Examples sheet 4 – Solutions – Linear Algebra

ISOMORPHISMS.

1. **(a)** Let $B_V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. By definition,

$$\varphi_V(x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Hence,

$$\varphi_V(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \varphi_V(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \varphi_V(\mathbf{v}_n) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

so that the matrix representation is $A_{VW} = I_n$.

(b) If we change the canonical basis of $W := \mathbb{R}^n$ to B'_n, by one of the three change of matrix representation formulas (namely $A_{VW'} = M_{WW'}A_{VW}$), we find that the matrix representation is

$$A_{VW'} = M_{WW'}A_{VW} = M^{-1}_{W'W} = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n]^{-1},$$

where we used the fact that the transition matrix $M_{WW'}$ satisfies $M_{WW'} = M^{-1}_{W'W}$ and $M_{W'W} = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n]$ since

$$\mathbf{c}_i = [\mathbf{c}_i]_1\mathbf{e}_1 + [\mathbf{c}_i]_2\mathbf{e}_2 + \cdots + [\mathbf{c}_i]_n\mathbf{e}_n \Longrightarrow \mathbf{c}_i(M_{W'W}) = \begin{bmatrix} [\mathbf{c}_i]_1 \\ [\mathbf{c}_i]_2 \\ \vdots \\ [\mathbf{c}_i]_n \end{bmatrix} = \mathbf{c}_i.$$

Note that there is a clash of notation here: $\mathbf{c}_i(M_{W'W})$ denotes the ith column of $M_{W'W}$, while $\mathbf{c}_i$ is the ith basis element of B'_n. If you wish for better clarity, work with $B'_n = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$.

These matrices are invertible: the first is the identity, the second is a transition matrix, which was shown to be invertible in Lecture 12.

2. **(a)** We need to check the vector space axioms. This is a straightforward exercise using the standard field axioms for $\mathbb{R}$; for example, the closure axioms VA0 and VM0 follow from the definitions of addition of matrices and multiplication of matrices by a scalar by using the closure axioms FA0 and FM0 for $\mathbb{R}$:

$$[A+B]_{ij} := a_{ij} + b_{ij} \in \mathbb{R} \quad \text{for } i,j = 1,2, \Longrightarrow A+B \in \mathcal{M}_{2,2};$$

$$[a \cdot A]_{ij} := aa_{ij} \in \mathbb{R} \quad \text{for } i,j = 1,2, \Longrightarrow a \cdot A \in \mathcal{M}_{2,2},$$

where $A, B \in \mathcal{M}_{2,2}$ and $a \in \mathbb{R}$.
We also highlight the following relevant matrices (needed for VA2 and VA3):

- the additive identity Z is the 2×2 zero matrix $Z := O_{2,2}$;
- the additive inverse A^- is the 2×2 matrix $A^- := -A = (-1)A$.

(b) A basis for $A \in \mathcal{M}_{2,2}$ is given by

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} =: \{E_1, E_2, E_3, E_4\},$$

since this is a linearly independent set in $\mathcal{M}_{2,2}$ which is also a spanning set, as any $A \in \mathcal{M}_{2,2}$ satisfies $A \in \operatorname{span} B$:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) A one-to-one correspondence between B and the canonical basis for $\mathbb{R}^4$ is simply $E_i \mapsto \mathbf{e}_i$, for $i = 1, 2, 3, 4$. This allows us to define a map $f : \mathcal{M}_{2,2} \to \mathbb{R}^4$ via

$$f(A) = f(a_{11}E_1 + a_{21}E_2 + a_{12}E_3 + a_{22}E_4) = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{12}\mathbf{e}_3 + a_{22}\mathbf{e}_4 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix}.$$

It is straightforward to show that $f(aA + bB) = af(A) + bf(B)$ and that

- f is injective:

$$f(A) = f(B) \Longrightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{12} \\ b_{22} \end{bmatrix} \Longrightarrow [a_{ij}]_{i,j=1,2} = [b_{ij}]_{i,j=1,2} \Longrightarrow A = B;$$

- f is surjective:

$$\text{given any column vector } \mathbf{a} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{bmatrix} \in \mathbb{R}^4, \text{ there exists } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}_{2,2} \text{ such that } \mathbf{a} = f(A).$$

Hence f is an isomorphism and $\mathcal{M}_{2,2} \cong \mathbb{R}^4$.

