University of Birmingham
School of Mathematics

2RCA/2RCA3 Real and Complex Analysis

Part B: Complex Analysis Semester 2

Problem Sheet 5 (formative)
Model Solutions

Classification of singularities, Cauchy's residue theorem

Q1. For each of the following functions f, find and classify all the isolated singularities of f. Moreover, find the residue of f at each such singularity.

$$\text{(a)}\quad f(z) = \frac{e^z}{z(1+z)^2}, \qquad \text{(b)}\quad f(z) = \frac{1-\cos(z)}{z^5}, \qquad \text{(c)}\ f(z) = \frac{z}{e^z - 1},$$

$$\text{(d)}\quad f(z) = \sin\left(\frac{1}{(z-2)^2}\right)\cos(z-2).$$

Solution.

(a) f has (isolated) singularities at the points 0 and -1.

About 0: We have that

$$f(z) = \frac{1}{z} \cdot h(z),$$

where

$$h(z) = \frac{e^z}{1+z^2}$$

is holomorphic in a neighbourhood of 0 (e.g. in $B(0, 1/2)$) and $h(0) = 1 \neq 0$. So, 0 is a simple pole of f, and

$$\mathrm{Res}(f; 0) = h(0) = 1.$$

About -1: We have that

$$f(z) = \frac{1}{(z+1)^2} \cdot g(z),$$

where

$$g(z) = \frac{e^z}{z}$$

is holomorphic in a neighbourhood of -1 (e.g. in $B(-1, 1/2)$) and $g(-1) = -e^{-1} \neq 0$. So, -1 is a pole of order 2, and

$$\mathrm{Res}(f; -1) = g'(-1).$$

Since

$$g'(z) = \frac{1}{z}(e^z)' + e^z\left(\frac{1}{z}\right)' = \frac{e^z}{z} - \frac{e^z}{z^2},$$

it follows that

$$\mathrm{Res}(f; -1) = \frac{e^{-1}}{-1} - \frac{e^{-1}}{(-1)^2} = -e^{-1} - e^{-1} = -\frac{2}{e}.$$

(b) f a singularity only at 0. Since

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots,$$

we have that

$$f(z) = \frac{1}{z^5}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \dots\right) = \frac{1}{2!}\frac{1}{z^3} - \frac{1}{4!}\cdot\frac{1}{z} + \frac{1}{6!}z - \dots.$$

Therefore, 0 is a pole of order 3, and

$$\operatorname{Res}(f;0) = -\frac{1}{4!} = -\frac{1}{24}.$$

(c) Since $(e^z - 1)'(z_k) \neq 0$, we know $e^z - 1$ has zeros of order 1 at z_k for $k \in \mathbb{Z}$. The numerator has a zero of order one at 0. Therefore f has a removable singularity at 0 and simple poles at z_k for $k \in \mathbb{Z}, k \neq 0$.

Since f has a removable singularity at 0, then $\operatorname{Res}(f, 0) = 0$. To compute $\operatorname{Res}(f, z_k)$ for $k \in \mathbb{Z}, k \neq 0$ we use L'Hopital's rule:

$$\begin{aligned} \operatorname{Res}(f, z_k) &= \lim_{z \to z_k} (z - z_k)\frac{z}{e^z - 1} \\ &= \lim_{z \to z_k} \frac{(z - z_k) + z}{e^z} = z_k. \end{aligned}$$

(d) f has a singularity at 2 only. We know that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots,$$

so

$$\sin\left(\frac{1}{(z-2)^2}\right) = \frac{1}{(z-2)^2} - \frac{1}{3!}\frac{1}{(z-2)^6} + \frac{1}{5!}\frac{1}{(z-2)^{10}} - \frac{1}{7!}\frac{1}{(z-2)^{14}} + \dots. \tag{1}$$

Therefore, $\sin\left(\frac{1}{(z-2)^2}\right)$ has an essential singularity at 2. Now,

$$f(z) = \sin\left(\frac{1}{(z-2)^2}\right) \cdot h(z),$$

where $h(z) = \cos(z-2)$ is holomorphic. Therefore, f has an essential singularity at 2.

