1Mech — Mechanics

Mechanics solutions 5

This sheet's assessed question is #5.

1. A smooth sphere of radius a has its centre at the origin. If (ρ, θ, z) give cylindrical polar coordinates, with the z axis pointing vertically upwards, the surface of the sphere is given by $\rho^2 + z^2 = a^2$. A particle of mass m moves on the interior surface of the sphere under the action of gravity, with position vector $\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$, where $\mathbf{e}_\rho$, $\mathbf{e}_z$ are basis vectors pointing in the $\rho(t)$ and $z(t)$ direction respectively, with normal reaction $\mathbf{R}$ acting purely perpendicular to the surface.

   (a) Show that the particle's velocity satisfies $|\dot{\mathbf{r}}|^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$.

   (b) Hence show that

   $$\begin{aligned} \rho^2\dot{\theta} &= h, \\ \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) + mgz &= E, \end{aligned}$$

   where h and E are constants. What do these expressions represent physically?

   (c) Show that

   $$2\rho\dot{\rho} + 2z\dot{z} = 0,$$

   and hence

   $$\dot{\rho}^2 = \frac{z^2\dot{z}^2}{a^2 - z^2}.$$

   (d) If the particle is initially located at $\rho = a/\sqrt{2}$, $z = -a/\sqrt{2}$, moving with speed V in the direction of the horizontal tangent of the surface, find the values of h and E.

   (e) Hence show that the motion satisfies

   $$a^2\dot{z}^2 = \left(z + a/\sqrt{2}\right)\left(2g\left(z^2 - a^2\right) - V^2\left(z - a/\sqrt{2}\right)\right).$$

   (f) By differentiating to find an expression for $\ddot{z}$, show that the particle will initially rise if $V^2 > ag/\sqrt{2}$.

   (g) [**Optional extension**] Suppose instead that the particle is initially located at $\rho = a$, $z = 0$, and is projected vertically with speed V. Find the smallest value of V that will allow the particle to reach $z = a$.

Solution. (a) Since $\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$ and $\dot{\mathbf{e}}_\rho = \dot{\theta}\mathbf{e}_\theta$, $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_\rho$, $\dot{\mathbf{e}}_z = 0$, we have

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{\rho}\mathbf{e}_\rho + \rho\dot{\mathbf{e}}_\rho + \dot{z}\mathbf{e}_z, \\ &= \dot{\rho}\mathbf{e}_\rho + \rho\dot{\theta}\mathbf{e}_\theta + \dot{\mathbf{z}}\mathbf{e}_\mathbf{z}, \end{aligned}$$

and hence $|\dot{\mathbf{r}}|^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$ as required.

(b) Newton's second law gives

$$m\ddot{\mathbf{r}} = -mg\mathbf{e}_z + \mathbf{R},$$

where g gives the acceleration due to gravity, $\ddot{\mathbf{r}} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\mathbf{e}_\rho + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{e}_z$, and $\mathbf{R}$ is the reaction force. For a surface of revolution $\mathbf{R}$ has no component in the $\mathbf{e}_\theta$ direction, so we immediately find $\rho^2\dot{\theta} = h$ where h is constant (conservation of angular momentum). We then take the dot product with $\dot{\mathbf{r}}$ to find

$$m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -mg\mathbf{e}_z \cdot \dot{\mathbf{r}} + \mathbf{R} \cdot \dot{\mathbf{r}}.$$

Now, $\mathbf{R} \cdot \dot{\mathbf{r}} = 0$ since these are perpendicular vectors. We now integrate with respect to time to give

$$\begin{aligned} \frac{1}{2}m\dot{\mathbf{r}}^2 &= -mg\mathbf{e}_z \cdot \mathbf{r} + \text{const}, \\ \implies \frac{1}{2}m\dot{\mathbf{r}}^2 + mgz &= E, \end{aligned}$$

where E is the constant conserved energy. Since $\dot{\mathbf{r}} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\theta}\mathbf{e}_\theta + \dot{\mathbf{z}}\mathbf{e}_\mathbf{z}$, this gives

$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) + mgz = E,$$

for a constant E.

