

2DE/2DE3 Example sheet 2 solutions: Analytical solutions to ODEs

1. Show that $u_1 = e^{-x}$ is one solution of

$$xy'' + (x - 1)y' - y = 0, \quad x > 0,$$

and use the Reduction of Order method to obtain the general solution.

$$u_1 = e^{-x}, \quad u_1'(x) = -e^{-x}, \quad u_1''(x) = e^{-x},$$

Sub these into the equation to check that u_1 is a solution:

$$LHS = xe^{-x} + (x - 1)(-e^{-x}) - e^{-x} = 0 = RHS,$$

so u_1 is indeed a solution, and we look for a solution in the form $y = vu_1$. Then

$$\begin{aligned} y &= ve^{-x}, \\ y' &= v'e^{-x} - ve^{-x}, \\ y'' &= v''e^{-x} - 2v'e^{-x} + ve^{-x}. \end{aligned}$$

Subbing the above into the ODE with some rearranging gives

$$\begin{aligned} xe^{-x}v'' + (-2xe^{-x} + (x - 1)e^{-x})v' + (xe^{-x} - (x - 1)e^{-x} - e^{-x})v &= 0, \\ xe^{-x}v'' - (xe^{-x} + e^{-x})v' &= 0. \end{aligned}$$

Letting $w = v'$ and rearranging we have,

$$\begin{aligned} \frac{dw}{dx} &= \left(1 + \frac{1}{x}\right)w, \\ \ln|w| &= x + \ln(x) + \alpha_1, \\ w &= \alpha_2 e^x x, \quad \text{where } \alpha_2 = \pm e^{\alpha_1} \\ \implies v &= \int \alpha_2 e^x x \, dx, \\ &= \alpha_2 e^x (x - 1) + \alpha_3, \quad (\text{using integration by parts}). \end{aligned}$$

Hence

$$\begin{aligned} y &= vu_1, \\ \implies y &= \alpha_2(x - 1) + \alpha_3 e^{-x} \end{aligned}$$

is the general solution.

2. Show that $u_1 = x$ is one solution of

$$y'' + xy' - y = 0, \quad x > 0,$$

and use the Reduction of Order method to obtain the general solution in terms of an integral (note that you do not need to evaluate the final integral).

$$u_1(x) = x, \quad u'_1(x) = 1, \quad u''_1(x) = 0.$$

Sub these into the equation to check that u_1 is a solution:

$$LHS = 0 + x - x = 0 = RHS,$$

so $u_1 = x$ is a solution.

The equation is already in an appropriate format to use the Reduction of Order method as it is written in the lecture notes, i.e. the coefficient of y'' is 1 and $a(x) = x$. Looking for a solution in the form $y = vu_1$ will yield the first order differential equation for $w = v'$ (note that for a model solution you should derive this formula using the method in the lecture notes):

$$\begin{aligned} xw' + (2 + x^2)w &= 0, \\ \frac{dw}{dx} &= -\frac{(2 + x^2)}{x}w, \\ \ln |w| &= -\int \frac{2}{x} + x \, dx, \\ &= -2 \ln(x) - \frac{x^2}{2} + \alpha_1, \\ w &= \alpha_2 e^{\ln(x^{-2})} e^{-x^2/2}, \quad \text{where } \alpha_2 = \pm e^{\alpha_1} \\ &= \frac{\alpha_2 e^{-x^2/2}}{x^2}, \\ \implies v &= \alpha_2 \int \frac{e^{-x^2/2}}{x^2} \, dx + \alpha_3, \\ \implies y &= \alpha_2 x \int \frac{e^{-x^2/2}}{x^2} \, dx + \alpha_3 x. \end{aligned}$$

3. Find a solution to the following equation in the form of x^r for some constant r and use this solution to obtain the general solution using the Reduction of Order method.

