University of Birmingham
School of Mathematics
Vectors, Geometry and Linear Algebra
VGLA

Model Solutions

Remember that there are practise questions under the materials section for each week.

SUM **Q1**. Suppose that $z_1 = 1 + i$ and $z_2 = 3 - 2i$.

(i) Calculate $z_1 + z_2$ and $z_1 z_2$ giving your answer in the form $a + bi$ for appropriate real numbers a and b.
(ii) Calculate z_1^8 in both modulus-argument form and exponential form giving the principal value of the argument.
(iii) Find all the fifth roots of z_1. Present your answer in exponential form giving the principal value of the argument.
(iv) Calculate β such that $z_1^2 + \beta z_1 + \beta = 0$.
(v) Let $\beta \in \mathbb{C}$, find all $z \in \mathbb{C}$ such that $z^2 + \beta z + \frac{\beta^2}{4} = 0$.
(vi) The circle of radius r in the xy-plane centred at $(a, b) \in \mathbb{R}^2$ is described by the equation $(x - a)^2 + (y - b)^2 = r^2$.
Describe the solution set to the equation

$$|z - 1| = 4|z - 3|$$

as a geometrically defined subset of the Argand diagram.
(vii) Suppose that $n \geq 2$ and set $R_n = \{r_1, \dots, r_n\}$ to be the set of the n distinct roots of the equation $z^n + 1 = 0$. Let r be a root of $z^n - 1 = 0$. Show that $R_n = \{rr_1, \dots, rr_n\}$. Deduce that $\sum_{i=1}^n r_i = 0$.

Solution. (i) We have $z_1 + z_2 = 4 - i$ and $z_1 z_2 = 5 + i$.
(ii) $z_1 = \sqrt{2}e^{\pi/4i} = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$. Hence $z_1^8 = \sqrt{2}^8 e^{2\pi i} = 16e^{2\pi i} = 16e^0 = 16(\cos 0 + \sin 0) = 16$. Note that the principal value of the argument of z_1^8 is 0.
(iii) Let γ be the positive 5th root of $\sqrt{2}$. So $\gamma = 2^{1/10}$. Then the 5th roots of z_1 are

$$\{\gamma e^{\frac{\pi/4+2k\pi}{5}i} \mid k \in \mathbb{Z}\} = \{\gamma e^{\frac{\pi+8k\pi}{20}i} \mid k \in \mathbb{Z}\} = \{\gamma e^{\frac{\pi}{20}i}, \gamma e^{\frac{9\pi}{20}i}, e^{\frac{17\pi}{20}i}, \gamma e^{\frac{-7\pi}{20}i}, \gamma e^{\frac{-15\pi}{20}i}\}.$$

(iv) We have $\beta = -z_1^2(z_1 + 1)^{-1} = -\frac{2}{5}(1 + 2i)$.
(v) We find $z = \frac{-\beta \pm \sqrt{\beta^2 - 4\beta^2/4}}{2} = -\beta/2$ is the unique solution.
(vi) Write $z = x + iy$, then $|z - 1| = 4|z - 3|$ if and only if $|z - 1|^2 - (4|z - 3|)^2 = 0$. We simplify

$$\begin{aligned}
(x-1)^2 + y^2 - 16((x-3)^2 + y^2) &= 0 \\
x^2 - 2x + 1 - 16(x^2 - 6x + 9) - 15y^2 &= 0 \\
-15x^2 + 94x - 143 - 15y^2 &= 0 \\
x^2 - \frac{94}{15} + \frac{143}{15} + y^2 &= 0 \\
(x - \frac{47}{15})^2 - \frac{64}{225} + y^2 &= 0
\end{aligned}$$

Hence the complex numbers which satisfy the equation $|z - 1| = 4|z - 3|$ describe a circle of radius $\frac{8}{15}$ centred at the real number $\frac{47}{15}$.

(vii) Suppose that $r_i \in R_n$. Then $(rr_i)^n = r^n r_i^n = 1 \cdot (-1) = -1$ and so $rr_i \in R_n$. Hence

$$\{rr_i \mid 1 \leq i \leq n\} \subseteq R_n.$$

Suppose that $1 \leq i < j \leq n$ and $rr_i = rr_j$. Then $r(r_i - r_j) = 0$. Since $r_i \neq r_j$, $r_i - r_j \neq 0$. Hence $r = 0$, but $r \neq 0$ so this is a contradiction. We conclude that $rr_i \neq rr_j$ when $i \neq j$. Hence $\{rr_i \mid 1 \leq i \leq n\}$ has n elements and we conclude that $\{rr_i \mid 1 \leq i \leq n\} = R_n$.

