LECTURE 2

Subspaces. Spanning sets

2.1 Subspaces

Any vector space V contains at least one other vector space, assuming they are equipped with the same operations – we will refer to any such space as a subspace of V.

Definition 2.1 Let $(V, +, \cdot, \mathbb{F})$ be a vector space and let U be a subset of V, namely, $U \subseteq V$. If $(U, +, \cdot, \mathbb{F})$ is a vector space, we say that it is a subspace of $(V, +, \cdot, \mathbb{F})$. We write $U(\mathbb{F}) \leq V(\mathbb{F})$, or simply $U \leq V$.

R Note that we use different symbols $\subseteq$ and $\leq$ to establish relations between sets and vector spaces, respectively. Some references use the former symbol to mean both 'subset' and 'subspace'.

The example below considers three special cases of subsets of a given set V.

Example 2.1 Let $V(\mathbb{F})$ be a vector space. We highlight the following three special cases.

- The empty set is a subset of V, but is not a subspace of V since axiom VA2 is not satisfied: the empty set does not contain the zero vector.
- Let $Z = \{\mathbf{0}\}$. Then $Z(\mathbb{F})$ is a vector space, known as **the trivial vector space**. By the above definition, Z is a subspace of any vector space V: $Z \leq V$.
- Since $V \subseteq V$, by the above definition, V is a subspace of V: $V \leq V$.

The example above prompts the following definition.

Definition 2.2 Let $U(\mathbb{F}) \leq V(\mathbb{F})$. If the strict set inclusion $U \subset V$ holds , the subspace $U(\mathbb{F})$ is said to be a **proper subspace** of $V(\mathbb{F})$. We write $U(\mathbb{F}) < V(\mathbb{F})$, or simply $U < V$.

When establishing that a given set V affords a vector space structure, typically we need to check the axioms VA0–VA4 and VM0–VM4. However, in the case of a subset U of V, many of them already hold due to the set inclusion $U \subseteq V$. We can make this precise with the following result.

Proposition 2.1 — Subspace criterion 1: closure criterion. Let $V(\mathbb{F})$ be a vector space. A non-empty subset U of V is a subspace of V over $\mathbb{F}$ if and only if it satisfies the **closure criterion**, i.e., U is closed under vector addition and scalar-vector multiplication.

Proof. Let $V(\mathbb{F})$ be a vector space and let U be a non-empty subset of V.
$\boxed{\Longrightarrow}$ Assume that U is a subspace of V. Then U is a vector space in its own right, i.e., it satisfies the vector space axioms, in particular VA0 and VM0, i.e., U is closed under vector addition and scalar-vector multiplication.
$\boxed{\Longleftarrow}$ Assume that U satisfies the closure criterion. Now, except for VA2 and VA3, all the vector space axioms hold for all the elements of U due to the set inclusion $U \subseteq V$. By closure, $a \bullet \mathbf{u} \in U$ for all $a \in \mathbb{F}$ and all $\mathbf{u} \in U$. Hence, using the elementary properties 2 and 3 in Proposition 1.1,

- VA2 must hold: taking $a = o \in \mathbb{F}$, we find that $\mathbf{z} = o \bullet \mathbf{u} \in U$. Since $\mathbf{u} \in V$, $\mathbf{u} + \mathbf{z} = \mathbf{u}$.
- VA3 must hold: taking $a = e^- \in \mathbb{F}$, we find that $\mathbf{u}^- = e^- \bullet \mathbf{u} \in U$. Since $\mathbf{u}^- \in V$, $\mathbf{u} + \mathbf{u}^- = \mathbf{z}$.

■

Subspace criterion 1 can be recast as follows.

Proposition 2.2 — Subspace criterion 2: linear combination criterion. Let $V(\mathbb{F})$ be a vector space. A non-empty subset U of V is a subspace of V over $\mathbb{F}$ if and only if for any $\mathbf{u}, \mathbf{v} \in U$ and for any $a, b \in \mathbb{F}$, there holds $a\mathbf{u} + b\mathbf{v} \in U$.

