1Mech — Mechanics

Mechanics exercises 2 (weeks 3 and 4)

Mechanics solutions 2

This sheet's assessed question is number 4.

1. The force on a particle of mass m and charge e, moving with velocity $\dot{\mathbf{r}}$ under the influence of a constant magnetic field $\mathbf{B}$ is $e\dot{\mathbf{r}} \times \mathbf{B}$. This is the only force acting on the particle, which starts at $\mathbf{0}$ with velocity $\mathbf{V}$ initially.

   (a) Show that
   $$m\ddot{\mathbf{r}} = e\dot{\mathbf{r}} \times \mathbf{B},$$
   and give appropriate initial conditions.

   (b) Hence show that
   $$m\dot{\mathbf{r}} = e\mathbf{r} \times \mathbf{B} + m\mathbf{V}.$$

   (c) If $\mathbf{B} = (0, 0, B)$ in Cartesians, write down an expression for $\mathbf{r} \times \mathbf{B}$.

   (d) Hence find the position of the particle $x(t)$, $y(t)$, $z(t)$ where $\mathbf{r} = (x, y, z)$, for initial velocity $\mathbf{V} = (V_1, 0, V_2)$.

   (e) What shape is the particle path?

   [Hint: Don't panic - this question looks a lot harder than it is initially, you should be able to get started OK! Recall the definition of the vector cross product and split into components. Combine your x and y equations to form a single second order ODE you can then solve.]

   Solution. (a) The force applied to the particle is $e\dot{\mathbf{r}} \times \mathbf{B}$ and hence, using Newton's second law, we have
   $$m\ddot{\mathbf{r}} = e\dot{\mathbf{r}} \times \mathbf{B}.$$
   The particle is initially at the origin, and so $\mathbf{r}(0) = \mathbf{0}$, moving with velocity $\dot{\mathbf{r}}(0) = \mathbf{V}$.

   (b) Since e and $\mathbf{B}$ are constants, the equation can be integrated directly to find
   $$m\dot{\mathbf{r}} = e\mathbf{r} \times \mathbf{B} + \mathbf{c}_1,$$
   where $\mathbf{c}_1$ is a constant of integration. Now, we know $\dot{\mathbf{r}}(0) = \mathbf{V}$, and $\mathbf{r}(0) = \mathbf{0}$ and hence
   $$\begin{aligned} m\mathbf{V} &= e\mathbf{0} \times \mathbf{B} + \mathbf{c}_1, \\ \Rightarrow \mathbf{c}_1 &= m\mathbf{V}. \end{aligned}$$
   Therefore
   $$m\dot{\mathbf{r}} = e\mathbf{r} \times \mathbf{B} + m\mathbf{V},$$
   as required.

(c) If $\mathbf{B} = (0, 0, B)$ then

$$\begin{aligned} \mathbf{r} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 0 & 0 & B \end{vmatrix}, \\ &= yB\mathbf{i} - xB\mathbf{j}. \end{aligned}$$

(d) Hence

$$m\dot{x}\mathbf{i} + m\dot{y}\mathbf{j} + m\dot{z}\mathbf{k} = eyB\mathbf{i} - exB\mathbf{j} + mV_1\mathbf{i} + mV_2\mathbf{k}.$$

We now equate components to find

$$\begin{aligned} m\dot{x} &= eyB + mV_1, \\ m\dot{y} &= -exB, \\ m\dot{z} &= mV_2. \end{aligned}$$

We can solve the z equation immediately by integrating

$$mz = mV_2 t + c_2,$$

and using $z(0) = 0$ to find $c_2 = 0$. Now the x and y equations give a system to solve. If we differentiate the x equation, and then substitute in from the y equation we have:

$$\begin{aligned} m\ddot{x} &= e\dot{y}B, \\ &= -\frac{(eB)^2}{m}x. \end{aligned}$$

