LA - Sum2

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1a) We have $\varphi_B(\vec{v}) = \underline{v} = v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}$

$\Rightarrow f(\underline{v}) = \underline{c} \times \underline{v} = (c_2 v_3 - c_3 v_2)\underline{i} + (c_3 v_1 - c_1 v_3)\underline{j} + (c_1 v_2 - c_2 v_1)\underline{k}$

$A = [a_{ji}] \in \mathbb{R}^{3\times 3}$ and let $e_1 = \underline{i}$, $e_2 = \underline{j}$, and $e_3 = \underline{k}$ $\Rightarrow$ we have,

$f(\underline{i}) = \sum_{j=1}^{3} a_{j1} e_j = 0\underline{i} + c_3 \underline{j} - c_2 \underline{k}$,

$f(\underline{j}) = \sum_{j=1}^{3} a_{j2} e_j = -c_3 \underline{i} + 0\underline{j} + c_1 \underline{k}$, and

$f(\underline{k}) = \sum_{j=1}^{3} a_{j3} e_j = c_2 \underline{i} - c_1 \underline{j} + 0\underline{k}$.

Using this, we can see that,

$$A = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}$$

1b) ker A is the set of all points $\underline{x}$ such that $A\underline{x} = \underline{0}$, which gives us the system of simultaneous linear equations:

$$\begin{aligned} -c_3 x_2 + c_2 x_3 &= 0 \\ c_3 x_1 - c_1 x_3 &= 0 \Rightarrow x_3 = \frac{c_3}{c_1} x_1 \\ -c_2 x_1 + c_1 x_2 &= 0 \Rightarrow x_2 = \frac{c_2}{c_1} x_1 \end{aligned}$$

(We can divide here because $c_1, c_2, c_3 \neq 0$).

Take $x_1 = \lambda$,

$$\Rightarrow \underline{x} = \begin{pmatrix} \lambda \\ \lambda \frac{c_2}{c_1} \\ \lambda \frac{c_3}{c_1} \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \frac{c_2}{c_1} \\ \frac{c_3}{c_1} \end{pmatrix} = \frac{\lambda}{c_1} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \forall \lambda \in \mathbb{R}$$

Since $\frac{\lambda}{c_1} \in \mathbb{R}$, we can let $\lambda' = \frac{\lambda}{c_1}$,

$\Rightarrow \underline{x} = \lambda' \underline{c} \quad \forall \lambda' \in \mathbb{R}$

$\Rightarrow \ker A = \text{span}\{\underline{c}\}$

Now deriving ker f, we have $\underline{c} = \varphi_B(\vec{c}) \Rightarrow \ker f = \text{span}\{\vec{c}\}$

1c) We have $\underline{c}^T = (c_1 \; c_2 \; c_3)$ and $\underline{x}^T = (x_1 \; x_2 \; x_3)$.

$$A\underline{x} = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -c_3 x_2 + c_2 x_3 \\ c_3 x_1 - c_1 x_3 \\ -c_2 x_1 + c_1 x_2 \end{pmatrix}$$

$$\underline{c}^T A \underline{x} = (c_1 \; c_2 \; c_3) \begin{pmatrix} -c_3 x_2 + c_2 x_3 \\ c_3 x_1 - c_1 x_3 \\ -c_2 x_1 + c_1 x_2 \end{pmatrix} = -c_1 c_3 x_2 + c_1 c_2 x_3 + c_2 c_3 x_1 - c_1 c_2 x_3 - c_2 c_3 x_1 + c_1 c_3 x_2$$

$$= (c_2 c_3 - c_2 c_3) x_1 + (c_1 c_3 - c_1 c_3) x_2 + (c_1 c_2 - c_1 c_2) x_3$$

