2DE/2DE3 Example sheet 5 Solutions: Partial Differential Equations

Please note that in some of the solutions, I have missed out steps in the working that can be found in the lecture notes. In submitted work, you must complete all steps to demonstrate that you have understood the method.

1. *Find the separable solution to the heat equation*

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0 \tag{2}$$

and the initial condition

$$u(x,0) = \sin(x) - 2\sin(2x) + 7\sin(10x) \tag{3}$$

when $\alpha = 2$. By substituting your solution back into the original equation, check that it is indeed a solution and that it satisfies the boundary conditions.

We know from lectures that the separable solution to (1) subject to (2) is given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n \exp\left(-\left(\frac{\alpha\pi n}{L}\right)^2 t\right) \sin\left(\frac{n\pi x}{L}\right), \\ &= \sum_{n=1}^{\infty} b_n \exp\left(-4n^2 t\right) \sin(nx), \end{aligned}$$

if $\alpha = 2$ and $L = \pi$ (note that you are expected to be able to derive this yourself and not simply repeat results from lecture notes). Therefore,

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} b_n \sin(nx), \\ &= \sin(x) - 2\sin(2x) + 7\sin(10x). \end{aligned}$$

$\implies$ the only terms that appear in the series can be $n = 1, 2, 10$ and $b_1 = 1$, $b_2 = -2, b_{10} = 7$. Therefore the solution is given by

$$u(x,t) = e^{-4t}\sin(x) - 2e^{-16t}\sin(2x) + 7e^{-400t}\sin(10x).$$

Check that this satisfies the boundary conditions (2):

$$\begin{aligned} u(0,t) &= e^{-4t}\sin(0) - 2e^{-16t}\sin(0) + 7e^{-400t}\sin(0) = 0, \\ u(\pi,t) &= e^{-4t}\sin(\pi) - 2e^{-16t}\sin(2\pi) + 7e^{-400t}\sin(10\pi) = 0. \end{aligned}$$

Check that it satisfies the PDE:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -4e^{-4t}\sin(x) + 32e^{-16t}\sin(2x) - 2800e^{-400t}\sin(10x),\\
\frac{\partial u}{\partial x} &= e^{-4t}\cos(x) - 4e^{-16t}\cos(2x) + 70e^{-400t}\cos(10x),\\
\frac{\partial^2 u}{\partial x^2} &= -e^{-4t}\sin(x) + 8e^{-16t}\sin(2x) - 700e^{-400t}\sin(10x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{LHS} &= \alpha^2\frac{\partial^2 u}{\partial x^2},\\
&= 4\left(-e^{-4t}\sin(x) + 8e^{-16t}\sin(4x) - 700e^{-400t}\sin(10x)\right),\\
&= -4e^{-4t}\sin(x) + 32e^{-16t}\sin(4x) - 2800e^{-400t}\sin(10x),\\
&= \frac{\partial u}{\partial t},\\
&= \text{RHS}.
\end{aligned}$$

2. *Write down the PDE, boundary conditions and initial condition representing the temperature of a straight, thin wire of length 1m, initially at 10°C across the wire, except at one end which is suddenly heated to 17°C. Assume that the wire is insulated across the length of the wire and the temperature at the ends is held fixed. (You do not need to solve the resulting model).*

$$\begin{aligned}
\alpha^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, && 0 < x < 1, t > 0,\\
u(0,t) &= 10, && t > 0,\\
u(1,t) &= 17, && t > 0,\\
u(x,0) &= 10, && 0 < x < 1,
\end{aligned}$$

where α^2 is the thermal diffusivity constant of the material. [Note that the boundary conditions could be swapped around.]

3. *What are the PDE, boundary conditions and initial condition representing the temperature of a straight, thin wire of length 1m, with both ends held at 10°C, the mid-point 5°C initially, and the initial distribution of heat across the wire given by a quadratic function? Assume that the wire is insulated across the length of the wire and the temperature at the ends is held fixed. (You do not need to solve the resulting model).*

$$\begin{aligned}
\alpha^2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, && 0 < x < 1, t > 0,\\
u(0,t) &= 10, && t > 0,\\
u(1,t) &= 10, && t > 0,\\
u(x,0) &= ax^2 + bx + c, && 0 < x < 1,
\end{aligned}$$

where α^2 is the thermal diffusivity constant of the material. We must have that

$$\begin{aligned}
u(0,0) = 10 &\implies c = 10,\\
u(1,0) = 10 &\implies a + b + 10 = 10 \implies a = -b,\\
u(0.5,0) = 5 &\implies 0.25a - 0.5a + 10 = 5 \implies a = 20, b = -20.
\end{aligned}$$

So $u(x,0) = 20x^2 - 20x + 10$.

