Mathematical Language

24 October 2022 13:14

Definitions - Tell us precisely what a mathematical concept means.

Theorem, Propositions, Lemmas and Corollaries:

All are mathematical statements that are proved

Theorem - Important big ideas

Propositions -

Lemmas - Usually used to prove a theorem or proposition

Corollaries - Easy to prove following a theorem or proposition

"Set up a notation. Suppose a hypothesis is true, then a conclusion is true".

Proof - Any logical arguments that show a mathematical statement is true

Counterexample - Show that certain statements are not true.

The negation of statement 'p' is the statement p is false.

The converse of "if P then Q" is "if Q then P"

The contrapositive of "if P, then Q" is "if not Q then not P".

↳ the contrapositive of a true statement is true

01: Sets

27 September 2022 14:02

Real analysis is the foundation of calculus.

Any differentiation + integration can be written as a limit.

- $\lim_{x \to x_0} f(x) = C$
- $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x)$ (Derivative of function).

Set: A collection of objects, which are called **elements**. These are commonly numbers but they don't have to be (eg sets or points).

Set Notation:

- $x \in A$ (x is an element of A).
- $x \notin A$ // $x \not\in A$ (x is not an element of A).
- $\emptyset$ (empty set)
- $A \subseteq B$ (**Subset** - when $x \in A$, then $x \in B$).
- $A = B$ ($A \subseteq B$, $B \subseteq A$)
- $A \subset B$ // $A \subsetneq B$ (**Proper subset** - $A \subseteq B$ and $A \neq B$)

Sets of Numbers:

$\mathbb{N}$ Natural Numbers $\{1, 2, 3, \dots\}$ ← Some people include 0, in this case we won't.

$\mathbb{N}_0$ Non Negative Integers $\{0, 1, 2, 3, \dots\}$

$\mathbb{Z}$ Integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{Q}$ Rational Numbers $\{p/q,\ p, q \in \mathbb{Z},\ q \neq 0\}$

$\mathbb{R}$ Real Numbers - Points of a straight line (rational + irrational).

↑ Real Analysis.

$$\mathbb{N} \overset{0}{\subseteq} \mathbb{N}_0 \overset{-}{\subseteq} \mathbb{Z} \overset{\div}{\subseteq} \mathbb{Q} \overset{\lim}{\subseteq} \mathbb{R}$$

Real numbers are the completion of $\mathbb{Q}$.

Representing sets:

① Listing Elements , Can't have repeats (because of $A=B$ definition)

• $A = \{1,2,3\} = \{3,2,1\} = \{1,2,3,2\}$ ←

② Set Builder Notation

• $B = \{x \in A;\ x \geq 2\} = \{2,3\}$

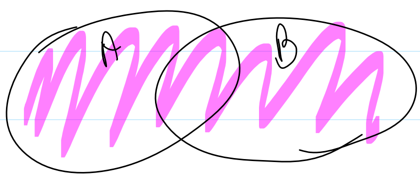
↳ set ↳ Range

③ By Replacement (shifts/translates another set)

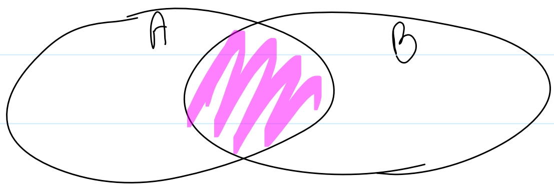
• $C = \{2x+3;\ x \in A\} = \{5,7,9\}$

Set Operators:

Union - $A \cup B = \{x;\ x \in A \text{ or } x \in B\}$

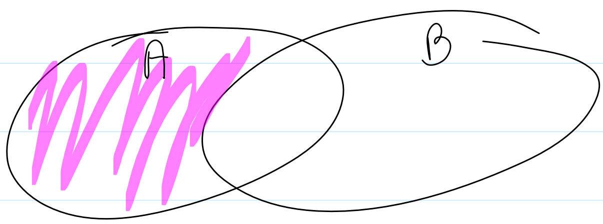


Intersection - $A \cap B = \{x;\ x \in A \text{ and } x \in B\}$

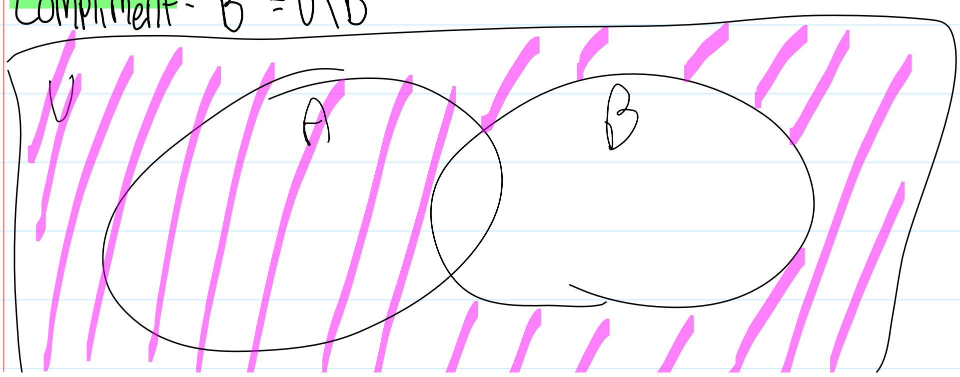


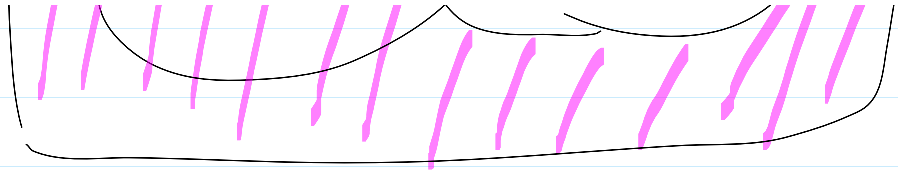
Disjoint - If $A \cap B = \emptyset$, then A and B are disjoint.

Difference - $A \setminus B = \{x;\ x \in A \text{ and } x \notin B\}$



Compliment - $B^c = U \setminus B$





Universal Set U - A set containing all elements being discussed.

Cartesian Products and ordered Pairs:

As $\{x,y\}$ and $\{y,x\}$ are the same, if we want to keep track of the order, instead of $\{x,y\}$, we instead use the **ordered Pair** (x,y)

$(x,y) \neq (y,x)$ unless $x=y$.

1st Element 2nd Element/component.

For example, using cartesian coordinates the Plane can be thought of as the set of all ordered Pairs of real numbers.

