2RCA Complex Analysis

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Foreword

These notes will form the basis of my lectures for the Complex Analysis part of the module 2RCA Real & Complex Analysis, given at the University of Birmingham in Spring Semester 2024. The notes are based on the Complex Analysis Lecture Notes for this module in previous years. These notes will be updated throughout the semester. I will usually upload a provisional version of the latest material to be covered before each lecture, which may be updated and finalised after the lecture. Any feedback on these lecture notes, including reports of typos, would be gratefully received and may be sent to me by email.

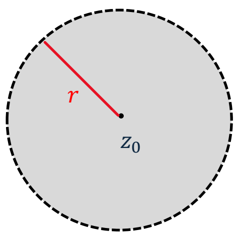
Michael Dymond, Birmingham, February 2024

Lecture 1

1. Sets and Topology

Notation Given a point $z_0 \in \mathbb{C}$ and a real number $r > 0$ we let $B(z_0, r)$ denote the open ball with centre z_0 and radius r, defined by

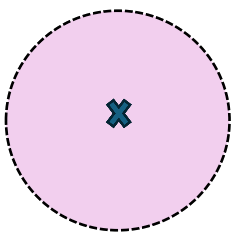
$$B(z_0, r) \coloneqq \{z \in \mathbb{C}: |z - z_0| < r\}$$



We further let $B'(z_0, r)$ denote the punctured ball with centre z_0 and radius r, defined by

$$B'(z_0, r) = \{z \in \mathbb{C}: 0 < |z - z_0| < r\},$$

so that $B'(z_0, r) = B(z_0, r) \setminus \{z_0\}$.



Definition 1.1

Let $S \subseteq \mathbb{C}$ be a set. A point $z \in \mathbb{C}$ is called

i. an ***interior point*** of S if there exists $r > 0$ such that $B(z, r) \subseteq S$.
ii. a ***boundary point*** of S if for every $r > 0$ we have $B(z, r) \cap S \neq \emptyset$ and $B(z, r) \cap (\mathbb{C} \setminus S) \neq \emptyset$.

Remark Interior points of S are always inside the set S: in fact they are "safely" inside the set S. On the other hand, boundary points of S may or may not be in S. We will see some examples to show this.

Definition 1.2

A set $S \subseteq \mathbb{C}$ is called

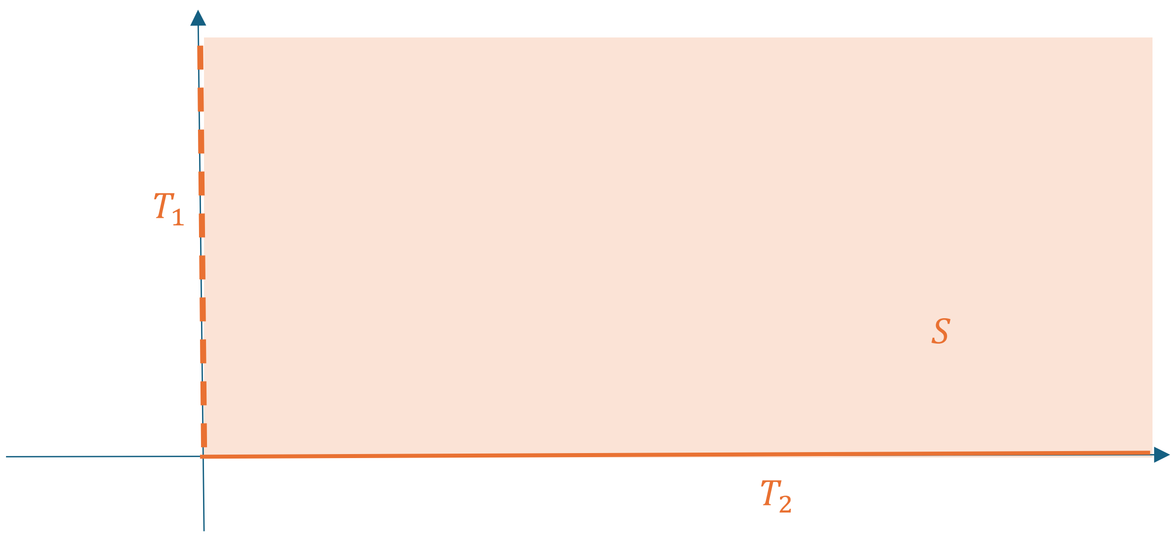
i. ***open*** if every point in S is an interior point of S.
ii. ***closed*** if $\mathbb{C} \setminus S$ is open.
iii. ***bounded*** if there exists $R > 0$ such that $S \subseteq B(0, R)$.
iv. ***compact*** if S is closed and bounded.

Remark (Warning) Closed is not the same thing as "not open". It is possible for a set to be neither open or closed, or both open and closed.

Example

Let

$$S = \{z \in \mathbb{C} : Re(z) > 0, Im(z) \geq 0\}.$$



We describe the interior and the boundary points of S. The set of interior points of S is

$$\{z \in \mathbb{C} : Re(z) > 0, Im(z) > 0\}.$$

The set of boundary points of S is a union of two sets T_1 and T_2 where

$$T_1 = \{z \in \mathbb{C} : Re(z) = 0, Im(z) \geq 0\}$$

and

$$T_2 = \{z \in \mathbb{C} : Re(z) > 0, Im(z) = 0\}.$$

Notice that we have $T_1 \cap S = \emptyset$ (i.e. $T_1 \subseteq \mathbb{C} \setminus S$) and $T_2 \subseteq S$. Hence some boundary points of S belong to S and other boundary points of S do not belong to S.

The set S is not open because not every point of S is an interior point of S. The points in T_2 belong to S and are not interior points of S.

The set S is not closed. This can be seen using the next Proposition.

Proposition 1.3

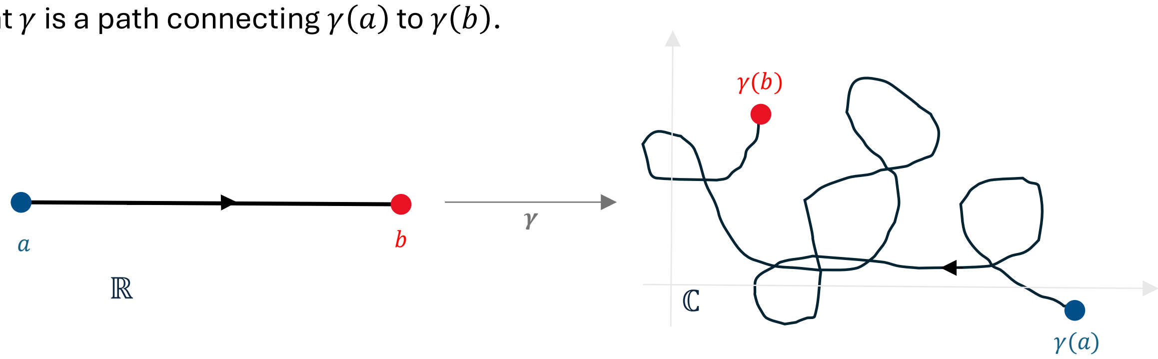
i. For each $z \in \mathbb{C}$ and $r > 0$ the set $B(z, r)$ is open.
ii. A set $S \subseteq \mathbb{C}$ is closed if and only if it contains all of its boundary points.

Proof

i. Let $w \in B(z, r)$. We need to check that w is an interior point of $B(z, r)$. Set $\delta = r - |w - z| > 0$. Then for all $w' \in B(w, \delta)$ we have $|w' - z| \leq |w' - w| + |w - z| < \delta + |w - z| = r$. So we have found $\delta > 0$ such that $B(w, \delta) \subseteq B(z, r)$. Hence w is an interior point of $B(z, r)$.
ii. Suppose S is closed and let $z \in \mathbb{C} \setminus S$. Since $\mathbb{C} \setminus S$ is open, z is an interior point of $\mathbb{C} \setminus S$, so there exists $r > 0$ so that $B(z, r) \subseteq \mathbb{C} \setminus S$, or put differently, $B(z, r) \cap S = \emptyset$. Hence, z is not a boundary point of S. Since $z \in \mathbb{C} \setminus S$ was arbitrary, we have shown that S contains all its boundary points.
Conversely, assume that S is not closed. Then $\mathbb{C} \setminus S$ is not open, which means that not every point of $\mathbb{C} \setminus S$ is an interior point of $\mathbb{C} \setminus S$. In other words, there exists $z \in \mathbb{C} \setminus S$ such that for every $r > 0$ the set $B(z, r)$ fails to be contained in $\mathbb{C} \setminus S$. Put differently, we have $B(z, r) \cap S \neq \emptyset$ for every $r > 0$. Since $z \in B(z, r) \cap (\mathbb{C} \setminus S)$ for every $r > 0$, we also have $B(z, r) \cap (\mathbb{C} \setminus S) \neq \emptyset$ for all $r > 0$. Therefore $z \in \mathbb{C} \setminus S$ is a boundary point of S and so S does not contain all its boundary points. ■

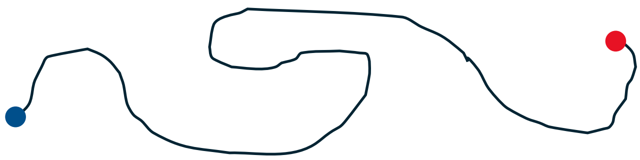
Definition 1.4

A ***path*** is a continuous function $\gamma: [a, b] \to \mathbb{C}$, where $[a, b]$ is an interval in the real line. The point $\gamma(a)$ is called the ***start point*** of γ and the point $\gamma(b)$ the ***end point***. We say that γ is a path connecting $\gamma(a)$ to $\gamma(b)$.

