Example sheet 1: Existence and Uniqueness and the Wronskian

1. A 1st order nonlinear ordinary differential equation initial value problem is given by

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

   (a) If $f(x, y) = \dfrac{x-y}{2x+5y}$, establish for which values of (x, y) the functions $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous (hint: in this case f and $\dfrac{\partial f}{\partial y}$ are not continuous on a line; find the equation of this line).

   (b) Use the appropriate Existence and Uniqueness Theorem to establish the region of the x-y plane in which a unique solution may exist to this differential equation for the given $f(x, y)$.

2. Explain why a unique solution to the differential equation

$$\frac{dy}{dx} = (x^2 + y^2)^{\frac{3}{2}}$$

   may exist throughout the x-y plane (state which Existence-Uniqueness Theorem you are applying).

3. (a) Solve the initial value problem

$$\frac{dy}{dx} = -\frac{4x}{y}, \quad y(0) = y_0.$$

   (b) Determine how the interval in which the solution exists depends on the initial value y_0 (i.e. determine for which values of x the solution is real in terms of y_0).

4. Solve the initial value problem

$$\frac{dy}{dx} = 2xy^2, \quad y(0) = y_0$$

   and determine how the interval in which the solution exists depends on the initial value y_0 (consider the cases $y_0 = 0, y_0 > 0$ and $y_0 < 0$).

5. (a) Verify that both $y_1(x) = 1 - x$ and $y_2(x) = -\dfrac{x^2}{4}$ are solutions of the initial value problem

$$\frac{dy}{dx} = \frac{-x + (x^2 + 4y)^{\frac{1}{2}}}{2}, \quad y(2) = -1$$

   and determine where these solutions are valid.

   (b) Explain why the existence of multiple solutions does not invalidate the Existence-Uniqueness (Picard's) theorem for nonlinear 1st order ODEs.

6. For the equation

$$x^2 y'' - 2y = 0, \quad x > 0,$$

   (a) verify that $u_1 = x^2$ and $u_2 = x^{-1}$ are linearly independent solutions;

(b) write down the general solution;

(c) find the solution that satisfies the initial conditions $y(1) = -2$ and $y'(1) = -7$.

7. For the equation

$$xy'' - (x+2)y' + 2y = 0, \quad x > 0,$$

(a) verify that $u_1 = e^x$ and $u_2 = x^2 + 2x + 2$ are linearly independent solutions;

(b) write down the general solution;

(c) find the solution that satisfies the initial conditions $y(1) = 0$ and $y'(1) = 1$.

8. This question is designed to give you some insight into methods used in the proof of the Existence and Uniqueness theorem. You can use software such as Wolfram Alpha, Maple or Matlab to help with plotting the functions.

(a) Use the method of successive approximations to solve the initial value problem

$$\frac{dy}{dx} = 2(y+1), \quad y(0) = 0$$

with $u_0(x) = 0$, by obtaining a general $u_n(x)$.

(b) Plot $u_n(x)$ for $n = 1, ..., 4$. Does it look like the u_n are converging to a solution?

(c) Express

$$\lim_{n\to\infty} u_n(x) = u(x)$$

in terms of elementary functions (i.e. solve the initial value problem). Add $u(x)$ to your plot to see how well the $u_i(x)$ approximate the solution.

(d) On a new graph, plot $\left|u(x) - u_n(x)\right|$ for $n = 1, ..., 4$ to visualise the error between the $u_i(x)$ and the solution. Use this to estimate the interval in which each $u_i(x)$ is a reasonably good approximation to the solution.