Cartesian Product of A, B - The set of all ordered Pairs,

$A \times B = \{(x,y);\, x \in A,\, y \in B\}$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y);\, x,y \in \mathbb{R}\}$

↳ Points of a Plane (2 dimensional)

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z);\, x,y,z \in \mathbb{R}\}$

↳ Points of 3D Space.

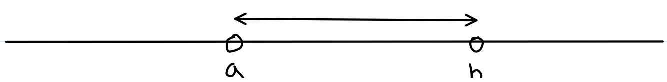
Intervals:

Intervals - Subsets of real numbers $\mathbb{R}$.

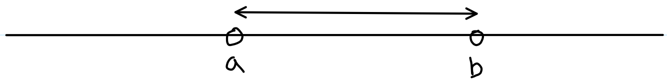
Bounded Intervals - Elements are real numbers between two numbers $a,b \in \mathbb{R}$ called end Points of the interval.

Unbounded Intervals - End Points are extended real numbers (includes infinity).

Open Interval - $(a,b) = \{x \in \mathbb{R};\ a < x < b\}$

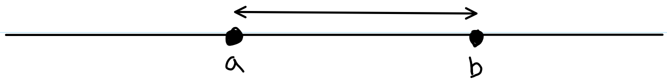


NOTE: Its notation clashes with ordered Pairs. Sometimes fixed with $]a,b[$



Clashes with ordered Pairs. Sometimes fixed with $]a,b[$

Closed Intervals - $[a,b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$ (Inclusive)



Half-Open Intervals - $(a,b]$, $[a,\infty)$

$\mathbb{R} = (-\infty, \infty)$ (Does not include infinities)

Extended Real Numbers, $\overline{\mathbb{R}} = \{-\infty, \infty\} \cup \mathbb{R}$

Logical Symbols:

Logical Implication	$\Rightarrow$	LHS implies RHS // if LHS then RHS
Reverse Logical Implication	$\Leftarrow$	RHS implies LHS
Logical Equivalence	$\Leftrightarrow$	LHS if and only if RHS
Universal Quantifier	$\forall$	For each // For all
Existential Quantifier	$\exists$	There exists
	$!\exists$	Exists only one

Def^n: A function from a set A to a set B is a rule that assigns to each element of A exactly one element of B.

$f: A \to B$

Rule (f), Domain (A), Codomain (B)

$x \mapsto f(x)$

Preimage of y [if $y = f(x)$] — 'maps to' — Image of x - Value of f at x

Two functions are equal if their domain, codomain and rule (or graph) are the same.

Image/Range of function - Subset of codomain whose elements are the images via f for all elements in the domain.

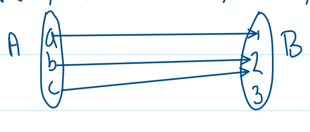
$f(A) = \{f(x) ; x \in A\}$

Defining Functions:

① Defined by listing cases

$A = \{a, b, c\}$ $\quad B = \{1, 2, 3\}$

$f(a) = 1 \quad f(b) = f(c) = 2$



(Range - $f(A)$, Codomain - B)

② Defined by formula

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^3 - 2$ $\quad$ OR $\quad x \mapsto x^3 - 2$

③ Defined by Piecewise Function

$S: \mathbb{R} \to \mathbb{R}$ by $S(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

Sign Function

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Modulus Function

Graphs of Functions:

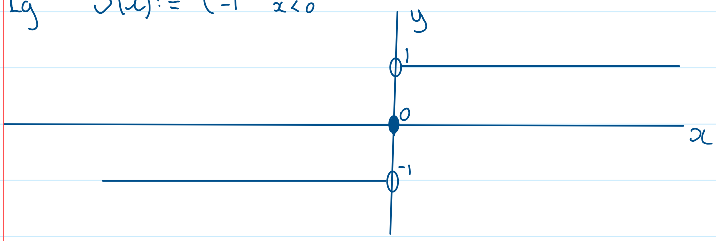
The graph of the function, Γ_f - the subset of the cartesian product $A \times B$, which itself is a subset of $\mathbb{R}^2$.

The graph of the function, is - the subset of the Cartesian Product $A \times B$,
Which itself is a Subset of $\mathbb{R}^2$.

$\Gamma f = \{(x,y) \in A \times B;\ y = f(x)\}$ where $f: A \to B$

Capital Gamma

Eg $S(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$



Variables

Eg 2 $g: \mathbb{R}^2 \to \mathbb{R}$ $g(x,y) := x^2 + y^2$

Can be thought of as a subset of $\mathbb{R}^3$, $\Gamma g = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}$
and therefore plotted as a surface in 3D space.

Restriction:

Restriction - a smaller domain, The codomain will remain the same, but range will change.

If $f: A \to B$ and $S \subseteq A$,
The restriction of f to S is $f|_S : S \to B$ $x \mapsto f(x)$

$$[f|_S(x) = f(x) \quad \forall x \in S]$$

Eg $f: \mathbb{R} \to \mathbb{R}$ $x \mapsto x$ $g: \mathbb{R} \to \mathbb{R}$ $x \mapsto |x|$

$f \neq g$ however $f|_{[0,\infty)} = g|_{[0,\infty)}$

Composition:

$f: A \to B$ $g: C \to D$

$f(a)$ Must be a subset of C for valid composition $g \circ f(x)$

$g \circ f : A \to D$ $x \mapsto g \circ f(x) = g(f(x))$ $\forall x \in A$

Domain of f Codomain of g

Eg $f: \mathbb{R} \to \mathbb{R}$ $g: \mathbb{R} \to \mathbb{R}$ $f(x) := x + 2$ $g(x) := x^2$

$f \circ g(x) = f(g(x)) = f(x^2) = x^2 + 2$

$g \circ f(x) = g(f(x)) = g(x+2) = (x+2)^2$

Composition is not Commutative
i.e. $f \circ g(x) \neq g \circ f(x)$ other than specific f, g cases.

