University of Birmingham
School of Mathematics

1RA Differentiation Autumn 2022

Problem Sheet 3

issued Week 6

You have approximately 10 working days to complete and submit the $\boxed{\text{SUM}}$ questions (**Q4** and **Q5**) and you may begin working on it immediately.

Assignment available from: **4 November** Submission due: **1700 on Wednesday 16 November 2022**	
Pre-submission	Post-submission
• Your Guided Study Support Class in Weeks 6-8. • Tutor meetings in Weeks 6-8. • PASS from Week 7 • Library MSC from Week 7 • Office Hours: Wednesday 1300-1430 and Friday 1000-1130.	• Written feedback on your submission. • Generic feedback (24 November). • Model solutions (24 November). • Tutor meetings in Week 9. • Office Hours: Wednesday 1300-1430 and Friday 1000-1130

Instructions:

You will spend the next two weeks (including your Guided Study Support Class in weeks 4 and 5 working on the $\boxed{\text{SUM}}$ questions (**Q4** and **Q5**).

The **deadline** for submission is as follows:

- **By 1700 on Wednesday 16 November 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

Important:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

Q1. Prove the following statements by directly using the definition of derivative.

(i) If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x^2 + 5$ for all $x \in \mathbb{R}$, then $f'(x) = 6x$.

(ii) If $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by $g(x) = 3/(1-x)$ for all $x \in \mathbb{R} \setminus \{1\}$, then $g'(x) = 3/(1-x)^2$.

(iii) If $k : \mathbb{R} \to \mathbb{R}$ is given by $k(x) = \sin(2x)$ for all $x \in \mathbb{R}$, then $k'(0) = 2$.

(iv) If $r : \mathbb{R} \to \mathbb{R}$ is given by

$$r(x) = \begin{cases} x^3 \sin \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

then $r'(0) = 0$.

Q2. Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $\displaystyle\lim_{x\to 0} \frac{17^x - 1}{2x}$.

(ii) $\displaystyle\lim_{x\to 2} \frac{\log(2x-3)}{\tan(x-2)}$.

(iii) $\displaystyle\lim_{x\to 0} \frac{x^2}{e^x - 1 - x}$.

(iv) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^4}$.

Q3. Differentiate the following functions (i.e., find a formula for the derivative of each of the functions). You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $f(x) = \dfrac{\sin(x^2)\sin^2 x}{2 + \sin x}$.

(ii) $g(x) = e^{\sin x} + \cos(x + \sin x)$.

(iii) $k(x) = (5 + \sin x)^{\log x}$.

(SUM) **Q4**. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x^4 - 26x^3 + 60x^2 - 11$.

(i) Find the stationary points of f, and determine whether they are local maximum or minimum points of f.

(ii) Determine the intervals in $\mathbb{R}$ where f is increasing and those where f is decreasing.

(iii) Determine the intervals in $\mathbb{R}$ where f is convex and those where f is concave.

(iv) Find the limits $\displaystyle\lim_{x\to -\infty} f(x)$ and $\displaystyle\lim_{x\to\infty} f(x)$.

(v) Find $\inf f$ and $\sup f$, and determine whether f has a (global) maximum and/or a minimum.

Justify your answers.

(SUM) **Q5**. (a) Find, with proof, the supremum of the set

$$A := \{f(x) : x \in [1,2]\} \cup \{x : x \in [1,2]\},$$

where $f : [0,4] \to \mathbb{R}$ is given by $f(x) := x^2 + 2$ for all $x \in [0,4]$.

(b) Find, with proof, the infimum of the set

$$B := \left\{ \frac{4n^3 + 3n + 1}{n^3} : n \in \mathbb{N} \right\}.$$

(c) Suppose that $P = \{-10, -2, 0, 1, 5\}$ and $g : [-10, 5] \to \mathbb{R}$ is given by

$$g(x) := \begin{cases} \dfrac{x + 2|x|}{|x|}, & \text{if } x \neq 0; \\ 4, & \text{if } x = 0. \end{cases}$$

(i) Calculate the Riemann–Darboux sums $L(g,P)$ and $U(g,P)$.

(ii) Find a partition Q of $[-10, 5]$ such that $U(g,Q) - L(g,Q) < 0.001$.

Q6. (a) Calculate the Riemann–Darboux sums $L(f,P)$ and $U(f,P)$ for each of the following functions f and partitions P:

(i) $f : [0,10] \to \mathbb{R},\ f(x) = x^2,\ P = \{0, 2, 7, 10\}$

(ii) $f : [0,10] \to \mathbb{R},\ f(x) = e^{-x},\ P = \{0, 1, 5, 8, 9, 10\}$

(iii) $f : [0,\pi] \to \mathbb{R},\ f(x) = \sin(x),\ P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$

(b) Use the Riemann–Darboux sums calculator (Week 7 Materials on Canvas) to visualize and check your answers above (note that $\frac{3\pi}{4}$ is entered as 3*Pi/4).

