2DE/2DE3 Example sheet 3 solutions: Power series solutions of ODEs

1. *Find the intervals of convergence of*

   (a)
$$\sum_{n=0}^{\infty} \frac{3^n}{n!} x^n,$$

   We can use the Ratio Test:

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)!} x^{n+1} \frac{n!}{3^n x^n} \right|, \\
&= |x| \lim_{n\to\infty} \left| \frac{3}{n+1} \right|, \\
&= |x| \lim_{n\to\infty} \left(\frac{\frac{3}{n}}{1+\frac{1}{n}} \right), \\
&= 0.
\end{aligned}$$

   Hence, $L < 1$ for all x and the interval of convergence is $(-\infty, \infty)$.

   (b)
$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} (x+2)^n.$$

   We again use the Ratio Test:

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{(n+1)^2 (x+2)^{n+1} 2^n}{2^{n+1} n^2 (x+2)^n} \right|, \\
&= |x+2| \lim_{n\to\infty} \left(\frac{(n+1)^2}{2n^2} \right), \\
&= |x+2| \lim_{n\to\infty} \left(\frac{n^2+2n+1}{2n^2} \right), \\
&= |x+2| \lim_{n\to\infty} \left(\frac{1+\frac{2}{n}+\frac{1}{n^2}}{2} \right), \\
&= \frac{1}{2} |x+2| .
\end{aligned}$$

   Therefore the series converges if

$$\begin{aligned}
\frac{1}{2} |x+2| &< 1, \\
|x+2| &< 2, \\
-2 < x+2 &< 2, \\
-4 < x &< 0.
\end{aligned}$$

We also need to check what happens when $\frac{1}{2}|x+2| = 1$. This happens if $x = 0$ or $x = -4$.
If $x = 0$,

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x+2)^n = \sum_{n=0}^{\infty} n^2,$$

and this diverges.
If $x = -4$,

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n}(x+2)^n = \sum_{n=0}^{\infty} \frac{n^2}{2^n}(-2)^n = \sum_{n=0}^{\infty}(-1)^n n^2,$$

and this diverges (the individual terms do not tend to zero).
Therefore, the interval of convergence is $(-4, 0)$.

2. *Let*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}x^n \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} 2^{-n}x^{n-1}.$$

Find $f(x) + g(x)$ as a power series with a single sum.

We first rewrite $g(x)$ as a series from $n = 0$, i.e. $n' = n - 1 \iff n = n' + 1$:

$$\begin{aligned} g(x) &= \sum_{n'=0}^{\infty} 2^{-(n'+1)}x^{n'}, \\ &= \sum_{n=0}^{\infty} 2^{-(n+1)}x^n \quad \text{(dropping primes)}. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) + g(x) &= \sum_{n=0}^{\infty} \frac{1}{n+1}x^n + 2^{-(n+1)}x^n, \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} + 2^{-(n+1)} \right) x^n. \end{aligned}$$

3. *Find $f'(x)$ and $g'(x)$ given*

(a)

$$f(x) = \sum_{n=0}^{\infty}(-1)^n x^n,$$

$$f'(x) = \sum_{n=0}^{\infty}(-1)^n n x^{n-1},$$

which is the same as

$$\sum_{n=1}^{\infty}(-1)^n n x^{n-1}.$$

(b)

$$g(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

$$\begin{aligned} g'(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)!}x^{2n}, \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}. \end{aligned}$$

4. *In the above example, the series given by $f(x)$ and $g(x)$ converge to $f(x)=(1+x)^{-1}$ and $g(x)=\sin(x)$ (you can check this by deriving the Taylor series of these functions and showing that they are the same as the series above). Verify therefore that your answers for $f'(x)$ and $g'(x)$ are correct by differentiating $f(x)=(1+x)^{-1}$ and $g(x)=\sin(x)$ directly and calculating the first few terms of the Taylor series of $f'(x)$ and $g'(x)$ about $x=0$.*

(a)

$$\begin{aligned} f(x) &= (1+x)^{-1}, \\ f'(x) &= -(1+x)^{-2} = F(x), \\ f''(x) &= 2(1+x)^{-3} = F'(x), \\ f'''(x) &= -6(1+x)^{-4} = F''(x), \\ &\vdots \end{aligned}$$

