Examples sheet 5 – Linear Algebra

The exercises below correspond to material from Lectures 18–20. Selected exercises will be covered in the Examples class scheduled in week 10. Solutions will be available on Canvas.

DIAGONALISATION

1. Let $A \in \mathbb{C}^{n\times n}$ and assume that it has $k < n$ eigenpairs $(\lambda_i, \mathbf{v}_i)$. Show that

$$AV = VD,$$

where

$$V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k], \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{bmatrix}.$$

2. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and define $A := \mathbf{v}\mathbf{u}^T$. Note that this product makes sense as a matrix-matrix product: an $n \times 1$ matrix multiplies a $1 \times n$ matrix to produce a $n \times n$ matrix.

(a) Compute $A = \mathbf{v}\mathbf{u}^T$, where

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

i. Find $\ker A$. State rank A.

ii. Find the eigenvalues and eigenvectors of A and, if diagonalisable, write down its eigenvalue decomposition.

(b) Let now $A = \mathbf{v}\mathbf{u}^T \in \mathbb{R}^{n\times n}$.

i. Check that $\mathbf{v}$ is an eigenvector and find the corresponding eigenvalue.

ii. Let $U = \operatorname{span}\{\mathbf{u}\}$ and let $U^\perp := \left\{\mathbf{w} \in \mathbb{R}^n : \mathbf{w}^T\mathbf{u} = 0\right\}$ Show that $\ker A = U^\perp$. Deduce that $\lambda = 0$ is an eigenvalue of A with $\gamma(\lambda) = n - 1$.

iii. Write down the eigenvalue decomposition of A.

3. Let $U = \{\mathbf{u}_1, \mathbf{u}_2\}$ denote an orthonormal set with respect to the Euclidean inner product on $\mathbb{R}^n$ and define

$$A := \lambda_1 \mathbf{u}_1\mathbf{u}_1^T + \lambda_2 \mathbf{u}_2\mathbf{u}_2^T.$$

(a) Show that $(\lambda_1, \mathbf{u}_1), (\lambda_2, \mathbf{u}_2)$ are eigenpairs of A.

(b) Show that $\ker A = U^\perp := \left\{\mathbf{v} \in \mathbb{R}^n : \mathbf{v}^T\mathbf{u} = 0, \mathbf{u} \in U\right\}$. Deduce that $\lambda = 0$ is an eigenvalue with multiplicity $n - 2$.

(c) Write down the eigenvalue decomposition of A.

4. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 & 4 \\ -3 & 3 & -1 \\ 1 & -4 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Write down the diagonal canonical form of A over $\mathbb{C}$.

(c) Write down the block-diagonal canonical form of A over $\mathbb{R}$.

ADJOINT MAPS

5. Let V, W be inner-product spaces and let $f \in \mathcal{L}(V, W)$. Let $f^* : W \to V$ be the adjoint map defined via

$$\langle f^*(\mathbf{w}), \mathbf{v} \rangle_V = \langle \mathbf{w}, f(\mathbf{v}) \rangle_W .$$

- **(a)** Show that f^* is linear.
- **(b)** Show that f^* is the unique map satisfying the above definition.
- **(c)** Show that $(f^*)^* = f$.

6. Let $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be defined via

$$f(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$$

Let $\mathcal{P}_2(\mathbb{R})$ be equipped with the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x) \mathrm{d}x$$

and assume that $\mathbb{R}^3$ is equipped with the Euclidean inner product.

- **(a)** Find the adjoint map f^* by using the explicit expression given in Proposition 18.5.
- **(b)** Find the matrix representation of f^* with respect to the orthonormal bases used in part **(a)** and check that it is the transpose of the matrix representation of f with respect to the same bases.

7. Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be defined via

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix}.$$

- **(a)** Find f^*, assuming Euclidean inner products for $\mathbb{R}^3, \mathbb{R}^4$.
- **(b)** Verify that $\ker f^* = (\operatorname{im} f)^\perp$ and that $\ker f = (\operatorname{im} f^*)^\perp$.

8. Let $\mathbb{E}^3$ denote the usual Euclidean space with basis $B = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ which is orthonormal with respect to the Euclidean dot product. A generic element of $\mathbb{E}^3$ is denoted by $\vec{a}$; its representation in the basis B is $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, with the vector of coordinates denoted by $\mathbf{a}$, i.e., $\varphi_B(\vec{a}) = \mathbf{a}$, where φ_B is the coordinate map with respect to the basis B.

For any vectors $\vec{a}, \vec{b}$ in $\mathbb{E}^3$, we define the following standard operations:

- the dot product:
$$\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a};$$
- the cross product (or vector product):
$$\vec{a} \times \vec{b} := (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}.$$

Let $f \in \mathcal{L}(\mathbb{E}^3)$ be defined via $f(\vec{v}) = \vec{c} \times \vec{v}$, where $\vec{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ is a non-zero vector.

