University of Birmingham
School of Mathematics

2RCA/2RCA3 Real and Complex Analysis

Part A: Real Analysis Semester 2

Problem Sheet 2
Model Solutions

The questions indicated with $\boxed{\text{SUM}}$ below constitute the second of the summative assessments of this module, and will contribute to the overall mark of the module. Please submit your answers to the questions indicated with a $\boxed{\text{SUM}}$ below by the deadline of **17:00, Thursday 15 February 2024**, as a single pdf file into 2RCA/2RCA3 Assignment 2: Real Analysis in the Canvas page of the module.

Please note that it is the student's responsibility to make sure that their submission has been uploaded correctly into Canvas and that the uploaded file contains the submission of their assessment (eg, the uploaded file is not corrupted and contains all the pages of their answers).

Please be also aware that where assessments are submitted late without an extension being granted that has been confirmed by the Wellbeing Officer, the standard University penalty of a 5% will be imposed for each working day that the assignment is late. Any work submitted after five working days passed the deadline of submission, with no extension granted by the Wellbeing Officer/s, will be awarded a 0% mark.

In addition to the $\boxed{\text{SUM}}$-questions, Problem Sheet 2 also contains exercises that will not contribute to your module mark. You are strongly encouraged to attempt these before the relevant Guide Study sessions and/or during the course of the semester. Solutions to all exercises will be provided.

The examples/feedback classes (Guided Study) and the lecturer's office hours should be used to ask about the problem sheets.

Q1. Using the definition, show that the sequence $\{f_n\}_{n=1}^{\infty}$ defined by

$$f_n(x) = \begin{cases} 1 - nx, & \text{if} \quad 0 \le x \le \frac{1}{n}, \\ 0, & \text{if} \quad \frac{1}{n} < x \le 1, \end{cases}$$

converges pointwise to the function $f : [0,1] \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 < x \le 1, \\ 1, & \text{if} \quad x = 0. \end{cases}$$

Solution. First notice that, from the definition of the functions f_n, we have that $f_n(0) = 1$ for all $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1 = 1.$$

On the other hand, if we fix $0 < x \leq 1$, since $\frac{1}{n} \longrightarrow 0$ as $n \to \infty$, it follows that there exists $N \in \mathbb{N}$ such that

$$\frac{1}{n} < x \qquad \text{whenever} \qquad n \geq N.$$

Therefore, from the definition of f_n, it follows that

$$f_n(x) = 0 \qquad \text{whenever} \qquad n \geq N,$$

and hence, for $0 < x \leq 1$,

$$\lim_{n\to\infty} f_n(x) = 0.$$

$\square$

Q2. Using the definition of uniform convergence, show that the sequence $\{f_n\}_{n=1}^{\infty}$ defined by

$$f_n(x) = \frac{x}{1+xn}, \qquad x > 0$$

converges uniformly to the function f given by

$$f(x) = 0 \qquad x > 0.$$

Solution. Let $\varepsilon > 0$ be given. Observe that for all $x > 0$, we have that

$$|f_n(x) - f(x)| = \left|\frac{x}{1+xn} - 0\right| = \frac{x}{1+xn} \leq \frac{x}{xn} = \frac{1}{n} \tag{1}$$

since $1 + xn \geq xn$ for all $x > 0$. Therefore, taking $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$ (independently of $x > 0$), from (1) we have that, whenever $n \geq N$,

$$|f_n(x) - f(x)| \underbrace{=}_{(1)} \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus the sequence $(f_n)_{n=1}^{\infty}$ converges uniformly to the function f given by $f(x) = 0$, $\forall x > 0$. $\square$

Q3. Find the pointwise limit (if exists) of the sequence $\{f_n\}$ of functions defined from X into $\mathbb{R}$ in each of the following cases. In each of the examples determine whether the convergence is uniform.

(i) $X = [0, 1]$,

$$f_n(x) = \begin{cases} 1 - nx, & \text{if} \quad 0 \leq x \leq \frac{1}{n}, \\ 0, & \text{if} \quad \frac{1}{n} < x \leq 1. \end{cases}$$

(ii) $X = [0, 1]$, $f_n(x) = x^n$.

(iii) $X = [0, 1]$, $f_n(x) = \dfrac{x^n}{n}$.

Remark: In the above exercises, if a pointwise limit f of the sequence $\{f_n\}$ exists, you are not require to justify why. You just need to find a explicit formula for the pointwise limit.

Solution. (i) We have seen in a previous exercise that the pointwise limit of sequence of function (f_n) is the function

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 < x \leq 1 \\ 1, & \text{if} \quad x = 0 \end{cases}$$

The sequence (f_n) <u>does not</u> converge uniformly to f. There are (at least) two ways of justifying this assertion: On the one hand, we can argue by contradiction and assume that (f_n) converges unformly to f. If this is the case, since the function f_n are continuous on $X = [0,1]$, we can conclude that f is a continuous function on $[0,1]$. This is a contradiction. Notice that the function f is not continuous at 0 ($\lim_{x\to 0^+} f(x) = 0$, $f(0) = 1$ and $0 \neq 1$).

Alternatively, notice that for fixed $k \in \mathbb{N}$ and $k \geq 2$, definying $n_k = k$ and $x_k = \frac{1}{k^2}$, for the subsequence $(f_{n_k})_{k=2}^{\infty}$ and the sequence of points $(x_k) \subseteq [0,1]$, we have that

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= \left| f_k\left(\frac{1}{k^2}\right) - f\left(\frac{1}{k^2}\right) \right| = \left| (1 - k\left(\frac{1}{k^2}\right)) - 0 \right| \\ &= 1 - \frac{1}{k} \geq 1 - \frac{1}{2} = \frac{1}{2}. \qquad \forall k \geq 2, \end{aligned}$$

Hence, from the definiton of uniform convergence, it follows that (f_n) does not converge uniformly to f.

(ii) We have seen in the lecture notes that the pointwise limit of sequence of function (f_n) is the function

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 \leq x < 1 \\ 1, & \text{if} \quad x = 1 \end{cases}$$

The sequence (f_n) <u>does not</u> converge uniformly to f. Indeed, arguing by contradiction, if one assumes that (f_n) converges uniformly to f. If this is the case, since the function f_n are continuous on $X = [0,1]$, we can conclude that f is a continuous function on $[0,1]$. This is a contradiction. Notice that the function f is not continuous at 1 ($\lim_{x\to 1^-} f(x) = 0$, $f(1) = 1$ and $0 \neq 1$).

(iii) The pointwise limit of the sequence (f_n) is the function $f : [0,1] \longrightarrow \mathbb{R}$ given by

$$f(x) = 0 \qquad , \quad \forall x \in [0,1].$$

Indeed, if $x = 0$, then

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0.$$

If $x = 1$, then

$$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Finally, if $0 < x < 1$, using the algebra of limits for sequences of real numbers, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{n} = \left(\lim_{n\to\infty} x^n\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) = 0 \times 0 = 0.$$

The converge of (f_n) to f is uniform: Given any $\varepsilon > 0$, we first observe that for any $x \in [0,1]$

$$|f_n(x) - f(x)| = \left| \frac{x^n}{n} - 0 \right| = \frac{|x|^n}{n} \leq \frac{1}{n}. \tag{2}$$

Therefore choosing $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$ (independently of $x \in [0,1]$), from (2) it follows that, if $n \geq N$ then

$$|f_n(x) - f(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$\square$

Q4. Let $(f_n)_{n=1}^{\infty}$ be the sequence of functions $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f_n(x) = \left(\sin\left(\frac{x}{\pi}\right)\right)^{2n}, \qquad \text{for all} \qquad x \in \mathbb{R}.$$

(i) Find the pointwise limit of the sequence $(f_n)_{n=1}^{\infty}$. Justify your answer.

(ii) Does the sequence $(f_n)_{n=1}^{\infty}$ converge uniformly on $\mathbb{R}$? Justify your answer.

Solution. (i) We distinguish the following two cases:

- If for all $k \in \mathbb{Z}$ $x \neq \frac{\pi^2}{2} + k\pi^2$, then $|\sin(x/\pi)| < 1$, so that $0 \leq (\sin(x/\pi))^2 < 1$, and in this case

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} [(\sin(x/\pi))^2]^n = 0.$$

- If $x = \frac{\pi^2}{2} + k\pi^2$ for some $k \in \mathbb{Z}$, then $(\sin(x/\pi))^2 = 1$, and in this case

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} [\left(\sin\left(\frac{x}{\pi}\right)\right)^2]^n = \lim_{n\to\infty} 1^n = 1.$$

The above argument shows that the sequence (f_n) converges pointwise to the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \neq \frac{\pi^2}{2} + k\pi^2 \quad \forall\, k \in \mathbb{Z} \\ 1, & \text{if} \quad x = \frac{\pi^2}{2} + k\pi^2 \quad \text{for some } k \in \mathbb{Z} \end{cases}$$

(ii) No, the sequence (f_n) does not converge uniformly on $\mathbb{R}$. Indeed, arguing by contradiction, if we assume that (f_n) converges uniformly to some f on $\mathbb{R}$, since the sequence (f_n) is a sequence of continuous functions, then we know that f is a continuous function. However, on the other hand, since (f_n) converges uniformly to f, then we know that (f_n) converges pointwise to f, and from the uniqueness of the limit of convergent sequences, the function f is the function found in part (a), that is the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \neq \frac{\pi^2}{2} + k\pi^2 \quad \forall\, k \in \mathbb{Z}, \\ 1, & \text{if} \quad x = \frac{\pi^2}{2} + k\pi^2 \quad \text{for some } k \in \mathbb{Z}. \end{cases}$$

Notice that f is not a continuous function (for any $k \in \mathbb{Z}$, defining $x_k = \frac{\pi^2}{2} + k\pi^2$ we have that $\lim_{x\to x_k} f(x) = 0 \neq 1 = f(x_k)$, so f is not continuous at x_k). This gives the contradiction.