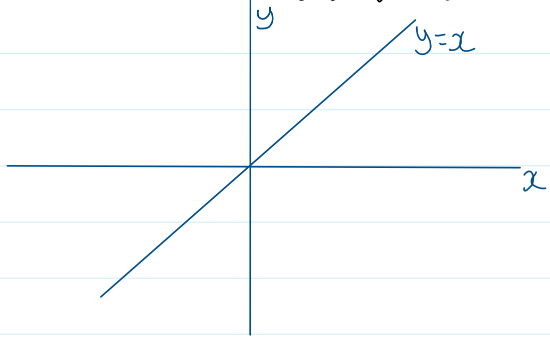
Identity:

Identity:

Identity Function $id_A: A \to A$, $id_A(x) = x \quad \forall x \in A$

$id_A \circ f = f \circ id_A = f$

If $f: A \to B$ and $S \subseteq A$ then $f|_S = f \circ id_S$



Constant function $f: A \to B$, $f(x) = b \quad \forall x \in A$ (image has 1 element)

Inverse of functions:

Def^n: $f:A\to B$ is invertible if $\exists g: B\to A$ such that $\forall x\in A$ and $y\in B$
$f(x)=y \Leftrightarrow g(y)=x$

NOTE: 'real-valued inverse' on next page has different definition.

Def^n: If $f:A\to B$ is invertible, then g is the inverse of f, $g=f^{-1}$

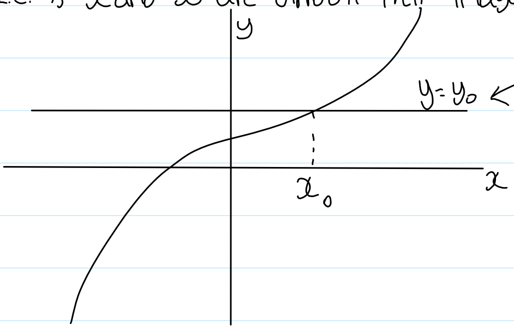
$$\left.\begin{array}{l} f\circ g(y)=f(g(y))=f(x)=y \Rightarrow f\circ g=id_B \\ g\circ f(x)=g(f(x))=g(y)=x \Rightarrow g\circ f=id_A \end{array}\right\}$$ $f\circ g(y)$ and $g\circ f$ provide different identity functions.

Eg Prove $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^3$ and $g:\mathbb{R}\to\mathbb{R}$, $g(x)=\sqrt[3]{x}$ are invertible.
$$\left.\begin{array}{l} f\circ g(x)=f(g(x))=f(x^{1/3})=(x^{1/3})^3=x \\ g\circ f(x)=g(f(x))=g(x^3)=(x^3)^{1/3}=x \end{array}\right\} (A=B=\mathbb{R})$$
Hence f and g are invertible $\forall x\in\mathbb{R}$.

Inverse functions reflect over $y=x$ graphically.

Injective, Surjective and Bijective:

Injective - $f:A\to B$ $\forall x,x'\in A$, such that $x\neq x'$ then $f(x)\neq f(x')$
↳ I.e. if x and x' are different their images will be different // 1-1 function.



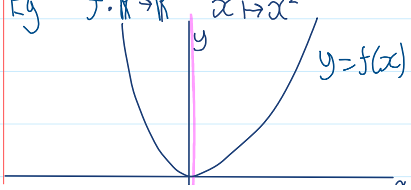
Only one intersection with each horizontal line so injective.

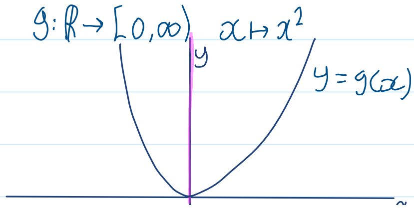
Prove injective property by $f(x)=f(y)$ then show $x=y$. Disprove with counterexample.

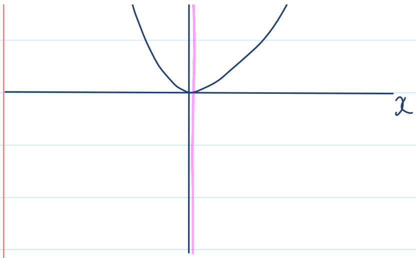
f is injective $\Leftrightarrow$ Every element of B has at most 1 preimage in A. $[f:A\to B]$

Surjective - $f:A\to B$, If $f(A)=B$.
↳ I.e. if the range and codomain are the same.

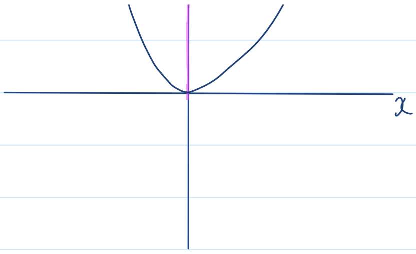
Eg $f:\mathbb{R}\to\mathbb{R}$, $x\mapsto x^2$ $\qquad$ $g:\mathbb{R}\to[0,\infty)$, $x\mapsto x^2$







Range $[0,\infty)$, codomain $\mathbb{R}$



Range $[0,\infty)$, codomain $[0,\infty)$

Here only g is surjective.

f is surjective $\iff$ Every element of B has at least 1 Preimage in A. $[f: A \to B]$

Bijective - If f is both injective and surjective, it is bijective.
↳ I.e if every element of B has exactly 1 Preimage in A $[f: A \to B]$

Eg $f: \mathbb{R} \to \mathbb{R}$ $x \mapsto x^3$.

f is bijective $\iff$ f is invertible

Image / Preimage of Sets:

$f: A \to B$

If $S \subseteq A$, then the image of S, $f(S) = \{f(x); x \in S\} \subseteq B$

If $T \subseteq B$, then the Preimage of T, $f^{-1}(T) = \{x \in A : f(x) \in T\} \subseteq A$

Taking the Preimage of a set in general is not the inverse operation to taking the image of a set unless the function satisfies additional Properties (bijective).

$f: \mathbb{R} \to \mathbb{R}$ $x \mapsto x^2$

$f([9,10]) = [81,100]$ ← Image

$f^{-1}([81,100]) = [-10,-9] \cup [9,10]$ ← Preimage

$g: \mathbb{R} \to \mathbb{R}$ $x \mapsto \sqrt{x}$

$g^{-1}([-10,-9] \cup [9,10]) = [81,100]$ ← Preimage

$g([81,100]) = [9,10] \subseteq [-10,-9] \cup [9,10]$

↑ Image.

- If $S \subseteq A$, then $S \subseteq f^{-1}(f(S))$ (left example)
- f is injective $\iff f^{-1}(f(S)) = S$ $\forall S \subseteq A$
- If $T \subseteq B$ then $f(f^{-1}(T)) \subseteq T$ (right example)
- f is surjective $\iff f(f^{-1}(T)) = T$ $\forall T \subseteq B$

Real-valued functions of a real variable

$f : A \to B$ where $A \subseteq \mathbb{R}$ then f is a function with a real variable.