Therefore the Taylor series for $f'(x)=F(x)$ is given by

$$\begin{aligned} F(x) &= F(0)+F'(0)x+\frac{F''(0)}{2!}x^2+... \\ &= -1+2x-3x^2+... \\ &= \sum_{n=0}^{\infty}(-1)^n n x^{n-1}. \end{aligned}$$

(b)

$$
\begin{aligned}
g(x) &= \sin(x), \\
g'(x) &= \cos(x) = G(x), \\
g''(x) &= -\sin(x) = G'(x), \\
g'''(x) &= -\cos(x) = G''(x), \\
&\vdots
\end{aligned}
$$

Therefore the Taylor series for $g'(x) = G(x)$ is given by

$$
\begin{aligned}
G(x) &= G(0) + G'(0)x + \frac{G''(0)}{2!}x^2 + ... \\
&= 1 - \frac{x^2}{2} + \frac{x^4}{24} - ... \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.
\end{aligned}
$$

5. *Rewrite the following as sums from $n = 0$ and simplify the resulting series wherever possible:*

(a)

$$
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},
$$

Let $n' = n - 2 \iff n = n' + 2$, then

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{n'=0}^{\infty} (n'+2)(n'+1)a_{n'+2}x^{n'}, \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.
\end{aligned}
$$

(b)

$$
\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1}.
$$

$$
\begin{aligned}
\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{k=1}^{\infty} k a_k x^k,\\
&= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{k=0}^{\infty} k a_k x^k,\\
&= \sum_{m'=0}^{\infty} (m+2)(m+1)a_{m+2} x^{m} + \sum_{k=0}^{\infty} k a_k x^k\\
&\quad (\text{using } m' = m-2 \iff m = m'+2),\\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + n a_n x^n,\\
&= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + n a_n]x^n.
\end{aligned}
$$

6. *Find the Taylor series of $f(x)$ around $x = x_0$ for the following functions and determine the interval of convergence in each case.*

(a)

$$f(x) = e^x, \quad x_0 = 0$$

$$
\begin{aligned}
f(x) &= e^x,\\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\end{aligned}
$$

Then the Ratio Test gives:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} \right|,\\
&= |x| \lim_{n\to\infty} \left(\frac{1}{n+1} \right),\\
&= 0
\end{aligned}
$$

for all x so the interval of convergence is $(-\infty, \infty)$.

(b)

$$f(x) = \ln(x), \quad x_0 = 1.$$

$$
\begin{aligned}
f(x-x_0) &= f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \ldots\\
\implies f(x-1) &= 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \ldots,\\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}.
\end{aligned}
$$

Then the Ratio Test gives:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1} \frac{n}{(-1)^{n+1}(x-1)^{n}} \right|, \\
&= |x-1| \lim_{n\to\infty} \left(\frac{n}{n+1} \right), \\
&= |x-1| .
\end{aligned}
$$

This series therefore converges if

$$
\begin{aligned}
|x-1| &< 1, \\
-1 < x-1 &< 1, \\
0 < x &< 2.
\end{aligned}
$$

We also need to check what happens when $|x-1| = 1$, i.e. $x = 0, 2$.
If $x = 0$,

$$
\begin{aligned}
f(0-1) &= f(-1), \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{n}}{n}, \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}, \\
&= \sum_{n=1}^{\infty} -\frac{1}{n},
\end{aligned}
$$

which diverges.
If $x = 2$,

$$
\begin{aligned}
f(2-1) &= f(1), \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^{n}}{n}, \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},
\end{aligned}
$$

which converges.
Therefore the interval of convergence is $(0, 2]$.

7. *Determine the a_n (in terms of a_0) such that*

$$
\sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0,
$$

is satisfied.

To shift the sums so that both sums are in terms of x^n, let $n = n' + 1 \iff n' = n - 1$

in the first sum. Then the above is equivalent to

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty}a_nx^n = 0,$$
$$\sum_{n=0}^{\infty}[(n+1)a_{n+1} + 2a_n]x^n = 0.$$

Matching coefficients of x^n gives us

$$(n+1)a_{n+1} = -2a_n,$$
$$a_{n+1} = \frac{-2}{n+1}a_n.$$

This will generate the coefficients

$$a_{n+1} = \frac{(-2)^{n+1}}{(n+1)!}a_0 \quad \text{or} \quad a_n = \frac{(-2)^n}{n!}a_0.$$

8. *Show that the power series solution about $x = 0$ to*

$$2y'' + (x+1)y' + y = 0$$

is given by

$$y = a_0\left(1 - \frac{x^2}{4} + \frac{x^3}{24} + ...\right) + a_1\left(x - \frac{x^2}{4} - \frac{x^3}{8} + ...\right),$$

where a_0 and a_1 are constants.

