University of Birmingham
School of Mathematics

2RCA/2RCA3 Real and Complex Analysis

Part A: Real Analysis Semester 2

Summative Problem Sheet 1
Marking Scheme (marked out of **40**)

SUM **Question 9.** Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a real function. Suppose that there exists $M > 0$ and $\delta > 0$ such that

$$|f(x)| \leq M \qquad \text{whenever} \quad 0 < |x| < \delta.$$

Define the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} x^2 f(x), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

Show that g is continuous at 0.

Solution. To show that g is continuous at 0, we will show that

$$\lim_{x \to 0} g(x) = g(0).$$

We will show that the above limit exists and equals $g(0) = 0$ using the Sandwich Theorem. Indeed, using the hypothesis on f we observe that

$$|g(x)| = |x^2 f(x)| \leq x^2 M \qquad \text{whenever} \qquad 0 < |x| < \delta,$$

and since $|g(0)| = 0$, we note the above inequality also holds for $x = 0$. Hence, we have

$$|g(x)| \leq x^2 M \qquad \text{whenever} \qquad |x| < \delta,$$

or equivalently,

$$(1) \qquad -Mx^2 \leq g(x) \leq Mx^2, \qquad \text{whenever} \qquad |x| < \delta.$$

Since $\lim_{x \to 0} \pm Mx^2 = 0$ (since x^2 is continuous at 0), and (1) holds, using the Sandwich Theorem we conclude that

$$\lim_{x \to 0} g(x) = 0 = g(0).$$

Therefore, g is continuous at 0.

Remark: Alternatively, the converge of the above limit to zero can be shown using the definition the $\varepsilon - \delta$ definition of limit of a function at a point.

□

Marking scheme. [**Total of 7 marks**]

- **1 mark** for the definition of the continuity of g at 0, that is g is continuous at 0 if

$$\lim_{x \to 0} g(x) = g(0) = 0.$$

- **3 marks** for the use of the hypothesis on f to deduce the inequality
$$|g(x)| \leq Mx^2, \qquad \text{whenever} \quad |x| < \delta.$$
Note: Do not penalize if they forget to include the value at 0, but make a comment about this in your feedback.
- **1 mark** for the inequality
$$-Mx^2 \leq g(x) \leq Mx^2, \qquad \text{whenever} \quad |x| < \delta.$$
- **2 marks** for the correct use of the Sandwich Theorem to conclude that
$$\lim_{x \to 0} g(x) = 0.$$

Note: Some students may try to prove that g is continuous at 0 using the $\varepsilon - \delta$ definition of continuity, using similar arguments to the ones provided in the model solution. Please, award a total of 7 marks for a valid argument of proof.

$\boxed{\text{SUM}}$ **Question 13.** Let $\alpha \in \mathbb{N} \cup \{0\}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by
$$f(x) = \begin{cases} |x|^\alpha \cos\left(\frac{1}{x}\right), & \text{if} \quad x \neq 0, \\ 0, & \text{if} \quad x = 0. \end{cases}$$

(i) Show that f is a continuous at 0 for all $\alpha \in \mathbb{N}$.
(ii) Show that f is not continuous at 0 when $\alpha = 0$.
(iii) Is f a continuous at $x \neq 0$ for all $\alpha \in \mathbb{N} \cup \{0\}$? Justify your answer.

Solution.

(i) Fix $\alpha \in \mathbb{N}$. In order to show that f is continuous at 0, we need to show that
$$\lim_{x \to 0} f(x) = f(0),$$
or equivalently
$$\lim_{x \to 0} |x|^\alpha \cos\left(\frac{1}{x}\right) = 0,$$
since $f(0) = 0$. Notice that for $x \neq 0$
$$\left| |x|^\alpha \cos\left(\frac{1}{x}\right) \right| \leq |x|^\alpha$$
and thus
$$-|x|^\alpha \leq |x|^\alpha \cos\left(\frac{1}{x}\right) \leq |x|^\alpha, \qquad x \neq 0$$
Since the absolute function is continuous on $\mathbb{R}$, using the algebra of continuous functions, $\pm|x|^\alpha$ is continuous at 0 for any $\alpha \in \mathbb{N}$, and consequently $\lim_{x \to 0} \pm|x|^\alpha = 0$. From this remark, the above inequality, and using the Sandwich Theorem, we conclude that
$$\lim_{x \to 0} |x|^\alpha \cos\left(\frac{1}{x}\right) = 0,$$
as required.

