1Mech — Mechanics

Mechanics exercises 4 (weeks 7 and 8)

This sheet's assessed question is question 5.

1. If we throw a particle of mass m upwards from the top of a wall of height h with velocity v, how high will it go and what will its velocity be when it hits the ground?

2. Let a particle of mass m be attached to two springs, both with spring constant k and natural length a. The end of one spring (denoted α) is attached at a point A, with the end of the other spring (denoted β) attached at a point B, a distance $4a$ directly above A. The mass is attached to the free end of both springs. The particle is at a location $x(t)$, where x is measured upwards such that $x = 0$ at point A.

   (a) Show that

      i. the extension in spring α is given by $x - a$.

      ii. the extension in spring β is given by $3a - x$.

   (b) Hence write down the equation for conservation of energy.

   (c) If the particle is initially at rest at $x = a$, find the value of the constant energy.

   (d) Find the height at which the particle will next come to rest.

3. A smooth (i.e. no friction) wire is in the shape of a helix so that $x = a\cos\theta(t)$, $y = a\sin\theta(t)$, $z = b\theta(t)$, with the central (z) axis pointing vertically upwards. A small bead of mass m moves along the wire, starting from height $z = b$ at rest.

   (a) By writing down the position vector and differentiating, show that

   $$\dot{\mathbf{r}} = -a\dot{\theta}\sin\theta\mathbf{i} + a\dot{\theta}\cos\theta\mathbf{j} + b\dot{\theta}\mathbf{k},$$

   and hence that the kinetic energy of the bead is given by

   $$\frac{1}{2}m\left(a^2 + b^2\right)\dot{\theta}^2.$$

   (b) Write down the potential energy of the bead in terms of θ, choosing the potential to be zero at $z = 0$.

   (c) Hence show that conservation of energy gives

   $$\frac{1}{2}m\left(a^2 + b^2\right)\dot{\theta}^2 + mgb\theta = mgb.$$

   (d) By rearranging to find an equation for $\dot{\theta}^2$, find the maximum value that θ can attain. What does this mean physically?

   (e) **Optional (hard!) extension:** How long will it take for the bead to reach $z = 0$?

4. A comet (mass m) which is travelling with speed V, approaches a stationary planet from a great distance. If the path of the comet was not affected by the planet, the distance of closest approach would be p. The comet experiences an attractive force GMm/r^2 towards the planet where r is the distance between them, G is the gravitational constant and M is the mass of the planet.

   (a) **Briefly** explain why

$$\begin{aligned} r^2\dot{\theta} &= \text{constant}, \\ \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r} &= \text{constant}. \end{aligned}$$

   (b) Using the initial conditions, find the values of the constants in part (a).

   (c) By eliminating $\dot{\theta}$, show that

$$\dot{r}^2 = V^2 + \frac{2GM}{r} - \frac{p^2V^2}{r^2}.$$

   (d) Hence calculate the actual distance of closest approach.

5. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

   A frictionless slide in the vertical plane takes the shape

$$x = a(\theta - \sin\theta), \quad y = a(1 + \cos\theta), \tag{1}$$

   where $a > 0$ is some constant, and $0 \leq \theta \leq \pi$ (see Figure 1). A ball of mass m moves along the slide under the force of gravity, which is $-mg\mathbf{j}$. The ball's position is $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

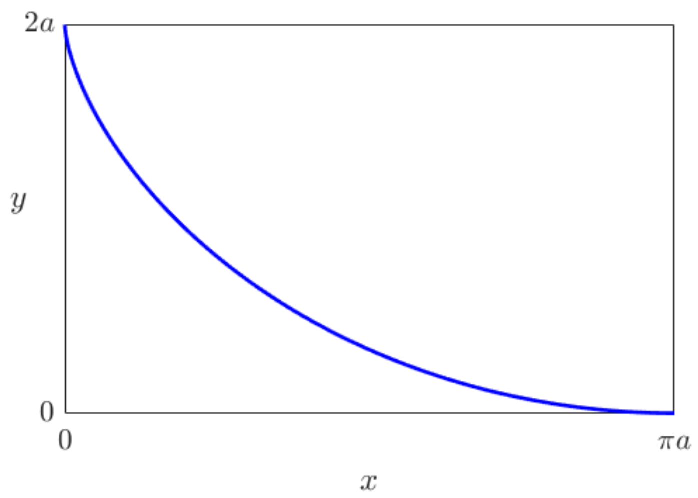

Figure 1: The curve given by Equation (1), with θ ranging from 0 to π.

(a) Find an expression for the ball's velocity, $\dot{\mathbf{r}}$.

(b) Let the ball's gravitational potential energy be zero at $y = 0$. Write down an expression for the ball's gravitational energy at generic position $\mathbf{r}$, expressing your answer in terms of θ.

(c) Suppose the ball is released from rest, from the top of the slide ($\theta = 0$). Show that

$$a\dot{\theta}^2(1 - \cos\theta) + g(1 + \cos\theta) = 2g.$$

Hence, find an expression for $\dot{\theta}$. Show that the time it takes the ball to reach the bottom of the slide ($\theta = \pi$) is

$$T = \pi\sqrt{\frac{a}{g}}.$$

(d) Suppose the ball is released from rest, from some point $\theta = \theta_0$ with $0 < \theta_0 < \pi$. Show that the time it takes the ball to reach the bottom of the slide ($\theta = \pi$) is

$$T' = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1 - \cos\theta}{\cos\theta_0 - \cos\theta}}\, \mathrm{d}\theta.$$

Use the trigonometric identity $\cos\theta = 2\cos^2(\theta/2) - 1$ to show that

$$T' = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sin(\theta/2)}{\sqrt{\cos^2(\theta_0/2) - \cos^2(\theta/2)}}\, \mathrm{d}\theta.$$

Finally, use the substitution

$$s = \frac{\cos(\theta/2)}{\cos(\theta_0/2)},$$

to show that

$$\sin(\theta/2)\mathrm{d}\theta = -2\cos(\theta_0/2)\mathrm{d}s,$$

and hence calculate T'. [*Hint:* $\int \frac{1}{\sqrt{1-s^2}}\, \mathrm{d}s = \arcsin(s) + \text{constant}$.]