- **(a)** Check that $f^* = -f$.
- **(b)** Deduce that $\operatorname{im} f \perp \ker f$ and therefore $\mathbb{E}^3 = \operatorname{im} f \oplus \ker f$.

SELF-ADJOINT MAPS

9. Indicate which of the following maps $f \in \mathcal{L}(V)$ is self-adjoint with respect to the inner-product given. In each case, write down the matrix representation with respect to the power basis.

(a) $V = \mathcal{P}_2([-1,1])$, $f(p) = -((1-x^2)p')'$, $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\mathrm{d}x$.

(b) $V = \mathcal{P}_2([-1,1])$, $f(p) = -((1-x^2)p')'$, $\langle p, q\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} p(x)q(x)\mathrm{d}x$.

10. Establish which of the following maps are self-adjoint with respect to the inner products indicated.

(a) $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{bmatrix}, \quad \langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{3} x_i y_i.$$

(b) $f : \mathbb{R}^{1\times 3} \to \mathbb{R}^{1\times 3}$,

$$f([x_1, x_2, x_3]) = [4x_1 + 2x_2 + 4x_3, 2x_1 + 3x_2 + 4x_3, 4x_1 + 4x_2 + 6x_3],$$

with the inner product given by

$$\langle [x_1, x_2, x_3], [y_1, y_2, y_3]\rangle = \sum_{i=1}^{3} x_i y_i.$$

11. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and let U be a subspace of V. By Theorem 6.7, given any $\mathbf{v} \in V$, there exists a unique orthogonal projection $\mathbf{v}_U^{\parallel} \in U$ such that $\mathbf{v} - \mathbf{v}_U^{\parallel} \perp U$. We define the orthogonal projector to be the map $\pi_U(\mathbf{v}) = \mathbf{v}_U^{\parallel}$.

(a) Show that $\pi_U^2 = \pi_U$.

(b) Show that π_U is self-adjoint.

SPECTRAL RESULTS

12. Show that A^T has the same eigenvalues as A.

13. Let $H = H^T \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of H are real and that the eigenvectors of H can be taken to be real.

14. Let $S = -S^T \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of S are purely imaginary if the dimension of S is even. Show further that if the dimension of S is odd, then S must be singular. [A matrix S satisfying the above property is called anti-symmetric.]

15. A vector $\mathbf{y} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ is a left eigenvector for $A \in \mathbb{C}^{n\times n}$ if $\mathbf{y}^* A = \lambda \mathbf{y}^*$. We say $(\lambda, \mathbf{y})$ form a left eigenpair.

(a) Show that if $(\lambda, \mathbf{y})$ is a left eigenpair for A then $(\bar{\lambda}, \mathbf{y})$ is a (right) eigenpair for A^* and $(\lambda, \bar{\mathbf{y}})$ is a (right) eigenpair for A^T. Deduce that if A is real symmetric, then a left eigenpair is also a right eigenpair.

(b) Show that if $(\lambda_1, \mathbf{y}_1)$, $(\lambda_2, \mathbf{y}_2)$ are distinct left and right eigenpairs for A, respectively, then $\langle \mathbf{y}_1, \mathbf{y}_2\rangle = 0$. Deduce that if A is real symmetric, the eigenvectors corresponding to distinct eigenvalues are orthogonal.

16. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

Find the eigenpairs of A and check that the spectral decomposition of A holds.

QUADRATIC FORMS

17. Let $\mathcal{B} : V \times V \to \mathbb{R}$ be a bilinear form and define the quadratic form $\mathcal{Q}(\mathbf{v}) = \mathcal{B}(\mathbf{v}, \mathbf{v})$.

(a) Let $\mathcal{B}$ be symmetric. Show that

$$\mathcal{B}(\mathbf{v}, \mathbf{w}) = \frac{1}{2} \left(\mathcal{Q}(\mathbf{v} + \mathbf{w}) - \mathcal{Q}(\mathbf{v}) - \mathcal{Q}(\mathbf{w}) \right).$$

(b) Find $\mathcal{Q}(\mathbf{v})$ for the case where $\mathcal{B}$ is anti-symmetric.

18. Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Check that for all $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, there holds $\lambda_{\min} \leq R_A(\mathbf{x}) \leq \lambda_{\max}$, where $R_A(\mathbf{x})$ is the Rayleigh quotient of A.

(c) Check further that $R_A(\mathbf{x})$ attains its extreme values at the eigenvectors of A.

19. Consider the following quadratic equation

$$3x^2 - 10xy + 3y^2 + 8 = 0.$$

Write this in matrix form and hence identify the conic it represents.

20. Consider the following quadratic equation

$$5x^2 - 6xy + 5y^2 - 14x + 2y + 5 = 0$$

Write this in matrix form and hence identify the conic it represents.

21. Consider the following quadratic equation

$$10x^2 + 10y^2 + 12xy + z^2 = 16.$$

Write this in matrix form and hence identify the quadric it represents.