University of Birmingham
School of Mathematics

1AC Algebra: Feedback Sheet 1

Marking guidance
AQ3 and AQ4 are the questions to be marked using the guidance after those questions. There are a total of 20 marks for these algebra questions. The mark out of 20 for the algebra questions should be combined with the mark for the combinatorics questions, and then a total mark given as a percentage.
More importantly than giving the mark, you should provide detailed written feedback. There should be comments to explain where improvements can be made, even when it is just minor improvements. This could include: places where there are mathematical errors or inaccuracies; places where mathematics should be explained more clearly or in more detail; places where the mathematics should be set out better; or places where the solution is longer than necessary or overly verbose. You should ensure that feedback is given to explain reasons why marks have not been gained.

AQ1. True or false?

(a) $7 \mid 28$

(b) $9 \mid 15$

(c) $6 \mid 3$

(d) $0 \mid 13$

(e) $11 \mid 0$

You should justify your answers.

Solution

(a) True, because $28 = 7 \cdot 4$ and $4 \in \mathbb{Z}$.

(b) False, because if $15 = 9z$, then $z = \frac{15}{9}$, which is not an integer.

(c) False, because if $3 = 6z$, then $z = \frac{1}{2}$, which is not an integer.

(d) False, because $13 = 0z$ is not possible for any $z \in \mathbb{Z}$.

(e) True, because $0 = 11 \cdot 0$ and $0 \in \mathbb{Z}$.

Feedback

Hopefully, this question will not have caused any serious difficulties. It is just included here to make sure that you have fully understood the definitions of factors. This is central to this course, so if you have made any mistakes, then you should look through it again to make sure you understand where have gone wrong. Often errors are made in (d) and (e), where you have to consider the definition carefully from which you can see that (d) is false and (e) it is true.

AQ2. $3, 5, 7$ is a list of three primes of the form $p, p + 2, p + 4$.
Prove that there are no other "triplet primes"?

Solution

First it is a good idea to write out by clearly what you would like to prove.

Claim. *Let $p \in \mathbb{N}$ with $p \neq 3$. Then one of $p, p + 2, p + 4$ is not prime.*

Proof. Since 1 is not prime, we can assume that $p \neq 1$, and since $2 + 2 = 4$ is not prime, we can assume that $p \neq 2$. Thus we can assume that $p \geq 4$.

Then we can write p in the form $3k$, $3k + 1$ or $3k + 2$ for some $k \in \mathbb{N}$. We consider these three cases separately.

Case 1: $p = 3k$ for some $k \in \mathbb{N}$. Then $3 \mid p$ so p is not prime, because $p \neq 3$.

Case 2: $p = 3k + 1$ for some $k \in \mathbb{N}$. Then $p + 2 = 3k + 3 = 3(k + 1)$, so $3 \mid p + 2$. Also $p + 2 \neq 3$, as $p \neq 1$, so $p + 2$ is not prime.

Case 3: $p = 3k + 2$ for some $k \in \mathbb{N}$. Then $p + 4 = 3k + 6 = 3(k + 2)$, so $3 \mid p + 4$. Also clearly $p + 4 \neq 3$, so $p + 4$ is not prime.

So in all three possible cases $p, p + 2, p + 4$ are not triplet primes. □

Feedback

This question is different from anything that you have seen in lectures and is included here as a more challenging exercise for you to think about. You may not have been sure how to start with this question and hopefully the hint provided on on the 1AC Canvas course will have helped you to get started.

Probably the most challenging part is to set out your proof well. It really helps to write down clearly what you're going to prove, then you can refer back to this when you are writing out the proof. The key step here is to break the proof in to the three cases and deal with each separately. Here it is important to explain what you are writing clearly so that you can convince the marker that you know what you are doing. You need to ensure that everything is explained. The main aim of this question is to challenge you to work out how to prove something and then write it out properly.

We've seen in this problem that 3, 5, 7 is the only triplet prime. Pairs of primes of the form p, $p + 2$ are known as twin primes, and the theory of twin primes is a really interesting area of number theory. There is some discussion about this at the end of Chapter 1 in the notes.

AQ3. (SUM)

(a) Prove Lemma 2.3:

Lemma. *Let $a, b, c, k, l \in \mathbb{Z}$.*

(a) *Suppose that $a \mid b$ and $a \mid c$. Then $a \mid (kb + lc)$.*

(b) *Suppose that $a \mid b$ and $b \mid c$. Then $a \mid c$.*

(c) *Suppose that $a \mid b$ and $b \mid a$. Then $a = \pm b$.*

(b) Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid b^2$ and $a^2 \mid c$. Prove that $a^5 \mid b^3c^2$.