3. (a) We trivially have that $\mathcal{M}_{2,2}^{\text{sym}}(\mathbb{R})$ is non-empty and that $\mathcal{M}_{2,2}^{\text{sym}}(\mathbb{R}) \subseteq \mathcal{M}_{2,2}(\mathbb{R})$. We can use therefore use a subspace criterion. We have, for any $A, B \in \mathcal{M}_{2,2}^{\text{sym}}(\mathbb{R})$ and for any $a, b \in \mathbb{R}$,

$$aA + bB = a \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} + b \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} aa_{11} + bb_{11} & aa_{12} + bb_{12} \\ aa_{12} + bb_{12} & aa_{22} + bb_{22} \end{bmatrix} \in \mathcal{M}_{2,2}^{\text{sym}}(\mathbb{R}).$$

By Subspace Criterion 2, $\mathcal{M}_{2,2}^{\text{sym}}(\mathbb{R})$ is a subspace of $\mathcal{M}_{2,2}(\mathbb{R})$.

(b) A basis for $\mathcal{M}_{2,2}^{\text{sym}}$ is

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

as it is a linearly independent spanning set for $\mathcal{M}_{2,2}^{\text{sym}}$. Hence $\dim \mathcal{M}_{2,2}^{\text{sym}} = 3$.

(c) Let us denote the elements of B as E_1, E_2, E_3. Then a one-to-one correspondence between the basis of $\mathcal{M}_{2,2}^{\text{sym}}(\mathbb{R})$ and the canonical basis of $\mathbb{R}^3$ is given by $E_i \xrightarrow{f} \mathbf{e}_i$, where $\mathbf{e}_i$ denotes the ith canonical basis element in $\mathbb{R}^3$. By Proposition 13.5, f is an isomorphism and therefore $\mathcal{M}_{2,2}^{\text{sym}} \cong \mathbb{R}^3$. Explicitly,

$$f\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = f(aE_1 + bE_2 + cE_3) = af(E_1) + bf(E_2) + cf(E_3) = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

The above can be generalised in a straightforward way to the case of n-by-n matrices; the details are rather technical, but included for completeness. The key observation is that any symmetric matrix is entirely determined by the entries below (and including) the main diagonal, i.e., by $n(n+1)/2$ parameters. This is indeed the dimension of $\mathcal{M}_{n,n}^{\text{sym}}$. The detailed description is included below.

A basis for $\mathcal{M}_{n,n}^{\text{sym}}$ is given by

$$B = \left\{ E_{s(k,\ell)} \in \mathcal{M}_{2,2}^{\text{sym}} : [E_{s(k,\ell)}]_{ij} = \delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell} - \delta_{ijk\ell},\ k = 1, \dots, n,\ \ell = 1, \dots, k,\ \ 1 \le i, j \le n \right\},$$

where $s(k,\ell)$ maps the labels (k,ℓ) of the entries in a symmetric matrix located below (and including) the main diagonal, listed column by column, to the index set $\{1,2,\ldots,n(n+1)/2\}$:

$$s(k,\ell) := k + (l-1)\left(n - \frac{\ell}{2}\right), \quad k = 1,\ldots,n,\ \ell = 1,\ldots,k.$$

Then

$$\dim \mathcal{M}^{\text{sym}}_{n,n} = |B| = \sum_{k=1}^{n}\sum_{\ell=1}^{k} 1 = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

A possible isomorphism is then $E_{s(k,\ell)} \xrightarrow{f} \mathbf{e}_{s(k,\ell)}$, for $k = 1,\ldots,n,\ \ell = 1,\ldots,k$.
For example, when $n = 3$, the map f indicated above is

$$f\left(\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix},$$

while the corresponding basis for $\mathcal{M}^{\text{sym}}_{3,3}$ has elements

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\quad E_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad E_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\quad E_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. **(a)** $V \ncong W$ since $\dim V \neq \dim W$, so $f : V \to W$ is not an isomorphism.

