

1Mech — Mechanics

Mechanics exercises 4 (weeks 7 and 8)

Mechanics solutions 4

This sheet's assessed question is question 5.

1. If we throw a particle of mass m upwards from the top of a wall of height h with velocity v , how high will it go and what will its velocity be when it hits the ground?

Solution. Let $z(t)$ denote the vertical position of the mass (measured upwards), where $z = 0$ denotes ground level. Then conservation of energy gives

$$\frac{1}{2}m\dot{z}^2 + mgz = E,$$

where we take the gravitational potential energy, mgz to be zero at ground level. Here $\frac{1}{2}m\dot{z}^2$ gives the kinetic energy.

At $t = 0$, we know $z = h$ and $\dot{z} = v$, hence

$$E = \frac{1}{2}m\dot{z}^2 + mgz = \frac{mv^2}{2} + mgh.$$

The maximum height of the particle occurs when $\dot{z} = 0$, as at this point the particle is no longer moving upwards. Hence the maximum height H satisfies

$$\begin{aligned} mgH &= \frac{mv^2}{2} + mgh, \\ \implies H &= \frac{v^2}{2g} + h. \end{aligned}$$

The particle hits the ground when $z = 0$. The speed at this point is \dot{z} such that

$$\begin{aligned} \frac{1}{2}m\dot{z}^2 &= \frac{mv^2}{2} + mgh, \\ \implies \dot{z} &= \sqrt{v^2 + 2gh}. \end{aligned}$$



Feedback: This question is hopefully not too challenging, but aims to get you thinking in terms of conservation of energy.

2. Let a particle of mass m be attached to two springs, both with spring constant k and natural length a . The end of one spring (denoted α) is attached at a point A , with the end of the other spring (denoted β) attached at a point B , a distance $4a$ directly above A . The mass is attached to the free end of both springs. The particle is at a location $x(t)$, where x is measured upwards such that $x = 0$ at point A .

- (a) Show that

- i. the extension in spring α is given by $x - a$.
- ii. the extension in spring β is given by $3a - x$.

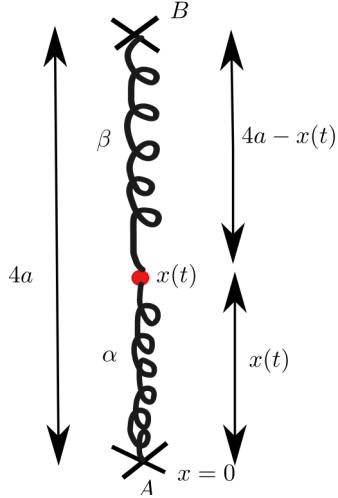


Figure 1: The setup for Question 4, showing two springs stretched between points A and B , with the particle at location $x(t)$ in between them. The top spring has length $4a - x(t)$, and the bottom has length $x(t)$.

- (b) Hence write down the equation for conservation of energy.
- (c) If the particle is initially at rest at $x = a$, find the value of the constant energy.
- (d) Find the height at which the particle will next come to rest.

Solution. (a) The setup is shown in Figure 1.

- i. Spring α has current total length $x(t)$, with natural length a , so the extension is $x - a$
- ii. Spring β has current total length $4a - x(t)$, with natural length a , so the extension is

$$4a - x - a = 3a - x.$$

- (b) The total energy in the system consists of four parts:

- The kinetic energy pf the particle $\frac{1}{2}m\dot{x}^2$.
- The potential energy in spring α , $\frac{k(x-a)^2}{2}$, where k is the spring constant.
- The potential energy in spring β , $\frac{k(3a-x)^2}{2}$, where k is the spring constant.
- The gravitational potential energy of the particle, mgx , where we take the zero of the GPE to be at $x = 0$.