This gives a second order ODE to solve:

$$\begin{aligned} \ddot{x} + \left(\frac{eB}{m}\right)^2 x &= 0, \\ \Rightarrow x &= A\cos\frac{eBt}{m} + C\sin\frac{eBt}{m}. \end{aligned}$$

Using $x(0) = 0$ gives $A = 0$, and $\dot{x}(0) = V_1$ gives

$$\begin{aligned} \dot{x} &= \frac{eBC}{m}\cos\frac{eBt}{m}, \\ \Longrightarrow V_1 &= \frac{eBC}{m}, \\ \Longrightarrow C &= \frac{mV_1}{eB}, \end{aligned}$$

We now substitute this back into the y equation:

$$\begin{aligned} m\dot{y} &= -eBx, \\ &= -mV_1\sin\frac{eBt}{m}, \end{aligned}$$

which we integrate to find

$$
\begin{aligned}
my &= \frac{-mV_1}{-eB/m}\cos\frac{eBt}{m} + D, \\
&= \frac{m^2V_1}{eB}\cos\frac{eBt}{m} + D.
\end{aligned}
$$

Finally, using $y(0) = 0$ we find

$$
D = -\frac{m^2V_1}{eB}.
$$

Therefore our overall solution is

$$
\begin{aligned}
x &= \frac{mV_1}{eB}\sin\frac{eBt}{m}, \\
y &= \frac{mV_1}{eB}\cos\frac{eBt}{m} - \frac{mV_1}{eB}, \\
z &= V_2 t.
\end{aligned}
$$

This gives a helix, since the x and y components gives a circle, with an additional constant motion in the z direction).

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Feedback: *This is a more challenging question which pulls together several elements we've covered. The question deliberately looks more complex than it is - the fact that the particle is moving in an electric field is not actually relevant to how you go about solving it! When you do modelling you are often given more information than you need which you learn to ignore. The technical elements are slightly complex, but the key is keeping calm and being very clear about what each stage is doing.*

2. (a) Starting from expressions for the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ in the r and θ directions, where r and θ are polar coordinates, derive the radial and transverse components of acceleration.

   (b) If a particle of mass m moves under a central force of the form $\mathbf{F} = F(r)\mathbf{e}_r$, prove that $r^2\dot{\theta} = h$ is constant, and find the governing equation for the particle path in terms of $u = 1/r$ and θ.

Solution. (a) Since

$$
\begin{aligned}
\mathbf{e}_r &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \\
\mathbf{e}_\theta &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j},
\end{aligned}
$$

then

$$
\begin{aligned}
\dot{\mathbf{e}}_r &= \frac{d}{dt}(\cos\theta)\,\mathbf{i} + \frac{d}{dt}(\sin\theta)\,\mathbf{j}, \\
&= -\dot{\theta}\sin\theta\mathbf{i} + \dot{\theta}\cos\theta\mathbf{j}, \\
&= \dot{\theta}\mathbf{e}_\theta, \\
\dot{\mathbf{e}}_\theta &= \frac{d}{dt}(-\sin\theta)\,\mathbf{i} + \frac{d}{dt}(\cos\theta)\,\mathbf{j}, \\
&= -\dot{\theta}\cos\theta\mathbf{i} - \dot{\theta}\sin\theta\mathbf{j}, \\
&= -\dot{\theta}\mathbf{e}_r.
\end{aligned}
$$

Now since $\mathbf{r} = r\mathbf{e}_r$ gives the position vector of the particle, we have the velocity of the particle given by

$$\begin{aligned}\dot{\mathbf{r}} &= \frac{d}{dt}\left(r\mathbf{e}_r\right), \\ &= r\frac{d\mathbf{e}_r}{dt} + \frac{dr}{dt}\mathbf{e}_r, \\ &= r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r,\end{aligned}$$

and the acceleration

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{d}{dt}\left(\dot{\mathbf{r}}\right), \\ &= \frac{d}{dt}\left(r\dot{\theta}\mathbf{e}_\theta + \dot{r}\mathbf{e}_r\right), \\ &= \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta + \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r, \\ &= \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r + \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta, \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta, \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta.\end{aligned}$$