(b) Let us find the dimension of $W = \mathcal{P}^0_{n+1}(\mathbb{R})$: since any $p \in \mathcal{P}^0_{n+1}$ satisfies $p(0) = 0$, it must have the form $p(x) = a_1 x + a_2 x^2 + \cdots + a_{n+1}x^{n+1}$, i.e., $p \in \operatorname{span}\left\{x, x^2, \ldots, x^{n+1}\right\} =: \operatorname{span} B_W$. This spanning set is linearly independent, so a basis for W. Hence, $\dim W = n+1 = \dim V$, so $V \cong W$. To see that f is an isomorphism, consider the matrix representation with respect to the power basis and B_W:

$$f\left(x^{j-1}\right) = \int \left(x^{j-1}\right) \mathrm{d}x = \frac{1}{j}x^j, \quad j = 1,\ldots,n+1 \Longrightarrow A_{VW} = \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & \frac{1}{n} & \\ & & & & \frac{1}{n+1} \end{bmatrix}.$$

Since A_{VW} is invertible, f is an isomorphism. Alternatively, the above expression indicates that a basis of V is mapped by f to a basis for W:

$$\{1, x, \ldots, x^n\} \xrightarrow{f} \left\{x, \frac{1}{2}x^2, \ldots, \frac{1}{n+1}x^{n+1}\right\},$$

so that $V \overset{f}{\cong} W$.

(c) $V \ncong W$ since $n+1 = \dim V \neq \dim W = n+3$, so $f : V \to W$ is not an isomorphism.

(d) Consider the dimension of W: since any $p \in W = \mathcal{P}^0_{n+2}$ satisfies $p(0) = 0$, it must have the form $p(x) = a_1 x + a_2 x^2 + \cdots + a_{n+2}x^{n+2}$, i.e., $p \in \operatorname{span}\left\{x, x^2, \ldots, x^{n+2}\right\} =: \operatorname{span} B_W$. This spanning set is linearly independent, so a basis for W. Hence, $\dim W = n+2 \neq \dim V = n+1$. Therefore, $V \ncong W$ so $f : V \to W$ is not an isomorphism.

(e) Consider the dimension of W: since any $p \in W = \mathcal{P}_{n+2}^{00}$ satisfies $p(0) = p'(0) = 0$, it must have the form $p(x) = a_2x^2 + \cdots + a_{n+2}x^{n+2}$, i.e., $p \in \operatorname{span}\left\{x^2, \ldots, x^{n+2}\right\} =: \operatorname{span} B_W$. This spanning set is linearly independent, so a basis for W. Hence, $\dim W = n+1 = \dim V$, so $V \cong W$. To see that f is an isomorphism, consider the matrix representation with respect to the power basis and B_W:

$$f\left(x^{j-1}\right) = \int \left(\int \left(x^{j-1}\right) \mathrm{d}x\right) \mathrm{d}x = \frac{1}{j(j+1)} x^{j+1}, \quad j = 1, \ldots, n+1 \Longrightarrow A_{VW} = \begin{bmatrix} \frac{1}{1\cdot 2} & & & & \\ & \frac{1}{2\cdot 3} & & & \\ & & \ddots & & \\ & & & \frac{1}{n(n+1)} & \\ & & & & \frac{1}{(n+1)(n+2)} \end{bmatrix}.$$

Since A_{VW} is invertible, f is an isomorphism. Alternatively, the above expression indicates that a basis of V is mapped by f to a basis for W:

$$\{1, x, \ldots, x^n\} \xrightarrow{f} \left\{\frac{1}{1\cdot 2}x^2, \frac{1}{2\cdot 3}x^3, \ldots, \frac{1}{(n+1)(n+2)}x^{n+2}\right\},$$

so that $V \overset{f}{\cong} W$.

