Mechanics revision exercises

1. Consider the equation

$$U\frac{d\rho}{dx} \quad = \quad D\frac{d^2\rho}{dx^2} + \frac{F}{V},$$

where U is speed, ρ is density, x is location, D is diffusivity (SI units m^2 s^{-1}), F is force and V is volume. Determine whether the equation is dimensionally consistent.

Solution. We have

- $[U] = [\mathrm{LT}^{-1}]$,
- $[\rho] = [\mathrm{ML}^{-3}]$,
- $[x] = [\mathrm{L}]$,
- $[V] = [\mathrm{L}^3]$,
- $[F] = [\mathrm{MLT}^{-2}]$, since force is mass times acceleration,
- $[D] = [\mathrm{L}^2\mathrm{T}^{-1}]$ since it has SI units of m^2 s^{-1}.

Each additive term therefore has dimensions:

- $[U\frac{d\rho}{dx}] = [U\rho]/[x] = [\mathrm{LT}^{-1}\mathrm{ML}^{-3}\mathrm{L}^{-1}] = [\mathrm{ML}^{-3}\mathrm{T}^{-1}]$.
- $[D\dfrac{d^2\rho}{dx^2}] = [D\rho]/[x]^2 = [\mathrm{L}^2\mathrm{T}^{-1}\mathrm{ML}^{-3}\mathrm{L}^{-2}] = [\mathrm{ML}^{-3}\mathrm{T}^{-1}]$,
- $[\dfrac{F}{V}] = [F]/[V] = [\mathrm{MLT}^{-2}\mathrm{L}^{-3}] = [\mathrm{ML}^{-2}\mathrm{T}^{-2}]$.

These are not the same, so the expression is not dimensionally consistent. ◀

2. If the acceleration of a particle is given by

$$\ddot{\mathbf{r}} \quad = \quad a\mathbf{i} + be^{-\omega t}\mathbf{j},$$

where a, b and ω are constants and t is time, find the velocity and position of a particle that starts from rest at $\mathbf{r} = 0$.

Solution. Integrating twice gives

$$\begin{aligned}\dot{\mathbf{r}} &= at\mathbf{i} - \frac{b}{\omega}e^{-\omega t}\mathbf{j} + \mathbf{c}_1, \\ \mathbf{r} &= \frac{at^2}{2}\mathbf{i} + \frac{b}{\omega^2}e^{-\omega t}\mathbf{j} + \mathbf{c}_1 t + \mathbf{c}_2,\end{aligned}$$

where $\mathbf{c}_1$, $\mathbf{c}_2$ are constants. Since $\mathbf{r} = 0$ we have

$$0 = \mathbf{r}(0) \quad = \quad \frac{b}{\omega^2}\mathbf{j} + \mathbf{c}_2,$$

and hence

$$\mathbf{c}_2 = -\frac{b}{\omega^2}\mathbf{j}.$$

Finally, $\dot{\mathbf{r}}(0) = 0$ since the particle starts at rest, so

$$0 = \dot{\mathbf{r}}(0) = -\frac{b}{\omega}\mathbf{j} + \mathbf{c}_1,$$

giving

$$\mathbf{c}_1 = \frac{b}{\omega}\mathbf{j}.$$

The complete solution is therefore

$$\begin{aligned} \mathbf{r} &= \frac{at^2}{2}\mathbf{i} + \frac{b}{\omega^2}e^{-\omega t}\mathbf{j} + \frac{b}{\omega}t\mathbf{j} - \frac{b}{\omega^2}\mathbf{j}, \\ &= \frac{at^2}{2}\mathbf{i} + \left(\frac{b}{\omega^2}e^{-\omega t} + \frac{b}{\omega}t - \frac{b}{\omega^2}\right)\mathbf{j}. \end{aligned}$$

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3. Consider a particle of mass m hanging vertically from a spring (with spring constant k) under its own weight.

   (a) If x gives the displacement of the particle measured downwards from the natural length of the spring, show that the equation of motion is

   $$\ddot{x} + \omega^2 x = g,$$

   where $\omega^2 = k/m$.