Proof. Use the result of Exercise 1.4. ■

(R) The expression $a\mathbf{u} + b\mathbf{v}$ is called a **linear combination** of the vectors $\mathbf{u}$ and $\mathbf{v}$. Thus, subspace criterion 2 above requires that any linear combination of any two vectors in U should also be in U.

2.1.1 Examples

Let us consider subspaces of the three vector space examples presented in Lecture 2.

Example 2.2 — Column vectors. Let $V = \mathbb{R}^3$ and define the subset $U \subset V$

$$U = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} : v_1, v_2 \in \mathbb{R} \right\}.$$

Given any two vectors $\mathbf{u}, \mathbf{v} \in U$, there holds for any $a, b \in \mathbb{R}$,

$$a\mathbf{u} + b\mathbf{v} = a \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} + b \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} au_1 + bv_1 \\ au_2 + bv_2 \\ 0 \end{bmatrix} \in U,$$

so that, by Subspace criterion 2 (Proposition 2.2), U is a subspace of V.

Remark This example can be extended to $V = \mathbb{R}^n$, with $U \subset V$ being the set of vectors with $k < n$ non-zero entries in fixed locations.

Example 2.3 — Polynomials. Recall that $\mathcal{P}_n$ is the space of polynomials of degree at most n:

$$\mathcal{P}_n = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{R}, i = 0, 1, \ldots, n\}.$$

Let $k < n$. Then $\mathcal{P}_k \subset \mathcal{P}_n$ is a subspace of $\mathcal{P}_n$ when equipped with the operations of polynomial addition and multiplication of polynomials by scalars.

The above two examples are related: since any polynomial $p_n(x)$ of degree n can be uniquely identified by the vector of its coefficients:

$$p_n(x) \longleftrightarrow [a_0, a_1, \cdots, a_n],$$

we find, given $p_k \in \mathcal{P}_n$,

$$p_k(x) \longleftrightarrow [a_0, a_1, \cdots, a_k, 0, \cdots, 0].$$

The latter row vector contains $k < n$ non-zero entries: see final remark in Example 2.2

Example 2.4 — Continuous functions. Let $V = C^0(\Omega)$ denote the space of functions $f : \Omega \to \mathbb{R}$ which are continuous. The space $U = C^1(\Omega)$ was defined as the space of continuous functions which are differentiable $k = 1$ times. Since a differentiable function is continuous, $U = C^1(\Omega) \subset C^0(\Omega) = V$. One can immediately use Proposition 2.2 to show that U is a subspace of V.

We consider now two set operations which preserve the vector space structure: intersection and sum.

2.1.2 Intersection of subspaces

The intersection of subspaces is also a subspace, as the following result shows.

Theorem 2.3 — Intersection of subspaces. Let $U_1(\mathbb{F}), U_2(\mathbb{F})$ be subspaces of $V(\mathbb{F})$. Then the set intersection $U_1 \cap U_2$ is also a subspace of V when equipped with the vector operations defined on V.

Proof. Let $U_1, U_2 \leq V$. Since $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$, the set intersection $U_1 \cap U_2$ is always non-empty. Let $\mathbf{u}, \mathbf{v} \in U_1 \cap U_2$. Then $\mathbf{u}, \mathbf{v} \in U_1$ and also $\mathbf{u}, \mathbf{v} \in U_2$. Since U_1, U_2 are subspaces, they satisfy the closure criterion, so that $a\mathbf{u} + b\mathbf{v} \in U_1$ and also $a\mathbf{u} + b\mathbf{v} \in U_2$. Hence $a\mathbf{u} + b\mathbf{v} \in U_1 \cap U_2$ and by Proposition 2.2 $U_1 \cap U_2$ is a subspace of V. ∎

(R) It is a standard convention to denote the intersection of subspaces using the same symbol as that for intersection of sets, with the vector space operations and the field assumed to be clear from the context. For example, the result of the above theorem can be written as $U_1(\mathbb{F}) \cap U_2(\mathbb{F}) \leq V(\mathbb{F})$, or simply $U_1 \cap U_2 \leq V$.