(c) Since $\rho^2 + z^2 = a^2$, by differentiating with respect to t we have

$$2\rho\dot{\rho} + 2z\dot{z} = 0,$$

and hence

$$\begin{aligned} \dot{\rho} &= -\frac{z\dot{z}}{\rho}, \\ \implies \dot{\rho}^2 &= \frac{z^2\dot{z}^2}{\rho^2}, \\ &= \frac{z^2\dot{z}^2}{a^2 - z^2}, \end{aligned}$$

again using $\rho^2 + z^2 = a^2$.

(d) At $t = 0$ we have $\rho = a/\sqrt{2}$, $z = -a/\sqrt{2}$ from the initial location of the particle, and $\dot{\rho} = 0$, $\rho\dot{\theta} = V$, $\dot{z} = 0$ since the velocity is entirely in the horizontal tangent to the surface. Hence

$$\begin{aligned} h = \rho^2\dot{\theta} &= \rho \cdot \rho\dot{\theta}, \\ &= \frac{a}{\sqrt{2}}V, \end{aligned}$$

and

$$\begin{aligned} E = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) + mgz &= \frac{1}{2}m\left(0^2 + V^2 + 0^2\right) - \frac{mga}{\sqrt{2}}, \\ &= \frac{1}{2}mV^2 - \frac{mga}{\sqrt{2}}, \end{aligned}$$

initially and hence for all time.

(e) We want to rewrite conservation of energy just in terms of z. We know $\dot{\rho}^2 = \dfrac{z^2\dot{z}^2}{a^2 - z^2}$, and

$$\begin{aligned}
\rho^2\dot{\theta}^2 &= \frac{\left(\rho^2\dot{\theta}\right)^2}{\rho^2}, \\
&= \frac{h^2}{\rho^2}, \\
&= \frac{a^2V^2}{2\rho^2}, \\
&= \frac{a^2V^2}{2\left(a^2 - z^2\right)}.
\end{aligned}$$

Hence

$$\frac{1}{2}m\left(\frac{z^2\dot{z}^2}{a^2 - z^2} + \frac{a^2V^2}{2\left(a^2 - z^2\right)} + \dot{z}^2\right) + mgz = \frac{1}{2}mV^2 - \frac{mga}{\sqrt{2}},$$

and rearranging gives

$$\begin{aligned}
\frac{z^2\dot{z}^2}{a^2 - z^2} + \frac{a^2V^2}{2\left(a^2 - z^2\right)} + \dot{z}^2 + 2gz &= V^2 - \frac{2ga}{\sqrt{2}}, \\
\implies \dot{z}^2\left(\frac{z^2}{a^2 - z^2} + 1\right) &= V^2 - \frac{a^2V^2}{2\left(a^2 - z^2\right)} - \sqrt{2}ga - 2gz, \\
\implies \dot{z}^2\left(\frac{z^2 + a^2 - z^2}{a^2 - z^2}\right) &= V^2 - \frac{a^2V^2}{2\left(a^2 - z^2\right)} - \sqrt{2}ga - 2gz, \\
\implies a^2\dot{z}^2 &= \left(a^2 - z^2\right)\left(V^2 - \frac{a^2V^2}{2\left(a^2 - z^2\right)} - \sqrt{2}ga - 2gz\right), \\
&= \left(a^2 - z^2\right)\left(V^2 - \sqrt{2}ga - 2gz\right) - \frac{a^2V^2}{2}, \\
&= \frac{a^2V^2}{2} - V^2z^2 + \sqrt{2}ga\left(z^2 - a^2\right) + 2gz\left(z^2 - a^2\right).
\end{aligned}$$

