Problem sheet 1

Question 1.

(a) Let A, B and C be three events in a sample space Ω. Express in symbols the event

(i) that only A occurs;

(ii) that at least two of the events occur;

(iii) that exactly one event occurs.

(b) A six-sided dice is rolled three times. Describe a sample space to model this experiment and formally describe the events

(i) that all rolls are equal;

(ii) that consecutive rolls are strictly increasing;

(iii) that the total sum of all rolls is either 7 or 8.

Assuming the dice is fair, calculate the corresponding probabilities.

Question 2. (SUM)

(a) A random number generator selects one number from $\{1, 2, 3, 4\}$. It is known that: the number '2' will be selected with probability $1/4$; an odd number will be selected with probability $1/2$; a perfect square will be selected with probability $7/12$.

(i) Find the probability of selecting each possible outcome.

(ii) What is the probability that the number selected is at most 2?

(b) Let $\mathbb{P}$ be a probability distribution on a sample space Ω and let $A, B \subseteq \Omega$ be events with $\mathbb{P}(A) = 0.6$ and $\mathbb{P}(B) = 0.8$.

(i) Suppose that $\mathbb{P}(A \cap B) = 0.5$. What is $\mathbb{P}(A \cup B)$?

(ii) Suppose instead that the value $\mathbb{P}(A \cap B)$ is unknown. Show that $\mathbb{P}(A \cap B) \in [0.4, 0.6]$.

(c) Let $\mathbb{P}$ be a probability distribution on the sample space $\Omega = \{1, \ldots, 100\}$.

(i) Suppose that there is $c > 0$ so that $\mathbb{P}(k) = \frac{c}{k(k+1)}$ for all $k \in \Omega$. Find c.

(ii) Suppose instead that for all $k \in \{1, 2, \ldots, 50\}$ we have

$$\mathbb{P}(2k - 1) = 3 \cdot \mathbb{P}(2k). \tag{1}$$

Show that $\mathbb{P}(A) = 1/4$, where $A = \{2, 4, 6, \ldots, 100\} \subseteq \Omega$. [1]

Question 3. A knockout tennis tournament is organised for 2^k competitors, so that there are k rounds with the last round being the final. At the end of the tournament, two players are selected at random from the competitors. What is the probability that the two players:

(a) are the finalists of the tournament;

(b) played each other in the first round of the tournament;

(c) did not play each other in any round of the tournament?

Question 4. Let n be an even natural number.

(a) Show that among the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, the value $\binom{n}{n/2}$ is maximal.

Hint: When is $\binom{n}{i} \geq \binom{n}{i-1}$ for $i = 1, \ldots, n$?

(b) Show that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$. Deduce that $\binom{n}{n/2} \geq \frac{2^n}{n+1}$.

(c) Use Stirling's approximation to estimate $\binom{n}{n/2}$.

[1] Optional extra: Show that $\mathbb{P}(B)$ is not determined by (1), where $B = A \cup \{1\}$. (Find two prob. distributions which satisfy (1) but give different probabilities to B.)