Note that the above result can be immediately generalised to the intersection of several subsets of V:

$$\bigcap_{i=1}^{m} U_i \leq V \qquad (U_i \leq V).$$

Example 2.5 Consider the subspaces $U_1, U_2 \leq \mathbb{R}^2$:

$$U_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = y \right\}, \quad U_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = -y \right\}.$$

Then $U_1 \cap U_2 = \{\mathbf{0}\}$. Note that both sets can be identified as lines in the plane through the origin.

2.1.3 Sums. Direct sums

Recall that the sum of two sets U_1, U_2 is the set W given by

$$W := U_1 + U_2 := \{\mathbf{u}_1 + \mathbf{u}_2 : \mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2\}.$$

Theorem 2.4 — Sum of subspaces. Let $U_1(\mathbb{F}), U_2(\mathbb{F})$ be subspaces of $V(\mathbb{F})$. Then the set sum $U_1 + U_2$ is also a subspace of V when equipped with the vector operations defined on V.

Proof. The proof is left as an exercise. ∎

As before, the above result can be immediately generalised to the sum of several subsets of V:

$$\sum_{i=1}^{m} U_i \leq V \qquad (U_i \leq V).$$

Example 2.6 Let $V = \mathbb{R}^3$ and consider the following subsets of V:

$$U_1 = \left\{ \begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix} : u_1 \in \mathbb{R} \right\}, \qquad U_2 = \left\{ \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} : u_2 \in \mathbb{R} \right\}.$$

Then $W = U_1 + U_2$ is the set

$$W = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} : u_1, u_2 \in \mathbb{R} \right\},$$

which is indeed a subspace of V (see Example 2.2).

In the above example, every $\mathbf{w} \in W$ can be written **uniquely** as a sum of elements in U_1 and U_2. Here is an example where this is not the case.

Example 2.7 Let $V = \mathbb{R}^3$ and consider the following subsets of V:

$$U_1 = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} : u_1, u_2 \in \mathbb{R} \right\}, \qquad U_2 = \left\{ \begin{bmatrix} u_3 \\ 0 \\ 0 \end{bmatrix} : u_3 \in \mathbb{R} \right\}.$$

We note that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

where the first element in each sum is in U_1, with the second in U_2.

These examples indicate that the sum of subspaces should be investigated further: this gives rise to the concept of direct sum.

A special case of a sum of subspaces is that of a direct sum. This concept is described in the following definition.

Definition 2.3 — Direct sum. Let $U_1, U_2 \leq V$. Then the subspace $W = U_1 + U_2$ is said to be the direct sum of U_1 and U_2 if for every non-zero $\mathbf{w} \in W$ there exist unique $\mathbf{u}_1 \in U_1$ and $\mathbf{u}_2 \in U_2$ such that $\mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2$. In this case, we write $W = U_1 \oplus U_2$.

The main interest in introducing the concept of direct sum concerns the representation of a vector space as a direct sum of subspaces U_i

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n = \bigoplus_{i=1}^{n} U_i.$$

In this case, the subspaces can be viewed as 'building blocks' for V. The uniqueness mentioned in the definition relates to the concept of basis which we will encounter later. For now, let us establish ways of identifying direct sums.

Proposition 2.5 — Direct sum criterion 1: trivial intersection. Let $U_1(\mathbb{F}), U_2(\mathbb{F}) \leq V(\mathbb{F})$. Then the subspace $W = U_1 + U_2$ is a direct sum if and only if $U_1 \cap U_2 = \{\mathbf{0}\}$.