$x \mapsto f(x)$ — $B \subseteq \mathbb{R}$ then f is a real-valued function.

x → real variable; $f(x)$ → real-valued

↳ With any $f : A \to B$ we can construct a new function $\tilde{f} = id_{\mathbb{R}} \circ f : A \to \mathbb{R}$ which has codomain $\mathbb{R}$. Hence the analysis works with other functions. Easier to assume codomain always $\mathbb{R}$.

Γ_f is the subset of the plane $\mathbb{R}^2$ that is the graph of the function.

$\Leftarrow\!\Rightarrow$ Every vertical line intersects Γ_f in at most 1 point.

Dirichlet Function - $d : \mathbb{R} \to \mathbb{R}$ $\quad d(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

Difficult to graph.

Domain convention

Domain convention/Natural Domain - The domain is the largest subset of $\mathbb{R}$ for which the given expression Makes s (when not specified).

For real valued functions we also require that its codomain is $\mathbb{R}$ (by definition).

Ex ① $f(x) = \dfrac{x^2+1}{x-2}$

Assume real variable function, so by domain convention Domain of $f = \mathbb{R} \setminus \{2\}$ and codomain of $f = \mathbb{R}$.

② $g(x) = \sqrt{x+2}$

Domain of $g = [-2, \infty)$

③ $h(x) = \sqrt[n]{x} = x^{1/n}$ $\quad (n \in \mathbb{N})$

Domain: $\begin{cases} n \text{ is even} & [0, \infty) \\ n \text{ is odd} & \mathbb{R} \end{cases}$

④ $\tan x = \dfrac{\sin x}{\cos x}$ $\quad$ Domain $\mathbb{R} \setminus \{\frac{\pi}{2} + n\pi\}$ $\quad n \in \mathbb{Z}$

⑤ $\cot x = \dfrac{\cos x}{\sin x}$ $\quad$ Domain $\mathbb{R} \setminus \{n\pi\}$ $\quad n \in \mathbb{Z}$

Real-Valued Inverse Function

$f : A \to \mathbb{R}$ $\qquad g : B \to \mathbb{R}$ $\qquad A, B \subseteq \mathbb{R}$

If $f(A) = B$ and $g(B) = A$

AND $f(x) = y \iff g(y) = x \quad \forall x \in A, y \in B$

Then g is the real-valued inverse of f $\quad (g = f^{-1})$

With our previous definition, if we have a real-valued function whose inverse is also a real-valued function (and we are using the assumption the codomain is $\mathbb{R}$) then its domain, codomain and image would all have to be $\mathbb{R}$. Hence this adjusted definition relaxes that constraint.

'Inverse' and 'real-valued inverse' is the same thing if only considering graphs and you forget codomains (or f is surjective).

Let $\hat{f} = id_{f(A)} \circ f : A \to f(A)$, the function with the same graph as

f, but whose codomain is the image/range of f (so surjective). $\tilde{g}$ similarly. Then g is the 'real-valued inverse' of f and $\tilde{g}$ is the inverse of $\tilde{f}$. Proof todo

Proposition: f has a real-valued inverse $\iff f$ is injective   Proof as exercise

Example: Let $a \in (0,\infty)$

Base-a exponential function, $\exp_a : \mathbb{R} \to \mathbb{R}$ $\exp_a(x) = a^x$

If $a=1$, then $\exp_1(x) = 1^x = 1$ $\forall x \in \mathbb{R}$ so not injective. (from Prop. no inverse)

If $a \neq 1$ then $\exp_a(x) = a^x$ is injective with image $(0,\infty)$.

Then its (real-valued) inverse is $g(x) = \log_a x$ (base-a logarithm function)

$(0,\infty) \to \mathbb{R}$

$\log_a(a^x) = x$ $a^{\log_a y} = y$

An inverse graph is obtained by reflecting the graph of f along the line $\{(x,y) \in \mathbb{R}^2 : x = y\}$.

Arithmetic Operations of functions

$(+,-,\times,\div)$

Let $f: A \to \mathbb{R}$ $g: A \to \mathbb{R}$ $A \subseteq \mathbb{R}$ $\lambda \in \mathbb{R}$

- Sum – $(f+g)(x) = f(x) + g(x)$
- Difference – $(f-g)(x) = f(x) - g(x)$
- Product (2 functions) – $(\lambda f)(x) = \lambda f(x)$
- Product (function + scalar) – $(f \cdot g)(x) = f(x) \cdot g(x)$

$\Bigg\}$ $\forall x \in A$

- Quotient – $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\forall x \in D$ where $D = \{x \in A : g(x) \neq 0\}$

Examples

Monomial – A function from $\mathbb{R} \to \mathbb{R}$ of the form $x \mapsto ax^n$ $n \in \mathbb{N}_0$ and $a \in \mathbb{R}$. This is product of a with function $x \mapsto x^n$, the latter is the product of the identity function with itself.

Polynomial – The sum of finitely many monomials. and coefficients.

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

coefficients (a_0)   degree n (if $a_n \neq 0$)

Rational Functions – The quotient of 2 polynomials.

$$g(x) = \frac{a_0 + a_1 x + \dots + a_n x^n}{b_0 + b_1 x + \dots + b_m x^m} \qquad m,n \in \mathbb{N}$$

Domain of $g = \{x \in \mathbb{R} ; b_0 + b_1 x + \dots + b_m x^m \neq 0\}$

Elementary Functions

Elementary Functions

Elementary Function: A function of a single variable, defined with arithmetic operations, and composition of finitely many of the following basic functions:

- Constant function: $f(x)=a$
- Identity function $f(x)=x$
- Power function $f(x)=x^n$ $\quad n \in \mathbb{N}$
- Exponential function $f(x)=a^x$ $\quad a>0$
- Trig functions
- Hyperbolic trig functions
- Inverses of the above if they exist.

$$f(x)=\frac{e^{\tan x}}{1+x^2+\sin x}\arccos\left(\sqrt{1+(\log_a x)^{10}}\right)$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x<0 \end{cases} = \sqrt{x^2} \quad \left(\text{so composition}\right)$$

are both elementary functions following this definition

Absolute Value, inequalities, sign analysis:

$\forall x, y \in \mathbb{R}$

$x \leq |x|$ (consider two cases)

Triangle Inequality - $|x+y| \leq |x|+|y|$

Triangle because geometric interpretation, but better understanding needed (some in lecture notes)

Proof: If $x+y \geq 0$ then $|x+y| = x+y \leq |x|+|y|$

If $x+y<0$ then $|x+y| = -(x+y) = (-x)+(-y) \leq |-x|+|-y| = |x|+|y|$

Reverse Triangle Inequality - $\big||x|-|y|\big| \leq |x-y|$

$|xy|=|x||y|$

Proof as exercise.

