University of Birmingham
School of Mathematics

1RA Differentiation Autumn 2022

Problem Sheet 3

Model Solutions

You have approximately 10 working days to complete and submit the $\boxed{\text{SUM}}$ questions (**Q4** and **Q5**) and you may begin working on it immediately.

<table>
<tr><td colspan="2">Assignment available from: 4 November
Submission due: 1700 on Wednesday 16 November 2022</td></tr>
<tr><td>Pre-submission</td><td>Post-submission</td></tr>
<tr><td>• Your Guided Study Support Class in Weeks 6-8.
• Tutor meetings in Weeks 6-8.
• PASS from Week 7
• Library MSC from Week 7
• Office Hours: Wednesday 1300-1430 and Friday 1000-1130.</td><td>• Written feedback on your submission.
• Generic feedback (24 November).
• Model solutions (24 November).
• Tutor meetings in Week 9.
• Office Hours: Wednesday 1300-1430 and Friday 1000-1130</td></tr>
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Instructions:

You will spend the next two weeks (including your Guided Study Support Class in weeks 4 and 5 working on the $\boxed{\text{SUM}}$ questions (**Q4** and **Q5**).

The **deadline** for submission is as follows:

- **By 1700 on Wednesday 16 November 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

Important:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

Q1. Prove the following statements by directly using the definition of derivative.

(i) If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x^2 + 5$ for all $x \in \mathbb{R}$, then $f'(x) = 6x$.

(ii) If $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by $g(x) = 3/(1-x)$ for all $x \in \mathbb{R} \setminus \{1\}$, then $g'(x) = 3/(1-x)^2$.

(iii) If $k : \mathbb{R} \to \mathbb{R}$ is given by $k(x) = \sin(2x)$ for all $x \in \mathbb{R}$, then $k'(0) = 2$.

(iv) If $r : \mathbb{R} \to \mathbb{R}$ is given by

$$r(x) = \begin{cases} x^3 \sin \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

then $r'(0) = 0$.

Solution. (i). Let $x, h \in \mathbb{R}$, and observe that (provided $h \neq 0$) the Newton quotient of f at x is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{(3(x+h)^2 + 5) - (3x^2 + 5)}{h} = 6x + 3h.$$

Hence, by the Algebra of Limits,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h) = 6x.$$

As this limit exists and is finite, by the definition of derivative we conclude that $f'(x) = 6x$.

(ii). Let $x, h \in \mathbb{R}$, and observe that (provided $x \neq 1$, $h \neq 0$, and $x + h \neq 1$) the Newton quotient of g at x is given by

$$\frac{g(x+h) - g(x)}{h} = \frac{3/(1-(x+h)) - 3/(1-x)}{h} = \frac{3}{(1-x)(1-(x+h))}.$$

Hence, by the Algebra of Limits,

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{3}{(1-x)(1-(x+h))} = \frac{3}{(1-x)^2}.$$

As this limit exists and is finite, by the definition of derivative we conclude that $g'(x) = \frac{3}{(1-x)^2}$.

(iii). Let $h \in \mathbb{R} \setminus \{0\}$, and observe that the Newton quotient of k at 0 is given by

$$\frac{k(h) - k(0)}{h} = \frac{\sin(2h)}{h} = 2s(2h),$$

where $s(y) = \frac{\sin(y)}{y}$. Note that $2h$ tends to 0 as $h \to 0$ (by the Algebra of Limits), and $2h \neq 0$ whenever $h \neq 0$. So, by a change of variables and a result from lectures ("notable limits"),

$$\lim_{h \to 0} s(2h) = \lim_{y \to 0} s(y) = 1,$$

and consequently, by the Algebra of Limits,

$$\lim_{h \to 0} \frac{k(h) - k(0)}{h} = \lim_{h \to 0} (2s(2h)) = 2 \cdot 1 = 2.$$

As this limit exists and is finite, by the definition of derivative we conclude that $k'(0) = 2$.

(iv). Let $h \in \mathbb{R} \setminus \{0\}$, and observe that the Newton quotient of r at 0 is given by

$$\frac{r(h) - r(0)}{h} = h^2 \sin \frac{1}{h^2}.$$

As $|\sin y| \leq 1$ for all $y \in \mathbb{R}$, we deduce that

$$0 \leq \left| h^2 \sin \frac{1}{h^2} \right| \leq h^2;$$

moreover clearly $\lim_{h\to 0} 0 = 0$ and $\lim_{h\to 0} h^2 = 0^2 = 0$ (by the Algebra of Limits), hence

$$\lim_{h\to 0} \left| h^2 \sin \frac{1}{h^2} \right| = 0$$

by the Sandwich Theorem, and therefore, by the Absolute Rule for Null Limits,

$$\lim_{h\to 0} \frac{r(h) - r(0)}{h} = \lim_{h\to 0} h^2 \sin \frac{1}{h^2} = 0$$

too. As this limit exists and is finite, by the definition of derivative we conclude that $r'(0) = 0$. $\square$

Q2. Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $\displaystyle\lim_{x\to 0} \frac{17^x - 1}{2x}$.

(ii) $\displaystyle\lim_{x\to 2} \frac{\log(2x-3)}{\tan(x-2)}$.

(iii) $\displaystyle\lim_{x\to 0} \frac{x^2}{e^x - 1 - x}$.

(iv) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{x^4}$.

Solution. (i). Note that

$$\frac{17^x - 1}{2x} = \frac{\exp(x\log 17) - 1}{2x} = \frac{\log 17}{2} \frac{\exp(x \log 17) - 1}{x \log 17}.$$

Note moreover that $x \log 17 \to 0$ if $x \to 0$, and $x \log 17 \neq 0$ when $x \neq 0$; so, by a change of variable,

$$\lim_{x\to 0} \frac{\exp(x\log 17) - 1}{x \log 17} = \lim_{y\to 0} \frac{\exp(y) - 1}{y} = 1,$$

where the latter equality is due to a notable limit discussed in lectures. Finally, by the Algebra of Limits,

$$\lim_{x\to 0} \frac{17^x - 1}{2x} = \frac{\log 17}{2} \lim_{x\to 0} \frac{\exp(x\log 17) - 1}{x\log 17} = \frac{\log 17}{2}.$$

(ii). Note that

$$\frac{\log(2x-3)}{\tan(x-2)} = 2\cos(x-2) \frac{\log(1 + 2(x-2))}{2(x-2)} \frac{x-2}{\sin(x-2)}.$$

Note also that $2(x-2) \to 0$ as $x \to 2$, and $2(x-2) \neq 0$ when $x \neq 2$; so, by a change of variable,

$$\lim_{x\to 2} \frac{\log(1+2(x-2))}{2(x-2)} = \lim_{y\to 0} \frac{\log(1+y)}{y} = 1,$$

where the latter equality is due to a notable limit discussed in lectures. Similarly, by another change of variable and notable limit, we obtain that

$$\lim_{x\to 2} \frac{\sin(x-2)}{x-2} = \lim_{z\to 0} \frac{\sin z}{z} = 1.$$

Since cos is continuous, we also deduce that

$$\lim_{x\to 2} 2\cos(x-2) = 2\cos(0) = 2.$$

Finally, by the Algebra of Limits,

$$\lim_{x\to 2}\frac{\log(2x-3)}{\tan(x-2)}=\left(\lim_{x\to 2}2\cos x\right)\left(\lim_{x\to 2}\frac{\log(1+2(x-2))}{2(x-2)}\right)\left(\lim_{x\to 2}\frac{\sin(x-2)}{x-2}\right)^{-1}=2.$$

(iii). Note that the functions $f, g:\mathbb{R}\to\mathbb{R}$ given by $f(x)=x^2$ and $g(x)=e^x-1-x$ are both differentiable, hence continuous, and

$$\lim_{x\to 0}f(x)=f(0)=0,\qquad \lim_{x\to 0}g(x)=g(0)=0,$$

while

$$f'(x)=2x,\qquad g'(x)=e^x-1,$$

whence, by the Algebra of Limits,

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=2\left(\lim_{x\to 0}\frac{e^x-1}{x}\right)^{-1}=2\cdot 1=2,$$

where in the latter equality a notable limit discussed in lectures was used. Hence, we can apply L'Hôpital Rule and deduce that

$$\lim_{x\to 0}\frac{x^2}{e^x-1-x}=\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=2.$$

(iv). Note that the functions $f, g:\mathbb{R}\to\mathbb{R}$ given by $f(x)=\cos x-1+x^2/2$ and $g(x)=x^4$ are both differentiable, hence continuous, and

$$\lim_{x\to 0}f(x)=f(0)=0,\qquad \lim_{x\to 0}g(x)=g(0)=0,$$

while

$$f'(x)=-\sin x+x,\qquad g'(x)=4x^3.$$

In particular, f' and g' are differentiable too, hence continuous, and

$$\lim_{x\to 0}f'(x)=f'(0)=0,\qquad \lim_{x\to 0}g'(x)=g'(0)=0.$$

Moreover

$$f''(x)=1-\cos x,\qquad g''(x)=12x^2,$$

and therefore, by the Algebra of Limits,

$$\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\frac{1}{12}\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{12}\cdot\frac{1}{2}=\frac{1}{24},$$

where in the latter equality a notable limit discussed in lectures was used. Hence, we can apply L'Hôpital Rule to the functions f' and g', and deduce that

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\frac{1}{24}.$$

Consequently, we can apply L'Hôpital Rule to the functions f and g, and deduce that

$$\lim_{x\to 0}\frac{\cos x-1+x^2/2}{x^4}=\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\frac{1}{24}.$$

□

Q3. Differentiate the following functions (i.e., find a formula for the derivative of each of the functions). You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $f(x)=\dfrac{\sin(x^2)\sin^2 x}{2+\sin x}$.

(ii) $g(x)=e^{\sin x}+\cos(x+\sin x)$.

(iii) $k(x)=(5+\sin x)^{\log x}$.