   (b) If the particle starts from rest at the point $x = h$, find $x(t)$ and describe the particle's motion.

Solution. (a) If x is measured downwards, then the particle will be acted on by two forces: gravity downwards in the positive direction, and tension from the spring upwards in the negative direction. Hooke's law gives the tension force to be $-kx$ since x will give the extension in the spring. Hence the total force acting on the particle is $mg - kx$. Thus Newton's second law gives

$$\begin{aligned} m\ddot{x} &= mg - kx, \\ \Rightarrow m\ddot{x} + kx &= mg, \\ \Rightarrow \ddot{x} + \frac{k}{m}x &= g, \\ \Rightarrow \ddot{x} + \omega^2 x &= g, \end{aligned}$$

where we set $\omega^2 = k/m$ since k and m are both positive.

(b) The particle is initially at rest, so $\dot{x}(0) = 0$, and at location such that $x(0) = h$. We first look for a solution to the homogeneous problem

$$\ddot{x}_c + \omega^2 x_c = 0.$$

The characteristic equation is

$$\lambda^2 + \omega^2 = 0,$$

giving $\lambda = \pm i\omega$ and hence

$$x_c = A\cos\omega t + B\sin\omega t,$$

where A and B are constants. We next look for a particular integral of the form $x_p = C$ where C is a constant to be found. Since

$$\begin{aligned} \ddot{x}_p + \omega^2 x_p &= g, \\ \Rightarrow \omega^2 C &= g, \\ \Rightarrow C &= \frac{g}{\omega^2}. \end{aligned}$$

This gives the complete solution

$$x = A\cos\omega t + B\sin\omega t + \frac{g}{\omega^2}.$$

We now use the initial conditions to determine A and B. Since $x(0) = h$, this gives

$$x(0) = h = A + \frac{g}{\omega^2}, \quad \Rightarrow \quad A = h - \frac{g}{\omega^2}.$$

Then since

$$\dot{x} = -\omega A\sin\omega t + \omega B\cos\omega t,$$

the condition $\dot{x}(0) = 0$ gives

$$\dot{x}(0) = \omega B, \quad \Rightarrow \quad B = 0.$$

Hence

$$x = \left(h - \frac{g}{\omega^2}\right)\cos\omega t + \frac{g}{\omega^2}.$$

The particle therefore oscillates about g/ω^2 with amplitude $h - g/\omega^2$.

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4. A particle of mass m is thrown from ground level, with initial speed V at an angle α to the horizontal. The particle is subject to gravity, and an additional force of the form $mg(1 - \omega t)$ acting vertically upwards. Find the time at which the particle hits the ground, and the horizontal distance from its initial location.

Solution. Let the origin be the initial location of the particle, with Cartesian vectors $\mathbf{i}$, $\mathbf{k}$ pointing in the (horizontal) x and (upwards vertical) z directions respectively. The forces acting on the particle are gravity downwards of the form $-mg\mathbf{k}$ and the additional force $mg(1-\omega t)\mathbf{k}$. Then Newton's second law gives

$$\begin{aligned} m\ddot{\mathbf{r}} &= mg(1-\omega t)\mathbf{k} - mg\mathbf{k}, \\ &= -mg\omega t\mathbf{k}, \end{aligned}$$

for position vector $\mathbf{r}$. The particle is initially located at the origin, $\mathbf{r}(0) = 0$, and is moving with velocity $\dot{\mathbf{r}}(0) = \mathbf{V} = V\cos\alpha\mathbf{i} + V\sin\alpha\mathbf{k}$.