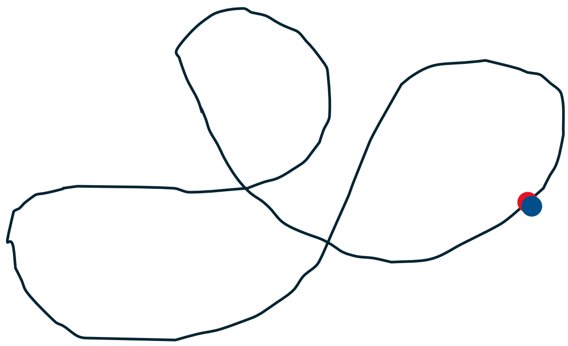


The path γ is called ***simple*** if for $s, t \in [a, b]$ we have

$$\gamma(s) = \gamma(t) \Rightarrow t = s \text{ or } \{s, t\} = \{a, b\}$$

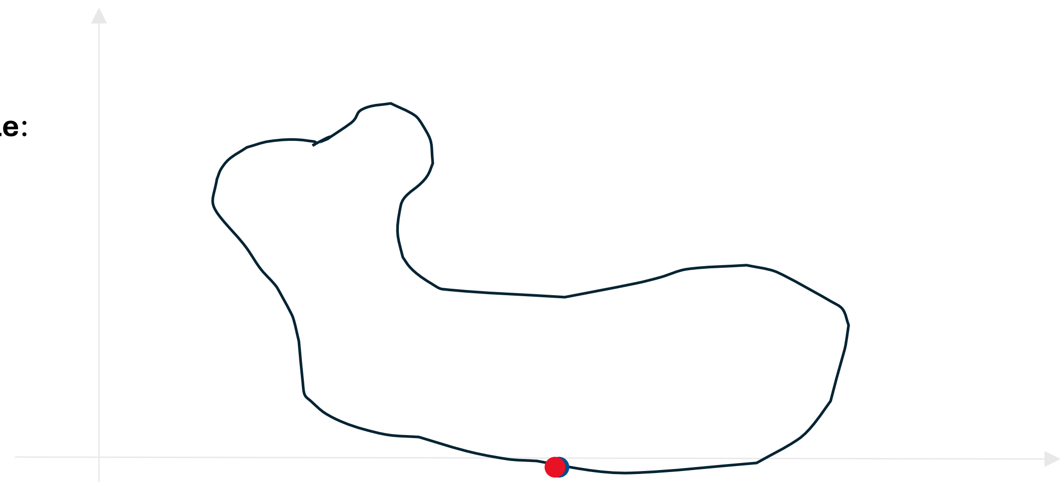


The path γ is called **closed** if $\gamma(a) = \gamma(b)$.



We say that γ is a path in a set $S \subseteq \mathbb{C}$ if $\gamma(t) \in S$ for all $t \in [a, b]$.

Example:



A simple closed path.

Lecture 2

Definition 1.5

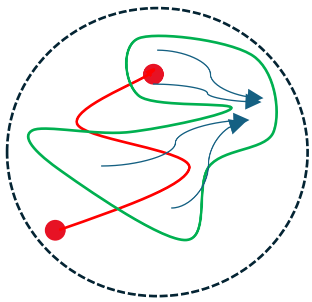
A set $S \subseteq \mathbb{C}$ is called

i. ***connected*** if for any $x, y \in S$ there exists a path γ connecting x and y whose image lies entirely in S.
ii. a ***domain*** if S is open and connected.
iii. ***simply connected*** if it is connected and any simple closed path whose image lies entirely in S can be shrunk to a point whilst staying inside S.

Examples

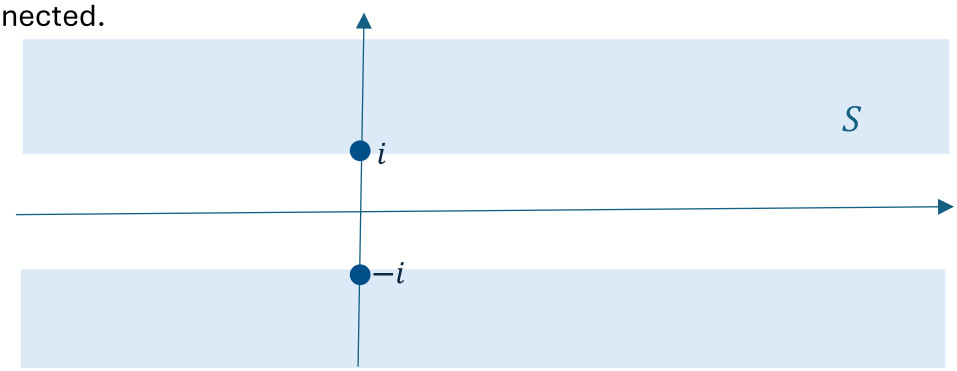
i. $B(0,1)$ is connected and open, therefore a domain. $B(0,1)$ is also simply connected.

$B(0,1)$ is connected. Any two points (the red points) in $B(0,1)$ may be connected by a path in $B(0,1)$ (e.g. the red path).

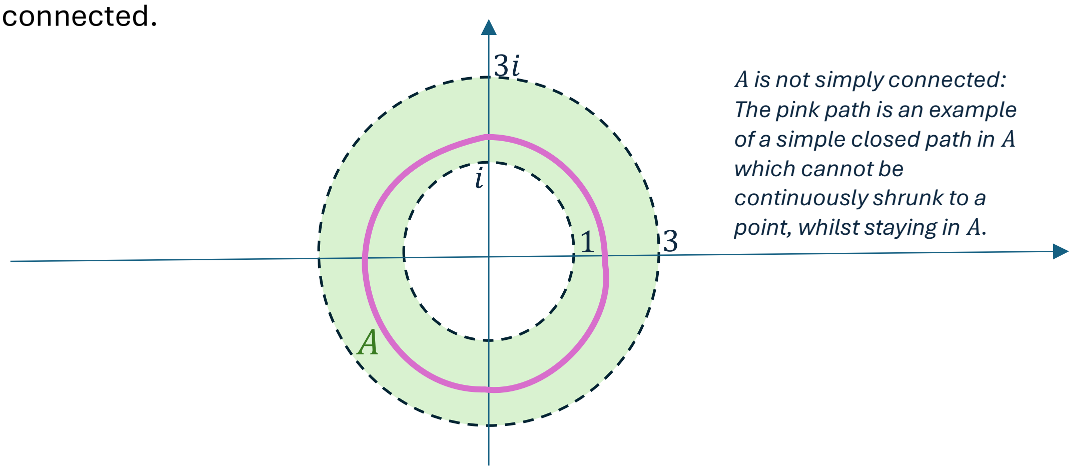


$B(0,1)$ is simply connected: Any simple closed path in $B(0,1)$ (e.g. the green path) may be continuosly shrunk inside $B(0,1)$ to a point.

ii. Consider the set $S = \{z \in \mathbb{C} \colon |Im(z)| \geq 1\}$. S is not connected because for example there is no path in S which connects the two points i and $-i$, which belong to S. Since S is not connected, it is not a domain and not simply connected.



iii. Consider the set $A = \{z \in \mathbb{C} : 1 < |z| < 3\}$. A is connected, but not simply connected.



Remark The key property of A which prevents it from being simply connected is that it has a "hole". In general, any subset of $\mathbb{C}$ which has a hole is not simply connected.

2. Complex Functions

Terminology By a **complex function** we mean a function whose domain or codomain is a subset of $\mathbb{C}$.

Definition 2.1

A **complex polynomial** is a function $p: \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0,$$

where $m \in \mathbb{N} \cup \{0\}$, $a_0, a_1, \dots, a_m \in \mathbb{C}$ and $a_m \neq 0$ if $m \geq 1$ (we allow a_m to be zero if $m = 0$). The number m is called the **degree** of the polynomial p.

Theorem 2.2 (The Fundamental Theorem of Algebra)

Suppose $p: \mathbb{C} \to \mathbb{C}$ is a complex polynomial of degree m. Then there exist complex numbers $c, \alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{C}$ such that

$$p(z) = c(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_m)$$

for all $z \in \mathbb{C}$.

