Mechanics: Introduction and crib sheet

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In this document we will define some general notation and recap/preview some of the fundamental mathematics we will use in this module. **This is not necessarily intended as something for you to work through now, but more as a reference for when needed.** This module will rely heavily on being able to solve second order linear ODEs, so if you are less confident on these it may be worth working through that section as revision!

1 Notation

Vectors I will use the printed notation $\mathbf{a}$ for a vector, with handwritten notation $\underline{a}$.

Differentiation I will use "dot" to mean differentiation with respect to time, so that

$$\dot{x} = \frac{dx}{dt},$$

and

$$\ddot{x} = \frac{d^2x}{dt^2}.$$

2 The quadratic formula

The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

3 Solving differential equations

Many problems in mechanics ultimately come down to solving a (system of) differential equation(s) - calculus was originally invented in order to solve questions in mechanics. You will have learnt some techniques for solving ODEs in 1RA(C), we will use these frequently during the mechanics module, so I summarise the key points here:

3.1 Separable first order differential equations

Consider the first order differential equation given by

$$\dot{x} = \frac{dx}{dt} = f(x,t).$$

If $f(x,t)$ can be factored as $f(x,t) = g(x)h(t)$ then

$$\frac{dx}{dt} = g(x)h(t).$$

This equation can be solved by bringing all of the x's onto one side and all of the t's onto the other and then integrating

$$\int \frac{1}{g(x)} dx = \int h(t) dt.$$

The method of partial fractions may be useful. Note that it is not always possible to obtain a closed form solution to a separable differential equation (i.e. it might not be possible to calculate the x integral or the t integral analytically). Also, it not always possible to calculate x explicitly as a function of t.

3.2 Second-order linear ordinary differential equations with constant coefficients

A linear, second-order differential equation with constant coefficients is an equation of the form

$$a\ddot{x} + b\dot{x} + cx = f(t),$$

where a, b, c are constants, and $x(t)$ is a function of time t. This type of equation very often crops up in mechanics examples, so being able to solve them is essential! Recall from RA:

3.2.1 Homogeneous equations

We start by considering the homogeneous case, this is where $f(t) = 0$. Hence we solve

$$a\ddot{x} + b\dot{x} + cx = 0.$$

This is a linear equation, so the solution will be the sum of all possible solutions. We "guess" these will be exponentials such that

$$x = C_1 e^{\lambda t},$$

for some constant C_1 (found using the initial conditions) and some value(s) of λ. We substitute this guess into the equation to find the value(s) of λ that work to solve the equation. As this is a second order equation we expect two values of λ. Hence

$$\begin{aligned} \dot{x} &= \lambda e^{\lambda t}, \\ \ddot{x} &= \lambda^2 e^{\lambda t}, \end{aligned}$$

which gives

$$\begin{aligned} a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} &= 0, \\ \implies a\lambda^2 + b\lambda + c &= 0. \end{aligned}$$

This is called the characteristic or auxillary equation of the ODE. This can be solved using the quadratic formula to give two solutions for λ such that

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Depending on the values of λ we find, the solutions are of different forms:

1. If we have **two real distinct** roots λ_1, λ_2 ($b^2 - 4ac > 0$) the the solution will be of the form

   $$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

   [Note that $\lambda_i = 0$ gives a constant term.]

2. If we have **one repeated root** λ ($b^2 - 4ac = 0$), then the solution will be of the form

   $$x = C_1 e^{\lambda t} + C_2 t e^{\lambda t}.$$

3. If we have **two complex conjugate** roots of the form $\lambda = \alpha \pm i\beta$, ($b^2 - 4ac < 0$) the the solution will be of the form

   $$x = e^{\alpha_1 t} \left(C_1 \cos\left(\beta t\right) + C_2 \sin\left(\beta t\right) \right).$$

Once we have found λ_1, λ_2, α, β and the appropriate form of the solution, we use the initial conditions to find C_1, C_2.

Example 1: Homogeneous ODE

Solve $\ddot{x} + \dot{x} - 2x = 0$ subject to $x(0) = 4$, $\dot{x}(0) = -5$.