$$x^2y'' - 3xy' + 4y = 0, \quad x > 0.$$

$$u_1 = x^r, \quad u'_1 = rx^{r-1}, \quad u''_1 = r(r-1)x^{r-2}.$$

Sub these into the equation:

$$\begin{aligned} x^2r(r-1)x^{r-2} - 3rxr x^{r-1} + 4x^r &= 0, \\ r(r-1)x^r - 3rx^r + 4x^r &= 0, \\ r^2 - r - 3r + 4 &= 0, \\ r^2 - 4r + 4 &= 0, \\ (r-2)^2 &= 0, \implies r = 2 \implies u_1 = x^2. \end{aligned}$$

Looking for a solution in the form $y = u_1 v$, we have

$$\begin{aligned} y &= x^2 v, \\ y' &= 2xv + x^2 v', \\ y'' &= 2v + 4xv' + x^2 v''. \end{aligned}$$

Subbing the above into the ODE with some rearranging gives

$$\begin{aligned} x^4 v'' + (4x^3 - 3x^3)v' + (2x^2 - 6x^2 + 4x^2)v &= 0, \\ x^4 v'' + x^3 v' &= 0, \\ xv'' + v' &= 0, \end{aligned}$$

(since $x \neq 0$). Letting $w = v'$, this is equivalent to

$$\begin{aligned} xw' + w &= 0, \\ x \frac{dw}{dx} &= -w, \\ \int \frac{1}{w} dw &= - \int \frac{1}{x} dx, \\ \ln |w| &= -\ln(x) + \alpha_1, \\ w &= \frac{\alpha_2}{x}, \quad \text{with } \alpha_2 = \pm e^{\alpha_1}, \\ \implies v &= \alpha_2 \ln(x) + \alpha_3, \\ \implies y &= \alpha_2 x^2 \ln(x) + \alpha_3 x^2. \end{aligned}$$

4. You have seen in previous modules that the solution to a 2nd order linear ODE with constant coefficients whose characteristic equation has a repeated root, a say, is of the form $y = \alpha_1 x e^{ax} + \alpha_2 e^{ax}$. By looking for a solution of the form $u_1 = e^{rx}$ to

$$y'' - 2ay' + a^2 y = 0,$$

and then using the Reduction of Order method with this first solution, verify that this is the case.

$$u_1 = e^{rx}, \quad u'_1 = re^{rx}, \quad u''_1 = r^2 e^{rx}.$$

Sub these into the equation to give,

$$\begin{aligned} r^2 e^{rx} - 2ar e^{rx} + a^2 e^{rx} &= 0, \\ r^2 - 2ar + a^2 &= 0, \\ (r - a)^2 = 0 &\implies r = a. \end{aligned}$$

Hence $u_1 = e^{ax}$ is one solution. Looking for a solution in the form $y = vu_1$, the Reduction of Order method (make sure you can reproduce this for yourself) will give

the following 1st order linear ODE in $w = v'$:

$$\begin{aligned} e^{ax}w' + (2ae^{ax} - 2ae^{ax})w &= 0, \\ \frac{dw}{dx} &= 0, \\ w &= \alpha_1, \\ \implies v &= \alpha_1 x + \alpha_2, \\ \implies y &= \alpha_1 x e^{ax} + \alpha_2 e^{ax}, \end{aligned}$$

which is the solution given in the question.

5. Given that $u_1 = 1 + x$ and $u_2 = e^x$ form a fundamental set of solutions to the homogeneous version of

$$xy'' - (1+x)y' + y = x^2 e^{2x}, \quad x > 0,$$

use the Variation of Parameters method to obtain the general solution to the inhomogeneous equation. What is the particular solution of this equation?

Rearrange the equation into the format used in the lecture notes ($y'' + a(x)y' + b(x) = c(x)$):

$$y'' - \left(\frac{1}{x} + 1\right)y' + \frac{1}{x}y = xe^{2x},$$

so $c(x) = xe^{2x}$.