Definition 2.3

Let $S \subseteq \mathbb{C}$ and $f: S \to \mathbb{C}$ be a complex function. The **real part** of f is defined as the function $S \to \mathbb{R}$ given by

$$z \mapsto Re(f(z)).$$

The **imaginary part** of f is defined as the function $S \to \mathbb{R}$ given by

$$z \mapsto Im\big(f(z)\big).$$

We often express the real and imaginary parts of f in the following way. For $z = x + iy \in S$, we let

$$u(x, y) = Re(f(x + iy) \text{ and } v(x, y) = Im(f(x + iy).$$

Thus, u and v are functions $\{(x, y) \in \mathbb{R}^2 : x + iy \in S\} \to \mathbb{R}$ and we have

$$f(x + iy) = u(x, y) + iv(x, y)$$

for all $z = x + iy \in S$.

Example

Conside the function $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z}.$$

For $z = x + iy \in \mathbb{C} \setminus \{0\}$ we have

$$f(x + iy) = \frac{1}{x + iy} = \frac{x - iy}{(x - iy)(x + iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right),$$

and so defining $u, v: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$u(x, y) = \frac{x}{x^2+y^2} \text{ and } v(x, y) = \frac{-y}{x^2+y^2},$$

we have

$$f(x + iy) = u(x, y) + iv(x, y).$$

Definition 2.4

Let $S \subseteq \mathbb{C}$ and $f: S \to \mathbb{C}$ be a complex function. We say that f is **bounded** if there exists $R > 0$ such that

$$|f(z)| \leq R \text{ for all } z \in S.$$

Non-Example

Consider again the function $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z}.$$

Then f is not bounded because $|f(z)|$ attains arbitrarily high values near to 0. Indeed, given any $R > 0$ we may consider any point $z \in \mathbb{C} \setminus \{0\}$ with $|z| < \frac{1}{R}$, for example $z = \frac{1}{2R}$, and observe that

$$|f(z)| = \left|\frac{1}{z}\right| = \frac{1}{|z|} > R.$$

Lecture 3

3. Limits

Definition 3.1

Let $S \subseteq \mathbb{C}$ be a set, $f: S \to \mathbb{C}$ be a complex function and $z_0, l \in \mathbb{C}$.

i. If z_0 is an interior point of $S \cup \{z_0\}$ we say that $\boldsymbol{f(z)}$ **tends (or converges) to $\boldsymbol{l}$ as $\boldsymbol{z}$ tends to $\boldsymbol{z_0}$**, and write

$$\lim_{z \to z_0} f(z) = l,$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - l| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

ii. If $T \subseteq S$ *is a subset of* S and z_0 is a boundary point of the set $T \setminus \{z_0\}$, we say that $\boldsymbol{f(z)}$ **tends (or converges) to $\boldsymbol{l}$ as $\boldsymbol{z}$ tends to $\boldsymbol{z_0}$ in $\boldsymbol{T}$**, and write

$$\lim_{\substack{z \to z_0 \\ z \in T}} f(z) = l,$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - l| < \varepsilon \text{ whenever } z \in T \text{ and } 0 < |z - z_0| < \delta.$$

Remark In part i. there are no restrictions on how z may approach z_0. In part ii. we only allow z to approach z_0 inside the subset T of S.

Example

We will show using the definition that

$$\lim_{z\to 2+i} Re(z) = 2.$$

Fix $\varepsilon > 0$ and let $\delta > 0$ be a number that we will specify later. We study $|Re(z) - 2|$ for complex numbers $z \in \mathbb{C} \setminus \{2 + i\}$ with $0 < |z - (2 + i)| < \delta$. For such z we observe that

$$|Re(z) - 2| = |Re(z) - Re(2 + i)| = \left|Re\big(z - (2 + i)\big)\right| \leq |z - (2 + i)| < \delta.$$

In the second last inequality we used that $|Re(w)| \leq |w|$ for all complex numbers w. Therefore, taking $\delta = \varepsilon$ we get that

$$|Re(z) - 2| < \delta = \varepsilon \text{ whenever } z \in \mathbb{C} \text{ and } 0 < |z - (2 + i)| < \delta.$$

Example

We show using the definition that

$$\lim_{z\to 2+i} |z| = \sqrt{5}.$$

Fix $\varepsilon > 0$ and let $\delta > 0$ be number that we will specify later. We study $||z| - \sqrt{5}|$ for complex numbers $z \in \mathbb{C}$ with $0 < |z - (2 + i)| < \delta$. For such z we have

$$\left||z| - \sqrt{(5)}\right| = ||z| - |2 + i|| \leq |z - (2 + i)| < \delta.$$

In the second last inequality we applied the inequality $||v| - |w|| \leq |v - w|$, which is true for all complex numbers v, w by the triangle inequality. Therefore, taking $\delta = \varepsilon$, we have

$$||z| - \sqrt{5}| < \delta = \varepsilon \text{ whenever } z \in \mathbb{C} \text{ and } 0 < |z - (2 + i)| < \delta.$$

Definition 3.2

Let $S \subseteq \mathbb{C}$ be a set, $f: S \to \mathbb{C}$ be a complex function and $z_0 \in \mathbb{C}$.

i. If z_0 is an interior point of $S \cup z_0$ we say that **$f(z)$ tends (or converges) to ∞ as z tends to z_0**, and write

$$\lim_{z\to z_0} f(z) = \infty,$$

if for every $A > 0$ there exists $\delta > 0$ such that

$$|f(z)| > A \text{ whenever } 0 < |z - z_0| < \delta.$$

iii. If $T \subseteq S$ *is a subset of* S and z_0 is a boundary point of the set $T \setminus \{z_0\}$, we say that **$f(z)$ tends (or converges) to ∞ as z tends to z_0 in T**, and write

$$\lim_{\substack{z \to z_0 \\ z \in T}} f(z) = \infty,$$

if for every $A > 0$ there exists $\delta > 0$ such that

$$|f(z)| > A \text{ whenever } z \in T \text{ and } 0 < |z - z_0| < \delta.$$

Lemma 3.3

Let $S \subseteq \mathbb{C}$ be a set, $f: S \to \mathbb{C}$ be a complex function and $z_0 \in \mathbb{C}$ be an interior point of the set $S \cup z_0$. Let $l \in \mathbb{C} \cup \infty$ be a complex number or be ∞. Then the following are equivalent:

i. $\lim_{z \to z_0} f(z) = l$

ii. $\lim_{\substack{z \to z_0 \\ z \in T}} f(z) = l$ for every set $T \subseteq S$ for which z_0 is a boundary point of $T \setminus \{z_0\}$

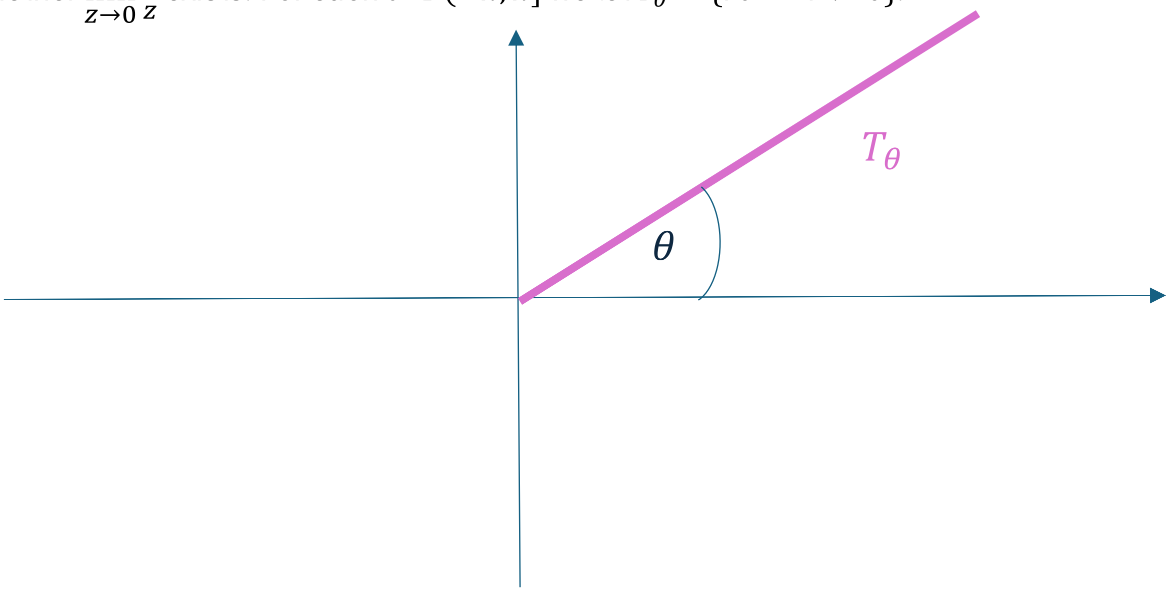
Proof Left as an exercise.

Remark To show that $\lim_{z \to z_0} f(z)$ does not exist it suffices, using Lemma 3.3 to find $T_1, T_2 \subseteq S$ such that

$$\lim_{\substack{z \to z_0 \\ z \in T_1}} f(z) = l_1 \quad \text{and} \quad \lim_{\substack{z \to z_0 \\ z \in T_2}} f(z) = l_2 \quad \text{with } l_1 \neq l_2.$$

Example

We show that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist. First note that the function $z \mapsto \frac{\bar{z}}{z}$ is defined on the set $S = \mathbb{C} \setminus \{0\}$ and 0 is an interior point of $S \cup \{0\}$, so it makes sense to investigate whether $\lim_{z \to 0} \frac{\bar{z}}{z}$ exists. For each $\theta \in (-\pi, \pi]$ we let $T_\theta = \{re^{i\theta} : r > 0\}$.