To calculate $\operatorname{Res}(f; 2)$, we also expand

$$h(z) = \cos(z-2) = 1 - \frac{1}{2!}(z-2)^2 + \frac{1}{4!}(z-2)^4 - \frac{1}{6!}(z-2)^6 + \dots \tag{2}$$

and observe that the coefficient of $\frac{1}{z-2}$ when we multiply the two series in (1) and (2) is 0. Indeed, both these series involve even powers of $\frac{1}{z-2}$. Therefore, when we mutiply them out, we will get a series with again only even powers of $\frac{1}{z-2}$. This means that the coefficient of $\frac{1}{z-2}$ in the resulting Laurent series will be 0, and

$$\operatorname{Res}(f; 0) = 0.$$

□

Q2. Evaluate the following contour integrals

(a)

$$\int_C \frac{\sin(3z)}{z + \frac{\pi}{2}} dz,$$

where C is the circle of centre 0 and radius 5 traversed in the anticlockwise direction.

(b)

$$\int_C \frac{e^z}{z(z-7)} dz,$$

where C is the circle of centre 0 and radius 2 traversed in the anticlockwise direction.

(c)
$$\int_C \frac{z^2}{z^2+4}dz,$$
where C is the rectangle with vertices $-2, 2, -2+4i, 2+4i$ traversed in the anticlockwise direction.

(d)
$$\int_C \frac{\sinh(z)}{(z-i\pi)^4}dz,$$
where $C : |z-2i| = 3$ traversed in the anticlockwise direction.

(e)
$$\int_C \frac{e^z}{z^2-2z}dz,$$
where $C : |z| = 4$ traversed in the anticlockwise direction.

(f)
$$\int_C \frac{z+1}{z^2(z-1)}dz,$$
where $C : |z-2| = \sqrt{2}$ traversed in the anticlockwise direction.

(g)
$$\int_\Gamma \frac{\cos(z)}{(z+1)^2(z+10)}\,dz,$$
where the contour Γ is parametrised by $\gamma : [-\pi, \pi] \to \mathbb{C}$ given by $\gamma(\theta) = 3e^{i\theta} + 1$.

Solution.

(a) Notice that the function $f(z) = \sin(3z)$ is holomorphic in $\mathbb{C}$ (we have proved this in class), $\mathbb{C}$ is a simply connected domain, C is a simple closed curve and $\frac{-\pi}{2}$ is in the interior of C. Since all hypothesis are verified, we are in a position to use Cauchy's integral formula to obtain:
$$\int_C \frac{\sin(3z)}{z+\frac{\pi}{2}}dz = \int_C \frac{\sin(3z)}{z-\frac{-\pi}{2}}dz = 2\pi i \sin(\frac{-3\pi}{2}) = 2\pi i.$$

(b) Notice that the function $f(z) = \frac{e^z}{z-7}$ is holomorphic in $B(0,7)$ (we are dividing two holomorphic functions, therefore it is holomorphic where the denominator is not zero), $B(0,7)$ is a simply connected domain, C is a simple closed curve and 0 is in the interior of C. Since all hypothesis are verified, we are in a position to use Cauchy's integral formula to obtain:
$$\int_C \frac{e^z}{z(z-7)}dz = \int_C \frac{f(z)}{z}dz = 2\pi i f(0) = \frac{-2\pi i}{7}.$$

(c) Notice that $z^2+4 = (z+2i)(z-2i)$, the zeros are in $2i$ and $-2i$, but only $2i$ is inside the rectangle C. Choose Ω to be the inside of the larger rectangle $-2-\epsilon-i\epsilon, 2+\epsilon-i\epsilon, -2-\epsilon+(4+\epsilon)i, 2+\epsilon+(4+\epsilon)i$ with $< 0\epsilon < 2$, then Ω is simply connected domain in $\mathbb{C}$, the function $f(z) = \frac{z^2}{z+2i}$ is holomorphic in Ω. All the hypothesis of Cauchy's integral formula are satisfied, so we get
$$\int_C \frac{z^2}{z^2+4}dz = 2\pi i f(2i) = 2\pi i \frac{(2i)^2}{4i} = -2\pi.$$

(d) Notice that the function $f(z) = \sinh(z)$ is holomorphic in $\mathbb{C}$ (we saw this in class), $\mathbb{C}$ is simply connected domain and πi is in the interior of C. We apply the generalized version of Cauchy's integral formula.
$$\int_C \frac{\sinh(z)}{(z-i\pi)^4}dz = 2\pi i \frac{f^{3)}(\pi i}{3!} = \frac{\pi i \cosh(\pi i)}{3} = \frac{\pi i}{3}\left[\frac{e^{i\pi}+e^{-i\pi}}{2}\right] = \frac{-\pi i}{3}.$$