(b) Newton's second law for a central force of the form $F\mathbf{e}_r$ gives

$$\begin{aligned}m\ddot{\mathbf{r}} &= F(r)\mathbf{e}_r, \\ \Longrightarrow m\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta &= F(r)\mathbf{e}_r.\end{aligned}$$

Equating coefficients gives

$$\begin{aligned}m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r), \\ \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) &= 0, \\ \Rightarrow r^2\dot{\theta} = h,\end{aligned}$$

where h is constant, and thus

$$\begin{aligned}m\left(\ddot{r} - r\left(h/r^2\right)^2\right) &= F(r), \\ \Rightarrow m\left(\ddot{r} - h^2/r^3\right) &= F(r).\end{aligned}$$

Since

$$\begin{aligned}\dot{r} &= \frac{dr}{d\theta}\frac{d\theta}{dt} \\ &= \frac{h}{r^2}\frac{dr}{d\theta}, \\ &= -h\frac{d}{d\theta}\left(\frac{1}{r}\right), \\ &= -h\frac{du}{d\theta},\end{aligned}$$

and

$$\begin{aligned}
\ddot{r} &= \frac{d\dot{r}}{dt}, \\
&= \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt}, \\
&= \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\dot{\theta}, \\
&= -h\dot{\theta}\frac{d^2u}{d\theta^2}, \\
&= -h^2u^2\frac{d^2u}{d\theta^2},
\end{aligned}$$

we find

$$\begin{aligned}
m\left(\ddot{r} - h^2/r^3\right) &= F(r), \\
\Longrightarrow m\left(-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) &= F(1/u), \\
\Longrightarrow \frac{d^2u}{d\theta^2} + u &= -\frac{F(1/u)}{mh^2u^2},
\end{aligned}$$

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Feedback: *This is bookwork, but it is very important that you understand and learn it! The critical points are that* $\mathbf{e}_r$, $\mathbf{e}_\theta$ *are functions of time, so the velocity and acceleration are more complicated than you might expect. Learn how to change between* $r(t)$, $\theta(t)$ *and* $u(\theta)$, *and to show that* $r^2\dot{\theta}$ *is constant. Be careful of the sign of the right hand side - this depends on which direction you define your force to be, and can be inconsistent in textbook!*

3. Find the value of the constant $h = r^2\dot{\theta}$, and suitable initial conditions for $u(\theta) = 1/r$, $du/d\theta$ for the following particles under the action of a central force.

   (a) The particle is initially at $r = a$, moving with radial velocity v and transverse (also known as angular) velocity $a\omega$.

   (b) The particle is initially at $r = b$, moving away from the origin with speed V in a direction which makes an angle $\pi/4$ with the outward pointing radial vector.

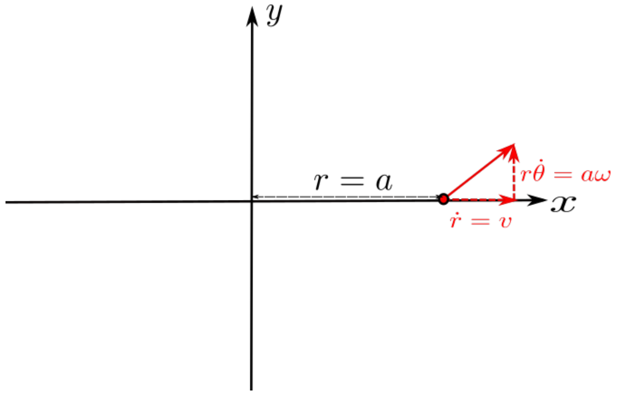
   (c) The particle is initially at $r = c$, moving with speed w in a direction making an angle $\pi/3$ with the **inward** pointing radial vector.

