Examples sheet 5 – Solutions – Linear Algebra

DIAGONALISATION

1. This question is motivated by the case where $\gamma(\lambda_i) < \alpha(\lambda_i)$ for one or more indices i. In this case, A is defective and therefore not diagonalizable. However, one can still write down a similar relation to connecting A and D, although an eigenvalue factorisation of A does not exist. Using the eigenvalue equation for $i = 1, \ldots, n$, we find

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \Longrightarrow [A\mathbf{v}_1,\ A\mathbf{v}_2,\ \cdots, A\mathbf{v}_k] = [\lambda_1\mathbf{v}_1,\ \lambda_2\mathbf{v}_2,\ \cdots, \lambda_k\mathbf{v}_k].$$

We show that the term on the left is AV, while the term on the right is VD. First, recall the following definitions and notation:

$$[\mathbf{c}_j(A)]_i = a_{ij}, \quad [AB]_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad [A\mathbf{x}]_i = \sum_{k=1}^{n} a_{ik}x_k.$$

Using these definitions and notation, we find

$$[\mathbf{c}_j(AV)]_i = [AV]_{ij} = \sum_{k=1}^{n} a_{ik}v_{kj} = \sum_{k=1}^{n} a_{ik}[\mathbf{c}_j(V)]_k = [A\mathbf{c}_j(V)]_i.$$

Hence, $\mathbf{c}_j(AV) = A\mathbf{c}_j(V)$, for $j = 1, \ldots, k$, i.e., the jth column of AV is $A\mathbf{c}_j(V)$. Hence,

$$AV = [A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k].$$

Let now D be diagonal with $d_{ij} = \lambda_j\delta_{ij}$

$$[\mathbf{c}_j(VD)]_i = [VD]_{ij} = \sum_{k=1}^{n} v_{ik}d_{kj} = \sum_{k=1}^{n} v_{ik}\lambda_j\delta_{kj} = \lambda_j v_{ij} = \lambda_j[\mathbf{c}_j(V)]_i.$$

Hence, $\mathbf{c}_j(VD) = \lambda_j\mathbf{c}_j(V)$, for $j = 1, \ldots, k$, i.e., the jth column of VD is $\lambda_j\mathbf{c}_j(V)$. Hence,

$$VD = [\lambda_1\mathbf{v}_1,\ \lambda_2\mathbf{v}_2,\ \cdots, \lambda_k\mathbf{v}_k].$$

The result then follows.

2. **(a)** We find

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

i. We have

$$A\mathbf{x} = \mathbf{0} \Longrightarrow \begin{cases} 2x + 2y + 2z & = 0 \\ x + y + z & = 0 \end{cases} \overset{x=\alpha, y=\beta}{\Longrightarrow} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ -\alpha - \beta \end{bmatrix} \Longrightarrow \ker A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Hence $\operatorname{rank} A = 3 - \dim\ker A = 1$.

ii. We first compute $p_A(t)$:

$$p_A(t) = \det \begin{bmatrix} t-2 & -2 & -2 \\ 0 & t & 0 \\ -1 & -1 & t-1 \end{bmatrix} = t[(t-2)(t-1) - 2] = t(t^2 - 3t) \Longrightarrow \operatorname{sp}A = \{0, 3\}.$$

Let $\lambda = 0$. In this case, $E_\lambda = \ker A$, so $\gamma(\lambda) = 2$ and we can use as eigenvectors the spanning set for $\ker A$ found in part i. We also note that $\gamma(\lambda) = \alpha(\lambda) = 2$.
Let $\lambda = 3$. We have

$$(3I - A)\mathbf{x} = \mathbf{0} \Longrightarrow \begin{bmatrix} 1 & -2 & -2 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{cases} x - 2z & = 0 \\ y & = 0 \end{cases} \overset{z=1}{\Longrightarrow} \begin{cases} x & = 2, \\ y & = 0, \\ z & = 1. \end{cases}$$

Hence, we have a full set of eigenvectors, since $\sum_{\lambda \in \mathrm{sp}(A)} \gamma(\lambda) = 3$, so that the matrix is diagonalisable. The corresponding eigenvalue factorisation is

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) Let $A = \mathbf{v}\mathbf{u}^T$.

i. We have

$$A\mathbf{v} = \mathbf{v}\mathbf{u}^T\mathbf{v} = \lambda\mathbf{v}, \quad \lambda := \mathbf{u}^T\mathbf{v}.$$

ii. We show $\ker A \subseteq U^\perp$ and $U^\perp \subseteq \ker A$.
First, let $\mathbf{x} \in \ker A$. Then

$$\mathbf{0} = A\mathbf{x} = \mathbf{v}\mathbf{u}^T\mathbf{x} = (\mathbf{u}^T\mathbf{x})\mathbf{v} \Longrightarrow \mathbf{u}^T\mathbf{x} = 0 \Longrightarrow \mathbf{x} \in U^\perp \Longrightarrow \ker A \subseteq U^\perp.$$