4. *Find the separable (non-trivial) solution to*

$$\begin{aligned}
8\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, && 0 < x < \pi,\\
u(0,t) &= u(\pi,t) = 0, && t > 0,\\
u(x,0) &= x^2, && 0 < x < \pi.
\end{aligned}$$

Reproducing the method given in your lecture notes with $L = \pi$ and $\alpha^2 = 8$, you should be able to arrive at

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-8n^2 t} \sin(nx),$$

where the b_n are the coefficients of the Fourier sine series for $f(x) = x^2$, i.e.

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} x^2 \sin(nx)\, dx,\\
&= \frac{2}{\pi}\left[\frac{(2 - n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3}\right]_0^{\pi} \qquad \text{(using integration by parts twice)},\\
&= \frac{2}{\pi}\left(\frac{(2 - n^2\pi^2)(-1)^n - 2}{n^3}\right),\\
&= \frac{2}{n^3\pi}((2 - n^2\pi^2)(-1)^n - 2)).
\end{aligned}$$

$$\implies u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^3\pi}((2 - n^2\pi^2)(-1)^n - 2)e^{-8n^2 t}\sin(nx).$$

Note that you should provide the steps in the above integration by parts.

5. *Find the separable solution to the previous question using the boundary conditions:*

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0,$$

instead.

Reproducing the method given in your lecture notes, you should be able to arrive at

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-8n^2 t} \cos(nx)$$

(see lecture notes – Chapter 5 §1.2), with

$$\begin{aligned} u(x,0) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \\ &= x^2, \end{aligned}$$

i.e. the a_n are the coefficients of the Fourier cosine series for x^2. Note that x^2 is an even function so that its full Fourier series would also be its Fourier cosine series.

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} x^2 \, dx, \\ &= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}, \\ &= \frac{2}{\pi} \left[\frac{\pi^3}{3} \right], \\ &= \frac{2\pi^2}{3}. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) \, dx, \\ &= \frac{2}{\pi} \left[\frac{(n^2x^2 - 2)\sin(nx) + 2nx\cos(nx)}{n^3} \right]_0^{\pi} \qquad \text{(using integration by parts twice)} \\ &= \frac{2}{\pi} \left(\frac{2\pi n(-1)^n}{n^3} \right), \\ &= \frac{4(-1)^n}{n^2}. \end{aligned}$$

$$\begin{aligned} \Longrightarrow u(x,0) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx), \\ \Longrightarrow u(x,t) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} e^{-8n^2 t} \cos(nx). \end{aligned}$$

As always, please detail the steps required to carry out the integration by parts.

6. *Find the separable solution to the following wave problem:*

$$\begin{aligned} 2\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2}, && 0<x<\pi, \quad t>0,\\ u(0,t) &= 0, \quad u(\pi,t)=0, && t>0,\\ u(x,0) &= x, && 0<x<\pi,\\ \frac{\partial u}{\partial t}(x,0) &= 0, && 0<x<\pi. \end{aligned}$$

From the lecture notes (Chapter 5, §2) we know that the separable solution will take the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\cos\left(\sqrt{2}nt\right),$$

(remember that you are expected to be able to derive this), where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx,\\ &= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \int \frac{1}{n}\cos(nx)\,dx\right]_0^{\pi}, \quad \text{(using integration by parts)},\\ &= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^{\pi},\\ &= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n\right],\\ &= \frac{2}{n}(-1)^{n+1}.\\ \implies \quad & u(x,t) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx)\cos\left(\sqrt{2}nt\right). \end{aligned}$$

7. *Find the separable solution to the following wave problem:*

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial t^2}, && 0<x<1, \quad t>0,\\ u(0,t) &= 0, \quad u(1,t)=0, && t>0,\\ u(x,0) &= x(1-x), && 0<x<1,\\ \frac{\partial u}{\partial t}(x,0) &= \sin(7\pi x), && 0<x<1. \end{aligned}$$

From the lecture notes (Chapter 5, §2) we know that the solution will take the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(b_n\cos\left(\frac{an\pi}{L}t\right) + c_n\sin\left(\frac{an\pi}{L}t\right)\right),$$

(note that this is different to the previous question because the initial velocity is non-zero – as always make sure you can derive this for yourself).

