LA - Sum1

Wednesday, 5 November 2025 19:12

1a) $\underline{u}_{k+1} = \underline{v}_{k+1} - \sum_{j=1}^{k} \frac{\langle \underline{v}_{k+1}, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j$

$$= \underline{v}_{k+1} - \left(\frac{\langle \underline{v}_{k+1}, \underline{u}_1 \rangle}{\langle \underline{u}_1, \underline{u}_1 \rangle} \underline{u}_1 + \frac{\langle \underline{v}_{k+1}, \underline{u}_2 \rangle}{\langle \underline{u}_2, \underline{u}_2 \rangle} \underline{u}_2 + \dots + \frac{\langle \underline{v}_{k+1}, \underline{u}_k \rangle}{\langle \underline{u}_k, \underline{u}_k \rangle} \underline{u}_k \right)$$

As $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space, $\langle \underline{u}, \underline{v} \rangle \in \mathbb{R} \ \forall \ \underline{u}, \underline{v} \in V$

$\Rightarrow \frac{\langle \underline{u}, \underline{v} \rangle}{\langle \underline{u}', \underline{v}' \rangle} \in \mathbb{R} \ \forall \ \underline{u}, \underline{v}, \underline{u}', \underline{v}' \in V.$

$\Rightarrow$ we can let $\frac{\langle \underline{v}_{k+1}, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} = a_j \in \mathbb{R}$, for $j = 1, 2, \dots, k$.

This allows us to rewrite our equation for $\underline{u}_{k+1}$ as follows:

$\underline{u}_{k+1} = \underline{v}_{k+1} - (a_1 \underline{u}_1 + a_2 \underline{u}_2 + \dots + a_k \underline{u}_k)$, where $\frac{\langle \underline{v}_{k+1}, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} = a_j \in \mathbb{R}$ for $j = 1, 2, \dots, k$.

$\Rightarrow \underline{v}_{k+1} = a_1 \underline{u}_1 + a_2 \underline{u}_2 + \dots + a_k \underline{u}_k + \underline{u}_{k+1}$

Here we have shown that $\underline{v}_{k+1}$ can be expressed as a linear combination of the vectors in the set $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_{k+1}\}$

Therefore by the definition of a spanning set, we have $\underline{v}_{k+1} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_{k+1}\}$

1b) Let $P(m)$ for $m \in \{1, 2, \dots, n\}$ be the statement

$\text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m\} = \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m\}$

We first establish whether $P(m)$ holds for $m=1$.

By the Gram-Schmidt procedure, we have $\underline{u}_1 = \underline{v}_1 \Rightarrow \text{span}\{\underline{u}_1\} = \text{span}\{\underline{v}_1\}$.

Now assume $P(k)$ holds for some $k \in \{1, 2, \dots, n-1\}$, i.e.

$\text{Span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\} = \text{Span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$.

We now wish to show that $P(k+1)$ holds for some $k \in \{1, 2, \dots, n-1\}$ (assuming that $P(k)$ holds). To show this, it is sufficient to prove that for some vector $\underline{w} \in V$,

$\underline{w} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{u}_{k+1}\} \iff \underline{w} \in \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{v}_{k+1}\}$

We first prove that $\underline{w} \in \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{v}_{k+1}\} \Rightarrow \underline{w} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{u}_{k+1}\} \ \forall \ \underline{w} \in \text{Span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{v}_{k+1}\}$

Let $\underline{w} \in \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{v}_{k+1}\}$

$\Rightarrow \underline{w} \in \text{Span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{v}_{k+1}\}$, from our assumption that $P(k)$ holds.

$\Rightarrow \underline{w} = b_1 \underline{u}_1 + b_2 \underline{u}_2 + \dots + b_k \underline{u}_k + b_{k+1} \underline{v}_{k+1}$, where $b_i \in \mathbb{R}$ for $i = 1, 2, \dots, k, k+1$

In question 1a we showed that $\underline{v}_{k+1}$ can be written as a linear combination of the elements in $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_{k+1}\}$ such that,

$\underline{v}_{k+1} = a_1 \underline{u}_1 + a_2 \underline{u}_2 + \dots + a_k \underline{u}_k + \underline{u}_{k+1}$, where $\frac{\langle \underline{v}_{k+1}, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} = a_j \in \mathbb{R}$ for $j = 1, 2, \dots, k$.