$\square$

$\boxed{\text{SUM}}$ **Q5**. Let $(f_n)_{n=4}^{\infty}$ be the sequence of functions $f_n : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} \dfrac{n^4}{16}\left(x - \dfrac{4}{n^2}\right)^2, & \text{if} \quad x \in \left[0, \dfrac{4}{n^2}\right], \\ 0, & \text{if} \quad x \in \left(\dfrac{4}{n^2}, \dfrac{9}{n^2}\right), \\ \dfrac{n^2}{n^2-9}\left(x - \dfrac{9}{n^2}\right), & \text{if} \quad x \in \left[\dfrac{9}{n^2}, 1\right]. \end{cases}$$

(i) Find the pointwise limit of the sequence $(f_n)_{n=4}^{\infty}$.

(ii) Does the sequence $(f_n)_{n=4}^{\infty}$ converge uniformly on $[0,1]$ to its pointwise limit?

Justify your answers. (In particular, state any result you use in the justification of your answer and justify its application.)

Solution. (i) We distinguish the following two cases:

- If for $x = 0$, by definition of f_n we have that $f_n(0) = 1$ for all n. Therefore

$$f_n(0) = 1 \longrightarrow 1 \qquad \text{as} \qquad n \to \infty.$$

- If $x \in (0,1]$, since $\frac{9}{n^2}$ tends to 0 as $n \to \infty$, we have that there exists $N \in \mathbb{N}$ such that

$$\frac{9}{n^2} < x, \qquad \text{whenever} \qquad n \geq N.$$

Therefore, from the definition of f_n, it follows that

$$f_n(x) = \frac{n^2}{n^2-9}\left(x - \frac{9}{n^2}\right) \qquad \text{whenever} \qquad n \geq N,$$

and we have that

$$f_n(x) == \frac{n^2}{n^2-9}\left(x - \frac{9}{n^2}\right) \longrightarrow x \qquad \text{as} \qquad n \to \infty.$$

The above argument shows that the sequence (f_n) converges pointwise to the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if} \quad x = 0 \\ x, & \text{if} \quad x \in (0,1] \end{cases} \tag{3}$$

(ii) No, the sequence (f_n) does not converge uniformly to f (its pointwise limit) on $[0,1]$. Indeed, arguing by contradiction, assume that (f_n) converges uniformly f on $[0,1]$, where the function f is the function given in (3).

Notice that, since polynomials are continuous functions, for each $n \in \mathbb{N}$ f_n is continuous on $[0, \frac{4}{n^2}) \cup (\frac{4}{n^2}, \frac{9}{n^2}) \cup (\frac{9}{n^2}, 1]$. Moreover, using the continuity of polynomials, we have that

$$\lim_{x \to \frac{4}{n^2}^-} f_n(x) = \lim_{x \to \frac{4}{n^2}^-} \frac{n^4}{16}\left(x - \frac{4}{n^2}\right)^2 = 0,$$

$$\lim_{x \to \frac{4}{n^2}^+} f_n(x) = \lim_{x \to \frac{4}{n^2}^+} 0 = 0,$$

and $f_n(4/n^2) = \frac{n^4}{16}\left(\frac{4}{n^2} - \frac{4}{n^2}\right)^2 = 0$. Since,

$$\lim_{x\to \frac{4}{n^2}^-} f_n(x) = \lim_{x\to \frac{4}{n^2}^+} f_n(x) = f_n\left(\frac{4}{n^2}\right),$$

we have that f_n is continuous at $4/n^2$. A similar argument shows that f_n is continuous at $9/n^2$ (we omit the details here). Consequently, we have that f_n is continuous on $[0,1]$.

Since the sequence (f_n) is a sequence of continuous functions and we are assuming that f_n converges uniformly to f, then we know that f is a continuous function (result from lectures). However, the function f in (3) is not continuous on $[0,1]$ (indeed $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$ which is different from $f(0) = 1$). This gives the contradiction. Therefore the sequence (f_n) does not converge uniformly on $[0,1]$ to its pointwise limit.

Note: Alternatively, one can show that the sequence $(f_n)_{n=4}^{\infty}$ does not converge uniformly to its pointwise limit using the criterion for non-uniform convergence. To this end, consider for $k \in \mathbb{N}$

$$n_k = k \qquad \text{and} \qquad x_k = \frac{1}{k^2}.$$

Using the definition of f_n and the pointwise limit f, observe that for $k \geq 2$ we have

$$\begin{aligned}|f_{n_k}(x_k) - f(x_k)| &= \left|f_k\left(\frac{1}{k^2}\right) - f\left(\frac{1}{k^2}\right)\right| \leq \left|\frac{k^4}{16}\left(\frac{1}{k^2} - \frac{4}{k^2}\right)^2 - \frac{1}{k^2}\right| \\ &= \left|\frac{9}{16} - \frac{1}{k^2}\right| = \frac{9}{16} - \frac{1}{k^2} \geq \frac{9}{16} - \frac{1}{4} = \frac{5}{16} > 0.\end{aligned}$$

Consequently the sequence (f_n) does not converge uniformly to f on $[0,1]$.

□

Q6. Suppose that $\alpha \in \mathbb{R}$ and the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x-y|^{\alpha}, \qquad \text{for all} \qquad x \neq y. \tag{4}$$

(i) Show that if $\alpha > 1$ then f is differentiable at every $x_0 \in \mathbb{R}$.

(ii) Give an example of a function satisfying (4) for $\alpha = 1$, which fails to be differentiable at 0.

Solution. (i) Suppose that $\alpha > 1$. We wish to show that f is differentiable at x_0. Let $\epsilon > 0$ be arbitrary and choose $\delta = \epsilon^{1/(\alpha-1)}$. Now, by condition (4),

$$\left|\frac{f(x_0+h) - f(x_0)}{h} - 0\right| \leq \frac{|(x_0+h) - x_0|^{\alpha}}{|h|} = |h|^{\alpha-1} < (\epsilon^{1/(\alpha-1)})^{\alpha-1} = \epsilon$$

whenever $0 < |h| < \delta$. Hence the limit

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists (and equals 0) if $\alpha > 1$, as claimed.

(ii) An example would be $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$. We have seen that this function is not differentiable at 0, but satisfies the condition with $\alpha = 1$.

□

Q7. Let $\alpha, \beta \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} \alpha x^2 + \beta & \text{if } x \leq 1 \\ 4x & \text{if } x > 1. \end{cases}$$

For which values of α and β is g continuous at $x = 1$? For which values of α and β is g differentiable at $x = 1$? Justify any assertions that you make.

Solution. The function f is continuous at $x = 1$ if and only if the left and right hand limits

$$\lim_{x \to 1-} f(x) \quad \text{and} \quad \lim_{x \to 1+} f(x)$$

exist and equal $f(1) = \alpha + \beta$.

Now,

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (\alpha x^2 + \beta) = \alpha + \beta$$

by the algebra of limits or by the continuity of the function $\alpha x^2 + \beta$ at the point $x = 1$.

On the other hand,

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (4x) = 4,$$

(by the continuity of the function $f(x) = 4x$). Thus f is continuous at 1 if and only if $\alpha + \beta = 4$.

For differentiability we also need to consider the existence and equality of certain left and right hand limits. The function f is differentiable at 1 if and only if the left and right hand limits

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} \quad \text{and} \quad \lim_{h \to 0+} \frac{f(1+h) - f(1)}{h}$$

exist and are equal.

Also, note that since differentiability at a point implies continuity at a point, we must have $\alpha + \beta = 4$ from the above continuity considerations. However, this will be apparent from the considerations below as well.

Now,

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0-} \frac{\alpha(1+h)^2 + \beta - (\alpha + \beta)}{h} = \lim_{h \to 0-} \alpha(h+2) = 2\alpha.$$

(since the function $x + 2$ is continuous on the real line).

