University of Birmingham
School of Mathematics

1RA Integration Autumn 2022

**Problem Sheet 5**
issued Week 10

You have approximately 10 working days from the release of this problem sheet to complete and submit your answers to the $\boxed{\text{SUM}}$ questions (**Q3** and **Q5**) via the Assignments tab on the 1RA Canvas page. You are strongly encouraged to attempt all of the remaining formative questions, and as many of the extra questions as you can, to prepare for the final exam. But only your solutions to the $\boxed{\text{SUM}}$ questions should be submitted to Canvas.

<table>
<tr><td colspan="2">Assignment available from: 2 December<br>Submission due: 1700 on Wednesday 14 December 2022</td></tr>
<tr><td>Pre-submission</td><td>Post-submission</td></tr>
<tr><td>• Your Guided Study Support Class in Weeks 11.<br>• Tutor meetings in Weeks 11.<br>• PASS from Week 10<br>• Library MSC from Week 10<br>• Office Hours (Watson 208):<br>Friday 1000-1130.</td><td>• Written feedback on your submission (22 December).<br>• Generic feedback (22 December).<br>• Model solutions (22 December).<br>• Revision Lectures (Week 12)<br>• Office Hours (Watson 208):<br>Friday 1000-1130.</td></tr>
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**Instructions**:

The **deadline** for submission of the two $\boxed{\text{SUM}}$ questions (**Q3** and **Q5**) is as follows:

- **By 1700 on Wednesday 14 December 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

**Important**:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

**Q1**. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \sin^3(3x)\cos^5(3x)\ \mathrm{d}x$

(b) $\displaystyle\int \frac{\sin^7(x)}{\cos^4(x)}\ \mathrm{d}x$

(c) $\displaystyle\int_0^{\frac{\pi}{4}} (\tan(x)\sec(x))^8\ \mathrm{d}x$

**Q2**. (a) For each $n \in \mathbb{N} \cup \{0\}$, suppose that $T_n(x) = \int \tan^n(4x)\ \mathrm{d}x$:

(i) Prove that

$$T_n(x) = \frac{\tan^{n-1}(4x)}{4(n-1)} - T_{n-2}(x)$$

for all $x \in (-\frac{\pi}{8}, \frac{\pi}{8})$ and all integers $n > 2$.

(ii) Use the reduction formula above to find $\int \tan^3(4x)\ \mathrm{d}x$.

(b) Find the following antiderivatives or integrals:

(i) $\displaystyle\int \tan^5(x)\sec^6(x)\ \mathrm{d}x$

(ii) $\displaystyle\int_0^{\pi/4} \cos^2(6x)\sin^2(6x)\ \mathrm{d}x$

(iii) $\displaystyle\int \frac{x^2+5x-4}{x^3-x}\ \mathrm{d}x$

SUM **Q3**. Find, or prove divergence of, the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{6x+1}{x^2+3x+5}\ \mathrm{d}x$

(b) $\displaystyle\int \sec^4(3x)\tan^4(3x)\ \mathrm{d}x$

(c) $\displaystyle\int_2^{\infty} \frac{1}{\sqrt{2x-3}}\ \mathrm{d}x$

(d) $\displaystyle\int_5^6 \frac{1}{\sqrt{x^2-25}}\ \mathrm{d}x$

**Q4**. Find the value of the following improper integrals or prove that they are divergent:

(a) $\displaystyle\int_0^1 x\log(x)\ \mathrm{d}x$

(b) $\displaystyle\int_0^{10} \frac{x}{x-5}\ \mathrm{d}x$

(c) $\displaystyle\int_{-\frac{\pi}{2}}^{0} \sec(x)\ \mathrm{d}x$

$\boxed{\text{SUM}}$ **Q5**. (a) Suppose that $f : [0,4] \to [0,1]$ is given by

$$f(x) := \begin{cases} x, & \text{if } 0 \le x \le 1; \\ 1, & \text{if } 1 < x \le 4. \end{cases}$$

Find a solution $y : [0,4) \to \mathbb{R}$ of the initial value problem

$$y' = f(x), \quad y(0) = 1.$$

You must prove that your solution is indeed differentiable on $(0,4)$.

