Chapter 4 – Separation of Variables and Fourier series

(a method for solving certain PDEs, including approximating functions with combinations of sine and cosine functions)

Dr S Jabbari, 2DE/2DE3

So far we have only considered differential equations in one independent variable, i.e. ordinary differential equations. Many physical problems that we might want to describe, however, vary in two or more independent variables, e.g. time and space. We will look at ways of solving such systems in the following chapter. Here we will look at methods required to seek a particular type of solution (a separable solution) to certain types of PDEs: the Separation of Variables method and representation of solutions in the form of Fourier series (sums of functions of sines and/cosines). We begin with an example of the Separation of Variables method to motivate why Fourier series is used in problems of this kind.

1 Motivation: the heat equation

We wish to develop a model of the flow of heat through a thin, insulated straight wire or rod with fixed temperature at the ends. The temperature of the wire, u, will depend both on the position on the wire (x) and the time (t), i.e. $u = u(x, t)$. We therefore require a partial differential equation (PDE) to represent this.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \quad t > 0, \tag{1}$$

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0, \qquad t > 0, \tag{2}$$

$$u(x,0) = f(x), \qquad 0 < x < L. \tag{3}$$

Equation (1) and the boundary (2) and initial (3) conditions describe the temperature, $u(x,t)$, across a straight wire or rod of uniform cross section and of length L with:

- perfectly insulated sides (so no heat is lost from the rod/wire);
- sufficiently small cross sections that we can assume the temperature is uniform across a cross section (i.e. the rod/wire is really thin and we can work in one dimension);
- fixed temperature 0°C at each end of the rod/wire;
- the initial temperature across the rod/wire prescribed by the function $f(x)$;

see Figure 1.

α^2 is called the thermal diffusivity constant and depends only on the material from which the rod is made (this is known for many materials). We have that $\alpha^2 = \kappa/\rho s$ where κ is the thermal conductivity, ρ is the density and s is the specific heat capacity (amount of heat per unit mass required to raise the temperature by one degree Celsius) of the material.

This is a linear 2nd order homogeneous equation. It is an initial value problem in t and a boundary value problem in x.

In order to solve this problem, we can seek what we call separable solutions using the **Separation of Variables** approach. This involves converting a PDE in n independent variables into n ODEs. It is easiest to understand this by following an example. Look for nontrivial solutions to (1) in the form

$$u(x,t) = X(x)T(t),$$

$$\begin{aligned} \implies \frac{\partial u}{\partial x} &= X'(x)T(t), \\ \frac{\partial^2 u}{\partial x^2} &= X''(x)T(t), \\ \frac{\partial u}{\partial t} &= X(x)T'(t). \end{aligned}$$

If we substitute these into (1) we get

$$\begin{aligned} \alpha^2 X''(x)T(t) &= X(x)T'(t), \\ \implies \frac{X''(x)}{X(x)} &= \frac{T'(t)}{\alpha^2 T(t)}. \end{aligned}$$

Notice that the LHS depends only on x and the RHS depends only on t. For this to be possible, both sides must be equal to a constant $-\sigma$, say, i.e.

$$\frac{X''(x)}{X(x)} = -\sigma, \qquad \text{and} \qquad \frac{T'(t)}{\alpha^2 T(t)} = -\sigma,$$

$$\implies X''(x) + \sigma X(x) = 0, \qquad (4) \qquad\qquad T'(t) + \alpha^2 \sigma T(t) = 0. \qquad (5)$$

We now have two ODEs to solve instead of the original PDE.

If we can solve these ODEs to obtain $X(x)$ and $T(t)$ (subject to the boundary and initial conditions given by (2) and (3)) we can put these together to get a solution to the original PDE $u(x,t) = X(x)T(t)$. This method is called **Separation of Variables** and is applicable

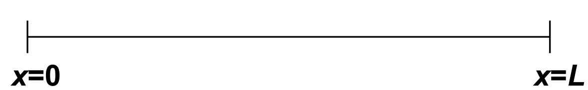

Figure 1: The temperature at any point x on the horizontal line between $x = 0$ and $x = L$ at time t is given by $u(x,t)$.

to a variety of PDEs.