Proof. Let $U_1(\mathbb{F}), U_2(\mathbb{F}) \leq V(\mathbb{F})$ and define $W = U_1 + U_2$.
$\boxed{\Longrightarrow}$ Let W be a direct sum: $W = U_1 \oplus U_2$. Assume, by contradiction, that there exists a non-zero $\mathbf{w}$ such that $\mathbf{w} \in U_1 \cap U_2$. Since the intersection is a vector space, we must have $-\mathbf{w} \in U_1 \cap U_2$. Choosing first $\mathbf{w} \in U_1, -\mathbf{w} \in U_2$ and then $-\mathbf{w} \in U_1, \mathbf{w} \in U_2$, we can write

$$\mathbf{0} = \mathbf{w} + (-\mathbf{w}) = (-\mathbf{w}) + \mathbf{w},$$

which contradicts the fact that W is a direct sum (due to the non-unique representation of $\mathbf{0}$ as sum of elements in U_1, U_2). The only possibility is that $\mathbf{w} = \mathbf{0}$, which is a contradiction.
$\boxed{\Longleftarrow}$ Let now $U_1 \cap U_2 = \{\mathbf{0}\}$. We show that $W = U_1 \oplus U_2$ by contradiction. Assume that there exists $\mathbf{w} \in W$ which can be written non-uniquely in two distinct ways as

$$\mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2 = \mathbf{u}_1' + \mathbf{u}_2',$$

where $\mathbf{u}_1, \mathbf{u}_1' \in U_1, \mathbf{u}_2, \mathbf{u}_2' \in U_2$, with $\mathbf{u}_1 \neq \mathbf{u}_1'$ and/or $\mathbf{u}_2 \neq \mathbf{u}_2'$. We can now manipulate the above relation:

$$\mathbf{u}_1 - \mathbf{u}_1' = \mathbf{u}_2' - \mathbf{u}_2 =: \mathbf{v}$$

and since $\mathbf{u}_1 - \mathbf{u}_1' \in U_1$ and $\mathbf{u}_2' - \mathbf{u}_2 \in U_2$ we must have $\mathbf{v} \in U_1 \cap U_2 = \{\mathbf{0}\}$. Hence $\mathbf{v} = \mathbf{0}$ and therefore $\mathbf{u}_1 = \mathbf{u}_1'$ *and* $\mathbf{u}_2 = \mathbf{u}_2'$, which is a contradiction. ■

R The previous result confirms that direct sums allow us to represent a vector space as sums of almost disjoint sets U_1 and U_2, with the only common element being zero. This type of representation of vectors in a vector space is very useful in practice as it avoids 'redundancy'; it also arises quite naturally in contexts that we will consider in the second half of this course.

We end with the following result which describes an alternative characterisation of direct sums.

Proposition 2.6 — Direct sum criterion 2: trivial zero sum. Let $U_1(\mathbb{F}), U_2(\mathbb{F}) \leq V(\mathbb{F})$. Then $W = U_1 + U_2$ is a direct sum if and only if the zero representation given by $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$ implies $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$.

Proof. Let $U_1(\mathbb{F}), U_2(\mathbb{F}) \leq V(\mathbb{F})$ and define $W = U_1 + U_2$.
$\boxed{\Longrightarrow}$ Let W be a direct sum: $W = U_1 \oplus U_2$. Assume by contradiction that $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$, with $\mathbf{u}_1 \neq \mathbf{0} \neq \mathbf{u}_2$. By definition of direct sum, this should be the only representation of $\mathbf{0}$. However, $\mathbf{0} = \mathbf{0} + \mathbf{0}$, with $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$. This is a contradiction.
$\boxed{\Longleftarrow}$ Let the zero representation $\mathbf{0} = \mathbf{u}_1 + \mathbf{u}_2$ imply $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$. Assume by contradiction that W is not a direct sum. Then there exists $\mathbf{w} \in W$ which can be written non-uniquely in two distinct ways as

$$\mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2 = \mathbf{u}_1' + \mathbf{u}_2',$$

where $\mathbf{u}_1, \mathbf{u}_1' \in U_1, \mathbf{u}_2, \mathbf{u}_2' \in U_2$, with $\mathbf{u}_1 \neq \mathbf{u}_1'$ and/or $\mathbf{u}_2 \neq \mathbf{u}_2'$. Then, taking the difference we get

