LH: Homework sheet 2 – Solutions – Linear Algebra

1. (a) To find the matrix representation A of f, we evaluate f at the elements of B and express the result in terms of B to generate each column of A. We have

$$\begin{cases} f(\mathbf{i}) = \vec{c}\times\mathbf{i} = & c_3\mathbf{j} - c_2\mathbf{k} \\ f(\mathbf{j}) = \vec{c}\times\mathbf{j} = & -c_3\mathbf{i} + c_1\mathbf{k} \\ f(\mathbf{k}) = \vec{c}\times\mathbf{k} = & c_2\mathbf{i} - c_1\mathbf{j} \end{cases} \implies \quad A = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}.$$ [**8**]

(b) To find $\ker A$ we solve $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \implies \begin{cases} -c_3y + c_2z & = 0 \\ c_3x - c_1z & = 0 \\ -c_2x + c_1y & = 0 \end{cases} \overset{z=\alpha}{\implies} \begin{cases} y = & \alpha c_2/c_3, \\ x = & \alpha c_1/c_3, \\ z = & \alpha, \end{cases}$$

so that

$$\ker A = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{\alpha}{c_3}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},\ \alpha \in \mathbb{R}. \right\} = \text{span}\left\{ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right\} = \text{span}\,\{\mathbf{c}\}.$$

Hence, by Proposition 11.1, $\ker f = \text{span}\,\{\vec{c}\}$. [**4**]

(c) We have

$$A\mathbf{x} = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_2z - c_3y \\ c_3x - c_1z \\ c_1y - c_2x \end{bmatrix},$$ [**2**]

so that

$$\begin{cases} \mathbf{c}^TA\mathbf{x} = \mathbf{c}^T(A\mathbf{x}) & = c_1(c_2z - c_3y) + c_2(c_3x - c_1z) + c_3(c_1y - c_2x) = 0, \\ \mathbf{x}^TA\mathbf{x} = \mathbf{x}^T(A\mathbf{x}) & = x(c_2z - c_3y) + y(c_3x - c_1z) + z(c_1y - c_2x) = 0. \end{cases}$$ [**2**]

Using the diagram from part **(a)**, $\mathbf{x} = \varphi_B(\vec{v})$ and $A\mathbf{x} = \varphi_B(f(\vec{v}))$. By the definition of the inner product provided, we have

$$\vec{c}\cdot f(\vec{v}) = \mathbf{c}^TA\mathbf{x} = 0, \quad \vec{v}\cdot f(\vec{v}) = \mathbf{x}^TA\mathbf{x} = 0.$$ [**2**]

Hence, $f(\vec{v})\cdot(a\vec{c} + b\vec{v}) = 0$ for any $a, b \in \mathbb{R}$, so that $f(\vec{v}) \perp \text{span}\,\{\vec{c}, \vec{v}\}$. [**1**]

(d) We have, using the fact that $\vec{c}$ is a unit vector,

$$A^2\mathbf{x} = A(A\mathbf{x}) = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}\begin{bmatrix} c_2z - c_3y \\ c_3x - c_1z \\ c_1y - c_2x \end{bmatrix} = \begin{bmatrix} c_1c_3z - c_3^2x + c_1c_2y - c_2^2x \\ c_3c_2z - c_3^2y - c_1^2y + c_1c_2x \\ c_2c_3y - c_2^2z + c_1c_3x - c_1^2z \end{bmatrix}$$

$$= \begin{bmatrix} c_1(c_1x + c_2y + c_3z) \\ c_2(c_1x + c_2y + c_3z) \\ c_3(c_1x + c_2y + c_3z) \end{bmatrix} - \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (\mathbf{c}^T\mathbf{x})\mathbf{c} - \mathbf{x}.$$ [**6**]

(e) The characteristic polynomial is $p_A(t) = \det(tI_3 - A)$. We find, using the fact that $\vec{c}$ is a unit vector,

$$p_A(t) = \det \begin{bmatrix} t & c_3 & -c_2 \\ -c_3 & t & c_1 \\ c_2 & -c_1 & t \end{bmatrix} = t(t^2+c_1^2)+c_3(c_1c_2+c_3t)-c_2(c_3c_1-c_2t) = t^3+(c_1^2+c_2^2+c_3^2)t = t^3+t.$$

Hence, the eigenvalues of A are $\lambda_1 = 0, \lambda_2 = i, \lambda_3 = -i$. [**6**]

Since the eigenvalues are distinct, the matrix is diagonalisable and we have 3 f-invariant subspaces over $\mathbb{C}$. Since the characteristic polynomial factorises over $\mathbb{R}$ into one linear term and one irreducible quadratic term, there are only two f-invariant subspaces over $\mathbb{R}$. Alternatively, since the real canonical form includes 2 blocks (with sizes 1×1 and 2×2), there are only 2 A-invariant (and hence f-invariant) subspaces over $\mathbb{R}$. [**2**]

By part **(b)**, since $\ker f$ is nontrivial, we find that span $\{\vec{c}\}$ is one of the two f-invariant subspaces. This can also be claimed by noting that $(0, \vec{c})$ is an eigenpair for f. [**1**]

(f) Using the hint, we note that

$$A^2\mathbf{n} = (\mathbf{c}^T\mathbf{n})\mathbf{c} - \mathbf{n} = -\mathbf{n},$$

since $0 = \vec{c} \cdot \vec{n} = \mathbf{c}^T\mathbf{n}$. Hence, $\varphi_B(U) = \text{span}\{\mathbf{n}, A\mathbf{n}\}$ is mapped by the matrix representation of f to span $\{A\mathbf{n}, A^2\mathbf{n}\} = \text{span}\{A\mathbf{n}, -\mathbf{n}\} = \text{span}\{\mathbf{n}, A\mathbf{n}\} = \varphi_B(U)$. Hence, U is f-invariant. [**4**]

Finally, note that $\vec{c} \perp \vec{n}$ and also, by part **(c)** with $\vec{v} = \vec{n}$, $\vec{c} \perp f(\vec{n}) = \vec{c} \times \vec{n}$. Hence, $\vec{c} \perp U$ and the geometric interpretation of U is that of a plane with normal vector $\vec{c}$. [**2**]