Solution. We calculate:

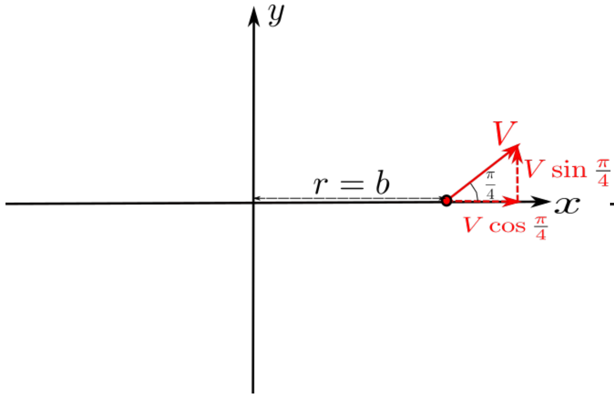
(a)
- At $t = 0$ we know $r = a$, $\dot{r} = v$, $r\dot{\theta} = a\omega$ and we **choose** $\theta = 0$.
- Then $h = r^2\dot{\theta} = r \cdot r\dot{\theta} = a^2\omega$ is **always constant** throughout the motion.
- We then have the initial conditions

$$u = 1/r = 1/a, \quad \text{and} \quad \frac{du}{d\theta} = -\frac{\dot{r}}{h} = -\frac{v}{a^2\omega} \qquad \text{at} \qquad \theta = 0,$$

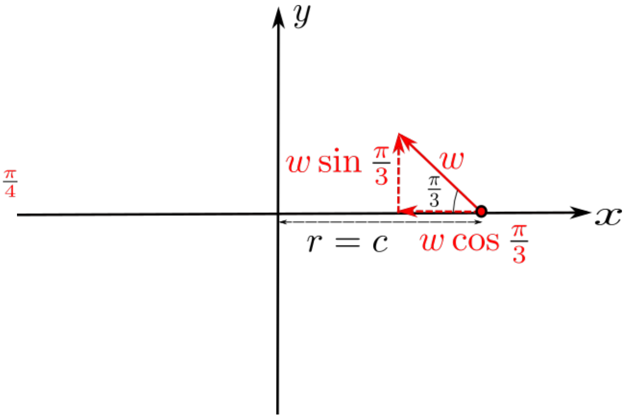
  for u as a function of θ.



(a) Example (a)



(b) Example (b)



(c) Example (c)

Figure 1: Sketch decomposing the initial velocity vector into components for each example

(b) • At $t = 0$ we know $r = b$, and

$$\begin{aligned} \dot{\mathbf{r}} &= V \cos \frac{\pi}{4}\mathbf{e}_r + V \sin \frac{\pi}{4}\mathbf{e}_\theta, \\ &= \frac{V}{\sqrt{2}}\mathbf{e}_r + \frac{V}{\sqrt{2}}\mathbf{e}_\theta, \end{aligned}$$

giving $\dot{r} = \dfrac{V}{\sqrt{2}}$ and $r\dot{\theta} = \dfrac{V}{\sqrt{2}}$. We again choose $\theta = 0$ at $t = 0$.

- Then $h = r^2\dot{\theta} = r \cdot r\dot{\theta} = b\dfrac{V}{\sqrt{2}}$ is **always constant** throughout the motion.
- We then have the initial conditions

$$u = 1/r = 1/b, \quad \text{and} \quad \frac{du}{d\theta} = -\frac{\dot{r}}{h} = -\frac{\frac{V}{\sqrt{2}}}{b\frac{V}{\sqrt{2}}} = -\frac{1}{b} \qquad \text{at} \qquad \theta = 0,$$

for u as a function of θ.