$\Rightarrow \underline{w} = b_1 \underline{u}_1 + b_2 \underline{u}_2 + \dots + b_k \underline{u}_k + b_{k+1} (a_1 \underline{u}_1 + a_2 \underline{u}_2 + \dots + a_k \underline{u}_k + \underline{u}_{k+1})$

$= (b_1 + b_{k+1} a_1) \underline{u}_1 + (b_2 + b_{k+1} a_2) \underline{u}_2 + \dots + (b_k + b_{k+1} a_k) \underline{u}_k + b_{k+1} \underline{u}_{k+1}$, which is a linear combination of vectors in the set $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{u}_{k+1}\}$

$\Rightarrow \underline{w} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{u}_{k+1}\}$

We now prove that $\underline{w} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{u}_{k+1}\} \Rightarrow \underline{w} \in \text{Span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{v}_{k+1}\} \quad \forall \, \underline{w} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{u}_{k+1}\}$

Let $\underline{w} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{u}_{k+1}\}$

$\Rightarrow \underline{w} \in \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{u}_{k+1}\}$, from our assumption that $P(k)$ holds.

$\Rightarrow \underline{w} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k + c_{k+1}\underline{u}_{k+1}$ for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, k, k+1$

As given above, we have,

$\underline{v}_{k+1} = a_1\underline{u}_1 + a_2\underline{u}_2 + \dots + a_k\underline{u}_k + \underline{u}_{k+1}$

$\Rightarrow \underline{u}_{k+1} = \underline{a} + \underline{v}_{k+1}$, where $\underline{a} = -a_1\underline{u}_1 - a_2\underline{u}_2 - \dots - a_k\underline{u}_k$

We can see that $\underline{a} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\} \Rightarrow \underline{a} \in \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$

$\Rightarrow \underline{a}$ can be expressed as a linear combination of the vectors in $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$.

$\Rightarrow \underline{a} = d_1\underline{v}_1 + d_2\underline{v}_2 + \dots + d_k\underline{v}_k$, for some $d_j \in \mathbb{R}$, $j = 1, 2, \dots, k$

$\Rightarrow \underline{u}_{k+1} = d_1\underline{v}_1 + d_2\underline{v}_2 + \dots + d_k\underline{v}_k + \underline{v}_{k+1}$

$\Rightarrow \underline{w} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k + c_{k+1}(d_1\underline{v}_1 + d_2\underline{v}_2 + \dots + d_k\underline{v}_k + \underline{v}_{k+1})$

$= (c_1 + c_{k+1}d_1)\underline{v}_1 + (c_2 + c_{k+1}d_2)\underline{v}_2 + \dots + (c_k + c_{k+1}d_k)\underline{v}_k + c_{k+1}\underline{v}_{k+1}$, which is a linear combination of vectors in the set $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{v}_{k+1}\}$

$\Rightarrow \underline{w} \in \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{v}_{k+1}\}$

Therefore we have $\underline{w} \in \text{span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k, \underline{u}_{k+1}\} \Leftrightarrow \underline{w} \in \text{Span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k, \underline{v}_{k+1}\}$, assuming that $P(k)$ holds.

Hence we have that if $P(k)$ holds, $P(k+1)$ holds $\forall \, k \in 1, 2, \dots, n-1$ and as $P(1)$ is true, we have that $P(m)$ holds $\forall \, m \in 1, 2, \dots, n$.

1c) $\langle \underline{u}_{k+1}, \underline{u}_{k+1} \rangle = \left\langle \underline{u}_{k+1}, \underline{v}_{k+1} - \sum_{j=1}^{k} \frac{\langle \underline{v}_{k+1}, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right\rangle$, using the Gram-Schmidt procedure.

$$= \langle \underline{u}_{k+1}, \underline{v}_{k+1} \rangle - \left\langle \underline{u}_{k+1}, \sum_{j=1}^{k} \frac{\langle \underline{v}_{k+1}, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right\rangle$$

$$= \langle \underline{u}_{k+1}, \underline{v}_{k+1} \rangle - \left(\left\langle \underline{u}_{k+1}, \frac{\langle \underline{v}_{k+1}, \underline{u}_1 \rangle}{\langle \underline{u}_1, \underline{u}_1 \rangle} \underline{u}_1 \right\rangle + \left\langle \underline{u}_{k+1}, \frac{\langle \underline{v}_{k+1}, \underline{u}_2 \rangle}{\langle \underline{u}_2, \underline{u}_2 \rangle} \underline{u}_2 \right\rangle + \dots + \left\langle \underline{u}_{k+1}, \frac{\langle \underline{v}_{k+1}, \underline{u}_k \rangle}{\langle \underline{u}_k, \underline{u}_k \rangle} \underline{u}_k \right\rangle \right)$$

By the definition of the Gram-Schmidt procedure, $\underline{u}_{k+1} \perp \underline{u}_j \quad \forall \, j \in \{1, 2, \dots, k\}$.

$\Rightarrow \langle \underline{u}_{k+1}, \alpha\underline{u}_j \rangle = 0 \quad \forall \, j \in \{1, 2, \dots, k\}$, $\alpha \in \mathbb{R}$.

