Example sheet 0: Recap of key techniques

Derive a solution to the following ordinary differential equations for $y = y(x)$, $x > 0$, applying initial conditions where given:

1. $$\frac{dy}{dx} = \frac{y-1}{x+3}; \tag{1}$$

   This is a separable equation:

   $$\begin{aligned}
   \frac{1}{y-1}\frac{dy}{dx} &= \frac{1}{x+3}, \\
   \int \frac{1}{y-1}\,dy &= \int \frac{1}{x+3}\,dx, \\
   \ln|y-1| &= \ln|x+3| + c_1, \\
   y - 1 &= c(x+3), \quad (c = \pm e^{c_1}), \\
   y(x) &= 1 + c(x+3).
   \end{aligned}$$

2. $$\frac{dy}{dx} = \frac{x^2-1}{y^2}; \tag{2}$$

   This is a separable equation:

   $$\begin{aligned}
   \int y^2\,dy &= \int x^2 - 1\,dx, \\
   \frac{y^3}{3} &= \frac{x^3}{3} - x + c_1, \\
   y^3 &= x^3 - 3x + c, \quad (c = 3c_1), \\
   y(x) &= (x^3 + c - 3x)^{1/3}.
   \end{aligned}$$

3. $$\frac{dy}{dx} = \frac{1}{xy^3}; \tag{3}$$

   This is a separable equation:

   $$\begin{aligned}
   \int y^3\,dy &= \int \frac{1}{x}\,dx, \\
   \frac{y^4}{4} &= \ln|x| + c_1, \\
   y^4 &= 4\ln(x) + c, \quad (x > 0, c = 4c_1), \\
   y(x) &= \pm(4\ln(x) + c)^{1/4}.
   \end{aligned}$$

4.
$$\frac{dy}{dx} + y^2 = 2y; \tag{4}$$

This is a separable equation:

$$\begin{aligned}\frac{dy}{dx} &= 2y - y^2,\\ &= y(2-y),\\ \int \frac{1}{y(2-y)}\,dy &= \int dx.\end{aligned}$$

We now exploit partial fractions:

$$\begin{aligned}\frac{1}{y(2-y)} &=\equiv \frac{A}{y} + \frac{B}{2-y},\\ 1 &=\equiv A(2-y) + By.\end{aligned}$$

Matching constant terms:

$$1 = 2A, \implies A = \frac{1}{2}.$$

Matching coefficients of y:

$$0 = -A + B, \implies B = \frac{1}{2}.$$

Subbing these back in:

$$\begin{aligned}\int \frac{1}{2y} + \frac{1}{2(2-y)}\,dy &= x + c_1,\\ \frac{1}{2}\int \frac{1}{y} + \frac{2-y}{}dy &= x + c_1,\\ \frac{1}{2}\left[\ln|y| - \ln|2-y|\right] &= x + c_1,\\ \ln|y| - \ln|2-y| &= 2x + c_2, \quad (c_2 = 2c_1),\\ \ln\left|\frac{y}{2-y}\right| &= 2x + c_2,\\ \frac{y}{2-y} &= ce^{2x}, \quad (c = \pm e^{c_2}),\\ y &= 2ce^{2x} - yce^{2x},\\ y(1 + c_3e^{2x}) &= 2ce^{2x},\\ y(x) &= \frac{2ce^{2x}}{1 + ce^{2x}}.\end{aligned}$$

5.
$$\frac{dy}{dx} + 2y = 3e^x; \tag{5}$$

This is a linear first order differential equation so we can use the integrating factor method:

$$\begin{aligned}\mu(x) &= e^{\int 2\,dx},\\ &= e^{2x}.\end{aligned}$$

Multiplying the ODE through by the integrating factor gives:

$$\begin{aligned}\frac{d}{dx}(e^{2x}y) &= 3e^{3x},\\ e^{2x}y &= \int 3e^{3x}\,dx,\\ &= e^{3x}+c,\\ y(x) &= e^{x}+ce^{-2x}.\end{aligned}$$