(ii) We will show that f is not continuos at 0 when $\alpha = 0$ by showing that the following limit
$$\lim_{x \to 0} f(x)$$
does <u>not</u> exist.

Indeed, consider (for example) the following sequences

$$a_n = \frac{1}{2\pi n} \qquad \text{and} \qquad b_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$$

We have that $a_n, b_n \neq 0$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0,$$

however

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \cos(1/a_n) = \lim_{n\to\infty} \cos(2\pi n) = \lim_{n\to\infty} 1 = 1,$$

and

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \cos(1/b_n) = \lim_{n\to\infty} \cos\left(\frac{\pi}{2} + 2\pi n\right) = \lim_{n\to\infty} 0 = 0,$$

since $1 \neq 0$, we conclude that the limit $\lim_{x\to 0} f(x)$ does not exists. Hence f is not continuous at 0 when $\alpha = 0$.

(iii) Yes, for any given $\alpha \in \mathbb{N} \cup \{0\}$, f is continous at any $x \neq 0$. This follows from the Algebra of continuous functions, and the fact that the functions $|x|$ and cosine function are continuous functions on the real line, $1/x$ is continuous at any $x \neq 0$.

□

Marking scheme. [**Total of 14 marks**]

Part (i):

- **1 mark** for the definition of continuity at 0.
- **4 marks** for a valid argument showing that f is continuous at 0 for all $\alpha \in \mathbb{N}$. Students need to indicate where they have used the fact that $\alpha \in \mathbb{N}$, so penalise with 1 mark if they do not do this.

Part (ii):

- **2 marks** for the choice of appropriate sequences (a_n) and (b_n).
- **1 mark** for observing that the chosen sequences tend to 0 and $n \to \infty$;
- **2 marks** for the convergence of $(f(a_n))$ and $(f(b_n))$;
- **1 mark** for observing that the limits of of the sequences of images through f are different;
- **1 mark** for concluding (using results seen in lectures) that the limit of f at 0 does not exists, and consequently the function is not continuous at 0.

Note: Please, note that students may consider other valid sequences (a_n) and (b_n) converging to 0 to show that f is not continuous at 0.

Part (iii): **2 marks** the justification of the fact that f is continuous at $x \neq 0$ for any $\alpha \in \mathbb{N} \cup \{0\}$.

$\boxed{\text{SUM}}$ **Question 15.** Suppose that the function $f : [-1, 1] \to [-1, 1]$ is continuous. Use the Intermediate Value Theorem to prove that there exists $c \in [-1, 1]$ such that

$$f(c) = c^5.$$

Note: You should carefully justify each of the hypotheses of the theorem.

Solution. We argue very much as in the proof of the fixed point theorem in the lecture notes.

If $f(-1) = -1$ then we choose $c = -1$ and there is nothing else to prove. Similarly, if $f(1) = 1$ then we choose $c = 1$.

So suppose that $f(-1) \neq -1$ and $f(1) \neq 1$. Since $f(x) \in [-1, 1]$ for all $x \in [-1, 1]$ then we must have that

$$f(-1) > -1 \qquad \text{and} \qquad f(1) < 1. \tag{2}$$

We now define a new function $g : [-1, 1] \to \mathbb{R}$ by

$$g(x) = x^5 - f(x).$$

Since f is continuous on $[-1, 1]$, polynomials are continuous, and sums of continuous functions are continuous, then the function g is continuous on $[-1, 1]$. Furthermore, from (2) we have that $g(-1) = -1 - f(-1) < 0$ and $g(1) = 1 - f(1) > 0$ so that

$$g(-1) < 0 < g(1).$$

Hence, Hence, applying the Intermediate Value Theorem to the function g on the interval $[-1, 1]$, we conclude that there exists $c \in (-1, 1)$ such that $g(c) = 0$; i.e. such that $f(c) = c^5$. $\square$