Hence conservation of energy gives

$$\frac{1}{2}m\dot{x}^2 + \frac{k(x-a)^2}{2} + \frac{k(3a-x)^2}{2} + mgx = \text{constant.}$$

(c) Initially $\dot{x} = 0$ and $x = a$ and hence

$$\begin{aligned}\frac{1}{2}m\dot{x}^2 + \frac{k(x-a)^2}{2} + \frac{k(3a-x)^2}{2} + mgx &= 0 + 0 + \frac{k(3a-a)^2}{2} + mga, \\ &= \frac{k(2a)^2}{2} + mga, \\ &= 2ka^2 + mga.\end{aligned}$$

(d) The particle will be (instantaneously) at rest when $\dot{x} = 0$, hence

$$\frac{k(x-a)^2}{2} + \frac{k(3a-x)^2}{2} + mgx = 2ka^2 + mga.$$

This is a quadratic equation in x , where we know that $x - a$ must be a root (since $\dot{x} = 0$ initially). So we can write

$$\begin{aligned}0 &= \frac{k(x-a)^2}{2} + \frac{k(3a-x)^2}{2} - 2ka^2 + mg(x-a), \\ &= \frac{k(x-a)^2}{2} + \frac{k(9a^2 - 6ax + x^2)}{2} - 2ka^2 + mg(x-a), \\ &= \frac{k(x-a)^2}{2} + \frac{k(9a^2 - 4a^2 - 6ax + x^2)}{2} + mg(x-a), \\ &= \frac{k(x-a)^2}{2} + \frac{k(5a^2 - 6ax + x^2)}{2} + mg(x-a), \\ &= \frac{k(x-a)^2}{2} + \frac{k(x-a)(5a-x)}{2} + mg(x-a), \\ &= \frac{k(x-a)^2}{2} + \frac{k(x-a)(5a-x)}{2} + mg(x-a), \\ &= (x-a) \left(\frac{k(x-a)}{2} + \frac{k(5a-x)}{2} + mg \right), \\ &= (x-a)(kx - 3ka + mg), \\ &= k(x-a) \left(x - \left(3a - \frac{mg}{k} \right) \right).\end{aligned}$$

Therefore the particle moves between $x = a$ and $x = 3a - mg/k$, and the particle will next come to rest as $x = 3a - mg/k$.



Feedback: This question is more complicated, using ideas from energy and some more technically challenging elements. Take care to think about what's happening in the system as you answer the question. Remember to present your solution carefully, fully explaining what you're doing.

3. A smooth (i.e. no friction) wire is in the shape of a helix so that $x = a \cos \theta(t)$, $y = a \sin \theta(t)$, $z = b\theta(t)$, with the central (z) axis pointing vertically upwards. A small bead of mass m moves along the wire, starting from height $z = b$ at rest.

- (a) By writing down the position vector and differentiating, show that

$$\dot{\mathbf{r}} = -a\dot{\theta}\sin\theta\mathbf{i} + a\dot{\theta}\cos\theta\mathbf{j} + b\dot{\theta}\mathbf{k},$$

and hence that the kinetic energy of the bead is given by

$$\frac{1}{2}m(a^2 + b^2)\dot{\theta}^2.$$

- (b) Write down the potential energy of the bead in terms of θ , choosing the potential to be zero at $z = 0$.
(c) Hence show that conservation of energy gives

$$\frac{1}{2}m(a^2 + b^2)\dot{\theta}^2 + mgb\theta = mgb.$$

- (d) By rearranging to find an equation for $\dot{\theta}^2$, find the maximum value that θ can attain. What does this mean physically?
(e) **Optional (hard!) extension:** How long will it take for the bead to reach $z = 0$?

Solution. (a) The particle moves along the wire, so the position vector is

$$\mathbf{r} = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j} + b\theta\mathbf{k},$$

and hence

$$\dot{\mathbf{r}} = -a\dot{\theta}\sin\theta\mathbf{i} + a\dot{\theta}\cos\theta\mathbf{j} + b\dot{\theta}\mathbf{k}.$$

This gives the kinetic energy as

$$\begin{aligned}\frac{1}{2}m|\dot{\mathbf{r}}|^2 &= \frac{1}{2}m(a^2\dot{\theta}^2\sin^2\theta + a^2\dot{\theta}^2\cos^2\theta + b^2\dot{\theta}^2), \\ &= \frac{1}{2}m(a^2 + b^2)\dot{\theta}^2.\end{aligned}$$

- (b) The potential energy depends on the height of the bead. If the potential is zero when $z = 0$, then the potential is given by

$$mgz = mgb\theta.$$

- (c) Conservation of energy therefore gives

$$\frac{1}{2}m(a^2 + b^2)\dot{\theta}^2 + mgb\theta = \text{constant},$$

by summing the kinetic and potential energies.