Solving Inequality

Ex: Solve $x^3-3x^2+2x>0$

↳ determine all $x \in \mathbb{R}$ such that the inequality holds.

Sol: $x^3-3x^2+2x = x(x-1)(x-2)$ $\quad$ 1. Factorise

2. **Sign Analysis**

	$x<0$	$x=0$	$0<x<1$	$x=1$	$1<x<2$	$x=2$	$x>2$
x							

	$x<0$	$x=0$	$0<x<1$	$x=1$	$1<x<2$	$x=2$	$x>2$
x	$-$	0	$-$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$		0	$+$	$+$	$+$
$x-2$	$-$	$-$	$+$	$-$	$-$	0	$+$
$x(x-1)(x-2)$	$-$	0		0	$-$	0	$+$

Hence $\{x ; x^3-3x^2+2x>0\} = (0,1) \cup (2,\infty)$

Ex2: Solve $\dfrac{x^2-3x+2}{x^3+x} \geq 0$

Sol2: Factorise to $\dfrac{(x-1)(x-2)}{x(x^2+1)} \geq 0$ ← x^2+1 always positive

	$x<0$	$x=0$	$0<x<1$	$x=1$	$1<x<2$	$x=2$	$2<x$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
x	$-$	0	$+$	$+$	$+$	$+$	$+$
x^2+1	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\dfrac{x^2-3x+2}{x^3+x}$	$-$	n.d.	$+$	0	$-$	0	$+$

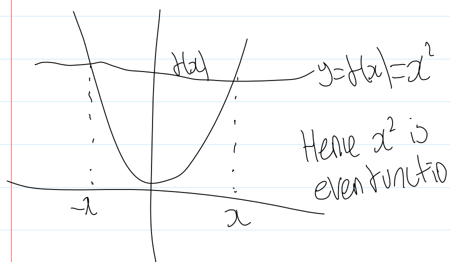
Hence $(0,1] \cup [2,\infty)$

Parity and Periodicity:

Parity - Even/odd Function $f: A \to \mathbb{R}$ $\forall x \in A$, then $-x \in A$

Even function - If $f(-x) = f(x)$ and

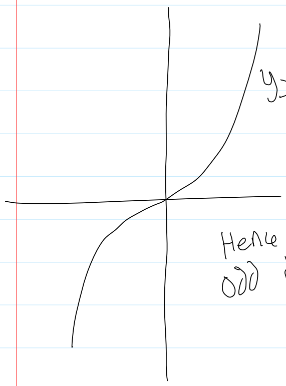
odd Function - If $f(-x) = -f(x)$ and



Hence x^2 is even function.

Even function $\Leftrightarrow$ its graph is symmetric in the vertical axis

$f(-x) = (-x)^2 = x^2 = f(x)$ hence even.



Hence x^3 is odd function.

Odd functions $\Leftrightarrow$ its graph is symmetric with respect to the origin.

$\hookrightarrow$ Rotational symmetry by $180°$.

$f(-x) = (-x)^3 = -x^3 = -f(x)$ hence odd

Periodic - If there exists a $\mathbb{R}$ $\omega > 0$ such that, $\forall x \in A$, we have $x + \omega \in A$ and $f(x + \omega) = f(x)$. (For $f: A \to \mathbb{R}$ for $A \subseteq \mathbb{R}$).

ω is the Period of f, f is ω-Periodic.

If f has a minimum period ω, then ω is the fundamental period.

A function is ω Periodic $\Leftrightarrow$ Its graph is invariant under a horizontal translation by a distance ω.

Eg $h(x) = 2\cos(37x) - 2$ has fundamental Period $2\pi/37$.

05: Real Valued Functions Continued

04 October 2022 14:03

Monotonicity

$f: A \subseteq \mathbb{R}$

Increasing	$x \leq x' \Rightarrow f(x) \leq f(x')$
Strictly Increasing	$x < x' \Rightarrow f(x) < f(x')$
Decreasing	$x \leq x' \Rightarrow f(x) \geq f(x')$
Strictly Decreasing	$x < x' \Rightarrow f(x) > f(x')$
Monotone	Increasing or decreasing

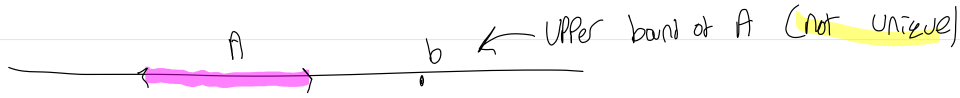
Eg $\sin(x)$ is neither increasing nor decreasing on $\mathbb{R}$, but strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

So Monotone functions have a derivative with a constant sign/equation with no roots.
Strictly increasing/decreasing $\Rightarrow$ Function is injective.

Boundedness

$A \subseteq \mathbb{R}$

- Upper Bound of A - $b \in \mathbb{R}$, such that $x \leq b \ \forall x \in A$



- A is bounded above if A has an upper bound.
- Can define a lower bound and bounded below similarly.

Bounded Sets

Bounded - A is both bounded above and below.
Unbounded - A is not bounded.

Maximum and Minimum

Max A: $x \in A$, such that $y \leq x \ \forall y \in A$
(maximum of A)

Min A: $x \in A$, such that $y \geq x \ \forall y \in A$
(minimum of A)

Can only have at most 1 Max and 1 Min.

Example

$A = [2, 3]$
Max $A = 3$
Min $A = 2$

$A\ (2, 3]$
Max $A = 3$
Min $A =$ does not exist

Supremum and Infimum

Supremum - The least upper bound,

$A \subseteq \mathbb{R}$

$\sup A = \min\{b \in \mathbb{R} : x \leq b \ \forall x \in A\}$

Infimum - The greatest lower bound,

$A \subseteq \mathbb{R}$

$\inf A = \max\{b \in \mathbb{R} ; b \leq x \ \forall x \in A\}$

If $\max A$ exists, then $\sup A = \max A$. Similarly if $\min A$ exists, $\inf A = \min A$.
However sets without a max or min may still have a sup and inf.

Eg $A = (2,3)$, $\sup A = 3$, $B = (5,7]$, $\sup B = 7$

Completeness Axiom

Completeness Axiom - "If A is bounded above, then a $\sup A$ exists."
"If A is bounded below, then A has an infimum
For $A \subseteq \mathbb{R}$ where A is nonempty.

Informally, on the real line there is no gaps, (completeness Property)

No Proof needed, beyond course depth.