We start by imposing the boundary conditions (2) on $u(x,t) = X(x)T(t)$:

$$u(0,t) = 0 \implies X(0)T(t) = 0 \implies \text{either} \quad X(0) = 0 \quad \text{or} \quad T(t) \equiv 0,$$
$$u(L,t) = 0 \implies X(L)T(t) = 0 \implies \text{either} \quad X(L) = 0 \quad \text{or} \quad T(t) \equiv 0.$$

If $T(t) \equiv 0$ then $u(x,t) \equiv 0$ for all x, t so we are not interested in this (trivial) solution. Therefore we take

$$X(0) = 0 \quad \text{and} \quad X(L) = 0 \tag{6}$$

to be boundary conditions of the ODE in x given by (4).

We now attempt to solve (4) subject to (6) by considering the three cases for σ: $\sigma = 0, \sigma < 0, \sigma > 0$. By applying the boundary conditions we can identify the possible values that σ can take.

$\underline{\sigma = 0}$

(4) becomes $X'' = 0 \implies X(x) = k_1 x + k_0$. Imposing the boundary conditions (6):

$$X(0) = 0 \implies k_0 = 0,$$
$$X(L) = 0 \implies k_1 = 0,$$

i.e. $X(x) \equiv 0$ giving $u(x,t) \equiv 0$ for all x, t, and, again, we are not interested in this trivial solution. Thus $\sigma \neq 0$.

$\underline{\sigma < 0}$

We let $\sigma = -\lambda^2$ $(\lambda > 0)$.

Then (4) becomes $X'' - \lambda^2 X = 0$, which is a 2nd order, linear, homogeneous ODE with constant coefficients.

Looking for a solution in the form $X = e^{rx}$ (see Chapter 0) gives the auxiliary equation

$$r^2 - \lambda^2 = 0 \implies r = \pm\lambda,$$

i.e. the general solution is $X(x) = \alpha_1 e^{\lambda x} + \alpha_2 e^{-\lambda x}$. It is helpful to write this solution in terms of

$$\cosh(\lambda x) = \frac{e^{\lambda x} + e^{-\lambda x}}{2}$$

and

$$\sinh(\lambda x) = \frac{e^{\lambda x} - e^{-\lambda x}}{2}$$

(it makes imposing the boundary conditions a little more straightforward).

We know that $e^{\lambda x} = \cosh(\lambda x) + \sinh(\lambda x)$ and $e^{-\lambda x} = \cosh(\lambda x) - \sinh(\lambda x)$, so

$$\begin{aligned} X(x) &= \alpha_1(\cosh(\lambda x) + \sinh(\lambda x)) + \alpha_2(\cosh(\lambda x) - \sinh(\lambda x)), \\ &= (\alpha_1 + \alpha_2)\cosh(\lambda x) + (\alpha_1 - \alpha_2)\sinh(\lambda x), \\ &= k_1\cosh(\lambda x) + k_2\sinh(\lambda x), \quad (k_1 = \alpha_1 + \alpha_2, k_2 = \alpha_1 - \alpha_2). \end{aligned}$$

Therefore, imposing the boundary conditions (6) gives:

$$\begin{aligned} X(0) = 0 &\implies k_1\cosh(0) + k_2\sinh(0) = 0 \implies k_1 = 0, \\ X(L) = 0 &\implies k_2\sinh(\lambda L) = 0. \end{aligned}$$

However, if $\lambda L > 0$ (which it is), then $\sinh(\lambda L) > 0$ (look at a plot of $\sinh(x)$ to convince yourself of this). Therefore we must have that $k_2 = 0$ and $X(x) \equiv 0$, i.e. we only have the trivial solution again. Thus σ cannot be negative: $\sigma \not< 0$.

<u>$\sigma > 0$</u>

Let $\sigma = \lambda^2$, $(\lambda > 0)$.