5. We need to check the vector space axioms. This is a straightforward exercise using the standard properties of addition and multiplication in $\mathbb{R}$; for example, the closure axioms follow from the definitions of addition of matrices and multiplication of matrices by a scalar:

$$[A+B]_{ij} := a_{ij} + b_{ij} \in \mathbb{R} \quad \text{for all } i = 1, \ldots, m,\ j = 1, \ldots, n \Longrightarrow A + B \in \mathbb{R}^{m\times n};$$

$$[a \cdot A]_{ij} := a a_{ij} \in \mathbb{R} \quad \text{for all } i = 1, \ldots, m,\ j = 1, \ldots, n \Longrightarrow a \cdot A \in \mathbb{R}^{m\times n},$$

where $A, B \in \mathbb{R}^{m\times n}$ and $a \in \mathbb{R}$.
We also highlight the following relevant matrices:

- the additive identity is the $m \times n$ zero matrix $O_{m,n}$;
- the additive inverse of $A \in \mathbb{R}^{m\times n}$ is $-A$, which is also in $\mathbb{R}^{m\times n}$.

The dimension of $\mathbb{R}^{m\times n}$ is mn. We can derive this in two different ways:

- find a basis and establish its cardinality; in our case, we can choose as a basis the set of matrices containing a single non-zero value (equal to one) in each of the mn locations/entries of an $m \times n$ matrix:

  $$B = \left\{E_{k+m(\ell-1)} : [E_{k+m(\ell-1)}]_{ij} = \delta_{ik}\delta_{j\ell}, k = 1, \ldots, m, \ell = 1, \ldots, n\right\}.$$

  Then $\dim \mathbb{R}^{m\times n} = |B| = mn$.
- define an isomorphism $f : \mathbb{R}^{m\times n} \to \mathbb{R}^N$, where $N = mn$, via the one-to-one correspondence (i.e., bijection) given below:

  $$[\mathbf{x}]_{i+m(j-1)} := [f(A)]_{i+m(j-1)} = a_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

  Note that the action of f is to stack the columns of A vertically (with the first column at the top) in the vector $\mathbf{x}$. Then $\mathbb{R}^{m\times n} \cong \mathbb{R}^{mn}$, so $\dim \mathbb{R}^{m\times n} = \dim \mathbb{R}^{mn} = mn$.

6. Let $f \in \mathcal{L}(V, W)$ be an isomorphism. Let us introduce the notation in this question:

- $S_V = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}, \mathbf{v}_i \in V$;
- $S_W = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$, $\mathbf{w}_i \in W$;
- $\mathbf{w}_i = f(\mathbf{v}_i), i = 1, \ldots, k$.

(a) Let S_V be a spanning set for V. Let $\mathbf{w} \in W$. Then

$$V \ni f^{-1}(\mathbf{w}) = \sum_{i=1}^{k} a_i \mathbf{v}_i \iff f\left(f^{-1}(\mathbf{w})\right) = f\left(\sum_{i=1}^{k} a_i \mathbf{v}_i\right) = \sum_{i=1}^{k} a_i f(\mathbf{v}_i) = \sum_{i=1}^{k} a_i \mathbf{w}_i \in \operatorname{span} S_W.$$

Since $\mathbf{w}$ is arbitrary in W, S_W is a spanning set for W. Note that the reverse implication follows by noting that $f^{-1} \in \mathcal{L}(V)$, so that swapping f and f^{-1} and also S_V and S_W yields the required statement.

(b) Consider the following equivalent statements:

$$\mathbf{0} = \sum_{i=1}^{k} a_i \mathbf{v}_i \iff \mathbf{0} = f(\mathbf{0}) = f\left(\sum_{i=1}^{k} a_i \mathbf{v}_i\right) = \sum_{i=1}^{k} a_i f(\mathbf{v}_i) \iff \mathbf{0} = \sum_{i=1}^{k} a_i \mathbf{w}_i.$$

Hence, S_V is linearly independent (i.e., $a_i = 0$ for all $i = 1, \ldots, k$) if and only if S_W is linearly independent.

(c) Follows from **(a)** and **(b)**.

(d) Let $U \leq V$. Then $f(U) \leq W$ (see **Q7**, Examples 3). Let B_U denote a basis for U. Then $f(B_U)$ is a linearly independent set in $f(U)$. Moreover, it is a spanning set for $f(U)$, and hence a basis. Therefore, $\dim U = |B_U| = |f(B_U)| = \dim f(U)$.