Let now $\mathbf{x} \in U^\perp$, so that $\mathbf{u}^T\mathbf{x} = 0$. Then

$$A\mathbf{x} = \mathbf{v}\mathbf{u}^T\mathbf{x} = \mathbf{v}(\mathbf{u}^T\mathbf{x}) = 0 \cdot \mathbf{v} = \mathbf{0} \Longrightarrow \mathbf{x} \in \ker A \Longrightarrow U^\perp \subseteq \ker A.$$

We conclude $\ker A = U^\perp$. Since $\dim \ker A > 0$, $\lambda = 0$ is an eigenvalue of A, with geometric multiplicity $\gamma(\lambda) = \dim \ker A = \dim U^\perp = n - \dim U = n - 1$.

iii. We found $\mathrm{sp} A = \{0, \mathbf{u}^T\mathbf{v}\}$, with $\sum_{\lambda \in \mathrm{sp}(A)} \gamma(\lambda) = (n-1) + 1 = n$, so the matrix is diagonalisable. Let $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}\}$ be a basis for $U^\perp$. Then $A\mathbf{w}_i = \mathbf{0}$, so $B \cup \{\mathbf{v}\}$ is a linearly independent set of eigenvectors. Hence,

$$A = VDV^{-1} := [\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}, \mathbf{v}] \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \mathbf{u}^T\mathbf{v} \end{bmatrix} [\mathbf{w}_1, \ldots, \mathbf{w}_{n-1}, \mathbf{v}]^{-1}.$$

3. **(a)** Let

$$A := \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T.$$

Then

$$A\mathbf{u}_1 = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T \mathbf{u}_1 = \lambda_1 \mathbf{u}_1 (\mathbf{u}_1^T \mathbf{u}_1) + 0 = \lambda_1 \mathbf{u}_1.$$

We similarly find $A\mathbf{u}_2 = \lambda_2 \mathbf{u}_2$.

(b) We show $\ker A \subseteq U^\perp$ and $U^\perp \subseteq \ker A$.
First, let $\mathbf{x} \in \ker A$. Then $\mathbf{x} \in U^\perp$ since

$$\mathbf{0} = A\mathbf{x} = (\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T)\mathbf{x} = \lambda_1(\mathbf{u}_1^T\mathbf{x})\mathbf{u}_1 + \lambda_2(\mathbf{u}_2^T\mathbf{x})\mathbf{u}_2 \Longrightarrow \begin{cases} \lambda_1(\mathbf{u}_1^T\mathbf{x}) & = 0 \\ \lambda_2(\mathbf{u}_2^T\mathbf{x}) & = 0 \end{cases} \Longrightarrow \begin{cases} \mathbf{u}_1^T\mathbf{x} & = 0 \\ \mathbf{u}_2^T\mathbf{x} & = 0 \end{cases} \Longrightarrow \mathbf{x} \perp U.$$

Let now $\mathbf{x} \in U^\perp$, so that $\mathbf{u}_1^T\mathbf{x} = \mathbf{u}_2^T\mathbf{x} = 0$. Then

$$A\mathbf{x} = (\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T)\mathbf{x} = \lambda_1(\mathbf{u}_1^T\mathbf{x})\mathbf{u}_1 + \lambda_2(\mathbf{u}_2^T\mathbf{x})\mathbf{u}_2 = \mathbf{0} \Longrightarrow \mathbf{x} \in \ker A \Longrightarrow U^\perp \subseteq \ker A.$$

We conclude $\ker A = U^\perp$. Since $\dim \ker A > 0$, $\lambda = 0$ is an eigenvalue of A, with geometric multiplicity $\gamma(\lambda) = \dim \ker A = \dim U^\perp = n - \dim U = n - 2$.

(c) We found sp$A = \{0, \lambda_1, \lambda_2\}$, with $\sum_{\lambda \in \mathrm{sp}(A)} \gamma(\lambda) = (n-2) + 2 = n$, so the matrix is diagonalisable. Let $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_{n-2}\}$ be a basis for $U^\perp$. Then $A\mathbf{w}_i = \mathbf{0}$, so $B \cup U$ is a linearly independent set of eigenvectors. Hence,

$$A = VDV^{-1} := [\mathbf{w}_1, \ldots, \mathbf{w}_{n-2}, \mathbf{u}_1, \mathbf{u}_2] \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & \mathbf{u}_1 & \\ & & & \mathbf{u}_2 \end{bmatrix} [\mathbf{w}_1, \ldots, \mathbf{w}_{n-2}, \mathbf{u}_1, \mathbf{u}_2]^{-1}.$$