Integrating twice gives

$$\begin{aligned} \dot{\mathbf{r}} &= -\frac{g\omega t^2}{2}\mathbf{k} + \mathbf{c}_1, \\ \mathbf{r} &= -\frac{g\omega t^3}{6}\mathbf{k} + t\mathbf{c}_1 + \mathbf{c}_2, \end{aligned}$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are constants. Using the initial conditions we find

$$0 = \mathbf{r}(0) = \mathbf{c}_2,$$

and

$$\mathbf{V} = \dot{\mathbf{r}}(0) = \mathbf{c}_1.$$

The location of the particle is hence

$$\mathbf{r} = -\frac{g\omega t^3}{6}\mathbf{k} + t\mathbf{V}.$$

The particle hits the ground when $z = 0$. Since

$$z = Vt\sin\alpha - \frac{g\omega t^3}{6}$$

this occurs for $t = \tau$ such that

$$0 = \tau\left(V\sin\alpha - \frac{g\omega\tau^2}{6}\right).$$

Hence $\tau = 0$, i.e. the initial location of the particle, or

$$\begin{aligned} \tau^2 &= \frac{6V\sin\alpha}{g\omega}, \\ \Rightarrow \tau &= \sqrt{\frac{6V\sin\alpha}{g\omega}}, \end{aligned}$$

taking the positive root so the time is positive. The horizontal distance travelled at this time is

$$\begin{aligned} x(\tau) &= V\tau\cos\alpha, \\ &= V\cos\alpha\sqrt{\frac{6V\sin\alpha}{g\omega}}. \end{aligned}$$

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5. A particle of mass m is attracted to the origin of an inertial frame by a central force c/r^3, where r gives the distance between the particle and the origin, and $c > 0$ is a constant. The particle is initially a distance d from the origin and moving with velocity $v\mathbf{e}_r + \sqrt{c/(md^2)}\mathbf{e}_\theta$.

   (a) Find the value of the constant $h = r^2\dot{\theta}$ and explain what it represents physically.

   (b) Show that the particle path satisfies $\dfrac{d^2u}{d\theta^2} = 0$, where $u = 1/r$.

   (c) Show that at $\theta = 0$, we have $u = 1/d$ and $\dfrac{du}{d\theta} = -v\sqrt{m/c}$.

   (d) Find the particle path, $r(\theta)$.

   (e) In each of the cases: $v = 0$, $v < 0$ and $v > 0$, describe the particle motion. You may assume that $\theta(t)$ strictly increases over time.

Solution. (a) The initial conditions are $r = d$, $\dot{r} = v$, $r\dot{\theta} = \sqrt{c/(md^2)}$ at $t = 0$. If $h = r^2\dot{\theta}$ is constant then

$$\begin{aligned} h &= r \cdot r\dot{\theta}, \\ &= d\sqrt{c/(md^2)}, \\ &= \sqrt{c/m}, \end{aligned}$$

initially, and hence for all time. Physically, h is the particle's conserved angular momentum per unit mass.

(b) Since $u = 1/r$ satisfies

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2},$$

for a central force F, and $F = -c/r^3$ (note the minus sign for an attractive force), we have

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u &= \frac{cu^3}{mh^2u^2}, \\ &= \frac{cu^3}{m\,(c/m)\,u^2}, \\ &= u, \end{aligned}$$

and therefore $\dfrac{d^2u}{d\theta^2} = 0$ as required.

(c) At $t = 0$ we choose $\theta = 0$. Then $u = 1/r = 1/d$ at $t = 0$ and hence $\theta = 0$, and

$$\begin{aligned} \frac{du}{d\theta} &= -\frac{\dot{r}}{h}, \\ &= -\frac{v}{\sqrt{c/m}}, \\ &= v\sqrt{m/c} \end{aligned}$$

also at $\theta = 0$.

(d) We solve to find

$$\begin{aligned} \frac{d^2u}{d\theta^2} &= 0, \\ \implies u &= A\theta + B, \end{aligned}$$

where A and B are constants. Now, $u = 1/d$ at $t = 0$ gives $B = 1/d$, and

$$\frac{du}{d\theta} = A = -v\sqrt{m/c}.$$

Hence

$$u = \frac{1}{d} - v\sqrt{m/c}\,\theta,$$

and so

$$r = \frac{1}{1/d - v\sqrt{m/c}\,\theta}$$

gives the particle path.

(e) When $v = 0$, the path is $r = d$ and hence the particle will move in a circle. When $v > 0$, $1/d - v\sqrt{m/c}\,\theta(t)$ strictly decreases over time and so $r(t)$ strictly increases, so the particle will spiral out. When $v < 0$, $1/d - v\sqrt{m/c}\,\theta$ strictly increases so the particle will spiral in. See https://www.geogebra.org/m/tgdeyhh4.