Because $x = 0$ is an ordinary point of the equation, we look for a solution in the form

$$y = \sum_{n=0}^{\infty}a_nx^n,$$
$$\implies y' = \sum_{n=1}^{\infty}na_nx^{n-1},$$
$$y'' = \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}.$$

Substituting these into the equation gives

$$2\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2} + (x+1)\sum_{n=1}^{\infty}na_nx^{n-1} + \sum_{n=0}^{\infty}a_nx^n = 0,$$
$$\sum_{n=2}^{\infty}2n(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty}na_nx^n + \sum_{n=1}^{\infty}na_nx^{n-1} + \sum_{n=0}^{\infty}a_nx^n = 0.$$

Shift the indices of the sums so that all are in terms of powers of x^n, i.e. let $n' = n - 2 \iff n = n' + 2$ in the first sum and $n^* = n - 1 \iff n = n^* + 1$ in the third sum to give

$$\sum_{n'=0}^{\infty} 2(n'+2)(n'+1)a_{n'+2}x^n + \sum_{n=1}^{\infty} na_n x^n + \sum_{n^*=0}^{\infty}(n^*+1)a_{n^*+1}x^{n^*} + \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

(dropping $'$ and $*$).

(Note that in the second sum we can shift the start point to $n = 0$). Then gather the sums into one:

$$\sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2} + na_n + (n+1)a_{n+1} + a_n]x^n = 0,$$

$$\sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + (n+1)a_n]x^n = 0.$$

Matching coefficients of x^0 (i.e. looking at $n = 0$):

$$\begin{aligned} 2 \times 2 \times 1a_2 + a_1 + a_0 &= 0, \\ 4a_2 + a_1 + a_0 &= 0, \\ a_2 &= -\frac{(a_1 + a_0)}{4}. \end{aligned}$$

Matching the coefficients of x (i.e. looking at $n = 1$):

$$\begin{aligned} 2 \times 3 \times 2a_3 + 2a_2 + 2a_1 &= 0, \\ a_3 &= -\frac{(a_2 + a_1)}{6}, \\ &= -\frac{1}{6}\left(-\frac{(a_1 + a_0)}{4} + a_1\right), \\ &= \frac{a_1 + a_0}{24} - \frac{a_1}{6}, \\ &= \frac{-3a_1 + a_0}{24}. \end{aligned}$$

$$\begin{aligned} \implies y &= \sum_{n=0}^{\infty} a_n x^n, \\ &= a_0 + a_1 x - \left(\frac{a_1 + a_0}{4}\right)x^2 + \left(\frac{-3a_1 + a_0}{24}\right)x^3 + ..., \\ &= a_0\left(1 - \frac{x^2}{4} + \frac{x^3}{24} + ...\right) + a_1\left(x - \frac{x^2}{4} - \frac{x^3}{8} + ...\right). \end{aligned}$$

9. *Show that the power series solution about $x = 1$ to*

$$(x^2 - 2x)y'' + 2y = 0$$

is given by

$$y = a_0\left(1 + (x-1)^2 + \frac{(x-1)^4}{3} + ...\right) + a_1\left((x-1) + \frac{(x-1)^3}{3} + \frac{2(x-1)^5}{15} + ...\right),$$

where a_0 and a_1 are constants. [Hint: express $x^2 - 2x$ in terms of $x - 1$]. Use the Ratio Test on the recurrence relation to show that the series converges in an interval around $x = 1$.