$$\mathbf{0} = (\mathbf{u}_1 - \mathbf{u}_1') + (\mathbf{u}_2 - \mathbf{u}_2') = \mathbf{u}_1'' + \mathbf{u}_2'',$$

where, by closure, $\mathbf{u}_1'' \in U_1$ and $\mathbf{u}_2'' \in U_2$. Since this is a representation of zero, by our assumption we must have $\mathbf{u}_1'' = \mathbf{u}_2'' = \mathbf{0}$, which in turn implies that $\mathbf{u}_1 = \mathbf{u}_1'$ *and* $\mathbf{u}_2 = \mathbf{u}_2'$, which is a contradiction. ■

2.2 Linear combinations. Span

Subspaces can be defined using the concept of linear combinations. We have already come across these when discussing subspace criterion 2 (see remark after Proposition 2.2). Given their obvious relevance, linear combinations can be defined more generally.

Definition 2.4 — Linear combination. Let $V(\mathbb{F})$ be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in V. A linear combination of these vectors is any vector $\mathbf{v}$ of the form

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k,$$

where $a_1, a_2, \ldots, a_k \in \mathbb{F}$.

Any vector in V can be expressed **non-uniquely** as a linear combination of some vectors in V; for example, in the above definition one can choose $a_1 = 1, \mathbf{v}_1 = \mathbf{v}$ and $a_i = 0, i = 2, \ldots, k$. We also note here the special case of the zero vector which can be seen to be **the trivial linear combination** with all scalars a_i equal to zero.

Let us consider now the case of a linear combination that involves a single vector. Let V be a vector space over $\mathbb{F}$ and let $\mathbf{v} \in V$ be given. Then $a\mathbf{v} \in V$ for any $a \in \mathbb{F}$, so that we can identify the following subset of V:

$$U := \{a\mathbf{v} : a \in \mathbb{F}\}.$$

The set U contains just multiples of the given vector $\mathbf{v}$; in this sense, it can be seen as being entirely determined, or generated, by $\mathbf{v}$. We say that U is spanned by $\mathbf{v}$ and write

$$U = \operatorname{span}\{\mathbf{v}\} := \{a\mathbf{v} : a \in \mathbb{F}\}.$$

One can show that U is a subspace, as the exercise below indicates.

Exercise 2.1 Let V be a vector space and let $\mathbf{v} \in V$. Show that $\operatorname{span}\{\mathbf{v}\}$ is a subspace of V.

Can we use more than one vector to define a span? Are spans always subspaces? The answer to the both questions is yes: first, let us generalise the concept of span with the following definition.

Definition 2.5 — Span of a finite set. Let $V(\mathbb{F})$ be a vector space and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ denote a non-empty finite set of vectors in V. The span of S is the set of all linear combinations of vectors in S:

$$\operatorname{span} S := \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} := \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k : a_i \in \mathbb{F}, i = 1, 2, \ldots, k\}.$$

If $S = \emptyset$, we define $\operatorname{span} S = \{\mathbf{0}\}$.

R One may argue that the concept of span when S is empty does not make sense. A possible justification is given by the following standard 'computer' evaluation of a sum using a *for loop*:

$sum = \mathbf{0}$ (set the sum to be zero initially)
for $i = 1 : |S|$
$sum = sum + a_i\mathbf{v}_i$;
end

In the case $S = \emptyset$, the loop will not run, as $|S| = 0$, so the calculation will return the zero vector.

Let us consider some examples to get a feel for the concept of span.

Example 2.8 — Span of column vectors (1). Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\} = \{a_1\mathbf{e}_1 + a_2\mathbf{e}_2 : a_1, a_2 \in \mathbb{R}\} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} : a_1, a_2 \in \mathbb{R} \right\},$$

which is a subspace of $\mathbb{R}^3$ (see Example 2.2).