Then (4) becomes $X'' + \lambda^2 X = 0$. Looking for a solution in the form $X = e^{rx}$ gives the auxiliary equation

$$r^2 + \lambda^2 = 0 \implies r = \pm\lambda i,$$

i.e. the general solution is $X(x) = \alpha_1 e^{\lambda i x} + \alpha_2 e^{-\lambda i x}$. This time it is more convenient to write the solution in terms of

$$\cos(\lambda x) = \frac{e^{\lambda i x} + e^{-\lambda i x}}{2}$$

and

$$\sin(\lambda x) = \frac{e^{\lambda i x} - e^{-\lambda i x}}{2i}.$$

We know that $e^{\lambda i x} = \cos(\lambda x) + i\sin(\lambda x)$ and $e^{-\lambda i x} = \cos(\lambda x) - i\sin(\lambda x)$, thus

$$\begin{aligned} X(x) &= \alpha_1(\cos(\lambda x) + i\sin(\lambda x)) + \alpha_2(\cos(\lambda x) - i\sin(\lambda x)), \\ &= (\alpha_1 + \alpha_2)\cos(\lambda x) + i(\alpha_1 - \alpha_2)\sin(\lambda x), \\ &= k_1\cos(\lambda x) + k_2\sin(\lambda x), \quad (k_1 = \alpha_1 + \alpha_2, k_2 = i(\alpha_1 - \alpha_2)). \end{aligned}$$

Therefore, imposing the boundary conditions (6) gives:

$$\begin{aligned} X(0) = 0 &\implies k_1\cos(0) + k_2\sin(0) = 0 \implies k_1 = 0, \\ X(L) = 0 &\implies k_2\sin(\lambda L) = 0, \end{aligned}$$

so either $k_2 = 0$ (which we do not want as this will give the trivial solution again) or $\sin(\lambda L) = 0$, i.e.

$$\begin{aligned} \lambda L &= n\pi \quad \text{for} \quad n = 1, 2, 3, ...(n \nleq 0 \text{ because } \lambda, L > 0), \\ \lambda &= \frac{n\pi}{L}, \\ \lambda^2 &= \left(\frac{n\pi}{L}\right)^2. \end{aligned}$$

Since $\sigma = \lambda^2$, we must have $\sigma = \left(\frac{n\pi}{L}\right)^2$ to obtain a nontrivial solution to (4) and (6). We call these values of σ the **eigenvalues** of (4) and (6), and the corresponding solutions $X(x) = k_2 \sin\left(\frac{n\pi x}{L}\right)$ the eigenfunctions. Thus

$$X_n(x) = k_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, ...$$

are all solutions to (4) subject to (6).

Letting $\sigma = \left(\frac{n\pi}{L}\right)^2$ in the ODE for $T(t)$ (5) we have

$$\begin{aligned} T' + \left(\frac{\alpha n\pi}{L}\right)^2 T = 0,& \\ \implies \frac{dT}{dt} &= -\left(\frac{\alpha n\pi}{L}\right)^2 T, \\ \frac{1}{T}\frac{dT}{dt} &= -\left(\frac{\alpha n\pi}{L}\right)^2, \\ \int \frac{1}{T}\, dT &= -\left(\frac{\alpha n\pi}{L}\right)^2 \int dt, \\ \ln|T| &= -\left(\frac{\alpha n\pi}{L}\right)^2 t + \beta_1 \quad (\beta_1 \text{ constant}), \\ T(t) &= \beta_2 e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}, \quad n = 1, 2, 3, ... \quad (\beta_2 = \pm e^{\beta_1}). \end{aligned}$$

Hence,

$$T_n(t) = \beta_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}, \quad n = 1, 2, 3, ...$$

and

$$\begin{aligned} u_n(x, t) &= X_n(x) T_n(t), \\ &= k_n \beta_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \\ &= b_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right), \quad (b_n = k_n \beta_n). \end{aligned}$$

All $u_n(x,t)$ for $n = 1, 2, 3, ...$ will satisfy (1) and (2) so by the superposition principle, so does a linear combination of all of these u_n, i.e.

$$u(x,t) = \sum_{n=1}^{m} u_n(x,t), \tag{7}$$

$$= \sum_{n=1}^{m} b_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right) \quad (m, n \in \mathbb{N}), \tag{8}$$

where the b_n are chosen to satisfy the initial condition (3).