(e) Notice that $z^2 - 2z = z(z-2)$, and the zeros of the denominator 0 and 2 are both in the interior of C. We need to separate those singularities using partial fractions.

$$\frac{1}{z^2 - 2z} = \frac{-1/2}{z} + \frac{1/2}{z-2}.$$

Therefore

$$\int_C \frac{e^z}{z^2-2z} dz = \int_C \frac{-1}{2}\frac{e^z}{z} dz + \frac{1}{2}\int_C \frac{e^z}{z-2} dz.$$

Since e^z is holomorphic in $\mathbb{C}$, which is simply connected domain. We can apply Cauchy's integral formula

$$\int_C \frac{e^z}{z^2-2z} dz = \frac{-1}{2}\int_C \frac{e^z}{z} dz + \frac{1}{2}\int_C \frac{e^z}{z-2} dz = \frac{-1}{2} 2\pi i e^0 + \frac{1}{2} 2\pi i e^2 = (e^2-1)\pi i.$$

(f) $z = 0$ is not in the interior of C, but $z = 1$ is. We consider $f(z) = \frac{z+1}{z^2}$ is holomorphic in a neighbourhood of C, for instance in the ball $B(2, \sqrt{2}+\epsilon)$ for $\epsilon > 0$ sufficiently small so that the ball doesn't touch the point 0. $B(2, \sqrt{2}+\epsilon)$ is simply connected domain. We are in position to apply the generalized Cauchy's integral formula,

$$\int_C \frac{z+1}{z^2(z-1)} dz = 2\pi i f(1) = 4\pi i.$$

(g) The range of Γ is the circle centred at $z = 1$ and of radius 3. Since -1 is in the interior of C but not -10, we consider the function $f(z) = \frac{\cos z}{z+10}$. Choose $\Omega = B(1, 3+\epsilon)$, then Ω is simply connected domain in $\mathbb{C}$ containing C. Moreover, if ϵ is sufficiently small, Ω will not contain -10 and f will be holomorphic in Ω. We can apply the generalized form of Cauchy's theorem and get

$$\int_\Gamma \frac{\cos(z)}{(z+1)^2(z+10)} dz = 2\pi i f^{1)}(-1) = 2\pi i \left[\frac{-9\sin(-1) - \cos(-1)}{81}\right] = 2\pi i \left[\frac{9\sin(1) - \cos(1)}{81}\right].$$

$\square$

Q3. Use the residue theorem to evaluate the following integrals:

(a)
$$\int_\Gamma \frac{e^{\pi z}}{z^2(z^2+2z+2)} dz,$$
where Γ is the circle of centre 0 and radius 3, traversed in the anticlockwise direction.

(b)
$$\int_\Gamma \frac{\sin z}{z^6} dz,$$
where Γ is the circle of centre 0 and radius 1, traversed in the anticlockwise direction.

(c)
$$\int_\Gamma z e^{1/z} dz,$$
where Γ is the circle of centre 0 and radius 1, traversed in the anticlockwise direction.

(d)
$$\int_\Gamma \frac{z+1}{z(z^2+4)^2} dz,$$
where Γ is the circle of centre 0 and radius 5, traversed in the anticlockwise direction.

Solution. (a) $f(z) = \frac{e^{\pi z}}{z^2(z^2+2z+2)}$ has isolated singularities at $z = 0$, $z = -1+i$ and $z = -1-i$, it is otherwise holomorphic. Since $e^{\pi z} \neq 0$ for any $z \in \mathbb{C}$, we see that f has a pole of order two at 0 and simple poles at $z = -1+i$ and $z = -1-i$. Since all of the singularities lie in the interior or Γ, the residue theorem applies and the value of the integral depends of the residues associated to the three singularities. We compute them in turn.

$$\begin{aligned}\operatorname{Res}(f,0) &= \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left(\frac{e^{\pi z}}{z^2+2z+2}\right)\\ &= \lim_{z\to 0}\frac{\pi e^{\pi z}}{z^2+2z+2}-\frac{(2z+2)e^{\pi z}}{(z^2+2z+2)^2}=\frac{\pi-1}{2},\end{aligned}$$

$$\operatorname{Res}(f,-1+i)=\lim_{z\to -1+i}\frac{e^{\pi z}}{z^2(z-(-1-i))}=\frac{-e^{-\pi}}{4},$$

and

$$\operatorname{Res}(f,-1-i)=\lim_{z\to -1-i}\frac{e^{\pi z}}{z^2(z-(-1+i))}=\frac{-e^{-\pi}}{4}.$$