The Wronskian of u_1 and u_2 is given by

$$\begin{aligned} W(u_1, u_2) &= \begin{vmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{vmatrix} \\ &= \begin{vmatrix} 1+x & e^x \\ 1 & e^x \end{vmatrix} \\ &= (1+x)e^x - e^x, \\ &= xe^x. \end{aligned}$$

If we look for a solution in the form $y = v_1 u_1 + v_2 u_2$ and use the Variation of Parameters

method, we obtain (see lecture notes to complete the intermediate steps):

$$\begin{aligned}
 v'_1 &= \frac{-c(x)u_2(x)}{W(u_1, u_2)} = -\frac{e^x xe^{2x}}{xe^x} = -e^{2x} \implies v_1 = -\frac{1}{2}e^{2x} + \alpha_1, \\
 v'_2 &= \frac{c(x)u_1(x)}{W(u_1, u_2)} = \frac{(1+x)xe^{2x}}{xe^x} = (1+x)e^x \implies v_2 = xe^x + \alpha_2, \quad (\text{integration by parts}), \\
 \implies y &= \left(-\frac{1}{2}e^{2x} + \alpha_1 \right)(1+x) + (xe^x + \alpha_2)e^x, \\
 &= \alpha_1(1+x) + \alpha_2e^x - \frac{1}{2}e^{2x}(1+x) + xe^{2x}, \\
 &= \alpha_1(1+x) + \alpha_2e^x + e^{2x}\left(x - \frac{1}{2} - \frac{x}{2}\right), \\
 &= \alpha_1(1+x) + \alpha_2e^x + e^{2x}\left(\frac{x}{2} - \frac{1}{2}\right), \\
 y &= \alpha_1(1+x) + \alpha_2e^x + \frac{e^{2x}}{2}(x-1).
 \end{aligned}$$

The particular solution is therefore $y_p = \frac{e^{2x}}{2}(x-1)$.

6. Find the general solution of

$$y'' + 4y' + 4y = x^{-2}e^{-2x}, \quad x > 0$$

using the Variation of Parameters method. What is the particular solution of this equation?

We first need to find the general solution of the corresponding homogeneous equation

$$y'' + 4y' + 4y = 0.$$

Looking for a solution in the form $u = e^{rx}$ (using the standard method for 2nd order linear ODEs with constant coefficients) gives $(r+2)^2$, i.e. we have a repeated root $r = -2$ so that $u_1 = e^{-2x}$ and $u_2 = xe^{-2x}$ form a fundamental set of solutions of the corresponding homogeneous equation (see Q4).

The Wronskian is given by

$$\begin{aligned}
 W(u_1, u_2) &= \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (e^{-2x} - 2xe^{-2x}) \end{vmatrix} \\
 &= e^{-2x}(e^{-2x} - 2xe^{-2x}) + 2e^{-2x}xe^{-2x}, \\
 &= e^{-4x}(1 - 2x) + 2xe^{-4x}, \\
 &= e^{-4x}(1 - 2x + 2x), \\
 &= e^{-4x}.
 \end{aligned}$$

Therefore (using intermediate steps given in the lecture notes)

$$\begin{aligned} v'_1 &= \frac{-xe^{-2x}x^{-2}e^{-2x}}{e^{-4x}} = -x^{-1} \implies v_1 = -\ln(x) + \alpha_1, \\ v'_2 &= \frac{e^{-2x}x^{-2}e^{-2x}}{e^{-4x}} = x^{-2} \implies v_2 = -x^{-1} + \alpha_2, \\ \implies y &= e^{-2x}(-\ln(x) + \alpha_1) + xe^{-2x}\left(-x^{-1} + \alpha_2\right), \\ &= -e^{-2x}(\ln(x) + 1) + \alpha_1 e^{-2x} + \alpha_2 x e^{-2x}, \\ &= -e^{-2x} \ln(x) + \tilde{\alpha}_1 e^{-2x} + \alpha_2 x e^{-2x}, \quad (\tilde{\alpha}_1 = \alpha_1 - 1). \end{aligned}$$

Hence the particular solution is $-e^{-2x} \ln(x)$ as the other two terms make up the general solution to the corresponding homogeneous equation.