Observe that 0 is a boundary point of each $T_\theta \setminus \{0\}$ because for any $s > 0$ we have

$$B(0,s) \cap (T_\theta \setminus \{0\}) = \{re^{i\theta} : 0 < r < s\} \neq \emptyset.$$

Now, observe that for each $\theta \in (-\pi, \pi]$ and $z \in T_\theta$ we have

$$\frac{\bar{z}}{z} = \frac{\overline{re^{i\theta}}}{re^{i\theta}} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}.$$

Hence

$$\lim_{\substack{z\to 0\\ z\in T_\theta}} \frac{\bar{z}}{z} = \lim_{\substack{z\to 0\\ z\in T_\theta}} e^{-2i\theta} = e^{-2i\theta},$$

for each $\theta \in (-\pi, \pi]$. In particular, for each set T_θ we get a different limit $\lim\limits_{\substack{z\to 0\\ z\in T_\theta}} \frac{\bar{z}}{z}$.

Therefore, by Lemma 3.3 the limit $\lim\limits_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

Definition 3.4

Let $S \subseteq \mathbb{C}$, $f: S \to \mathbb{C}$ be a complex function and $l \in \mathbb{C}$.

i. If there exists $R > 0$ such that $\mathbb{C} \setminus B(0,R) \subseteq S$ we say that **$f(z)$ tends (or converges) to l as z tends to** ∞, and write

$$\lim_{z\to\infty} f(z) = l$$

if for every $\varepsilon > 0$ there exists $M > 0$ such that

$$|f(z) - l| < \varepsilon \text{ whenever } |z| > M.$$

ii. If S is unbounded we say that **$f(z)$ tends (or converges) to l as z tends to ∞ in S**, and write

$$\lim_{\substack{z\to\infty\\ z\in S}} f(z) = l,$$

if for every $\varepsilon > 0$ there exists $M > 0$ such that

$$|f(z) - l| < \varepsilon \text{ whenever } z \in S \text{ and } |z| > M.$$

Lecture 4

Theorem 3.5 (Algebra of Limits)

Let $S \subseteq \mathbb{C}$, $f, g: S \to \mathbb{C}$ be complex functions, $z_0 \in \mathbb{C}$ be an interior point of $S \cup \{z_0\}$ and $c \in \mathbb{C}$. Then

$$\lim_{z\to z_0} cf(z) = c \lim_{z\to z_0} f(z),$$

$$\lim_{z\to z_0} f(z) + g(z) = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z),$$

$$\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z),$$

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{\lim\limits_{z\to z_0} f(z)}{\lim\limits_{z\to z_0} g(z)},$$

in the sense that if the right hand side exists, so does the left hand side and they are equal. The analogous equalities hold with $\lim\limits_{z\to z_0}$ replaced by $\lim\limits_{\substack{z\to z_0\\ z\in T}}$ if we have $T\subseteq S$ and that z_0 is instead a boundary point of $T\setminus z_0$. There are also similar statements of the theorem for limits of the type $\lim\limits_{z\to\infty}$ and $\lim\limits_{\substack{z\to\infty\\ z\in S}}$.

4. Continuity

Definition 4.1

Let $S\subseteq\mathbb{C}$, $f: S\to\mathbb{C}$ be a complex function and $z_0\in S$. We say that f is continuous at z_0 if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|f(z)-f(z_0)|<\varepsilon \text{ whenever } 0\le|z-z_0|<\delta.$$

We say that f is continuous if f is continuous at every point of S.

Proposition 4.2

Let $S\subseteq\mathbb{C}$, $f: S\to\mathbb{C}$ be a complex function and z_0 be a boundary point of $S\setminus\{z_0\}$. Then f is continuous at z_0 if and only if

$$\lim_{\substack{z\to z_0\\ z\in S}} f(z)=f(z_0).$$

Proof Similar to in Real Analysis. ■

Examples

- Each of the functions $\mathbb{C}\to\mathbb{C}$, $z\mapsto z, z\mapsto\bar{z}, z\mapsto|z|, z\mapsto Re(z), z\mapsto Im(z)$ are continuous.
- Complex polynomials $p:\mathbb{C}\to\mathbb{C}$ are continuous.

Example (An important discontinuous complex function).

Recall that every non-zero complex number z may be written as $z=|z|e^{i\theta}$ with for a unique $\theta\in(-\pi,\pi]$. We may therefore define a mapping $Arg:\mathbb{C}\setminus\{0\}\to(-\pi,\pi]$, which sends each $z\in\mathbb{C}\setminus\{0\}$ to its corresponding argument $\theta\in(-\pi,\pi]$. So for example, we have $Arg(1)=0$, $Arg(i)=\frac{\pi}{2}$, $Arg(-1)=\pi$ and $Arg(-i)=-\frac{\pi}{2}$. Then, letting $T_+=\{z\in\mathbb{C}: |z|=1, Im(z)>0\}$ and $T_-=\{z\in\mathbb{C}: |z|=1, Im(z)<0\}$ we may check that

$$\lim_{\substack{z \to -1 \\ z \in T_+}} Arg(z) = \pi \text{ and } \lim_{\substack{z \to -1 \\ z \in T_-}} Arg(z) = -\pi.$$

Therefore, by Lemma 3.3, the limit $\lim_{z \to -1} Arg(z)$ does not exist. By Proposition 4.2 we conclude that the complex function Arg is not continuous at the point -1. We may show similarly that Arg is not continuous at each point on the negative real axis, that is, at each point r with $r < 0$. We may also check that the complex function Arg is continuous at every other point of $\mathbb{C} \setminus \{0\}$.

Theorem 4.3

Let $S \subseteq \mathbb{C}$, $f, g: S \to \mathbb{C}$ be continuous complex functions and $c \in \mathbb{C}$. Then each of the functions cf, $f + g$, $\overline{f}$, $Re(f)$, $Im(f)$, fg, and f/g (provided $g(z) \neq 0$ for every $z \in S$) is continuous.

Proof Similar to in Real Analysis. ■

Theorem 4.4

Let $S, T \subseteq \mathbb{C}$, $f: S \to \mathbb{C}$, $g: T \to \mathbb{C}$ be continuous complex functions and suppose that $f(S) \subseteq T$. Then the composition $g \circ f$ is continuous.

Proof Similar to in Real Analysis. ■

5. Complex Differentiability

Definition 5.1

Let $S \subseteq \mathbb{C}$, $f: S \to \mathbb{C}$ be a complex function and z_0 be an interior point of S. We say that f is **differentiable at z_0** if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

If f is differentiable at z_0 we write $f'(z_0)$ for the above limit, i.e.

$$f'(z_0) \coloneqq \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

$f'(z_0)$ is called the **derivative of f at z_0.**

If S is open we say that f is **differentiable** if f is differentiable at every point in S. In this case we call the complex function $f': S \to \mathbb{C}$ the **derivative** of f.

Example

We show that for the complex polynomial $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$ we have that f is differentiable and its derivative is given by $f'(z) = 2z$ for all $z \in \mathbb{C}$.

Fix $z_0 \in \mathbb{C}$ and observe, for $z \in \mathbb{C} \setminus \{z_0\}$, that

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{z^2 - z_0^2}{z - z_0} = \frac{(z - z_0)(z + z_0)}{z - z_0} = z + z_0.$$

Therefore, we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (z + z_0) = \lim_{z \to z_0} z + \lim_{z \to z_0} z_0 = 2z_0.$$

In the second last equation we applied Theorem 3.5 (Algebra of Limits) and in the last equation we used Proposition 4.2. This shows that f is differentiable at z_0 and that $f'(z_0) = 2z_0$.

Lecture 5

Lemma 5.2

Let $S \subseteq \mathbb{C}$, $f: S \to \mathbb{C}$ be a complex function and z_0 be an interior point of $S \cup z_0$. Then $\lim_{z \to z_0} g(z)$ exists if and only if $\lim_{w \to 0} g(z_0 + w)$ exists and

$$\lim_{z \to z_0} g(z) = \lim_{w \to 0} g(z_0 + w).$$

Proof Exercise.

Remark Using Lemma 5.2 the limits in the definition of differentiability of f at z_0 can be rewritten as

$$\lim_{w \to 0} \frac{f(z_0 + w) - f(z_0)}{w}.$$

Theorem 5.3

Let $S \subseteq \mathbb{C}$, $f, g: S \to \mathbb{C}$ be complex functions and $z_0 \in S$ be an interior point of S and suppose that both f and g are differentiable at z_0. Then each of the functions cf, $f + g$, fg, and f/g (provided $g(z_0) \neq 0$) is differentiable at z_0 and

$$(cf)'(z_0) = cf'(z_0),$$
$$(f + g)'(z_0) = f'(z_0) + g'(z_0),$$
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0),$$
$$(f/g)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof Similar to in Real Analysis.