Notice that

$$u(x,0) = \sum_{n=1}^{m} b_n \sin\left(\frac{n\pi x}{L}\right)$$

so if $u(x,0) = f(x)$ is a linear combination of functions in the form $\sin\left(\frac{n\pi x}{L}\right)$ where $n \in \mathbb{N}$, we can read off the coefficients b_n.

Example: Find the separable solution of the heat equation (1) subject to the boundary conditions (2) and the initial condition

$$u(x,0) = f(x) \quad \text{where} \quad f(x) = 4\sin\left(\frac{3\pi x}{L}\right) - 7\sin\left(\frac{8\pi x}{L}\right).$$

Answer:

See Canvas for an animation of this solution.

In general, however, the initial condition $f(x)$ is unlikely to be given explicitly as a sum of functions of this kind. Therefore, it will not be possible to satisfy the initial condition with the finite sum (8). Instead, we must consider the infinite series

$$u(x,t)=\sum_{n=1}^{\infty} b_n \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2 t\right)\sin\left(\frac{n\pi x}{L}\right). \tag{9}$$

Then

$$u(x,0)=f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

and we need to choose the coefficients b_n such that this series converges to $f(x)$.

Thus if we can express $f(x)$ as an infinite sum of sine functions, we can identify the coefficients b_n. Notice that this is similar to the idea of a Taylor series, but terms in the series take a different form. A large class of functions can be represented by infinite sums of sine and cosine terms; we call these **Fourier series**.

2 Properties of the sine and cosine functions

Before we consider Fourier series, it is important to remind ourselves of certain properties of the sine and cosine functions.

2.1 Periodicity

A function $f(x)$ is periodic if $f(x+P)=f(x)$ for some period $P>0$ and all x in the domain. Note that the domain of f must contain $x+P$ whenever it contains x. By definition, f is also periodic with period nP where $n\in\mathbb{N}$, and so the smallest value of P for which $f(x+P)=f(x)$ for all x is called the **fundamental period** of f.

If f and g are two periodic functions with period P then any linear combination of f and g, $h_1(x)=\alpha_1 f(x)+\alpha_2 g(x)$ is also periodic with period P (α_1,α_2 constants).

Proof

$$\begin{aligned} h_1(x+P) &= \alpha_1 f(x+P)+\alpha_2 g(x+P),\\ &= \alpha_1 f(x)+\alpha_2 g(x),\\ &= h_1(x). \end{aligned}$$

Indeed, the sum (including an infinite sum) of any number of functions with period P is also periodic with period P.

Additionally, the product of two functions of period P, $h_2(x)=f(x)g(x)$ will also be periodic with period P.

Proof

$$\begin{aligned} h_2(x+P) &= f(x+P)g(x+P),\\ &= f(x)g(x),\\ &= h_2(x). \end{aligned}$$

Recall that the functions $\sin(x)$ and $\cos(x)$ are periodic with fundamental period 2π, and the functions $\sin(\alpha x)$ and $\cos(\alpha x)$ are periodic with period $2\pi/\alpha$ (to see this let $y = \alpha x$, then if $\sin(y)$ repeats itself every $y = 2\pi$, then it must repeat itself every $x = 2\pi/\alpha$).

In general, Fourier series consider functions of the form $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$, $n \in \mathbb{N}$. By the above, these are periodic with fundamental period $2\pi \times (L/n\pi) = 2L/n$.

Since any positive integer multiple of a period is also a period, the functions $\sin(n\pi x/L)$ and $\cos(m\pi x/L)$ will always have common period $2L$ for any $n, m \in \mathbb{N}$.

2.2 Orthogonality

The standard **inner product** (u, v) of two real-valued functions u and v on the interval $\alpha \leq x \leq \beta$ is given by

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)\, dx.$$

u and v are said to be **orthogonal** on $\alpha \leq x \leq \beta$ if their inner product $(u, v) = 0$, i.e. if

$$\int_{\alpha}^{\beta} u(x)v(x)\, dx = 0.$$

A set of functions is said to be mutually orthogonal if each (distinct) pair of functions within that set is orthogonal, i.e. if $f_i(x)$ and $f_j(x)$ are functions within a mutually orthogonal set, then $(f_i(x), f_j(x)) = 0$ for all i, j.