Because $x = 1$ is an ordinary point of the equation, we look for a solution in the form

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x-1)^n, \\ \implies y' &= \sum_{n=1}^{\infty} a_n n (x-1)^{n-1}, \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}. \end{aligned}$$

Substitute these into the equation:

$$\begin{aligned} (x^2-2x)\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + 2\sum_{n=0}^{\infty} a_n(x-1)^n &= 0, \\ [(x-1)^2 - 1]\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + 2\sum_{n=0}^{\infty} a_n(x-1)^n &= 0, \\ \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^n - \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} 2a_n(x-1)^n &= 0, \\ \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=0}^{\infty} 2a_n(x-1)^n &= 0, \\ \sum_{n=0}^{\infty}[n(n-1)a_n - (n+2)(n+1)a_{n+2} + 2a_n](x-1)^n &= 0, \\ \sum_{n=0}^{\infty}[(n(n-1)+2)a_n - (n+2)(n+1)a_{n+2}](x-1)^n &= 0. \end{aligned}$$

Match the coefficients of $(x-1)^n$:

$$\begin{aligned}
n &= 0, & 2a_0 - 2a_2 = 0 &\implies a_2 = a_0,\\
n &= 1, & 2a_1 - 6a_3 = 0 &\implies a_3 = \frac{a_1}{3},\\
n &= 2, & 4a_2 - 12a_4 = 0 &\implies a_4 = \frac{a_2}{3} = \frac{a_0}{3},\\
n &= 3, & 8a_3 - 20a_5 = 0 &\implies a_5 = \frac{2a_3}{5} = \frac{2a_1}{15},\\
&\vdots & \vdots &
\end{aligned}$$

Hence

$$y = a_0\left(1 + (x-1)^2 + \frac{(x-1)^4}{3} + ...\right) + a_1\left((x-1) + \frac{(x-1)^3}{3} + \frac{2(x-1)^5}{15} + ...\right).$$

In general,

$$\begin{aligned}
a_{n+2} &= \frac{(n(n-1)+2)}{(n+2)(n+1)}a_n,\\
&= \frac{n^2-n+2}{n^2+3n+2}a_n,
\end{aligned}$$

and we can use this to check that the series converge by the Ratio Test:

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{a_{n+2}(x-1)^{n+2}}{a_n(x-1)^n}\right|,\\
&= (x-1)^2\lim_{n\to\infty}\left|\frac{n^2-n+2}{n^2+3n+2}\right|,\\
&= (x-1)^2\lim_{n\to\infty}\left|\frac{1-\frac{1}{n}+\frac{2}{n^2}}{1+\frac{3}{n}+\frac{2}{n^2}}\right|,\\
&= (x-1)^2.
\end{aligned}$$

Hence the series converge if

$$\begin{aligned}
(x-1)^2 &< 1,\\
-1 < x-1 &< 1,\\
0 < x &< 2.
\end{aligned}$$

This confirms that both series should converge in an interval around $x = 1$.

10. *Find the solution of the previous question if* $y(1) = 2$ *and* $y'(1) = 4$.

$$\begin{aligned}
y(1) &= a_0 = 2,\\
y'(1) &= a_1 = 4.
\end{aligned}$$

Therefore

$$y(x) = 2\left(1 + (x-1)^2 + \frac{(x-1)^4}{3} + ...\right) + 4\left((x-1) + \frac{(x-1)^3}{3} + \frac{2(x-1)^5}{15} + ...\right).$$

11. *Find the general solution of an inhomogeneous version of the above equation (with general initial conditions):*

$$(x^2 - 2x)y'' + 2y = x^2 - 2x + 1$$

[Hint: the only thing that changes is the RHS so you can use most of your working from before. The particular solution will also be a series solution].

We now have

$$\begin{aligned}\sum_{n=0}^{\infty}[(n(n-1)+2)a_n - (n+2)(n+1)a_{n+2}](x-1)^n &= x^2 - 2x + 1, \\ &= (x-1)^2.\end{aligned}$$

Matching the coefficients:

$$\begin{aligned}
n &= 0, & 2a_0 - 2a_2 &= 0 \implies a_2 = a_0, \\
n &= 1, & 2a_1 - 6a_3 &= 0 \implies a_3 = \frac{a_1}{3}, \\
n &= 2, & 4a_2 - 12a_4 &= 1 \implies a_4 = -\frac{1}{12} + \frac{a_2}{3} = -\frac{1}{12} + \frac{a_0}{3}, \\
n &= 3, & 8a_3 - 20a_5 &= 0 \implies a_5 = \frac{2a_3}{5} = \frac{2a_1}{15}, \\
n &= 4, & 14a_4 - 30a_6 &= 0 \implies a_6 = \frac{7a_4}{15} = \frac{7}{15}\left(\frac{a_0}{3} - \frac{1}{12}\right), \\
&\vdots & &\vdots
\end{aligned}$$