We know that $\dot{z} = 0$ at $z = -a/\sqrt{2}$ by the initial condition. So $z + a/\sqrt{2}$ must be a factor of the right hand side, and hence

$$a^2\dot{z}^2 = \left(z + a/\sqrt{2}\right)\left(2g\left(z^2 - a^2\right) - V^2\left(z - a/\sqrt{2}\right)\right).$$

(f) Differentiating with respect to t we find

$$\begin{aligned}
2a^2\dot{z}\ddot{z} &= \dot{z}\left(2g\left(z^2 - a^2\right) - V^2\left(z - a/\sqrt{2}\right)\right) + \left(z + a/\sqrt{2}\right)\left(4gz\dot{z} - V^2\dot{z}\right), \\
\implies 2a^2\ddot{z} &= 2g\left(z^2 - a^2\right) - V^2\left(z - a/\sqrt{2}\right) + \left(z + a/\sqrt{2}\right)\left(4gz - V^2\right)
\end{aligned}$$

At $t = 0$, $z = -a/\sqrt{2}$, so

$$a^2\ddot{z} = \left(g\left(\frac{a^2}{2} - a^2\right) + aV^2/\sqrt{2}\right) = -\frac{ga^2}{2} + aV^2/\sqrt{2}.$$

If $V^2 > ag/\sqrt{2}$, then $aV^2/\sqrt{2} > a^2g/2$, and so the initial value of $\ddot{z}$ will be positive. This means the particle has an upward acceleration initially, i.e. it rises initially.

(g) **Optional extension:** If we instead take initial conditions $\rho = a$, $z = 0$, $\dot{\rho} = 0$, $\dot{z} = V$, $\rho\dot{\theta} = 0$ at $t = 0$, we will have $h = 0$, so there will be no change in the θ-coordinate ($\dot{\theta} = 0$) and conservation of energy will become

$$\frac{1}{2}m\left(\dot{\rho}^2 + \dot{z}^2\right) + mgz = \frac{1}{2}mV^2.$$

We will still have $\rho^2 = \dfrac{z^2\dot{z}^2}{a^2 - z^2}$, and therefore

$$\begin{aligned} \frac{1}{2}m\left(\frac{z^2\dot{z}^2}{a^2 - z^2} + \dot{z}^2\right) + mgz &= \frac{1}{2}mV^2, \\ \implies \frac{\dot{z}^2a^2}{a^2 - z^2} + 2gz &= V^2. \end{aligned}$$

The smallest value of V which allows the particle to reach $z = a$ occurs when $\dot{z} = 0$ at this point. Hence the minimum value satisfies

$$2ga = V_{\min}^2,$$

and

$$V_{\min} = \sqrt{2ga},$$

choosing the positive square root so that the particle moves upwards (it is not allowed to move downwards on the sphere from the point $z = 0$).

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Feedback: *This question takes you through the method of using conservation of energy to investigate a particle moving on a surface of revolution, although parts of the question are quite technically challenging! The key is to keep in mind what you know, and taking care at each step. If you can't do a "show that" type part you can still carry on with the later parts using what has been given in the question!*

2. A smooth surface of revolution has equation $z = a^2/\rho$ where $a > 0$ is a constant, (ρ, θ, z) are cylindrical polar coordinates, with z pointing downwards. A small particle of mass m slides on the interior of the surface.

(a) Briefly explain why

$$\begin{aligned} \rho^2\dot{\theta} &= h, \\ \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz &= E, \end{aligned}$$

are both constants.

(b) If the particle initially moves horizontally with velocity $\rho\dot{\theta} = a\omega > 0$ at depth $z = a$ below the origin, show that

$$\rho^2\dot{\theta} = a^2\omega, \tag{1}$$

$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz = \frac{1}{2}m\omega^2a^2 - mga. \tag{2}$$

(c) By rewriting every term in (2) in terms of z and/or $\dot{z}$, show that the particle moves between $z = a$ and $z = 2g/\omega^2 - a$. Hence show that the particle moves in a circle if $g = a\omega^2$.