7. Sketch proof:

- First show that m is a linear map.
- Then we show $\ker m$ is trivial.
- Finally, if $\ker m$ is trivial, then $\dim \mathcal{L}(V, W) = \operatorname{rank} m$, by the rank-nullity theorem. We are only left to show that $\dim \operatorname{im} m = \dim \mathbb{F}^{m\times n}$; we can then apply the isomorphism criterion to deduce that m is an isomorhism. This part follows by considering the maps with the matrix representations E_i in **Q5**.

Note. It is evident that if $\ker m = \{\mathbf{0}\}$ and $\dim \operatorname{im} m = \dim \mathbb{F}^{m\times n}$, then the map m is injective and surjective, therefore bijective, i.e., an isomorphism. In fact, this observation can be used to provide an isomorphism criterion: **Isomorphism criterion**: A map $f \in \mathcal{L}(V, W)$ is an isomorphism if and and only if $\ker f = \{\mathbf{0}_V\}$ and $\operatorname{rank} f = \dim W$.

ENDOMORPHISMS.

8. **(a)** Let $\{\mathbf{i}, \mathbf{j}\}$ be a basis for $\mathbb{E}^2$. Then, assuming anti-clockwise rotation, we have

$$f(\mathbf{i}) = \cos\theta \mathbf{i} + \sin\theta \mathbf{j}, \quad f(\mathbf{j}) = -\sin\theta \mathbf{i} + \cos\theta \mathbf{j} \implies A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} =: R_\theta$$

Hence, A is invertible (it is a rotation matrix R_θ with inverse $R_{-\theta}$) and f is an isomorphism.

(b) Let $\{\mathbf{i}, \mathbf{j}\}$ be a basis for $\mathbb{E}^2$. Then

$$f(\mathbf{i}) = \mathbf{j}, \quad f(\mathbf{j}) = \mathbf{i} \implies A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, A is invertible (it is a reflection matrix with inverse $A^{-1} = A$) and f is an isomorphism.

(c) Let $\{\mathbf{i}, \mathbf{j}\}$ be a basis for $\mathbb{E}^2$. Then

$$f(\mathbf{i}) = \mathbf{i}, \quad f(\mathbf{j}) = \mathbf{0} \implies A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, A is not invertible and f is not an isomorphism.

(d) Let $\{1, x, x^2\}$ be a basis for $\mathcal{P}_2(\mathbb{R})$. Then

$$f(1) = 1 + 0 = 1, \quad f(x) = x + 1, \quad f(x^2) = x^2 + 2x \implies A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, A is invertible (since $\det A = 1 \neq 0$) and f is an isomorphism.

(e) Let $\{1, x, x^2\}$ be a basis for $\mathcal{P}_2(\mathbb{R})$. Then

$$f(1) = 0, \quad f(x) = x, \quad f(x^2) = 2x^2 \implies A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Hence, A is not invertible and f is not an isomorphism.

(f) Let $\{1, x, x^2\}$ be a basis for $\mathcal{P}_2(\mathbb{R})$. Then

$$f(1) = 0, \quad f(x) = 0, \quad f(x^2) = 2x^2 \implies A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Hence, A is not invertible and f is not an isomorphism.

9. Let V be a vector space over $\mathbb{F}$. Recall that $\mathrm{Aut}(V)$ is the set of invertible endomorphisms $f \in \mathcal{L}(V)$. Hence, the following strict set inclusion holds: $\mathrm{Aut}V \subset \mathcal{L}(V)$, since not all endomorphisms are invertible (see examples in lectures). However, $\mathrm{Aut}V$ is not a vector space under the standard operations of function addition and scalar-function multiplication, as it does not have an additive identity: the zero function is not invertible, therefore not in $\mathrm{Aut}(V)$. If function addition is replaced with function composition, then the composition identity exists (it is the identity map), but commutativity fails.

10. Recall that A and B are similar (and write $A \approx B$) if there exists an invertible matrix M such that $B = M^{-1}AM$. We show that this is an equivalence relation on $\mathbb{R}^{n\times n}$.