$$= 0$$

$$\underline{x}^T A \underline{x} = (x_1 \; x_2 \; x_3) \begin{pmatrix} -c_3 x_2 + c_2 x_3 \\ c_3 x_1 - c_1 x_3 \\ -c_2 x_1 + c_1 x_2 \end{pmatrix} = -c_3 x_1 x_2 + c_2 x_1 x_3 + c_3 x_1 x_2 - c_1 x_2 x_3 - c_2 x_1 x_3 + c_1 x_2 x_3$$

$$= (x_2 x_3 - x_2 x_3) c_1 + (x_1 x_3 - x_1 x_3) c_2 + (x_1 x_2 - x_1 x_2) c_3$$

$$= 0$$

Now deducing that $f(\vec{v}) \perp \text{span}\{\vec{c}, \vec{v}\}$:

Let $\varphi_B(\vec{v})$ be denoted by $\underline{x} \Rightarrow f(\vec{v}) = A\underline{x}$.
By the defined standard dot product operation, we have,

$$\vec{c} \cdot f(\vec{v}) = \underline{c}^T A \underline{x} \quad \text{and} \quad \vec{v} \cdot f(\vec{v}) = \underline{x}^T A \underline{x}.$$

Above, we have already shown that $\underline{c}^T A \underline{x} = \underline{x}^T A \underline{x} = 0$
$\Rightarrow f(\vec{v})$ is orthogonal to both $\vec{c}$ and $\vec{v}$

$\Rightarrow f(\vec{v}) \perp \text{span}\{\vec{c}, \vec{v}\}$

1d)
$$A^2 \underline{x} = A(A\underline{x}) = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix} \begin{pmatrix} -c_3 x_2 + c_2 x_3 \\ c_3 x_1 - c_1 x_3 \\ -c_2 x_1 + c_1 x_2 \end{pmatrix} = \begin{pmatrix} -c_3^2 x_1 + c_1 c_3 x_3 - c_2^2 x_1 + c_1 c_2 x_2 \\ -c_3^2 x_2 + c_2 c_3 x_3 + c_1 c_2 x_1 - c_1^2 x_2 \\ c_2 c_3 x_2 - c_2^2 x_3 + c_1 c_3 x_1 - c_1^2 x_3 \end{pmatrix}$$

$$(\underline{c}^T \underline{x})\underline{c} - \underline{x} = (\vec{c} \cdot \vec{v})\underline{c} - \underline{x} = (c_1 x_1 + c_2 x_2 + c_3 x_3)\underline{c} - \underline{x}$$

$$= \begin{pmatrix} c_1^2 x_1 + c_1 c_2 x_2 + c_1 c_3 x_3 - x_1 \\ c_1 c_2 x_1 + c_2^2 x_2 + c_2 c_3 x_3 - x_2 \\ c_1 c_3 x_1 + c_2 c_3 x_2 + c_3^2 x_3 - x_3 \end{pmatrix}$$

Equating the $\underline{i}, \underline{j}$, and $\underline{k}$ parts of $A^2 \underline{x} = (\underline{c}^T \underline{x})\underline{c} - \underline{x}$:

$\underline{i}$ part:

$$-c_3^2 x_1 + c_1 c_3 x_3 - c_2^2 x_1 + c_1 c_2 x_2 = c_1^2 x_1 + c_1 c_2 x_2 + c_1 c_3 x_3 - x_1$$
$\Rightarrow (c_1^2 + c_2^2 + c_3^2 - 1)x_1 + (c_1 c_2 - c_1 c_2)x_1 + (c_1 c_3 - c_1 c_3)x_3 = 0$
$\Rightarrow (c_1^2 + c_2^2 + c_3^2 - 1)x_1 = 0$ which is true because $\vec{c}$ is a unit vector $\Rightarrow \vec{c} \cdot \vec{c} = c_1^2 + c_2^2 + c_3^2 = 1$
$\Rightarrow$ Statement is true for $\underline{i}$ component.