Here, $L = 1$ and $a = 1$, hence

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\big(b_n \cos(n\pi t) + c_n \sin(n\pi t)\big).$$

Imposing the initial condition

$$u(x,0) = x(1-x), \implies \sum_{n=1}^{\infty} b_n \sin(n\pi x) = x(1-x),$$

$$\begin{aligned}
\implies b_n &= 2\int_0^1 x(1-x)\sin(n\pi x)\,dx,\\
&= 2\left[-\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\sin(n\pi x) + \frac{x^2}{n\pi}\cos(n\pi x) - \frac{2x}{n^2\pi^2}\sin(n\pi x) - \frac{2}{n^3\pi^3}\cos(n\pi x)\right]_0^1,\\
&= 2\left(-\frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{2(-1)^n}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right),\\
&= 2\left(-\frac{2(-1)^n}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right),\\
&= \begin{cases} \dfrac{8}{n^3\pi^3}, & n \text{ odd},\\[2ex] 0, & n \text{ even}. \end{cases}
\end{aligned}$$

Imposing the final initial condition:

$$\frac{\partial u}{\partial t}(x,0) = \sin(7\pi x) \implies \sum_{n=1}^{\infty} c_n n\pi \sin(n\pi x) = \sin(7\pi x),$$

$$\implies c_7 = \frac{1}{7\pi}$$

and all other $c_i = 0$. Hence

$$u(x,t) = \frac{1}{7\pi}\sin(7\pi t)\sin(7\pi x) + \sum_{n \text{ odd}}^{\infty} \frac{8}{n^3\pi^3}\cos(n\pi t)\sin(n\pi x).$$

8. *Find the separable solution to Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

subject to the Neumann boundary conditions

$$\begin{aligned}
&u_x(0,y) = 0, \qquad && u_x(a,y) = f(y), \qquad && 0 < y < b,\\
&u_y(x,0) = 0, && u_y(x,b) = 0, && 0 < x < a.
\end{aligned}$$

Derive expressions for the coefficients and show that

$$\int_0^b f(y)\,dy = 0$$

is a necessary condition for the problem to be solvable.

Looking for solution in the form $u(x,t) = X(x)Y(y)$ and employing the Separation of Variables methods will give the ODEs:

$$X''(x) - \sigma X(x) = 0,$$
$$Y''(y) + \sigma Y(y) = 0,$$

where σ is an unknown constant.

Imposing the homogeneous boundary conditions we have:

$$u_x(0,y) = 0 \implies X'(0)Y(y) = 0 \implies X'(0) = 0,$$
$$u_y(x,0) = 0 \implies X(x)Y'(0) = 0 \implies Y'(0) = 0,$$
$$u_y(x,b) = 0 \implies X(x)Y'(b) = 0 \implies Y'(b) = 0,$$

to avoid the trivial solution.

Solving

$$Y''(y) + \sigma Y(y) = 0,$$

subject to $Y'(0) = Y'(b) = 0$ will give us a condition on σ and the following solutions for $Y(y)$:

$$Y_n(y) = \alpha_n \cos\left(\frac{n\pi y}{b}\right), \quad n = 0, 1, 2, ...$$

α_n constants (and $\sigma = \left(\frac{n\pi}{b}\right)^2$ – see §1.2 of Chapter 5).

Then we must solve

$$X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x) = 0,$$
$$\implies X(x) = k_1 \cosh\left(\frac{n\pi x}{b}\right) + k_2 \sinh\left(\frac{n\pi x}{b}\right),$$
$$\implies X'(x) = \frac{k_1 n\pi}{b} \sinh\left(\frac{n\pi x}{b}\right) + \frac{k_2 n\pi}{b} \cosh\left(\frac{n\pi x}{b}\right).$$

So

$$X'(0) = 0 \implies \frac{k_2 n\pi}{b} \cosh\left(\frac{n\pi x}{b}\right) = 0,$$
$$\implies k_2 = 0.$$

So

$$\begin{aligned}
X_n(x) &= k_n \cosh\left(\frac{n\pi x}{b}\right), \quad n = 0, 1, 2, ... \\
\implies u_n(x,t) &= c_n \cosh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right), \quad (c_n = k_n \alpha_n) \\
\implies u(x,t) &= \sum_{n=0}^{\infty} c_n \cosh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right), \\
&= c_0 + \sum_{n=1}^{\infty} c_n \cosh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right).
\end{aligned}$$

Imposing the final boundary condition:

$$u_x(a,y) = f(y) \implies \sum_{n=1}^{\infty} \frac{c_n n\pi}{b} \sinh\left(\frac{n\pi a}{b}\right) \cos\left(\frac{n\pi y}{b}\right) = f(y),$$

i.e. we need $\frac{c_n n\pi}{b} \sinh\left(\frac{n\pi a}{b}\right)$ to be the Fourier cosine coefficients of $f(y)$ with $c_0 = 0$ – we must be able to express $f(y)$ as a Fourier cosine series with period $2b$ and no constant term. Hence, extending $f(y)$ to be even over $(-b, b)$:

$$\begin{aligned}
\frac{1}{2b}\int_{-b}^{b} f(y)\,dy = 0, \\
\frac{1}{b}\int_{0}^{b} f(y)\,dy = 0,
\end{aligned}$$

and

$$\begin{aligned}
\frac{c_n n\pi}{b} \sinh\left(\frac{n\pi a}{b}\right) &= \frac{2}{b}\int_0^b f(y) \cos\left(\frac{n\pi y}{b}\right) dy, \quad n = 1, 2, 3, ... \\
c_n &= \frac{2}{n\pi \sinh(n\pi a/b)} \int_0^b f(y) \cos\left(\frac{n\pi y}{b}\right) dy \quad n = 1, 2, 3, ...
\end{aligned}$$

Note that we have no way of obtaining c_0!