Solution. (i). Recall that the derivative of sin is cos, and the derivative of $x \mapsto x^2$ is $x \mapsto 2x$. By the Chain Rule,

$$\frac{d}{dx}(\sin(x^2)) = \cos(x^2)\frac{d}{dx}(x^2) = 2x\cos(x^2),$$
$$\frac{d}{dx}(\sin^2 x) = 2\sin x\cos x,$$

whence (by the Leibniz Rule and the Sum Rule),

$$\frac{d}{dx}(\sin(x^2)\sin^2 x) = 2x\cos(x^2)\sin^2 x + 2\sin x\cos x\sin(x^2),$$
$$\frac{d}{dx}(2+\sin x) = \cos x,$$

and finally, by the Quotient Rule,

$$\begin{aligned} f'(x) &= \frac{(2x\cos(x^2)\sin^2 x + 2\sin x\cos x\sin(x^2))(2+\sin x) - (\sin(x^2)\sin^2 x)\cos x}{(2+\sin x)^2} \\ &= \frac{4x\cos(x^2)\sin^2 x + 4\sin x\cos x\sin(x^2) + 2x\cos(x^2)\sin^3 x + \sin^2 x\cos x\sin(x^2)}{(2+\sin x)^2}. \end{aligned}$$

(ii). Recall that

$$\frac{d}{dx}x = 1, \qquad \frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x.$$

Hence, by the Sum Rule,

$$\frac{d}{dx}(x+\sin x) = 1+\cos x,$$

and therefore, by the Chain Rule,

$$\begin{aligned} \frac{d}{dx}(e^{\sin x}) &= e^{\sin x}\frac{d}{dx}\sin x = e^{\sin x}\cos x, \\ \frac{d}{dx}(\cos(x+\sin x)) &= -\sin(x+\sin x)\frac{d}{dx}(x+\sin x) \\ &= -(1+\cos x)\sin(x+\sin x), \end{aligned}$$

so

$$g'(x) = e^{\sin x}\cos x - (1+\cos x)\sin(x+\sin x).$$

(iii). Observe that

$$k(x) = \exp((\log x)\log(5+\sin x)),$$

and recall that

$$\frac{d}{dx}\exp(x) = \exp(x), \qquad \frac{d}{dx}\sin x = \sin x, \qquad \frac{d}{dx}\log x = \frac{1}{x}.$$

Hence, by the Chain Rule,

$$\frac{d}{dx}(\log(5+\sin x)) = \frac{1}{5+\sin x}\frac{d}{dx}(5+\sin x) = \frac{\cos x}{5+\sin x},$$

so, by the Leibniz Rule,

$$\frac{d}{dx}((\log x)\log(5+\sin x)) = \frac{\log(5+\sin x)}{x} + \frac{\log x\cos x}{5+\sin x},$$

and finally, again by the Chain Rule,

$$\begin{aligned} k'(x) &= \exp((\log x)\log(5+\sin x))\left(\frac{\log(5+\sin x)}{x} + \frac{\log x\cos x}{5+\sin x}\right) \\ &= (5+\sin x)^{\log x}\left(\frac{\log(5+\sin x)}{x} + \frac{\log x\cos x}{5+\sin x}\right). \end{aligned}$$

□

(SUM) **Q4**. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x^4 - 26x^3 + 60x^2 - 11$.

(i) Find the stationary points of f, and determine whether they are local maximum or minimum points of f.
(ii) Determine the intervals in $\mathbb{R}$ where f is increasing and those where f is decreasing.
(iii) Determine the intervals in $\mathbb{R}$ where f is convex and those where f is concave.
(iv) Find the limits $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.
(v) Find $\inf f$ and $\sup f$, and determine whether f has a (global) maximum and/or a minimum.

Justify your answers.

Solution. (i). Note that f is differentiable and

$$f'(x) = 12x^3 - 78x^2 + 120x = 6x\,(x-4)\,(2x-5)$$

for all $x \in \mathbb{R}$, where we used the fact that $\frac{d}{dy}x^n = nx^{n-1}$ for $n \in \mathbb{N}$. Therefore, $0, 5/2, 4$ are stationary points of f.

We now note that,

$$f''(x) = 36x^2 - 156x + 120 = 12(x-1)(3x-10).$$

In particular, we have $f''(0) > 0$, $f''(5/2) < 0$ and $f''(4) > 0$, which means that 0 and 4 are local minimum points and 5/2 is a local maximum point by using the second derivative test.

(ii). By the sign analysis, we note that $f'(x) \geq 0$ when $x \in [0, \frac{5}{2}] \cup [4, \infty)$, and $f'(x) \leq 0$ for $x \in (-\infty, 0] \cup [\frac{5}{2}, 4]$. From this we have that f is increasing on the intervals $[0, \frac{5}{2}]$ and $[4, \infty)$, while f is decreasing on the intervals $(-\infty, 0]$ and $[\frac{5}{2}, 4]$.

(iii). From the above formula, we see that $f''(x) \geq 0$ for $x \in (-\infty, 1] \cup [\frac{10}{3}, \infty)$, and $f''(x) \leq 0$ if $x \in [1, \frac{1}{3}]$. From this we deduce that f is convex on $(-\infty, 1]$ and $[\frac{10}{3}, \infty)$, while f is concave on $[1, \frac{1}{3}]$.

(iv). We can write, for all $x \in \mathbb{R} \setminus \{0\}$,

$$f(x) = x^4(3 - 26/x + 60/x^2 - 11/x^4).$$

We know that $\lim_{x\to\pm\infty} x = \pm\infty$, and therefore $\lim_{x\to\pm\infty} 1/x = 0$ by the Algebra of Limits. Again by the Algebra of Limits we deduce that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^4(3-26/x+60/x^2-11/x^4) = \infty^4 \cdot (3-26\cdot 0+60\cdot 0-11\cdot 0) = \infty$$

and similarly

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^4(3-26/x+60/x^2-11/x^4) = (-\infty)^4 \cdot (3-26\cdot 0+60\cdot 0-11\cdot 0) = \infty.$$

(v). From the discussion of monotonicity in part (ii), we deduce that $x = 0$ and $x = 4$ are local minimum points of f, actually one of them should be a global minimum point of f. We have $f(0) = -11$ and $f(4) = 53$, which implies that 0 is the global minimum point of f and thus $\inf f = \min f = -11$.

On the other hand, from part (iv) we know that $\lim_{x\to\infty} f(x) = \infty$; by the definition of limit, this implies that f is unbounded above, that is, $\sup f = \infty$, and f has no maximum. □

(SUM) **Q5**. (a) Find, with proof, the supremum of the set

$$A := \{f(x) : x \in [1,2]\} \cup \{x : x \in [1,2]\},$$

where $f : [0,4] \to \mathbb{R}$ is given by $f(x) := x^2 + 2$ for all $x \in [0,4]$.

(b) Find, with proof, the infimum of the set

$$B := \left\{ \frac{4n^3 + 3n + 1}{n^3} : n \in \mathbb{N} \right\}.$$

(c) Suppose that $P = \{-10, -2, 0, 1, 5\}$ and $g : [-10, 5] \to \mathbb{R}$ is given by

$$g(x) := \begin{cases} \dfrac{x + 2|x|}{|x|}, & \text{if } x \neq 0; \\ 4, & \text{if } x = 0. \end{cases}$$

(i) Calculate the Riemann–Darboux sums $L(g, P)$ and $U(g, P)$.
(ii) Find a partition Q of $[-10, 5]$ such that $U(g, Q) - L(g, Q) < 0.001$.

Solution. (a) Observe that $A = [1, 6] \cup [1, 2] = [1, 6]$, since

$$\{f(x) : x \in [1, 2]\} = \{x^2 + 2 : x \in [1, 2]\} = [1, 6] \qquad \text{and} \qquad \{x : x \in [1, 2]\} = [1, 2].$$

We claim that $\sup A = 6$. To prove this, first note that $x \leq 6$ for all $x \in A$, so 6 is an upper bound for A. Next, suppose that $\epsilon > 0$ and observe that $6 - \epsilon \leq 6 \in A$, so $6 - \epsilon$ is not an upper bound for A. Therefore, the least upper bound of A is 6 ($\sup A = 6$).

(b) The elements of the set $B = \left\{ \frac{4n^3 + 3n + 1}{n^3} : n \in \mathbb{N} \right\}$ satisfy

$$\frac{4n^3 + 3n + 1}{n^3} = 4 + \frac{3}{n^2} + \frac{1}{n^3} \geq 4$$

for all $n \in \mathbb{N}$, so 4 is a lower bound for B. Next, observe that the elements of B form a decreasing sequence with

$$\lim_{n \to \infty} \frac{4n^3 + 3n + 1}{n^3} = \lim_{n \to \infty} \left(4 + \frac{3}{n^2} + \frac{1}{n^3} \right) = 4.$$

If $\epsilon > 0$, then this limit implies that there exists $N \in \mathbb{N}$ such that $4 + \epsilon > 4 + \frac{3}{N^2} + \frac{1}{N^3} \in B$, so $4 + \epsilon$ is not a lower bound for B. Therefore, the greatest lower bound of B is 4 ($\inf B = 4$).

(c)(i) Observe that

$$g(x) := \begin{cases} 1, & \text{if } x \in [-10, 0); \\ 4, & \text{if } x = 0; \\ 3, & \text{if } x \in (0, 5]. \end{cases}$$

Using the notation $P = \{-10, -2, 0, 1, 5\} =: \{x_0, x_1, x_2, x_3, x_4\}$, we calculate

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{4} m_i (x_i - x_{i-1}) \\ &= 1(-2 - -10) + 1(0 - -2) + 3(1 - 0) + 3(5 - 1) \\ &= 8 + 2 + 3 + 12 = 25 \end{aligned}$$

and

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{4} m_i (x_i - x_{i-1}) \\ &= 1(-2 - -10) + 4(0 - -2) + 4(1 - 0) + 3(5 - 1) \\ &= 8 + 8 + 4 + 12 = 32. \end{aligned}$$

Alternatively, these calculations can be done using the Riemann–Darboux Sums Calculator on Canvas, but a screenshot as in Figure 1 below must be provided.