$\underline{j}$ part:

$$-c_3^2 x_2 + c_2 c_3 x_3 + c_1 c_2 x_1 - c_1^2 x_2 = c_1 c_2 x_1 + c_2^2 x_2 + c_2 c_3 x_3 - x_2$$
$\Rightarrow (c_1 c_2 - c_1 c_2)x_1 + (c_1^2 + c_2^2 + c_3^2 - 1)x_2 + (c_2 c_3 - c_2 c_3)x_3 = 0$
$\Rightarrow (c_1^2 + c_2^2 + c_3^2 - 1)x_2 = 0$ which is true because $\vec{c}$ is a unit vector $\Rightarrow \vec{c} \cdot \vec{c} = c_1^2 + c_2^2 + c_3^2 = 1$
$\Rightarrow$ Statement is true for $\underline{j}$ component.

$\underline{k}$ part:

$$c_2 c_3 x_2 - c_2^2 x_3 + c_1 c_3 x_1 - c_1^2 x_3 = c_1 c_3 x_1 + c_2 c_3 x_2 + c_3^2 x_3 - x_3$$
$\Rightarrow (c_1 c_3 - c_1 c_3)x_1 + (c_2 c_3 - c_2 c_3)x_2 + (c_1^2 + c_2^2 + c_3^2 - 1)x_3 = 0$
$\Rightarrow (c_1^2 + c_2^2 + c_3^2 - 1)x_3 = 0$ which is true because $\vec{c}$ is a unit vector $\Rightarrow \vec{c} \cdot \vec{c} = c_1^2 + c_2^2 + c_3^2 = 1$
$\Rightarrow$ Statement is true for $\underline{k}$ component.

Therefore, we have that $A^2 \underline{x} = (\underline{c}^T \underline{x})\underline{c} - \underline{x} \quad \forall \underline{x} \in \mathbb{R}^3$.

1e) Eigenvalues of A are values λ_i that satisfy $A\underline{v}_i = \lambda_i \underline{v}_i$, for some corresponding eigenvector $\underline{v}_i$.

$A\underline{x} = \lambda \underline{x} \Rightarrow (\lambda I - A)\underline{x} = \underline{0}$

$$\Rightarrow \begin{pmatrix} \lambda & c_3 & -c_2 \\ -c_3 & \lambda & c_1 \\ c_2 & -c_1 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ We have characteristic polynomial,

$$\lambda^3 + c_1 c_2 c_3 - c_1 c_2 c_3 + \lambda c_2^2 + \lambda c_1^2 + \lambda c_3^2 = 0$$

$$\Rightarrow \lambda^3 + (c_1^2 + c_2^2 + c_3^2)\lambda = 0$$

$$\Rightarrow \lambda^3 + \lambda = 0 \Rightarrow \lambda(\lambda^2 + 1) = 0$$

$\Rightarrow \lambda_1 = 0, \ \lambda_2 = i, \ \lambda_3 = -i \Rightarrow$ Spectrum of f is $\{0, i, -i\}$.

Reasoning that there are only two proper non-trivial f-invariant subspaces over $\mathbb{R}$:

We have that the characteristic polynomial only has 1 real root and so we have 1 f-invariant subspace of dimension 1 over $\mathbb{R}$.
The other 2 roots of the characteristic polynomial are a complex-conjugate pair, and we know that each of these eigenvalues are associated with a 1-dimensional invariant subspace over $\mathbb{C}$.
These 2 subspaces induce a real two-dimensional f-invariant subspace of $\mathbb{R}^3$.

Since we have a 1-dimensional f-invariant subspace (corresponding to $\lambda=0$) and a 2-dimensional f-invariant subspace (corresponding to $\lambda = \pm i$), we have that the only other subspaces can be the trivial subspace $\{0\}$ and the subspace equal to the domain itself, $\mathbb{R}^3$.

$\Rightarrow$ There are exactly two proper non-trivial f-invariant subspaces over $\mathbb{R}$.