On the other hand,

$$\lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{4(1+h) - (\alpha + \beta)}{h}$$

exists if and only if $\alpha + \beta = 4$, and in this case it equals 4. Hence the above left and right hand limits are equal if and only if $2\alpha = 4$. We therefore conclude that f is differentiable at $x = 1$ if and only if $\alpha = 2$ and $\beta = 4 - \alpha = 2$. $\square$

Q8. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x \sin(x) \sin\left(\frac{1}{x}\right) & \text{if } x \in (-1, 1) \text{ and } x \neq 0 \\ 0 & \text{if } x = 0 \\ x + 1 & \text{if } x \in (-\infty, -1] \cup [1, \infty). \end{cases}$$

Show that f is differentiable at 0 and give the value of $f'(0)$. Justify your answer.

Solution. For $0 < |h| < 1/2$ we have

$$\frac{f(h)-f(0)}{h} = \frac{h\sin(h)\sin(\frac{1}{h})}{h} = h\frac{\sin(h)}{h}\sin(\tfrac{1}{h}).$$

Now

$$\frac{\sin(h)}{h} \to 1$$

as $h \to 0$ by standard limits. Also

$$h\sin(\tfrac{1}{h}) \to 0$$

as $h \to 0$. This follows because $|h\sin(\frac{1}{h})| \le |h|$ for $h \neq 0$, and the Sandwich Theorem. Therefore

$$\frac{f(h)-f(0)}{h} \to 0$$

as $h \to 0$ by the algebra of limits. This means f is differentiable at 0 and $f'(0) = 0$. □

Q9. Let $\alpha \in \mathbb{N}$ be fixed and let the function $f : (-\frac{\pi}{4}, \frac{\pi}{4}) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{x^{3\alpha}\cos(x)}{\sin(x)}, & \text{if } x \neq 0 \quad \text{and} \quad x \in (-\frac{\pi}{4}, \frac{\pi}{4}), \\ 0, & \text{if } x = 0. \end{cases}$$

(i) Show that f is a differentiable function for all $\alpha \in \mathbb{N}$ and give the value of the derivative $f'(x)$ for all $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$.
(ii) Is f', the derivative of f, a continuous function at 0 when $\alpha = 2$?

Justify your answers.

Solution.
(i) Let $\alpha \in \mathbb{N}$. By definition, f is differentiable at 0 if the limit

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^{3\alpha}\cos(x)}{x\sin(x)} = \lim_{x\to 0} x^{3\alpha-2}\cos(x)\frac{1}{\frac{\sin(x)}{x}}$$

exists. Now, since $\lim_{x\to 0} x^{3\alpha-2} = 0$ for $\alpha \in \mathbb{N}$, $\lim_{x\to 0}\cos(x) = 1$ and $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, from the algebra of limits it follows that the above limit exists and

$$\lim_{x\to 0} x^{3\alpha-2}\cos(x)\frac{1}{\frac{\sin(x)}{x}} = 0.$$

Thus f is differentiable at 0 and $f'(0)$=0.

When $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$ and $x \neq 0$, since polynomials, sine and cosine functions are differentiabe on the real line and $\sin(x) \neq 0$ at these points, from the algebra of differentiable functions it follows that f is differentiable at x and

$$f'(x) = \frac{(3\alpha x^{3\alpha-1}\cos(x) - x^{3\alpha}\sin(x))\sin(x) - x^{3\alpha}(\cos(x))^2}{(\sin(x))^2}.$$

The above arguments show that f is dfferentiable on $(-\frac{\pi}{4}, \frac{\pi}{4})$ and

$$f'(x) = \begin{cases} \dfrac{(3\alpha x^{3\alpha-1}\cos(x) - x^{3\alpha}\sin(x))\sin(x) - x^{3\alpha}(\cos(x))^2}{(\sin(x))^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

(ii) When $\alpha = 2$ the derivative of f, ,f' is given by

$$f'(x) = \begin{cases} \dfrac{(6x^5\cos(x) - x^6\sin(x))\sin(x) - x^6(\cos(x))^2}{(\sin(x))^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

We will continue to show that f' is continuous at 0, that is

$$\lim_{x\to 0} f'(x) = f'(0),$$

or equivalently

$$\lim_{x\to 0} \frac{(6x^5\cos(x) - x^6\sin(x))\sin(x) - x^6(\cos(x))^2}{(\sin(x))^2} = 0.$$

Indeed, notice that, since $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$, and $[(6x^3\cos(x) - x^4\sin(x))\sin(x) - x^4(\cos(x))^2]$ is continuous at 0 with value zero at 0, using the algebra of limits it follows that

$$\lim_{x\to 0} [(6x^3\cos(x) - x^4\sin(x))\sin(x) - x^4(\cos(x))^2] \cdot \frac{1}{\left(\frac{\sin(x)}{x}\right)^2} = 0.$$

□

Q10. Let $\alpha, \beta \in \mathbb{Z}$ be fixed and let the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^\alpha(\sin(x))^\beta\cos\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(i) Show that f is differentiable at 0 if and only if $\alpha + \beta > 1$. In the case $\alpha + \beta > 1$, give the value of $f'(0)$.
(ii) In the case $\alpha + \beta > 1$, explain why f is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and give the value of $f'(x)$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
(iii) For which values of α and β is f continuous at $x = 0$?

Justify your answers.

Solution. (i) For $h \neq 0$ sufficiently small,

$$\begin{aligned}\frac{f(h) - f(0)}{h} &= h^{\alpha-1}(\sin(h))^\beta\cos(\tfrac{1}{h}) \\ &= h^{\alpha+\beta-1}\left(\frac{\sin(h)}{h}\right)^\beta\cos(\tfrac{1}{h}).\end{aligned}$$

We note that for any $\beta \in \mathbb{Z}$ we have

$$\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)^\beta = 1$$

using the algebra of limits and the standard limit $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$.

Now we claim that, if $\gamma \in \mathbb{Z}$, then

(5) $$\lim_{h\to 0} h^\gamma\cos(\tfrac{1}{h}) \quad \text{exists if and only if } \gamma > 0$$

and when $\gamma > 0$ we have $\lim_{h\to 0} h^\gamma\cos(\frac{1}{h}) = 0$.

To prove the claim, let $g(h) = h^\gamma\cos(\frac{1}{h})$ and note that

$$-|h|^\gamma \leq g(h) \leq |h|^\gamma$$

for all $h \neq 0$ and if $\gamma > 0$ then $|h|^\gamma \to 0$ as $h \to 0$.

Conversely, if $\gamma \le 0$ then we define the sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ by

$$a_n = \frac{1}{2n\pi}, \qquad b_n = \frac{1}{(2n+\frac{1}{2})\pi}.$$

Then $a_n, b_n \neq 0$ for each n and $a_n, b_n \to 0$ as $n \to \infty$. However

$$g(a_n) = (2n\pi)^{-\gamma}, \qquad g(b_n) = 0$$

for each n. So

$$g(a_n) \to \begin{cases} 1 & \text{if } \gamma = 0 \\ \infty & \text{if } \gamma < 0 \end{cases}$$

and $g(b_n) \to 0$ as $n \to \infty$. This means $\lim_{h\to0} g(h)$ does not exist.

Since $\alpha+\beta > 1$ if and only if $\gamma > 0$, where $\gamma = \alpha+\beta-1$, it follows from the above claim that f is differentiable at $x = 0$ if and only if $\alpha+\beta > 1$. In the case $\alpha+\beta > 1$ we have also shown that $f'(0) = 0$.

(ii) Suppose that $\alpha+\beta > 1$. From part (i) we know that f is differentiable at 0 and $f'(0) = 0$. For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $x \neq 0$, since $\sin(x) \neq 0$, $x \neq 0$, and sine, cosine and polynomial functions are differentiable on $\mathbb{R}$, using the algebra of differentiable functions we have that f is differentiable at any $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$. Moreover, for $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$,

$$\begin{aligned} f'(x) &= \alpha x^{\alpha-1}(\sin(x))^{\beta} \cos\left(\frac{1}{x}\right) + \beta x^{\alpha}(\sin(x))^{\beta-1}\cos(x)\cos\left(\frac{1}{x}\right) \\ &+ x^{\alpha-2}(\sin(x))^{\beta} \sin\left(\frac{1}{x}\right) \end{aligned}$$

The above argument shows that if $\alpha+\beta > 1$, then f is differentiable on $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$f'(x) = \begin{cases} \alpha x^{\alpha-1}(\sin(x))^{\beta}\cos\left(\frac{1}{x}\right) + \beta x^{\alpha}(\sin(x))^{\beta-1}\cos(x)\cos\left(\frac{1}{x}\right) & \text{if } \quad x \neq 0 \\ \qquad +x^{\alpha-2}(\sin(x))^{\beta}\sin\left(\frac{1}{x}\right), & \\ 0, & \text{if } \quad x = 0 \end{cases}$$

(iii) For $x \neq 0$ sufficiently small,

$$f(x) = x^{\alpha+\beta}\left(\frac{\sin(x)}{x}\right)^{\beta}\cos(\tfrac{1}{x})$$

and from the above claim in (5) we know $\lim_{x\to0} f(x)$ exists and is equal to 0 if and only if $\alpha+\beta > 0$. Since $f(0) = 0$, we get that f is continuous if and only if $\alpha+\beta > 0$.