Theorem 5.4

Let $S, T \subseteq \mathbb{C}$, $f: S \to \mathbb{C}$ and $g: T \to \mathbb{C}$ be complex functions $z_0 \in S$ be an interior point of S and suppose that $f(S) \subseteq T$ and that $f(z_0)$ is an interior point of T. Suppose further that f is differentiable at z_0 and g is differentiable at $f(z_0)$. Then $g \circ f$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'\big(f(z_0)\big)f'(z_0).$$

Proof Similar to in Real Analysis.

Theorem 5.5

Let $S \subseteq \mathbb{C}$, z_0 be an interior point of S, $f: S \to \mathbb{C}$ be a complex function and suppose that f is differentiable at z_0. Then defining, for $z \in S$,

$$E(z, z_0, f) = f(z) - f(z_0) - f'(z_0)(z - z_0)$$

We have, for $z \in S$,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z, z_0, f)$$

and

$$\lim_{z \to z_0} \frac{E(z, z_0, f)}{z - z_0} = 0.$$

Consequently, f is continuous at z_0.

Proof Similar to in Real Analysis.

Example

We investigate differentiability of the complex function $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = \overline{z}$. Fix $z_0 \in \mathbb{C}$. To check differentiability of f at z_0 we study the quotient

$$\frac{f(z_0 + w) - f(z_0)}{w} = \frac{\overline{z_0 + w} - \overline{z_0}}{w} = \frac{(\overline{z_0} + \overline{w} - \overline{z_0})}{w} = \frac{\overline{w}}{w},$$

as $w \to 0$. However, in an example in Lecture 3 we have already seen that $\lim\limits_{w \to 0} \frac{\overline{w}}{w}$ does not exist. Hence $\lim\limits_{w \to 0} \frac{f(z_0+w)-f(z_0)}{w}$ does not exist and f is not differentiable at z_0. Since $z_0 \in \mathbb{C}$ was arbitrarily, we have shown that f is nowhere differentiable in $\mathbb{C}$.

This example reveals something surprising about complex differentiability. Observe that f may be written as $f(x + iy) = u(x, y) + iv(x, y)$ for functions $u, v: \mathbb{R}^2 \to \mathbb{R}$ given by $u(x, y) = x$ and $v(x, y) = -y$ for each $(x, y) \in \mathbb{R}^2$. Both u and v are C^∞ smooth functions – as real functions they are infinitely many times differentiable and all their derivatives of any order are continuous. Despite this, the complex function f fails to be differentiable.

6. The Cauchy Riemann Equations

Theorem 6.1

Let $S \subseteq \mathbb{C},\ z_0 = x_0 + iy_0 \in S$ be an interior point of S and $f: S \to \mathbb{C}$ be a complex function which is differentiable at the point z_0. Let $u, v: \{(x, y) \in \mathbb{R}^2 : x + iy \in S\} \to \mathbb{R}$ be the functions defined by

$$f(x + iy) = u(x, y) + iv(x, y)$$

whenever $z = x + iy \in S$. Then the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist at the point (x_0, y_0) and satisfy the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and}$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

Moreover, the derivative of f at z_0 is given by

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Before proving Theorem 6.1, we show in an example how Theorem 6.1 and studying the Cauchy Riemann equations can show the non-differentiability of a complex function.

Example

We revisit the example of $f: \mathbb{C} \to \mathbb{C}, f(z) = \overline{z}$. Recall that we have

$$f(x + iy) = u(x, y) + iv(x, y)$$

where $u, v: \mathbb{R}^2 \to \mathbb{R}$ are given by

$$u(x, y) = x \text{ and } v(x, y) = -y.$$

We compute the partial derivatives of u and v and check whether they satisfy the Cauchy Riemann equations at any points (x, y). Observe that

$$\frac{\partial u}{\partial x}(x, y) = 1, \qquad \frac{\partial u}{\partial y}(x, y) = 0, \qquad \frac{\partial v}{\partial x}(x, y) = 0, \qquad \frac{\partial v}{\partial y}(x, y) = -1,$$

for all $(x, y) \in \mathbb{R}^2$. Hence,

$$\frac{\partial u}{\partial x}(x, y) = 1 \neq -1 = \frac{\partial v}{\partial y}(x, y)$$

for each $(x, y) \in \mathbb{R}^2$. Therefore, u and v fail the Cauchy Riemann equations of Theorem 6.1at every point $(x, y) \in \mathbb{R}^2$. By Theorem 6.1we conclude that f is not differentiable at any point $x + iy \in \mathbb{C}$.

Lecture 6

In the proof of Theorem 6.1 we will make use of the following useful statement:

Lemma 6.2

Let $T \subseteq \mathbb{C}$, z_0 be a boundary point of $T \setminus 0$ and $g: T \setminus \{0\} \to \mathbb{C}$ be a complex function. Then $\lim_{z \to z_0, z \in T} g(z)$ exists if and only if both $\lim_{z \to z_0, z \in T} Re(g(z))$ and $\lim_{z \to z_0, z \in T} Im(g(z))$ exist and $\lim_{z \to z_0, z \in T} g(z) = \lim_{z \to z_0, z \in T} Re\big(g(z)\big) + i \lim_{z \to z_0, z \in T} Im(g(z))$.

Proof The lemma follows from the fact that $w \mapsto Re(w)$ and $w \mapsto Im(w)$ are continuous functions, Theorem 3.5 (Algebra of Limits) and Proposition 4.2. ∎

Proof of Theorem 6.1

Since f is differentiable at z_0 we have that $f'(z_0) = \lim_{w \to 0} \frac{f(z_0+w)-f(z_0)}{w}$ exists. It follows that for any set $T \subseteq \mathbb{C}$ for which 0 is a boundary point of $T \setminus \{0\}$ we have

$$\lim_{w \to 0, w \in T} \frac{f(z_0 + w) - f(z_0)}{w} = f'(z_0).$$

We will investigate limits of this form for two specific sets $T_1 = \mathbb{R}$ and $T_2 = i\mathbb{R}$. For $h \in \mathbb{R} \setminus \{0\}$ we have

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0 + h, y) - v(x_0, y)}{h}.$$

Letting $h \to 0$ in $\mathbb{R}$ in this equation and applying Lemma 4.2 we deduce that the partial derivatives $\frac{\partial u}{\partial x}(x_0, y_0)$ and $\frac{\partial v}{\partial y}(x_0, y_0)$ exist and

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Equation 1

On the other hand, we also have

$$\frac{f(z_0 + ih) - f(z_0)}{ih} = \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{ih} + i\frac{v(x_0, y + h) - v(x_0, y)}{ih}.$$

We may manipulate the right hand side of this equation, to get

$$\frac{f(z_0 + ih) - f(z_0)}{ih} = \frac{v(x_0, y + h) - v(x_0, y)}{h} - i\frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h}.$$

Letting $h \to 0$ in $\mathbb{R}$ in this equation and applying Lemma 4.2 we deduce that the partial derivatives $\frac{\partial v}{\partial y}(x_0, y_0)$ and $\frac{\partial u}{\partial y}(x_0, y_0)$ exist and

$$f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Equation 2

We have derived two equations Equation 1 and Equation 2 for $f'(z_0)$. Equating real and imaginary parts in these two expressions, we obtain the Cauchy Riemann equations. ■

If $S \subseteq \mathbb{C}$ is open and $f: S \to \mathbb{C}$ is differentiable, then Theorem 6.1 tells us that the Cauchy Riemann equations are satisfied at every point of S. The next Theorem is a type of converse to this statement.

Theorem 6.3

Let $S \subseteq \mathbb{C}$ be open, $f: S \to \mathbb{C}$, $f(x+iy) = u(x,y) + iv(x,y)$ be a complex function and suppose that the functions $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist, are defined and continuous on the set of $(x,y) \in \mathbb{R}^2$ with $x + iy \in S$ and that the Cauchy Riemann equations are satisfied at every point $(x,y) \in \mathbb{R}^2$ with $x + iy \in S$. Then f is differentiable (at every point of S) and for each $z_0 = x_0 + iy_0 \in S$ its derivative is given by

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Proof Omitted.

7. Differentiable Functions

Definition 7.1

Let $\Omega \subseteq \mathbb{C}$ be an open set. We call a complex function $f: \Omega \to \mathbb{C}$ **holomorphic** if f is differentiable. A complex function is called **entire** if it is defined on the whole of $\mathbb{C}$ and is holomorphic.

Lecture 7

Theorem 7.2

Let $\Omega \subseteq \mathbb{C}$ be a domain and $f: \Omega \to \mathbb{C}$ be holomorphic. Then **any** of the following conditions imply that f is constant:

a) $f'(z) = 0$ for all $z \in \Omega$.
b) $|f|$ is constant.
c) Either $Re(f)$ or $Im(f)$ are constant.

Proof Let $u, v: \{(x, y): x + iy \in \Omega\} \to \mathbb{C}$ be the real and imaginary parts of f so that $f(x + iy) = u(x, y) + iv(x, y)$ whenever $z = x + iy \in \Omega$.

By Theorem 6.1 the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at every point of Ω and satisfy the Cauchy Riemann equations.

