Mathematical Induction Practice Questions

The following questions relate to Mathematical Induction. Questions are ranked in difficulty from A (basic) to C (challenging).

(A) Question 1. Prove by mathematical induction that for all $n \in \mathbb{N}$:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

(B) Question 2. Let $P(n)$ be a statement for all $n \in \mathbb{Z}$ such that $n \leq m$. State the base step and induction step, in terms of $P(n)$, that could potentially prove that $P(n)$ is true for all $n \in \mathbb{Z}$ with $n \leq m$.

(B) Question 3. Prove by mathematical induction that:

(a) $n^2 - n$ is even, for all $n \in \mathbb{N}$;

(b) $n^3 - n$ is divisible by 3, for all $n \in \mathbb{N}$; and

(c) $(n^2 + n)^2$ is divisible by 4, for all $n \in \mathbb{N}$.

(B) Question 4. Prove by mathematical induction that:

(a) $(n + 1)! > 2^{n+3}$, for all $n \geq 5$; and

(b) $n! > 2^n$ for all $n \geq 4$.

(B) Question 5. Use the Principle of Mathematical Induction to show:

(a) $\left(\bigcup_{k=1}^{n} A_k\right)' = \bigcap_{k=1}^{n} A'_k, \quad \forall n \in \mathbb{N},$

(b) $2^n > n, \quad \forall n \in \mathbb{N},$

(c) $\sum_{k=1}^{n} k^3 = \left[\frac{1}{2}n(n+1)\right]^2, \quad \forall n \in \mathbb{N}.$

(B) Question 6. Prove the following:
Theorem: Let $P(n)$ be a statement for each $n \in \mathbb{N}$ with $n \geq m$. Additionally, suppose that both of the following statements are satisfied:

(i) $P(m)$ is true; and

(ii) for each $k \in \mathbb{N}$, we have

$$P(n) \text{ is true } \forall m \leq n \leq k \implies P(k+1) \text{ is true.}$$

Then $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq m$. *Hint - Consider Theorem 2.2.*

(B) Question 7. For all $n \in \mathbb{N}$, using the PMI, prove that $5^n - 1$ is divisible by 4. *Recall that $n \in \mathbb{N}$ is divisible by a $m \in \mathbb{N}$ if there exists $k \in \mathbb{N}$ such that $n = mk$.*

(C) Question 8. Consider the following argument:
"**Theorem:** $n^2 + n$ is odd for all $n \in \mathbb{N}$.
Proof: Let $P(n)$ be the statement $n^2 + n$ is odd for each $n \in \mathbb{N}$. Certainly it is true that $n = 1$ is odd. Now, suppose that $P(k)$ is true, i.e. $k^2 + k$ is odd. Then,

$$(k+1)^2 + (k+1) = (k^2 + 2k + 1) + (k + 1)$$

$$
\begin{aligned}
&= k^2 + k + (2k+2) \\
&= \text{odd} \ + \ \text{even} \\
&= \text{odd.}
\end{aligned}
$$

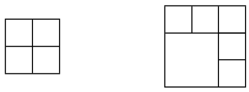
Therefore, by the principle of mathematical induction,

$$n^2 + n \text{ is odd for all } n \in \mathbb{N}.''$$

State whether or not the above proof is correct, and justify your answer.

(C) Question 9. Let $P(n)$ for $n \geq 3$ be the statement, ‘The sum of the internal angles of a convex n-sided polygon is $(n-2)\pi$ radians". Explain in one sentence why $P(3)$ is true, and then use mathematical induction to establish that $P(n)$ is true for all $n \geq 3$.

(C) Question 10. A natural number $n \in \mathbb{N}$ is *fulfilling* if it is possible to fill a square with n sub-squares (not necessarily of the same size). For instance, 4 and 6 are fulfilling since:



(a) Show that 7 and 8 are fulfilling.

(b) Show that if k is fulfilling, then so is $k+3$.

(c) Hence, prove that that all natural numbers greater than 5 are fulfilling.

(d) State whether or not 1, 2, 3, 4 and 5 are fulfilling, and hence describe all fulfilling numbers.