Ex, $A = \{x \in \mathbb{Q}, x^2 < 2\}$ $\quad -2 < x^2 < 2 \ \forall x \in A$ $\quad$ So can't only consider $\mathbb{Q}$

No max or min, but must have Supremum and Infimum.

$\sup A = \sqrt{2} \notin \mathbb{Q}$ $\quad \sqrt{2} \in \mathbb{R}$ by the Completeness Axiom

Above definition only deals when nonempty and bounded above or below.

- If unbounded above, $\sup A = \infty$
- If unbounded below, $\inf A = -\infty$
- If A empty, we say $\sup A = -\infty$ and $\inf A = \infty$

Boundedness in the Case of functions:

Function f is said to be bounded, unbounded, bounded above, unbounded above etc
if its image $f(A)$ is bounded, unbounded, ... respectively.

A function $f : [a,b] \to \mathbb{R}$ is bounded if there exists $m, M \in \mathbb{R}$ such that if its image $f(A)$ is bounded, unbounded, ... respectively.

A function $f:[a,b]\to\mathbb{R}$ is bounded if there exists $m, M \in \mathbb{R}$ such that
$m \le f(x) \le M \quad \forall\ x \in [a,b]$

Let $S \subseteq \mathbb{R}$,

- $\min_S f = \min_{x\in S} f(x) = \min\{f(x);\ x \in A\}$
- $\max_S f = \max_{x\in S} f(x) = \max\{f(x);\ x\in A\}$
- $\inf_S f = \inf_{x\in S} f(x) = \inf\{f(x);\ x\in A\}$
- $\sup_S f = \sup_{x\in S} f(x) = \sup\{f(x);\ x\in A\}$

or if $S = A$ (the whole domain of f) we simply write $\sup f$, $\inf f$ etc.

$\max_{\mathbb{R}} \sin x = \sup_{\mathbb{R}} \sin(x) = 1$ ← Bounded $[-1,1]$

$\min_{\mathbb{R}} e^x$ does not exist $\inf_{\mathbb{R}} e^x = 0$

$\max_{\mathbb{R}} \arctan x$ does not exist, $\sup_{\mathbb{R}} \arctan x = \pi/2$

$f(x) = x^2 \quad f:\mathbb{R}\to\mathbb{R}$

$\min f = \inf f = 0 \qquad \sup f = \infty$, $\max f$ does not exist.

<u>Maximum/Minimum Points</u>

(Global) Maximum Point- $x_0 \in A$, $f(x_0) = \max f$

(Global) Minimum Point- $x_0 \in A$, $f(x_0) = \min f$

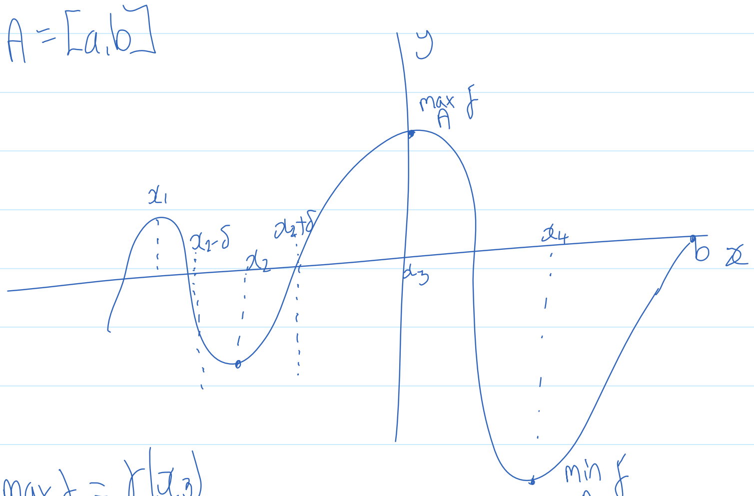
Local Maximum Points- $x_0 \in A$, $f(x_0) = \max_{A\cap(x_0-\delta,\, x_0+\delta)} f$ where there exists any $\delta > 0$.

Local Minimum Points- $x_0 \in A$, $f(x_0) = \min_{A\cap(x_0-\delta,\, x_0+\delta)} f$

$A = [a,b]$

y

$A = [a,b]$



$\max f = f(x_3)$
$\min f = f(x_4)$
local max $= x_1, x_3$ as $\max_{(-5,-4)} f = f(x_1)$
local min $= x_2, x_4$

arbitrary points, showing local is a max between an interval.

06: Limits Definition I

05 October 2022 10:04

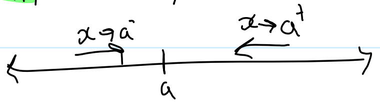
$f: A \to \mathbb{R}$ $\quad A \subseteq \mathbb{R}$

$\lim_{x \to a} f(x) = L$ OR $f(x) \to L$ as $x \to a$.

$a, L \in \overline{\mathbb{R}}$

↳ "$f(x)$ tends to L as x tends to a"

one-sided limit, $x \to a^+, x \to a^-$, $a \in \mathbb{R}$



Types of limits:

$\lim_{x \to a} f(x) = L$

	$L \in \mathbb{R}$	$L = \infty$	$L = -\infty$
$a \in \mathbb{R}$	(a, L) Ex iv	(a, ∞) Ex v	$(a, -\infty)$
	(a^+, L) Ex vi	(a^+, ∞)	$(a^+, -\infty)$
	(a^-, L)	(a^-, ∞)	$(a^-, -\infty)$
$a = \infty$	(∞, L) Ex i	Ex ii (∞, ∞)	$(\infty, -\infty)$ Ex iii
$a = -\infty$	$(-\infty, L)$	$(-\infty, \infty)$	$(-\infty, -\infty)$

If $A = \mathbb{N}$, this one can also be thought of as a sequence limit (in other modules, not relevant here).

We split our definition into a number of cases on whether a and L are $\mathbb{R}$ or $\pm\infty$.

Limit of $f(x)$ as x tends to $\pm\infty$

These can all be proved in a similar way:

Let $f: \Omega \to \mathbb{R}$ for some $\Omega \subseteq \mathbb{R}$. Assume Ω not bounded above/below.

i) $\lim_{(x \to \infty / x \to -\infty)} f(x) = L$ where $L \in \mathbb{R}$

If the limit is valid, $\forall \varepsilon > 0$, there exists $N \in \mathbb{R}$ such that:

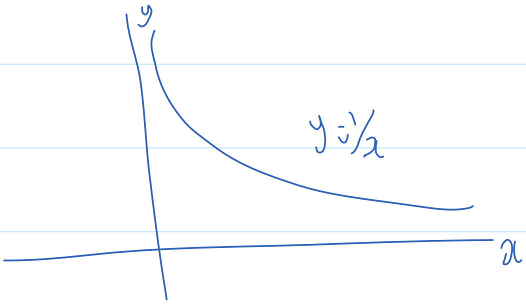
$\forall x \in \Omega$ and $x > N / x < N \Rightarrow |f(x) - L| < \varepsilon$.