□

Q11. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} (x-1)^2\cos(\frac{1}{x-1}), & \text{if } x \neq 1 \\ 0, & \text{if } x = 1. \end{cases}$$

Determine those $x_0 \in \mathbb{R}$ for which f is differentiable at x_0. Give the value of $f'(x_0)$. If $f'(x_0)$ does not exist, you must justify why it does not exists..

(ii) For the function f given in Part (i), determine whether the statement

$$f'(1) = \lim_{x\to 1} f'(x)$$

is true or false. Justify your answer.

Solution. (i) We know that the function g given by $g(x) = \frac{1}{x-1}$ is differentiable at x_0 whenever $x_0 \neq 1$ with

$$g'(x_0) = -\frac{1}{(x_0-1)^2}.$$

It now follows from the Product Rule and the Chain Rule that whenever $x_0 \neq 1$ the function f is differentiable at x_0 with

$$f'(x_0) = 2(x_0-1)\cos(\tfrac{1}{x_0-1}) + \sin(\tfrac{1}{x_0-1}).$$

The case $x_0 = 1$ must be treated differently. We claim that f is differentiable at x_0 with $f'(x_0) = 0$. To prove this, note that whenever $h \neq 0$ we have

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{f(1+h)-f(1)}{h} = h\cos(\tfrac{1}{h}).$$

This means

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = 0$$

and thus $f'(x_0) = 0$ when $x_0 = 1$.

Note that $h\cos(\frac{1}{h}) \to 0$ as $h \to 0$ because, for each $h \neq 0$, we have

$$-|h| \leq h\cos(\tfrac{1}{h}) \leq |h|$$

and the claim now follows from the Sandwich Theorem.

(ii) We showed in Part (i) that $f'(1) = 0$. We also showed that whenever $x \neq 1$ we have

$$f'(x) = 2(x-1)\cos(\tfrac{1}{x-1}) + \sin(\tfrac{1}{x-1})$$

and therefore

$$\lim_{x\to 1} f'(x)$$

does not exist. To justify this, suppose that $\lim_{x\to 1} f'(x)$ did exist. Since

$$\lim_{x\to 1}(x-1)\cos(\tfrac{1}{x-1}) = 0$$

(we showed this above in Part (i), since $\lim_{x\to 1}(x-1)\cos(\frac{1}{x-1}) = \lim_{y\to 0} y\cos(\frac{1}{y})$ which we argued is equal to zero) it follows from the Algebra of Limits that

$$\lim_{x\to 1}(f'(x) - 2(x-1)\cos(\tfrac{1}{x-1})) = \lim_{x\to 1} f'(x).$$

Since, for $x \neq 1$, we have

$$\sin(\tfrac{1}{x-1}) = f'(x) - 2(x-1)\cos(\tfrac{1}{x-1})$$

it follows that

$$\lim_{x\to 1}\sin(\tfrac{1}{x-1})$$

exists, which is a contradiction. This means $\lim_{x\to 1} f'(x)$ cannot exist, and hence the statement

$$f'(1) = \lim_{x\to 1} f'(x)$$

is false.

Finally, we continue to show that $\lim_{x\to 1}\sin(\frac{1}{x-1})$ does not exist. To this end, we observe that the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ defined by:

$$a_n = 1 + \frac{1}{2\pi n} \qquad \text{and} \qquad b_n = 1 + \frac{1}{\frac{\pi}{2} + 2\pi n} \qquad \text{for each} \quad n \in \mathbb{N}$$

satisfy $a_n \neq 1$, $b_n \neq 1$ for all $n \in \mathbb{N}$,

$$a_n \longrightarrow 1 \qquad \text{and} \qquad b_n \longrightarrow 1, \qquad \text{as} \qquad n \to \infty.$$

Also,

$$\sin\left(\frac{1}{a_n - 1}\right) = \sin(2\pi n) = 0 \longrightarrow 0 \qquad \text{as} \qquad n \to \infty,$$

and

$$\sin\left(\frac{1}{b_n - 1}\right) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1 \longrightarrow 1 \qquad \text{as} \qquad n \to \infty.$$

Since $1 \neq 0$, we conclude that $\lim_{x\to 1} \sin(\frac{1}{x-1})$ does not exist.

□

Q12. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} (\tan x)^2 \sin\left(\frac{1}{x}\right), & \text{if } x \in (-1,1) \text{ and } x \neq 0, \\ 0, & \text{if } x = 0, \\ x + 1, & \text{if } x \in (-\infty, -1] \cup [1, \infty). \end{cases}$$

Show that f is differentiable at 0 and give the value of $f'(0)$.

Solution. For $0 < |h| < 1/2$ we have

$$\frac{f(h) - f(0)}{h} = \frac{\tan^2(h)\sin(\frac{1}{h})}{h} = \frac{\sin^2(h)}{h^2}\frac{1}{\cos^2(h)} h \sin(\tfrac{1}{h}).$$

Now

$$\frac{\sin^2(h)}{h^2}\frac{1}{\cos^2(h)} \to 1$$

as $h \to 0$ by standard limits and the algebra of limits. Also

$$h \sin(\tfrac{1}{h}) \to 0$$

as $h \to 0$. This follows because $|h \sin(\frac{1}{h})| \leq |h|$ for $h \neq 0$, and the Sandwich Theorem. Therefore

$$\frac{f(h) - f(0)}{h} \to 0$$

as $h \to 0$ by the algebra of limits. This means f is differentiable at 0 and $f'(0) = 0$.

□

Q13. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \max\{x, x^3\}$. Determine those $x_0 \in \mathbb{R}$ for which f is differentiable at x_0. Give the value of $f'(x_0)$ when it exists, and prove any assertions that you make. If $f'(x_0)$ does not exist, you must prove why.

Solution. Note that $x \geq x^3$ if and only if $x(1-x)(1+x) \geq 0$ and hence $x \geq x^3$ if and only if $x \in (-\infty, -1] \cup [0, 1]$. Therefore

$$f(x) = \begin{cases} x & \text{if } x \leq -1 \\ x^3 & \text{if } -1 < x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ x^3 & \text{if } x > 1. \end{cases}$$

It follows immediately that $f'(x_0)$ exists for $x_0 \in (-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,\infty)$ with

$$f'(x_0) = \begin{cases} 1 & \text{if } x_0 < -1 \\ 3x_0^2 & \text{if } -1 < x_0 < 0 \\ 1 & \text{if } 0 < x < 1 \\ 3x_0^2 & \text{if } x_0 > 1. \end{cases}$$

It remains to consider $x_0 \in \{-1,0,1\}$. We claim that $f'(x_0)$ does not exist for all such x_0. First, if $x_0 = -1$ then

$$\lim_{h\to 0-} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0-} \frac{-1+h-(-1)}{h} = 1$$

whilst

$$\lim_{h\to 0+} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0+} \frac{(-1+h)^3-(-1)}{h} = \lim_{h\to 0+}(3-3h+h^2) = 3.$$

Hence $\lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$ does not exist and f is not differentiable at -1.

For $x_0 = 0$ we have

$$\lim_{h\to 0-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0-} \frac{h^3-0}{h} = 0$$

whilst

$$\lim_{h\to 0+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0+} \frac{h-0}{h} = 1.$$

Hence $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ does not exist and f is not differentiable at 0.

Finally, if $x_0 = 1$ then

$$\lim_{h\to 0-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0-} \frac{1+h-1}{h} = 1$$

whilst

$$\lim_{h\to 0+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0+} \frac{(1+h)^3-1}{h} = \lim_{h\to 0+}(3+3h+h^2) = 3.$$

Hence $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ does not exist and f is not differentiable at 1.

□

Q14. You are given that the function $f : [0,\pi] \to [-1,1]$ given by

$$f(x) = \cos(x) \qquad \text{for each } x \in [0,\pi]$$

is continuous and strictly decreasing on $[0,\pi]$, and therefore the inverse function $f^{-1} : [-1,1] \to [0,\pi]$ is well-defined.

(i) Fix $x_0 \in (0,\pi)$. Prove that f is differentiable at x_0 and

$$f'(x_0) = -\sin(x_0).$$

(ii) Fix $y_0 \in (-1,1)$. Prove that f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = -\frac{1}{\sqrt{1-y_0^2}}.$$

Solution. Let $y_0 \in (-1,1)$ and x_0 such that $f(x_0) = y_0$ (that is, such that $\cos(x_0) = y_0$). Since, f is differentiable on $(0,\pi)$, and $f'(x_0) = -\sin(x_0) \neq 0$ for all $x_0 \in (0,\pi)$, it follows from a theorem from lectures (Inverse Function Theorem for differentiable functions) that f^{-1} is differentiable at y_0 and moreover

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = -\frac{1}{\sin(x_0)} = -\frac{1}{\sqrt{1-\cos^2(x_0)}} = -\frac{1}{\sqrt{1-y_0^2}},$$

since x_0 is such that $\cos(x_0) = y_0$. Note, we take the positive square root because the sine function is positive on $(0, \pi)$.