Solution. (a) Angular momentum is conserved, since the surface of revolution is smooth, and hence $r^2\dot{\theta}$ is constant. Energy conservation gives

$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \quad = \quad E,$$

where the first term on the left hand side is kinetic energy and the last term is the potential energy due to gravity, where we define the potential to be zero when $z = 0$.

(b) Initially $z = a$, and since $\rho = a^2/z$ on the surface, this means $\rho = a$ also. The initial velocity of the particle is also given by $\rho\dot{\theta} = a\omega$, $\dot{\rho} = 0$, $\dot{z} = 0$. Hence

$$\begin{aligned} \rho^2\dot{\theta} &= \rho \cdot \rho\dot{\theta}. \\ &= a^2\omega, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz &= \frac{1}{2}m\left(0^2 + a^2\omega^2 + 0^2\right) - mga, \\ &= \frac{1}{2}m\omega^2a^2 - mga. \end{aligned}$$

(c) Now, since $\rho = a^2/z$, by differentiating with respect to t we find

$$\dot{\rho} \quad = \quad -\frac{a^2}{z^2}\dot{z}.$$

We also have

$$\begin{aligned} \rho^2\dot{\theta}^2 &= \left(\rho^2\dot{\theta}\right)^2/\rho^2, \\ &= a^4\omega^2/\rho^2, \\ &= a^4\omega^2\left(z/a^2\right)^2, \\ &= \omega^2z^2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz &= \frac{1}{2}m\omega^2a^2 - mga, \\ \implies \frac{1}{2}m\left(\left(-\frac{a^2}{z^2}\dot{z}\right)^2 + \omega^2z^2 + \dot{z}^2\right) - mgz &= \frac{1}{2}m\omega^2a^2 - mga, \\ \implies \frac{1}{2}m\left(\frac{a^4}{z^4}\dot{z}^2 + \omega^2z^2 + \dot{z}^2\right) - mgz &= \frac{1}{2}m\omega^2a^2 - mga, \\ \implies \left(\frac{a^4}{z^4} + 1\right)\dot{z}^2 + \omega^2z^2 - 2gz &= \omega^2a^2 - 2ga. \end{aligned}$$

Now, we want to rearrange to get the derivative on the left hand side, in such a way that it is ≥ 0, and then try to factorise the right hand side. In general you may already know one of the roots which will help! In this case, $z = a$ makes $\dot{z}$ zero (due to the initial condition), so $(z - a)$ must be a factor. Hence

$$\begin{aligned}\left(\frac{a^4}{z^4}+1\right)\dot{z}^2 &= 2gz-\omega^2 z^2+\omega^2 a^2-2ga, \\ &= \omega^2\left(a^2-z^2\right)+2g\left(z-a\right), \\ &= \omega^2\left(a-z\right)\left(a+z\right)+2g\left(z-a\right), \\ &= \left(a-z\right)\left(\omega^2\left(a+z\right)-2g\right), \\ &= -\omega^2\left(z-a\right)\left(z-\left(\frac{2g}{\omega^2}-a\right)\right).\end{aligned}$$

Now since the left hand side is ≥ 0, the right hand side must also be ≥ 0, so location of the particle must obey

$$\omega^2\left(z-a\right)\left(z-\left(\frac{2g}{\omega^2}-a\right)\right) \leq 0,$$

and thus the particle must move between $z = a$ and $z = 2g/\omega^2 - a$. When these values coincide, i.e. when $2g/\omega^2 - a = a$, i.e. $g = a\omega^2$, the particle never changes its z-coordinate; it remains at the initial depth $z = a$ for all time. This means the particle's ρ-coordinate also remains constant at $\rho = a^2/z = a$, and so its angular velocity $\dot{\theta} = h/\rho^2$ remains constant too, at the initial value $\dot{\theta} = \omega \neq 0$. Therefore the particle must move in the circle of depth a with constant speed. In particular, the constant angular velocity means that the period of the circular motion is $T = 2\pi/\omega$; this is the time it takes the particle to move from $\theta = 0$ to $\theta = 2\pi$, and can be found from $\int_0^{2\pi} d\theta = \int_0^T \omega dt$, or simply from $T = 2\pi/\dot{\theta}$.