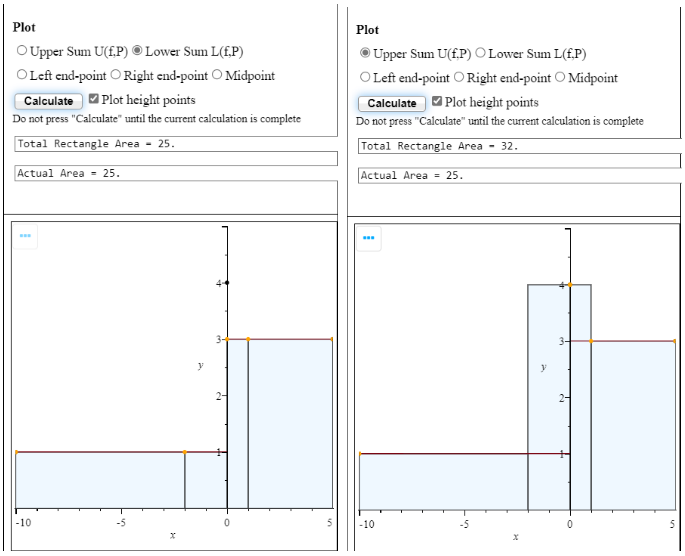

FIGURE 1. Two graphs of $g(x) := \begin{cases} 1, & \text{if } x \in [-10, 0); \\ 4, & \text{if } x = 0; \\ 3, & \text{if } x \in (0, 5]. \end{cases}$ showing the areas covered by $L(g, Q)$ and $U(g, Q)$ for $Q = \{-10, -2, 0, 1, 5\}$.

(c)(ii) The key point is that partition points near the origin reduce the difference $(U - L)$. Suppose that $\delta \in (0, 1)$ and consider $P_\delta := \{-10, -\delta, \delta, 5\} =: \{x_0, x_1, x_2, x_3\}$. Observe that

$$\begin{aligned} U(g, P_\delta) - L(g, P_\delta) &= \sum_{i=1}^{3} (M_i - m_i)(x_i - x_{i-1}) \\ &= (1-1)(-\delta - -10) + (4-1)(2\delta) + (3-3)(5-\delta) \\ &= 6\delta, \end{aligned}$$

so we need to find δ such that $6\delta < 0.001$, which is the same as requiring that $\delta < \frac{1}{6000}$. We now choose $\delta_0 := \frac{1}{7000}$ and define $Q := P_{\delta_0} = \{-10, -\frac{1}{7000}, \frac{1}{7000}, 5\}$. The preceding computation shows that $U(g, Q) - L(g, Q) = 6\delta_0 = \frac{6}{7000} < 0.001$, as required.

$\square$

Q6. (a) Calculate the Riemann–Darboux sums $L(f, P)$ and $U(f, P)$ for each of the following functions f and partitions P:

(i) $f : [0, 10] \to \mathbb{R},\ f(x) = x^2,\ P = \{0, 2, 7, 10\}$

(ii) $f : [0, 10] \to \mathbb{R},\ f(x) = e^{-x},\ P = \{0, 1, 5, 8, 9, 10\}$

(iii) $f : [0, \pi] \to \mathbb{R},\ f(x) = \sin(x),\ P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$

(b) Use the Riemann–Darboux sums calculator (Week 7 Materials on Canvas) to visualize and check your answers above (note that $\frac{3\pi}{4}$ is entered as 3*Pi/4).

(c) For each of the functions f above, use the Riemann–Darboux sums calculator to calculate $L(f, P_n)$ and $U(f, P_n)$ when $n = 5$, $n = 10$ and $n = 100$, where P_n is the partition of the domain of f into n subintervals of equal width.

(d) For each $\delta \in (0, 1/10)$, find an expression for the sums $L(f, P_\delta)$ and $U(f, P_\delta)$ when $f : [-2, 2] \to \mathbb{R}$, $f(x) = \begin{cases} 5, & x \in \mathbb{N} \\ 3, & x \notin \mathbb{N} \end{cases}$ and $P_\delta = \{-2, 1-\delta, 1+\delta, 2-\delta, 2\}$.

Solution. (a)(i) For $f : [0, 10] \to \mathbb{R},\ f(x) = x^2,\ P = \{0, 2, 7, 10\}$, we calculate

$$\begin{aligned} L(f, P) &= \textstyle\sum_{i=1}^{3} m_i(x_i - x_{i-1}) \\ &= 0(2-0) + 4(7-2) + 49(10-7) \\ &= 167 \end{aligned}$$

and

$$\begin{aligned} U(f, P) &= \textstyle\sum_{i=1}^{3} M_i(x_i - x_{i-1}) \\ &= 4(2-0) + 49(7-2) + 100(10-7) \\ &= 553. \end{aligned}$$

(a)(ii) For $f : [0, 10] \to \mathbb{R},\ f(x) = e^{-x},\ P = \{0, 1, 5, 8, 9, 10\}$, we calculate

$$\begin{aligned} L(f, P) &= \textstyle\sum_{i=1}^{5} m_i(x_i - x_{i-1}) \\ &= e^{-1}(1-0) + e^{-5}(5-1) + e^{-8}(8-5) \\ &\quad + e^{-9}(9-8) + e^{-10}(10-9) \\ &= e^{-1} + 4e^{-5} + 3e^{-8} + e^{-9} + e^{-10} \end{aligned}$$

and

$$\begin{aligned} U(f, P) &= \textstyle\sum_{i=1}^{5} M_i(x_i - x_{i-1}) \\ &= e^{-0}(1-0) + e^{-1}(5-1) + e^{-5}(8-5) \\ &\quad + e^{-8}(9-8) + e^{-9}(10-9) \\ &= 1 + 4e^{-1} + 3e^{-5} + e^{-8} + e^{-9}. \end{aligned}$$

(a)(iii) For $f : [0, \pi] \to \mathbb{R},\ f(x) = \sin(x),\ P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$, we calculate

$$\begin{aligned} L(f, P) &= \textstyle\sum_{i=1}^{3} m_i(x_i - x_{i-1}) \\ &= \sin(0)(\tfrac{\pi}{2} - 0) + \sin(\tfrac{3\pi}{4})(\tfrac{3\pi}{4} - \tfrac{\pi}{2}) + \sin(\pi)(\pi - \tfrac{3\pi}{4}) \\ &= \tfrac{\sqrt{2}}{2}\tfrac{\pi}{4} \end{aligned}$$

and

$$\begin{aligned} U(f, P) &= \textstyle\sum_{i=1}^{3} M_i(x_i - x_{i-1}) \\ &= \sin(\tfrac{\pi}{2})(\tfrac{\pi}{2} - 0) + \sin(\tfrac{\pi}{2})(\tfrac{3\pi}{4} - \tfrac{\pi}{2}) + \sin(\tfrac{3\pi}{4})(\pi - \tfrac{3\pi}{4}) \\ &= \tfrac{\pi}{2} + \tfrac{\pi}{4} + \tfrac{\sqrt{2}}{2}\tfrac{\pi}{4} \\ &= (3 + \tfrac{\sqrt{2}}{2})\tfrac{\pi}{4}. \end{aligned}$$

(b)(i) For $f : [0, 10] \to \mathbb{R},\ f(x) = x^2,\ P = \{0, 2, 7, 10\}$, the Riemann–Darboux Sums Calculator depicts the areas covered by $L(f, P)$ and $U(f, P)$ in Figure 2 below.

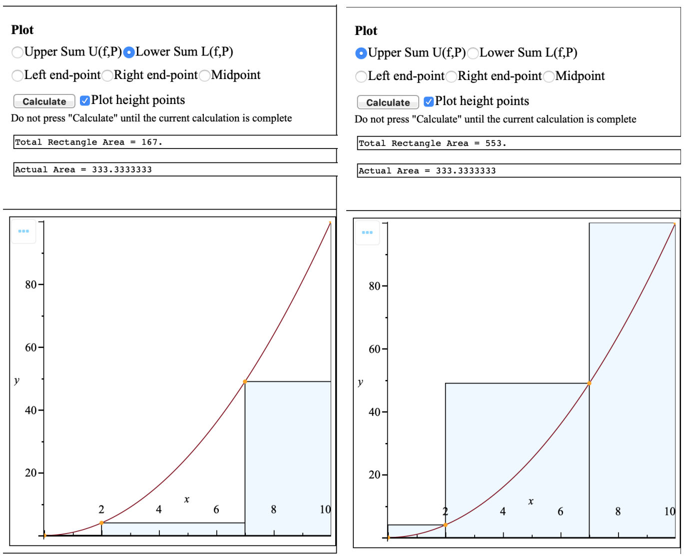

FIGURE 2. Two graphs of $f : [0, 10] \to \mathbb{R}$, $f(x) = x^2$, showing the areas covered by $L(f, P)$ and $U(f, P)$ for $P = \{0, 2, 7, 10\}$.

(b)(ii) For $f : [0, 10] \to \mathbb{R}$, $f(x) = e^{-x}$, $P = \{0, 1, 5, 8, 9, 10\}$, the Riemann–Darboux Sums Calculator depicts the areas covered by $L(f, P)$ and $U(f, P)$ in Figure 3 below.

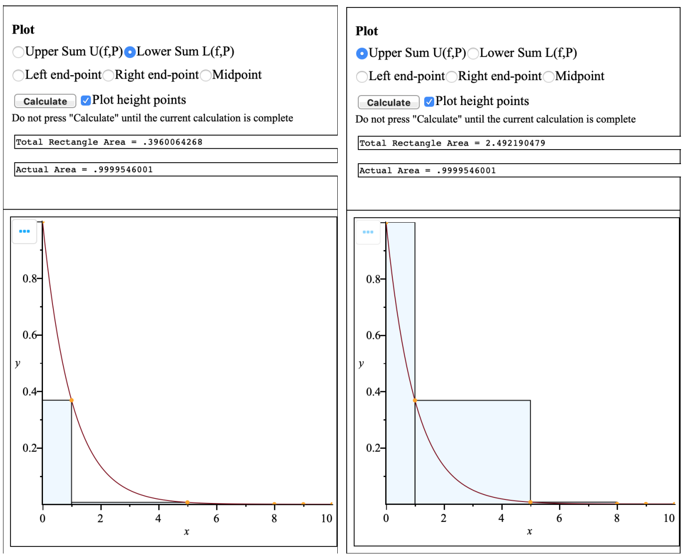

FIGURE 3. Two graphs of $f : [0, 10] \to \mathbb{R}$, $f(x) = e^{-x}$, showing the areas covered by $L(f, P)$ and $U(f, P)$ for $P = \{0, 1, 5, 8, 9, 10\}$.