Now, initially at $t = 0$ we have $\dot{\theta} = 0$, and $z = b = b\theta$ gives $\theta = 1$. Hence

$$\frac{1}{2}m(a^2 + b^2)\dot{\theta}^2 + mgb\theta = mgb,$$

as required.

(d) Therefore we rearrange to find

$$\dot{\theta}^2 = \frac{2gb(1-\theta)}{a^2+b^2}.$$

Since $\dot{\theta}^2$ must be positive, the right hand side must also be positive. This means the maximum value of θ is 1 to ensure $\theta - 1 \geq 0$. Therefore the maximum height of the particle is $z = b$ and it only falls.

(e) **optional extension:** Now

$$\begin{aligned}\dot{\theta} &= \pm \sqrt{\frac{2gb(1-\theta)}{a^2+b^2}}, \\ &= \pm \sqrt{\frac{2gb}{a^2+b^2}} \sqrt{(1-\theta)},\end{aligned}\quad (1)$$

and we choose the negative sign to ensure that $\dot{\theta}$ is negative so that the particle is falling downwards.

We need to find the value of time, T say, at which the particle reaches $z = 0$. This is equivalent to $b\theta = 0$, and so $\theta = 0$. We could solve (1), apply the initial conditions and then calculate T , but it's easier to do this all at once by integrating with definite limits:

$$\frac{1}{\sqrt{(1-\theta)}} \frac{d\theta}{dt} = -\sqrt{\frac{2gb}{a^2+b^2}}, \quad (2)$$

$$\Rightarrow \int_{t=0}^T \frac{1}{\sqrt{(1-\theta)}} \frac{d\theta}{dt} dt = -\sqrt{\frac{2gb}{a^2+b^2}} \int_{t=0}^T dt, \quad (3)$$

by integrating with respect to t between $t = 0$ and T . Hence

$$\int_{\theta(0)=1}^{\theta(T)=0} \frac{1}{\sqrt{(1-\theta)}} d\theta = -\sqrt{\frac{2gb}{a^2+b^2}} \int_{t=0}^T dt, \quad (4)$$

$$\Rightarrow \int_1^0 (1-\theta)^{-1/2} d\theta = -\sqrt{\frac{2gb}{a^2+b^2}} [t]_0^T, \quad (5)$$

$$\Rightarrow \left[\frac{(1-\theta)^{1/2}}{1/2} \right]_1^0 d\theta = -\sqrt{\frac{2gb}{a^2+b^2}} T, \quad (6)$$

$$\Rightarrow -2 = -\sqrt{\frac{2gb}{a^2+b^2}} T, \quad (7)$$

$$\Rightarrow T = \frac{2\sqrt{a^2+b^2}}{\sqrt{2gb}}. \quad (8)$$

Note that the earlier choice of sign has ensured that T is positive.



Feedback: This gives some practice at using energy conservation along a one dimensional (but not straight) line. Notice that some elements of the central forces framework are also needed.

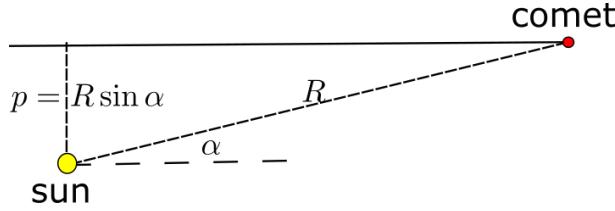


Figure 2: Initial set up, showing comet and sun

4. A comet (mass m) which is travelling with speed V , approaches a stationary planet from a great distance. If the path of the comet was not affected by the planet, the distance of closest approach would be p . The comet experiences an attractive force GMm/r^2 towards the planet where r is the distance between them, G is the gravitational constant and M is the mass of the planet.

(a) **Briefly** explain why

$$\begin{aligned} r^2\dot{\theta} &= \text{constant}, \\ \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r} &= \text{constant}. \end{aligned}$$

- (b) Using the initial conditions, find the values of the constants in part (a).
(c) By eliminating $\dot{\theta}$, show that

$$\dot{r}^2 = V^2 + \frac{2GM}{r} - \frac{p^2V^2}{r^2}.$$

- (d) Hence calculate the actual distance of closest approach.