$\forall x \in \Omega$ and $x > N \,/\, x < N \Rightarrow |f(x) - L| < \varepsilon$.

Explaination in words: The limit is valid if you can find a real number N (in terms of ε) that makes the statement hold. The statement being if an element in the domain is greater/less than N, then the distance between its image and the proposed limit is less than any $\varepsilon > 0$.

<u>Ex i:</u> Prove, using the definition, that $\lim_{x\to\infty} \frac{1}{x} = 0$.

<u>Pf i:</u>



$f(x) = 1/x$, $\Omega = \mathbb{R}\setminus\{0\}$ (domain convention), and $L = 0$.

- Ω is indeed unbounded above
- We need to show $\forall \varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $\forall x \in \mathbb{R}\setminus\{0\}$ and $x > N \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$.

Let an arbitrary $\varepsilon > 0$ be given.

Note that $\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| = \frac{1}{|x|}$

So for $\frac{1}{|x|} < \varepsilon \Leftrightarrow |x| > 1/\varepsilon$.

2. This decision designed to get ε

If we take $N = 1/\varepsilon \in \mathbb{R}$

Then $\forall x \in \mathbb{R}\setminus\{0\}$ and $x > N$ we have $\left|\frac{1}{x} - 0\right| = \frac{1}{|x|} < \frac{1}{N} = \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon$

flipped numerator, denominator and sign.

What about

Flipped numerator, denominator and sign

1. (Rearrange $x > N$ to look like inequality) What about the modulus?

In other words, for every given $\varepsilon > 0$, we have found an $N \in \mathbb{R}$ such that $x > N \Rightarrow |\frac{1}{x} - 0| < \varepsilon$ $(x \in \Omega)$ holds and so by definition proves that $\lim_{x\to\infty} \frac{1}{x} = 0$.

ii) $\lim_{x\to\infty / x\to-\infty} f(x) = \infty$

If the limit is valid, $\forall M > 0$, there exists $N \in \mathbb{R}$ such that:

$\forall x \in \Omega$ and $x > N / x < N \Rightarrow f(x) > M$

Explaination in words: The limit is valid if you can find a real number N (in terms of M) that makes the statement hold. The statement being if an element in the domain is greater/less than N, then its image will always be greater than an arbitrary M $(\forall M > 0)$.

Ex ii: Prove $\lim_{x\to\infty} \sqrt{x} = \infty$ by the definition.

Pf ii: $f(x) = \sqrt{x}$, $\Omega = [0, \infty)$, $L = \infty$

- Ω is unbounded above
- We need to show that $\forall M > 0$, there exists $N \in \mathbb{R}$ such that $\forall x \in \Omega$ and $x > N \Rightarrow \sqrt{x} > M$.

Let an arbitrary $M > 0$ be given.

Note that $\sqrt{x} > M \Rightarrow x > M^2$

2. This decision designed to get M.

If we take $N = M^2 \in \mathbb{R}$
Then $\forall x \in \Omega$ and $x > N$ we have $\sqrt{x} > \sqrt{N} = \sqrt{M^2} = M$

1. (Rearrange $x > N$ to look like inequality)

In other words, for every given $M > 0$, we have found an $N \in \mathbb{R}$ such that $x > N \Rightarrow \sqrt{x} > M$ $(x \in \Omega)$ holds and so by definition proves that $\lim_{x \to \infty} \sqrt{x} = \infty$

iii) $\lim_{(x \to \infty / x \to -\infty)} f(x) = -\infty$

If the limit is valid, $\forall M > 0$, there exists $N \in \mathbb{R}$ such that:

$\forall x \in \Omega$ and $x > N \,/\, x < N \Rightarrow f(x) < -M$

Explaination in words: The limit is valid if you can find a real number N (in terms of M) that makes the statement hold. The statement being if an element in the domain is greater/less than N, then its image will always be <u>less</u> than an arbitrary M $(\forall M > 0)$.

<u>Ex iii:</u> Prove $\lim_{x \to \infty} (x - x^2) = -\infty$ by the definition.

Pt iii: $f(x) = x - x^2$, $\Omega = \mathbb{R}$, $L = -\infty$

- Ω is unbounded below

We need to show that $\forall M > 0$, there exists $N \in \mathbb{R}$ such that $\forall x \in \Omega$ and $x < N \Rightarrow x - x^2 < -M$

Note from $x < N$ we deduce that $x^2 < N^2$ (so $-x^2 < N^2$)
hence $x - x^2 < N^2 + N <$

Let an arbitrary $M > 0$ be given.
If we take $N = M$
Then $\forall x \in \Omega$ and $x < N$ we have $x - x^2 <$

In other words, for every given $M > 0$, we have found an $N \in \mathbb{R}$ such that $x < N \Rightarrow x - x^2 < -M$ $(x \in \Omega)$ holds and so by definition proves that $\lim_{x \to \infty} x - x^2 = -\infty$

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Accumulation Points:

Let $\Omega \subseteq \mathbb{R}$.

We say that real number a is an accumulation point of Ω if, $\forall \delta > 0$ there exists $x \in \Omega$ such that $0 < |x-a| < \delta$

Doesn't have to accumulate on both sides? ↳ one-sided limits

↳ In other words we must be able to find elements of Ω which are arbitrarily close to a (but not a itself)

The following properties are all true:

- If $A \subseteq B \subseteq \mathbb{R}$ then any accumulation point of A is also an accumulation point of B.
- Let $\Omega \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then a is an accumulation point of Ω $\Leftrightarrow a = \inf(\Omega \cap (a, \infty))$ or $a = \sup(\Omega \cap (-\infty, a))$ ← Don't understand proofs. P31. Mainly ii
- Let $b, c \in \mathbb{R}$ with $b < c$. Every real number $a \in [b,c]$ is an accumulation point of (b,c)
- Let $b \in \mathbb{R}$. No real number $a < b$ is an accumulation point of $[b, \infty)$. Similarly no real number $a > b$ is an accumulation point of $(-\infty, b]$.
- Let Ω be any of the intervals $(b,c), [b,c), (b,c], [b,c]$. All of these have the same accumulation points.
- No real number is an accumulation point of $\mathbb{Z}$. Every real number is an accumulation point of $\mathbb{Q}$. What?