Therefore we conclude

$$\int_\Gamma \frac{e^{\pi z}}{z^2(z^2+2z+2)}\,dz=2\pi i(\operatorname{Res}(f,0)+\operatorname{Res}(f,-1+i)+\operatorname{Res}(f,-1-i))=\pi i(\pi-1-e^{-\pi}).$$

(b) The function $f(z)=\frac{\sin z}{z^6}$ has an isolated singularity at 0, otherwise the function is holomorphic. To see the type of singularity, notice that $\sin z$ has a zero of order one at 0 and z^6 has a zero of order six at 0, therefore f has a pole of order five at 0. Since 0 belongs to the interior of Γ, simple closed curve, we are in a position to apply the residue theorem

$$\int_\Gamma \frac{\sin z}{z^6}\,dz=2\pi i\operatorname{Res}(f,0)$$

Since it is a pole of a rather large order, we will consider the Laurent expansion of f instead of using the formula that will involve four derivatives.

We know that for any $z\in\mathbb{C}$, $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots$, for $z\neq 0$ we have

$$\frac{\sin z}{z^6}=z^{-5}-\frac{z^{-3}}{3!}+\frac{z^{-1}}{5!}-\frac{z}{7!}+\ldots$$

Therefore $\operatorname{Res}(f,0)=\frac{1}{5!}$, then

$$\int_\Gamma \frac{\sin z}{z^6}\,dz=2\pi i\operatorname{Res}(f,0)=\frac{2\pi i}{5!}.$$

(c) The function $f(z)=ze^{1/z}$ has an isolated singularity at 0, otherwise the function is holomorphic. To see the type of singularity, notice that e^z is holomorphic and evaluated at $1/z$ is going to give an essential singularity at 0. Let us compute the Laurent series to verify that.

$$e^z=\sum_{n\geq 0}\frac{z^n}{n!}$$

and it is convergent in all of $\mathbb{C}$. In particular, for $z\neq 0$,

$$e^{1/z}=\sum_{n\geq 0}\frac{1}{n!}\left(\frac{1}{z}\right)^n$$

and

$$ze^{1/z}=\sum_{n\geq -1}\frac{1}{(n+1)!}\left(\frac{1}{z}\right)^n.$$

The Laurent series allow us to conclude that f has an essential singularity at 0. Since 0 belongs to the interior of Γ, simple closed curve, we are in a position to apply the residue theorem

$$\int_\Gamma ze^{1/z}\,dz=2\pi i\operatorname{Res}(f,0)=2\pi i$$

where the last inequality follows from the fact that $\operatorname{Res}(f,0)=1$ as can be seen from the Laurent expansion.

(d) Consider $f(z) = \frac{z+1}{z(z^2+4)^2}$, then f has isolated singularities at $z = 0, 2i, -2i$ and it is otherwise holomorphic. The denominator of f, $g(z) = z(z^2+4)^2$, has a zero of order 1 at $z = 0$ (because $g(0) = 0$ but $g'(0) \neq 0$) and zeros of order 2 at $2i$ and $-2i$ (because of the same reasons). The numerator of f however does not vanish at $z = 0, 2i, -2i$. So $f(z)$ has a simple pole at $z = 0$ and poles or order 2 at $z = 2i$ and $z = -2i$. Since $0, 2i, -2i$ belong to the interior of Γ, simple closed curve, we are in a position to apply the residue theorem

$$\int_\Gamma \frac{z+1}{z(z^2+4)^2}\, dz = 2\pi i \left(\mathrm{Res}(f,0) + \mathrm{Res}(f,2i) + \mathrm{Res}(f,-2i)\right).$$

Let us compute each of these residues.