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Feedback: *This problem gives you practice with using energy conservation for a particle moving on a surface. The algebra looks quite complicated initially, but don't panic and try and keep clear what you're trying to achieve. The approaches to try tend to be fairly standard.*

3. A rocket of mass m launches from rest and expels exhaust gases at speed u relative to the rocket's motion. The rocket burns fuel such that the total mass of the rocket over time is given by $m = m_0 e^{-bt}$ where $b > g/u$ is a constant, and moves under the action of gravity g and air resistance assumed to be of the form kmv^2 where v is the velocity of the rocket.

   (a) Find the velocity of the rocket as a function of time.

   (b) Hence find the limiting velocity of the rocket as time tends to infinity.

Solution. (a) We choose the coordinate system such that gravity acts in the negative directions. The external force acting on the rocket will therefore be given by

$$F = -mg - kmv^2,$$

where the first term gives the effect of gravity, and the second is air resistance. Then the rocket equation gives

$$-mg - mkv^2 = m\frac{dv}{dt} + u\frac{dm}{dt}.$$

Since $m = m_0 e^{-bt}$, the mass will satisfy

$$\frac{dm}{dt} = -bm,$$

and hence

$$-mg - mkv^2 = m\frac{dv}{dt} - bum.$$

Rearranging gives

$$\frac{dv}{dt} = bu - g - kv^2.$$

This is a separable equation and hence

$$\begin{aligned}\frac{1}{bu - g - kv^2}\frac{dv}{dt} &= 1, \\ \Longrightarrow \int \frac{1}{bu - g - kv^2}\frac{dv}{dt}dt &= \int dt.\end{aligned}$$

We solve this using partial fractions, first noticing this will be easier is we define $\gamma^2 = (bu - g)/k (> 0)$. Then

$$\begin{aligned}\frac{1}{bu - g - kv^2} &= \frac{1}{k}\frac{1}{(bu - g)/k - v^2}, \\ &= \frac{1}{k}\frac{1}{\gamma^2 - v^2}, \\ &= \frac{1}{k}\frac{1}{(\gamma - v)(\gamma + v)}, \\ &= \frac{1}{2k\gamma}\left(\frac{1}{(\gamma - v)} + \frac{1}{(\gamma + v)}\right).\end{aligned}$$

Hence

$$\begin{aligned}\int \frac{1}{bu - g - kv^2}\frac{dv}{dt}dt &= \int dt, \\ \Longrightarrow \int \frac{1}{2k\gamma}\left(\frac{1}{(\gamma - v)} + \frac{1}{(\gamma + v)}\right) dv &= \int dt, \\ \Longrightarrow \frac{1}{2k\gamma}\left(-\ln(\gamma - v) + \ln(\gamma + v)\right) &= t + \text{constant}.\end{aligned}$$

Using $v = 0$ at $t = 0$ we find

$$t = \frac{1}{2k\gamma}\ln\left(\frac{\gamma + v}{\gamma - v}\right).$$

This is rearranged to give

$$\begin{aligned}
\frac{\gamma+v}{\gamma-v} &= e^{2k\gamma t}, \\
\implies \gamma+v &= (\gamma-v)\,e^{2k\gamma t}, \\
\implies v\left(1+e^{2k\gamma t}\right) &= \gamma\left(e^{2k\gamma t}-1\right), \\
\implies v &= \gamma\frac{e^{2k\gamma t}-1}{1+e^{2k\gamma t}}, \\
&= \gamma\tanh(\gamma k t).
\end{aligned}$$

As $t \to \infty$, $v \to \gamma$ and hence the rocket reaches the limiting velocity $\sqrt{(bu-g)/k}$.