Solution. (a) Angular momentum is conserved, since the comet is moving under the action of a central force and hence $r^2\dot{\theta}$ is constant. Energy conservation gives

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r} = \text{constant},$$

where the first term on the left hand side is kinetic energy and the last term is the potential energy.

- (b) At $t = 0$ we have $\dot{r} = -V \cos \alpha$, $r = R$, $r\dot{\theta} = V \sin \alpha$, where $R \gg 1$ is the initial distance, and $\alpha \ll 1$ is the initial angle. We also know $p = R \sin \alpha$. See Figure 2. Hence

$$\begin{aligned} r^2\dot{\theta} &= r \cdot r\dot{\theta}, \\ &= RV \sin \alpha, \\ &= R \sin \alpha V, \\ &= pV, \end{aligned}$$

and

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r} = \frac{1}{2}mV^2 - \frac{GMm}{R},$$

which can be approximated as $\frac{1}{2}mV^2$ since R is big and hence $1/R$ is negligible.

(c) If $r^2\dot{\theta} = pV$, then

$$\begin{aligned} r^2\dot{\theta}^2 &= \frac{(r^2\dot{\theta})^2}{r^2}, \\ &= \frac{p^2V^2}{r^2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r} &= \frac{1}{2}mV^2, \\ \Rightarrow \frac{1}{2}m\left(\dot{r}^2 + \frac{p^2V^2}{r^2}\right) - \frac{GMm}{r} &= \frac{1}{2}mV^2, \\ \Rightarrow \dot{r}^2 &= V^2 + \frac{2GM}{r} - \frac{p^2V^2}{r^2}. \end{aligned}$$

(d) The point of closest approach will be when $\dot{r} = 0$ (as you have to stop in order to turn around). Therefore we have a quadratic equation for the maximum/minimum distance r_m :

$$\begin{aligned} V^2 + \frac{2GM}{r_m} - \frac{p^2V^2}{r_m^2} &= 0, \\ \Rightarrow V^2r_m^2 + 2GMr_m - p^2V^2 &= 0, \end{aligned}$$

which we can solve using the quadratic formula to find

$$\begin{aligned} r_m &= \frac{-2GM \pm \sqrt{4G^2M^2 + 4p^2V^4}}{2V^2}, \\ &= \frac{-GM}{V^2} \pm \sqrt{\frac{G^2M^2}{V^4} + p^2}. \end{aligned}$$

Since r_m must be positive, this gives the distance of closest approach as

$$r_m = \frac{-GM}{V^2} + \sqrt{\frac{G^2M^2}{V^4} + p^2}.$$

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Feedback: This problem combines elements from Central Forces and Energy. These problems can be technically quite challenging, but are generally quite formulaic, you need to do the same thing each time!

5. **Assessed, marked out of 20.** To earn full marks, your answer must be well presented with clear explanations of key steps.

A frictionless slide in the vertical plane takes the shape

$$x = a(\theta - \sin \theta), \quad y = a(1 + \cos \theta), \tag{9}$$

where $a > 0$ is some constant, and $0 \leq \theta \leq \pi$ (see Figure 3). A ball of mass m moves along the slide under the force of gravity, which is $-mg\mathbf{j}$. The ball's position is $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

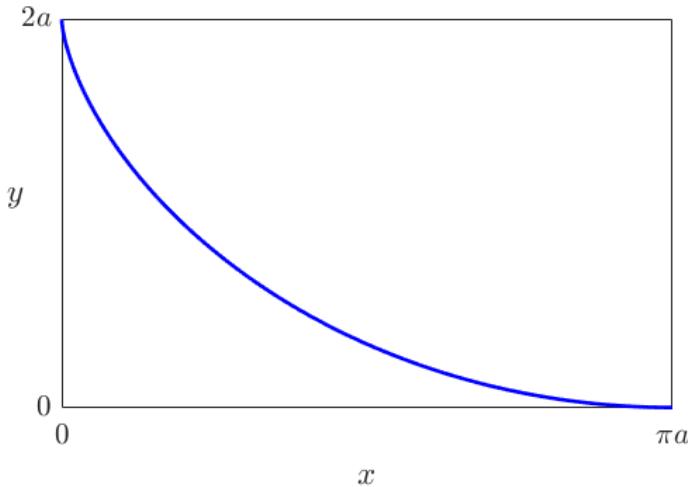


Figure 3: The curve given by Equation (3), with θ ranging from 0 to π .