$$\mathrm{Res}(f,0) = \lim_{z\to 0} z\frac{z+1}{z(z^2+4)^2} = \frac{1}{16},$$

$$\begin{aligned}
\mathrm{Res}(f,2i) &= \frac{1}{1!}\lim_{z\to 2i}\frac{d}{dz}\left((z-2i)^2\frac{z+1}{z(z^2+4)^2}\right)\\
&= \lim_{z\to 2i}\frac{d}{dz}\left(\frac{z+1}{z(z+2i)^2}\right)\\
&= \lim_{z\to 2i}\frac{1}{z(z+2i)^2} - \frac{2(z+1)}{z(z+2i)^3} - \frac{(z+1)}{z^2(z+2i)^2} = \frac{-1-i}{32},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Res}(f,-2i) &= \frac{1}{1!}\lim_{z\to -2i}\frac{d}{dz}\left((z+2i)^2\frac{z+1}{z(z^2+4)^2}\right)\\
&= \lim_{z\to -2i}\frac{d}{dz}\left(\frac{z+1}{z(z-2i)^2}\right)\\
&= \lim_{z\to -2i}\frac{1}{z(z-2i)^2} - \frac{2(z+1)}{z(z-2i)^3} - \frac{(z+1)}{z^2(z-2i)^2} = \frac{-1+i}{32}.
\end{aligned}$$

We conclude

$$\int_\Gamma \frac{z+1}{z(z^2+4)^2}\, dz = 2\pi i\left(\frac{1}{16} + \frac{-1-i}{32} + \frac{-1+i}{32}\right) = 0.$$

□

Q4. Using residue theory, evaluate the integrals

$$\text{(a)} \quad \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+9)}\, dx \qquad\qquad \text{(b)} \quad \int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+4}\, dx$$

Solution.

(a) We look at the complex integral

$$\int_{C_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz,$$

for all large $R > 0$, where C_R is the contour traversed in the counterclockwise direction formed by the segment in the real axis $[-R, R]$ and Γ_R, where Γ_R is the half-circle $|z| = R$ that lies in the upper half-plane. Then

$$\int_{C_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz = \int_{-R}^{R} \frac{x^2}{(x^2+1)^2(x^2+9)} dx + \int_{\Gamma_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz. \tag{3}$$

First we compute $\displaystyle\int_{C_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz$. The function f has isolated singularities at $z = i$, $z = -i$, $z = -3i$, $z = 3i$, it is otherwise holomorphic. Moreover, since the numerator doesn't annihilate at any of those points, f has simple poles at both $z = -3i$ and $z = 3i$ and poles of order 2 at $z = i$ and $z = -i$. Of these four poles, only two of them lie inside C_R for R sufficiently large, that is $z = i$ and $z = 3i$. We are in a position to apply the residue theorem:

$$\int_{C_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz = 2\pi i(\text{Res}(f,3i) + \text{Res}(f,i)).$$

Let us compute the residues:

$$\begin{aligned}\text{Res}(f,i) &= \frac{1}{1!}\lim_{z\mapsto i}\frac{d}{dz}\left(\frac{z^2}{(z+i)^2(z^2+9)}\right)\\ &= \lim_{z\to i}\frac{2z}{(z+i)^2(z^2+9)} - \frac{2z^2}{(z+i)^3(z^2+9)} - \frac{2z^3}{(z+i)^2(z^2+9)^2}\\ &= \frac{-5i}{128}\end{aligned}$$

and

$$\text{Res}(f,3i) = \lim_{z\to 3i}\frac{z^2}{(z^2+1)^2(z+3i)} = \frac{3i}{128}.$$

Therefore we conclude

$$\int_{C_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz = \frac{\pi}{32}.$$

And since the right hand-side does not depend on R, we can take limits as $R \to \infty$ and deduce that

$$\lim_{R\to\infty}\int_{C_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz = \frac{\pi}{32}.$$

Let us prove that $\lim_{R\to\infty}\int_{\Gamma_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz = 0$. For that we will use the ML Lemma. First, note that the length of Γ_R is πR. For $z \in \Gamma_R$, that is $|z| = R$ and $\text{Im}(z) > 0$, we have that

$$\begin{aligned}|z^2+1|^2 &\geq (|z|^2-1)^2 = (R^2-1)^2\\ |z^2+9| &\geq |z|^2 - 9 = R^2 - 9.\end{aligned}$$

Therefore,

$$|f(z)| = \left|\frac{z^2}{(z^2+1)^2(z^2+9)}\right| \leq \frac{R^2}{(R^2-1)^2(R^2-9)}.$$

Using the ML Lemma we conclude

$$\left|\int_{\Gamma_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz\right| \leq \frac{\pi R^3}{(R^2-1)^2(R^2-9)} \longrightarrow 0 \quad \text{as } R \to \infty.$$