(c) • At $t = 0$ we know $r = c$, and

$$\begin{aligned}\dot{\mathbf{r}} &= -w\cos\frac{\pi}{3}\mathbf{e}_r + w\sin\frac{\pi}{3}\mathbf{e}_\theta, \\ &= -\frac{w}{2}\mathbf{e}_r + \frac{\sqrt{3}w}{2}\mathbf{e}_\theta,\end{aligned}$$

giving $\dot{r} = -\frac{w}{2}$ and $r\dot{\theta} = \frac{\sqrt{3}w}{2}$. We again choose $\theta = 0$ at $t = 0$.

• Then $h = r^2\dot{\theta} = r \cdot r\dot{\theta} = c\frac{\sqrt{3}w}{2}$ is **always constant** throughout the motion.

• We then have the initial conditions

$$u = 1/r = 1/c, \quad \text{and} \quad \frac{du}{d\theta} = -\frac{\dot{r}}{h} = -\frac{-\frac{w}{2}}{c\frac{\sqrt{3}w}{2}} = \frac{1}{\sqrt{3}c} \qquad \text{at} \qquad \theta = 0,$$

for u as a function of θ.

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Feedback: *This gives some practice in finding h and the initial conditions for the $u(\theta)$ system, which can be one of the more challenging elements. Make sure you read the question carefully, draw a picture and take care with the trigonometry. A really important point to take care over is what is always constant (i.e. h) and what are initial conditions (i.e. only hold at the beginning of the motion).*

4. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

A particle of mass m is moving in the **i**-**j** plane. Let the polar coodinates (r, θ) and basis vectors $\mathbf{e}_r, \mathbf{e}_\theta$ be defined in the usual way.

(a) Show that
$$\mathbf{i} = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta, \quad \mathbf{j} = \sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta.$$

(b) The particle is subject to a force $F(r,\theta)\mathbf{e}_r$, exerted along a massless tether with one end fixed at the origin; and the force of gravity, $m\mathbf{g} = -mg\mathbf{j}$. Show that the equations of motion for the particle are:

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r,\theta) - mg\sin\theta, \tag{1}$$
$$\frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = -mg\cos\theta. \tag{2}$$

You may assume the following expression for the particle's acceleration:

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta.$$

(c) Assume that the force is central, meaning $F(r,\theta) = F(r)$ with no θ-dependence. Assume also that the particle follows a circular path, meaning $r = a > 0$ is constant and $\dot{\theta} > 0$ for all t. Differentiate (1) with respect to t, to obtain

$$\ddot{\theta} = f(\theta), \tag{3}$$

where f is a function you should determine.

(d) Show that equations (2) and (3) contradict each other.

(e) The following is an interpretation of the result in part (d). Whenever we derive a model that is inconsistent, somewhere along the line we have made an invalid assumption. Here, only two assumptions could possibly be invalid: that the force is central, or that the particle follows a circular path. In conclusion, if F depends only on r then the particle cannot follow a circular path, and if the particle must follow a circular path then F must have a θ-dependence.

Let us insist that the particle follows a circular path of radius $a > 0$ and allow F to depend on θ. If $F(r,\theta) = mrG(\theta)$ for some function G, differentiate (1) with respect to t to obtain

$$\ddot{\theta} = -\frac{G'(\theta)}{2} + f(\theta), \tag{4}$$

where f is the same as in part (c). Hence, find a function G that makes the model consistent, i.e., making (2) equivalent to (4).

Solution. (a) It is easier to start from the right-hand side,

$$\begin{aligned}
\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta &= \cos\theta(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) - \sin\theta(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) \\
&= (\cos^2\theta + \sin^2\theta)\mathbf{i} + (\sin\theta\cos\theta - \sin\theta\cos\theta)\mathbf{j} = \mathbf{i}, \\
\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta &= \sin\theta(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + \cos\theta(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) \\
&= (\sin\theta\cos\theta - \sin\theta\cos\theta)\mathbf{i} + (\cos^2\theta + \sin^2\theta)\mathbf{j} = \mathbf{j}.
\end{aligned}$$

(b) Newton's Second Law gives $m\ddot{\mathbf{r}} = \mathbf{F} - mg\mathbf{j}$. Hence,

$$m\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta = \left(F(r,\theta) - mg\sin\theta\right)\mathbf{e}_r - mg\cos\theta\mathbf{e}_\theta. \tag{5}$$

Equating coefficients yields the desired result.