Example 2.9 — Span of column vectors (2). Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}1\\2\\0\end{bmatrix}.$$

Then

$$\operatorname{span}\{\mathbf{u},\mathbf{v}\} = \{a\mathbf{u}+b\mathbf{v} : a,b\in\mathbb{R}\} = \left\{\begin{bmatrix}a+b\\b\\0\end{bmatrix} : a,b\in\mathbb{R}\right\} = \left\{\begin{bmatrix}a'\\b\\0\end{bmatrix} : a',b\in\mathbb{R}\right\},$$

while

$$\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\} = \{a\mathbf{u}+b\mathbf{v}+c\mathbf{w} : a,b,c\in\mathbb{R}\} = \left\{\begin{bmatrix}a+b+c\\b+2c\\0\end{bmatrix} : a,b,c\in\mathbb{R}\right\} =: \left\{\begin{bmatrix}a'\\b'\\0\end{bmatrix} : a',b'\in\mathbb{R}\right\}.$$

Both spans therefore represent the same subspace of $\mathbb{R}^3$ as in the previous example.

At this stage, it is natural to ask some 'obvious' questions:

- Is a span a subspace?
- Is a subspace a span?
- Can we 'build up' a vector space V using spans?
- Can we characterise the intersection/sum of subspaces in terms of some/any spans?

The following results answers the first two questions.

Proposition 2.7 — A span is a subspace. Let $V(\mathbb{F})$ be a vector space and let $S\subseteq V$. Then span$S$ is a subspace of $V(\mathbb{F})$.

Proof. If $S=\emptyset$, then, by definition, span$S=\{\mathbf{0}\}$. By Exercise 2.1, this is a subspace of $V(\mathbb{F})$. Assume now that S is a non-empty set of vectors in V: $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$. Let $\mathbf{u},\mathbf{v}\in$ span$S$. By definition,

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k, \qquad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k,$$

for some $a_i, b_i\in\mathbb{F}$, $i=1,2,\ldots k$. Then

$$a\mathbf{u}+b\mathbf{v} = a(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k) + b(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k)$$

$$=: c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \in \operatorname{span}S,$$

where $c_i := aa_i + bb_i \in \mathbb{F}$ for $i=1,2,\ldots k$. By the subspace criterion 2 (Proposition 2.2), span$S$ is a subspace of V. ■

Before we answer the second question above, we need to make precise the notion of a **span of an infinite set**. More precisely, if the set S in Definition 2.5 is infinite, then we might be tempted to consider linear combinations which involve all the elements in S. However, not all infinite sums of vectors are well defined, as the following example shows.

Example 2.10 Let $V=\mathbb{R}^2$ and consider the following infinite set of vectors in V:

$$S = \left\{\mathbf{v}_k := \begin{bmatrix}k\\0\end{bmatrix} : k\in\mathbb{N}\right\}.$$

Then
$$\sum_{k=1}^{\infty} \mathbf{v}_k = \begin{bmatrix} \sum_{k=1}^{\infty} k \\ 0 \end{bmatrix} \notin V,$$
since the first entry is not a real number. Note, however, that any finite sum yields a vector in V.

The above example indicates how we can generalise our definition of span to the case where the set is infinite.

Definition 2.6 — Span of an infinite set. Let $V(\mathbb{F})$ be a vector space and let S denote an infinite set of vectors in V. The span of S is the set of all finite linear combinations of vectors in S.

R It is evident that the above definition coincides with Definition 2.5 when S is a finite set. However, it is important to highlight this distinction by giving two separate definitions.

Proposition 2.8 — A subspace is a span. Let $U(\mathbb{F})$ be a subspace of $V(\mathbb{F})$. Then $U = \text{span}U$.