Now identifying one of the subspaces:

E_λ corresponding to $\lambda = 0$ is given by,

$$E_\lambda = \ker(A - \lambda I) = \ker A.$$

From part b, we know that $\ker A = \text{Span}\{\underline{c}\}$

$\Rightarrow$ The one-dimensional, non-trivial f-invariant subspace corresponding to $\lambda = 0$ is given by,

$$\text{span}\{\vec{c}\}.$$

1f) Let $\underline{n} = \varphi_B(\vec{n})$. $f(\vec{n}) = \vec{c} \times \vec{n} \Rightarrow \varphi_B(\vec{c} \times \vec{n}) = A\underline{n}$.

Therefore, we have, $\text{span}\{\vec{n}, \vec{c} \times \vec{n}\} = \text{span}\{\underline{n}, A\underline{n}\}$

and so if we let $\underline{u} = a_1 \underline{n} + a_2 A\underline{n} \in U$, $a_1, a_2 \in \mathbb{R}$. We have,

$$\begin{aligned} U \text{ is invariant} &\Leftrightarrow A\underline{u} \in U & \forall\, \underline{u} \in U \\ &\Leftrightarrow A(a_1 \underline{n}) + A(a_2 A \underline{n}) \in U & \forall\, a_1, a_2 \in \mathbb{R} \\ &\Leftrightarrow a_1(A\underline{n}) + a_2(A A \underline{n}) \in U & \forall\, a_1, a_2 \in \mathbb{R} \end{aligned}$$

$\Rightarrow$ To show that U is f-invariant, it is sufficient to show that A maps both $\underline{n}$ and $A\underline{n}$ back into U.

First, we try for $\underline{n}$:

By definition we have $A(\underline{n}) = A\underline{n} \in \text{Span}\{\underline{n}, A\underline{n}\}$

Now, for $A\underline{n}$:

$$A(A\underline{n}) = A^2 \underline{n}.$$

Using the equation from part (d), $A^2 \underline{x} = (\underline{c}^T \underline{x})\underline{c} - \underline{x}$, we get,

$$\begin{aligned} A^2 \underline{n} &= (\underline{c}^T \underline{n})\underline{c} - \underline{n} \\ &= (\vec{c} \cdot \vec{n})\underline{c} - \underline{n}. \end{aligned}$$

Since $\vec{n} \perp \vec{c}$, $\vec{c} \cdot \vec{n} = 0 \Rightarrow A^2 \underline{n} = -\underline{n}$.

Therefore we have $A(A\underline{n}) = -\underline{n} \in \text{Span}\{\underline{n}, A\underline{n}\} = U$, and so we have shown that A maps both $\underline{n}$ and $A\underline{n}$ back into U.

$\Rightarrow U = \text{Span}\{\vec{n}, \vec{c} \times \vec{n}\}$ is f-invariant.

Now, a geometric interpretation of U:

$\vec{n} \perp \vec{c} \times \vec{n} \Rightarrow U = \text{span}\{\underline{n}, A\underline{n}\}$ forms a two-dimensional f-invariant plane in $\mathbb{R}^3$.

In part (e), we found that one of two proper non-trivial f-invariant subspaces over $\mathbb{R}$ was the one-dimensional subspace, $\text{span}\{\vec{c}\}$.

It can be seen that $\text{span}\{\underline{c}\}$ forms a one-dimensional f-invariant line in $\mathbb{R}^3$.

Since $\vec{n} \perp \vec{c}$, and $\vec{c} \times \vec{n} \perp \vec{c}$,

the two-dimensional f-invariant plane, formed by $U = \text{span}\{\underline{n}, A\underline{n}\}$,
is the plane perpendicular to the f-invariant line formed by $\text{span}\{\underline{c}\}$

(i.e., $U = \text{span}\{\vec{n}, \vec{c} \times \vec{n}\}$ is the other of the two proper non-trivial f-invariant subspaces over $\mathbb{R}$.)