We first show that a) implies that f is constant. At each $z = x + iy \in \Omega$ we have

$$0 = f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y).$$

Equating real and imaginary parts in this equation, we deduce that all the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are zero at (x, y). Hence all the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are constant zero on $\{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\}$.

Let Γ be a horizontal line segment inside the set Ω. Then there is $y_0 \in \mathbb{R}$ so that $Im(z) = y_0$ for all $z \in \Gamma$. Let $z_1 = x_1 + iy_0$ and $z_2 = x_2 + iy_0$ be two points in Γ and assume without loss of generality that $x_1 \leq x_2$. Define a real function $u_{y_0}: [x_1, x_2] \to \mathbb{R}$ by $u_{y_0}(x) = u(x, y_0) = Re(f(x + iy_0)$. Then u_{y_0} is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2) and $u'_{y_0}(x) = \frac{\partial u}{\partial x}(x, y_0) = 0$ for all $x \in (x_1, x_2)$. Real differentiable functions defined on an open interval with derivative constant zero are constant. Hence u_{y_0} is constant on (x_1, x_2) and by continuity, also on $[x_1, x_2]$. So $Re\big(f(x_1 + iy_0)\big) = Re\big(f(x_2 + iy_0)\big)$. We conclude that $Re(f)$ is constant on Γ. Similarly, we may show that $Im(f)$ is constant on Γ. Therefore, f is constant on every horizontal line segment contained in the set Ω.

By an analogous argument, we may show that f is constant on every vertical line segment contained in Ω.

To finish the proof, we use that any two points in Ω may be connected by path consisting entirely of horizontal and vertical line segments.

We now prove that b) implies that f is constant. Let $C \in \mathbb{C}$ denote the constant value of $|f|$. If $C = 0$ then f is the constant 0 function and we are done. So we may assume that $C \neq 0$. We may express $|f|^2$ as

$$C^2 = |f(x, y)|^2 = u(x, y)^2 + v(x, y)^2.$$

The left hand side of this equation is constant and the right hand side is differentiable with respect to both x and y. Differentiating with respect to x, we get

$$0 = 2u(x, y)\frac{\partial u}{\partial x}(x, y) + 2v(x, y)\frac{\partial v}{\partial x}(x, y),$$

and with respect to y gives

$$0 = 2u(x,y)\frac{\partial u}{\partial y}(x,y) + 2v(x,y)\frac{\partial v}{\partial y}(x,y).$$

We apply the Cauchy Riemann equations in the first of these equations to replace the partial derivatives with respect to x with partial derivatives with respect to y. Then, the two above equations rearrange to

$$u\frac{\partial v}{\partial y} = v\frac{\partial u}{\partial y} \qquad \text{and} \qquad u\frac{\partial u}{\partial y} = -v\frac{\partial v}{\partial y}.$$

Multiplying the latter equation by u and then applying the former we get

$$u^2\frac{\partial u}{\partial y} = -vu\frac{\partial v}{\partial y} = -v^2\frac{\partial u}{\partial y},$$

so

$$C^2\frac{\partial u}{\partial y} = (u^2+v^2)\frac{\partial u}{\partial y} = 0.$$

A similar argument, where we use the Cauchy Riemann equations to get rid of the partial derivatives with respect to y instead of those with respect to x gives the equation

$$C^2\frac{\partial u}{\partial y} = (u^2+v^2)\frac{\partial u}{\partial x} = 0.$$

Since $C^2 \neq 0$, we conclude that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are constant 0, implying, by Theorem 6.1, that f' is constant zero. Hence, the proof is finished by part a).

Finally, we show that c) implies that f is constant. Suppose $Re(f)$ is constant. Then we have that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are constant zero, implying, via the Cauchy Riemann equations that also $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are constant zero. Hence, a) applies again to complete the proof. ∎

Lecture 8

8. Harmonic Functions

Definition 8.1

Let $\Omega \subseteq \mathbb{R}^2$ be an open set. A function $u : \Omega \to \mathbb{R}$ is called **harmonic** if all of the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$, and $\frac{\partial^2 u}{\partial x \partial y}$ exist everywhere in Ω, are continuous as functions $\Omega \to \mathbb{R}$ and the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is satisfied everywhere in Ω.

Example

Consider the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by $u(x, y) = x^2 - y^2$. Then we may check that all of the partial derivatives $\frac{\partial u}{\partial x}(x, y) = 2x$, $\frac{\partial u}{\partial y}(x, y) = -2y$, $\frac{\partial^2 u}{\partial x^2}(x, y) = 2$, $\frac{\partial^2 u}{\partial y^2}(x, y) = -2$, and $\frac{\partial^2 u}{\partial x \partial y}(x, y) = 0$ exist everywhere and are continuous as functions $\mathbb{R}^2 \to \mathbb{R}$. Moreover, we have $\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 2 - 2 = 0$ for every $(x, y) \in \mathbb{R}^2$. Therefore u is harmonic.

Theorem 8.2

Let $\Omega \subseteq \mathbb{C}$ be an open set, $f: \Omega \to \mathbb{C}$ be a holomorphic function and $u, v: \{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\} \to \mathbb{R}$ be the functions satisfying

$$f(x + iy) = u(x, y) + iv(x, y)$$

whenever $x + iy \in \Omega$. Then u and v are harmonic functions.

Proof We will see later in the course that both u and v are C^∞ smooth functions, that is, all partial derivatives of u and v (of any order) exist and are continuous. We compute

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y},$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y}.$$

The last equality, where we swap the order of differentiation, is a general fact for C^2 functions v. Summing the two equations derived above, we verify that u is harmonic. The proof that v is harmonic is similar. ■

Theorem 8.3

Let $\Omega \subseteq \mathbb{C}$ be a simply connected, open set. We identify Ω with a subset of $\mathbb{R}^2$ via the identification $x + iy \leftrightarrow (x, y)$. Let $u: \Omega \to \mathbb{R}$ be a harmonic function. Then there exists a harmonic function $v: \Omega \to \mathbb{R}$ such that the complex function $f: \Omega \to \mathbb{C}$ given by

$$f(x + iy) = u(x, y) + iv(x, y),$$

is holomorphic. Such a function v is called a **harmonic conjugate** of f. Moreover, any two such functions v differ only by a constant.

Example

We return to the example of the harmonic function $u: \mathbb{R}^2 \to \mathbb{R}$, $u(x, y) = x^2 - y^2$. We construct the harmonic conjugates of u, which exist by Theorem 8.3, since $\mathbb{R}^2$ is simply connected. Suppose $v: \mathbb{R}^2 \to \mathbb{R}$ is a harmonic conjugate of u. Then, by Theorem 8.3 the complex function $f: \mathbb{C} \to \mathbb{C}$, $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic. Therefore, by Theorem 6.1, u and v satisfy the Cauchy Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, everywhere in $\mathbb{R}^2$. So, looking at the first of these equations, we have

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) = 2x,$$

which we may integrate with respect to y to obtain

$$v(x,y) = 2xy + c(x).$$

For some function $c(x)$. If we now differentiate this equation with respect to x we get

$$\frac{\partial v}{\partial x}(x,y) = 2y + c'(x),$$

whilst we also know from the second Cauchy Riemann equation that

$$\frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) = 2y.$$

We conclude that $c'(x) = 0$, so $c(x) = c \in \mathbb{C}$ is a constant. Hence, the harmonic conjugates of u are the functions v of the form $v(x,y) = 2xy + c$, where $c \in \mathbb{C}$ is a constant. We can verify that for $v(x,y) = 2xy + c$, the complex function $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic, using Theorem 6.3. We have already demonstrated that u and v satisfy the Cauchy Riemann equations at every $(x,y) \in \mathbb{R}^2$ and we have that $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all defined and continuous on the whole of $\mathbb{R}^2$. Therefore, the conditions of Theorem 6.3 are satisfied.

9. Power Series

Definition 9.1

Given $z_0 \in \mathbb{C}$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers, the complex power series with centre z_0 and coefficients a_n is the expression

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

considered for complex variable $z \in \mathbb{C}$. Here, $(z - z_0)^0$ should be understood as being equal to 1 for every $z \in \mathbb{C}$, even $z = z_0$.

For each $z \in \mathbb{C}$ we say that this power series converges at z if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n (z - z_0)^n$ exists.

The **radius of convergence** of this power series is defined by

$$R = \frac{1}{limsup_{n\to\infty}|a_n|^{\frac{1}{n}}},$$

and can be equal to either 0 or ∞. In this equation we interpret $1/0$ as ∞ and $1/\infty$ as 0.

Lecture 9

Proposition 9.2

The radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

has the following properties:

a) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges if $|z-z_0| < R$.
b) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ does not converge if $|z-z_0| > R$.
c) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ may or may not converge if $|z-z_0| = R$.

Proposition 9.3

The radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

Is given by

a) (The ratio test) $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ if this limit exists.
b) (The root test) $R = \lim_{n\to\infty} \frac{1}{|a_n|^{\frac{1}{n}}}$ if this limit exists.