4. (a) We have

$$p_A(t) = \det(tI_3 - A) = \det \begin{bmatrix} t-3 & -4 & -4 \\ 3 & t-3 & 1 \\ -1 & 4 & t \end{bmatrix} = t^3 - 6t^2 + 13t - 20 = (t-4)(t^2 - 2t + 5).$$

Hence, the roots of $p(t)$ are $\lambda_1 = 4, \lambda_{2,3} = 1 \pm 2i$.
The corresponding eigenvectors can be found in the usual way.

$$(\lambda_1 I - A)\mathbf{x} = \mathbf{0} \iff \begin{bmatrix} 1 & -4 & -4 \\ 3 & 1 & 1 \\ -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \overset{x_3 = a}{\iff} \begin{cases} x_1 - 4x_2 & = 4a \\ 3x_1 + x_2 & = -a \end{cases} \iff \begin{cases} x_1 & = 0 \\ x_2 & = -a \end{cases} \implies \mathbf{x} = \begin{bmatrix} 0 \\ -a \\ a \end{bmatrix}$$

which means that the eigenspace of A associated with λ_1 is

$$E_{\lambda_1} = \mathrm{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Let now $\lambda_2 = 1 - 2i$. We find

$$(\lambda_2 I - A)\mathbf{x} = \mathbf{0} \iff \begin{bmatrix} -2-2i & -4 & -4 \\ 3 & -2-2i & 1 \\ -1 & 4 & 1-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}.$$

Adding rows 1 and 3, we find $x_1 = -x_3$. Setting $x_2 = a$, we find

$$(-2 + 2i)x_3 = 4a \implies x_3 = -(1+i)a = -x_1,$$

which means that the eigenspace of A associated with λ_2 is

$$E_{\lambda_2} = \mathrm{span} \left\{ \begin{bmatrix} 1+i \\ 1 \\ -(1+i) \end{bmatrix} \right\}.$$

Finally, the last eigenspace can be obtained by taking the complex conjugate of the expression for E_{λ_3}:

$$E_{\lambda_3} = \mathrm{span} \left\{ \begin{bmatrix} 1-i \\ 1 \\ -(1-i) \end{bmatrix} \right\}.$$

(b) The diagonal canonical form is the eigenvalue decomposition of A over $\mathbb{C}$:

$$V^{-1}AV = D, \quad V = \begin{bmatrix} 0 & 1+i & 1-i \\ -1 & 1 & 1 \\ 1 & -(1+i) & -(1-i) \end{bmatrix}, \quad D = \begin{bmatrix} 4 & & \\ & 1-2i & \\ & & 1+2i. \end{bmatrix}$$

(c) The block-diagonal canonical form represents a decomposition over $\mathbb{R}$

$$V^{-1}AV = D, \qquad V = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & & \\ & 1 & -2 \\ & 2 & 1 \end{bmatrix}.$$

One can check that $AV = VD$

$$AV = \begin{bmatrix} 3 & 4 & 4 \\ -3 & 3 & -1 \\ 1 & -4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ -4 & 1 & -2 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 1 & -2 \\ & 2 & 1 \end{bmatrix} = VD.$$

ADJOINT MAPS

5. (a) We have, for all $\mathbf{v} \in V$,

$$\langle f^*(a\mathbf{w}_1 + b\mathbf{w}_2), \mathbf{v} \rangle_V = \langle a\mathbf{w}_1 + b\mathbf{w}_2, f(\mathbf{v}) \rangle_W = a \langle \mathbf{w}_1, f(\mathbf{v}) \rangle_W + b \langle \mathbf{w}_2, f(\mathbf{v}) \rangle_W$$

$$= a \langle f^*(\mathbf{w}_1), \mathbf{v} \rangle_V + b \langle f^*(\mathbf{w}_2), \mathbf{v} \rangle_V = \langle af^*(\mathbf{w}_1) + bf(\mathbf{w}_2), \mathbf{v} \rangle_V \Longrightarrow f^*(a\mathbf{w}_1 + b\mathbf{w}_2) = af^*(\mathbf{w}_1) + bf(\mathbf{w}_2),$$

for all $a, b \in \mathbb{F}$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$. Hence f^* is linear.