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6. A particle of mass m is attracted to the origin of an inertial frame by a force mk/r^2, where r gives the distance between the particle and the origin, and $k > 0$ is a constant. The particle is initially a distance R from the origin and moving with velocity $V\mathbf{e}_\theta$.

   (a) Find a second-order ordinary differential equation for $u(\theta)$, where $u = 1/r$.

   (b) Find expressions for the initial conditions for the equation in part (a), and the value of the constant $h = r^2\dot{\theta}$.

   (c) Find the particle path $r(\theta)$. Show that the path is an circle when $V^2 = k/R$, and an ellipse when $V^2 < 2k/R$ and $V^2 \neq k/R$.

Solution. (a) Since $u = 1/r$ satisfies

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2},$$

for a central force F, and $F = -mk/r^2$, we have

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u &= \frac{mku^2}{mh^2u^2}, \\ &= \frac{k}{h^2}, \end{aligned}$$

where $h = r^2\dot{\theta}$.

(b) The initial conditions are $r = R$, $\dot{r} = 0$, $r\dot{\theta} = V$ at $t = 0$. We choose $\theta = 0$ at $t = 0$. Hence

$$\begin{aligned} h &= r \cdot r\dot{\theta}, \\ &= RV, \end{aligned}$$

initially and hence for all time. The initial conditions become $u = 1/R$ and $du/d\theta = 0$ at $t = 0$ and hence $\theta = 0$.

(c) The governing equation becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{R^2V^2},$$

which is a second order linear ODE with constant coefficients. We first solve the homogeneous equation

$$\frac{d^2u_c}{d\theta^2} + u_c = 0,$$

giving the characteristic equation

$$\lambda^2 + 1 = 0,$$

and hence $\lambda = \pm 1$, and so $u_c = A\cos\theta + B\sin\theta$ where A and B are constants. We then find the particular solution $u_p = C$, where $C = \dfrac{k}{R^2V^2}$. Hence the complete solution is

$$u = A\cos\theta + B\sin\theta + \frac{k}{R^2V^2}.$$

Now, using the initial conditions we find

$$\begin{aligned} u(0) &= A + \frac{k}{R^2V^2} = \frac{1}{R}, \\ \Longrightarrow A &= \frac{1}{R} - \frac{k}{R^2V^2}, \end{aligned}$$

and

$$\frac{du}{d\theta}(0) = B = 0.$$

Hence

$$u = \left(\frac{1}{R} - \frac{k}{R^2V^2}\right)\cos\theta + \frac{k}{R^2V^2}.$$

This gives

$$\begin{aligned} r &= \frac{1}{\left(\dfrac{1}{R} - \dfrac{k}{R^2V^2}\right)\cos\theta + \dfrac{k}{R^2V^2}}, \\ &= \frac{\dfrac{R^2V^2}{k}}{1 + \left(\dfrac{RV^2}{k} - 1\right)\cos\theta}. \end{aligned}$$

This is a conic section $r = e/(1 \pm e\cos\theta)$ with eccentricity

$$e = \left|\frac{RV^2}{k} - 1\right|.$$

For the path to be a circle, we must have $e = 0$, so

$$\frac{RV^2}{k} - 1 = 0,$$

and rearranging gives $V^2 = k/R$ as required.
For the path to be an ellipse, we must have $0 < e < 1$, and so

$$-1 < \frac{RV^2}{k} - 1 < 1 \text{ and } \frac{RV^2}{k} - 1 \neq 0.$$

Rearranging the first inequality gives $0 < \dfrac{RV^2}{k} < 2$, and hence the two inequalities combine to give $V^2 < 2k/R$ and $V^2 \neq k/R$ as required.

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7. We will revisit question 6. A particle of mass m is attracted to the origin of an inertial frame by a force mk/r^2, where r gives the distance between the particle and the origin, and $k > 0$ is a constant. The particle is initially located a distance R from the origin and moving with velocity $\sqrt{k/R}\mathbf{e}_\theta$.