Put $w = \sum_{i=1}^{n} r_i$. Since $n \geq 1$, we can choose r to be an nth root of 1 with $r \neq 1$.

Then $w = \sum_{x \in R_n} x = \sum_{x \in R_n} rx = r(\sum_{x \in R_n} x) = rw$. Then $w - rw = 0$ and so $(1 - r)w = 0$. Since $1 - r \neq 0$ by the choice of r, we deduce $w = 0$ as claimed.

$\square$

SUM **Q2**. (i) Using the augmented matrix and row operations, find the solution set of the following system of real simultaneous linear equations.

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 5 \\ x & + & 3y & + & 2z & = & 1 \\ 3x & + & 4y & + & 3z & = & 6. \end{array}$$

(ii) Suppose that $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 5 \end{smallmatrix}\right)$.

(a) Determine $\mathbf{A}^{-1}$.

(b) Solve the system of simultaneous linear equation

$$\mathbf{A} \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 8 \\ 2 \end{smallmatrix}\right).$$

Solution. (i) We first make the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 3 & 2 & 1 \\ 3 & 4 & 3 & 6 \end{array}\right)$$

Now we perform row operations to obtain the reduced row echelon form. Notice that we have selected some row operations so as to reduce a proliferation of rational numbers in the calculations.

$$\boxed{r_1 \circlearrowleft r_2} \qquad \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & 1 & 1 & 1 \\ 3 & 4 & 3 & 6 \end{array}\right)$$

$$\boxed{r_2 \mapsto r_2 - 2r_1} \qquad \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -5 & -3 & 3 \\ 3 & 4 & 3 & 6 \end{array}\right)$$

$$\boxed{r_3 \mapsto r_3 - 3r_1} \qquad \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -5 & -3 & 3 \\ 0 & -5 & -3 & 3 \end{array}\right)$$

$$\boxed{r_3 \mapsto r_3 - r_2} \qquad \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -5 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We now know that there isn't a unique solution, we do a couple more row operations

$$\boxed{r_1 = 5r_1 \text{ and} r_2 = -r_2} \qquad \left(\begin{array}{ccc|c} 5 & 15 & 10 & 5 \\ 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\boxed{r_1 = r_1 - 3r_2} \qquad \left(\begin{array}{ccc|c} 5 & 0 & 1 & 14 \\ 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence the solution set is

$$\{(\frac{1}{5}(14-\lambda), -\frac{3}{5}(\lambda+1), \lambda) \in \mathbb{R}^3 \mid \lambda \in \mathbb{R}\}.$$

(ii) We have $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 4 & 9 \end{smallmatrix}\right)$. Construct the augmented matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right).$$

Check your answer by back substitution.

Now reduce the left hand side to reduced echelon form (so that it becomes the 3×3 identity matrix) by using elementary row operations.

$$\boxed{r_2 \mapsto r_2 - r_1} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\boxed{r_3 \mapsto r_3 - 3r_1} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & -4 & -4 & -3 & 0 & 1 \end{array}\right)$$

$$\boxed{r_3 \mapsto -r_3 \text{ and } r_2 \mapsto -r_2} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 4 & 4 & 3 & 0 & -1 \end{array}\right)$$

$$\boxed{r_3 \mapsto r_3 - 4r_2} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -4 & -1 & 4 & -1 \end{array}\right)$$

$$\boxed{r_3 \mapsto -\tfrac{1}{4}r_3} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -1 & \frac{1}{4} \end{array}\right)$$

$$\boxed{r_2 \mapsto r_2 - 2r_3} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -1 & \frac{1}{4} \end{array}\right)$$

$$\boxed{r_1 \mapsto r_1 - 3r_3} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{4} & 3 & -\frac{3}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -1 & \frac{1}{4} \end{array}\right)$$

$$\boxed{r_1 \mapsto r_1 - 2r_2} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{4} & 1 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -1 & \frac{1}{4} \end{array}\right)$$

Now we read that

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{3}{4} & 1 & \frac{1}{4} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -1 & \frac{1}{4} \end{pmatrix}.$$

Check that $\mathbf{A}\mathbf{A}^{-1}$ is the identity matrix.

It follows that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A} \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & 1 & \frac{1}{4} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{31}{4} \\ \frac{15}{2} \\ \frac{-29}{4} \end{pmatrix}.$$

Hence the solution set is $\{(\frac{31}{4}, \frac{15}{2}, \frac{-29}{4})\}$.

Check the soluton by substituting into the original equation. $\square$