- symmetry: $A \approx B \iff B = M^{-1}AM \iff A = MBM^{-1} \iff B \approx A$.
- reflexivity: let $M = I_n$; then $A = M^{-1}AM \iff A \approx A$.
- transitivity:

$$\begin{cases} A \approx B \\ B \approx C \end{cases} \iff \begin{cases} B = M^{-1}AM \\ C = N^{-1}BN \end{cases} \iff C = N^{-1}M^{-1}AMN = (MN)^{-1}A(MN) \iff A \approx C,$$

where we note that M, N are invertible matrices and so is MN.

11. First, let f be an automorphism. Then f is invertible, hence a bijection; being injective, its kernel is trivial. Let now f have trivial kernel, so that $\mathrm{nullity} f = 0$ and also f is injective. By the rank-nullity formula, $\mathrm{rank}\ f = \dim V - \mathrm{nullity}(f) = \dim V$. Since $f : V \to V$, we conclude that its codomain has the same dimension as its image, so that f is surjective. Hence f is a bijection and therefore invertible – an automorphism.

INVARIANCE

12. **(a)** $f(a \cdot \mathbf{0}) = a \cdot \mathbf{0} = \mathbf{0} \in U \implies f(U) \subseteq U$.

(b) $f(V) = \mathrm{im}\,(f) \subseteq V$.

(c) For any $\mathbf{v} \in \ker f$, $f(\mathbf{v}) = \mathbf{0} \in \ker f$, so that $f(\ker f) \subseteq \ker f$.

(d) For any $\mathbf{v} \in \mathrm{im}\, f$, $f(\mathbf{v}) \in \mathrm{im}\, f$, so that $f(\mathrm{im}\, f) \subseteq \mathrm{im}\, f$.

13. Let $f \in \mathcal{L}(V)$, $V = U \oplus W$, with U, W f-invariant subspaces of V. Let $\dim V = n$, $\dim U = k$, $\dim W = \ell$.

(a) Let $B_U = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}, B_W = \{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$ denote bases for U and W. Then $B := \{B_U, B_W\}$ is a basis for V since $V = U \oplus W$. Note that $f(\mathbf{u}_i) \in U$ and $f(\mathbf{w}_j) \in W$, due to the f-invariance stated in the question. The matrix representation of f with respect to B is therefore

$$\begin{cases} f(\mathbf{u}_1) = & \sum_{j=1}^{k} a_{j1}^U \mathbf{u}_j + \sum_{s=1}^{\ell} 0 \cdot \mathbf{w}_s \\ f(\mathbf{u}_2) = & \sum_{j=1}^{k} a_{j2}^U \mathbf{u}_j + \sum_{s=1}^{\ell} 0 \cdot \mathbf{w}_s \\ \dots & \dots \qquad \dots \\ f(\mathbf{u}_k) = & \sum_{j=1}^{k} a_{jk}^U \mathbf{u}_j + \sum_{s=1}^{\ell} 0 \cdot \mathbf{w}_s \\ f(\mathbf{w}_1) = & \sum_{j=1}^{k} 0 \cdot \mathbf{u}_j + \sum_{s=1}^{\ell} a_{s1}^W \cdot \mathbf{w}_s \\ f(\mathbf{w}_2) = & \sum_{j=1}^{k} 0 \cdot \mathbf{u}_j + \sum_{s=1}^{\ell} a_{s2}^W \cdot \mathbf{w}_s \\ \dots & \dots \qquad \dots \\ f(\mathbf{w}_\ell) = & \sum_{j=1}^{k} 0 \cdot \mathbf{u}_j + \sum_{s=1}^{\ell} a_{s\ell}^W \cdot \mathbf{w}_s, \end{cases}$$

which yields the block form of the matrix representation

$$A = \begin{bmatrix} A_U & O \\ O & A_W \end{bmatrix}, \quad A_U \in \mathbb{F}^{k \times k}, \quad A_W \in \mathbb{F}^{\ell \times \ell},$$

with $[A_U]_{ji} = a_{ji}^U, \quad [A_W]_{sr} = a_{sr}^W$, and with $i, j = 1, \dots, k,\ s, r, = 1, \dots, \ell$, for $k + \ell = n$.