Proof. To show this set equality, we need to show that $U \subseteq \text{span}U$ and $\text{span}U \subseteq U$. By closure, any finite linear combination of elements of U is in U. Hence $\text{span}U \subseteq U$. However, there also holds (trivially, by the definition of span) that $U \subseteq \text{span}U$. Hence, $U = \text{span}U$. ■

Note that since any vector space is its own subspace, we have that $V = \text{span}V$. The remaining questions above can also be answered in the affirmative; the details will be provided in the next lecture. We end this discussion with one last question: how do we establish if a non-zero vector is in a given span? The answer is: via a calculation. We illustrate this with an example.

Example 2.11 Let $U = \text{span}\{\mathbf{u}, \mathbf{w}\}$, where
$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}.$$
A non-zero vector $\mathbf{v}$ is in U if it can be written as a linear combination of $\mathbf{u}$ and $\mathbf{w}$:
$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in U \iff \mathbf{v} = a\mathbf{u} + b\mathbf{w} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ a+4b \\ a+3b \end{bmatrix},$$
where a, b are not both zero. This holds provided that $a = v_1$ and $b = v_2 - v_3$ and that $v_1 \neq 0, v_2 \neq v_3$.

2.3 Spanning sets

We have seen how a set of vectors can be used to generate other vectors as linear combinations. Given $S \subset V$, we have that $\text{span}S \subseteq V$. A natural question to ask is when do we have $\text{span}S = V$? For example, working in $\mathbb{Z}_3^2$, one can establish that
$$\text{span}\{(1,2),(2,1)\} = \{(0,0),(1,2),(2,1)\},$$
while (check this)
$$\text{span}\{(1,2),(1,1)\} = \mathbb{Z}_3^2.$$
In other words, one can generate the whole set $\mathbb{Z}_3^2$ using two elements, provided they are chosen suitably. This is an example of a **spanning set** for a vector space.

Definition 2.7 — Spanning set. Let $V(\mathbb{F})$ be a vector space and let S be a set of vectors in V. We say S is a spanning set for V if $\text{span}S = V$.

R Since $S \subseteq V$, if follows by closure that $\text{span}S \subseteq V$. Hence, to check the set equality $\text{span}S = V$ we only need to check that $V \subseteq \text{span}S$.

Note that every non-trivial vector space has at least one spanning set since $\text{span}V = V$. While they always exist, spanning sets are not unique, with some having considerably fewer vectors than the set V itself.

Example 2.12 The set $V = \mathbb{R}^3$ contains infinitely-many vectors. The following sets contain finitely-many vectors and are both spanning sets for $\mathbb{R}^3$:

$$S_1 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}.$$

2.3.1 Minimal spanning sets

Note that if we remove any vector from S_1 in the above example, then the set is not a spanning set for V anymore, while if we remove any vector from S_2, then the resulting set is still a spanning set for V. These observations seem to suggest that one may be able to trim down a spanning set, but up to a point! To make this more precise, let us provide the following definition.

Definition 2.8 — Minimal spanning set. Let S be a spanning set for a vector space $V(\mathbb{F})$.

i. If $V(\mathbb{F})$ is trivial, the minimal spanning set if defined to be the empty set.
ii. If $V(\mathbb{F})$ is non-trivial, S is called **a minimal spanning set** if

$$\text{span}S \setminus \{\mathbf{v}\} \subset V \quad \text{for any } \mathbf{v} \in S.$$

In other words, S is a minimal spanning set if removing any of its elements also removes the spanning property of S. This is indeed possible as the above example shows in the case of S_1.

R The above definition considers separately the case where $V(\mathbb{F})$ is the trivial vector space, as we want to exclude the possibility that S is the empty set (see Definition 2.5); in this case there is no $\mathbf{v}$ in S and the statement in the above definition does not make sense.

At this stage, the usual existence and uniqueness questions arise:

- does a minimal spanning set always exist?
- if a minimal spanning set exists, is it unique?

We answer the first question for the case where the spanning set is a finite set. The answer to the second question is in the negative (see counter-example below).

Proposition 2.9 Every vector space $V(\mathbb{F})$ has a minimal spanning set.