(c) For each of the functions f above, use the Riemann–Darboux sums calculator to calculate $L(f,P_n)$ and $U(f,P_n)$ when $n = 5$, $n = 10$ and $n = 100$, where P_n is the partition of the domain of f into n subintervals of equal width.

(d) For each $\delta \in (0, 1/10)$, find an expression for the sums $L(f,P_\delta)$ and $U(f,P_\delta)$ when $f : [-2,2] \to \mathbb{R}$, $f(x) = \begin{cases} 5, & x \in \mathbb{N} \\ 3, & x \notin \mathbb{N} \end{cases}$ and $P_\delta = \{-2, 1-\delta, 1+\delta, 2-\delta, 2\}$.

Q7. It is proved in Lecture 7.3 of the Integration Lecture Notes that if $f : [a,b] \to [0,\infty)$ is a bounded function, where $-\infty < a < b < \infty$, and P, Q are partitions of $[a,b]$ such that $P \subseteq Q$, then $L(f,P) \leq L(f,Q)$. List the changes that are needed to prove that $U(f,P) \geq U(f,Q)$, and in particular, explain how the inequalities $m_1 \leq m_1'$ and $m_1 \leq m_1''$ need to be modified.

Q8. Let $f : [0,b] \to [0,\infty)$ be defined by $f(x) = x^2$, where $b \in (0,\infty)$:

(a) For each $n \in \mathbb{N}$, let $P_n = \{x_i : i = 0, 1, \ldots, n\}$ denote the partition of $[0,b]$ into n subintervals of equal width. Express x_i in terms of i and b.

(b) Use the formula $\sum_{j=1}^{k} j^2 = \frac{1}{6}k(k+1)(2k+1)$ to prove that

$$L(f,P_n) = \frac{b^3}{6n^3}(n-1)n(2n-1) \quad \text{and} \quad U(f,P_n) = \frac{b^3}{6n^3}n(n+1)(2n+1).$$

(c) Find, with proof, the values $\sup\{L(f,P_n) : n \in \mathbb{N}\}$ and $\inf\{U(f,P_n) : n \in \mathbb{N}\}$.

(d) Find, with proof, the lower integral $\underline{\int_0^b} f$ and the upper integral $\overline{\int_0^b} f$.

(e) Prove that f is bounded and integrable, then find $\int_0^b f$ (without using calculus).

Extra Questions

[Questions marked with a * may be more challenging than others.]

EQ1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x|x|$ for all $x \in \mathbb{R}$.

(i) Prove that the function f is differentiable, and find a formula for its derivative $f' : \mathbb{R} \to \mathbb{R}$.
(ii) Is f twice differentiable? Justify your answer.

EQ2. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) If $f : (0, 1) \to \mathbb{R}$ is continuous, then f is bounded.
(ii) If $g : (0, 1) \to \mathbb{R}$ is continuous, then g is differentiable.
(iii) If $k : [0, 1] \to \mathbb{R}$ is differentiable, then k is bounded.

EQ3. Prove that the function $\arcsin : [-1, 1] \to \mathbb{R}$ is not differentiable at any of the points 1 and -1. [Hint: apply the Chain Rule to the identity $\sin \arcsin x = x$.]

* **EQ4**. Let $A \subseteq \mathbb{R}$. Recall that, by the Leibniz rule, if $f, g : A \to \mathbb{R}$ are differentiable, then their product fg is differentiable too, and the formula

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

holds for all $x \in \mathbb{R}$.

(i) Let $f_1, f_2, f_3 : A \to \mathbb{R}$ be differentiable functions. Prove that their product $f_1f_2f_3$ is differentiable too, and find a formula for its derivative $(f_1f_2f_3)'$. [Hint: $f_1f_2f_3 = (f_1f_2)f_3$.]
(ii) Let $n \in \mathbb{N}$, and let $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$ be differentiable functions. Prove that their product $f_1f_2 \cdots f_n$ is differentiable. [Hint: induction on n.]
(iii) Can you write a formula for the derivative $(f_1f_2 \cdots f_n)'$?

EQ5. A factory has received an order for open-top metal cans of a given volume V, having the shape of a cylinder with circular base. What should the height of each of these cans be, so as to minimise the amount of metal used to produce them?

EQ6. For each of the following statements, give a proof if it is true, or provide a counterexample if it is false.

(i) Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be differentiable. If $f'(x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$, then f is constant.
(ii) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and strictly increasing. Then $g'(x) > 0$ for all $x \in \mathbb{R}$.
(iii) Let $h : \mathbb{R} \to \mathbb{R}$ be strictly increasing. Then h is differentiable.
(iv) Let $k : \mathbb{R} \to \mathbb{R}$ be convex. Then k is twice differentiable.