   (a) Briefly explain why the problem can be formulated as

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{mk}{r} = \text{constant}, \tag{1}$$
$$r^2\dot{\theta} = \text{constant}. \tag{2}$$

   Write down the physical interpretation of each equation.

   (b) Using the initial values for location and velocities, find the constants in the expressions from part (a).

   (c) Using (2), find an expression for $\dot{\theta}$ in terms of k, R and r.

   (d) Hence, rewrite (1) as an expression for $\dot{r}^2$ in terms of k, R and r.

   (e) By considering the signs of each side of the resulting expression in (d), show that the particle moves in a circle. Determine the period of the circular motion.

Solution. (a) Conservation of energy gives

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{mk}{r} = \text{constant} = E, \tag{3}$$

where the first term gives the kinetic energy and the second the potential energy. Conservation of angular momentum gives

$$r^2\dot{\theta} = \text{constant} = h, \tag{4}$$

since the force is central.

(b) Initially the particle is located at $r = R$, and we choose $\theta = 0$. The initial velocity gives $r\dot{\theta} = \sqrt{k/R}$ and $\dot{r} = 0$, also at $t = 0$. Then

$$\begin{aligned} h = r^2\dot{\theta} &= r \cdot \dot{\theta} && (5) \\ &= R\sqrt{k/R}, && (6) \\ &= \sqrt{kR}, && (7) \end{aligned}$$

initially and hence for all time. Similarly

$$\begin{aligned} E &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{mk}{r}, && (8) \\ &= \frac{1}{2}m\frac{k}{R} - \frac{mk}{R}, && (9) \\ &= -\frac{mk}{2R}, && (10) \end{aligned}$$

initially and hence for all time.

(c) Since $r^2\dot{\theta} = \sqrt{kR}$, we immediately find

$$\dot{\theta} = \frac{\sqrt{kR}}{r^2}.$$

(d)

$$\begin{aligned} \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{mk}{r} &= -\frac{mk}{2R}, && (11) \\ \Longrightarrow \left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{2k}{r} &= -\frac{k}{R}, && (12) \\ \Longrightarrow \left(\dot{r}^2 + r^2\left(\frac{\sqrt{kR}}{r^2}\right)^2\right) - \frac{2k}{r} &= -\frac{k}{R}, && (13) \\ \Longrightarrow \dot{r}^2 + \frac{kR}{r^2} - \frac{2k}{r} &= -\frac{k}{R}. && (14) \end{aligned}$$

This is rearranged to give

$$\begin{aligned} \dot{r}^2 &= -\frac{kR}{r^2} + \frac{2k}{r} - \frac{k}{R}, && (15) \\ &= -\frac{kR^2 - 2kRr + kr^2}{Rr^2}, && (16) \\ &= -\frac{k\left(r^2 - 2Rr + R^2\right)}{Rr^2}, && (17) \\ &= -\frac{k\left(r - R\right)^2}{Rr^2}. && (18) \end{aligned}$$

(e) Since the left hand side $\dot{r}^2 \geq 0$ and the right hand side is ≤ 0, they must both equal zero for the equation to be satisfied. Therefore $r = R$, and the particle moves in a circle with constant angular speed $\dot{\theta} = \sqrt{kR}/R^2 = \sqrt{k/R^3}$. For a circular motion with constant $\dot{\theta}$, the period is given by $T = 2\pi/\dot{\theta} = 2\pi\sqrt{R^3/k}$. Note that we have reached the same conclusion as in question 6: the particle moves in a circle when $V = \sqrt{k/R}$.

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8. A smooth sphere of radius $2a$ has its centre at the origin. If ρ, θ, z give cylindrical polar coordinates, with the z axis pointing vertically downwards, the surface of the sphere is given by $\rho^2 + z^2 = 4a^2$. The particle starts at $\rho = 2a$, moving with horizontal velocity V.

   (a) Briefly explain why

$$\rho^2\dot{\theta} = h, \tag{19}$$
$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz = E, \tag{20}$$

   where h and E are constants. What are the physical interpretations of each term?