(b) When $V = \bigoplus_k U_k$, where U_k are f-invariant subspaces of V, the matrix representation will also have block-diagonal structure, with the kth block having size equal to $\dim U_k$. If all U_k are one-dimensional, then A is diagonalisable, as the resulting matrix representation will be a diagonal matrix.

14. Let U be f-invariant. Then $f(U) \subseteq U$. Since f is invertible, $f(U) = \operatorname{im} f = U$. Hence,

$$f^{-1}(f(U)) = f^{-1}(U) \Longrightarrow U = f^{-1}(U).$$

Therefore, U is f^{-1}-invariant.

EIGENVALUES AND EIGENVECTORS.

15. The algebraic multiplicity of $\lambda = 1$ is $\alpha(\lambda) = 3$. Both matrices in this questions are defective as $\gamma(\lambda) < \alpha(\lambda)$, as shown below.

i. Let us find the eigenspace associated with $\lambda = 1$:

$$(1 \cdot I - A)\mathbf{x} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \Longleftrightarrow \begin{cases} -x_2 - x_3 & = 0 \\ -x_3 & = 0 \end{cases} \Longleftrightarrow x_2 = x_3 = 0, x_1 \in \mathbb{R},$$

so that the eigenspace of $\lambda = 1$ is

$$E_\lambda = \operatorname{span}\{\mathbf{e}_1\}.$$

Hence $\gamma(\lambda) = \dim E_\lambda = 1$.

ii. We proceed similarly:

$$(1 \cdot I - B)\mathbf{x} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \Longleftrightarrow x_3 = 0, x_1, x_2 \in \mathbb{R},$$

so that the eigenspace of $\lambda = 1$ is

$$E_\lambda = \text{span}\,\{\mathbf{e}_1, \mathbf{e}_2\}.$$

Hence $\gamma(\lambda) = \dim E_\lambda = 2$.

16. (a) We have

$$p_A(t) = \det(tI_3 - A) = \det \begin{bmatrix} t-3 & -4 & -4 \\ 3 & t-3 & 1 \\ -1 & 4 & t \end{bmatrix} = t^3 - 6t^2 + 13t - 20 = (t-4)(t^2 - 2t + 5).$$

Hence, the roots of $p(t)$ are $\lambda_1 = 4, \lambda_{2,3} = 1 \pm 2i$.
The corresponding eigenvectors can be found in the usual way.

$$(\lambda_1 I - A)\mathbf{x} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} 1 & -4 & -4 \\ 3 & 1 & 1 \\ -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \overset{x_3 = a}{\Longleftrightarrow} \begin{cases} x_1 - 4x_2 & = 4a \\ 3x_1 + x_2 & = -a \end{cases} \Longleftrightarrow \begin{cases} x_1 & = 0 \\ x_2 & = -a \end{cases} \Longrightarrow \mathbf{x} = \begin{bmatrix} 0 \\ -a \\ a \end{bmatrix}$$

which means that the eigenspace of A associated with λ_1 is

$$E_{\lambda_1} = \text{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Let now $\lambda_2 = 1 - 2i$. We find

$$(\lambda_2 I - A)\mathbf{x} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} -2-2i & -4 & -4 \\ 3 & -2-2i & 1 \\ -1 & 4 & 1-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}.$$

Adding rows 1 and 3, we find $x_1 = -x_3$. Setting $x_2 = a$, we find

$$(-2+2i)x_3 = 4a \Longrightarrow x_3 = -(1+i)a = -x_1,$$

which means that the eigenspace of A associated with λ_2 is

$$E_{\lambda_2} = \text{span} \left\{ \begin{bmatrix} 1+i \\ 1 \\ -(1+i) \end{bmatrix} \right\}.$$

Finally, the last eigenspace can be obtained by taking the complex conjugate of the expression for E_{λ_3}:

$$E_{\lambda_3} = \text{span} \left\{ \begin{bmatrix} 1-i \\ 1 \\ -(1-i) \end{bmatrix} \right\}.$$