(c) Since $r = a$ is constant, we have $\dot{r} = \ddot{r} = 0$, and $F(r) = F(a)$ is a constant. Equation (1) therefore becomes

$$-ma\dot{\theta}^2 = F(a) - mg\sin\theta.$$

Differentiating gives

$$-2ma\dot{\theta}\ddot{\theta} = -mg\dot{\theta}\cos\theta. \tag{6}$$

Since $\dot{\theta}$ is never zero by assumption, we are allowed to divide by $-2ma\dot{\theta}$, giving

$$\ddot{\theta} = \frac{g}{2a}\cos\theta = f(\theta). \tag{7}$$

(d) Equation (2) now reads

$$\frac{m}{a}\frac{d}{dt}(a^2\dot{\theta}) = ma\frac{d}{dt}(\dot{\theta}) = ma\ddot{\theta} = -mg\cos\theta.$$

Dividing by ma yields

$$\ddot{\theta} = -\frac{g}{a}\cos\theta, \tag{8}$$

which contradicts (7) (unless $\cos\theta = 0$ for all t, which is ruled out by the assumption that the particle follows a circular path).

(e) Equation (1) now reads

$$-ma\dot{\theta}^2 = maG(\theta) - mg\sin\theta.$$

Differentiating gives

$$-2ma\dot{\theta}\ddot{\theta} = maG'(\theta)\dot{\theta} - mg\dot{\theta}\cos\theta. \tag{9}$$

Dividing by $-2ma\dot{\theta}$ yields

$$\ddot{\theta} = -\frac{G'(\theta)}{2} + \frac{g}{2a}\cos\theta.$$

For the model to be consistent, we need this to be identical to (8), i.e.

$$-\frac{G'(\theta)}{2} + \frac{g}{2a}\cos\theta = -\frac{g}{a}\cos\theta \quad\Leftrightarrow\quad G'(\theta) = \frac{3g}{a}\cos\theta, \tag{10}$$

and solving this gives

$$G(\theta) = \frac{3g}{a}\sin\theta + c,$$

where c is any constant.

The following is interesting but not required for credit. The constant c is determined by the initial condition for $\dot{\theta}$. *Recall that the general expression for the particle's velocity is* $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$. *Given the constraint that* $r = a$ *is constant, the velocity is simply* $\dot{\mathbf{r}} = a\dot{\theta}\mathbf{e}_\theta$, *always tangential to the circular path. For a physically realistic situation, let us assume that at* $t = 0$, $\theta = 0$ *and* $\dot{\theta} = \omega > 0$. *Putting this into* $-ma\dot{\theta}^2 = maG(\theta) - mg\sin\theta$ *gives* $-ma\omega^2 = maG(0) - mg\sin 0 = mac$, *from which we find* $c = -\omega^2$. *Note that the radial force* $maG(\theta) = 3mg\sin\theta - ma\omega^2$ *is maximised when* $\theta = \pi/2$ *and minimised when* $\theta = -\pi/2$. *Physical intuition tells us that if the force is exerted along something like a piece of string, the string is not allowed to go slack, meaning the radial force must always be negative, pulling the particle rather than pushing it. This means that in order for the model to be truly consistent, we need* $\omega^2 > 3g/a$. *In other words, the particlar must initially be moving sufficiently fast; otherwise it will not make it over the top! Finally, imagine that the fixed end of the tether is fixed onto you, and you are swinging the particle in a circle. Newton's Third Law says that the force on you is equal and opposite to the force on the particle. So, when the particle is at the bottom, it is maximally pulling you towards it, as you might expect from your intuition.*

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