7. Given that $u_1(x) = x$ is a solution to

$$x^2y'' - 2xy' + 2y = 0, \quad x > 0,$$

- (a) find the general solution to this equation;
- (b) and find the particular solution to

$$x^2y'' - 2xy' + 2y = 4x^2, \quad x > 0.$$

(a) Use the Reduction of Order method to find a second solution. Let $y = vu_1 = xv$, then

$$\begin{aligned} y' &= xv' + v, \\ y'' &= xv'' + v' + v' = xv'' + 2v'. \end{aligned}$$

Subbing these into the equation gives

$$\begin{aligned} x^2(2v' + xv'') - 2x(v + xv') + 2xv &= 0, \\ x^3v'' + (2x^2 - 2x^2)v' + v(-2x + 2x) &= 0, \\ x^3v'' &= 0, \\ v'' &= 0 \implies v' = \alpha_1 \implies v = \alpha_1 x + \alpha_2, \\ \implies y &= \alpha_1 x^2 + \alpha_2 x, \end{aligned}$$

i.e. $\{x, x^2\}$ form a fundamental set of solutions to the homogeneous equation.

(b) Use the Variation of Parameters method to find the general solution of the inhomogeneous equation. To use the formulae from the notes, rearrange the inhomogeneous ODE into the appropriate format:

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 4.$$

Note that $\{x, x^2\}$ will also form a fundamental set of solutions to the corresponding homogeneous ODE of the rearranged equation – make sure you are convinced of this!

The Wronskian is given by

$$\begin{aligned} W(x, x^2) &= 2x^2 - x^2, \\ &= x^2. \end{aligned}$$

Then (using intermediate steps given in the lecture notes)

$$\begin{aligned} v'_1 &= -\frac{4x^2}{x^2} = -4 \implies v_1 = -4x + \alpha_1, \\ v'_2 &= \frac{4x}{x^2} = \frac{4}{x} \implies v_2 = 4 \ln(x) + \alpha_2, \\ \implies y &= -4x^2 + \alpha_1 x + 4x^2 \ln(x) + \alpha_2 x^2, \\ &= 4x^2 \ln(x) + \alpha_1 x + \tilde{\alpha}_2 x^2, \quad (\tilde{\alpha}_2 = \alpha_2 - 4). \end{aligned}$$

Hence the particular solution to the inhomogeneous equation is $4x^2 \ln(x)$.

8. Given that $u_1(x) = e^x$ is a solution to

$$(1-x)y'' + xy' - y = 0, \quad 0 < x < 1,$$

- (a) find the general solution to this equation;
- (b) and find the particular solution to

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}, \quad 0 < x < 1.$$

This time, we can start by rearranging the equations into the format used in lecture notes (note: you do not *need* to do this unless you're memorising relevant formulae – see solution to previous question, for example):

$$\begin{aligned} y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y &= 0, \\ y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y &= -2(x-1)e^{-x}. \end{aligned}$$

- (a) Use the Reduction of Order method to find a second solution. Let $y = vu_1 = ve^x$, then

$$\begin{aligned} y' &= ve^x + v'e^x, \\ y'' &= ve^x + v'e^x + v''e^x + v'e^x = ve^x + 2v'e^x + v''e^x. \end{aligned}$$

Subbing these into the equation (rearranged into the appropriate form) gives

$$\begin{aligned} ve^x + 2v'e^x + v''e^x + \frac{x}{(1-x)}(ve^x + v'e^x) - \frac{1}{(1-x)}ve^x &= 0, \\ v''e^x + v'\left(2e^x + \frac{xe^x}{1-x}\right) + v\left(e^x + \frac{xe^x}{1-x} - \frac{e^x}{1-x}\right) &= 0, \\ v'' + v'\left(\frac{2-2x+x}{1-x}\right) + v\left(\frac{e^x - xe^x + xe^x - e^x}{1-x}\right) &= 0, \\ v'' + v'\left(\frac{2-x}{1-x}\right) &= 0. \end{aligned}$$