(b) Assume there is another map $\hat{f}^*$ satisfying the same definition. Then, for all $\mathbf{v} \in V$,

$$\begin{cases} \langle f^*(\mathbf{w}), \mathbf{v} \rangle_V & = \langle \mathbf{w}, f(\mathbf{v}) \rangle_W \\ \left\langle \hat{f}^*(\mathbf{w}), \mathbf{v} \right\rangle_V & = \langle \mathbf{w}, f(\mathbf{v}) \rangle_W \end{cases} \Longrightarrow \left\langle f^*(\mathbf{w}) - \hat{f}^*(\mathbf{w}), \mathbf{v} \right\rangle_V = 0 \Longrightarrow f^*(\mathbf{w}) - \hat{f}^*(\mathbf{w}) =: \mathbf{z} \perp B_V.$$

where we choose B_V to be an orthonormal basis of V. Note that $\mathbf{z} \notin B_V$ (otherwise $\|\mathbf{z}\| = 0$, which is not possible for a basis element). Assume $\mathbf{z} \neq \mathbf{0}$. Then the set $\{\mathbf{z}\} \cup B_V$ would be linearly independent and have cardinality $n + 1$, which means B_V is not maximal. This is a contradiction, so $\mathbf{z} = \mathbf{0}$ and therefore $f^*(\mathbf{w}) = \hat{f}^*(\mathbf{w})$. Since this holds for arbitrary $\mathbf{w}$, we conclude that $f^* = \hat{f}^*$.

(c) The adjoint of f^* is a map $(f^*)^* : V \to W$ satisfying

$$\langle (f^*)^*(\mathbf{v}), \mathbf{w} \rangle_W = \langle \mathbf{v}, f^*(\mathbf{w}) \rangle_V = \langle f^*(\mathbf{w}), \mathbf{v} \rangle_V = \langle \mathbf{w}, f(\mathbf{v}) \rangle_W = \langle f(\mathbf{v}), \mathbf{w} \rangle_W .$$

Since this holds for all $\mathbf{v} \in V, \mathbf{w} \in W$, we conclude $(f^*)^* = f$.

6. (a) First, we need an orthonormal basis with respect to the inner product provided. Since $\{1, x, x^2 - 1/3\}$ is orthogonal, we need to normalise this set to obtain the required basis. We let

$$p_1(x) = 1/\|1\| = 1/\sqrt{2}, \quad p_2(x) = x/\|x\| = x/\sqrt{2/3}, \quad p_3(x) = \left(x^2 - \frac{1}{3}\right)/\sqrt{8/45}.$$

The explicit expression for f^* given in Proposition 18.5, uses the above orthonormal basis for $\mathcal{P}_2$:

$$f^*(\mathbf{w}) = \sum_{i=1}^{3} \langle \mathbf{w}, f(p_i) \rangle \, p_i.$$

We have

$$f(p_1) = \begin{bmatrix} p_1(0) \\ p_1(1) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad f(p_2) = \begin{bmatrix} p_2(0) \\ p_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3/2} \end{bmatrix}, \quad f(p_3) = \begin{bmatrix} p_3(0) \\ p_3(1) \end{bmatrix} = \begin{bmatrix} -\sqrt{5}/\sqrt{8} \\ \sqrt{5}/\sqrt{2} \end{bmatrix}.$$

Hence

$$\mathbf{w}^T f(p_1) p_1 = \frac{1}{\sqrt{2}}(w_1 + w_2) \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}(w_1 + w_2),$$

$$\mathbf{w}^T f(p_2)p_2 = \frac{\sqrt{3}}{\sqrt{2}}w_2 \cdot \frac{\sqrt{3}}{\sqrt{2}}x = \frac{3}{2}w_2 x,$$

$$\mathbf{w}^T f(p_3)p_3 = \frac{\sqrt{5}}{\sqrt{2}}\left(-\frac{1}{2}w_1 + w_2\right) \cdot \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right) = \frac{15}{4}\left(-\frac{1}{2}w_1 + w_2\right)\left(x^2 - \frac{1}{3}\right).$$

Hence,

$$f^*(\mathbf{w}) = \frac{1}{2}(w_1 + w_2) + \frac{3}{2}w_2 x + \frac{15}{4}\left(-\frac{1}{2}w_1 + w_2\right)\left(x^2 - \frac{1}{3}\right).$$

(b) We have

$$f^*\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{2} + 0 + \frac{15}{4}\cdot\left(-\frac{1}{2}\right)\left(x^2 - \frac{1}{3}\right) = \frac{1}{\sqrt{2}}p_1 + 0\cdot p_2 - \frac{\sqrt{5}}{\sqrt{8}}p_3,$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2} + \frac{3}{2}x + \frac{15}{4}\cdot\left(x^2 - \frac{1}{3}\right) = \frac{1}{\sqrt{2}}p_1 + \frac{\sqrt{3}}{\sqrt{2}}p_2 + \frac{\sqrt{5}}{\sqrt{2}}p_3.$$

Hence, the matrix representation of f^* with respect to the orthonormal bases considered is

$$A_{f^*} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 0 & \sqrt{3}\\ -\sqrt{5}/2 & \sqrt{5}\end{bmatrix} = A_f^T.$$