Hence,

$$\begin{aligned}y(x) = a_0&\left(1 + (x-1)^2 + \frac{(x-1)^4}{3} + ...\right) + a_1\left((x-1) + \frac{(x-1)^3}{3} + \frac{2(x-1)^5}{15} + ...\right) \\ &- \frac{(x-1)^4}{12} - \frac{7(x-1)^6}{180} + ...,\end{aligned}$$

and the particular solution must be

$$y_p(x) = -\frac{(x-1)^4}{12} - \frac{7(x-1)^6}{180} + ...$$

12. *Show that the power series solution about $x = 0$ to*

$$y'' + xy' + e^x y = 0, \quad y(0) = 0, \quad y'(0) = -1$$

is given by

$$y = -\left(x - \frac{x^3}{3} - \frac{x^4}{12} + ...\right).$$

[Hint: can you express e^x as a power series?]

First we need to express e^x as a power series. We use its Taylor series expansion for this:

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... \\ \implies e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!}. \end{aligned}$$

Then substituting this and $y = \sum_{n=0}^{\infty} a_n x^n$ ($x = 0$ is an ordinary point) into our equation, we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} a_n x^n = 0.$$

To handle the multiplicative term, it's easiest to write out the first few terms in each sum, e.g.

$$\begin{aligned} &2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + ... \\ &+ a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + ... \\ &+ \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...\right)\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right) = 0. \end{aligned}$$

Then, matching coefficients we have:

$$\begin{aligned} x^0 :&\quad 2a_2 + a_0 = 0 \implies a_2 = -\frac{a_0}{2}, \\ x^1 :&\quad 6a_3 + a_1 + a_1 + a_0 = 0 \implies 6a_3 + 2a_1 + a_0 = 0 \implies a_3 = -\frac{a_1}{3} - \frac{a_0}{6}, \\ x^2 :&\quad 12a_4 + 2a_2 + a_2 + a_1 + \frac{a_0}{2} = 0 \implies 12a_4 + 3a_2 + a_1 + \frac{a_0}{2} = 0 \\ &\quad \implies a_4 = \frac{-1}{12}\left(3a_2 + a_1 + \frac{a_0}{2}\right) = \frac{a_0}{12} - \frac{a_1}{12}, \\ x^3 :&\quad 20a_5 + 3a_3 + a_3 + a_2 + \frac{a_1}{2} + \frac{a_0}{6} = 0 \implies 20a_5 = -\left(4a_3 + a_2 + \frac{a_1}{2} + \frac{a_0}{6}\right), \end{aligned}$$

etc.

This gives us

$$y = a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + ...\right) + a_1\left(x - \frac{x^3}{3} - \frac{x^4}{12} + ...\right).$$

Imposing the initial conditions tells us

$$y(0) = a_0 = 0,$$
$$y'(0) = a_1 = -1,$$

so that

$$y = -\left(x - \frac{x^3}{3} - \frac{x^4}{12} + ...\right).$$

13. *Show that the power series solution about $x = 0$ to*

$$(x+2)x^2y'' - xy' + (1+x)y = 0, \quad x > 0,$$

is given by

$$y = \alpha_1\left(x - \frac{x^2}{3} + \frac{x^3}{10} - \frac{x^4}{30} + ...\right) + \alpha_2\left(x^{\frac{1}{2}} - \frac{3x^{\frac{3}{2}}}{4} + \frac{7x^{\frac{5}{2}}}{32} - \frac{133x^{\frac{7}{2}}}{1920} + ...\right),$$

where α_1 and α_2 are constants.

