LH: Homework sheet 2 – Linear Algebra

Instructions. Please upload your solutions to the Canvas assignment page by 5pm, 10 December.

Plagiarism check: Your submission should be your own work and should not be identical or substantially similar to other submissions. A check for plagiarism will be performed on all submissions.

Assessment: This sheet is assessed, with a maximum contribution to your final mark of 5%.

Note. You should make sure that you state clearly any results you use in your proofs or derivations.

1. Let $\mathbb{E}^3$ denote the usual Euclidean space with basis $B = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ which is orthonormal with respect to the Euclidean dot product. A generic element of $\mathbb{E}^3$ is denoted by $\vec{a}$; its representation in the basis B is $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, with the vector of coordinates denoted by $\mathbf{a}$, i.e., $\varphi_B(\vec{a}) = \mathbf{a}$, where φ_B is the coordinate map with respect to the basis B.

For any vectors $\vec{a}, \vec{b}$ in $\mathbb{E}^3$, we define the following standard operations:

- the dot product:
$$\vec{a} \cdot \vec{b} := a_1b_1 + a_2b_2 + a_3b_3 = \mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a};$$
- the cross product (or vector product):
$$\vec{a} \times \vec{b} := (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Let $f \in \mathcal{L}(\mathbb{E}^3)$ be defined via $f(\vec{v}) = \vec{c} \times \vec{v}$, where $\vec{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ is a unit vector with $c_i \neq 0, i = 1, 2, 3$.

(a) Find the matrix representation A of f with respect to the basis B.

(b) Find $\ker A$ and hence derive $\ker f$.

(c) Check directly (through calculations) that, for all $\mathbf{x} \in \mathbb{R}^3$,
$$\mathbf{c}^T A\mathbf{x} = \mathbf{x}^T A\mathbf{x} = 0.$$
Deduce that $f(\vec{v}) \perp \operatorname{span}\{\vec{c}, \vec{v}\}$.

(d) Check further that, for all $\mathbf{x} \in \mathbb{R}^3$,
$$A^2\mathbf{x} = (\mathbf{c}^T\mathbf{x})\mathbf{c} - \mathbf{x}.$$

(e) Find the eigenvalues of A. Hence, explain why there are only two proper non-trivial f-invariant subspaces over $\mathbb{R}$. Use part **(b)** to identify one of them.

(f) Let $\vec{n} \perp \vec{c}$. Define $U = \operatorname{span}\{\vec{n}, \vec{c} \times \vec{n}\}$. Show that U is f-invariant by using the matrix representation of f and the coordinates of $\vec{n}$ and $\vec{c} \times \vec{n}$. [Hint: consider applying A to the elements of the set $\{\mathbf{n}, A\mathbf{n}\}$.] Give a geometric interpretation of U.