EQ7. Let $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be given by

$$f(x) = e^x, \qquad g(x) = 1 + x + x^2/2$$

for all $x \in \mathbb{R}$.

(i) Prove that $f''(x) \geq g''(x)$ for all $x \in [0, \infty)$.
(ii) Prove that $f' - g'$ is an increasing function. [Hint: consider the sign of its derivative.]
(iii) Prove that $f'(x) - g'(x) \geq 0$ for all $x \in [0, \infty)$. [Hint: evaluate the l.h.s. at $x = 0$, and then use part (ii).]
(iv) Prove that $f(x) - g(x) \geq 0$ for all $x \in [0, \infty)$. [Hint: repeat the above steps with f and g instead of f' and g'.]
(v) Deduce that

$$e^x \geq 1 + x + x^2/2$$

for all $x \in [0, \infty)$.

EQ8. Let $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$. Recall that, by the Sum Rule for differentiation, if f and g are differentiable, then their sum $f + g$ is differentiable as well and the formula

$$(f + g)'(x) = f'(x) + g'(x)$$

holds for all $x \in A$.

(i) Assume that f and g are twice differentiable. Prove that $f + g$ is twice differentiable and find a formula for the second derivative $(f + g)''$.
(ii) Let $n \in \mathbb{N}$, and assume that f and g are n times differentiable. Prove that $f + g$ is n times differentiable and find a formula for its nth derivative. [Hint: induction.]

* **EQ9**. Let $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$. Recall that, by the Leibniz Rule, if f and g are differentiable, then their product fg is differentiable as well and the formula

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

holds for all $x \in A$.

(i) Assume that f and g are twice differentiable. Prove that fg is twice differentiable and find a formula for the second derivative $(fg)''$.
(ii) Let $n \in \mathbb{N}$, and assume that f and g are n times differentiable. Prove that fg is n times differentiable. [Hint: use induction and **EQ8**.]
(iii) Under the same assumptions as in part (ii), can you write a formula for the nth derivative $(fg)^{(n)}$?

EQ10. Let $a \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{ax^2} & \text{if } x < 0, \\ \cosh x & \text{if } x \geq 0. \end{cases}$$

(i) Determine all the values of $a \in \mathbb{R}$ such that the function f defined above is twice differentiable.
(ii) For each of the values found in part (i), determine whether the function f is convex.

EQ11. (i) Prove that the function $\log : (0,\infty) \to \mathbb{R}$ is concave.
(ii) Prove that, for all $a, b \in (0,\infty)$ and $t \in [0,1]$, the inequality

$$a^t b^{1-t} \le ta + (1-t)b.$$

holds. [Hint: take the logarithm of both sides, and use part (i).]
(iii) Prove that the inequality in part (ii) holds more generally for all $a, b \in [0,\infty)$ and $t \in [0,1]$.
(iv) Prove that, for all $x, y \in [0,\infty)$ and all $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$,

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}.$$

[Hint: apply part (iii) with $t = 1/p$, $a = x^p$, $b = y^q$.]

EQ12. (a) Find, with proof, the infimum and supremum of each of the following sets:
(i) $A = (0,1] \cup (2,3]$
(ii) $B = \{x^2 - 4x + 5 : x \in (1,3]\}$
(iii) $C = \{(2n+3)/n : n \in \mathbb{N}\}$
(iv) $D = \{n^2 - 6n + 10 : n \in \mathbb{N}\}$
(b) Let X denote a nonempty bounded subset of $\mathbb{R}$. Prove that for each $\epsilon > 0$, there exists x in X such that $\inf X \le x < \inf X + \epsilon$.

EQ13. For each $n \in \mathbb{N}$, let P_n denote the partition of $[0,1]$ into n subintervals of equal width. Find an expression, possibly involving summation notation, for the sums $L(f, P_n)$ and $U(f, P_n)$ for $f : [0,1] \to \mathbb{R}$ in each of the following cases:
(a) $f(x) = x^2 + x$
(b) $f(x) = \cos(x)$
(c) $f(x) = \begin{cases} 5, & x \in \mathbb{Q} \\ 3, & x \notin \mathbb{Q} \end{cases}$
(d) $f(x) = \begin{cases} 1, & x \in \{0,1\}, \\ 0, & x \notin \{0,1\}. \end{cases}$

EQ14. Let P_1 denote a partition of $[a,b]$ and $P_2 = P_1 \cup \{c\}$, where $-\infty < a < c < b < \infty$. Use results from lectures to prove that $U(f,P_2) - L(f,P_2) \le U(f,P_1) - L(f,P_1)$.

EQ15. Let $f : [a,b] \to \mathbb{R}$ be defined by $f(x) = x^3$, where $0 \le a < b < \infty$. Use the procedure outlined in **Q8** and the formula $\sum_{j=1}^{k} j^3 = \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2$ to prove that f is integrable with $\int_a^b f = \frac{1}{4}b^4 - \frac{1}{4}a^4$.