Theorem 9.4 (Algebra of Power Series)

Let $z_0 \in \mathbb{C}, r > 0$, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of complex numbers and suppose that both power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge inside the ball $B(z_0, r)$. Then each of the following power series also converge inside $B(z_0, r)$ and they have the following formulae.

a) $\sum_{n=0}^{\infty} ca_n(z-z_0)^n = c\sum_{n=0}^{\infty} a_n(z-z_0)^n$.
b) $\sum_{n=0}^{\infty}(a_n+b_n)(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=0}^{\infty} b_n(z-z_0)^n$.
c) $\sum_{n=0}^{\infty} c_n(z-z_0)^n = (\sum_{n=0}^{\infty} a_n(z-z_0)^n)(\sum_{n=0}^{\infty} b_n(z-z_0)^n)$, where

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}.$$

Theorem 9.5 (Differentiability of Power Series)

Let $z_0 \in \mathbb{C}$, $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, $r > 0$ and suppose that the complex power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in the ball $B(z_0, r)$. Then the function $f: B(z_0, r) \to \mathbb{C}$ is infinitely many times differentiable and for each $k \in \mathbb{N}$ its k-th derivative $f^{(k)}$ is given on $B(z_0, r)$ by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) a_n (z - z_0)^{n-k}.$$

Moreover, writing $f(x + iy) = u(x, y) + iv(x, y)$, we have that both u and v are infinitely many times differentiable and all their partial derivatives (of any order) are continuous.

10. Important complex functions defined using power series.

Definition 10.1

The exponential function is defined $\mathbb{C} \to \mathbb{C}$ by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We sometimes write e^z instead of $\exp(z)$.

Theorem 10.2

The exponential function has the following properties:

a) $\exp(0) = 1.$
b) $\exp(z + w) = \exp(z)\exp(w)$ for all $z, w \in \mathbb{C}$.
c) $\exp(-z) = \frac{1}{\exp(z)}$ for all $z \in \mathbb{C}$.
d) $\exp(z) \neq 0$ for all $z \in \mathbb{C}$.
e) $\exp(x + iy) = e^x(\cos y + i \sin y)$ for all $x, y \in \mathbb{R}$.
f) $\exp(z + 2\pi ik) = \exp(z)$ for all $z \in \mathbb{C}$.
g) $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.
h) $\exp$ is holomorphic and $\exp'(z) = \exp(z)$ for all $z \in \mathbb{C}$.

Moreover, the exponential function is the unique function $\mathbb{C} \to \mathbb{C}$ with properties (a) and (g).

Proof (a) follows by plugging $z = 0$ into the power series expression for $\exp(z)$ and recalling that $0^0 = 0! = 1$.

(b) may be verified by writing $\exp(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}$ and then applying the binomial theorem to rewrite $(z+w)^n$ as $\sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$. We may then manipulate the expression, relabelling indices to arrive at the desired right hand side $\exp(z)\exp(w)$. Since the proof is no different to the corresponding identity for the real exponential function, we leave this part as an exercise.

c) follows from a) and b) because we have $1 = \exp(0) = \exp(-z+z) = \exp(-z)\exp(z)$ for all $z \in \mathbb{C}$, and then d) is implied by c).

e) is an application of b) to $z = x$ and $w = iy$. We get $\exp(x+iy) = \exp(x)\exp(iy)$. Finally we apply the identity $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$.

To prove g), let $w \in \mathbb{C} \setminus \{0\}$. We find $z \in \mathbb{C}$ so that $\exp(z) = w$. Recall that we may write $w = |w|\exp(i\,Arg(w))$, where $|w| > 0$ and $Arg(w) \in (-\pi, \pi]$. Since the real exponential function $\exp|_{\mathbb{R}}: \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is surjective, we know there exists $x \in \mathbb{R}$ such that $e^x = |w|$. Setting $z = x + i\,Arg(w)$ we get $\exp(z) = e^x \exp\big(i\,Arg(w)\big) = |w|\exp\big(i\,Arg(w)\big) = w$.

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The last property h) is verified by applying the formula for $\exp'(z)$ given by Theorem 9.5 (Differentiability of Power Series). We get

$$\exp'(z) = \sum_{n=1}^{\infty} n\frac{1}{n!}z^{n-1} = \sum_{n=1}^{\infty} \frac{(z^{n-1})}{\big((n-1)!\big)} = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Finally we prove the `Moreover' statement. Suppose $f: \mathbb{C} \to \mathbb{C}$ is another holomorphic complex function satisfying $f(0) = 1$ and $f'(z) = f(z)$ for all $z \in \mathbb{C}$. We show that $f = \exp$. We may consider the complex function $\frac{f}{exp}: \mathbb{C} \to \mathbb{C}$ which is well-defined because, by (d) we have that $\exp(z) \neq 0$ for all $z \in \mathbb{C}$. By the Algebra of Differentiability, this function is holomorphic and its derivative is given by

$$\left(\frac{f}{exp}\right)'(z) = \frac{\exp(z)\,f'(z) - f(z)\exp'(z)}{\exp(z)^2} = \frac{\exp(z)\,f(z) - f(z)\exp(z)}{\exp(z)^2} = 0,$$

Where in the second equality we applied the property (h) of exp and f. Hence f/exp is a holomorphic function with constant derivative zero on $\mathbb{C}$. We conclude, by Theorem 7.2 that f/exp is constant. Hence, for all $z \in \mathbb{C}$ we have $\frac{f(z)}{\exp(z)} = \frac{f(0)}{\exp(0)} = 1$. ■

Definition 10.3

The hyperbolic sine and cosine functions are defined $\mathbb{C} \to \mathbb{C}$ by

$$\cosh(z) = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!},\ \sinh(z) = \frac{(e^z - e^{-z})}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.$$

Theorem 10.4

The hyperbolic sine and cosine functions satisfy the following identities for all $z, w \in \mathbb{C}$.

a) $\cosh(-z) = \cosh(z)$.
b) $\sinh(-z) = -\sinh(z)$.
c) $\cosh'(z) = \sinh(z)$.
d) $\sinh'(z) = \cosh(z)$.
e) $\cosh(z + 2\pi ik) = \cosh(z)$.
f) $\sinh(z + 2\pi ik) = \sinh(z)$.
g) $\cosh(z + w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$.
h) $\sinh(z + w) = \sinh(z)\cosh(w) + \cosh(w)\sinh(w)$.
i) $\cosh^2(z) - \sinh^2(z) = 1$.

Definition 10.5

The sine and cosine functions are defined $\mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad \text{and} \quad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

Remark There is an analogous list of identities to Theorem 10.5 for the sine and cosine functions, which you will already be familiar with.

Theorem 10.6

The functions $sin, \cos: \mathbb{C} \to \mathbb{C}$ are surjective, i.e. we have $\sin(\mathbb{C}) = \mathbb{C}$ and $\cos(\mathbb{C}) = \mathbb{C}$. In particular $sin, \cos: \mathbb{C} \to \mathbb{C}$ are not bounded!

Proof We show that $\sin: \mathbb{C} \to \mathbb{C}$ is surjective; the proof for cos is similar. Let $w \in \mathbb{C}$. We find $z \in \mathbb{C}$ such that $\sin z = w$. For $z \in \mathbb{C}$ we have

$$\sin z = w \quad \Leftrightarrow \quad \frac{e^{iz} - e^{-iz}}{2i} = w \quad \Leftrightarrow \quad \left(e^{iz}\right)^2 - 2iwe^{iz} - 1 = 0$$

and the last equation is equivalent to stating that e^{iz} is a root of the complex polynomial $p(u) = u^2 - 2iu - 1, u \in \mathbb{C}$. By Theorem 2.2 (The Fundamental Theorem of Algebra) we know that there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $p(u) = u^2 - 2iu - 1 = (u - \alpha_1)(u - \alpha_2)$. It also follows from this equation that $\alpha_1\alpha_2 = 1$, so in particular $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. We conclude that $\sin z = w$ if and only if $e^{iz} = \alpha_1$ or $e^{iz} = \alpha_2$. Writing $z = x + iy$ these equations may be rewritten as $e^{-y}e^{ix} = \alpha_1 = |\alpha_1|e^{iArg(\alpha_1)}$ or $e^{-y}e^{ix} = \alpha_2 = |\alpha_2|e^{iArg(\alpha_2)}$. So it suffices to define $z = x + iy$ where $e^{-y} = |\alpha_1|$ and $x = Arg(\alpha_1)$. Since $|\alpha_1| > 0$ we know that there exists $y \in \mathbb{R}$ satisfying $e^{-y} = |\alpha_1|$. ■

11. Contour Integrals

Definition 11.1

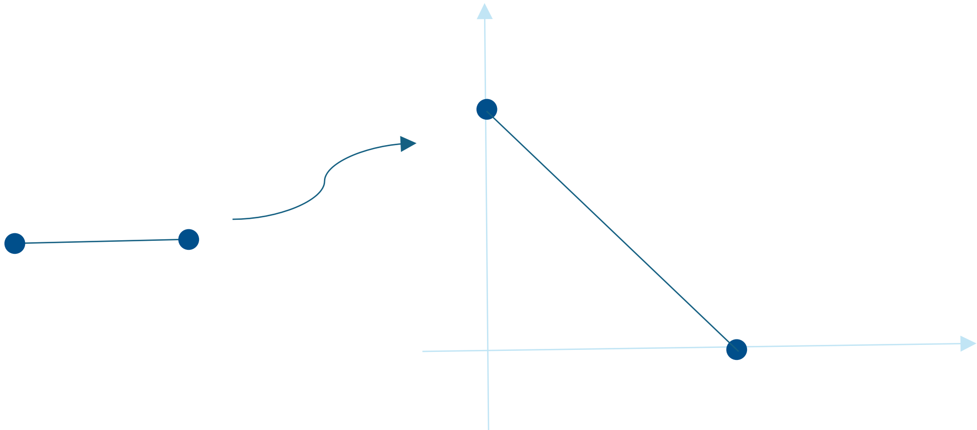
A set $\Gamma \subseteq \mathbb{C}$ is called a **curve** if there exists a path $\gamma: [a, b] \to \mathbb{C}$ such that

$$\Gamma = \gamma([a, b]) = \{\gamma(t) : t \in [a, b]\}.$$

Any such path γ is called a **parameterisation** of Γ.