Let $w = v'$ to give the following 1st order linear equation in w :

$$\begin{aligned} \frac{dw}{dx} + w\left(\frac{2-x}{1-x}\right) &= 0, \\ \frac{dw}{dx} &= \frac{(x-2)}{(1-x)}w, \\ \int \frac{1}{w} dw &= \int \frac{x-2}{1-x} dx, \\ \ln|w| &= \int \frac{u+1}{u} du, \quad (\text{using the substitution } u = 1-x, \text{ for example}), \\ &= \int 1 + \frac{1}{u} du, \\ &= u + \ln|u| + \alpha_1, \\ &= 1-x + \ln(1-x) + \alpha_1, \\ &= \alpha_2 - x + \ln(1-x), \quad (\alpha_2 = \alpha_1 + 1), \\ w &= \alpha_3 e^{-x} e^{\ln(1-x)}, \quad (\alpha_3 = \pm e^{\alpha_2}), \\ &= \alpha_3 e^{-x}(1-x), \\ \implies v &= \int \alpha_3 e^{-x}(1-x) dx, \\ &= \alpha_3 x e^{-x} + \alpha_4, \quad (\text{integration by parts}), \\ \implies y &= e^x(\alpha_3 x e^{-x} + \alpha_4), \\ &= \alpha_3 x + \alpha_4 e^x. \end{aligned}$$

i.e. $\{x, e^x\}$ form a fundamental set of solutions to the homogeneous equation.

(b) Use the Variation of Parameters method to find the general solution of the inhomogeneous equation. The Wronskian is given by

$$\begin{aligned} W(x, e^x) &= xe^x - e^x, \\ &= e^x(x-1). \end{aligned}$$

Then (using intermediate steps given in the lecture notes)

$$\begin{aligned}
 v'_1 &= -\frac{2(1-x)e^{-x}e^x}{e^x(x-1)} = -\frac{2(1-x)}{e^x(x-1)} = 2e^{-x} \implies v_1 = -2e^{-x} + \alpha_5, \\
 v'_2 &= \frac{2(1-x)e^{-x}x}{e^x(x-1)} = -2xe^{-2x} \implies v_2 = xe^{-2x} + \frac{e^{-2x}}{2} + \alpha_6, \\
 \implies y &= x(\alpha_5 - 2e^{-x}) + e^x(xe^{-2x} + \frac{e^{-2x}}{2} + \alpha_6), \\
 &= \alpha_5x + \alpha_6e^x - 2xe^{-x} + xe^{-x} + \frac{e^{-x}}{2}, \\
 &= \alpha_5x + \alpha_6e^x - xe^{-x} + \frac{e^{-x}}{2}, \\
 &= \alpha_5x + \alpha_6e^x - e^{-x}\left(x - \frac{1}{2}\right).
 \end{aligned}$$

Hence the particular solution to the inhomogeneous equation is $-e^{-x}\left(x - \frac{1}{2}\right)$.

9. (a) By looking for solutions to

$$y''' - y'' = 0,$$

in the form $y = e^{rx}$, find a fundamental set of solutions to the above equation.

(b) Use the Variation of Parameters method to find the particular solution to

$$y''' - y'' = e^x.$$

(a) The solutions $y = e^{rx}$ must satisfy $r^3 - r^2 = 0 \implies r^2(r-1) = 0 \implies r = 0, 0, 1$. Hence $\{e^0, xe^0, e^x\}$ or $\{1, x, e^x\}$ form a fundamental set of solutions.