9. *Find the separable solution to Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

subject to the mixed boundary conditions

$$\begin{aligned}
u(0,y) &= 0, & u(a,y) &= 0, & 0 < y < b, \\
u_y(x,0) &= 0, & u_y(x,b) &= kx(a-x), & 0 < x < a.
\end{aligned}$$

Looking for a solution in the form $u(x,t) = X(x)T(t)$ and using separation of variables gives the following ODEs:

$$\begin{aligned}
X''(x) + \sigma X(x) = 0, \\
Y''(y) - \sigma Y(y) = 0,
\end{aligned}$$

(note that we have reversed the sign of σ from the answer to the previous solution for ease).

Imposing the homogeneous boundary conditions gives the new boundary conditions

$$X(0) = X(a) = Y'(0) = 0,$$

(to avoid the trivial solution). Taken together these will give the following solutions for $X(x)$:

$$X_n(x) = \alpha_n \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, ...$$

and $\sigma = \left(\frac{n\pi}{a}\right)^2$ (see §1 of Chapter 4 of the lecture notes). Hence we must solve

$$\begin{aligned} Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) = 0, \\ \implies Y(y) = k_1 \cosh\left(\frac{n\pi y}{a}\right) + k_2 \sinh\left(\frac{n\pi y}{a}\right). \end{aligned}$$

Imposing the boundary condition on $Y(y)$:

$$\begin{aligned} Y'(0) = 0 \implies \frac{k_2 n\pi}{a} = 0 \implies k_2 = 0 \\ \implies Y_n(y) = k_n \cosh\left(\frac{n\pi y}{a}\right), \quad n = 1, 2, 3, ... \\ \implies u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \cosh\left(\frac{n\pi y}{a}\right), \quad (c_n = \alpha_n k_n) \\ \implies u_y(x,y) = \sum_{n=1}^{\infty} \frac{c_n n\pi}{a} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right), \\ \implies u_y(x,b) = \sum_{n=1}^{\infty} \frac{c_n n\pi}{a} \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right). \end{aligned}$$

Hence we require

$$\sum_{n=1}^{\infty} \frac{c_n n\pi}{a} \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) = kx(a-x),$$

i.e. $\frac{c_n n\pi}{a} \sinh\left(\frac{n\pi b}{a}\right)$ are the coefficients of the Fourier sine series for $kx(a-x)$ where

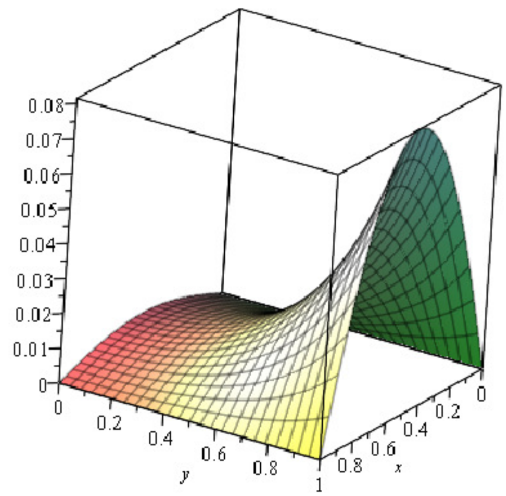

Figure 1: A surface plot illustration of the solution to Question 9 with $a = b = k = 1$ taking terms in the series up to and including $n = 10$.

$kx(a-x)$ is extended to be an odd function over $(-a, a)$. This means that

$$
\begin{aligned}
\frac{c_n n\pi}{a}\sinh\left(\frac{n\pi b}{a}\right) &= \frac{2}{a}\int_0^a kx(a-x)\sin\left(\frac{n\pi x}{a}\right)dx,\\
&= \frac{2}{a}\left(-\frac{a^3k}{n^3\pi^3}(2\cos(n\pi)-2)\right), \quad \text{(integration by parts)}\\
&= -\frac{4a^2k}{n^3\pi^3}((-1)^n-1),\\
&= \frac{4a^2k}{n^3\pi^3}(1-(-1)^n),\\
\implies c_n &= \frac{4a^3k}{n^4\pi^4\sinh(n\pi b/a)}(1-(-1)^n),\\
\implies u(x,y) &= \sum_{n=1}^{\infty}\frac{4a^3k(1-(-1)^n)}{n^4\pi^4\sinh(n\pi b/a)}\sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right),
\end{aligned}
$$

see Figure 1 for an example solution with $a = b = k = 1$.