Therefore,

$$\lim_{R\to\infty}\int_{\Gamma_R} \frac{z^2}{(z^2+1)^2(z^2+9)} dz = 0.$$

Using (3) and the integrals computed above, we conclude that

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+9)}\, dx = \frac{\pi}{32}.$$

(b) We look at the complex integral

$$\int_{C_R} \frac{ze^{iz}}{z^2+4} dz,$$

for a sufficiently large $R > 0$, where C_R is the contour traversed in the counterclockwise direction formed by the segment in the real axis $[-R, R]$ and Γ_R, where Γ_R is the half-circle $|z| = R$ that lies in the upper half-plane. Then

$$\int_{C_R} \frac{ze^{iz}}{z^2+4}dz = \int_{-R}^{R} \frac{x(\cos x + i\sin x)}{x^2+4}dx + \int_{\Gamma_R} \frac{ze^{iz}}{z^2+4}dz. \tag{4}$$

First we compute $\int_{C_R} \frac{ze^{iz}}{z^2+4}dz$. The function f has isolated singularities at $z = 2i$ and $z = -2i$, it is otherwise holomorphic. Moreover, since the numerator doesn't annihilate at any of those points, f has simple poles at both $z = 2i$ and $z = -2i$. Of these two poles, only one of them lies inside C_R for R sufficiently large, that is $z = 2i$. We are in a position to apply the residue theorem

$$\int_{C_R} \frac{ze^{iz}}{z^2+4}dz = 2\pi i \mathrm{Res}(f, 2i)$$

We compute $\mathrm{Res}(f, 2i)$.

$$\mathrm{Res}(f, 2i) = \lim_{z\to 2i}(z-2i)\frac{ze^{iz}}{z^2+4} = \lim_{z\to 2i} \frac{ze^{iz}}{z+2i} = \frac{e^{-2}}{2}.$$

Therefore we conclude

$$\int_{C_R} \frac{ze^{iz}}{z^2+4}dz = \pi i e^{-2}.$$

And since the right hand-side does not depend on R, we can take limits as $R \to \infty$ and say

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{iz}}{z^2+4}dz = \pi i e^{-2}.$$

Let us prove that $\displaystyle\lim_{R\to\infty} \int_{\Gamma_R} \frac{ze^{iz}}{z^2+4}dz = 0.$

Parametrizing the half circle as $z = Re^{i\theta}$, where $0 \le \theta \le \pi$, we can write

$$\int_{\Gamma_R} \frac{ze^{iz}}{z^2+4}dz = \int_0^{\pi} \frac{iR^2e^{2i\theta}e^{iR(\cos\theta + i\sin\theta)}}{R^2e^{2i\theta}+4}d\theta.$$

We bound the modulus of the right hand-side. We will need to use that $\sin\theta$ is an odd function and that for $0 \le \theta \le \pi/2$, $\sin\theta \ge \frac{2\theta}{\pi}$.

$$\begin{aligned}
\left|\int_0^{\pi} \frac{iR^2e^{2i\theta}e^{iR(\cos\theta + i\sin\theta)}}{R^2e^{2i\theta}+4}d\theta\right| &\le \frac{R^2}{R^2-4}\int_0^{\pi} e^{-R\sin\theta}d\theta \\
&= \frac{2R^2}{R^2-4}\int_0^{\pi/2} e^{-R\sin\theta}d\theta \\
&\le \frac{2R^2}{R^2-4}\int_0^{\pi/2} e^{-2R\pi^{-1}\theta}d\theta \\
&= \frac{2R^2}{R^2-4}\left.\frac{e^{-2R\pi^{-1}\theta}}{-2R\pi^{-1}}\right|_{\theta=0}^{\theta=\pi/2} \\
&= \frac{2R^2}{R^2-4}\frac{\pi}{2R}(1-e^{-R}) \to 0 \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Therefore, we can conclude

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{ze^{iz}}{z^2+4}dz = 0.$$

Remark: Another way of proving the integral above is 0 uses Jordan's lemma. You are welcome to do it however way is more comfortable from you.

Using (4) and the integrals computed above, we see that

$$\int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)}{x^2+4}dx = \pi i e^{-2}$$

In particular,

$$\mathrm{Im}\left(\int_{-\infty}^{\infty}\frac{x(\cos x+i\sin x)}{x^2+4}dx\right)=\mathrm{Im}\left(\pi i e^{-2}\right),$$

what allows us to conclude that

$$\int_{-\infty}^{\infty}\frac{x\sin x)}{x^2+4}dx=\pi e^{-2}$$

which is the integral we were looking for.