6.

$$\frac{dy}{dx} - y = e^{3x}; \tag{6}$$

This is a linear first order differential equation so we can use the integrating factor method:

$$\begin{aligned}\mu(x) &= e^{\int -1\,dx},\\ &= e^{-x}.\end{aligned}$$

Multiplying the ODE through by the integrating factor gives:

$$\begin{aligned}\frac{d}{dx}(e^{-x}y) &= e^{2x},\\ e^{-x}y &= \frac{1}{2}e^{2x}+c,\\ y(x) &= \frac{e^{3x}}{2}+ce^{x}.\end{aligned}$$

7.

$$x\frac{dy}{dx} + 3y + 2x^2 = x^3 + 4x; \tag{7}$$

This is a linear first order differential equation so we can use the integrating factor method. First we rearrange the ODE into the standard format:

$$\frac{dy}{dx} + \frac{3}{x}y = x^2 - 2x + 4.$$

Next we calculate the integrating factor:

$$\begin{aligned}\mu(x) &= e^{\int \frac{3}{x}\,dx},\\ &= e^{3\ln|x|},\\ &= x^3, \quad (x>0).\end{aligned}$$

Multiplying the ODE through by the integrating factor gives:

$$\begin{aligned}\frac{d}{dx}(x^3y) &= x^5 - 2x^4 + 4x^3,\\ x^3y &= \frac{x^6}{6} - \frac{2x^5}{5} + x^4 + c,\\ y(x) &= \frac{x^3}{6} - \frac{2x^2}{5} + x + \frac{c}{x^3}.\end{aligned}$$

8.

$$y'' - 5y' + 6y = 0; \tag{8}$$

This is a linear second order homogeneous ODE with constant coefficients so we look for a solution in the form $y = e^{rx}$. Substituting in gives the characteristic equation:

$$\begin{aligned}r^2 - 5r + 6 &= 0,\\ (r-2)(r-3) &= 0, \implies r_{1,2} = 2, 3.\end{aligned}$$

Hence

$$y(x) = c_1e^{2x} + c_2e^{3x}$$

is a solution by the superposition principle.

9.

$$y'' - 5y' = 0; \tag{9}$$

This is a linear second order homogeneous ODE with constant coefficients so we look for a solution in the form $y = e^{rx}$. Substituting in gives the characteristic equation:

$$\begin{aligned}r^2 - 5r &= 0,\\ r(r-5) &= 0, \implies r_{1,2} = 0, 5.\end{aligned}$$

Hence

$$\begin{aligned}y(x) &= c_1e^0 + c_2e^{5x},\\ &= c_1 + c_2e^{5x}\end{aligned}$$

is a solution by the superposition principle.

10.

$$y'' - 2y' + 5y = 0; \tag{10}$$

[can you write your solution using real coefficients only?]

This is a linear second order homogeneous ODE with constant coefficients so we look for a solution in the form $y = e^{rx}$. Substituting in gives the characteristic equation:

$$r^2 - 2r + 5 = 0.$$

Solving this gives $r_{1,2} = 1 \pm 2i$. Hence

$$
\begin{aligned}
y &= \alpha_1 e^{(1+2i)x} + \alpha_2 e^{(1-2i)x}, \\
&= \alpha_1 e^x e^{2i} + \alpha_2 e^x e^{-2i}, \\
&= \alpha_1 e^x (\cos(2x) + i\sin(2x)) + \alpha_2 e^x (\cos(2x) - i\sin(2x)), \\
y(x) &= c_1 e^x \cos(2x) + c_2 e^x \sin(2x), \quad (c_1 = \alpha_1 + \alpha_2, c_2 = \alpha_1 i - \alpha_2 i).
\end{aligned}
$$

11.