We first need to check whether $x = 0$ is an ordinary point or a singular point. Rearranging the equation into the standard form, we have:

$$y'' - \frac{1}{x(x+2)}y' + \frac{(1+x)}{x^2(x+2)}y = 0,$$

so that

$$p(x) = \frac{-1}{x(x+2)} \implies xp(x) = \frac{-1}{x+2},$$
$$q(x) = \frac{1+x}{x^2(x+2)} \implies x^2q(x) = \frac{1+x}{x+2}.$$

Neither $p(x)$ nor $q(x)$ are analytic at $x = 0$ so this is a singular point. However, both $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ therefore this is a regular singular point, so we must look for a solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$
$$\implies y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1},$$
$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

It is now easier to proceed with the original equation, rather than the standard form. Thus, substituting the above into the original equation gives

$$(x+2)x^2\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2} - x\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1} + (1+x)\sum_{n=0}^{\infty}a_nx^{n+r} = 0,$$

$$\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r+1} + \sum_{n=0}^{\infty}2a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty}a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r+1} = 0.$$

We next want all sums to be in terms of x^{n+r}, so let $n' = n+1$ in the first and final sums (and drop primes):

$$\sum_{n=1}^{\infty}a_{n-1}(n+r-1)(n+r-2)x^{n+r} + \sum_{n=0}^{\infty}2a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty}a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty}a_nx^{n+r} + \sum_{n=1}^{\infty}a_{n-1}x^{n+r} = 0.$$

Next, for the middle three sums, take the $n = 0$ terms outside their sums so that all sums can then be combined:

$$2a_0r(r-1)x^r - a_0rx^r + a_0x^r + \sum_{n=1}^{\infty}\left(a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) - a_n(n+r) + a_n + a_{n-1}\right)x^{n+r} = 0,$$

$$[2r(r-1) - r + 1]a_0x^r + \sum_{n=1}^{\infty}\left[((n+r-1)(n+r-2)+1)a_{n-1} + (2(n+r)(n+r-1) - (n+r) + 1)a_n\right]x^{n+r} = 0.$$

Setting the coefficients of x^r to be zero to match the RHS we see that $2r^2 - 3r + 1 = 0$, i.e. $r = 1$ or $r = \frac{1}{2}$.

Setting the coefficients of x^{n+r} to be zero gives

$$\begin{aligned} a_n &= -\frac{(n+r-1)(n+r-2)+1}{2(n+r)(n+r-1)-(n+r)+1}a_{n-1}, \\ &= -\frac{(n+r-1)(n+r-2)+1}{(n+r-1)(2(n+r)-1)}a_{n-1}, \\ &= -\frac{(n+r-1)(n+r-2)+1}{(n+r-1)(2n+2r-1)}a_{n-1}. \end{aligned}$$

Let $r = 1$, then

$$\begin{aligned} a_1 &= \frac{-a_0}{3}, \\ a_2 &= \frac{-3a_1}{10} = \frac{a_0}{10}, \\ a_3 &= \frac{-7a_2}{21} = \frac{-7}{21}\left(\frac{a_0}{10}\right) = \frac{-a_0}{30}, \\ &\vdots \end{aligned}$$

Let $r = \frac{1}{2}$, then

$$\begin{aligned} a_1 &= \frac{-3a_0}{4}, \\ a_2 &= \frac{-\frac{7}{4}a_1}{6} = \frac{7a_0}{32}, \\ a_3 &= \frac{-\frac{19}{4}a_2}{15} = \frac{-133a_0}{1920}, \\ &\vdots \end{aligned}$$

Hence,

$$\begin{aligned} y &= \alpha_1 x\left(1 - \frac{x}{3} + \frac{x^2}{10} - \frac{x^3}{30} + ...\right) + \alpha_2 x^{\frac{1}{2}}\left(1 - \frac{3x}{4} + \frac{7x^2}{32} - \frac{133x^3}{1920} + ...\right), \\ &= \alpha_1\left(x - \frac{x^2}{3} + \frac{x^3}{10} - \frac{x^4}{30} + ...\right) + \alpha_2\left(x^{\frac{1}{2}} - \frac{3x^{\frac{3}{2}}}{4} + \frac{7x^{\frac{5}{2}}}{32} - \frac{133x^{\frac{7}{2}}}{1920} + ...\right). \end{aligned}$$

Notice that we must replace a_0 with α_1 and α_2 because the a_0 in the two series may differ.

