Example sheet 7 – formative

1. Consider the dynamical system

$$\dot{x} = \mu x - y + x^2,$$
$$\dot{y} = x - \sigma y + y^2,$$

where $(x, y) \in \mathbb{R}^2$ and μ, σ are constants.

(a) Determine the nature of the equilibrium point $(0, 0)$ for each $\mu, \sigma \geqslant 0$.

(b) Sketch the (σ, μ)-plane and indiate what the equilibrium is in each defined area.

2. Consider the 2-dimensional dynamical system

$$\begin{aligned} \dot{x} &= (1 - x - y)x, \\ \dot{y} &= (4 - 7x - 3y)y, \end{aligned} \tag{1}$$

where $x, y \geqslant 0$.

(a) Find the equilibrium points of (1).

(b) Determine the horizontal and vertical isoclines for dynamical system (1). and find the direction of the flow on them.

(c) Consider the region $D = \left\{(x, y) : 0 < x < 1, 0 < y < \frac{3}{2}\right\}$ with boundary $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where:

$$C_1 = \{(x, y) : y = 0, x \in [0, 1]\}$$
$$C_2 = \left\{(x, y) : x = 1, y \in \left[0, \frac{3}{2}\right]\right\}$$
$$C_3 = \left\{(x, y) : y = \frac{3}{2}, x \in [0, 1]\right\}$$
$$C_3 = \left\{(x, y) : x = 0, y \in \left[0, \frac{3}{2}\right]\right\}$$

and establish that $D \cup C$ is a positively invariant set for dynamical system (1).

(d) Using parts b and c locate any positively invariant sets for (1) within $D \cup C$.

3. Show that the nonlinear system

$$\dot{x} = -y + x\left(1 - \sqrt{x^2 + y^2}\right),$$
$$\dot{y} = x + y\left(1 - \sqrt{x^2 + y^2}\right),$$

has a limit cycle given by $x^2 + y^2 = 1$.

4. Use the Poincaré-Benedixson theorem to establish that the dynamical system

$$\begin{aligned}
\dot{x} &= x + y - x^3 + xy^2 - x\left(x^2+y^2\right)^2, \\
\dot{y} &= y - x - x^2y + y^3 - y\left(x^2+y^2\right)^2,
\end{aligned}$$

where $(x, y) \in \mathbb{R}^2$ has at least one periodic orbit surrounding the origin. Note that $(0, 0)$ is the only equilibrium point.