The functions $\sin(n\pi x/L)$ and $\cos(m\pi x/L)$, $(n, m \in \mathbb{N})$ form a mutually orthogonal set of functions on $-L \leq x \leq L$. In fact

$$\begin{aligned}
\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx &= 0 \quad \text{for all } m, n \in \mathbb{N}, \\
\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx &= \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n, \end{cases} \\
\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx &= \begin{cases} 0 & \text{if } m \neq n, \\ L & \text{if } m = n. \end{cases}
\end{aligned}$$

We can obtain these by directly integrating.

Example: Find

$$I = \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx, \quad m, n \in \mathbb{N}.$$

Answer:

The orthogonality properties given above are crucial in determining the Fourier series of a given function.

2.3 Even and odd functions

A function $f(x)$ is even if $f(x) = f(-x)$ and odd if $f(x) = -f(-x)$ for all x. Note that $\sin(x)$ is an odd function and $\cos(x)$ is an even function.

If $f(x)$ is even, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

If $f(x)$ is odd then,

$$\int_{-L}^{L} f(x)\,dx = 0.$$

Recall also that

- the product of two odd functions is an even function,
- the product of two even functions is an even function,
- the product of an even function and an odd function is an odd function,

(you can verify these statments yourself by directly appealing to the definitions of even and odd functions).

2.4 Integer multiples of π

Remember that

$$\begin{aligned} \sin(n\pi) &= 0 \quad \forall \quad n \in \mathbb{Z}, \\ \cos(n\pi) &= (-1)^n \quad \forall \quad n \in \mathbb{Z}, \end{aligned}$$

(plot $\sin(x), \cos(x)$ to convince yourself of this).

3 Determining the coefficients of a Fourier series

Let's assume that the function $f(x)$ has a convergent series expansion given by a Fourier series on $[-L, L]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right). \tag{10}$$

We can obtain the coefficients a_n and b_n by exploiting the orthogonality properties given in the previous section.

Firstly, to obtain a_0 we can simply integrate $f(x)$:

$$\begin{aligned} \int_{-L}^{L} f(x)\,dx &= \frac{a_0}{2}\int_{-L}^{L} dx + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\,dx + \sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)} dx, \\ &= \frac{a_0}{2}\Big[x\Big]_{-L}^{L} + \sum_{n=1}^{\infty} a_n \left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}, \\ &= \frac{a_0}{2}(L+L) + \sum_{n=1}^{\infty} a_n \left(\cancelto{0}{\frac{L}{n\pi}\sin(n\pi)} - \cancelto{0}{\frac{L}{n\pi}\sin(-n\pi)}\right), \\ &= a_0 L, \\ \implies a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,dx. \end{aligned} \tag{11}$$

To obtain the a_n we must multiply (10) by $\cos(m\pi x/L)$ and then integrate:

$$
\begin{aligned}
\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)\,dx &= \frac{a_0}{2}\cancelto{0}{\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)}\,dx \\
&\quad+\sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx \\
&\quad+\sum_{n=1}^{\infty} b_n \cancelto{0}{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)}\,dx, \\
&= a_m L \quad \text{(the only term that doesn't vanish is when } n=m\text{)}, \\
\implies a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx. \qquad (12)
\end{aligned}
$$

To obtain the b_n, we multiply (10) by $\sin(m\pi x/L)$ and then integrate:

$$
\begin{aligned}
\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,dx &= \frac{a_0}{2}\cancelto{0}{\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)}\,dx \\
&\quad+\sum_{n=1}^{\infty} a_n \cancelto{0}{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)}\,dx \\
&\quad+\sum_{n=1}^{\infty} b_n \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx, \\
&= b_m L \quad \text{(the only term that doesn't vanish is when } n=m\text{)}, \\
\implies b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \qquad (13)
\end{aligned}
$$

(11), (12), (13) are known as the **Euler-Fourier formulae** for the coefficients in a Fourier series. If a Fourier series does converge to a given function $f(x)$, we can use these formulae to obtain the coefficients of the series (assuming that we can evaluate the integrals). Notice that (11) is (12) with $n = 0$ so we may simply take (12) for $n \geq 0$ and (13) for $n \geq 1$.