(b) The diagonal canonical form is the eigenvalue decomposition of A over $\mathbb{C}$:

$$V^{-1}AV = D, \quad V = \begin{bmatrix} 0 & 1+i & 1-i \\ -1 & 1 & 1 \\ 1 & -(1+i) & -(1-i) \end{bmatrix}, \quad D = \begin{bmatrix} 4 & & \\ & 1-2i & \\ & & 1+2i. \end{bmatrix}$$

(c) The block-diagonal canonical form represents a decomposition over $\mathbb{R}$

$$V^{-1}AV = D, \qquad V = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & & \\ & 1 & -2 \\ & 2 & 1 \end{bmatrix}.$$

One can check that $AV = VD$

$$AV = \begin{bmatrix} 3 & 4 & 4 \\ -3 & 3 & -1 \\ 1 & -4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ -4 & 1 & -2 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 1 & -2 \\ & 2 & 1 \end{bmatrix} = VD.$$

17. We have

$$A\mathbf{v} = \lambda \mathbf{v} \Leftrightarrow \frac{1}{\lambda}\mathbf{v} = A^{-1}\mathbf{v}.$$

Hence if $(\lambda, \mathbf{v})$ is an eigenpair for A, then $(\lambda^{-1}, \mathbf{v})$ is an eigenpair for A^{-1}.

18. Only (a) is true. For (b), (c), find counter-examples. Assuming B is non-singular, we have

(a) $(AB)\mathbf{v} = \lambda \mathbf{v} \Leftrightarrow (BAB)\mathbf{v} = \lambda B\mathbf{v} \Leftrightarrow (BA)\mathbf{w} = \lambda \mathbf{w}, (\mathbf{w} = B\mathbf{v})$.

19. (a) We have

$$A\mathbf{v} = \lambda \mathbf{v} \Longleftrightarrow (\lambda I - A)\mathbf{v} = 0 \Longleftrightarrow \det(\lambda I - A) = 0.$$

If A is lower triangular, the expression for the determinant simplifies to

$$\det(\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = \prod_{k=1}^{n} (\lambda - a_{kk}).$$

Hence,

$$\det(\lambda I - A) = 0 \Longleftrightarrow \lambda = a_{kk},$$

for some k. Since A has n eigenvalues, this must hold for all $k = 1, \ldots, n$, so that the spectrum (set of eigenvalues) of A is

$$\operatorname{sp}(A) = \{a_{kk}, k = 1, \ldots, n\}.$$

(b) By the properties of determinants, $\det A = \det A_1 \cdot \det A_2$. Since

$$\lambda I - A = \begin{bmatrix} \lambda I - A_1 & \\ & \lambda I - A_2 \end{bmatrix},$$

we find

$$\det(\lambda I - A) = \det(\lambda I - A_1)\det(\lambda I - A_2),$$

and the given set equality holds. The same applies to the block lower (or upper) triangular case.

20. (a) This can be shown by induction, with base case $m = 1$. Assuming the relation holds for $m = k - 1$, we find

$$A^k = AA^{k-1} = (VDV^{-1})(VD^{k-1}V^{-1}) = VD^kV^{-1}.$$

(b) Given $p(t) = a_0 + a_1 t + \cdots + a_n t^n$, let $p(A)$ denote the matrix

$$p(A) = a_o I + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

Then

$$p(A) = a_0 VV^{-1} + a_1 VDV^{-1} + a_2 VD^2V^{-1} + \cdots + a_n VD^nV^{-1} = V(a_0 I + a_1 D + a_2 D^2 + \cdots + a_n D^n)V^{-1} = Vp(D)V^{-1}.$$

(c) By part **(b)**,

$$p_A(A) = Vp_A(D)V^{-1} = V \begin{bmatrix} p_A(\lambda_1) & & & \\ & p_A(\lambda_2) & & \\ & & \ddots & \\ & & & p_A(\lambda_n) \end{bmatrix} V^{-1} = O_n,$$

since $p_A(\lambda_i) = 0$, by definition of characteristic polynomial of A.