$\square$

Q5. Using residue theory, evaluate the integrals

(a) $\displaystyle\int_0^{2\pi}\frac{d\theta}{2+\cos(\theta)}$,

(b) $\displaystyle\int_0^{2\pi}\frac{d\theta}{3-2\cos(\theta)+\sin(\theta)}d\theta$.

Solution.

(a) We convert this integral into a complex integral along the countour C which is the circle centre at 0 and radius 1 traversed in the anticlockwise direction by using the following change of variable: $z=e^{i\theta}$, then $dz=izd\theta$. Also $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}=\dfrac{z+z^{-1}}{2}$ and $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}=\dfrac{z-z^{-1}}{2i}$, then we have

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{2+\cos(\theta)}&=\int_C\frac{-i/zdz}{2+\frac{z+z^{-1}}{2}}\\&=-i\int_C\frac{2dz}{z^2+4z+1}.\end{aligned}$$

The function $f(z)=\frac{2}{z^2+4z+1}$ has isolated singularities at the points z such that $z^2+4z+1=0$, that is, at $z=-2+\sqrt{3}$ and $z=-2-\sqrt{3}$ which are simple poles. The function is otherwise holomorphic. Of the two singularities, only $z=-2+\sqrt{3}$ lies inside the interior of C, which is a closed simple curve. We are in a position to use the residue theorem:

$$\int_C\frac{2dz}{z^2+4z+1}=2\pi i\mathrm{Res}(f,-2+\sqrt{3}).$$

Let us find $\mathrm{Res}(f,-2+\sqrt{3})$ using the formula for simple poles:

$$\mathrm{Res}(f,-2+\sqrt{3})=\lim_{z\to-2+\sqrt{3}}(z-(-2+\sqrt{3}))\frac{2}{z^2+4z+1}=\lim_{z\to-2+\sqrt{3}}\frac{2}{z-(-2-\sqrt{3})}=\frac{1}{\sqrt{3}}.$$

Therefore we conclude

$$\int_0^{2\pi}\frac{d\theta}{2+\cos(\theta)}=-i\int_C\frac{2dz}{z^2+4z+1}=\frac{2\pi}{\sqrt{3}}.$$

(b) We convert this integral into a complex integral along the countour C which is the circle centre at 0 and radius 1 traversed in the anticlockwise direction by using the following change of variable: $z=e^{i\theta}$, then $dz=izd\theta$. Also $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}=\dfrac{z+z^{-1}}{2}$ and $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}=\dfrac{z-z^{-1}}{2i}$, then we have

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{3-2\cos(\theta)+\sin(\theta)}d\theta&=\int_C\frac{\frac{dz}{iz}}{3-(z+z^{-1})+\frac{z-z^{-1}}{2i}}\\&=\int_C\frac{2dz}{(1-2i)z^2+6iz-1-2i}.\end{aligned}$$

The function $f(z) = \frac{2}{(1-2i)z^2+6iz-1-2i}$ has isolated singularities at the points z such that $(1-2i)z^2 + 6iz - 1 - 2i = 0$, that is, at $z = 2 - i$ and $z = \frac{2-i}{5}$ which are simple poles. The function is otherwise holomorphic. Of the two singularities, only $z = \frac{2-i}{5}$ lies inside the interior of C, which is a closed simple curve. We are in a position to use the residue theorem:

$$\int_C \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i} = 2\pi i \mathrm{Res}(f, \frac{2-i}{5}).$$

Let us find $\mathrm{Res}(f, \frac{2-i}{5})$ using the formula and L'Hopitals rule,

$$\mathrm{Res}(f, \frac{2-i}{5}) = \lim_{z\to\frac{2-i}{5}} (z - \frac{2-i}{5})\frac{2}{(1-2i)z^2 + 6iz - 1 - 2i} = \lim_{z\to\frac{2-i}{5}} \frac{2}{2(1-2i)z + 6i} = \frac{1}{2i}.$$

Therefore we conclude

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos(\theta) + \sin(\theta)} d\theta = \int_C \frac{2dz}{(1-2i)z^2 + 6iz - 1 - 2i} = 2\pi i \mathrm{Res}(f, \frac{2-i}{5}) = \pi.$$

$\square$