(b)(iii) For $f : [0, \pi] \to \mathbb{R}$, $f(x) = \sin(x)$, $P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$, the Riemann–Darboux Sums Calculator depicts the areas covered by $L(f, P)$ and $U(f, P)$ in Figure 4 below.

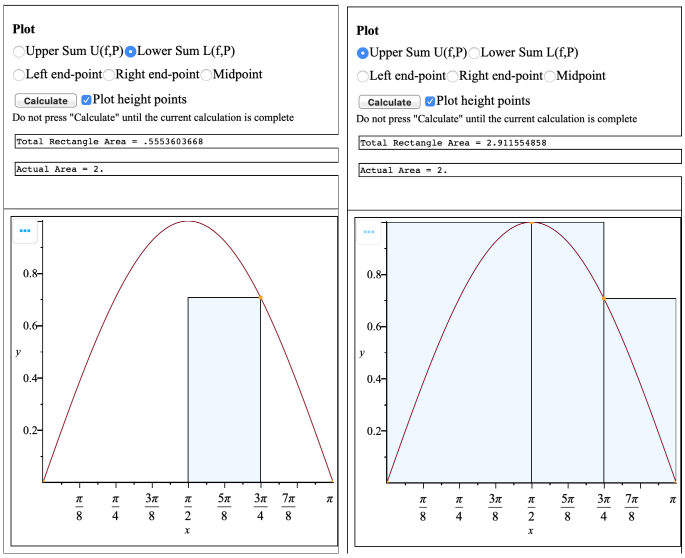

FIGURE 4. Two graphs of $f : [0, \pi] \to \mathbb{R},\ f(x) = \sin(x)$, showing the areas covered by $L(f, P)$ and $U(f, P)$ for $P = \{0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}$.

(c) Table 1 shows the values (correct to three decimal places) for $L(f, P_n)$ and $U(f, P_n)$ when $n = 5$, $n = 10$ and $n = 100$ for each function f from part (a). These values were found using the Riemann–Darboux Sums Calculator:

	$L(f,P_5)$	$L(f,P_{10})$	$L(f,P_{100})$	Area	$U(f,P_{100})$	$U(f,P_{10})$	$U(f,P_5)$
(i)	240	285	328.35	333.333	338.35	385	440
(ii)	0.313	0.582	0.951	1.000	1.051	1.582	2.313
(iii)	1.336	1.669	1.968	2	2.030	2.298	2.193

TABLE 1. The values of $L(f, P_n)$ and $U(f, P_n)$ when $n = 5$, $n = 10$ and $n = 100$ for each function f from part (a).

(d) For $f : [-2, 2] \to \mathbb{R},\ f(x) = \begin{cases} 5, & x \in \mathbb{N} \\ 3, & x \notin \mathbb{N} \end{cases}$ and $P_\delta = \{-2, 1-\delta, 1+\delta, 2-\delta, 2\}$, we calculate

$$\begin{aligned} L(f, P_\delta) &= \textstyle\sum_{i=1}^{4} m_i(x_i - x_{i-1}) \\ &= 3(3-\delta) + 3(2\delta) + 3(1-2\delta) + 3(\delta) \\ &= 12 \end{aligned}$$

and

$$\begin{aligned} U(f, P_\delta) &= \textstyle\sum_{i=1}^{4} M_i(x_i - x_{i-1}) \\ &= 3(3-\delta) + 5(2\delta) + 3(1-2\delta) + 5(\delta) \\ &= 12 + 6\delta \end{aligned}$$

for each $\delta \in (0, 1/10)$. $\square$

Q7. It is proved in Lecture 7.3 of the Integration Lecture Notes that if $f : [a, b] \to [0, \infty)$ is a bounded function, where $-\infty < a < b < \infty$, and P, Q are partitions of $[a, b]$ such that $P \subseteq Q$, then $L(f, P) \leq L(f, Q)$. List the changes that are needed to prove that $U(f, P) \geq U(f, Q)$, and in particular, explain how the inequalities $m_1 \leq m_1'$ and $m_1 \leq m_1''$ need to be modified.

Solution. We follow the same procedure as in the Lecture Notes, except instead of considering the quantities m_1, m_1', and m_1'', we introduce

$$\begin{aligned} M_1 &:= \sup\{f(x) : x \in [x_0, x_1]\}, \\ M_1' &:= \sup\{f(x) : x \in [x_0, y]\}, \\ M_1'' &:= \sup\{f(x) : x \in [y, x_1]\}. \end{aligned}$$

The inequalities $m_1 \leq m_1'$ and $m_1 \leq m_1''$ are then reversed in that $M_1 \geq M_1'$ and $M_1 \geq M_1''$ to obtain

$$M_1(x_1 - x_0) \geq M_1'(y - x_0) + M_1''(x_1 - y)$$

and thus ultimately $U(f, P) \geq U(f, Q)$. □

Q8. Let $f : [0, b] \to [0, \infty)$ be defined by $f(x) = x^2$, where $b \in (0, \infty)$:

(a) For each $n \in \mathbb{N}$, let $P_n = \{x_i : i = 0, 1, \dots, n\}$ denote the partition of $[0, b]$ into n subintervals of equal width. Express x_i in terms of i and b.

(b) Use the formula $\sum_{j=1}^{k} j^2 = \frac{1}{6}k(k + 1)(2k + 1)$ to prove that

$$L(f, P_n) = \frac{b^3}{6n^3}(n - 1)n(2n - 1) \quad \text{and} \quad U(f, P_n) = \frac{b^3}{6n^3}n(n + 1)(2n + 1).$$

(c) Find, with proof, the values $\sup\{L(f, P_n) : n \in \mathbb{N}\}$ and $\inf\{U(f, P_n) : n \in \mathbb{N}\}$.

(d) Find, with proof, the lower integral $\underline{\int_0^b} f$ and the upper integral $\overline{\int_0^b} f$.

(e) Prove that f is bounded and integrable, then find $\int_0^b f$ (without using calculus).

Solution. (a) For each $n \in \mathbb{N}$, we have

$$P_n = \{0, \tfrac{b}{n}, \tfrac{2b}{n}, \dots, \tfrac{(n-1)b}{n}, 1\} = \{\tfrac{ib}{n} : i = 0, 1, \dots, n\},$$

so $x_i = \frac{ib}{n}$ for each $i \in \{0, 1, \dots, n\}$.

(b) For each $i \in \{0, 1, \dots, n\}$, we have

$$\begin{aligned} m_i &= \inf\{x^2 : x \in [\tfrac{(i-1)b}{n}, \tfrac{ib}{n}]\} = \left(\tfrac{(i-1)b}{n}\right)^2, \\ M_i &= \sup\{x^2 : x \in [\tfrac{(i-1)b}{n}, \tfrac{ib}{n}]\} = \left(\tfrac{ib}{n}\right)^2. \end{aligned}$$

We use these to obtain

$$\begin{aligned} L(f, P_n) &= \textstyle\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= \textstyle\sum_{i=1}^{n} \left(\tfrac{(i-1)b}{n}\right)^2 \tfrac{b}{n} \\ &= \tfrac{b^3}{n^3} \textstyle\sum_{j=0}^{n-1} j^2 \\ &= \tfrac{b^3}{n^3} \textstyle\sum_{j=1}^{n-1} j^2 \\ &= \tfrac{b^3}{6n^3}(n - 1)n(2n - 1), \end{aligned}$$

where we used the given sum with $k = n - 1$ in the penultimate equality, and

$$\begin{aligned} U(f, P_n) &= \textstyle\sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &= \textstyle\sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \frac{b}{n} \\ &= \frac{b^3}{n^3} \textstyle\sum_{i=1}^{n} i^2 \\ &= \frac{b^3}{6n^3}(n+1)n(2n+1), \end{aligned}$$

where we used the given sum with $k = n$ in the penultimate equality.

(c) We can now calculate

$$\sup\{L(f, P_n) : n \in \mathbb{N}\} = \sup\{\tfrac{b^3}{6}(2 - \tfrac{3}{n} + \tfrac{1}{n^2}) : n \in \mathbb{N}\} = \tfrac{b^3}{3}$$

and

$$\inf\{U(f, P_n) : n \in \mathbb{N}\} = \inf\{\tfrac{b^3}{6}(2 + \tfrac{3}{n} + \tfrac{1}{n^2}) : n \in \mathbb{N}\} = \tfrac{b^3}{3}.$$

To prove these facts, it suffices to observe that $a_n := \frac{b^3}{6}(2 - \frac{3}{n} + \frac{1}{n^2})$ is an increasing sequence with $\lim_{n\to\infty} a_n = \frac{b^3}{3}$, whilst $b_n := \frac{b^3}{6}(2 + \frac{3}{n} + \frac{1}{n^2})$ is a decreasing sequence with limit $\lim_{n\to\infty} b_n = \frac{b^3}{3}$.