Example: Find the Fourier series of

$$
f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}
$$

Answer:

Example: Find the Fourier series of

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi \end{cases}$$

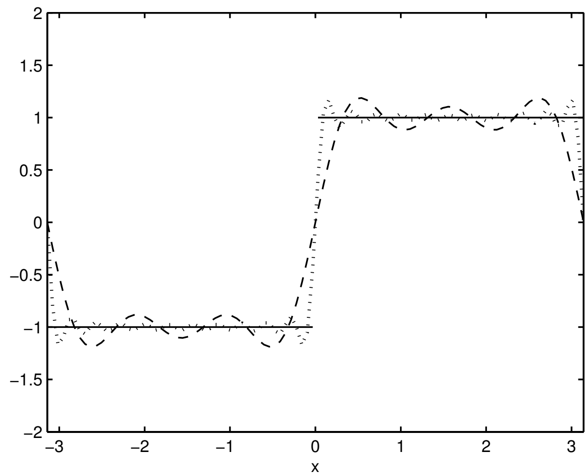

Figure 2: The solid line represents the function $f(x) = -1$ if $x \in (-\pi, 0)$ and $f(x) = 1$ if $x \in (0, \pi)$, while the dashed line represents its Fourier series up to $n = 5$ and the dotted line up to $n = 21$. The more terms we use in the Fourier series, the more accurate the approximation becomes.

Answer:

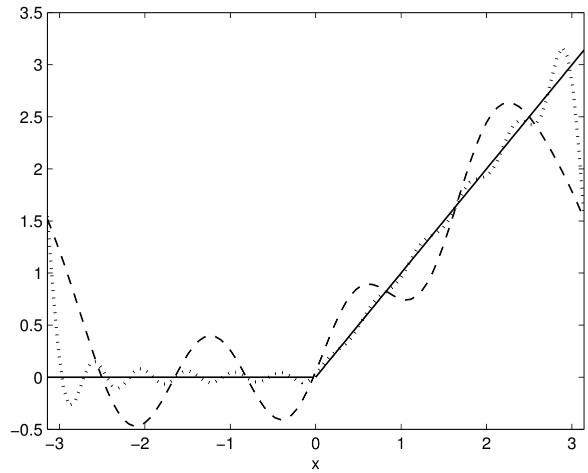

Figure 3: The solid line represents the function $f(x) = 0$ if $x \in (-\pi, 0)$ and $f(x) = x$ if $x \in (0, \pi)$, while the dashed line represents its Fourier series up to $n = 5$ and the dotted line up to $n = 11$. As in Figure 2, notice how the approximation becomes more accurate the more terms that are included.

4 Convergence of a Fourier series

The **Fourier Convergence Theorem** is given by the following.

Suppose that $f(x)$ and $f'(x)$ are piecewise continuous on the interval $-L \leq x \leq L$ and that $f(x)$ is defined outside the interval $-L \leq x \leq L$ so that it is periodic with period $2L$, then $f(x)$ has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

with the coefficients given by (11)-(13). The Fourier series converges to $f(x)$ at all points where f is continuous, and to $\left(\dfrac{f(x^+) + f(x^-)}{2}\right)$ at all points where $f(x)$ is discontinuous.

Recall that

- a function $f(x)$ is piecewise continuous on $a \leq x \leq b$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < ... < x_n = b$ such that
  - $f(x)$ is continuous on each (x_{i-1}, x_i), $i = 1, ..., n$,
  - $f(x)$ approaches a finite limit as x approaches the end points of an interval from within that interval.
- $f(a^+)$ is the limit as $x \to a$ from the right (if this exists).
- $f(a^-)$ is the limit as $x \to a$ from the left (if this exists).