$\Rightarrow \langle \underline{u}_{k+1}, \underline{u}_{k+1} \rangle = \langle \underline{u}_{k+1}, \underline{v}_{k+1} \rangle - (0 + 0 + \dots + 0)$

$= \langle \underline{u}_{k+1}, \underline{v}_{k+1} \rangle$.

Now we show that $\|\underline{u}_k\| \leq \|\underline{v}_k\|$ for $k = 1, \dots, n$.

Firstly we have that $\underline{u}_1 = \underline{v}_1 \Rightarrow \|\underline{u}_1\| = \|\underline{v}_1\| \Rightarrow \|\underline{u}_k\| \leq \|\underline{v}_k\|$ for $k=1$.

Now we consider when $k \in \{2, 3, \dots, n\}$,

$$\|\underline{u}_k\| = \left\| \underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right\|$$

$$\Rightarrow \sqrt{\langle \underline{u}_k, \underline{u}_k \rangle} = \sqrt{\left\langle \left(\underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right), \left(\underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right) \right\rangle}$$

$$\Rightarrow \langle \underline{u}_k, \underline{u}_k \rangle = \left\langle \left(\underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right), \left(\underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right) \right\rangle$$

$$= \left\langle \underline{v}_k, \left(\underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right) \right\rangle + \left\langle -\sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle}, \left(\underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j \rangle}{\langle \underline{u}_j, \underline{u}_j \rangle} \underline{u}_j \right) \right\rangle$$

$$= \left\langle \underline{v}_k, \left(\underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j\rangle}{\langle \underline{u}_j, \underline{u}_j\rangle} \underline{u}_j\right)\right\rangle + \left\langle -\sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j\rangle}{\langle \underline{u}_j, \underline{u}_j\rangle}, \left(\underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j\rangle}{\langle \underline{u}_j, \underline{u}_j\rangle} \underline{u}_j\right)\right\rangle$$

<u>Unfinished</u> - continue working from here

$$\|\underline{u}_k\| = \left\| \underline{v}_k - \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j\rangle}{\langle \underline{u}_j, \underline{u}_j\rangle} u_j \right\|$$

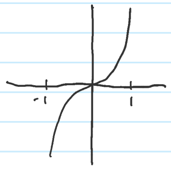
$$\leqslant \|\underline{v}_k\| + \left\| -\sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j\rangle}{\langle \underline{u}_j, \underline{u}_j\rangle} \underline{u}_j \right\| = \|\underline{v}_k\| + \left\| \sum_{j=1}^{k-1} \frac{\langle \underline{v}_k, \underline{u}_j\rangle}{\langle \underline{u}_j, \underline{u}_j\rangle} \underline{u}_j \right\| \quad (\text{Triangle inequality and } \|av\| = |a|\,\|\underline{v}\|)$$

$$\Rightarrow \sqrt{\langle \underline{u}_k, \underline{u}_k\rangle} \leqslant \sqrt{\langle \underline{v}_k, \underline{v}_k\rangle} + \cdots$$

$$\Rightarrow \sqrt{\langle \underline{u}_k, \underline{v}_k} \leqslant \sqrt{\langle \underline{v}_k, \underline{v}_k\rangle}$$

2a) By definition, a set, $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$, is orthogonal if $\langle \underline{v}_i, \underline{v}_j\rangle = 0 \ \forall\ i \neq j$ and $i, j \in \{1, \dots, k\}$

Therefore, in order to prove that S is orthogonal, it is sufficient to show that $\langle 1, x\rangle = 0$ since 1 and x are the only 2 elements in S.



$$\langle 1, x\rangle = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-} \int_0^b \frac{x}{\sqrt{1-x^2}}\,dx + \lim_{a\to -1^+} \int_a^0 \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $I = \int \frac{x}{\sqrt{1-x^2}}\,dx$, let $u = 1-x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow dx = -\frac{1}{2x}\,du$

$$\Rightarrow I = \int \frac{x}{\sqrt{u}} \cdot -\frac{1}{2x}\,du = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = -\frac{1}{2}\left(2u^{\frac{1}{2}}\right) + C = -u^{\frac{1}{2}} + C = -\sqrt{1-x^2} + C$$

Now we have,

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\left[-\sqrt{1-x^2}\right]_0^b + \lim_{a\to -1^+}\left[-\sqrt{1-x^2}\right]_a^0$$

$$= \lim_{b\to 1^-}\left(-\sqrt{1-b^2}\right) - \left(-\sqrt{1-(0)^2}\right) + \left(-\sqrt{1-(0)^2}\right) - \lim_{a\to -1^+}\left(-\sqrt{1-a^2}\right)$$

$= 0 + 1 - 1 - 0 = 0 \Rightarrow S$ is an orthogonal set in U with respect to $\langle \cdot, \cdot \rangle$.