(d) Now, as

$$\{L(f, P_n) : n \in \mathbb{N}\} \subseteq \{L(f, P) : P \text{ is a partition of } [0, b]\},$$

we have

$$\underline{\int_0^b} f := \sup_P L(f, P) \geq \sup_{n\in\mathbb{N}} L(f, P_n) = \tfrac{b^3}{3}.$$

Moreover, as

$$\{U(f, P_n) : n \in \mathbb{N}\} \subseteq \{U(f, P) : P \text{ is a partition of } [0, b]\},$$

we have

$$\overline{\int_0^b} f := \inf_P U(f, P) \leq \inf_{n\in\mathbb{N}} U(f, P_n) = \tfrac{b^3}{3}.$$

We also know from Proposition 7.4.2 in the Lecture Notes that the lower integrals is always less than or equal to the upper integral, hence

$$\tfrac{b^3}{3} \leq \underline{\int_0^b} f \leq \overline{\int_0^b} f \leq \tfrac{b^3}{3},$$

which implies that $\frac{b^3}{3} = \underline{\int_0^b} f = \overline{\int_0^b} f = \frac{b^3}{3}$.

(e) The preceding equality proves that f is integrable with $\int_0^b f = \frac{b^3}{6}$. $\square$

Extra Questions

EQ1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x|x|$ for all $x \in \mathbb{R}$.

(i) Prove that the function f is differentiable, and find a formula for its derivative $f' : \mathbb{R} \to \mathbb{R}$.

(ii) Is f twice differentiable? Justify your answer.

Solution. (i). Note that

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x \leq 0. \end{cases}$$

If we define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2$, then g is differentiable (it is a polynomial) and $g'(x) = 2x$ for all $x \in \mathbb{R}$. Moreover, $f(x) = g(x)$ for all $x \geq 0$. In particular, if $a > 0$, then, for all $h \in \mathbb{R}$ such that $|h| < a$,

$$\frac{f(a+h) - f(a)}{h} = \frac{g(a+h) - g(a)}{h},$$

and consequently, by taking the limit as $h \to 0$, we deduce that f is differentiable at a and

$$f'(a) = g'(a) = 2a;$$

a similar argument applies to the case where $a = 0$, provided we restrict to $h > 0$, so we also obtain that

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = g'(0) = 0. \tag{1}$$

In a similar way, by using the fact that $f(x) = -x^2$ for all $x \leq 0$, we deduce that f is differentiable at all $a < 0$, and

$$f'(a) = -2a,$$

and that moreover

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = 0. \tag{2}$$

Since the two one-sided limits in (1) and (2) exist and are equal (and finite), we conclude that f is differentiable at 0, and $f'(0) = 0$. In conclusion, f is differentiable (i.e., f is differentiable at every $a \in \mathbb{R}$), and

$$f'(x) = 2|x|$$

for all $x \in \mathbb{R}$.

(ii). No, f is not twice differentiable, because f' is not differentiable at 0. Indeed, for all $h \neq 0$,

$$\frac{f'(h) - f'(0)}{h} = \frac{2|h|}{h},$$

and the latter expression has no limit as $h \to 0$ (the one-sided limits are ± 2). □

EQ2. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) If $f : (0,1) \to \mathbb{R}$ is continuous, then f is bounded.
(ii) If $g : (0,1) \to \mathbb{R}$ is continuous, then g is differentiable.
(iii) If $k : [0,1] \to \mathbb{R}$ is differentiable, then k is bounded.

Solution. (i). The statement is false. For example, if $f : (0,1) \to \mathbb{R}$ is defined by $f(x) = 1/x$ for all $x \in (0,1)$, then f is continuous (by the Algebra of Continuous Functions), but f is unbounded (indeed $f((0,1)) = (1,\infty)$, and therefore $\sup f = \infty$).

(ii). The statement is false. For example, if $g : (0,1) \to \mathbb{R}$ is defined by $g(x) = |x - 1/2|$, then g is continuous (since $x \mapsto |x|$ is continuous and composition of continuous functions is continuous); however g is not differentiable at $1/2$, because, for all $h \in \mathbb{R} \setminus \{0\}$,

$$\frac{f(1/2+h) - f(1/2)}{h} = \frac{|h|}{h}$$

and the latter expression has no limit as $h \to 0$ (the one-sided limits are ± 1).

(iii). The statement is true. Indeed, by a result from lectures, if $k : [0,1] \to \mathbb{R}$ is differentiable, then k is continuous; moreover, by the Boundedness Theorem, if $k : [0,1] \to \mathbb{R}$ is continuous, then it is bounded. □

EQ3. Prove that the function $\arcsin : [-1,1] \to \mathbb{R}$ is not differentiable at any of the points 1 and -1. [Hint: apply the Chain Rule to the identity $\sin \arcsin x = x$.]

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : [-1,1] \to \mathbb{R}$ be defined by $f(x) = \sin x$ and $g(y) = \arcsin y$ for all $x \in \mathbb{R}$ and $y \in [-1,1]$. From lectures we know that f and g are continuous functions, and f is differentiable, with $f'(x) = \cos x$ for all $x \in \mathbb{R}$. Moreover, since g is the inverse of $f|_{[-\pi/2,\pi/2]}$, we have

$$f(g(y)) = y$$

for all $y \in [-1,1]$. For a contradiction, assume that g is differentiable at 1. By differentiating both sides of the above identity and evaluating at 1, by the Chain Rule (and the fact that $\frac{\partial}{\partial y} y = 1$) we then have that

$$f'(g(1))g'(1) = 1$$

On the other hand, $f'(g(1)) = \cos \arcsin 1 = \cos(\pi/2) = 0$, so we deduce

$$0 = 0 \cdot g'(1) = 1,$$

which is a contradiction. Hence the assumption that g is differentiable at 1 cannot be true, and we conclude that g is not differentiable at 1. The same argument can be repeated with -1 in place of 1 (note that $\cos \arcsin(-1) = \cos(-\pi/2) = 0$ too), thus proving that g is not differentiable at -1 too. $\square$

* **EQ4**. Let $A \subseteq \mathbb{R}$. Recall that, by the Leibniz rule, if $f, g : A \to \mathbb{R}$ are differentiable, then their product fg is differentiable too, and the formula

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

holds for all $x \in \mathbb{R}$.

(i) Let $f_1, f_2, f_3 : A \to \mathbb{R}$ be differentiable functions. Prove that their product $f_1 f_2 f_3$ is differentiable too, and find a formula for its derivative $(f_1 f_2 f_3)'$. [Hint: $f_1 f_2 f_3 = (f_1 f_2) f_3$.]

(ii) Let $n \in \mathbb{N}$, and let $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$ be differentiable functions. Prove that their product $f_1 f_2 \cdots f_n$ is differentiable. [Hint: induction on n.]

(iii) Can you write a formula for the derivative $(f_1 f_2 \cdots f_n)'$?

Solution. (i). Let $h = f_1 f_2$ be the product of f_1 and f_2. Since both f_1 and f_2 are differentiable, by the Leibniz rule their product h is differentiable too, and

$$h' = f_1' f_2 + f_1 f_2'.$$

Note now that $f_1 f_2 f_3 = (f_1 f_2) f_3 = h f_3$ is the product of f and f_3. Since both h and f_3 are differentiable, again by the Leibniz rule we conclude that their product $h f_3$ is differentiable, and

$$\begin{aligned}(f_1 f_2 f_3)' &= (h f_3)' = h' f_3 + h f_3' \\ &= (f_1' f_2 + f_1 f_2') f_3 + (f_1 f_2) f_3' \\ &= f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3'.\end{aligned}$$

In other words, for all $x \in A$,

$$(f_1 f_2 f_3)'(x) = f_1'(x) f_2(x) f_3(x) + f_1(x) f_2'(x) f_3(x) + f_1(x) f_2(x) f_3'(x).$$

(ii). Let $P(n)$ be the statement "for any differentiable functions $f_1, \ldots, f_n : A \to \mathbb{R}$, their product $f_1 \cdots f_n$ is differentiable". We prove that $P(n)$ is true for all $n \in \mathbb{N}$, by induction on n.

Base case: $P(1)$. In the case $n = 1$, the product $f_1 \cdots f_n$ reduces to the single function f_1, and the statement $P(1)$ becomes "if f_1 is differentiable, then f_1 is differentiable", which is trivially true.

Induction step: $\forall k \in \mathbb{N}\,(P(k) \Rightarrow P(k+1))$. Let $k \in \mathbb{N}$, and assume that $P(k)$ is true. To prove $P(k+1)$, let us consider any $k+1$ differentiable functions $f_1, \ldots, f_{k+1} : A \to \mathbb{R}$, and note that we can write their product as $f_1 f_2 \cdots f_k f_{k+1} = (f_1 f_2 \cdots f_k) f_{k+1} = h f_{k+1}$, where $h = f_1 \cdots f_k$ is the product of the first k functions. Since $f_1, \ldots, f_k$ are differentiable, by the induction hypothesis $P(k)$ we deduce that their product $h = f_1 \cdots f_k$ is differentiable too. Now we can apply the Leibniz rule to the product $h f_{k+1}$ and conclude that, since both factors h and f_{k+1} are differentiable, their product is too. Since $f_1 \cdots f_{k+1} = h f_{k+1}$, this proves that $f_1 \cdots f_{k+1}$ is differentiable. Since $f_1, \ldots, f_{k+1} : A \to \mathbb{R}$ were arbitrary differentiable functions, this proves $P(k+1)$. Since $k \in \mathbb{N}$ was arbitrary, this proves the induction step.

Conclusion. From the base case and the induction step, by the induction principle we deduce that $P(n)$ is true for all n, that is, the product of any n differentiable functions is differentiable, as desired.