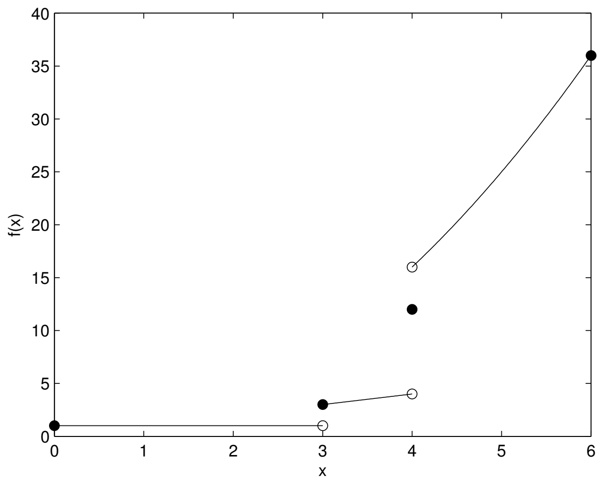

Figure 4: $f(x)$ as given by (14). This is a piecewise continuous function.

Example:

$$f(x) = \begin{cases} 1, & 0 \leq x < 3, \\ x, & 3 \leq x < 4, \\ 12, & x = 4, \\ x^2, & 4 < x \leq 6, \end{cases} \tag{14}$$

is piecewise continuous, see Figure 4.

Example:

$$f(x) = \frac{1}{x^2}, \quad -1 \leq x \leq 1, \tag{15}$$

is not piecewise continuous, see Figure 5.

The theorem provides sufficient conditions for convergence of a Fourier series, but not necessary conditions, i.e. functions not satisfying the conditions given in this theorem may also have convergent Fourier series. The proof of this theorem is fairly tricky and so is not included in this course (see advanced calculus textbooks for more details).

Example: To which function does the Fourier series for

$$f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi, \end{cases}$$

converge on $[-\pi, \pi]$?

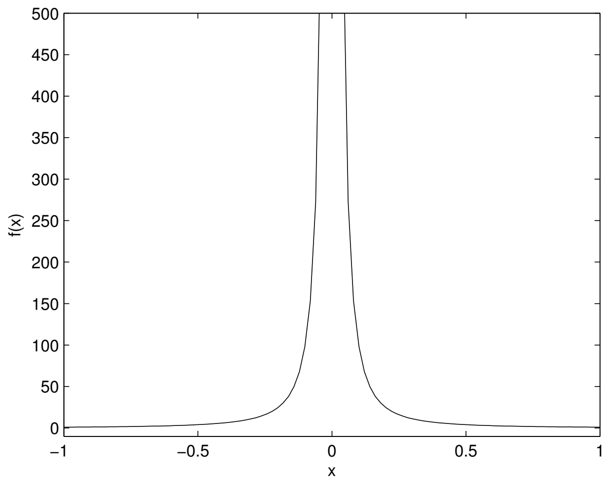

Figure 5: $f(x)$ as given by (15), i.e. $f(x) = \dfrac{1}{x^2}, \quad -1 \leq x \leq 1$. This function is not piecewise continuous.

Answer:

5 Approximating even and odd functions using Fourier series

If $f(x)$ and $f'(x)$ are piecewise continuous on $[-L, L]$ and $f(x)$ is an **even periodic function** with period $2L$, then $f(x)\cos(n\pi x/L)$ is even and $f(x)\sin(n\pi x/L)$ is odd. Since $f(x)\sin(n\pi x/L)$ is odd, all the coefficients $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx$ must be zero. Therefore, an even $f(x)$ has the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

i.e. an even function has a Fourier series that consists only of terms of even functions. We call this the **Fourier cosine series**.

If instead $f(x)$ is an **odd periodic function** of period $2L$, by the same reasoning, its Fourier series will only contain terms of odd functions:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

This is called the **Fourier sine series**.

Example:

In general, always check to see if the function that you are trying to approximate with a Fourier series is odd or even (remember that it may not be either) as this will allow you to set relevant coefficients to zero and significantly lower the amount of work you must do.