(b) Let $u_1 = 1, u_2 = x, u_3 = e^x$ and look for a solution in the form $y = v_1 + v_2x + v_3e^x$ satisfying (as an extension of Variation of Parameters for 2nd order equations – see lecture notes to fill in the intermediate steps)

$$\begin{aligned}
 v'_1u_1 + v'_2u_2 + v'_3u_3 &= 0, & \implies v'_1 + xv'_2 + e^xv'_3 &= 0, \\
 v'_1u'_1 + v'_2u'_2 + v'_3u'_3 &= 0, & \implies v'_2 + e^xv'_3 &= 0, \\
 v'_1u''_1 + v'_2u''_2 + v'_3u''_3 &= e^x, & \implies e^xv'_3 &= e^x.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 v'_3 &= 1 \implies v_3 = x + \alpha_3, \\
 v'_2 &= -e^x \implies v_2 = - \int e^x dx = -e^x + \alpha_2, \\
 v'_1 &= xe^x - e^x \implies v_1 = \int e^x(x-1) dx = e^x(x-2) + \alpha_1, \quad (\text{integration by parts}).
 \end{aligned}$$

$$\begin{aligned}\implies y &= e^x(x - 2) + \alpha_1 + x(\alpha_2 - e^x) + e^x(x + \alpha_3), \\ &= \alpha_1 + \alpha_2 x + \alpha_3 e^x + e^x(x - 2) - x e^x + x e^x, \\ &= \alpha_1 + \alpha_2 x + \tilde{\alpha}_3 e^x + x e^x, \quad (\tilde{\alpha}_3 - 2).\end{aligned}$$

The particular solution to the inhomogeneous equation is therefore $x e^x$.

10. Use the Reduction of Order method for **nonlinear** ODEs to obtain solutions to the following ODE for $y = y(x)$:

$$y'' + y(y')^3 = 0.$$

You do not need to express the solution explicitly for $y = y(x)$.

We first note that the ODE only contains the independent variable implicitly and so we can use the Reduction of Order method for nonlinear ODEs. We make the substitution $y' = w$ which yields $y'' = \frac{dw}{dx} = w \frac{dw}{dy}$ (see lectures notes for details).

Substituting the above into the ODE gives:

$$\begin{aligned}w \frac{dw}{dy} + yw^3 &= 0, \\ w \left(\frac{dw}{dy} + yw^2 \right) &= 0.\end{aligned}$$

So either $w = 0 \implies y = c_1$ where c_1 is an arbitrary constant, or

$$\frac{dw}{dy} = -yw^2.$$

This is a separable first order ODE:

$$\begin{aligned}\int w^{-2} dw &= \int -y dy, \\ -\frac{1}{w} &= -\frac{y^2}{2} + c_2, \\ w &= \frac{-2}{c_2 - y^2}.\end{aligned}$$

But we know that $w = \frac{dy}{dx}$. Hence,

$$\begin{aligned}\frac{dy}{dx} &= \frac{-2}{c_2 - y^2}, \\ \int -2 dx &= \int c_2 - y^2 dy, \\ -2x &= c_2 y - \frac{y^3}{3} + c_3, \\ \frac{y^3}{3} - c_2 y &= c_3 + 2x.\end{aligned}$$

Here c_1, c_2 and c_3 are arbitrary constants.

11. Use the Reduction of Order method for nonlinear ODEs to obtain solutions to the following ODE for $y = y(x)$:

$$yy'' + (y')^2 = yy', \quad y > 0.$$

We first note that the ODE only contains the independent variable implicitly and so we can use the Reduction of Order method for nonlinear ODEs. We make the substitution $y' = w$ which yields $y'' = \frac{dw}{dx} = w \frac{dw}{dy}$ (see lectures notes for details).