(iii). By proceeding as in part (i), if we apply the Leibniz rule iteratively we obtain, for example, that

$$
\begin{aligned}
(f_1 f_2)' &= f_1' f_2 + f_1 f_2', \\
(f_1 f_2 f_3)' &= f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3', \\
(f_1 f_2 f_3 f_4)' &= f_1' f_2 f_3 f_4 + f_1 f_2' f_3 f_4 + f_1 f_2 f_3' f_4 + f_1 f_2 f_3 f_4'
\end{aligned}
$$

in the cases $n = 2, 3, 4$. From this one can recognise a pattern: we may expect that the derivative of the n-fold product $f_1 \cdots f_n$ is a sum of n terms, each of which is a product of the form $f_1 \cdots f_{k-1} f_k' f_{k+1} \cdots f_n$, where the kth factor is differentiated, but the others are not, and k ranges from 1 to n. In other words,

$$
\begin{aligned}
(f_1 \cdots f_n)' = f_1' f_2 f_3 \cdots f_{n-2} f_{n-1} f_n \\
&+ f_1 f_2' f_3 \cdots f_{n-2} f_{n-1} f_n \\
&+ \ldots \\
&+ f_1 f_2 f_3 \cdots f_{n-2} f_{n-1}' f_n \\
&+ f_1 f_2 f_3 \cdots f_{n-2} f_{n-1} f_n'.
\end{aligned}
$$

In order to avoid so many "…" and write this in a more compact form, we could use the summation symbol:

$$
(f_1 \cdots f_n)' = \sum_{k=1}^{n} f_1 \cdots f_{k-1} f_k' f_{k+1} \cdots f_n.
$$

We could also get rid of the "$\cdots$" in the products by using the iterated product notation:

$$
\begin{aligned}
(f_1 \cdots f_n)' &= \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} f_j \right) f_k' \left(\prod_{j=k+1}^{n} f_j \right) \\
&= \sum_{k=1}^{n} f_k' \prod_{\substack{1 \le j \le n \\ j \ne k}} f_j.
\end{aligned}
$$

This formula (however written) can be proved by induction on n, by suitably adapting the arguments in parts (i) and (ii) above. □

EQ5. A factory has received an order for open-top metal cans of a given volume V, having the shape of a cylinder with circular base. What should the height of each of these cans be, so as to minimise the amount of metal used to produce them?

Solution. Let r and h be the base radius and the height of the cylinder can; then $V = \pi r^2 h$, and consequently $h = V/(\pi r^2)$; it is reasonable to assume here that $V > 0$ and therefore $r, h > 0$ too. In order to minimise the amount of metal, we must minimise the surface of the open-top can. On the other hand, the surface of the base of the cylinder is πr^2, while the lateral surface is $2\pi r h$, which means that the total surface of the open-top cylinder is $S = 2\pi r h + \pi r^2$. By plugging in the previous expression for h, we obtain

$$S = \frac{2V}{r} + \pi r^2.$$

Based on this formula, we can consider S as a function of r, with domain $(0, \infty)$ (due to the condition $r > 0$). By the given expression, we have that S is a rational function (fraction of polynomials) in the variable r, hence it is infinitely differentiable. Moreover

$$\frac{dS}{dr} = \frac{d}{dr}\left(\frac{2V}{r} + \pi r^2\right) = -\frac{2V}{r^2} + 2\pi r = 2\frac{\pi r^3 - V}{r^2}$$

for all $r \in (0, \infty)$. Since $r^2 > 0$ for all $r \in (0, \infty)$, we deduce that the derivative dS/dr has the same sign of $\pi r^3 - V$; in other words, dS/dr is positive (resp., negative or null) if and only if $r^3 > V/\pi$ (resp., $r^3 < V/\pi$ or $r^3 = V/\pi$) if and only if $r > \sqrt[3]{V/\pi}$ (resp., $r < \sqrt[3]{V/\pi}$ or $r = \sqrt[3]{V/\pi}$). Consequently, we deduce that S is decreasing on $(0, \sqrt[3]{V/\pi}]$ and increasing on $[\sqrt[3]{V/\pi})$, and that $r = \sqrt[3]{V/\pi}$ is a global minimum point of S. As a consequence, the minimum surface (and metal used) is attained by taking base radius $r = \sqrt[3]{V/\pi}$ and height $h = V/(\pi r^2) = V/(\pi \sqrt[3]{V^2/\pi^2}) = \sqrt[3]{V/\pi}$. □

EQ6. For each of the following statements, give a proof if it is true, or provide a counterexample if it is false.

(i) Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be differentiable. If $f'(x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$, then f is constant.

(ii) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and strictly increasing. Then $g'(x) > 0$ for all $x \in \mathbb{R}$.

(iii) Let $h : \mathbb{R} \to \mathbb{R}$ be strictly increasing. Then h is differentiable.

(iv) Let $k : \mathbb{R} \to \mathbb{R}$ be convex. Then k is twice differentiable.

Solution. The statements are all false. Here are counterexamples.

(i). Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Clearly f is not constant, because it takes both the values -1 and 1. Let $a \in (0, \infty)$. Note that, for all $x > 0$,

$$\frac{f(x) - f(a)}{x - a} = \frac{1 - 1}{x - a} = 0.$$

Hence, by locality of limits,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} 0 = 0.$$

This proves that f is differentiable at a and $f'(a) = 0$. Similarly, if $a \in (-\infty, 0)$, we note that, for all $x < 0$,

$$\frac{f(x) - f(a)}{x - a} = \frac{(-1) - (-1)}{x - a} = 0,$$

whence again, by locality of limits, we deduce that f is differentiable at a and $f'(a) = 0$.

(ii). Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3$ for all $x \in \mathbb{R}$. By the properties of powers, we know that, for all $x, y \in \mathbb{R}$, we have $x^3 < y^3$ whenever $x < y$, which means that g is strictly increasing. On the other hand, g is a polynomial (hence differentiable) and $g'(x) = 3x^2$ for all $x \in \mathbb{R}$, whence $g'(0) = 0$; consequently it is not true that $g'(x) > 0$ for all $x \in \mathbb{R}$ in this case.

(iii). Let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(x) = \sqrt[3]{x}.$$

Then h is strictly increasing; indeed, for all $x, y \in \mathbb{R}$, if $x < y$, then $\sqrt[3]{x} < \sqrt[3]{y}$. On the other hand, h is not differentiable at 0. Indeed, the Newton quotient of h at 0 is given, for all $x \neq 0$, by

$$\frac{h(x) - h(0)}{x - 0} = \frac{\sqrt[3]{x}}{x} = \frac{1}{\sqrt[3]{x^2}}.$$

Since $x \mapsto \sqrt[3]{x^2}$ is continuous (it is the composition of two continuous functions), $\lim_{x\to 0} \sqrt[3]{x^2} = \sqrt[3]{0^2} = 0$, and moreover $\sqrt[3]{x^2} > 0$ for all $x \neq 0$; hence by the Algebra of Limits

$$\lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x\to 0} \frac{1}{\sqrt[3]{x^2}} = \infty.$$

Since the limit of the Newton quotient is not a real number, we conclude that h is non differentiable at 0.

(iv). Let $k : \mathbb{R} \to \mathbb{R}$ be defined by $k(x) = |x|$ for all $x \in \mathbb{R}$. Then k is convex: indeed, for all $a, b \in \mathbb{R}$ and $t \in [0, 1]$, by the Triangle Inequality and other properties of the absolute value,

$$\begin{aligned} k((1-t)a + tb) &= |(1-t)a + tb| \leq |(1-t)a| + |tb| \\ &= (1-t)|a| + t|b| = (1-t)k(a) + tk(b), \end{aligned}$$

where we used that both $t, 1 - t \geq 0$. On the other hand, as discussed in lectures (see Example 5.7 in the lecture notes), k is not differentiable at 0; hence k is not differentiable and, *a fortiori*, k is not twice differentiable. □

EQ7. Let $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be given by

$$f(x) = e^x, \qquad g(x) = 1 + x + x^2/2$$

for all $x \in \mathbb{R}$.

(i) Prove that $f''(x) \geq g''(x)$ for all $x \in [0, \infty)$.

(ii) Prove that $f' - g'$ is an increasing function. [Hint: consider the sign of its derivative.]

(iii) Prove that $f'(x) - g'(x) \geq 0$ for all $x \in [0, \infty)$. [Hint: evaluate the l.h.s. at $x = 0$, and then use part (ii).]

(iv) Prove that $f(x) - g(x) \geq 0$ for all $x \in [0, \infty)$. [Hint: repeat the above steps with f and g instead of f' and g'.]

(v) Deduce that

$$e^x \geq 1 + x + x^2/2$$

for all $x \in [0, \infty)$.

Solution. (i). We compute

$$f'(x) = e^x, \qquad f''(x) = e^x, \qquad g'(x) = 1 + x, \qquad g''(x) = 1$$

for all $x \in [0, \infty)$. Since $e^x \geq e^0 = 1$ for all $x \geq 0$, we conclude that $f''(x) \geq g''(x)$ for all $x \in [0, \infty)$.

(ii). By the Difference Rule for differentiation, $(f' - g')'(x) = f''(x) - g''(x) \geq 0$ for all $x \in [0, \infty)$, where the last inequality follows from part (i). This tells us that $f' - g'$ is differentiable and its derivative is nonnegative; since $[0, \infty)$ is an interval, we deduce that $f' - g'$ is increasing.

(iii). Note that $(f' - g')(0) = f'(0) - g'(0) = e^0 - (1 + 0) = 0$. From part (ii) we know that $f' - g'$ is increasing, hence, for all $x \in [0, \infty)$, $f'(x) - g'(x) = (f' - g')(x) \geq (f' - g')(0) = 0$.

(iv). Note that $(f - g)' = f' - g'$ by the Difference Rule. Hence, by part (iii) we deduce that $(f' - g')(x)$ is nonnegative for all $x \in [0, \infty)$. Since $[0, \infty)$ is an interval, we conclude that $f - g$ is increasing. On the other hand, $(f - g)(0) = f(0) - g(0) = e^0 - (1 + 0 + 0^2/2) = 0$. Consequently, for all $x \in [0, \infty)$, $f(x) - g(x) = (f - g)(x) \geq (f - g)(0) = 0$.

(v). From part (iv) we deduce that $f(x) \geq g(x)$ for all $x \in [0, \infty)$, which is the required inequality. □

EQ8. Let $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$. Recall that, by the Sum Rule for differentiation, if f and g are differentiable, then their sum $f + g$ is differentiable as well and the formula

$$(f + g)'(x) = f'(x) + g'(x)$$

holds for all $x \in A$.

(i) Assume that f and g are twice differentiable. Prove that $f + g$ is twice differentiable and find a formula for the second derivative $(f + g)''$.