$$y'' - 5y' + 6y = x^2 \tag{11}$$

This is a linear second order inhomogeneous ODE with constant coefficients. We know from above that the solution of the corresponding homogeneous equation is

$$y_h(x) = c_1 e^{2x} + c_2 e^{3x}.$$

From Table 1 in the lecture notes for Chapter 0, we look for a particular solution in the following form:

$$
\begin{aligned}
y_p(x) &= A + Bx + Cx^2, \\
y_p'(x) &= B + 2Cx, \\
y_p''(x) &= 2C.
\end{aligned}
$$

Subbing into the inhomogeneous ODE gives:

$$2C - 5B - 10Cx + 6A + 6Bx + 6Cx^2 \equiv x^2.$$

Matching coefficients of:

$$
\begin{aligned}
x^2 : \quad 6C = 1 &\implies C = \frac{1}{6}, \\
x : -10C + 6B = 0 &\implies B = \frac{5}{18}, \\
\text{constants} : 2C - 5B + 6A = 0 &\implies A = \frac{19}{108}.
\end{aligned}
$$

Subbing these back into the particular solution and combining with the general solution of the corresponding homogeneous equation by the superposition principle gives

$$y(x) = c_1 e^{3x} + c_2 e^{2x} + \frac{x^2}{6} + \frac{5x}{18} + \frac{19}{108}.$$

12.

$$y'' - 5y' + 6y = e^{2x} \tag{12}$$

This is a linear second order inhomogeneous ODE with constant coefficients. We know from above that the solution of the corresponding homogeneous equation is

$$y_h(x) = c_1 e^{2x} + c_2 e^{3x}.$$

Notice that $d(x)$ is contained in $y_h(x)$. Therefore, from Table 1 in the lecture notes for Chapter 0, we look for a particular solution in the following form:

$$
\begin{aligned}
y_p(x) &= Axe^{2x}, \\
y_p'(x) &= Ae^{2x} + 2Axe^{2x}, \\
y_p''(x) &= 4e^{2x} + 2Axe^{2x}.
\end{aligned}
$$

Subbing into the inhomogeneous ODE gives:

$$
\begin{aligned}
4Ae^{2x} + 4Axe^{2x} - 5Ae^{2x} - 10Axe2x + 6Axe^{2x} &\equiv e^{2x}, \\
-Ae^{2x} &\equiv e^{2x}.
\end{aligned}
$$

Hence $A = -1$.

$$
y(x) = c_1 e^{3x} + c_2 e^{2x} - xe^{2x}
$$

13.

$$
y'' - 5y' = \cos(x). \tag{13}
$$

This is a linear second order inhomogeneous ODE with constant coefficients. We know from above that the solution of the corresponding homogeneous equation is

$$
y(x) = c_1 + c_2 e^{5x}.
$$

From Table 1 in the lecture notes for Chapter 0, we look for a particular solution in the following form:

$$
\begin{aligned}
y_p(x) &= A\cos(x) + B\sin(x), \\
y_p'(x) &= -A\sin(x) + B\cos(x), \\
y_p''(x) &= -A\cos(x) - B\sin(x).
\end{aligned}
$$

Subbing into the inhomogeneous ODE gives:

$$
\begin{aligned}
-A\cos(x) - B\sin(x) + 5A\sin(x) - 5B\cos(x) &\equiv \cos(x), \\
(-A - 5B)\cos(x) + (-B + 5A)\sin(x) &\equiv \cos(x).
\end{aligned}
$$

Matching coefficients of $\sin(x)$ gives $A = B/5$. Matching coefficients of $\cos(x)$ (and using $A = B/5$) gives $B = -5/26$ and hence $A = -1/26$. Subbing back into the particular solution and combining with the general solution of the corresponding homogeneous equation by the superposition principle gives

$$
y(x) = c_1 + c_2 e^{5x} - \frac{5\sin(x)}{26} - \frac{\cos(x)}{26}
$$