14. *Show that the power series solution about $x = 0$ to*

$$3x^2y'' + 2xy' + x^2y = 0, \quad x > 0,$$

is given by

$$y = \alpha_1 x^{\frac{1}{3}}\left(1 - \frac{x^2}{14} + \frac{x^4}{728} + ...\right) + \alpha_2\left(1 - \frac{x^2}{10} + \frac{x^4}{440} + ...\right),$$

where α_1 and α_2 are constants.

We first need to check whether $x = 0$ is an ordinary point or a singular point. Rearranging the equation into the standard form, we have:

$$y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0,$$

so that

$$\begin{aligned} p(x) &= \frac{2}{3x} \implies xp(x) = \frac{2}{3}, \\ q(x) &= \frac{1}{3} \implies x^2q(x) = \frac{x^2}{3}. \end{aligned}$$

$p(x)$ is not analytic at $x = 0$ so this is a singular point. However, both $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ therefore this is a regular singular point, so we must look for a solution in the form

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\ \implies y' &= \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}, \\ y'' &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}. \end{aligned}$$

It is now easier to proceed with the original equation, rather than the standard form. Thus, substituting the above into the original equation gives

$$3x^2\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2}+2x\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1}+x^2\sum_{n=0}^{\infty}a_nx^{n+r}=0,$$

$$\sum_{n=0}^{\infty}3a_n(n+r)(n+r-1)x^{n+r}+\sum_{n=0}^{\infty}2a_n(n+r)x^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}=0.$$

We next want all sums to be in terms of x^{n+r}, so let $n' = n+2$ in the final sum (and drop primes):

$$\sum_{n=0}^{\infty}3a_n(n+r)(n+r-1)x^{n+r}+\sum_{n=0}^{\infty}2a_n(n+r)x^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0.$$

Next, take the $n = 0$ and $n = 1$ terms from the first and second sums outside the sum so that all sums can be combined:

$$\begin{aligned}&3a_0r(r-1)x^r+3a_1(r+1)rx^{r+1}+2a_0rx^r+2a_1(r+1)x^{r+1}\\&\quad+\sum_{n=2}^{\infty}\left(3a_n(n+r)(n+r-1)+2a_n(n+r)+a_{n-2}\right)x^{n+r}=0.\end{aligned}$$

Matching the coefficients of x^r:

$$\begin{aligned}3a_0r(r-1)+2a_0r&=0,\\3r^2-3r+2r&=0\quad(a_0\neq 0),\\3r^2-r&=0,\\r(3r-1)&=0\implies r=0\quad\text{or } r=\frac{1}{3}.\end{aligned}$$

Matching the coefficients of x^{r+1} when $r = 0$:

$$2a_1=0\implies a_1=0.$$

Matching the coefficients of x^{r+1} when $r = \frac{1}{3}$:

$$3a_1\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)+2a_1\left(\frac{4}{3}\right)=0\implies 4a_1=0\implies a_1=0.$$

Matching the coefficients of x^{n+r} for general r:

$$\begin{aligned}a_n&=\frac{-a_{n-2}}{3(n+r)(n+r-1)+2(n+r)},\\&=\frac{-a_{n-2}}{(n+r)(3n+3r-1)}.\end{aligned}$$

Hence, $a_3 = a_5 = a_7 = ... = 0$ for either value of r.

If $r = 0$,

$$\begin{aligned} a_2 &= \frac{-a_0}{2 \times 5} = \frac{-a_0}{10}, \\ a_4 &= \frac{-a_2}{4 \times 11} = \frac{-a_2}{44} = \frac{a_0}{440}, \\ &\vdots \end{aligned}$$

If $r = \frac{1}{3}$,

$$\begin{aligned} a_2 &= \frac{-a_0}{\frac{7}{3} \times 6} = \frac{-a_0}{14}, \\ a_4 &= \frac{-a_2}{\frac{13}{3} \times 12} = \frac{-a_2}{52} = \frac{a_0}{728}, \\ &\vdots \end{aligned}$$

Hence,

$$y = \alpha_1 x^{\frac{1}{3}} \left(1 - \frac{x^2}{14} + \frac{x^4}{728} + ...\right) + \alpha_2 \left(1 - \frac{x^2}{10} + \frac{x^4}{440} + ...\right)$$

Notice that, as in the previous question, we replace a_0 with α_1 and α_2 because the a_0 in the two series may differ.