(b) Find a solution $y : [0,\infty) \to \mathbb{R}$ of the initial value problem

$$yy' = \log x, \quad y(0) = 2.$$

You must justify all limit computations.

**Q6**. A swimming pool by the sea has a capacity of 5 000 000 L and the concentration of salt in the seawater is 0.045 kg $\mathrm{L}^{-1}$. The pool is initially filled with pure water. The concentration of salt in the pool is then increased by pumping in seawater at a rate of 3 000 L $\mathrm{min}^{-1}$ whilst the pool is drained at the same rate. Assume that the mixture in the pool is instantly and uniformly mixed:

(a) Let $y(t)$ denote the mass (in kilograms) of salt in the pool at time $t$ (in minutes) after mixing begins. Formulate an initial value problem to model the flow $y'(t)$.

(b) Find a solution to your initial value problem and determine how long it will take for the salt concentration in the swimming pool to reach 0.0035 kg $\mathrm{L}^{-1}$?

(c) Suppose instead that the pump operates at 1 000 L $\mathrm{min}^{-1}$ whilst the pool is drained at 3 000 L $\mathrm{min}^{-1}$. Formulate and solve an initial value problem to determine the mass of salt in the swimming pool after $t$ minutes of mixing for all $t \in [0, +\infty)$.

**Q7**. Find the general solution of the following homogeneous equations on $\mathbb{R}$, and where specified, find a solution of the initial value problem or boundary value problem.

(a) $y'' - 7y' + 12y = 0$.

(b) $y'' = -64y, \quad y(0) = 0, \quad y'(0) = 3, \quad y : [0,\infty) \to \mathbb{R}$.

(c) $y'' - 2y' + y = 0, \quad y(0) = 1, \quad y(1) = 2, \quad y : [0,1] \to \mathbb{R}$.

**Q8**. Find the general solution of the following inhomogeneous equations on $\mathbb{R}$, and where specified, find a solution of the initial value problem or boundary value problem.

(a) $y'' - 2y' + 10y = e^x$.

(b) $y'' + 5y' + 4y = 3 - 2x, \quad y(0) = 0, \quad y'(0) = 0, \quad y : [0,\infty) \to \mathbb{R}$.

(c) $y'' + 9y = x\cos x, \quad y(0) = 1, \quad y(\frac{\pi}{2}) = \frac{1}{32}, \quad y : [0, \frac{\pi}{2}] \to \mathbb{R}$.

## Extra Questions

**EQ1**. A cylindrical water tank is mounted sideways, so it has a circular vertical cross-section. The tank is 10 meters in diameter. Calculate the percentage of the tank's capacity that is filled when the height of the water in the tank is 6 meters.

(a) Calculate the percentage of the tank's capacity that is filled with water when the height of the water in the tank is 6 meters.

(b) Calculate the height of the water in the tank when it is at 33% capacity.

**EQ2**. (a) For each $n \in \mathbb{N} \cup \{0\}$, suppose that $I_n = \int (\log(x))^n \, \mathrm{d}x$. Prove that

$$I_n(x) = x(\log(x))^n - nI_{n-1}(x)$$

for all $x > 0$ and all $n \in \mathbb{N}$, and use this formula to find $\int (\log(x))^5 \, \mathrm{d}x$.

(b) Find the following antiderivatives:

(i) $\displaystyle\int \frac{4x+1}{\sqrt{x-4}} \, \mathrm{d}x$

(ii) $\displaystyle\int \frac{\sqrt{x+1}}{x} \, \mathrm{d}x$

(iii) $\displaystyle\int \frac{1}{\sqrt{x^2+16}} \, \mathrm{d}x$

**EQ3**. A bounded function $f : \mathbb{R} \to \mathbb{R}$ is integrable if there exists $a \in \mathbb{R}$ such that both $\int_{-\infty}^{a} f$ and $\int_{a}^{\infty} f$ are convergent and finite, in which case $\int_{-\infty}^{\infty} f := \int_{-\infty}^{a} f + \int_{a}^{\infty} f$.