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4. **[Challenging; optional]** Two railway workers, each of mass m, are standing on a frictionless railway cart of mass M which is on a horizontal track and is initially stationary. The railway workers run from the front of the cart and jump off of the rear of the cart with a speed u relative to the cart.

   (a) What is the final speed of the cart if both workers jump simultaneously?

   (b) What is the final speed of the cart if the workers do not jump at the same time (i.e. the second worker only jumps after the first worker has already jumped off the cart).

   (c) If instead there are N workers, what is the final speed if they all jump sequentially?

Solution. (a) We measure speeds relative to an inertial frame fixed to the stationary ground, with the speed of the cart given by v_c and the speed of the workers given by v_w. These are related via the speed of jumping relative to the cart motion to give

$$v_w = v_c + u.$$

There are no external forces and hence momentum must be conserved. Since everything is initially stationary, the initial momentum is zero, and hence the final momentum must also be zero. Hence

$$2mv_w + Mv_c = 0.$$

Now, since $v_w = v_c + u$, we find

$$\begin{aligned}
2m(v_c+u) + Mv_c &= 0, \\
\implies v_c(M+2m) &= -2mu, \\
\implies v_c &= -\frac{2m}{M+2m}u.
\end{aligned}$$

(b) If alternatively the workers jump one at a time, they will have two different speeds, since the cart will be moving at a different speed when each jumps. We therefore need the speed of the cart when one worker has jumped v_{1c} and when both have jumped v_{2c}; we also define v_{1w}, v_{2w} to be the speed of the 1st and 2nd person at the point of jumping respectively. For the first jump the total momentum will be zero, giving

$$mv_{1w} + (M+m)\,v_{1c} = 0,$$

and hence

$$v_{1c} = -\frac{m}{M+2m}u,$$

as above.

Once the first person has jumped the cart is now moving, and thus the system with the cart and the remaining person has nonzero momentum. We have

$$(M+m)\,v_{1c} = mv_{2w} + Mv_{2c},$$

where the left hand side is the momentum between the jumps and the right hand side is the total momentum after the second jump. Since $v_{2w} = v_{2c} + u$ this gives

$$\begin{aligned}(M+m)\,v_{1c} &= m\,(v_{2c}+u) + Mv_{2c},\\ &= (M+m)\,v_{2c} + mu,\\ \implies v_{2c} &= v_{1c} - \frac{m}{M+m}u,\\ &= -\frac{m}{M+2m}u - \frac{m}{M+m}u.\end{aligned}$$

This is slightly larger in magnitude than in part (a).

(c) Generalising the expression above for N consecutive jumpers gives

$$v_{Nc} = -\frac{m}{M+Nm}u - \ldots - \frac{m}{M+m}u,$$

since each jumper in position n in the line adds velocity

$$-\frac{m}{M+(N+1-n)m}u,$$

to the current velocity of the cart by the same argument as above.

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5. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

A particle of mass m moves without friction on the inner surface of a bowl, whose shape is given in cylindrical polar coordinates (ρ, θ, z) by

$$z = \frac{\rho^2}{a},$$

where $a > 0$ is a constant and the z-axis points vertically upwards. This shape is known as a *paraboloid* since it is obtained by revolving a parabola around its axis. The particle is released from a point where $z = z_0$ for some $z_0 > 0$, with initial velocity $v\mathbf{e}_\theta$ for some $v \geq 0$.

You may assume the following expression for the particle's velocity:

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z.$$

(a) Using the conservation of angular momentum and of energy, show that the particle's z-coordinate obeys the following equation (which is valid for $z > 0$):

$$\dot{z}^2\left(1+\frac{a}{4z}\right) = -2gz + 2gz_0 + v^2 - \frac{v^2 z_0}{z}. \tag{3}$$

You may assume any generic forms of the conservation equations, as long as you state them clearly.