Marking scheme. [**Total of 10 marks**] assigned as follows:

- **1 mark** for distinguishing the cases when $f(-1) = -1$, $f(1) = 1$ and otherwise.
- **1 mark** for the case when $f(-1) = -1$.
- **1 mark** for the case when $f(1) = 1$.
- **2 marks** for deducing from the hypothesis that f takes values on $[-1, 1]$ that both $f(-1) > -1$ and $f(1) < 1$.
- **1 mark** for defining an appropriate function $g(x)$.
- **1 mark** for noting that this function is continuous on $[-1, 1]$ and **1 mark** for a correct justification.
- **1 mark** for correctly applying the Intermediate Value Theorem to g on $[0, 1]$.
- **1 mark** for a correct argument to conclude that there exists $c \in (-1, 1) \subset [-1, 1]$ such that $f(c) = c^5$. Please, if when applying the Intermediate Value Theorem a student states the existence of $c \in [-1, 1]$ instead of $c \in (-1, 1)$, please refer the student to look at the statement of the theorem to notice that the statement says $c \in (a, b)$.

$\boxed{\text{SUM}}$ **Question 17.** Use the Intermediate Value Theorem to prove that the equation

$$x^2 + 2\sin(x) - \cos(x) = 1$$

has at least two real solutions.

Note: You should carefully justify each of the hypotheses of the theorem.

Hint: You need to carefully choose your own intervals to apply the Intermediate Value Theorem.

Solution. Define $f(x) = x^2 + 2\sin(x) - \cos(x) - 1$. It suffices to show that $f(c_1) = f(c_2) = 0$ for some $c_1, c_2 \in \mathbb{R}$ with $c_1 \neq c_2$.

Since the sine function, cosine function and polynomials are continuous on $\mathbb{R}$, it follows that f is continuous on $\mathbb{R}$ (using the algebra of continuous functions, and composition of continuous functions).

Note $f(0) = -2 < 0$. Also

$$f(10) = 100 + 2\sin(10) - \cos(10) - 1 \geq 100 - 2 - 1 - 1 = 96 > 0$$

since $\sin(10) \geq -1$ and $\cos(10) \leq 1$. Hence, by the Intermediate Value Theorem applied to the function f on the interval $[-10, 0]$, there exists $c_1 \in (0, 10)$ such that $f(c_1) = 0$.

Similarly,

$$f(-10) = 100 - 2\sin(10) - \cos(10) - 1 \geq 100 - 2 - 1 - 1 = 96 > 0$$

since $\sin(10) \leq 1$ and $\cos(10) \leq 1$. By the Intermediate Value Theorem applied to the function f on the interval $[-10, 0]$, there exists $c_2 \in (-10, 0)$ such that $f(c_2) = 0$. Since $(0, 10) \cap (-10, 0) = \emptyset$ it follows that $c_1 \neq c_2$, as required. $\square$

Marking scheme. [**Total of 9 marks**]

- **1 mark** for defining an appropriate function - either $f(x) = x^2 + 2\sin(x) - \cos(x) - 1$, or minus that.
- **1 mark** for observing that this function is continuous and **1 mark** for a correct justification.
- **2 marks** for finding two suitable intervals of the form $[x_1, x_2]$ such that $f(x_1) > 0$ and $f(x_2) < 0$ or $f(x_1) < 0$ and $f(x_2) > 0$.
- **3 marks** for applying the Intermediate Value Theorem correctly to the function f in each of these intervals to conclude the existence of c_1 and c_2 such that $f(c_1) = 0$ and $f(c_2) = 0$.
- **1 mark** for a valid justification of why these two points are different.