

University of Birmingham  
School of Mathematics  
**1AC Algebra and Combinatorics**  
Problem Sheet 4: Algebra part

You should carefully write out your solutions to all the questions below.

Ensure you have read and understood the Canvas assignment for the problem sheet for instructions about submitting solutions to SUM questions.

**AQ1.** (SUM)

- (a) (i) Calculate the multiplication table of  $\mathbb{Z}_8$ .  
*You may wish to use the notation  $\bar{a}$  rather than  $[a]_8$  for elements of  $\mathbb{Z}_8$  and it would be ok for you to omit the row and column for  $[0]_8$ .*
- (ii) Determine all  $x \in \mathbb{Z}_8$  for which there exists  $x^{-1} \in \mathbb{Z}_8$  such that  $x \cdot x^{-1} = [1]_8$ .  
*You do not need to write too much for (a)(ii) and just explain how this can be determined from the multiplication table.*
- (b) Let  $n \in \mathbb{N}$ , and let  $x, y, z \in \mathbb{Z}_n$ .
- (i) Suppose that  $x + z = y + z$ . Prove that  $x = y$ .
- (ii) Prove that  $x \cdot (y \cdot z) + (y \cdot x) \cdot x = x \cdot (y \cdot (z + x))$ .

**AQ2.** (a) Let  $n, p \in \mathbb{N}$ . Suppose that  $p$  is an odd prime and that  $p \mid n^2 + 1$ . Prove that  $p \equiv 1 \pmod{4}$ .

- (b) Show that there are infinitely many primes  $p$  with  $p \equiv 1 \pmod{4}$ .

*This question is quite challenging, so you're likely to want to look at the hints that will be put on canvas.*

**AQ3.** Determine the cycle notation, cycle shape and order of each of the following permutations in  $S_9$ .

(a)  $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 4 & 1 & 8 & 9 & 3 & 2 & 6 \end{pmatrix}$

(b)  $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 6 & 9 & 4 & 7 & 1 & 2 \end{pmatrix}$

Please turn over

**AQ4.** (SUM) Let  $f = (1546)(37) \in S_7$  and  $g = (137)(245) \in S_7$  be permutations given in cycle notation.

Calculate the following permutations giving your solution in cycle notation.

(a)  $f \circ g$

(f)  $g^{-1}$

(b)  $g \circ f$

(g)  $g^{-1} \circ f^{-1}$

(c)  $f^2$

(h)  $(f \circ g)^{-1}$

(d)  $g^3$

(i) Find  $h \in S_7$  such that  $f \circ h = g$

(e)  $f^{-1}$

(j) Find  $k \in S_7$  such that  $k \circ f = g$