Now we find another monic polynomial in U that is orthogonal to S.

We choose initial monic polynomial, $P(x) = x^2$, as $S \cup \{x^2\} = \{1, x, x^2\}$ forms a basis for $P_2([a, b])$.

Now we use the Gram-Schmidt procedure to find a monic polynomial orthogonal to S:

Let $B = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ with $\underline{v}_1 = 1$, $\underline{v}_2 = x$, and $\underline{v}_3 = x^2$.

Since we know that 1 is orthogonal to x,

we can set $\underline{u}_1 = 1$ and $\underline{u}_2 = x$.

Now we have,

$$\underline{u}_{k+1} = \underline{v}_{k+1} - \sum_{i=1}^{k} \frac{\langle \underline{v}_{k+1}, \underline{u}_i\rangle}{\langle \underline{u}_i, \underline{u}_i\rangle} \underline{u}_i$$

Now we have,

$$\underline{u}_{k+1} = \underline{v}_{k+1} - \sum_{i=1}^{k} \frac{\langle \underline{v}_{k+1}, \underline{u}_i \rangle}{\langle \underline{u}_i , \underline{u}_i \rangle} \underline{u}_i$$

$$\Rightarrow \underline{u}_3 = x^2 - \sum_{i=1}^{2} \frac{\langle x^2, \underline{u}_i \rangle}{\langle \underline{u}_i, \underline{u}_i \rangle} \underline{u}_i$$

$$= x^2 - \left(\frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1 + \frac{\langle x^2, x \rangle}{\langle x, x \rangle} \cdot x \right)$$

$$\langle x^2, 1 \rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} \, dx$$

Let $I = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} \, dx$, let $x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta \, d\theta$

When $x=1$, $\theta = \arcsin(1) = \frac{\pi}{2}$. when $x=-1$, $\theta = \arcsin(-1) = -\frac{\pi}{2}$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta$$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} \, d\theta \quad \text{as} \quad \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta.$$

$$\Rightarrow I = \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta \right]_{-\pi/2}^{\frac{\pi}{2}} = \left(\frac{\pi}{4} - 0 \right) - \left(-\frac{\pi}{4} - \right) = \frac{\pi}{2}$$

$$\langle 1, 1 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= \lim_{b \to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}} \, dx + \lim_{a \to -1^+} \int_a^0 \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= \lim_{b \to 1^-} \left[\arcsin x \right]_0^b + \lim_{a \to -1^+} \left[\arcsin x \right]_a^0$$

$$= \lim_{b \to 1^-} \arcsin b - 0 + 0 - \lim_{a \to -1^+} \arcsin a$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi$$

$$\langle x^2, x \rangle = \int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}} \, dx = \lim_{b \to 1^-} \int_0^b \frac{x^3}{\sqrt{1-x^2}} \, dx + \lim_{a \to -1^+} \int_a^0 \frac{x^3}{\sqrt{1-x^2}} \, dx$$

Let $I = \int \frac{x^3}{\sqrt{1-x^2}} \, dx$, let $u = 1 - x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow dx = -\frac{1}{2x} du$

$$\Rightarrow I = \int \frac{x^3}{\sqrt{u}} \cdot \frac{-1}{2x} \, du = \int -\frac{x^2}{2\sqrt{u}} \, du = \frac{1}{2} \int \frac{u-1}{\sqrt{u}} \, du = \frac{1}{2} \int \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}} \, du = \frac{1}{2} \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} \, du$$

$$= \frac{1}{2} \left(\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} \right) = \sqrt{u} \left(\frac{u}{3} - 1 \right)$$

$$= \sqrt{1-x^2} \left(\frac{1-x^2}{3} - 1 \right)$$

Now we have,

$$\int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}} \, dx = \lim_{b \to 1^-} \left[\sqrt{1-x^2} \left(\frac{1-x^2}{3} - 1 \right) \right]_0^b + \lim_{a \to -1^+} \left[\sqrt{1-x^2} \left(\frac{1-x^2}{3} - 1 \right) \right]_a^0$$

$$= \left(0 - \left(-\frac{2}{3} \right) \right) + \left(\left(-\frac{2}{3} \right) - 0 \right) = \frac{2}{3} - \frac{2}{3} = 0$$

$$\langle x, x \rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} \, dx = \langle x^2, 1 \rangle = \frac{\pi}{2}$$

Now, we have $\underline{u}_3 = x^2 - \left(\frac{\pi}{2} \cdot \frac{1}{\pi} \cdot 1 + 0 \cdot \frac{2}{\pi} \cdot x\right)$
$= x^2 - \frac{1}{2}$

$\Rightarrow P(x) = x^2 - \frac{1}{2}$ is orthogonal to S.