(ii) Let $n \in \mathbb{N}$, and assume that f and g are n times differentiable. Prove that $f + g$ is n times differentiable and find a formula for its nth derivative. [Hint: induction.]

Solution. (i). If f and g are twice differentiable, then they are differentiable, so, by the Sum Rule, $f + g$ is also differentiable and $(f + g)' = f' + g'$. Now, since f and g are twice differentiable, we also know that f' and g' are both differentiable, so we deduce by the Sum Rule that $(f + g)' = f' + g'$ is differentiable (as a sum of differentiable functions) and

$$(f + g)'' = (f' + g')' = f'' + g''.$$

(ii). Let $P(n)$ be the statement "for all $f, g : A \to \mathbb{R}$, if f and g are both n times differentiable, then $f + g$ is n times differentiable, and $(f + g)^{(n)} = f^{(n)} + g^{(n)}$". We prove this statement for all $n \in \mathbb{N}$, by induction on n.

<u>Base case: $P(1)$.</u> In the case $n = 1$ the statement reduces to the Sum Rule for differentiation, hence it is true.

<u>Induction step: $\forall k \in \mathbb{N} : P(k) \Rightarrow P(k + 1)$.</u> Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. We must prove $P(k + 1)$. So, let $f, g : A \to \mathbb{R}$ be $k + 1$ times differentiable. Then they are differentiable, so by the Sum Rule their sum is differentiable and $(f + g)' = f' + g'$. We now note that, since f and g are $k + 1$ times differentiable, their derivatives f' and g' are k times differentiable, so by applying the inductive hypothesis $P(k)$ to f' and g' we deduce that $(f+g)' = f'+g'$ is k times differentiable (hence $f + g$ is $k + 1$ times differentiable) and

$$\begin{aligned}(f + g)^{(k+1)} &= ((f + g)')^{(k)} = (f' + g')^{(k)} \\ &= (f')^{(k)} + (g')^{(k)} = f^{(k+1)} + g^{(k+1)}.\end{aligned}$$

Since f and g were arbitrary, this proves $P(k + 1)$. Since k was arbitrary, this proves the induction step.

Conclusion. Having proved the base case and the induction step, by the induction principle we deduce that $P(n)$ is true for all $n \in \mathbb{N}$. Hence, if f and g are n times differentiable, then their sum $f+g$ is n times differentiable and the formula

$$(f+g)^{(n)} = f^{(n)} + g^{(n)}$$

holds. □

* **EQ9**. Let $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$. Recall that, by the Leibniz Rule, if f and g are differentiable, then their product fg is differentiable as well and the formula

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

holds for all $x \in A$.

(i) Assume that f and g are twice differentiable. Prove that fg is twice differentiable and find a formula for the second derivative $(fg)''$.

(ii) Let $n \in \mathbb{N}$, and assume that f and g are n times differentiable. Prove that fg is n times differentiable. [Hint: use induction and **EQ8**.]

(iii) Under the same assumptions as in part (ii), can you write a formula for the nth derivative $(fg)^{(n)}$?

Solution. (i). Under our assumptions, f and g are differentiable, hence, by the Leibniz Rule, fg is differentiable and

$$(fg)' = f'g + fg'.$$

Since f and g are twice differentiable, we also know that f' and g' are differentiable. Hence, by the Leibniz Rule, both products $f'g$ and fg' are differentiable and

$$(f'g)' = f''g + f'g', \qquad (fg')' = f'g' + fg'';$$

consequently, by the Sum Rule for differentiation, we conclude that $(fg)' = f'g + fg'$ is also differentiable and

$$\begin{aligned}(fg)'' &= (f'g + fg')' \\ &= (f'g)' + (fg')' \\ &= f''g + f'g' + f'g' + fg'' \\ &= f''g + 2f'g' + fg''.\end{aligned}$$

(ii). Let $P(n)$ be the statement "for all $f, g : A \to \mathbb{R}$, if f and g are n times differentiable, then fg is n times differentiable". We prove $P(n)$ is true for all $n \in \mathbb{N}$, by induction on n.

Base case: $P(1)$. For $n = 1$, the statement $P(n)$ reduces to "for all $f, g : A \to \mathbb{R}$, if f and g are differentiable, then fg is differentiable", which is true by the Leibniz Rule.

Induction step: $\forall k \in \mathbb{N} : P(k) \Rightarrow P(k+1)$. Let $k \in \mathbb{N}$. Assume that $P(k)$ is true. We now prove $P(k+1)$. Let f and g be $k+1$ times differentiable. Then they are differentiable, so, by the Leibniz Rule, fg is differentiable and

$$(fg)' = f'g + fg'.$$

Now, since f and g are $k+1$ times differentiable, their derivatives f' and g' are k times differentiable, and clearly f and g themselves are also k times differentiable. This means that, by the inductive hypothesis $P(k)$, the products $f'g$ and fg' are k times differentiable. So $(fg)' = f'g + fg'$ is the sum of two k times differentiable functions, and therefore it is k times differentiable as well, by **EQ8**. Since $(fg)'$ is k times differentiable, we conclude that fg is $k+1$ times differentiable.

Since f and g were arbitrary, this proves $P(k+1)$. Since k was arbitrary, this concludes the proof of the induction step.

Conclusion. From the base case and the induction step, by the induction principle we deduce that $P(n)$ is true for all $n \in \mathbb{N}$. Hence, if f and g are n times differentiable, then their product fg is n times differentiable too.

(iii). By applying iteratively the Leibniz Rule and the Sum Rule as in part (i), it is not difficult to work out the formulas

$$\begin{aligned}(fg)' &= f'g + fg',\\ (fg)'' &= f''g + 2f'g' + fg'',\\ (fg)''' &= f'''g + 3f''g' + 3f'g'' + fg''',\\ (fg)'''' &= f''''g + 4f'''g' + 6f''g'' + 4fg''' + fg'''',\end{aligned}$$

and so on. One may recognise a similar structure to the one in the expansion of $(a+b)^n$ given by the Binomial Theorem:

$$\begin{aligned}(a+b)^1 &= a^1b^0 + a^0b^1,\\ (a+b)^2 &= a^2b^0 + 2a^1b^1 + a^0b^2,\\ (a+b)^3 &= a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3,\\ (a+b)^4 &= a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + a^0b^4,\end{aligned}$$

and so on. For an arbitrary n, the expansion given by the Binomial Theorem can be written by using summation and binomial cofficients $\binom{n}{k}$ as follows:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

This suggests that the general formula for the nth derivative of fg should have a similar expression:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

This formula is indeed true and can be proved by induction on n (the details are left to the interested reader). □

EQ10. Let $a \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{ax^2} & \text{if } x < 0,\\ \cosh x & \text{if } x \geq 0.\end{cases}$$

(i) Determine all the values of $a \in \mathbb{R}$ such that the function f defined above is twice differentiable.

(ii) For each of the values found in part (i), determine whether the function f is convex.

Solution. (i). We know from lectures that the functions $g : x \mapsto e^{ax^2}$ and $h : x \mapsto \cosh x$ are twice differentiable on $\mathbb{R}$, irrespective of the value $a \in \mathbb{R}$, and

$$\begin{aligned} g'(x) &= 2axe^{ax^2}, \quad & g''(x) &= 2a(1+2ax^2)e^{ax^2},\\ h'(x) &= \sinh x, & h''(x) &= \cosh x \end{aligned} \tag{3}$$

for all $x \in \mathbb{R}$.

Now, for all $c, x < 0$,

$$\frac{f(x) - f(c)}{x - c} = \frac{g(x) - g(c)}{x - c},$$

which means, by locality of limits, that, for all $c < 0$,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c),$$

This proves that f is differentiable at any $c < 0$, and $f'(c) = g'(c)$. In a similar way, we prove that f is differentiable at any $c > 0$, and $f'(c) = h'(c)$. In other words, f is differentiable on $\mathbb{R} \setminus \{0\}$ and, for all $x \in \mathbb{R} \setminus \{0\}$,

$$f'(x) = \begin{cases} g'(x) & \text{if } x < 0, \\ h'(x) & \text{if } x > 0. \end{cases}$$

By repeating this argument with f', g', h' in place of f, g, h, we can also prove that f is twice differentiable on $\mathbb{R} \setminus \{0\}$ and, for all $x \in \mathbb{R} \setminus \{0\}$,

$$f''(x) = \begin{cases} g''(x) & \text{if } x < 0, \\ h''(x) & \text{if } x > 0. \end{cases}$$

It remains to discuss (first- and second-order) differentiability of f at 0. Let us first consider the Newton quotient of f at 0. Note that, for all $x > 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{h(x) - h(0)}{x - 0},$$

hence

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0^+} \frac{h(x) - h(0)}{x - 0} \\ &= \lim_{x \to 0^+} \frac{h(x) - h(0)}{x - 0} = h'(0). \end{aligned}$$

On the other hand, by observing that $h(0) = \cosh 0 = 1 = e^{a \cdot 0^2} = g(0)$, we also obtain that, for all $x < 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{g(x) - h(0)}{x - 0} = \frac{g(x) - g(0)}{x - 0},$$

and therefore, by arguing as before, we deduce that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = g'(0).$$

On the other hand, from (3) we know that $g'(0) = 0 = h'(0)$. This shows that the two one-sided limits of the Newton quotient $\frac{f(x) - f(0)}{x - 0}$ are equal, and therefore we conclude that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

as well, that is, f is differentiable at 0, and $f'(0) = 0$. To sum up, f is differentiable on $\mathbb{R}$ and, for all $x \in \mathbb{R}$,

$$f'(x) = \begin{cases} g'(x) & \text{if } x \leq 0, \\ h'(x) & \text{if } x \geq 0. \end{cases}$$

We now consider the Newton quotient of f' at 0. By arguing as before, we obtain

$$\lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = g''(0)$$

and

$$\lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = h''(0).$$

Hence, f is twice differentiable at 0 if and only if the two-sided limit $\lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0}$ exists (and is finite), which is true if and only if the one-sided limits $g''(0)$ and $h''(0)$ are equal. From (3) we deduce that

$$g''(0) = 2a, \qquad h''(0) = 1,$$

and these two quantities are equal if and only if $2a = 1$, that is, if and only if $a = 1/2$. In conclusion, f is twice differentiable if and only if $a = 1/2$.