$\square$

Q15. Using the Intermediate Value Theorem and Rolle's Theorem, show that the equation

$$e^x - x - 2 = 0$$

has exactly one positive real solution.
Hint: Use the Intermediate Value Theorem to prove that the equation has al least one positive real solution. Using proof by contradiction and Rolle's theorem show that the equation has exactly one positive real solution.

Solution. Let $f(x) = e^x - x - 2$. f is continuous and differentiable on the real line (since e^x and $x + 2$ are differentiable on the real line, and the algebra of continuous and differentiable functions). Notice that

$$f(0) = 1 - 0 - 2 = -1 < 0 \qquad \text{and} \quad f(4) = e^4 - 4 - 2 > 2^4 - 6 = 10 > 0.$$

Thus, by applying the Intermediate Value Theorem to the function f on the interval $[0, 4]$, we obtain that there exists $c \in (0, 4)$ such that $f(c) = 0$, i.e., the equation $f(x) = 0$ has at least a positive real solution (on the interval $(0, 4)$). We will prove that the equation $f(x) = 0$ has exactly one positive solution by using Rolle's Theorem. To this end, notice that

$$f'(x) = e^x - 1 > 0, \qquad \text{for all} \qquad x > 0.$$

We will argue by contradiction. Assume that $f(x) = 0$ has two or more positive real solutions, and let a and b with $0 < a < b$ be two solutions of the equation

$$e^x - x - 2 = 0.$$

Since a and b are two solutions of the above equation, $f(a) = f(b) = 0$. Then, applying Rolle's theorem to the function f on the interval $[a, b]$, there exists $\tilde{c} \in (a, b)$ such that

$$f'(\tilde{c}) = 0$$

On the other hand, we have observed that $f'(x) > 0$ for all $x > 0$, and in particular $f'(\tilde{c}) > 0$. This gives a contradiction. Therefore the equation has a unique positive real solution.

$\square$

$\boxed{\text{SUM}}$ **Q16**. Using the Intermediate Value Theorem and Rolle's Theorem, show that the equation

$$2x - 1 - \sin(x) = 0$$

has exactly one real solution.
Hint: Use the Intermediate Value Theorem to prove that the equation has al least one real solution. Using proof by contradiction and Rolle's theorem show that the equation has exactly one real solution.

Solution. Let $f(x) = 2x - 1 - \sin(x)$. f is continuous and differentiable on the real line (since $\sin(x)$ and $2x - 1$ are differentiable on the real line, and the algebra of continuous and differentiable functions). Notice that

$$f(0) = -1 < 0 \qquad \text{and} \quad f\left(\frac{\pi}{2}\right) = \pi - 2 > 0.$$

Thus, by applying the Intermediate Value Theorem to the function f on the interval $[0, \frac{\pi}{2}]$ (since in particular f is continuous on $[0, \pi/2]$), we obtain that there exists

$c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$, i.e., the equation $f(x) = 0$ has at least a real solution (on the interval $(0, \frac{\pi}{2})$).

We will show that the equation $f(x) = 0$ has exactly one real solution by using Rolle's Theorem. To this end, since $\cos(x) \in [-1, 1]$, we have that

$$f'(x) = 2 - \cos(x) > 0, \qquad \text{for all} \qquad x \in \mathbb{R}.$$

We will argue by contradiction. Assume that $f(x) = 0$ has two or more real solutions, and let a and b with $a < b$ be two solutions of the equation

$$f(x) = 2x - 1 - \sin(x) = 0.$$

Since a and b are two solutions of the above equation, $f(a) = f(b) = 0$. Then, by applying Rolle's theorem to the function f (notice that in particular f is continuous on the interval $[a, b]$ and differentiable on (a, b)), there exists $\tilde{c} \in (a, b)$ such that

$$f'(\tilde{c}) = 0$$

On the other hand, we have observed that $f'(x) > 0$ for all $x \in \mathbb{R}$, and in particular $f'(\tilde{c}) > 0$. This gives a contradiction. Therefore the equation has a unique real solution.

□

Q17. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable, and that $f'(x) > 1$ for all $x \in (a, b)$. Using Rolle's theorem, show that f has at most one fixed point; i.e. there exists at most one point $x_0 \in (a, b)$ such that $f(x_0) = x_0$.

Solution. Let $g : (a, b) \to \mathbb{R}$ be given by

$$g(x) = f(x) - x.$$

Now, fixed points of f correspond to solutions to the equation $g(x) = 0$.

Now, suppose that f has more than one fixed point; i.e. that there exist $x_1, x_2 \in (a, b)$, with $x_1 \neq x_2$, such that $g(x_1) = g(x_2) = 0$. Since f is differentiable on (a, b), and that differentiability implies continuity, f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). (Here we are assuming without loss of generality that $x_1 < x_2$.) Hence by Rolle's Theorem there exists $c \in (x_1, x_2) \subset (a, b)$ such that $g'(c) = 0$. Since $g'(x) = f'(x) - 1$ for all x, we must have that $f'(c) = 1$. This contradicts the hypothesis that $f'(x) > 1$ for all $x \in (a, b)$. Hence f can have at most one fixed point.

□

Q18. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \cos^2(3x) \qquad \text{for each } x \in \mathbb{R}.$$

Use the Mean Value Theorem to prove that for all $a, b \in [-\frac{\pi}{24}, \frac{\pi}{24}]$ we have

$$|f(a) - f(b)| \leq \frac{3}{\sqrt{2}}|a - b|.$$

Hint: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Solution. We first consider the case

$$-\frac{\pi}{24} \leq a < b \leq \frac{\pi}{24}.$$

Note that $|f(a) - f(b)| = |f(b) - f(a)|$ and $|a - b| = |b - a|$ so the case $-\frac{\pi}{24} \leq b < a \leq \frac{\pi}{24}$ follows from the case where $a > b$. Also, when $a = b$ the desired inequality reads $0 \leq 0$ which is clearly true.

We know f is continuous and differentiable on $\mathbb{R}$ because the sine function and polynomials are continuous and differentiable on $\mathbb{R}$, and because f is a composition of such functions. Moreover, by the Chain Rule,

$$f'(x) = -6\cos(3x)\sin(3x) = -3\sin(6x)$$

for each $x \in \mathbb{R}$. So, in particular, f is continuous on $[a, b]$ and differentiable on (a, b) and by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$\frac{f(a) - f(b)}{b - a} = f'(c) = -3\sin(6c).$$

This implies

$$\frac{|f(a) - f(b)|}{|b - a|} = 3|\sin(6c)|$$

and it suffices to show that $|\sin(6c)| \leq \frac{1}{\sqrt{2}}$. But, since $c \in (a, b)$ and $-\frac{\pi}{24} \leq a < b \leq \frac{\pi}{24}$, we have

$$-\frac{\pi}{4} \leq 6a < 6c < 6b \leq \frac{\pi}{4}$$

so (since the sine function is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$)

$$-\frac{1}{\sqrt{2}} = \sin\left(-\frac{\pi}{4}\right) < \sin(6c) < \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

which means $|\sin(6c)| < \frac{1}{\sqrt{2}}$, as required.

$\square$

Q19. Use the Mean Value Theorem to prove that for all $a, b \in (\frac{\pi}{8}, \frac{\pi}{4})$ we have

$$|\tan(2a) - \tan(2b)| \geq 4|a - b|.$$

Hint: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Solution. Fix $a, b \in (\frac{\pi}{8}, \frac{\pi}{4})$. If $a = b$ the inequality is clearly true, and there is nothing to be proved in this case. Assume that $a \neq b$ and without loss of generality that $a < b$.