Solution. The characteristic equation is given by

$$\begin{aligned} \lambda^2 + \lambda - 2 &= 0, \\ \implies (\lambda - 1)(\lambda + 2) &= 0, \end{aligned}$$

to give the roots $\lambda_1 = 1$, $\lambda_2 = -2$ [or use the quadratic formula]. This is two distinct real toots, so hence the solution is of the form

$$x = C_1 e^t + C_2 e^{-2t}.$$

We now use the initial conditions to find $C_{1,2}$. If

- $x(0) = 4$, then

$$C_1 + C_2 = 4. \tag{1}$$

- $\dot{x}(0) = -5$ then, since $\dot{x} = C_1 e^t - 2C_2 e^{-2t}$, we have

$$\dot{x}(0) = C_1 - 2C_2 = -5. \tag{2}$$

This gives two simultaneous equations for C_1, C_2. Equation (1)-(2) gives $3C_2 = 9$ and hence $C_2 = 3$. Substituting this into the (1) gives

$$\begin{aligned} C_1 + 3 &= 4, \\ \implies C_1 &= 1. \end{aligned}$$

The full solution is hence

$$x = e^t + 3e^{-2t}.$$

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3.2.2 Inhomogeneous equations

What about when $f(t) \neq 0$? This gives an inhomogeneous equation.

$$a\ddot{x} + b\dot{x} + cx = f(t).$$

The solution is now split into two parts:

- the solution to the homogeneous equation (as found above, called the "complementary function" x_c),
- a "particular solution" or "particular integral" which satisfies the whole equation, denoted x_p,

so that $x = x_c + x_p$. How to find x_p depends on the form of $f(t)$ - essentially we "guess" a x_p and then find values of the constants when it works. There are rules about what form to try:

	$f(t)$	Trial solution of the form x_p
polynomial of degree n:	$\alpha_0 + \alpha_1 t + \alpha_2 t^2$ $+ \ldots + \alpha_n t^n$	$A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n$
exponential:	$e^{\alpha t}$	$Ae^{\alpha t}$
trig functions:	$\cos \alpha t$ **or** $\sin \alpha t$	$A \cos \alpha t + B \sin \alpha t$ (**both!**)

then we substitute the trial solution into the equation and find the constants A_0, A_1, A, B etc by equating coefficients of the relevant terms in t.

There are two cases where this must be modified:

1. If $f(t)$ is several of the above functions multiplied together, then use a trial solution which is the relevant trial solutions multiplied together. For example, if $f(t) = t \cos(2t)$, then x_p is of the form

$$x_p = (A_0 + A_1 t)(B_1 \cos(2t) + B_2 \sin(2t)).$$

2. If your trial solution x_p is already contained in the complementary function x_c, multiply it by t. For example, if $f(t) = e^{2t}$ with $x_c = e^{2t}$, then $x_p = Ate^{2t}$.

Example 2: Inhomogeneous ODE

Solve the equation $\ddot{x} + 4x = 8t^2$.

Solution. The solution is $x = x_c + x_p$ where x_c is the complementary function (the solution to the homogeneous problem) and x_p is the particular integral.

We first solve the homogeneous function $\ddot{x}_c + 4x_c = 0$ by forming the characteristic equation

$$\lambda^2 + 4 = 0,$$

which has solutions $\lambda = \pm 2i$ and hence

$$y_c = C_1 \sin(2t) + C_2 \cos(2t),$$

where C_1, C_2 are constants.

We now look for a particular integral. Since $f(t) = 8t^2$ is a polynomial of degree 2, we guess a function of the form

$$x_p = A_0 + A_1 t + A_2 t^2, \tag{3}$$

where we now have to find the right A_i values to make it work. To substitute in we first calculate

$$\begin{aligned} \dot{x}_p &= A_1 + 2A_2 t, \\ \ddot{x}_p &= 2A_2. \end{aligned}$$

Hence we find

$$\begin{aligned} \ddot{x}_p + 4x_p &= 8t^2, \\ \implies 2A_2 + 4\left(A_0 + A_1 t + A_2 t^2\right) &= 8t^2. \end{aligned}$$

We now equate coefficients of the t terms to find the correct values of A_i to make the test solution actually solve the equation.

- constant terms: $2A_2 + 4A_0 = 0$
- linear t terms: $4A_1 = 0$
- quadratic t^2 terms: $4A_2 = 8$.

Hence $A_2 = 2$, $A_0 = -1$ and thus $x_p = -1 + 2t^2$ and the full solution is

$$\begin{aligned} x &= x_c + x_p, \\ &= C_1 \sin(2t) + C_2 \cos(2t) - 1 + 2t^2. \end{aligned}$$

We would then use initial conditions to find C_1 and C_2. ◀

4 Vectors

Another crucial tool for mechanics are concepts using vectors. Some parts of this you might not have seen yet, and will be covered in the first couple of weeks of VGLA, I've included it for completeness!

Vectors have both length/magnitude and direction. They are typically identified in written mathematics with a bold font in typeset work (e.g. $\mathbf{a}$) or underlined in handwritten work (e.g. $\underline{a}$). Distinguishing between vectors and scalars is crucial - r and $\mathbf{r}$ are not the same thing! Equations must be consistent, i.e. in a given equation every term is a scalar, or every term is a vector.

