Examples sheet 3 – Linear Algebra

The exercises below correspond to material from Lectures 8–12. Selected exercises will be covered in the Examples class scheduled in week 7. Solutions will be available on Canvas.

LINEAR MAPS.

1. Find the parameters $\alpha, \beta \in \mathbb{R}$ such that the maps indicated below are linear

(a) $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \alpha x + \beta.$$

(b) $f : \mathbb{R} \to \mathbb{R}^2$,

$$f(x) = \begin{bmatrix} \alpha \\ x^\beta \end{bmatrix}.$$

2. Consider the map $f : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$f(p) = \begin{bmatrix} p(-1) \\ 2p(1) \end{bmatrix}$$

(a) Check that f is linear.

(b) Find $\ker f$ and state its dimension. Is f injective?

(c) Find $\operatorname{im} f$ and state its dimension. Is f surjective?

3. Let $V_n = (\mathbb{R}^n, \boldsymbol{+}, \bullet, \mathbb{R})$ be the vector space of real column vectors equipped with the following operations:

$$\mathbf{v} \boldsymbol{+} \mathbf{w} := \mathbf{v} + \mathbf{w} + \mathbf{1}_n, \quad a \bullet \mathbf{v} := a\mathbf{v} + (a-1)\mathbf{1}_n,$$

where $\mathbf{1}_n \in \mathbb{R}^n$ is the vector of ones.
Let $f : V_n \to V_m$ be defined via

$$f(\mathbf{v}) = A\mathbf{v} \boldsymbol{+} \mathbf{b}.$$

(a) Find $A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$ such that f is a linear transformation.

(b) Find $\mathbf{b}$ such that $\ker f = \ker A$.

4. Let f be given by $f(\mathbf{v}) = A\mathbf{v}$, where $A \in \mathbb{R}^{m \times n}$.

(a) For what matrices A is f injective?

(b) For what matrices A is f surjective?

5. Let V, W be vector spaces over a field $\mathbb{F}$. Consider the following sets:

- $\mathcal{L}(V, W) = \{f : V \to W : f \text{ is linear}\}$;
- $\mathcal{L}_0(V, W) = \{f \in \mathcal{L}(V, W) : \ker f = \{\mathbf{0}_{\mathrm{V}}\}\}$.

(a) Show that $\mathcal{L}(V, W)$ is a vector space when equipped with the operations of function addition and scalar-function multiplication.

(b) Show that $\mathcal{L}_0(V, W) \not\leq \mathcal{L}(V, W)$.

6. Let B denote the basis set of a vector space V. Consider the coordinate map $\varphi_B : V \to \mathbb{R}^n$.

(a) Show that φ_B is linear.

(b) Show that φ_B is bijective.

7. Let $f \in \mathcal{L}(V, W)$. Show that if $U \leq V$, then $f(U) \leq W$.

8. Show that a 3×7 matrix A must have $4 \leq \text{nullity}(A) \leq 7$. Give an example where the upper and lower bounds are attained.

9. **(a)** Let $f(\mathbf{x}) = A\mathbf{x}$, where $A \in \mathbb{R}^{m \times n}$. Show that if $n > m$, then f has a non-trivial kernel.

(b) Let $f \in \mathcal{L}(V, W)$, where $\dim V = \dim W = n$. Show that f is injective if and only if f is surjective.

MATRIX REPRESENTATIONS. KERNEL AND RANK.

10. Consider the following bases for $V = \mathbb{R}^2, W = \mathbb{R}^3$:

$$B_V = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \quad B_W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Let $f : V \to W$ be given by the following mappings:

$$f\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad f\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Find the matrix representation of f with respect to B_V, B_W.

11. Let the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as follows:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} 3a + b - c \\ -a - 2b + 2c \\ 2a \end{bmatrix}$$

(a) Find the matrix representation of f relative to the canonical basis of $\mathbb{R}^3$.

(b) Find a basis and the dimension of the image of f.

(c) Find a basis and the dimension of the kernel of f.

12. The following matrices have orthogonal columns. In each case, find the rank and the kernel.

(a) $A \in \mathbb{R}^{3 \times 5}$;

(b) $B \in \mathbb{R}^{5 \times 3}$.

13. Let $f : V \to W$ be a linear map with matrix representation $A \in \mathbb{R}^{m \times n}$ relative to bases B_V, B_W.

(a) Show that $\varphi_V(\ker f) = \ker A$.

(b) Show that $\varphi_W(\operatorname{im} f) = \operatorname{col} A$.

14. Find the matrix representations of the maps f, g below with respect to the canonical bases of $\mathbb{R}^5, \mathbb{R}^3$:

(a) $f : \mathbb{R}^5 \to \mathbb{R}^3$

$$[f(\mathbf{v})]_k = v_{2k-1}, \qquad k = 1, \ldots, 3.$$

(b) $g : \mathbb{R}^3 \to \mathbb{R}^5$,

$$[g(\mathbf{v})]_\ell = \begin{cases} v_{(\ell+1)/2} & \text{if } \ell \text{ is odd,} \\ \frac{1}{2}(v_{\ell/2} + v_{\ell/2+1}) & \text{if } \ell \text{ is even,} \end{cases} \qquad \ell = 1, \ldots, 5$$

(c) Find the matrix representation of $h = f \circ g$.

(d) Describe the actions of f, g and h on column vectors.

15. Let $A \in \mathbb{R}^{m\times n}$, with $m \leq n$. Consider the linear system $A\mathbf{x} = \mathbf{b}$.

(a) Show that if $\mathbf{b} \in \operatorname{col} A$, then the system has at least one solution.

(b) Show that if $\ker A$ is non-trivial, then there are infinitely many solutions.

CHANGE OF BASIS

16. Let $f \in \mathcal{L}(V, V)$ have matrix representation A relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of V.

(a) Find the transition matrix from basis B to basis $B' = \{\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3\}$.

(b) Describe the change in the matrix representation of f relative to the basis B'.

17. **(a)** Let $V = \mathcal{P}_1(\mathbb{R})$. Find the transition matrix from the basis $B = \{2x, 4x - 2\}$ to the basis $B' = \{x - 1, x + 1\}$.

(b) Find the B'-coordinates of $p \in V$, if its B-coordinates are 3 and 1.

(c) Find p in two ways: (i) using its B-coordinates and (ii) using its B'-coordinates. Do you get the same result both times?

18. Let B and B' be the two bases of $V = \mathcal{P}_1(\mathbb{R})$ as introduced in **Q17**. Consider the linear map $f : V \to V$ defined by $f \mapsto f + f'$, where f' is the derivative of f.

(a) Find the matrix representation of f with respect to the basis B'.

(b) Now compute the matrix representation of f with respect to B using the transition matrix found in **Q17**.

19. Show that matrix equivalence is an equivalence relation on $\mathbb{F}^{m\times n}$.