Example

Let Γ be the straight line segment with endpoints 1 and i.



The Γ is a curve and two examples of parameterisations of Γ are

$$\gamma_1: [0,1] \to \mathbb{C},\ \gamma_1(t) = (1 - t)1 + ti, \text{ and}$$

$$\gamma_2: [0,1] \to \mathbb{C}, \gamma_2(t) = (1 - t)i + t1.$$

Remark The formula $\gamma(t) = (1 - t)z + tw$ for $t \in [0,1]$ is a useful one to remember. It is a standard parameterisation of the line segment with endpoints z and w.

Lecture 11

Definition 11.2

Suppose that $\gamma_1: [a, b] \to \mathbb{C}$ and $\gamma_2: [c, d] \to \mathbb{C}$ are two paths for which the end point of γ_1 is the same as the start point of γ_2, that is, $\gamma_1(b) = \gamma_2(c)$. Then we may define the **join** of γ_1 and γ_2, denoted $\gamma_1 + \gamma_2$, as the path $(\gamma_1 + \gamma_2) : [a, b + d - c] \to \mathbb{C}$ defined by

$$(\gamma_1 + \gamma_2)(t) = \gamma_1(t) \text{ if } t \in [a, b] \text{ and}$$

$$(\gamma_1 + \gamma_2)(t) = \gamma_2(t - b + c) \text{ if } t \in [b, b + d - c].$$

Given a path $\gamma: [a, b] \to \mathbb{C}$ we define the **reverse** of γ as the path, denoted γ^*, as the path $\gamma^*: [a, b] \to \mathbb{C}$ defined by

$$\gamma^*(t) = \gamma(b + a - t) \text{ for all } t \in [a, b].$$

Definition 11.3

A path $\gamma: [a, b] \to \mathbb{C}$ is said to be differentiable at a point $t \in (a, b)$ if both of the derivatives $(Re(\gamma))'(t)$ and $(Im(\gamma))'(t)$ exist. In this case we define the derivative of γ at t as

$$\gamma'(t) = \big(Re(\gamma)\big)'(t) + i\big(Im(\gamma)\big)'(t).$$

The path γ is called **smooth** if it is differentiable at every $t \in (a, b)$ and $\gamma'(t) \neq 0$ for all $t \in (a, b)$.

A path $\gamma: [a, b] \to \mathbb{C}$ is called a **contour** if it is the join of finitely many smooth paths.

Lemma 11.4

If a path $\gamma: [a, b] \to \mathbb{C}$ has the form $\gamma = g \circ \phi$ for functions $\phi: [a, b] \to S \subseteq \mathbb{C}$ and $g: S \to \mathbb{C}$. Then the chain rule formula

$$\gamma'(t) = g'(\gamma(t))\gamma'(t)$$

is valid, provided all derivatives on the right hand side exist.

Definition 11.5

Let $\phi: [a, b] \to \mathbb{C}$ be a function. We define the integral of ϕ on the interval $[a, b]$ by

$$\int_a^b \phi(t)dt = \int_a^b Re\big(\phi(t)\big)dt + i\int_a^b Im\big(\phi(t)\big)dt,$$

provided that both of the real integrals on the right hand side exist. Otherwise we say that the integral of ϕ on $[a, b]$ does not exist.

Lemma 11.6

a) Let $\phi_1, \phi_2: [a, b] \to \mathbb{C}$ be functions and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then
$$\int_a^b \lambda_1\phi_1(t) + \lambda_2\phi_2(t)dt = \lambda_1\int_a^b \phi_1(t)dt + \lambda_2\int_a^b \phi_2(t)dt.$$
b) If $\phi: [a, b] \to \mathbb{C}$ is continuous and $\Psi: [a, b] \to \mathbb{C}$ is a function for which $\Psi'(t) = \phi(t)$ for all $t \in [a, b]$ then
$$\int_a^b \phi(t)dt = \Psi(b) - \Psi(a).$$

Definition 11.7

Let $\gamma: [a, b] \to \mathbb{C}$ be a contour, $S \subseteq \mathbb{C}$ be a set with $\gamma([a, b]) \subseteq S$ and $f: S \to \mathbb{C}$ be a complex function. Then we define the **path integral** or **contour integral** of f on γ by

$$\int_\gamma f(z)dz = \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt,$$

provided that the integral on the right hand side exists. Otherwise we say that the contour integral of f on γ does not exist.

Lecture 12

Theorem 11.8 (Properties of contour integrals)

The contour integral satisfies the following identities whenever the expressions on both sides exist:

a) $\int_\gamma \lambda_1 f_1(z) + \lambda_2 f_2(z)dz = \lambda_1 \int_\gamma f_1(z)dz + \lambda_2 \int_\gamma f_2(z)dz.$
b) $\int_{\gamma^*} f(z)dz = -\int_\gamma f(z)dz.$
c) $\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz\,.$
d) Suppose that $\phi: [a, b] \to [c, d]$ is continuous, monotone increasing, bijective and piecewise differentiable and that $\gamma: [c, d] \to \mathbb{C}$ is a contour. Then $\gamma \circ \phi: [a, b] \to \mathbb{C}$ is a contour and
$$\int_{\gamma\circ\phi} f(z)dz = \int_\gamma f(z)dz.$$

Definition 11.9

A curve $\Gamma \subseteq \mathbb{C}$ is called **piecewise smooth** if it may be parameterised by a contour, that is, if there exists a contour $\gamma: [a, b] \to \mathbb{C}$ such that $\gamma([a, b]) = \Gamma$.

Γ is said to be **simple** if it may be parameterised by a simple path, i.e. if there exists a simple path $\gamma: [a, b] \to \mathbb{C}$ such that $\gamma([a, b]) = \Gamma$.

Γ is said to **closed** if it may be parameterised by a closed path, i.e. if there exists a closed path $\gamma: [a, b] \to \mathbb{C}$ such that $\gamma([a, b]) = \Gamma$.

A curve $\Gamma \subseteq \mathbb{C}$ is said to be **oriented** if it comes with an instruction which tells us which way to go along Γ. If Γ is an oriented curve we say that a parameterisation $\gamma: [a, b] \to \mathbb{C}$ of Γ respects the orientation on Γ if as t goes from a to b the image point $\gamma(t)$ goes along Γ from $\gamma(a)$ to $\gamma(b)$ in the direction given by the orientation of Γ.

Using Theorem 11.8 (Properties of contour integrals) it is possible to check that if Γ is a simple, piecewise smooth, oriented curve and f is a complex function defined on Γ then $\int_\gamma f(z)dz$ is the same for every parameterisation of Γ by a simple contour $\gamma: [a,b] \to \mathbb{C}$ which respects the orientation on Γ. Because of this, we may define the integral of f on Γ in the following way:

Definition 11.11

Let $\Gamma \subseteq \mathbb{C}$ be a simple, piecewise smooth, oriented curve, $S \subseteq \mathbb{C}$ be a set with $\Gamma \subseteq S$ and $f: S \to \mathbb{C}$ be a complex function. Then we define the integral of f on Γ by

$$\int_\Gamma f(z)dz = \int_\gamma f\big(\gamma(t)\big)\gamma'(t)dt = \int_a^b f\big(\gamma(t)\big)\gamma'(t)dt$$

where $\gamma: [a,b] \to \mathbb{C}$ is any simple, contour parameterising Γ and respecting the orientation on Γ.

Lemma 11.12 (The ML-Lemma)

Let $S \subseteq \mathbb{C}$, $\gamma: [a,b] \to \mathbb{C}$ be a contour such that $\gamma([a,b]) \subseteq S$ and $f: S \to \mathbb{C}$ be a continuous function. Suppose that $M > 0$ and that $|f\big(\gamma(t)\big)| \leq M$ for all $t \in [a,b]$. Then

$$|\int_\gamma f(z)dz\,| \leq ML,$$

where L denotes the length of γ. An analogous upper bound holds for $|\int_\Gamma f(z)dz|$, if $\Gamma \subseteq S$ is a simple, oriented, smooth curve.