Consider the function $f(x) = \tan(2x)$. Notice that $f(x) = \frac{\sin(2x)}{\cos(2x)}$, and since the function $2x$, sine and cosine functions are continuous and differentiable on the real line and $\cos(2x) \neq 0$ for all $x \in (\frac{\pi}{8}, \frac{\pi}{4})$, from the Algebra of continuous and differentiable functions, it follows that f is continuous on $[a, b]$ and differentiable on (a, b) (recall that $b < \frac{\pi}{4}$), therefore by applying the Mean Value Theorem we obtain that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c),$$

so in particular

$$|f(a) - f(b)| = |f'(c)|\,|b - a| = \frac{2}{\cos^2(2c)}\,|b - a|. \tag{6}$$

On the other hand, since $c \in (a, b)$ and $a, b \in (\frac{\pi}{8}, \frac{\pi}{4})$, we obtain $c \in (\frac{\pi}{8}, \frac{\pi}{4})$, from which we get

$$\cos(\frac{\pi}{2}) < \cos(2c) < \cos(\frac{\pi}{4}) \tag{7}$$

(since the cosine function is decreasing in $(\frac{\pi}{8}, \frac{\pi}{4})$). Now, using (7) into (6), we obtain that

$$|f(a) - f(b)| = \frac{2}{\cos^2(2c)}\,|b-a| \geq \frac{2}{\cos^2\frac{\pi}{4}}\,|b-a| = \frac{2}{\left(\frac{\sqrt{2}}{2}\right)^2}\,|b-a| = 4\,|b-a|,$$

as required. □

Q20. Using the Mean Value Theorem, prove that

$$\arctan(x) < x \quad \text{for all} \qquad x > 0.$$

Solution. Let $x > 0$. Consider (with some abuse of notation) the function

$$f(x) = \arctan(x) \quad \text{on the interval} \quad [0, x].$$

Notice that f is continuous on $[0, x]$ and differentiable on $(0, x)$ (since the arctan function is continuous an differentiable on $\mathbb{R}$). Thus, applying the Mean Value Theorem, there exists $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c),$$

or equivalently

$$\frac{\arctan(x) - \arctan(0)}{x - 0} = \frac{1}{1 + c^2}.$$

Finally, since $\frac{1}{1+c^2} < 1$ for all $c \in \mathbb{R}$, $\arctan(0) = 0$, and $\underline{x > 0}$ from the above identity it follows that

$$\arctan(x) < x \qquad \text{for all} \qquad x > 0.$$

□

Q21. Using the Mean Value Theorem, show that

$$|\arcsin(x)| \geq |x| \quad \text{for all} \qquad x \in (-1, 1).$$

Solution. First notice that, since $\arcsin(0) = 0$ the inequality is true for $x = 0$.

For fixed $\underline{0 < x < 1}$ consider (with some abuse of notation) the function $f(x) = \arcsin(x)$ on the interval $[0, x]$. Since f is continuous on $[0, x]$ and differentiable on $(0, x)$, by applying the Mean Value Theorem, there exists $c \in (0, x) \subseteq (0, 1)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

or, equivalently, since $\arcsin(0) = 0$,

$$\frac{\arcsin(x)}{x} = \frac{1}{\sqrt{1 - c^2}}. \tag{8}$$

Now, observe that since $c \in (0, x)$, then $\frac{1}{\sqrt{1-c^2}} > 1$. Therefore, from (8) we get that

$$\frac{\arcsin(x)}{x} > 1,$$

Since $\underline{x > 0}$, from the above inequality we obtain that

$$\arcsin(x) > x.$$

Therefore

$$|\arcsin(x)| = \arcsin(x) > x = |x|, \quad \text{for all} \quad x \in (0, 1),$$

since $\arcsin(x) > 0$ for all $x \in (0, 1)$.

For fixed $\underline{-1 < x < 0}$ consider (with some abuse of notation) the function $f(x) = \arcsin(x)$ on the interval $[x, 0]$. Since f is continuous on $[x, 0]$ and differentiable on

$(x,0)$, by applying the Mean Value Theorem, there exists $c \in (x,0) \subseteq (-1,0)$ such that

$$\frac{f(x)-f(0)}{x-0} = f'(c)$$

or, equivalently, since $\arcsin(0)=0$,

$$\frac{\arcsin(x)}{x} = \frac{1}{\sqrt{1-c^2}}. \tag{9}$$

Now, observe that since $c \in (x,0)$, then $\frac{1}{\sqrt{1-c^2}} > 1$. Therefore, from (9) we get that

$$\frac{\arcsin(x)}{x} > 1,$$

Since $\underline{x < 0}$, from the above inequality we obtain that

$$\arcsin(x) < x,$$

Therefore

$$|\arcsin(x)| = -\arcsin(x) > -x = |x|, \quad \text{for all} \quad x \in (-1,0),$$

since $\arcsin(x) < 0$ for all $x \in (-1,0)$.

The above argument shows that

$$|\arcsin(x)| \geq |x|, \qquad \text{for all} \qquad x \in (-1,1),$$

as required. $\square$

Q22. Let $f : (-\frac{\pi}{4}, \frac{\pi}{4}) \to (-1,1)$ be given by

$$f(x) = \sin(2x) \qquad \text{for each } x \in (-\tfrac{\pi}{4}, \tfrac{\pi}{4}).$$

Use the Inverse Function Theorem for differentiable functions and the Mean Value Theorem to prove that for all $a, b \in [-\frac{1}{10}, \frac{1}{10}]$ we have

$$|f^{-1}(a) - f^{-1}(b)| \leq \frac{5}{3\sqrt{11}}|a-b|.$$

[Here, $f^{-1} : (-1,1) \to (-\frac{\pi}{4}, \frac{\pi}{4})$ denotes the inverse function of f.]

Solution. We have $f'(x) = 2\cos(2x)$ for each $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$ (by the Chain Rule), and notice that $2\cos(2x) \neq 0$ for all $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$. Therefore, by the Inverse Function Theorem for differentiable functions, f^{-1} is differentiable on $(-1,1)$. Moreover, if $y \in (-1,1)$ and $x = f^{-1}(y)$ (that is $\sin(2x) = y$) then

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{2\cos(2x)} = \frac{1}{2\sqrt{1-y^2}}.$$

Now suppose that $a, b \in [-\frac{1}{10}, \frac{1}{10}]$. Firstly, note that if $a = b$ then there is nothing to do since both sides of the desired inequality are equal to zero. Furthermore, for the case $a \neq b$ it suffices to handle $a < b$ since the case $a > b$ follows by symmetry.

So, in the case where $a < b$ we apply the Mean Value Theorem to f^{-1} on $[a,b]$ to obtain the existence of some $c \in (a,b)$ satisfying

$$f^{-1}(b) - f^{-1}(a) = (f^{-1})'(c)(b-a).$$

Since $c \in [-\frac{1}{10}, \frac{1}{10}]$ it follows that $c^2 \leq \frac{1}{100}$ and therefore

$$(f^{-1})'(c) = \frac{1}{2\sqrt{1-c^2}} \leq \frac{1}{2}\sqrt{\frac{100}{99}} = \frac{5}{3\sqrt{11}}.$$

Hence

$$|f^{-1}(b) - f^{-1}(a)| = |(f^{-1})'(c)||b-a| \leq \frac{5}{3\sqrt{11}}|b-a|$$

as required.

□

$\boxed{\text{SUM}}$ **Q23**. Using the Mean Value Theorem show that

$$\arccos(x) - \frac{\pi}{2} \leq -x \qquad \text{for all} \qquad x \in [0,1].$$

Here, $\arccos : [-1,1] \longrightarrow [0,\pi]$ is the inverse of the cosine function.

Solution. Notice that, since $\arccos(0) = \frac{\pi}{2}$ and $\arccos(1) = 0$, then the inequality in the statement of the question is true when $x = 0$ or $x = 1$. We need to show that the inequality is also true for all $x \in (0,1)$.

Now, let $x \in (0,1)$. Consider the function $f(t) = \arccos(t)$ on the interval $[0,x]$. Notice that, since the arcosine function is continuous on $[-1,1]$ and differentiable on $(-1,1)$, and $[0,x] \subseteq [-1,1]$ for any $x \in (0,1)$, then f is differentiable on $(0,x)$ and continuous on $[0,x]$, and applying the Mean Value theorem we obtain that there exists $c \in (0,x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c),$$

or equivalently

$$\frac{\arccos(x) - \frac{\pi}{2}}{x} = -\frac{1}{\sqrt{1-c^2}}. \tag{10}$$

Finally notice that since $c \in (0,x)$ and $x \in (0,1)$, we have that $0 < c < x < 1$, and in particular $0 < 1 - c^2 < 1$. Hence

$$-\frac{1}{\sqrt{1-c^2}} < -1.$$

From the above inequality and (10), we have that

$$\frac{\arccos(x) - \frac{\pi}{2}}{x} = -\frac{1}{\sqrt{1-c^2}} < -1, \tag{11}$$

and, using that $\underline{x > 0}$, we have that

$$\arccos(x) - \frac{\pi}{2} < -x,$$

as desired. □

Q24. Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions on $\mathbb{R}$ and that

$$0 < e^x f'(x) \leq x g'(x) \qquad \text{for all} \quad x \in [1,10].$$

Show that for any $a, b \in [1,10]$ with $a < b$

$$\frac{f(b) - f(a)}{g(b) - g(a)} < 10.$$

Solution. The result will follow from an application of the Generalised Mean Value Theorem. Precisely, given any $a, b \in [1,10]$ with $a < b$, consider the functions f and g on the interval $[a,b]$.