(c) Let $a, b, c \in \mathbb{N}$. Give a counterexample to the following statement.
Suppose that $a^2 \mid bc$. Then $a \mid b$ or $a \mid c$.

Solution

(a)

Lemma. *Let $a, b, c, k, l \in \mathbb{Z}$.*

(a) *Suppose that $a \mid b$ and $a \mid c$. Then $a \mid (kb + lc)$.*

(b) *Suppose that $a \mid b$ and $b \mid c$. Then $a \mid c$.*

(c) *Suppose that $a \mid b$ and $b \mid a$. Then $a = \pm b$.*

Proof. (a) Since $a \mid b$, there exists $x \in \mathbb{Z}$ such that $b = ax$. Since $a \mid c$, there exists $y \in \mathbb{Z}$ such that $c = ay$. Then

$$kb + lc = kax + lay = a(kx + ly).$$

We have $kx + ly \in \mathbb{Z}$, so $a \mid (kb + lc)$.

(b) Since $a \mid b$, there exists $x \in \mathbb{Z}$ such that $b = ax$. Since $b \mid c$, there exists $y \in \mathbb{Z}$ such that $c = by$. Then substituting $b = ax$ in to $c = by$ we get

$$c = (ax)y = a(xy).$$

We have $xy \in \mathbb{Z}$, so $a \mid c$.

(c) Since $a \mid b$ there exists $x \in \mathbb{Z}$ such that $b = ax$. Since $b \mid a$ there exists $y \in \mathbb{Z}$ such that $a = by$. Combining these two equations we obtain

$$b = ax = byx.$$

Therefore,

$$b(1 - yx) = 0.$$

Hence, $b = 0$ or $yx = 1$.

If $b = 0$, then $a = 0$, so $a = b$.

If $yx = 1$, then we must have $x, y = 1$ or $x, y = -1$, so $a = \pm b$. $\square$

(b) Let $a, b, c \in \mathbb{Z}$. Suppose that $a \mid b^2$ and $a^2 \mid c$.

Claim. $a^5 \mid b^3c^2$.

Proof. Since $a \mid b^2$, there exists $x \in \mathbb{Z}$ such that $b^2 = ax$.

Since $a^2 \mid c$, there exists $y \in \mathbb{Z}$ such that $c = a^2y$.

Then by substituting we get that

$$\begin{aligned} b^3c^2 &= (ax)b(a^2y)^2 \\ &= a^5bxy^2 \end{aligned}$$

and $bxy^2 \in \mathbb{Z}$.

Therefore, $a^5 \mid b^3c^2$. $\square$

(c) Let $a, b, c \in \mathbb{N}$. We give a counterexample to the statement.

Suppose that $a^2 \mid bc$. Then $a \mid b$ or $a \mid c$.

Counterexample

Let $a = 6$, $b = 4$ and $c = 28$. We have $a^2 = 36 \mid 36 = bc$, but $a = 6 \nmid 4 = b$, and $a = 6 \nmid 9 = c$. Therefore, $a = 6$, $b = 4$ and $c = 9$ is a counterexample to the statement.

Feedback

(a) You should write out the lemma in your work, then it is easy to refer to what you are proving when you are writing out the proof.

We mainly just give feedback for the proof of (a), as the others are pretty similar. This proof is very similar to that of Lemma 2.2 in the notes. You can adapt that proof and just multiply by k and l at the relevant place. There are some common mistakes here. For example you should not write:

Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $b = ak$.

This is not allowed as k has already been used in the statement of the lemma. Another thing not to write is:

Since $a \mid b$, there exists $x \in \mathbb{Z}$ such that $kb = ax$.

Although this is actually true, you are jumping a step of the argument. Another way to lose marks is to not write enough to explain your answer, for example you should not write

$a \mid b$, $b = ax$, $a \mid c$, $c = ay$.

You need some words to explain what you're doing and to say what x and y are. Remember that should you write you should make sense if it is read aloud.

One further comment about (c) is that you do have to deal with the case $b = 0$ separately, as you cannot cancel the b if it is equal to zero. So this must be included in your proof.

The main point of this question is to give you practice of writing proofs correctly and well.

(b) The proof here has some similarity with the first proof required in (a), and you can set out the proof similarly making sure that you explained it fully. You have to be more careful as the algebraic manipulations are more complicated here, and you have to make multiple substitutions. So you should be careful not to make any errors and it is worth checking your answer once you have written it down.