- (a) Find an expression for the ball's velocity, $\dot{\mathbf{r}}$.
- (b) Let the ball's gravitational potential energy be zero at $y = 0$. Write down an expression for the ball's gravitational energy at generic position \mathbf{r} , expressing your answer in terms of θ .
- (c) Suppose the ball is released from rest, from the top of the slide ($\theta = 0$). Show that

$$a\dot{\theta}^2(1 - \cos \theta) + g(1 + \cos \theta) = 2g.$$

Hence, find an expression for $\dot{\theta}$. Show that the time it takes the ball to reach the bottom of the slide ($\theta = \pi$) is

$$T = \pi \sqrt{\frac{a}{g}}.$$

- (d) Suppose the ball is released from rest, from some point $\theta = \theta_0$ with $0 < \theta_0 < \pi$. Show that the time it takes the ball to reach the bottom of the slide ($\theta = \pi$) is

$$T' = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos \theta}{\cos \theta_0 - \cos \theta}} d\theta.$$

Use the trigonometric identity $\cos \theta = 2 \cos^2(\theta/2) - 1$ to show that

$$T' = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_0/2) - \cos^2(\theta/2)}} d\theta.$$

Finally, use the substitution

$$s = \frac{\cos(\theta/2)}{\cos(\theta_0/2)},$$

to show that

$$\sin(\theta/2)d\theta = -2 \cos(\theta_0/2)ds,$$

and hence calculate T' . [Hint: $\int \frac{1}{\sqrt{1-s^2}} ds = \arcsin(s) + \text{constant.}$]

Solution. (a)

$$\begin{aligned}
\dot{\mathbf{r}} &= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \\
&= a(\dot{\theta} - \dot{\theta} \cos \theta)\mathbf{i} + a\dot{\theta}(-\sin \theta)\mathbf{j} \\
&= a\dot{\theta}((1 - \cos \theta)\mathbf{i} - \sin \theta\mathbf{j}).
\end{aligned} \tag{10}$$

(b) The gravitational potential energy is $mgy = mga(1 + \cos \theta)$.

(c) Conservation of energy implies

$$\frac{1}{2}m|\dot{\mathbf{r}}|^2 + mga(1 + \cos \theta) = E, \tag{11}$$

where E is the constant total energy and can be evaluated at the initial time, when $\theta = 0$ and $\dot{\theta} = 0$ (released from rest). So, initially $\dot{\mathbf{r}} = \mathbf{0}$ and $\cos \theta = 1$, giving $E = 2mga$. Putting (10) and the value of E into (11) gives

$$\frac{1}{2}ma^2\dot{\theta}^2 |(1 - \cos \theta)\mathbf{i} - \sin \theta\mathbf{j}|^2 + mga(1 + \cos \theta) = 2mga,$$

and calculating the squared modulus gives

$$\begin{aligned}
&\frac{1}{2}ma^2\dot{\theta}^2 ((1 - \cos \theta)^2 + \sin^2 \theta) + mga(1 + \cos \theta) \\
&= \frac{1}{2}ma^2\dot{\theta}^2 (1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta) + mga(1 + \cos \theta) \\
&= ma^2\dot{\theta}^2 (1 - \cos \theta) + mga(1 + \cos \theta) = 2mga.
\end{aligned} \tag{12}$$

Dividing by ma on both sides yields the desired equation. Rearranging the equation for $\dot{\theta}$ yields

$$\dot{\theta}^2 = \frac{2g - g(1 + \cos \theta)}{a(1 - \cos \theta)} = \frac{g}{a}. \tag{13}$$

Since the ball moves downwards, increasing in θ , we must have the positive square root:

$$\dot{\theta} = \sqrt{\frac{g}{a}}.$$

From here, either: solve the ODE for θ and use the initial condition ($\theta(0) = 0$) to obtain $\theta(t) = \sqrt{g/a}t$, and then finding the value T that makes $\theta(T) = \pi$ is $T = \pi\sqrt{a/g}$; or (better), do

$$\int_0^\pi d\theta = \int_0^T \sqrt{\frac{g}{a}} dt,$$

giving the same conclusion.