(b) By writing the right-hand side of (3) as

$$-\frac{2g}{z}(z-z_1)(z-z_2),$$

for some constants z_1 and z_2 which you should determine, show that the particle moves in a circle if $v = \sqrt{2gz_0}$ and find the period of the circular motion.

(c) By differentiating (3) with respect to t, show that:

i. If $v > \sqrt{2gz_0}$, then the particle rises initially, and it will begin to fall again if and when it reaches the height $z = v^2/(2g)$.

ii. If $v < \sqrt{2gz_0}$, then the particle falls initially, and it will begin to rise again if and when it reaches the height $z = v^2/(2g)$.

(d) In the special case $v = 0$, describe the motion of the particle. Be as specific as you can, but you do not need to justify your answer.

Solution. (a) Conservation of angular momentum:

$$\rho^2\dot{\theta} = h,$$

where h is a constant determined by evaluating the left-hand side at $t = 0$:

$$h = [\rho]_{t=0}\,[\rho\dot{\theta}]_{t=0} = [\sqrt{az}]_{t=0}\,[\rho\dot{\theta}]_{t=0}.$$

The fact that the initial velocity is $v\mathbf{e}_\theta$ means $[\dot{\rho}]_{t=0} = 0$, $[\rho\dot{\theta}]_{t=0} = v$ and $[\dot{z}]_{t=0} = 0$. We also have $[z]_{t=0} = z_0$, so it follows that

$$h = v\sqrt{az_0}. \tag{4}$$

Conservation of energy:

$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) + mgz = E, \tag{5}$$

where E is a constant determined by evaluating the left-hand side at $t = 0$:

$$E = \frac{1}{2}m\left(0^2 + v^2 + 0^2\right) + mgz_0. \tag{6}$$

In the desired equation (3), everything is written in terms of the variables z and/or $\dot{z}$ (all else are constants); so we must rewrite $\dot{\rho}^2$ and $\rho^2\dot{\theta}^2$ on the left-hand side of (5). Differentiating $z = \rho^2/a$ with respect to t gives

$$\dot{z} = \frac{2\rho\dot{\rho}}{a} \quad\Longleftrightarrow\quad \dot{\rho} = \frac{a\dot{z}}{2\rho} = \frac{a\dot{z}}{2\sqrt{az}} = \frac{\sqrt{a}\dot{z}}{2\sqrt{z}}, \tag{7}$$

which is valid if and only if $z > 0$. Moreover, (4) implies

$$\rho^2\dot{\theta}^2 = \frac{\rho^4\dot{\theta}^2}{\rho^2} = \frac{v^2 a z_0}{\rho^2} = \frac{v^2 a z_0}{az} = \frac{v^2 z_0}{z}. \tag{8}$$

Putting (6), (7) and (8) into (5) yields

$$\frac{1}{2}m\left(\frac{a\dot{z}^2}{4z} + \frac{v^2 z_0}{z} + \dot{z}^2\right) + mgz = \frac{1}{2}mv^2 + mgz_0.$$

Multiplying by $2/m$, we find

$$\dot{z}^2\left(1 + \frac{a}{4z}\right) + \frac{v^2 z_0}{z} + 2gz = v^2 + 2gz_0,$$

and rearranging provides (3) as desired.

(b) From (3), we have

$$\dot{z}^2\left(1 + \frac{a}{4z}\right) = -2gz + 2gz_0 + v^2 - \frac{v^2 z_0}{z} = \frac{-2g}{z}\left(z^2 - z_0 z - \frac{v^2 z}{2g} + \frac{v^2 z_0}{2g}\right). \tag{9}$$

We know that one value of z that makes the left-hand side (specifically $\dot{z}$) zero is $z = z_0$ (this happens at time $t = 0$). So, $(z - z_0)$ must be a factor of the quadratic on the right-hand side of (9). Taking the factor out, we find

$$\dot{z}^2\left(1 + \frac{a}{4z}\right) = \frac{-2g}{z}(z - z_0)(z - \frac{v^2}{2g}). \tag{10}$$