Substituting the above into the ODE gives:

$$\begin{aligned} yw \frac{dw}{dy} + w^2 &= yw, \\ w \left(y \frac{dw}{dy} + w - y \right) &= 0. \end{aligned}$$

Hence, either $w = 0 \implies y = c_1$, or

$$\frac{dw}{dy} + \frac{1}{y}w = 1.$$

This is a first order linear ODE in $w = w(y)$ and hence we can use the integrating factor method (it is also possible to arrive at the below by inspection):

$$\mu(y) = \exp \left(\int \frac{1}{y} dy \right) = e^{\ln(y)} = y.$$

Multiplying through by the integrating factor gives:

$$\begin{aligned} \frac{d}{dy}(yw) &= y, \\ yw &= \frac{y^2}{2} + c_2, \\ w &= \frac{y}{2} + c_2 y^{-1}, \\ \frac{dy}{dx} &= \frac{y}{2} + \frac{c_2}{y}, \end{aligned}$$

which is a separable equation giving:

$$\int \frac{2y}{y^2 + 2c_2} dy = \int dx.$$

We can solve the above using the substitution $u = y^2 \implies \frac{du}{dy} = 2y$:

$$\begin{aligned} \int \frac{1}{u + 2c_2} du &= x + c_3, \\ \ln |u + 2c_2| &= x + c_3, \\ u + 2c_2 &= c_4 e^x, \quad (c_4 = \pm e^{c_3}), \\ y^2 &= c_4 e^x - 2c_2. \end{aligned}$$

12. If we have the 2nd order linear homogeneous ODE

$$a(x)y'' + b(x)y' + c(x)y = 0 \quad (1)$$

where

$$a''(x) - b'(x) + c(x) = 0 \quad (2)$$

then the equation is called **exact** and can be transformed into the 1st order (inhomogeneous) linear equation in y :

$$a(x)y' + (b(x) - a'(x))y = \alpha_1 \quad (3)$$

where α_1 is a constant. By differentiating (3) prove that the two equations are equivalent.

Differentiating (3) gives

$$\begin{aligned} a'(x)y' + a(x)y'' + (b'(x) - a''(x))y + (b(x) - a'(x))y' &= 0, \\ a(x)y'' + (a'(x) + b(x) - a'(x))y' + (b'(x) - a''(x))y &= 0, \\ a(x)y'' + b(x)y' + (b'(x) - a''(x))y &= 0. \end{aligned}$$

Imposing the condition (2) means this is equivalent to

$$a(x)y'' + b(x)y' + c(x)y = 0$$

which is our original equation (1).

13. Solve $x^2y'' + xy' - y = 0$ on $x > 0$.

First, check that the equation is exact:

$$\begin{aligned} a(x) &= x^2, & b(x) &= x, & c(x) &= -1, \\ a'(x) &= 2x, & b'(x) &= 1, \\ a''(x) &= 2, \end{aligned}$$

$a'' - b' + c = 2 - 1 - 1 = 0$ so the equation satisfies the condition (2) and is exact.

This means we can rewrite the equation in the form given by (3):

$$\begin{aligned} x^2y' + (x - 2x)y &= \alpha_1, \\ x^2y' - xy &= \alpha_1, \\ y' - \frac{1}{x}y &= \alpha_1 \frac{1}{x^2}, \quad (\text{solve using the integrating factor for 1st order linear equations}) \\ \frac{d}{dx} \left(\frac{1}{x}y \right) &= \alpha_1 \frac{1}{x^3}, \\ y(x) &= \alpha_1 x \int x^{-3} dx, \\ &= \alpha_1 x \left(-\frac{1}{2x^2} + \alpha_2 \right), \\ y(x) &= \tilde{\alpha}_1 \frac{1}{x} + \tilde{\alpha}_2 x, \end{aligned}$$

(where $\tilde{\alpha}_1 = -\frac{1}{2}\alpha_1$ and $\tilde{\alpha}_2 = \alpha_1\alpha_2$).