Since f and g are differentiable functions on $\mathbb{R}$, and in particular continuous functions on the real line, we have that f and g are continuous on $[a,b]$ and differentiable on (a,b) . Moreover, since we know that

$$0 < e^x f'(x) \leq x g'(x) \qquad \text{for all} \quad x \in [1,10], \tag{12}$$

then $0 < xg'(x)$ for all $x \in [1,10]$, and therefore $g'(x) > 0$ for all $x \in [1,10]$, and in particular $g'(x) \neq 0$ for all $x \in (a,b) \subseteq [1,10]$. Consequently, from the application of the Generalised Mean Value Theorem, it follows that there exists $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Finally notice that from (12), since $c \in (a,b) \subseteq [1,10]$ we have that

$$\frac{f'(c)}{g'(c)} \le \frac{c}{e^c} < c < 10$$

since $e^x > 1$ for $x > 0$ and $c < 10$.

□

L'Hôpital's Rule

(i) From lectures, we have the following case of L'Hôpital's Rule:

0/0 form of L'Hôpital's Rule. Suppose that f and g are differentiable on (a,b). Suppose further that $x_0 \in (a,b)$ and that $g'(x) \neq 0$ for all $x \in (a,b) \setminus \{x_0\}$. If $f(x_0) = g(x_0) = 0$ and

$$\frac{f'(x)}{g'(x)} \to A \quad \text{as } x \to x_0,$$

then

$$\frac{f(x)}{g(x)} \to A \quad \text{as } x \to x_0.$$

(ii) For verifying the hypothesis that $g'(x) \neq 0$ for x in a *punctured* interval around the point x_0, the following result from lectures is very useful.

Lemma from lecture notes. Suppose that g, g', g'', …, $g^{(n)}$ exist and are continuous on (a,b), and that for some point $x_0 \in (a,b)$,

$$g^{(k)}(x_0) = 0 \text{ whenever } 0 \le k \le n-1 \qquad \text{and} \qquad g^{(n)}(x_0) \neq 0.$$

Then there exists $\delta \in \mathbb{R}^+$ such that

$$g^{(k)}(x) \neq 0 \quad \text{whenever } 0 \le k \le n \text{ and } 0 < |x - x_0| < \delta.$$

SUM **Q25.** Using L'Hôpital's Rule, show that the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{1}{x^2} - \dfrac{1}{(\tan(x))^2}, & \text{if } x \neq 0, \\ \dfrac{2}{3}, & \text{if } x = 0 \end{cases}$$

is differentiable at $x_0 = 0$ and give the value of $f'(0)$.

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Solution. By the definition of differentiability of a function at a point, f is differentiable at 0 is the following limit exists

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}.$$

Now, for $h\in(-\frac{\pi}{2},\frac{\pi}{2})$ with $h\neq 0$ we have

$$\frac{f(h)-f(0)}{h}=\frac{1}{h^3}-\frac{1}{h(\tan(h))^2}-\frac{2}{3h}=\frac{1}{h^3}-\frac{(\cos(h))^2}{h(\sin(h))^2}-\frac{2}{3h}$$

$$=\frac{3(\sin(h))^2-3h^2(\cos(h)^2)-2h^2(\sin(h))^2}{3h^3(\sin(h))^2}$$

so we let $g_1(h)=3(\sin(h))^2-3h^2(\cos(h)^2)-2h^2(\sin(h))^2$ and $g_2(h)=3h^3(\sin(h))^2$. Observe that g_1 and g_2 are differentiable functions of any order on the real line (since sine, cosine and polynomial functions are and the algebra of differentiable functions).

We have that

$$\begin{aligned}
g_1'(h)&=-6h\cos^2(h)+2(3+h^2)\cos(h)\sin(h)-4h\sin^2(h)\\
g_1''(h)&=2h^2\cos^2(h)+8h\cos(h)\sin(h)-2(5+h^2)\sin^2(h)\\
g_1'''(h)&=-4(-3h\cos^2(h)+(3+2h^2)\cos(h)\sin(h)+3h\sin^2(h))\\
g_1^{(iv)}(h)&=-8h(h\cos^2(h)+8\cos(h)\sin(h)-h\sin^2(h))\\
g_1^{(v)}(h)&=16(-5h\cos^2(h)+2(-2+h^2)\cos(h)\sin(h)+5h\sin^2(h))\\
&=16((-9+2h^2)\cos^2(h)+24h\cos(h)\sin(h)+(9-2h^2)\sin^2(h))
\end{aligned}$$

and

$$\begin{aligned}
g_2'(h)&=3h^2\sin(h)(2h\cos(h)+3\sin(h))\\
g_2''(h)&=6h(h^2\cos^2(h)+6h\cos(h)sin(h)-(-3+h^2)\sin^2(h))\\
g_2'''(h)&=-6(-9h^2\cos^2(h)+2h(-9+2h^2)\cos(h)\sin(h)+3(-1+3h^2)\sin^2(h))\\
g_2^{(iv)}(h)&=-24(h(-9+h^2)\cos^2(h)+6(-1+2h^2)\cos(h)\sin(h)-h(-9+h^2)\sin^2(h))\\
g_2^{(v)}(h)&=24(-15(-1+h^2)\cos^2(h)+4h(-15+h^2)\cos(h)\sin(h)+15(-1+h^2)\sin^2(h))
\end{aligned}$$

So $g_1(0)=g_1'(0)=g_1''(0)=g_1^{(iii)}=g_1^{(iv)}(0)=g_1^{(v)}(0)=0$ and $g_2(0)=g_2'(0)=g_2''(0)=g_2'''(0)=g_2^{(iv)}(0)=0$, and $g_2^{(v)}(0)=360\neq 0$.

Since g_2 is differentiable to any order on $\mathbb{R}$ (using a lemma from the lectures) there exists (a,b) such that $g_2^{(j)}(x)\neq 0$ for all $x\in(a,b)\setminus\{0\}$ for $j=1,2,3,4$. Hence, by the 0/0 form of L'Hôpital's Rule (repeatedly) and the Algebra of Limits, it follows that

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{g_1(h)}{g_2(h)}=\lim_{h\to 0}\frac{g_1'(h)}{g_2'(h)}=\lim_{h\to 0}\frac{g_1''(h)}{g_2''(h)}$$

$$=\lim_{h\to 0}\frac{g_1'''(h)}{g_2'''(h)}=\lim_{h\to 0}\frac{g_1^{(iv)}(h)}{g_2^{(iv)}(h)}=\lim_{h\to 0}\frac{g_1^{(v)}(h)}{g_2^{(v)}(h)}=\frac{0}{360}=0.$$

Therefore f is differentiable at 0 with $f'(0)=0$. Here we have used the continuity of the functions $g_1^{(v)}$ and $g_2^{(v)}$ at 0 in the evaluation of the last limit. $\square$

Q26. Using L'Hôpital's Rule, show that the function $f : (-\frac{\pi}{3}, \frac{\pi}{3}) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{\sin x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable at $x_0 = 0$ and give the value of $f'(0)$.

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Solution. For $h \in (-\frac{\pi}{3}, \frac{\pi}{3})$ with $h \neq 0$ we have

$$\frac{f(h) - f(0)}{h} = \frac{1}{h}\left(\frac{1}{h} - \frac{1}{\sin(h)}\right) = \frac{\sin(h) - h}{h^2 \sin(h)},$$

so we let $g_1(h) = \sin(h) - h$ and $g_2(h) = h^2 \sin(h)$. Then

$$\begin{aligned} g_1'(h) &= \cos(h) - 1 \\ g_1''(h) &= -\sin(h) \\ g_1'''(h) &= -\cos(h) \end{aligned}$$

and

$$\begin{aligned} g_2'(h) &= h^2 \cos(h) + 2h \sin(h) \\ g_2''(h) &= -h^2 \sin(h) + 4h \cos(h) + 2 \sin(h) \\ g_2'''(h) &= -h^2 \cos(h) - 6h \sin(h) + 6 \cos(h). \end{aligned}$$

So $g_1(0) = g_1'(0) = g_1''(0) = 0$ and $g_2(0) = g_2'(0) = g_2''(0) = 0$, and $g_1'''(0) = -1$ and $g_2'''(0) = 6$.

Since g_2 is differentiable to any order on $\mathbb{R}$ (using a lemma from lectures) there exists (a, b) such that $g_2^{(j)}(x) \neq 0$ for all $x \in (a, b) \setminus \{0\}$ for $j = 1, 2, 3$. Hence, by the $0/0$ form of L'Hôpital's Rule (repeatedly) and the Algebra of Limits, it follows that

$$\lim_{h\to 0} \frac{g_1(h)}{g_2(h)} = \lim_{h\to 0} \frac{g_1'(h)}{g_2'(h)} = \lim_{h\to 0} \frac{g_1''(h)}{g_2''(h)} = \lim_{h\to 0} \frac{g_1'''(h)}{g_2'''(h)} = -\frac{1}{6}.$$

Here, we have used the Algebra of limits and the continuity of the functions g_2''' and g_2''' at 0 to evaluate the last limit in the above chain of identities.

