1Mech — Mechanics

Mechanics exercises 3 (weeks 5 and 6)

This sheet's assessed question is number 5.

1. A particle of mass m is attracted towards the origin by a force of the form mc/r^2, where r, θ are polar coordinates and $c > 0$ is a constant. The particle is initially located a distance a from the origin and moving with speed $\sqrt{c/a}$ perpendicular to the radial direction.

   (a) Show that $u = 1/r$ satisfies

   $$\frac{d^2u}{d\theta^2} + u = \frac{c}{h^2},$$

   where $h = r^2\dot{\theta}$ is constant.

   (b) Find expressions for the initial conditions for the $u(\theta)$ equation, and the value of h.

   (c) Hence solve to find the particle path. What shape is it?

2. Suppose that a particle of mass m is subject to a central force acting towards the origin of magnitude

   $$\mu m\left(\frac{1}{r^2} + \frac{3a}{4r^3}\right),$$

   with μ, a constant, and r the distance between the particle and the origin. The particle is initially at $r = a$ with initial velocity $\sqrt{\mu/a}$ in a direction perpendicular to the line joining the origin to the particle.

   (a) Show that $u = 1/r$ satisfies

   $$\frac{d^2u}{d\theta^2} + \frac{u}{4} = \frac{1}{a},$$

   subject to $u = 1/a$, $du/d\theta = 0$ at $\theta = 0$.

   (b) Solve the system to find the particle path. What is the distance of closest approach to the origin?

3. For the following question you may find it advantageous to convert into $u(\theta)$ where $u = 1/r$. Note the question does not ask about stability of these paths.

   (a) If a particle under a central force $F(r)$ moves along the spiral $r = e^{-k\theta}$ where k is a constant, show that $F(r) = -C/r^3$ where C is a constant.

   (b) If a particle under a central force $F(r)$ moves along a circular arc terminating at $r = 0$, show that $r = a\cos\theta$ gives the particle path, where a gives the diameter of the circle, and hence show that $F(r) = -D/r^5$ where D is a constant. [Hint: Sketch the semi circular part of the particle path and draw a line joining $r = 0$ to any point on the particle path to find a right angled triangle.]

4. A comet moves under the inverse square law attraction of the Sun. The force is given by

$$\mathbf{F} \quad = \quad -\frac{GMm}{r^2}\mathbf{e}_r = -\frac{GMm}{r^3}\mathbf{r},$$

where $\mathbf{r}$ is the position vector of the comet relative to the Sun, r gives the distance between the Sun and the comet such that $\mathbf{r} = r\mathbf{e}_r$, G is the gravitational constant and M is the mass of the sun. Starting from Newton's second law show that

(a) the moment of momentum of the comet with respect to the Sun is constant.

(b) the orbit of the comet lies in a plane containing the Sun.

(c) $r^2\dot{\theta}$ is constant.

5. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

Let (r, θ) be the standard polar coordinates in 2D and let the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ be defined in the usual way. A particle of mass m is attracted towards the origin by a force of magnitude

$$m\omega^2 \left(\frac{a^4}{r^3} + r \right),$$

where ω and a are positive constants. The particle is initially at distance a from the origin and moving with initial velocity $a\omega\mathbf{e}_\theta$.

(a) Write down Newton's Second Law for the particle in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Hence, show that the quantity $h = r^2\dot{\theta}$ is constant throughout the particle's motion, and find the value of h.

(b) Starting from Newton's Second Law, derive the following equation satisfied by $u = 1/r$:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = \frac{1}{a^4 u^3}. \qquad (1)$$

Determine the initial conditions for u and $\frac{\mathrm{d}u}{\mathrm{d}\theta}$ at $\theta = 0$.

(c) Verify that the function

$$u(\theta) = \frac{\sqrt{\theta^2 + c^4}}{ac},$$

where c is any constant, solves (1) and satisfies the initial condition for $\frac{\mathrm{d}u}{\mathrm{d}\theta}$. Use the initial condition for u to determine the value of c.

(d) Show that

$$\dot{\theta} = \omega(\theta^2 + 1). \qquad (2)$$

Verify that the function

$$\theta(t) = \tan(\omega t)$$

solves (2) and satisfies the initial condition for θ at $t = 0$. How much time does it take the particle to reach the origin? Justify your answer.