14. Check that the following equation is an exact 2nd order linear ODE and solve it in terms of an integral (note that you do not need to evaluate the integral).

$$y'' + xy' + y = 0$$

To establish if the equation is exact we need to show that, writing the equation in the form $a(x)y'' + b(x)y' + c(x)y = 0$,

$$a''(x) - b'(x) + c(x) = 0.$$

We have that

$$\begin{aligned} a(x) &= 1, & b(x) &= x, & c(x) &= 1, \\ a'(x) &= a''(x) = 0, & b'(x) &= 1. \end{aligned}$$

Therefore,

$$a'' - b' + c = 0 - 1 + 1 = 0$$

and the equation is exact.

To solve an exact equation, rewrite it in the form

$$a(x)y' + (b(x) - a'(x))y = \alpha_1$$

to give

$$\begin{aligned} y' + xy &= \alpha_1, \\ \frac{d}{dx}(e^{\frac{x^2}{2}}y) &= \alpha_1 e^{\frac{x^2}{2}}, \quad (\text{using an integrating factor}) \\ y &= \alpha_1 e^{-\frac{x^2}{2}} \left(\int e^{\frac{x^2}{2}} dx + \alpha_2 \right). \end{aligned}$$

15. If a 2nd order linear ODE is not exact, it can be made exact by multiplying through by an appropriate integrating factor $\mu(x)$:

$$\mu(x)a(x)y'' + \mu(x)b(x)y' + \mu(x)c(x)y = 0$$

where $\mu(x)$ is chosen to satisfy the condition

$$[\mu(x)a(x)]'' - [\mu(x)b(x)]' + \mu(x)c(x) = 0, \quad (\text{taken directly from (2)}).$$

Derive a 2nd order linear ODE that must be solved to obtain $\mu(x)$. This equation is called the **adjoint** of the original equation. [However, solving the adjoint equation to find $\mu(x)$ is usually just as hard as solving the original equation, so this approach is not often much use...]

Expanding the above gives

$$\begin{aligned} [\mu'a + \mu a']' - \mu'b - \mu b' + \mu c &= 0, \\ \mu''a + \mu'a' + \mu'a' + \mu a'' - \mu'b - \mu b' + \mu c &= 0, \\ a\mu'' + (2a' - b)\mu' + (a'' - b' + c)\mu &= 0, \end{aligned}$$

which is a second order linear ODE for $\mu = \mu(x)$.

16. A 2nd order linear equation is called **self-adjoint** if its adjoint is the same as the original equation. Show that Airy's equation (use in optics and astronomy, amongst other applications):

$$y'' - xy = 0$$

is self-adjoint.

The adjoint of a 2nd order linear equation is given by

$$a(x)\mu'' + (2a'(x) - b(x))\mu' + (a''(x) - b'(x) + c(x))\mu = 0.$$

We have that

$$\begin{aligned} a(x) &= 1, & b(x) &= 0, & c(x) &= -x, \\ a'(x) &= a''(x) = 0, & b'(x) &= 0. \end{aligned}$$

Hence, the adjoint of the Airy equation is

$$\mu'' - x\mu = 0$$

which is itself the Airy equation.

This is an example of when the adjoint does not help us at all!

17. Show that a necessary condition for the general 2nd order linear homogeneous ODE

$$a(x)y'' + b(x)y' + c(x)y = 0$$

to be self-adjoint is that

$$a'(x) = b(x).$$

The adjoint of the above 2nd order linear equation is given by

$$a(x)\mu'' + (2a'(x) - b(x))\mu' + (a''(x) - b'(x) + c(x))\mu = 0.$$

Matching coefficients between the original equation and the adjoint, we require

$$\begin{aligned} a(x) &= a(x), & b(x) &= 2a'(x) - b(x), & c(x) &= a''(x) - b'(x) + c(x), \\ a'(x) &= b(x), & a''(x) &= b'(x). \end{aligned}$$

Hence $a'(x) = b(x)$ is a necessary condition for the 2nd order linear homogeneous equation to be self-adjoint.