(ii). Based on the above discussion, if $a = 1/2$, then f is twice differentiable on $\mathbb{R}$ and, for all $x \in \mathbb{R}$,

$$f''(x) = \begin{cases} g''(x) & \text{if } x \leq 0, \\ h''(x) & \text{if } x \geq 0. \end{cases}$$

Since f is twice differentiable, by a result from lectures we know that f is convex if and only if $f''(x) \geq 0$ for all $x \in \mathbb{R}$. On the other hand, if $x \geq 0$, then

$$f''(x) = g''(x) = (1+x^2)e^{x^2/2} \geq 0,$$

(here we used that $e^y > 0$ for all $y \in \mathbb{R}$, and $1 + x^2 \geq x^2 \geq 0$ for all $x \in \mathbb{R}$), while, for all $x < 0$,

$$f''(x) = h''(x) = \cosh x \geq 0$$

(here we use that $\cosh x = \frac{e^x+e^{-x}}{2} > 0$ for all $x \in \mathbb{R}$). This proves that $f''(x) \geq 0$ for all $x \in \mathbb{R}$ and therefore that f is indeed convex. □

EQ11. (i) Prove that the function $\log : (0,\infty) \to \mathbb{R}$ is concave.

(ii) Prove that, for all $a, b \in (0,\infty)$ and $t \in [0,1]$, the inequality

$$a^t b^{1-t} \leq ta + (1-t)b.$$

holds. [Hint: take the logarithm of both sides, and use part (i).]

(iii) Prove that the inequality in part (ii) holds more generally for all $a, b \in [0,\infty)$ and $t \in [0,1]$.

(iv) Prove that, for all $x, y \in [0,\infty)$ and all $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

[Hint: apply part (iii) with $t = 1/p$, $a = x^p$, $b = y^q$.]

Solution. (i). We know that log is differentiable and

$$\frac{d}{dx}\log x = \frac{1}{x}.$$

for all $x \in (0,\infty)$. Since the derivative is a fraction of polynomials, it is differentiable too (i.e., log is twice differentiable) and

$$\frac{d^2}{dx^2}\log x = \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}.$$

Since $x^2 > 0$ for all $x \in (0,\infty)$, this proves that the second derivative of log is negative at every point of the domain, which implies (by a result from lectures) that log is concave.

(ii) Since log is concave, for all $a, b \in (0,\infty)$ and $t \in [0,1]$ we have

$$\log(ta + (1-t)b) \geq t\log a + (1-t)\log b.$$

From this we deduce

$$ta + (1-t)b = e^{\log(ta+(1-t)b)} \geq e^{t\log a+(1-t)\log b} = e^{t\log a}e^{(1-t)\log b} = a^t b^{1-t},$$

which is the desired inequality.

(iii). In light of part (ii), it only remains to check the inequality when one or both of a and b are zero. In this case, however, the left-hand side is zero, while the right-hand side is nonnegative, so the inequality is trivially true.

(iv). Let $t = 1/p$, so $1 - t = 1 - 1/p = 1/q$. If we apply the inequality from (iii) with $t = 1/p$, $a = x^p$, $b = y^q$, we deduce that

$$(x^p)^{1/p}(y^q)^{1/q} \leq (1/p)x^p + (1/q)y^q,$$

which is the desired inequality since $(x^p)^{1/p} = x$ and $(y^q)^{1/q} = y$. $\square$

EQ12. (a) Find, with proof, the infimum and supremum of each of the following sets:
- (i) $A = (0,1] \cup (2,3]$
- (ii) $B = \{x^2 - 4x + 5 : x \in (1,3]\}$
- (iii) $C = \{(2n+3)/n : n \in \mathbb{N}\}$
- (iv) $D = \{n^2 - 6n + 10 : n \in \mathbb{N}\}$

(b) Let X denote a nonempty bounded subset of $\mathbb{R}$. Prove that for each $\epsilon > 0$, there exists x in X such that $\inf X \leq x < \inf X + \epsilon$.

Solution. (a)(i) $A = (0,1] \cup (2,3]$: We claim that $\sup A = 3$ and $\inf A = 0$. To prove this, observe that $0 \leq x \leq 3$ for all $x \in A$. This proves that 0 is a lower bound for A and that 3 is an upper bound for A. It remains to prove that 0 is the *greatest* such lower bound and that 3 is the *least* such upper bound. To do this, we consider an arbitrary $\epsilon > 0$. Observe that $0 + \epsilon$ is never a lower bound for A, since either $\epsilon > \frac{\epsilon}{2} \in A$ or $\epsilon > 1 \in A$, hence $\inf A = 0$. Moreover, observe that $3 - \epsilon$ is never an upper bound for A, since $3 - \epsilon < 3 \in A$, hence $\sup A = 3$.

(a)(ii) $B = \{x^2 - 4x + 5 : x \in (1,3]\}$: We complete the square to obtain

$$f(x) := x^2 - 4x + 5 = (x-2)^2 + 1.$$

The graph of f is a parabola with minimum value $f(2) = 1$, whilst $f(1) = 2 = f(3)$, hence $B = [1,2]$. We claim that $\inf B = 1$ and $\sup B = 2$. To prove this, observe that $1 \leq x \leq 2$ for all $x \in B$, so 1 is a lower bound for B and 2 is an upper bound for B. Now let $\epsilon > 0$ and observe that $1 + \epsilon$ is never a lower bound for B, since $1 + \epsilon > 1 \in B$, hence $\inf B = 1$. Moreover, observe that $2 - \epsilon$ is never an upper bound for B, since $2 - \epsilon < 2 \in B$, hence $\sup B = 2$.

(a)(iii) $C = \{(2n+3)/n : n \in \mathbb{N}\}$: The elements of the set

$$C = \{2 + \tfrac{3}{n} : n \in \mathbb{N}\} = \{5, 3\tfrac{1}{2}, 3, 2\tfrac{3}{4}, \ldots\}$$

form a decreasing sequence with $\lim_{n\to\infty}(2 + \frac{3}{n}) = 2$. This shows that 2 is a lower bound for C and that 5 is an upper bound for C. Now let $\epsilon > 0$ and observe that $2 + \epsilon$ is never a lower bound for C, since we can always find $n \in \mathbb{N}$ such that $2 + \frac{3}{n} < 2 + \epsilon$ (e.g. choose $n \in \mathbb{N}$ with $n > \frac{3}{\epsilon}$; although it is enough here to note that the sequence is decreasing and tends to 2), hence $\inf C = 2$. Moreover, observe that $5 - \epsilon$ is never an upper bound for C, since $5 - \epsilon < 5 \in C$, hence $\sup C = 5$.

(a)(iv) $D = \{n^2 - 6n + 10 : n \in \mathbb{N}\}$: We complete the square to obtain

$$g(n) := n^2 - 6n + 10 = (n-3)^2 + 1$$

for all $n \in \mathbb{N}$. The graph of the function g is thus a (discrete) parabola with minimum value $g(3) = 1$. This shows that 1 is a lower bound for D. Now let $\epsilon > 0$ and observe that $1 + \epsilon$ is never a lower bound for D because $1 + \epsilon > 1 \in D$, hence $\inf D = 1$. Moreover, the set D is not bounded above, as $\lim_{n\to\infty}(n^2 - 6n + 10) = \infty$, hence $\sup D = \infty$ by definition. Alternatively, we can note that for any $M > 1$, there exists $n \in \mathbb{N}$ such that $n^2 - 6n + 10 > M$ (e.g. choose $n > 3 + \sqrt{M-1}$) to prove that D is not bounded above, hence $\sup D = \infty$ by definition.

(b) Let X denote a nonempty bounded subset of $\mathbb{R}$. Prove that for each $\epsilon > 0$, there exists x in X such that $\inf X \leq x < \inf X + \epsilon$.

The first inequality $\inf X \leq x$ actually holds for all x in X because $\inf X$ is a lower bound for X. Now let $\epsilon > 0$ and observe that $\inf X + \epsilon$ is *not* a lower bound for X because $\inf X$ is the *greatest* such lower bound and $\inf X + \epsilon > \inf X$. Therefore, there must exist x_ϵ in X such that $x_\epsilon < \inf X + \epsilon$ (as otherwise $\inf X + \epsilon$ would be a lower bound for X). We combine this result with our initial observation to obtain $\inf X \leq x_\epsilon < \inf X + \epsilon$, as required. (It is not essential to add the ϵ-subscript but it is included here to emphasise that the value x_ϵ depends on ϵ.) $\square$

EQ13. For each $n \in \mathbb{N}$, let P_n denote the partition of $[0, 1]$ into n subintervals of equal width. Find an expression, possibly involving summation notation, for the sums $L(f, P_n)$ and $U(f, P_n)$ for $f : [0, 1] \to \mathbb{R}$ in each of the following cases:

(a) $f(x) = x^2 + x$

(b) $f(x) = \cos(x)$

(c) $f(x) = \begin{cases} 5, & x \in \mathbb{Q} \\ 3, & x \notin \mathbb{Q} \end{cases}$

(d) $f(x) = \begin{cases} 1, & x \in \{0, 1\}, \\ 0, & x \notin \{0, 1\}. \end{cases}$

EQ14. Let P_1 denote a partition of $[a, b]$ and $P_2 = P_1 \cup \{c\}$, where $-\infty < a < c < b < \infty$. Use results from lectures to prove that $U(f, P_2) - L(f, P_2) \leq U(f, P_1) - L(f, P_1)$.

EQ15. Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = x^3$, where $0 \leq a < b < \infty$. Use the procedure outlined in **Q8** and the formula $\sum_{j=1}^{k} j^3 = \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2$ to prove that f is integrable with $\int_a^b f = \frac{1}{4}b^4 - \frac{1}{4}a^4$.