Since the left-hand side of (10) is ≥ 0, the right-hand side must be ≥ 0, i.e.,

$$(z - z_0)(z - \frac{v^2}{2g}) \leq 0.$$

For this upward-opening parabola to be ≤ 0, we must have z taking values between z_0 and $v^2/(2g)$. In particular, if $z_0 = v^2/(2g)$, i.e.

$$v = \sqrt{2gz_0} > 0,$$

then the particle can never change its z-coordinate; it remains at height $z = z_0$ for all time. In this case the particle's radial coordinate $\rho = \sqrt{az} = \sqrt{az_0}$ also remains constant, and hence its angular velocity $\dot{\theta} = h/\rho^2$ remains constant too. The constant angular velocity is non-zero, so the particle must move in the circle of height z_0 with period

$$T = \frac{2\pi}{\dot{\theta}} = \frac{2\pi\rho^2}{h} = \frac{2\pi a z_0}{v\sqrt{az_0}} = \frac{2\pi a z_0}{\sqrt{2gz_0}\sqrt{az_0}} = \pi\sqrt{\frac{2a}{g}}. \tag{11}$$

Note that this period does not depend on the initial height z_0!

(c) Differentiating (3) with respect to t gives

$$2\dot{z}\ddot{z}\left(1+\frac{a}{4z}\right)+\dot{z}^2\left(\frac{-a\dot{z}}{4z^2}\right)=-2g\dot{z}+\frac{v^2z_0\dot{z}}{z^2},$$

Dividing by $\dot{z}$ yields that wherever $\dot{z}\neq 0$, i.e. $z\neq z_0$ and $z\neq v^2/(2g)$, we have

$$2\ddot{z}\left(1+\frac{a}{4z}\right)=-2g+\frac{v^2z_0}{z^2}+\frac{a\dot{z}^2}{4z^2}. \tag{12}$$

Note that $1+a/(4z)$ is always positive. So, $\ddot{z}$ is positive if and only if the right-hand side of (12) is positive. Initially, $\dot{z}=0$ and $z=z_0$, so the sign of $[\ddot{z}]_{t=0}$ is the same as the sign of $-2g+v^2z_0/z^2=-2g+v^2/z_0$. On the other hand, when $z=v^2/(2g)$, (10) implies $\dot{z}=0$, so the sign of $[\ddot{z}]_{z=v^2/(2g)}$ is the same as the sign of

$$-2g+\frac{v^2z_0}{z^2}=-2g+\frac{4g^2z_0}{v^2}$$

.

i. If $v>\sqrt{2gz_0}$, then $-2g+v^2/z_0>0$, so $[\ddot{z}]_{t=0}>0$ which means the particle initially rises; it cannot stop rising until the next point at which $\dot{z}=0$, which occurs at $z=v^2/(2g)>z_0$; and

$$-2g+\frac{4g^2z_0}{v^2}<-2g+\frac{4g^2z_0}{2gz_0}=0,$$

so $[\ddot{z}]_{z=v^2/(2g)}<0$ which means the particle begins falling at $z=v^2/(2g)$.

ii. If $v<\sqrt{2gz_0}$, then $-2g+v^2/z_0<0$, so $[\ddot{z}]_{t=0}<0$ which means the particle initially falls; it cannot stop falling until the next point at which $\dot{z}=0$, which occurs at $z=v^2/(2g)<z_0$; and

$$-2g+\frac{4g^2z_0}{v^2}>-2g+\frac{4g^2z_0}{2gz_0}=0,$$

so $[\ddot{z}]_{z=v^2/(2g)}>0$ which means the particle begins rising at $z=v^2/(2g)$.

(d) If $v=0$, the particle will move along a single parabola in some vertical plane. Its motion will be periodic; within each period, it will start from $z=z_0$, fall to $z=0$ (the bottom of the bowl), rise to $z=z_0$ on the opposite side of the bowl, fall again to $z=0$ and then rise to the starting position $z=z_0$.

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