This means

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{g_1(h)}{g_2(h)} = -\frac{1}{6}$$

and therefore f is differentiable at 0 with $f'(0) = -\frac{1}{6}$. □

Q27. Use l'Hopital's Rule to evaluate the following limits:

(i)

$$\lim_{x\to 0} \frac{\sin^2 x}{1 - \cos x}.$$

(ii)

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{\tan(x)}.$$

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Solution. (i) Let $f(x) = \sin^2 x$ and $g(x) = 1 - \cos x$. First, observe that the functions f and g are differentiable on $(-\pi/2, \pi/2)$ (since the functions $\sin x$, $\cos x$ and constant functions are differentiable on $\mathbb{R}$ and the fact that sums and products of differentiable functions are differentiable), and $g'(x) = -\sin x \neq 0$ for all $x \in (-\pi/2, \pi/2)$ with $x \neq 0$. (Note that I specifically chose my open interval sufficiently small so that g' does not vanish.) Now, since

$$f(0) = g(0) = 0,$$

by the l'Hopital's Rule and the algebra of limits (since the function $\sin x$ and $\cos x$ are continuous), we have that

$$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2\sin x \cos x}{\sin x} = 2 \lim_{x\to 0} \cos x = 2.$$

(ii) For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $x \neq 0$ we have

$$\frac{1}{x} - \frac{1}{\tan(x)} = \frac{\sin(x) - x\cos(x)}{x\sin(x)}$$

so we let $g_1(x) = \sin(x) - x\cos(x)$ and $g_2(x) = x\sin(x)$. Then

$$\begin{aligned} g_1'(x) &= \cos(x) + x\sin(x) - \cos(x) = x\sin(x) \\ g_1''(x) &= \sin(x) + x\cos(x) \end{aligned}$$

and

$$\begin{aligned} g_2'(x) &= \sin(x) + x\cos(x) \\ g_2''(x) &= 2\cos(x) - x\sin(x). \end{aligned}$$

So $g_1(0) = g_1'(0) = 0$ and $g_2(0) = g_2'(0) = 0$, and $g_2''(0) = 2 \neq 0$.

Since g_2 is differentiable to any order on $\mathbb{R}$ (using a lemma from lectures) there exists (a, b) such that $g_2^{(j)}(x) \neq 0$ for all $x \in (a, b) \setminus \{0\}$ for $j = 1, 2$. Hence, by the $0/0$ form of L'Hôpital's Rule (twice) and the Algebra of Limits, it follows that

$$\lim_{x\to 0} \frac{g_1(x)}{g_2(x)} = \lim_{x\to 0} \frac{g_1'(x)}{g_2'(x)} = \lim_{x\to 0} \frac{g_1''(x)}{g_2''(x)} = \lim_{x\to 0} \frac{\sin(x) + \cos(x)}{2\cos(x) - x\sin(x)} = \frac{0}{2} = 0.$$

In the last identity we have use the continuity of the numerator and denominator functions at the point 0 and the fact that $g_2''(0) = 2 \neq 0$ to evaluate the last limit. This means

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{\tan(x)} = 0.$$

□

Q28. Using L'Hôpital's Rule evaluate the following limit

$$\lim_{x\to 0} \frac{\cos(x) - \cos(2x)}{\sin^2(x) + x\sin^2(x)}.$$

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Solution. Let $f(x) = \cos(x) - \cos(2x)$ and $g(x) = \sin^2(x) + x\sin^2(x) = (1 + x)\sin^2(x)$. Notice that that f and g are differentiable of any order on the real line. Also

$$f'(x) = -\sin(x) + 2\sin(2x)$$
$$f''(x) = -\cos(x) + 4\cos(2x),$$

and

$$g'(x) = \sin^2(x) + (1+x)\sin(2x)$$
$$g''(x) = 2\sin(2x) + 2(1+x)\cos(2x)$$

So $f(0) = f'(0) = 0$, and $g(0) = g'(0) = 0$, and $g''(0) = 2$.

Since g is differentiable to any order on $\mathbb{R}$, and $g(0) = g'(0) = 0$, and $g''(0) = 2 \neq 0$, there exists (a, b) such that $g_2^{(j)}(x) \neq 0$ for all $x \in (a, b) \setminus \{0\}$ for $j = 1, 2$. Hence, by the $0/0$ form of L'Hôpital's Rule (twice) and the Algebra of Limits, it follows that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{-\cos(x) + 4\cos(2x)}{2\sin(2x) + 2(1+x)\cos(2x)} = \frac{3}{2}.$$

Here, we have use the algebra of limits an the continuity of the functions $-\cos(x) + 4\cos(2x)$ and $2\sin(2x) + 2(1+x)\cos(2x)$ at 0 in the evaluation of the last limit.

□

Q29. Using Taylor's Theorem, show that

$$\sin x \geq x - \frac{x^3}{6}, \quad \text{for all} \quad x \geq 0.$$

Solution. First notice that if $x = 0$ the inequality in the statement is true. Hence we assume in what follows that $\underline{x > 0}$.

Consider the interval $[0, x]$, and the function $f(t) = \sin(t)$ on $[0, x]$.

Since the sine function is differentiable of any order on the real line in particular we have that, f', f'' exist and are continuous on $[0, x]$, and furthermore f''' exists on $(0, x)$. Hence by Taylor's Theorem there exists $c \in (0, x)$ such that

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(c)}{3!}x^3.$$

Since in this case $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$ and $f'''(c) = -\cos(c)$, the above formula rewrites

$$\sin(x) = x - \frac{\cos(c)}{6}x^3. \tag{13}$$

Finally, notice that $\cos(x) \leq 1$ for all $x \in \mathbb{R}$, so that

$$-\frac{\cos(c)}{6}x^3 \geq -\frac{x^3}{6}, \qquad \text{for all} \qquad x > 0. \tag{14}$$

Using (14) into (13), we conclude that

$$\sin(x) \geq x - \frac{x^3}{6}, \qquad \text{for all} \qquad x > 0,$$

as desired. □

Q30. Using Taylor's Theorem, show that for each $t \in [0,1)$,

$$\sqrt{1+t} = 1 + \sum_{k=1}^{\infty} c_k t^k,$$

where

$$c_k = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right)}{k!}.$$

Hint: Consider the behaviour of the remainder term $R_n(t)$ given by Taylor's Theorem as $n \to \infty$.

Solution. Let $t \in [0,1)$. If $f(x) = \sqrt{x}$ then $f', f'', \ldots, f^{(n)}$ exist and are continuous on $[1, 1+t]$, and furthermore $f^{(n+1)}$ exists on $(1, 1+t)$. Hence by Taylor's Theorem there exists $\theta \in (0,1)$ such that

$$\sqrt{1+t} = f(1+t) = 1 + \sum_{k=1}^{n} c_k t^k + \frac{f^{(n+1)}(1+\theta t)}{(n+1)!} t^{n+1},$$

where

$$c_k = \frac{f^{(k)}(1)}{k!} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right)}{k!}.$$

Now,

$$\left|\frac{f^{(n+1)}(1+\theta t)}{(n+1)!} t^{n+1}\right| = \left|\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!}(1+\theta t)^{-(n+1/2)} t^{n+1}\right| \leq t^{n+1} \to 0$$

as $n \to \infty$ as $0 \leq t < 1$. Therefore the sum

$$\sum_{k=1}^{\infty} c_k t^k$$

converges and

$$\sqrt{1+t} = 1 + \sum_{k=1}^{\infty} c_k t^k.$$

□

Q31. Use Taylor's Theorem to prove that for each $t \in \mathbb{R}$, the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

converges to $\cos(t)$. [You may use without proof the fact that for any $t \in \mathbb{R}$, $\frac{t^n}{n!} \to 0$ as $n \to \infty$.]

Solution. We did this example for the sine function in lectures. All that is required here is to amend that argument suitably.

For $f(x) = \cos(x)$ we first observe that f has derivatives of all orders since

$$\begin{aligned} f(x) &= \cos(x), \\ f'(x) &= -\sin(x), \\ f''(x) &= -\cos(x), \\ f'''(x) &= \sin(x), \\ f^{(4)}(x) &= \cos(x), \end{aligned}$$

and so on. Setting $x = 0$, gives

$$f^{(n)}(0) = \begin{cases} 0 & \text{for } n \text{ odd}, \\ 1 & \text{for } n \in 4\mathbb{N}, \\ -1 & \text{for } n \in 4\mathbb{N} + 2. \end{cases}$$

Now, taking the centre to be $x_0 = 0$ and for fixed $t \in \mathbb{R}$, we have

$$|R_n(t)| = \left| \frac{t^{n+1}}{(n+1)!} \right| \left| f^{(n+1)}(\theta t) \right| \le \frac{|t|^{n+1}}{(n+1)!},$$

since $|\sin(\theta t)| \le 1$ and $|\cos(\theta t)| \le 1$. Therefore, since

$$\frac{|t|^{n+1}}{(n+1)!} \to 0 \ \text{ as } \ n \to \infty,$$

(using the given fact), by the Sandwich Theorem we have

$$R_n(t) \to 0 \ \text{ as } \ n \to \infty,$$

and so

$$f(t) = \cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots .$$

$\square$