   (b) Using the initial conditions, find the values of E and h.

   (c) Hence show that the motion satisfies

$$2a^2\dot{z}^2 = -gz\left(z - z_1\right)\left(z - z_2\right),$$

   for some $z_{1,2}$ you should define. What does this tell you about the particle's motion?

   (d) Does the particle rise or fall initially? Justify your answer.

Solution. (a) Conservation of angular momentum gives

$$\rho^2\dot{\theta} = h,$$

since there is no force in the θ direction. Conservation of energy then gives

$$\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz = E,$$

where the first term gives the kinetic energy and the second is the gravitational potential energy, since z points downwards.

(b) The particle is initially located at $\rho = 2a$ and hence $z = 0$ at $t = 0$. The initial velocity is purely horizontal such that $r\dot{\theta} = V$, with $\dot{\rho} = 0$, $\dot{z} = 0$, also at $t = 0$. Hence

$$\begin{aligned} h &= \rho^2\dot{\theta}, \\ &= \rho \cdot \rho\dot{\theta}, \\ &= 2aV, \end{aligned}$$

initially and for all time. Similarly

$$\begin{aligned} E &= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz, \\ &= \frac{1}{2}m\left(0^2 + V^2 + 0^2\right) - mg \times 0, \\ &= \frac{1}{2}mV^2. \end{aligned}$$

(c) Now, since $\rho^2 + z^2 = 4a^2$, by differentiating with respect to t we have

$$2\rho\dot{\rho} + 2z\dot{z} = 0,$$

and hence

$$\begin{aligned}
\dot{\rho} &= -\frac{z\dot{z}}{\rho}, \\
\implies \dot{\rho}^2 &= \frac{z^2\dot{z}^2}{\rho^2}, \\
&= \frac{z^2\dot{z}^2}{4a^2 - z^2},
\end{aligned}$$

again using $\rho^2 + z^2 = 4a^2$.

In addition we know $\dot{\rho}^2 = \dfrac{z^2\dot{z}^2}{4a^2 - z^2}$, and hence

$$\begin{aligned}
\rho^2\dot{\theta}^2 &= \frac{\left(\rho^2\dot{\theta}\right)^2}{\rho^2}, \\
&= \frac{h^2}{\rho^2}, \\
&= \frac{4a^2V^2}{\rho^2}, \\
&= \frac{4a^2V^2}{(4a^2 - z^2)}.
\end{aligned}$$

This gives

$$\begin{aligned}
\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - mgz &= \frac{1}{2}mV^2, \\
\implies \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2 - 2gz &= V^2, \\
\implies \frac{z^2\dot{z}^2}{4a^2 - z^2} + \frac{4a^2V^2}{(4a^2 - z^2)} + \dot{z}^2 - 2gz &= V^2.
\end{aligned}$$

Rearranging we find

$$\begin{aligned}
\dot{z}^2\left(\frac{z^2}{4a^2 - z^2} + 1\right) + \frac{4a^2V^2}{(4a^2 - z^2)} - 2gz &= V^2, \\
\implies \dot{z}^2\left(z^2 + 4a^2 - z^2\right) + 4a^2V^2 - 2gz\left(4a^2 - z^2\right) &= V^2\left(4a^2 - z^2\right),
\end{aligned}$$

and hence we find an expression for $\dot{z}^2$,

$$\begin{aligned}
4a^2\dot{z}^2 &= -V^2z^2 + 2gz\left(4a^2 - z^2\right), \\
\implies 2a^2\dot{z}^2 &= -gz\left(z^2 + \frac{V^2}{2g}z - 4a^2\right), \\
&= -gz\left(z - z_1\right)\left(z - z_2\right),
\end{aligned}$$

where $z_{1,2}$ satisfy

$$z^2 + \frac{V^2}{2g}z - 4a^2 = 0.$$

Hence

$$z_{1,2} = \frac{-\frac{V^2}{2g} \pm \sqrt{\left(\frac{V^2}{2g}\right)^2 + 16a^2}}{2}.$$

Now, $z_1 < 0 < z_2$ without loss of generality, so the particle will remain between $z = 0$ and either $z = z_1$ if the particle rises initially or z_2 if the particle falls initially (recalling that z increasing is downwards).

