $\lim_{x\to 0} f(x)$ does not exists. Hence f is not continuous at 0 when $\alpha = 0$.

(iii) Yes, for any given $\alpha \in \mathbb{N} \cup \{0\}$, f is continous at any $x \neq 0$. This follows from the Algebra of continuous functions, and the fact that the functions $|x|$ and cosine function are continuous functions on the real line, $1/x$ is continuous at any $x \neq 0$.

$\square$

Q14. (i) Show that if $a, b \in \mathbb{R}$, then $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$.
(ii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Using part (i), prove/deduce that the function $h : \mathbb{R} \to \mathbb{R}$, given by

$$h(x) = \max\{f(x), g(x)\}$$

for each $x \in \mathbb{R}$, is continuous. Justify your answer.

Solution. (i) In order to show that

$$\max\{a, b\} = \tfrac{1}{2}(a + b + |a - b|), \tag{12}$$

we will consider two cases: $a \le b$, and $b \le a$. In the first case, the left hand side of (12) is equal to b. Now since $a - b \le 0$, we have that $|a - b| = -(b - a)$, and thus the right hand side of (12) equals $\frac{1}{2}(a + b - (a - b)) = b$, as required. The second case (when $b \le a$) now follows from the first case by symmetry (i.e. by interchanging a and b). Alternatively, arguing as before, if $b \le a$, then $\max\{a, b\} = a$ and, since $a - b \ge 0$, then the left hand side of (12) is equal to $\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+(a-b)) = a$. Thus, $\max\{a, b\} = \frac{1}{2}(a+b+|a-b|)$ in the second case, that is when $b \le a$.

(ii) We now deduce that $h(x) = \max\{f(x), g(x)\}$ is continuous. Since f and g are continuous $f - g$ is continuous. Since the modulus function is continuous, and compositions of continuous functions are continuous, $|f - g|$ is continuous. Applying once more the "Algebra of continuous functions"-Theorem, we find that $\frac{1}{2}(f + g + |f - g|) = \max\{f, g\}$ is continuous.

□

$\boxed{\text{SUM}}$ **Q15**. Suppose that the function $f : [-1, 1] \to [-1, 1]$ is continuous. Use the Intermediate Value Theorem to prove that there exists $c \in [-1, 1]$ such that

$$f(c) = c^5.$$

Note: You should carefully justify each of the hypotheses of the theorem.

Solution. We argue very much as in the proof of the fixed point theorem in the lecture notes.

If $f(-1) = -1$ then we choose $c = -1$ and there is nothing else to prove. Similarly, if $f(1) = 1$ then we choose $c = 1$.

So suppose that $f(-1) \ne -1$ and $f(1) \ne 1$. Since $f(x) \in [-1, 1]$ for all $x \in [-1, 1]$ then we must have that

$$f(-1) > -1 \qquad \text{and} \qquad f(1) < 1. \tag{13}$$

We now define a new function $g : [-1, 1] \to \mathbb{R}$ by

$$g(x) = x^5 - f(x).$$

Since f is continuous on $[-1, 1]$, polynomials are continuous, and sums of continuous functions are continuous, then the function g is continuous on $[-1, 1]$. Furthermore, from (13) we have that $g(-1) = -1 - f(-1) < 0$ and $g(1) = 1 - f(1) > 0$ so that

$$g(-1) < 0 < g(1).$$

Hence, Hence, applying the Intermediate Value Theorem to the function g on the interval $[-1, 1]$, we conclude that there exists $c \in (-1, 1)$ such that $g(c) = 0$; i.e. such that $f(c) = c^5$. □

Q16. Use the Intermediate Value Theorem to show that the equation

$$-\pi^2 \sin^2(x) + \sin(x) = -1$$

has at least two distinct real solutions on $(0, \infty)$. Justify your answer.

Solution. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = -\pi^2 \sin^2(x) + \sin(x) + 1.$$

Since the sine function is continuous on $\mathbb{R}$, from the Algebra of Continuous function it follows that f is continuous on $\mathbb{R}$, and in particular f is continuous on $[a, b]$ for all $a, b \in \mathbb{R}$ with $a < b$.

We consider the function f on the intervals $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$. First notice that f is continuous on $\left[0, \frac{\pi}{2}\right]$, and

$$f(0) = 1 > 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) = -\pi^2 + 2 < 0,$$

so that

$$f\left(\frac{\pi}{2}\right) < 0 < f(0).$$

Using the Intermediate Value Theorem we conclude that there exists $c_1 \in \left(0, \frac{\pi}{2}\right)$ such that $f(c_1) = 0$. Also, notice that f is continuous on $\left[\frac{\pi}{2}, \pi\right]$, and

$$f\left(\frac{\pi}{2}\right) = -\pi^2 + 2 < 0 \quad \text{and} \quad f(\pi) = 1 > 0$$

so that

$$f\left(\frac{\pi}{2}\right) < 0 < f(\pi).$$

Using the Intermediate Value Theorem we conclude that there exists $c_2 \in \left(\frac{\pi}{2}, \pi\right)$ such that $f(c_2) = 0$.