(a) Use the properties of integrals to prove that the value of $\int_{-\infty}^{\infty} f$ does not depend on the value of $a$ in this definition. (In other words, you need to prove that $\int_{-\infty}^{a} f + \int_{a}^{\infty} f = \int_{-\infty}^{b} f + \int_{b}^{\infty} f$ for all $a$, $b \in \mathbb{R}$ whenever each side is defined.)

(b) Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := (1 + x^2)^{-1}$ for all $x \in \mathbb{R}$ is integrable and calculate the value of $\int_{-\infty}^{\infty} f$.

(c) Prove that $g : [1, \infty) \to \mathbb{R}$ given by $g(x) := e^{-x}(4x^2 + 3x)^{-1}$ for all $x \in [1, \infty)$ is integrable.

**EQ4**. Find all real numbers $p$ such that the following properties hold:

(a) The integral $\int_0^1 x^p \, \mathrm{d}x$ is an improper integral.

(b) The improper integral $\int_0^1 x^p \, \mathrm{d}x$ is convergent.

**EQ5**. (a) Find three solutions of the differential equation $y' = \cos\left(3x + \frac{\pi}{3}\right)$ on $\mathbb{R}$.

(b) Find a solution $y : [0, \infty) \to \mathbb{R}$ of the initial value problem

$$x^2 y' = y^3, \quad y(0) = 0.$$

(c) Find solutions $y : [0, R) \to \mathbb{R}$, for some $R \in [0, \infty]$, of the initial value problem

$$y' = y^2 + y - 12, \quad y(0) = y_0$$

for each $y_0 \in \{2, 3, 5\}$.

**EQ6**. Find solutions $y : [0, \infty) \to \mathbb{R}$ of the following initial value problems:

(a) $y' + y = \cos(e^x), \quad y(0) = 9.$

(b) $y' + 2xy = 4x, \quad y(0) = y_0 \in \mathbb{R}.$

(c) $xy' = y + x^3 + 3x^2 - 2x, \quad y(0) = 0.$

**EQ7**. Suppose that $a$, $b$, $c \in \mathbb{R}$ with $a \neq 0$. Let $y_1 : [0,1] \to \mathbb{R}$ and $y_2 : [0,1] \to \mathbb{R}$ denote solutions of the respective boundary value problems below:

$$ay_1'' + by_1' + cy_1 = 0, \quad y_1(0) = 1, \quad y_1(1) = 3;$$
$$ay_2'' + by_2' + cy_2 = 0, \quad y_2(0) = 2, \quad y_2(1) = -5.$$

(a) Prove that $y_3 := 2y_1 + 3y_2$ is a solution of the boundary value problem

$$ay'' + by' + cy = 0, \quad y(0) = 8, \quad y(1) = -9.$$

(b) Let $\alpha$, $\beta \in \mathbb{R}$. Find a solution $y_4 : [0,1] \to \mathbb{R}$ of the boundary value problem

$$ay'' + by' + cy = 0, \quad y(0) = \alpha, \quad y(1) = \beta$$

in terms of $y_1$ and $y_2$.

**EQ8**. Hooke's Law states that a spring with spring constant $k > 0$ exerts a force $F = -ky$ (in Newtons) when stretched a distance $y$ (in metres) from its equilibrium position. Suppose that an object with mass $m = 5$ kg is attached to the end of a spring with spring constant $k = 100$. Combine Hooke's Law with Newton's Law $F = my''$ to formulate initial value problems or boundary value problems, as appropriate, to model each of the following scenarios. In each case, determine the distance $y(t)$ of the object from its equilibrium position at time $t$ (in seconds) after it is released:

(a) The object is released at rest at a distance of 1 m from its equilibrium position.

(b) The object is released at a distance of 1 m from its equilibrium position so that after 1 second it has travelled 0.7 m.

(c) The object is released at rest at a distance of 1 m from its equilibrium position and it is subject to an additional force of $5\sin(2\sqrt{5}t)$ N at time $t$.