7. (a) Using the explicit expression for f^* given in Proposition 18.5, we find:

$$f^*\left(\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}\right) = \sum_{i=1}^{3}\mathbf{y}^T f(\mathbf{e}_i)\mathbf{e}_i = \mathbf{y}^T\begin{bmatrix}0\\1\\1\\1\end{bmatrix}\mathbf{e}_1 + \mathbf{y}^T\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\mathbf{e}_2 + \mathbf{y}^T\begin{bmatrix}1\\1\\0\\1\end{bmatrix}\mathbf{e}_3 = \begin{bmatrix}y_2+y_3+y_4\\ y_1+y_3+y_4\\ y_1+y_2+y_4\end{bmatrix}.$$

Note that the matrix representations of f and f^* are

$$A = \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}, \qquad A^T = \begin{bmatrix}0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\end{bmatrix}.$$

(b) We need to show that $\ker A^T = (\operatorname{col} A)^\perp$. We have

$$A^T\mathbf{y} = \mathbf{0} \Longrightarrow \begin{cases} y_2 + y_3 + y_4 &= 0\\ y_1 + y_3 + y_4 &= 0\\ y_1 + y_2 + y_4 &= 0\end{cases} \Longrightarrow \begin{cases} y_2 - y_1 &= 0\\ y_2 - y_3 &= 0\end{cases} \Longrightarrow y_1 = y_2 = y_3 = a, y_4 = -2a, \quad a \in \mathbb{R}.$$

Hence,

$$\ker A^T = \operatorname{span}\{\mathbf{u}\}, \quad \mathbf{u} = \begin{bmatrix}1\\1\\1\\-2\end{bmatrix}.$$

Now $\dim \operatorname{col} A^\perp = 4 - \dim \operatorname{col} A = 4 - 3 = 1$, since $\operatorname{col} A$ is spanned by 3 linearly independent vectors (the columns of A). Hence, $\operatorname{col} A^\perp$ is spanned by a single vector and we note that $\mathbf{u} \perp \mathbf{c}_j(A)$ for all j, since $\mathbf{u}^T\mathbf{c}_j(A) = 0$, i.e., $\mathbf{u} \in \operatorname{col} A^\perp$, so we must have $\operatorname{col} A^\perp = \operatorname{span}\{\mathbf{u}\} = \ker A^T$.
The remaining relation requires us to check that $\ker A = (\operatorname{col} A^T)^\perp$, both of which can be shown to be $\mathbf{0}$.

8. **(a)** We use the definition of the adjoint:

$$\langle f^*(\vec{w}), \vec{v}\rangle = \langle \vec{w}, f(\vec{v})\rangle\,.$$

Let $f^*(\vec{w}) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$. We identify u_i from the above definition:

$$u_1v_1 + u_2v_2 + u_3v_3 = w_1(c_2v_3 - c_3v_2) + w_2(c_3v_1 - c_1v_3) + w_3(c_1v_2 - c_2v_1) \Longrightarrow \begin{cases} u_1 = & w_2c_3 - w_3c_2, \\ u_2 = & w_3c_1 - w_1c_3, \\ u_3 = & w_1c_2 - w_2c_1. \end{cases}$$

Hence,

$$f^*(\vec{w}) = (w_2c_3 - w_3c_2)\mathbf{i} + (w_3c_1 - w_1c_3)\mathbf{j} + (w_1c_2 - w_2c_1)\mathbf{k} = -\vec{c}\times\vec{w} = -f(\vec{w}) \Longrightarrow f^* = -f.$$

(b) Using part **(a)**, $\mathrm{im}\, f^* = \mathrm{im}\,(-f) = \mathrm{im}\, f$. By Proposition 18.7, $\mathrm{im}\, f = \mathrm{im}\, f^* = (\ker f)^\perp$, so that

$$\mathbb{E}^3 = (\ker f)^\perp + \ker f = \mathrm{im}\, f + \ker f.$$

SELF-ADJOINT MAPS

9. In each case, we need to check whether $\langle f(p), q\rangle = \langle p, f(q)\rangle$.

(a) We have

$$\langle f(p), q\rangle = -\int_{-1}^{1}((1-x^2)p'(x))'q(x)\mathrm{d}x = -\left[((1-x^2)p'(x))'q(x)\right]_{-1}^{1} + \int_{-1}^{1}(1-x^2)p'(x)q'(x)\mathrm{d}x = \int_{-1}^{1}(1-x^2)p'(x)q'(x)\mathrm{d}x,$$

which is symmetric in p and q, i.e., $\langle f(p), q\rangle = \langle f(q), p\rangle$. Hence $\langle f(p), q\rangle = \langle p, f(q)\rangle$, so that f is self-adjoint.