(c) This question is here to give you some practice giving counterexamples. It is important that you make sure you understand the statement before trying to come up with a counterexample. You should be aware that in mathematics when we write *or* we always mean inclusive-or, so that $a \mid b$ *or* $a \mid c$ means that at least one of these is true. Now you should realise that you are looking to find integers a, b and c such that $a^2 \mid bc$, but $a \nmid b$ and $a \nmid c$.

It is not obvious how to come up with a counterexample, and a bit of trial and error may be required to give you some idea. A good way to start may be to try some small values of a and then try to find b and c giving a counterexample. You may find it is not possible for $a = 2$, $a = 3a = 4$ or $a = 5$, so be led to try $a = 6$. Then you can realise that you're looking for $b, c \in \mathbb{Z}$ such that $36 \mid bc$ and that you want to choose b and c such that neither is divisible by 6. Then you may think that the easiest thing to try is b and c such that $bc = 36$ which leads to take b to be 4 of 9 and c being the other.

Once you have your counterexample you have to write out your solution well and make sure that you say why it is a counterexample.

Marking guidance.

12 marks

(a)(i) 2 marks. Award 2 marks if proof is fully correct and explained well. Award 1 mark if the structure of the proof is mainly correct and is explained fairly well. You should look out to make sure that x and y are introduced properly and it is stated that $kx + ly \in \mathbb{Z}$.
(a)((ii) 2 marks. Same as for (a)(i) though you can be generous and not penalise if similar errors are made as in (a)(i).
(a)(iii) 3 marks. Award 2 marks if proof is fully correct and explained fully. Award 2 marks if the structure of the prof is correct, it is explained fairly well and there are just minor errors. Award 1 mark in the general idea of the proof is correct and deserves some credit. Look at for whether the case $b = 0$ is considered as this is required.
(b) 3 marks. Same as for (a)(iii)
(c) 2 marks. 1 mark for a correct counterexample. 1 mark for the giving justification it is a counterexample.

AQ4. (SUM)

(a) Use the Euclidean algorithm to find $\mathrm{hcf}(671, 314)$.

(b) Use your working to find $x, y \in \mathbb{Z}$ such that

$$\mathrm{hcf}(671, 314) = 671x + 314y.$$

Solution

(a) We want to find $\mathrm{hcf}(671, 314)$. First we write

$$671 = 2 \cdot 314 + 43, \tag{1}$$

so $\mathrm{hcf}(671, 314) = \mathrm{hcf}(314, 43)$. Next

$$314 = 7 \cdot 43 + 13, \tag{2}$$

so $\mathrm{hcf}(314, 43) = \mathrm{hcf}(43, 13)$. Next

$$43 = 3 \cdot 13 + 4, \tag{3}$$

so $\mathrm{hcf}(43, 13) = \mathrm{hcf}(13, 4)$. Next

$$13 = 3 \cdot 4 + 1, \tag{4}$$

so $\mathrm{hcf}(13, 4) = \mathrm{hcf}(4, 1) = 1$.

Hence, $\mathrm{hcf}(671, 314) = 1$.

(b) From (4) we get

$$1 = 13 - 3 \cdot 4.$$

Substituting from (3) gives

$$\begin{aligned}1 &= 13 - 3(43 - 3 \cdot 13)\\ &= -3 \cdot 43 + 10 \cdot 13\end{aligned}$$

Substituting from (2) gives

$$\begin{aligned}1 &= -3 \cdot 43 + 10(314 - 7 \cdot 43)\\ &= 10 \cdot 314 - 73 \cdot 43\end{aligned}$$

Substituting from (1) gives

$$\begin{aligned}1 &= 10 \cdot 314 - 73(671 - 2 \cdot 314)\\ &= -73 \cdot 671 + 156 \cdot 314\end{aligned}$$

Therefore,

$$1 = 671x + 314y,$$

where $x = -73$ and $y = 156$.

Feedback

This question doesn't tend to cause many problems, and you proceed in the same way as we have done in examples. You should provide enough explanation of your calculations to ensure that someone reading it believes that you know what you're doing.

Also remember that you can check your answer to (b) just by working it out. For example, if you had made an error and got $x = -71$ and $y = 152$, then you would have

$$671x + 314y = 671 \cdot (-71) + 314 \cdot 152 = 87$$

so you would see that you had made a mistake.

Markng guidance.

8 marks

(a) 4 marks. Award 3 marks if the calculations are done correctly, and give partial credit if there are any calculational errors. There is 1 mark for presentation, which should be awarded if the calculations are set out clearly and there is some explanation to demonstrate that the method is understood.

(b) 4 marks. The same as (a).