1AC Algebra and Combinatorics

Problem Sheet 3: Algebra part

You should carefully write out your solutions to all the questions below.

Ensure you have read and understood the Canvas assignment for the problem sheet for instructions about submitting solutions to SUM questions.

AQ1. (SUM)

    (a) Use the extended Euclidean algorithm to find $y, z \in \mathbb{Z}$ such that

$$41y + 32z = 1.$$

    (b) Solve the linear congruence equation

$$32x \equiv 7 \bmod 41.$$

    (c) Solve the following pair of simultaneous congruences.

$$x \equiv 18 \bmod 32$$
$$x \equiv 31 \bmod 41.$$

AQ2. Prove Lemma 3.12

    Lemma. *Let $a, b, x \in \mathbb{Z}$, let $n \in \mathbb{N}$ and let $h = \mathrm{hcf}(a, n)$.*

    (a) *Suppose that $ax \equiv b \bmod n$. Then $h \mid b$.*

    (b) *Suppose that $h \mid b$, let $a' = \frac{a}{h}$, $b' = \frac{b}{h}$ and $n' = \frac{n}{h}$.*

        (i) *$ax \equiv b \bmod n$ if and only if $a'x \equiv b' \bmod n'$; and*

        (ii) *a' is coprime to n'.*

AQ3. (SUM) Let $a \in \mathbb{N}$ with digits $a_r a_{r-1} \ldots a_2 a_1 a_0$. So

$$a = a_0 + 10a_1 + 10^2 a_2 + \cdots + 10^{r-1} a_{r-1} + 10^r a_r.$$

(a) Prove that $9 \mid a$ if and only if $9 \mid a_0 + a_1 + a_2 + \cdots + a_{r-1} + a_r$.

(b) Prove that $11 \mid a$ if and only if $11 \mid a_0 - a_1 + a_2 - \cdots + (-1)^{r-1} a_{r-1} + (-1)^r a_r$.

AQ4. (a) Let A be a subset of $\mathbb{Z}$ with 5 elements. Show that there exists a subset $B = \{a, b, c\}$ of A with 3 elements such that $a + b + c$ is divisible by 3.

(b) Find a subset A of $\mathbb{Z}$ with 4 elements such that $a + b + c$ is not divisible by 3 for any subset $B = \{a, b, c\}$ of A with 3 elements.