(b) We have

$$\langle f(p), q\rangle = -\int_{-1}^{1}\frac{((1-x^2)p'(x))'q(x)}{\sqrt{1-x^2}}\mathrm{d}x = -\left[\frac{((1-x^2)p'(x))'q(x)}{\sqrt{1-x^2}}\right]_{-1}^{1} + \int_{-1}^{1}(1-x^2)p'(x)\left(\frac{q(x)}{\sqrt{1-x^2}}\right)'\mathrm{d}x$$

$$= \int_{-1}^{1}\sqrt{1-x^2}p'(x)q'(x)\mathrm{d}x - \int_{-1}^{1}\frac{xp'(x)q(x)}{\sqrt{1-x^2}}\mathrm{d}x,$$

while

$$\langle p, f(q)\rangle = \langle f(q), p\rangle = \int_{-1}^{1}\sqrt{1-x^2}q'(x)p'(x)\mathrm{d}x - \int_{-1}^{1}\frac{xq'(x)p(x)}{\sqrt{1-x^2}}\mathrm{d}x,$$

so that f is not self-adjoint, since $p'q \neq q'p$, in general.
Finally, the matrix representations are the same in each case, since only the inner product changes. We find

$$f(1) = 0, \quad f(x) = 2x, \quad f(x^2) = -2 + 6x^2 \Longrightarrow A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Note that the matrix is not symmetric because the basis used is not orthonormal with respect to either inner product.

10. In each case, we find the matrix representation with respect to the canonical bases and check its symmetry.

(a) We have

$$f(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad f(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad f(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = A^T,$$

so that f is self-adjoint.

(b) We have

$$f(\mathbf{e}_1) = \begin{bmatrix} 4 & 2 & 4 \end{bmatrix}, \quad f(\mathbf{e}_2) = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}, \quad f(\mathbf{e}_3) = \begin{bmatrix} 4 & 4 & 6 \end{bmatrix} \Longrightarrow A = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 3 & 4 \\ 4 & 4 & 6 \end{bmatrix} = A^T,$$

so that f is self-adjoint.

11. (a) We have, for all $\mathbf{v} \in V$,

$$\pi_U(\mathbf{v}) = \mathbf{v}_U^{\|} \Longrightarrow \pi_U(\pi_U(\mathbf{v})) = \pi_U(\mathbf{v}_U^{\|}) = \mathbf{v}_U^{\|} = \pi_U(\mathbf{v}) \Longrightarrow \pi_U^2(\mathbf{v}) = \pi_U(\mathbf{v}) \Longrightarrow \pi_U^2 = \pi_U.$$

(b) Let $\mathbf{v}, \mathbf{w} \in V$ and note that $\mathbf{v} = \mathbf{v}_U^{\perp} + \mathbf{v}_U^{\|}, \mathbf{w} = \mathbf{w}_U^{\perp} + \mathbf{w}_U^{\|}$.

$$\langle \pi_U(\mathbf{v}), \mathbf{w} \rangle = \left\langle \mathbf{v}_U^{\|}, \mathbf{w}_U^{\perp} + \mathbf{w}_U^{\|} \right\rangle = 0 + \left\langle \mathbf{v}_U^{\|}, \mathbf{w}_U^{\|} \right\rangle = \langle \pi_U(\mathbf{v}), \pi_U(\mathbf{w}) \rangle,$$

which is an expression symmetric in $\mathbf{v}, \mathbf{w}$. Hence, swapping $\mathbf{v}$ and $\mathbf{w}$ and using the symmetry of the inner product we get

$$\langle \pi_U(\mathbf{w}), \mathbf{v} \rangle = \langle \mathbf{v}, \pi_U(\mathbf{w}) \rangle = \langle \pi_U(\mathbf{v}), \pi_U(\mathbf{w}) \rangle,$$

so that $\langle \pi_U(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, \pi_U(\mathbf{w}) \rangle$ and therefore π_U is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$.

SPECTRAL RESULTS

12. We have

$$p_A(\lambda) = \det(A - \lambda I) = \det\left((A - \lambda I)^T\right) = \det\left(A^T - \lambda I\right) = p_{A^T}(\lambda),$$

i.e., A and A^T have the same characteristic polynomial and hence the same eigenvalues.

13. Let $H\mathbf{v} = \lambda \mathbf{v}$. Multiplying from the left by $\mathbf{v}^* = \bar{\mathbf{v}}^T$ we get $\mathbf{v}^* H \mathbf{v} = \lambda \mathbf{v}^* \mathbf{v}$. Then

$$(\mathbf{v}^* H \mathbf{v})^* = (\lambda \mathbf{v}^* \mathbf{v})^* \Leftrightarrow \mathbf{v}^* H^T \mathbf{v} = \bar{\lambda} \mathbf{v}^* \mathbf{v}$$

and hence

$$0 = \mathbf{v}^*(H - H^T)\mathbf{v} = (\lambda - \bar{\lambda})\mathbf{v}^*\mathbf{v} = 2Im(\lambda)\mathbf{v}^*\mathbf{v} \Rightarrow Im(\lambda) = 0$$

so that the eigenvalues of H are real.