We will use $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as the standard Cartesian vectors, i.e. unit vectors in the x, y, z directions respectively. The magnitude (i.e. length) of a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$ is denoted by $|\mathbf{a}|$ and given by $\sqrt{a_1^2 + a_2^2 + a_3^2}$.

The **scalar or dot product** between two Cartesian vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = (b_1, b_2, b_3)$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{4}$$

Hence $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$. We can also write this as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta,$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular then $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$.

The **vector or cross product** is calculated (in Cartesians) as

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \mathbf{i}\,(a_2 b_3 - b_2 a_3) - \mathbf{j}\,(a_1 b_3 - b_1 a_3) + \mathbf{k}\,(a_1 b_2 - b_1 a_2) && (5) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. && (6) \end{aligned}$$

(The second version is much easier to remember once you've learnt about determinants of matrices (i.e. the $|\ |$) which will also be covered later in VGLA.) The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{a} \times \mathbf{b} = 0$ when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

5 Geometry

Again, some of this may be unfamiliar until it is covered in VGLA later this term. We may use some of these ideas before they have been fully covered, but I'll give you enough detail for what you need at the time!

5.1 Equation of a plane

The equation of a plane consisting of points with position vector $\mathbf{r}$, with normal $\mathbf{n}$ is given by $\mathbf{r} \cdot \mathbf{n} = d$ for some constant d.

5.2 Equation of a circle

In (x, y) Cartesian coordinates, a circle, centre (a, b) of radius R is given by

$$(x - a)^2 + (y - b)^2 = R^2.$$

5.3 Parabola

In (x, y) Cartesian coordinates, an equation of the form

$$y = (x - a)^2 + b,$$

gives a parabola with a minimum at (a, b). Flip the sign to change from a positive parabola with a minimum to a negative parabola with a maximum.

5.4 Curve sketching

To sketch a curve you should think about (for example, as appropriate):

- The domain that is appropriate.
- Where the function intercepts the axes.
- What happens for large values?
- Does it have any maxima or minima or is it monotonic?
- Does the function increase or decrease?
- ...

You can also use curve sketching software such as GeoGebra, Desmos or Maple (plenty of others are available!).

5.5 Conics

A conic section is the shape you get when you take the intersection of a plane with a cone. Depending on the angle of your plane this could be an ellipse, a parabola or a hyperbola. A nice demonstration of these is available at https://www.geogebra.org/m/GmTngth7#material/T8TV2JqG. You will learn about these in VGLA, but not until the end of this semester!

Conics are constructed with a focus (which is a point), a directrix (a line) and have a property called eccentricity (a positive number). A general conic with focus at $(0,0)$ has a polar (i.e. radius r as a function of angle θ) representation

$$r(\theta) = \frac{e\hat{d}}{1 \pm e\cos\theta},$$

with a vertical directrix $x = \pm\hat{d}$ ($\hat{d} > 0$) or

$$r(\theta) = \frac{e\hat{d}}{1 \pm e\sin\theta},$$

with a horizontal directrix $y = \pm\hat{d}$ ($\hat{d} > 0$). The plus/minus sign is chosen to ensure that $\hat{d}$ is positive. The eccentricity e determines which shape the conic is, such that

- $0 < e < 1$ gives an ellipse,
- $e = 1$ gives a parabola,
- $e > 1$ gives a hyperbola.

When $e = 0$, we have the special case of a circle. In this case the polar representation is simply $r = R$ where R is a constant (i.e. the radius is constant).

You will learn about conic sections in VGLA, but at the moment you just need to be able to recognise the shape from the polar representation.

5.6 Taylor Series

The Taylor series of the function $y = f(x)$ about the point $x = c$ is

$$y = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2 + \frac{1}{3!}f'''(c)(x-c)^3 + \frac{1}{4!}f''''(c)(x-c)^4 + \ldots$$

where $'$ denotes d/dx.

6 Binomial approximation

The binomial approximation is given by

$$(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \ldots,$$

for small x. This can be shown by applying a Taylor series expansion about $x = 0$.

7 GCSE/High school science concepts

The following concepts from GCSE science will come up:

Hooke's law The tension in a spring is proportional to its extension, i.e. the difference between its current and natural length. The constant of proportionality is called the spring constant.

The moment of a force A moment gives the turning effect of a force and is taken about a point (typically a pivot - think about a seesaw). The moment is given by Fd where F is the force, and d is the perpendicular distance from the pivot to the line of action.