Since $\left(0, \frac{\pi}{2}\right) \bigcap \left(\frac{\pi}{2}, \pi\right) = \emptyset$, the above argument shows that there exist at least two distinct real solutions of the equation $f(x) = 0$ on the interval $(0, \infty)$.

$\square$

$\boxed{\text{SUM}}$ **Q17**. Use the Intermediate Value Theorem to prove that the equation

$$x^2 + 2\sin(x) - \cos(x) = 1$$

has at least two real solutions.

Note: You should carefully justify each of the hypotheses of the theorem.

Hint: You need to carefully choose your own intervals to apply the Intermediate Value Theorem.

Solution. Define $f(x) = x^2 + 2\sin(x) - \cos(x) - 1$. It suffices to show that $f(c_1) = f(c_2) = 0$ for some $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq c_2$.

Since the sine function, cosine function and polynomials are continuous on $\mathbb{R}$, it follows that f is continuous on $\mathbb{R}$ (using the algebra of continuous functions, and composition of continuous functions).

Note $f(0) = -2 < 0$. Also

$$f(10) = 100 + 2\sin(10) - \cos(10) - 1 \geq 100 - 2 - 1 - 1 = 96 > 0$$

since $\sin(10) \geq -1$ and $\cos(10) \leq 1$. Hence, by the Intermediate Value Theorem applied to the function f on the interval $[-10, 0]$, there exists $c_1 \in (0, 10)$ such that $f(c_1) = 0$.

Similarly,

$$f(-10) = 100 - 2\sin(10) - \cos(10) - 1 \geq 100 - 2 - 1 - 1 = 96 > 0$$

since $\sin(10) \leq 1$ and $\cos(10) \leq 1$. By the Intermediate Value Theorem applied to the function f on the interval $[-10, 0]$, there exists $c_2 \in (-10, 0)$ such that $f(c_2) = 0$. Since $(0, 10) \cap (-10, 0) = \emptyset$ it follows that $c_1 \neq c_2$, as required. $\square$

Q18. Use the Intermediate Value Theorem to show that if the function $f : [0, 1] \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x \in [0, 1]$, then *either* $f(x) > 0$ for all $x \in [0, 1]$, *or* $f(x) < 0$ for all $x \in [0, 1]$.

Hint: You may want to argue by contradiction.

Solution. Suppose for a contradiction that f may take both positive and negative values; i.e. there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) < 0$ and $f(x_2) > 0$. Since f is continuous on $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$), and

$$f(x_1) < 0 < f(x_2),$$

by the Intermediate Value Theorem there exists $c \in (x_1, x_2)$ (or (x_2, x_1) if $x_2 < x_1$) such that $f(c) = 0$. This contradicts the hypothesis that $f(x) \neq 0 \; \forall x \in [0, 1]$. Hence either $f(x) > 0 \; \forall x \in [0, 1]$ or $f(x) < 0 \; \forall x \in [0, 1]$, as required.

$\square$

Q19. Use the Boundedness Theorem to show that if the function $f : [0, 1] \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x \in [0, 1]$, then there exists $\delta > 0$ such that $|f(x)| > \delta$ for all $x \in [0, 1]$.

Solution. Since $f : [0, 1] \to \mathbb{R}$ is continuous and $f(x) \neq 0$ for all $x \in [0, 1]$, the function $g : [0, 1] \to \mathbb{R}$, given by

$$g(x) = \frac{1}{f(x)}$$

is continuous (by the algebra of continuous functions proved in lectures). By the Boundedness Theorem applied to the function g, there exists $R \in \mathbb{R}^+$ such that

$$|g(x)| < R \qquad \text{for all } x \in [0, 1].$$

Setting $\delta = \frac{1}{R}$ this may be rewritten as

$$|f(x)| > \delta \qquad \text{for all } x \in [0, 1],$$

as required.

Alternatively, we must have either (i) $f(x) > 0$ for all $x \in [0, 1]$ or (ii) $f(x) < 0$ for all $x \in [0, 1]$. This follows from the Intermediate Value Theorem: if $f(x_1) > 0$ and $f(x_2) < 0$ then, using the continuity of f, the Intermediate Value Theorem gives us $x_0 \in (0, 1)$ such that $f(x_0) = 0$ which is a contradiction. Now, by the Boundedness Theorem, there exist $c, d \in [0, 1]$ such that

$$f(c) \leq f(x) \leq f(d) \qquad \text{for all } x \in [0, 1].$$

In Case (i) we can take $\delta = \frac{1}{2} f(c) > 0$, and in Case (ii) we can take $\delta = -\frac{1}{2} f(d) > 0$. This is because in Case (i),

$$|f(x)| = f(x) \geq f(c) > \delta$$

and in Case (ii),

$$|f(x)| = -f(x) \geq -f(d) > \delta.$$

$\square$