(d) The sign of $\ddot{z}$ at $t = 0$ will tell us whether the particle initially rises or falls. Now

$$\begin{aligned} 4a^2\dot{z}\ddot{z} &= -g\dot{z}\left(z - z_1\right)\left(z - z_2\right) - gz\dot{z}\left(z - z_2\right) - gz\left(z - z_1\right)\dot{z}, \\ \implies 4a^2\ddot{z} &= -g\left(z - z_1\right)\left(z - z_2\right) - gz\left(z - z_2\right) - gz\left(z - z_1\right). \end{aligned}$$

Since $z = 0$ initially, we have

$$\begin{aligned} 4a^2\ddot{z} &= -g \times -z_1 \times -z_2, \\ &= -gz_1z_2. \end{aligned}$$

Note that $z_1 < 0$ and $z_2 > 0$, so $\ddot{z} > 0$ initially, so the particle will start by falling, as the vertical speed increases from zero.

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9. A raindrop falls through a cloud while accumulating mass at a rate λr^2 where r is its radius (assume that the raindrop remains spherical) and $\lambda > 0$ is a constant. Find its velocity v at time t if it starts from rest with radius a. (You should take the direction of positive v to be downwards.)

(a) If ρ is the density of rainwater (assumed constant), show that the mass of the raindrop is $m = \frac{4}{3}\pi r^3\rho$ and hence find an expression for $r(t)$. You may find it useful to define $\mu = \frac{\lambda}{4\rho\pi}$.

(b) Show that

$$\frac{dv}{dt} + \frac{3\mu}{\mu t + a}v = g.$$

(c) Find $v(t)$ and explains what happens in the limit $t \to \infty$.

Solution. (a) Since density is mass per unit volume, and the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, the mass of the raindrop will be $m = \frac{4}{3}\pi r^3\rho$. Now we have

$$\begin{aligned} \frac{dm}{dt} &= \frac{d}{dt}\left(\frac{4}{3}\pi\rho r^3\right), \\ &= 4\rho\pi r^2\frac{dr}{dt}, \\ &= \lambda r^2. \end{aligned}$$

Hence r satisfies

$$\begin{aligned}\frac{dr}{dt} &= \frac{\lambda}{4\rho\pi},\\ &= \mu.\end{aligned}$$

Hence $r = \mu t + \text{constant}$. Since $r = a$ at $t = 0$, this gives $r = \mu t + a$ for the evolving radius of the raindrop.

(b) Conservation of momentum gives

$$\begin{aligned}mg &= \frac{d(mv)}{dt},\\ \implies \frac{dv}{dt} + \frac{1}{m}\frac{dm}{dt}v &= g.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{m}\frac{dm}{dt} &= \frac{\lambda r^2}{\frac{4}{3}\pi\rho r^3},\\ &= \frac{3\lambda}{4\pi\rho r},\\ &= \frac{3\mu}{r},\\ &= \frac{3\mu}{\mu t + a},\end{aligned}$$

this becomes

$$\frac{dv}{dt} + \frac{3\mu}{\mu t + a}v = g.$$

This can be solved using the integrating factor $(a + \mu t)^3$ to give

$$\frac{d}{dt}\left(v\,(a + \mu t)^3\right) = g\,(a + \mu t)^3,$$

and hence

$$v\,(a + \mu t)^3 = \frac{g}{4\mu}\,(a + \mu t)^4 + \text{constant}.$$

Since $v(0) = 0$, this gives

$$v = \frac{g}{4\mu}\,(a + \mu t) - \frac{g a^4}{4\mu\,(a + \mu t)^3}.$$

As $t \to \infty$, the second term tends to 0, so the velocity becomes asymptotic to $\frac{g}{4\mu}\,(a + \mu t)$ and hence the acceleration approaches the constant $g/4$, which is smaller than the acceleration due to gravity for objects with constant mass. The accumulation of mass reduces the terminal acceleration.

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