Proof. If $V(\mathbb{F})$ is trivial, the minimal spanning set exists: it is the empty set, by the above definition. Assume now that $V(\mathbb{F})$ is non-trivial. Let S be a spanning set for $V(\mathbb{F})$ and let $k := |S| \in \mathbb{N}$. Define $S_k := S$. We consider two cases.

i. $k = 1$. In this case, $S_1 = \{\mathbf{v}\}$ for some vector $\mathbf{v} \in V$ and this is a minimal spanning set, since $S_0 := S_1 \setminus \{\mathbf{v}\} = \emptyset$ and $\text{span}S_0 = \{\mathbf{0}\} \subset \text{span}S_1 = V$.
ii. $k > 1$. Let $\mathbf{v} \in S_k$ and define $S_{k-1} := S_k \setminus \{\mathbf{v}\}$. Then $\text{span}S_{k-1} \subseteq \text{span}S_k = V$. If $\text{span}S_{k-1} \subset V$, then, by definition, S_k is a minimal spanning set. Otherwise, $\text{span}S_{k-1} = V$ and S_{k-1} is a spanning set for $V(\mathbb{F})$, with $|S_{k-1}| = k - 1$. We can now repeat the argument for S_{k-1}, S_{k-2} and so on, to deduce that S_r is a minimal spanning set for some $r \leq k$ (and with $r > 1$).

■

The previous result is related to the following property of spanning sets.

Corollary 2.10 Every spanning set contains a minimal spanning set.

To see that minimal spanning sets are not unique, consider the following examples.

Example 2.13 Let $S_1 = \{1, x\}$ and $S_2 = \{1, 1+x\}$. Then spanS_1 = spanS_2 = $\mathcal{P}_1$ and both S_1 and S_2 are minimal spanning sets for $\mathcal{P}_1$, as we cannot remove any elements from either without changing the span.

Example 2.14 Let

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}.$$

It is straightforward to see that both S_1 and S_2 are distinct minimal spanning sets for $V = \mathbb{R}^2$.

The concept of a spanning set S is useful in that it allows us to represent any vector in V as a linear combination of vectors in S, where the span employs the vector space operations associated with $V(\mathbb{F})$. Minimal spanning sets provide this representation in the most 'economical' way. Given a certain choice of minimial spanning set, is this representation unique? We provide an answer to this question in the next lecture.

2.4 Finite-dimensional vector spaces

We discuss the concept of dimension in the next lecture. However, at this stage we can provide an early definition regarding dimensionality based on the discussion on infinite spanning sets.

Definition 2.9 A vector space is said to be **finite-dimensional** if it has a finite spanning set. If there is no finite spanning set, the vector space is said to be **infinite-dimensional**.

Here is a typical example of an infinite-dimensional space.

Example 2.15 Let $V = \mathcal{P}(\mathbb{R})$ denote the vector space of polynomials of arbitrary degree with real coefficients. This vector space is infinite-dimensional. To see this, assume that there exists a finite spanning set containing polynomials $p_i(x)$, $1 \le i \le k$:

$$S = \{p_1(x), p_2(x), \ldots, p_k(x)\}.$$

Then

$$\text{span}S = \{p(x) := a_1 p_1(x) + a_2 p_2(x) + \cdots + a_k p_k(x) : a_i \in \mathbb{R}\},$$

where

$$\deg p = \max_{1 \le i \le k} \deg p_i =: n.$$

Then $x^{n+1} \in \mathcal{P}(\mathbb{R})$, but $x^{n+1} \notin \text{span}S$, which is a contradiction. In other words, for any choice of polynomials p_i and for any finite k (i.e., for any finite set S of vectors in V), there will always exist a polynomial which is not in the span of S. Hence, V must be infinite-dimensional.

Infinite-dimensional vector spaces are important in many areas of mathematics; they will certainly arise in courses you may study later. However, the focus of this course, and of Linear Algebra generally, is on finite-dimensional vector spaces. As such, the remainder of these notes will assume that spanning sets are finite sets and vector spaces are finite-dimensional.