(d) Equation (11) still holds, except now the value of E is different; initially we have $\theta = \theta_0$ and $\dot{\theta} = 0$, so $E = mga(1 + \cos \theta_0)$. The calculation of $|\dot{\mathbf{r}}|^2$ is also unchanged from before, so (12) now reads

$$ma^2\dot{\theta}^2 (1 - \cos \theta) + mga(1 + \cos \theta) = mga(1 + \cos \theta_0). \tag{14}$$

Dividing by ma , rearranging for $\dot{\theta}^2$ and taking the positive square root again, we find

$$\dot{\theta} = \sqrt{\frac{g(\cos \theta_0 - \cos \theta)}{a(1 - \cos \theta)}}. \quad (15)$$

Moving all of the RHS to the LHS, moving dt to the RHS, then integrating θ from initial position θ_0 to final position θ , while integrating t from initial time 0 to final time T' , gives

$$\int_{\theta_0}^{\pi} \sqrt{\frac{a(1 - \cos \theta)}{g(\cos \theta_0 - \cos \theta)}} d\theta = \int_0^{T'} dt = T', \quad (16)$$

and taking the constant $\sqrt{a/g}$ out of the integral yields the desired expression for T' . Using $\cos \theta = 2 \cos^2(\theta/2) - 1$, we find $\sqrt{1 - \cos \theta} = \sqrt{2(1 - \cos^2(\theta/2))} = \sqrt{2 \sin^2(\theta/2)}$, and taking the positive square root ($\theta/2$ is between 0 and $\pi/2$ in this problem and so $0 \leq \sin(\theta/2) \leq 1$), we find

$$\begin{aligned} T' &= \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{2} \sin(\theta/2)}{\sqrt{(2 \cos^2(\theta_0/2) - 1) - (2 \cos^2(\theta/2) - 1)}} d\theta \\ &= \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_0/2) - \cos^2(\theta/2)}} d\theta, \end{aligned} \quad (17)$$

as required. Differentiating the given substitution with respect to θ (remembering that $\cos(\theta_0/2)$ is a constant), gives

$$\frac{ds}{d\theta} = \frac{1}{2} \frac{-\sin(\theta/2)}{\cos(\theta_0/2)},$$

and rearranging yields

$$\sin(\theta/2)d\theta = -2 \cos(\theta_0/2)ds, \quad (18)$$

as required. Putting (18) into (17), we finally arrive at (remembering to change the integration limits from θ_0 and π to the corresponding limits for s)

$$\begin{aligned} T' &= \sqrt{\frac{a}{g}} \int_1^0 \frac{-2 \cos(\theta_0/2) ds}{\sqrt{\cos^2(\theta_0/2) - \cos^2(\theta/2)}} \\ &= \sqrt{\frac{a}{g}} \int_1^0 \frac{-2 \cos(\theta_0/2) ds}{\cos(\theta_0/2) \sqrt{1 - \cos^2(\theta/2)/\cos^2(\theta_0/2)}} \\ &= \sqrt{\frac{a}{g}} \int_1^0 \frac{-2 ds}{\sqrt{1 - s^2}} \\ &= \sqrt{\frac{a}{g}} [-2 \arcsin(s)]_1^0 \\ &= 2 \sqrt{\frac{a}{g}} (\arcsin(1) - \arcsin(0)) = 2 \sqrt{\frac{a}{g}} \frac{\pi}{2} = \pi \sqrt{\frac{a}{g}}. \end{aligned} \quad (19)$$

It is interesting to note that the answers to parts (c) and (d) are identical. This means that, as long as the ball is released from rest, the time taken to reach the bottom is always the same, regardless of where the ball is released from! This “equal-time” phenomenon is unique to the curve described by Equation (3). In fact, the curve takes its name from “equal-time”: it is the *tautochrone* curve, “tauto-” meaning “same”, and “chrono” meaning time. Note also that θ in Equation (3) is different from the standard polar angle; it begins due west and increases by π after only a quarter of a revolution. This non-standard definition makes the algebra a little more bearable! 

Feedback: *Following bookwork in parts (a) and (b), and some seen application in part (c), this problem becomes quite challenging in part (d). The key is to realise that the same method that allowed you to solve part (c) still applies (including the same principle of energy conservation), but now the value of E is different, leading to more difficult algebra. In particular, the algebra is only do-able if one does not attempt to solve (15) for $\theta(t)$, and instead opts for the “definite integrals between initial and final states” method.*