14. Let $S\mathbf{v} = \lambda \mathbf{v}$. Multiplying from the left by $\mathbf{v}^* = \bar{\mathbf{v}}^T$ we get $\mathbf{v}^* S \mathbf{v} = \lambda \mathbf{v}^* \mathbf{v}$. Then

$$(\mathbf{v}^* S \mathbf{v})^* = (\lambda \mathbf{v}^* \mathbf{v})^* \Leftrightarrow \mathbf{v}^* S^T \mathbf{v} = \bar{\lambda} \mathbf{v}^* \mathbf{v}$$

and hence

$$0 = \mathbf{v}^*(S + S^T)\mathbf{v} = (\lambda + \bar{\lambda})\mathbf{v}^*\mathbf{v} = 2Re(\lambda)\mathbf{v}^*\mathbf{v} \Rightarrow Re(\lambda) = 0$$

and hence the eigenvalues are purely imaginary or zero. If the dimension is even, we get only purely imaginary pairs (under the assumption that S is a general matrix and does not include a zero diagonal sub-block of size 2×2 or larger). If the dimension is odd, the matrix must be singular, as one eigenvalue must be zero, with the others appearing as conjugate pairs of imaginary numbers.

15. (a) Taking the conjugate transpose we find

$$\mathbf{y}^* A = \lambda \mathbf{y}^* \Leftrightarrow A^* \mathbf{y} = \bar{\lambda} \mathbf{y} \Rightarrow (\bar{\lambda}, \mathbf{y}) \text{ is an eigenpair for } A^*.$$

Taking the transpose we find

$$A^T \bar{\mathbf{y}} = \lambda \bar{\mathbf{y}} \Rightarrow (\lambda, \bar{\mathbf{y}}) \text{ is an eigenpair for } A^T.$$

Hence, if $A \in \mathbb{R}^{n \times n}$ is symmetric, then $\lambda = \bar{\lambda}$ and by the first result, $(\bar{\lambda}, \mathbf{y}) = (\lambda, \mathbf{y})$ is a (right) eigenpair for $A^* = \bar{A}^T = A^T = A$. Hence, for real symmetric matrices a left eigenpair is also a right eigenpair.

(b) Since $(\lambda_1, \mathbf{y}_1)$ is a left eigenpair for A, we have

$$(\mathbf{y}_1^* A)\mathbf{y}_2 = (\lambda_1 \mathbf{y}_1^*)\mathbf{y}_2.$$

Since $(\lambda_2, \mathbf{y}_2)$ is a right eigenpair for A, we have

$$\mathbf{y}_1^*(A\mathbf{y}_2) = \mathbf{y}_1^*(\lambda_2 \mathbf{y}_2).$$

Taking the difference we obtain

$$0 = (\lambda_1 - \lambda_2)\mathbf{y}_1^*\mathbf{y}_2 \Leftrightarrow \langle \mathbf{y}_1, \mathbf{y}_2 \rangle = 0.$$

If A is real symmetric, left eigenvectors are right eigenvectors and

$$\mathbf{y}_1^T(A\mathbf{y}_2) = \mathbf{y}_1^T(\lambda_2\mathbf{y}_2), \quad \mathbf{y}_2^T(A\mathbf{y}_1) = \mathbf{y}_2^T(\lambda_1\mathbf{y}_1)$$

and taking the difference we find $\mathbf{y}_1^T\mathbf{y}_2 = 0$. Hence, if A has n distinct eigenvalues, there will be n distinct (and orthogonal) eigenvectors, and therefore A is diagonalizable: $A = V\Lambda V^T$, where $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. Note that for real-symmetric matricesone can show that this result holds also if some eigenvalues are non-distinct.

16. We have

$$p_A(t) = t^3 - 6t^2 + 11t - 6 = (t-1)(t-2)(t-3) \Longrightarrow \begin{cases} \lambda_1 & = 1 \\ \lambda_2 & = 2 \\ \lambda_3 & = 3 \end{cases}$$

The corresponding orthonormal eigenvectors are found by solving $(\lambda_i I_3 - A)\mathbf{q}_i = (A - \lambda_i I_3)\mathbf{q}_i = \mathbf{0}$. We then normalise them via $\hat{\mathbf{q}}_i = \mathbf{q}_i / \|\mathbf{q}_i\|$. We find

$$(A - \lambda_1 I_3)\mathbf{q}_1 = \mathbf{0} \Longleftrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{21} \\ q_{31} \end{bmatrix} = \mathbf{0} \Longrightarrow q_{11} = q_{31} \Longrightarrow \mathbf{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \hat{\mathbf{q}}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$(A - \lambda_2 I_3)\mathbf{q}_1 = \mathbf{0} \Longleftrightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_{12} \\ q_{22} \\ q_{32} \end{bmatrix} = \mathbf{0} \Longrightarrow q_{12} = q_{32} = 0 \Longrightarrow \mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \hat{\mathbf{q}}_2.$$

$$(A - \lambda_3 I_3)\mathbf{q}_1 = \mathbf{0} \Longleftrightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix} = \mathbf{0} \Longrightarrow q_{13} = -q_{33} \Longrightarrow \mathbf{q}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Longrightarrow \hat{\mathbf{q}}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

It is now straightforward to check the spectral decomposition

$$\lambda_1\mathbf{q}_1\mathbf{q}_1^T + \lambda_2\mathbf{q}_2\mathbf{q}_2^T + \lambda_3\mathbf{q}_3\mathbf{q}_3^T = 1 \cdot \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 3 \cdot \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = A.$$

QUADRATIC FORMS

17. **(a)** We have, using the definition of $\mathcal{Q}$ and the symmetry of $\mathcal{B}$:

$$\mathcal{Q}(\mathbf{v} + \mathbf{w}) = \mathcal{B}(\mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) = \mathcal{B}(\mathbf{v}, \mathbf{v}) + \mathcal{B}(\mathbf{w}, \mathbf{w}) + 2\mathcal{B}(\mathbf{v}, \mathbf{w}) = \mathcal{Q}(\mathbf{v}) + \mathcal{Q}(\mathbf{w}) + 2\mathcal{B}(\mathbf{v}, \mathbf{w}),$$

and the result follows.

(b) If $\mathcal{B}$ is anti-symmetric, we have

$$\mathcal{B}(\mathbf{v}, \mathbf{v}) = -\mathcal{B}(\mathbf{v}, \mathbf{v}) \Longrightarrow 2\mathcal{B}(\mathbf{v}, \mathbf{v}) = 0 \Longrightarrow \mathcal{Q}(\mathbf{v}) = \mathcal{B}(\mathbf{v}, \mathbf{v}) = 0.$$

18. **(a)** We have

$$p_A(t) = (t-2)^2 - 1 \Longrightarrow p_A(\lambda) = 0 \Longleftrightarrow \lambda \in \{1, 3\}.$$

(b) We have to show that $\lambda_{\min} \leq R_A(\mathbf{x}) \leq \lambda_{\max}$, i.e., that

$$1 \leq \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq 3 \iff \mathbf{x}^T\mathbf{x} \leq \mathbf{x}^T A\mathbf{x} \leq 3\mathbf{x}^T A\mathbf{x}.$$

Now, writing $\mathbf{x}^T = [x, y]$, we find

$$\mathbf{x}^T A\mathbf{x} = 2x^2 - 2xy + 2y^2,$$

so that the inequality becomes

$$x^2 + y^2 \leq 2x^2 - 2xy + 2y^2 \leq 3(x^2 + y^2) \iff \begin{cases} x^2 - 2xy + y^2 & \geq 0 \\ x^2 + 2xy + y^2 & \geq 0 \end{cases}$$

which indeed holds for all $x, y \in \mathbb{R}$.

(c) The eigenvectors of A can be found as follows:

$$\lambda = 1: \quad (I - A)\mathbf{v} = \mathbf{0} \iff \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\lambda = 3: \quad (3I - A)\mathbf{v} = \mathbf{0} \iff \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0} \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Hence,

$$\frac{\mathbf{v}_1^T A\mathbf{v}_1}{\mathbf{v}_1^T\mathbf{v}_1} = \frac{1}{2}[1\ 1] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 = \lambda_1, \qquad \frac{\mathbf{v}_2^T A\mathbf{v}_2}{\mathbf{v}_2^T\mathbf{v}_2} = \frac{1}{2}[1\ -1] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3 = \lambda_2.$$

19. The matrix corresponding to the quadratic part is

$$A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix},$$

with spectrum $\operatorname{sp}(A) = \{-2, 8\}$, so the shape is a hyperbola (as the eigenvalues have opposite signs).

20. The matrix corresponding to the quadratic part is

$$A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix},$$

with spectrum $\operatorname{sp}(A) = \{2, 8\}$, so the shape is an ellipse (as both eigenvalues are positive).

21. The matrix corresponding to the quadratic part is

$$A = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with spectrum $\operatorname{sp}(A) = \{1, 2, 8\}$, so the shape is an ellipsoid (as all three eigenvalues are positive).