Limit of $f(x)$ as x tends to $\mathbb{R}$:

Let $f: \Omega \to \mathbb{R}$ for some $\Omega \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$ be an accumulation point of Ω.

iv) $\lim_{x \to a} f(x) = L$ where $L \in \mathbb{R}$.

If the limit is valid, $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $\forall x \in \Omega$, ← Most important

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Doesn't have to be defined at a, just around.

Ex iv: Prove by definition that $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$ $(x \neq 1)$

Pf iv
- By domain convention we see $f(x) = \frac{x^2-1}{x-1}$ has domain $\Omega = \mathbb{R} \setminus \{1\}$
- $L = 2$, $a = 1$. 1 is indeed an accumulation point of Ω. ← Need to make sure f is defined around a when $a \in \mathbb{R}$ ↳ $\forall \delta > 0, \Omega \cap \{x : 0 < |x-a| < \delta\} \neq \emptyset$

Need to show that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $\forall x \in \Omega$

$0 < |x-1| < \delta \Rightarrow \left| \frac{x^2-1}{x-1} - 2 \right| < \varepsilon$

Note that for $x \in \Omega$ $\frac{x^2-1}{x-1} - 2 = \frac{(x-1)(x+1)}{(x-1)} - 2 = x+1-2 = x-1$, hence

$0<|x-1|<\delta \Rightarrow |x-1|<\varepsilon$

For any arbitarily given $\varepsilon>0$, if we take $\delta=\varepsilon$ then

$\forall x\in\Omega$ and $0<|x-1|<\delta$ we have $\left|\frac{x^2-1}{x-1}-2\right|<\varepsilon$

By definition this proves $\lim_{x\to1}\frac{x^2-1}{x-1}=2$.

V) $\lim_{x\to a} f(x)=\infty$

If the limit is valid, $\forall M>0$ there exists $\delta>0$ such that $\forall x\in\Omega$,

$0<|x-a|<\delta \Rightarrow f(x)>M$

Ex V: Prove by definition that $\lim_{x\to0}\frac{1}{x^2}=\infty$

Pf V: • By domain convention we see $f(x)=1/x^2$ has domain $\Omega=\mathbb{R}\setminus\{0\}$

• 0 is indeed an accumulation point of Ω.

Need to show that $\forall M>0$ there exists $\delta>0$ such that $\forall x\in\Omega$,

$0<|x-0|<\delta \Rightarrow 1/x^2>M$

Note that $1/x^2>M \Leftrightarrow x^2<1/M \Leftrightarrow |x|<1/\sqrt{M}$

For any arbitarily given $M>0$, if we take $\delta=1/\sqrt{M}$ then

$\forall x\in\Omega$ and $0<|x-0|<\delta$ we have $1/x^2=1/|x|^2>1/\delta^2=\frac{1}{(1/\sqrt{M})^2}=\frac{1}{1/M}=M$ or $1/x^2>M$

By definition this proves $\lim_{x\to0}1/x^2=\infty$

Vi) $\lim_{x\to a} f(x)=-\infty$

If the limit is valid, $\forall M>0$ there exists $\delta>0$ such that $\forall x\in\Omega$,

$0<|x-a|<\delta \Rightarrow f(x)<-M$

Existence of limits:

Limits need not exist.

Existence of limits:

Limits need not exist.

Example Show $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

Proof: $\frac{x}{|x|} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$ Domain $\mathbb{R}\setminus\{0\}$ (D.C.)

We can prove by contradiction. Lets assume $\lim_{x\to 0}\frac{x}{|x|}$ exists and $\lim_{x\to 0}\frac{x}{|x|} = l \in \overline{\mathbb{R}}$.

We first note that $\left|\frac{x}{|x|}\right| \leq 1$ for all $x \in \Omega$ so $l \neq \pm\infty$. For $\varepsilon = 1/2$ there exists $\delta > 0$ such that $0 < |x-0| < \delta$, we have $\left|\frac{x}{|x|} - l\right| < 1/2$.

If $0 < x < \delta$ then the above reads $|1-l| < 1/2$ so $1/2 < l < 3/2$

If $-\delta < x < 0$ then the above reads $|-1-l| < 1/2$ $(|1+l| < 1/2)$ so $-3/2 < l < -1/2$

Which is a contradiction as no l satisfies both. Therefore $\lim_{x\to 0}\frac{x}{|x|}$ does not exist.

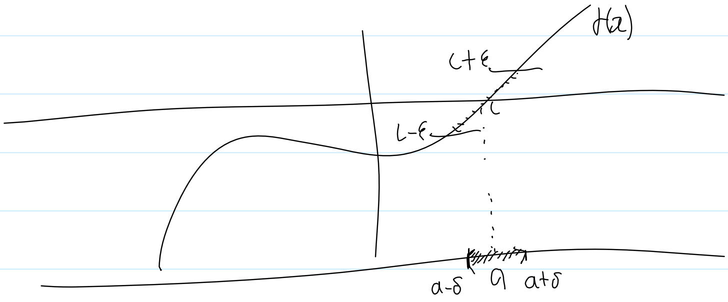
Example 2: Show $\lim_{x\to\infty} \sin x = l$ does not exist.

See answer on p34.

Uniqueness of limits:

$f:\Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}$ let $a, L_1, L_2 \in \overline{\mathbb{R}}$

$\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$ then $L_1 = L_2$



Can roughly see why, as δ shrinks the $f(x)$ values also shrink to a single value, but need to prove with mathematical detail.

Proof: We prove by contradiction. (only doing case where $a, L_1, L_2 \in \mathbb{R}$)

Assume $L_1 \neq L_2$. Then $|L_1 - L_2| > 0$. So we can take $\varepsilon = \frac{|L_1 - L_2|}{2}$

Since $\lim_{x\to a} f(x) = L_1$ there exists $\delta_1 > 0$ such that $x \in \Omega$ and $0 < |x-a| < \delta_1$ we have $|f(x) - L_1| < \varepsilon$.

Similarly since $\lim_{x\to a} f(x) = L_2$ there exists $\delta_2 > 0$ such that $x \in \Omega$ and $0 < |x-a| < \delta_2$ we have $|f(x) - L_2| < \varepsilon$.

If we take $\delta = \min(\delta_1, \delta_2)$ then $0 < |x-a| < \delta$ $x \in \Omega$ we have $|f(x) - L_1| < \varepsilon$ and $|f(x) - L_2| < \varepsilon$.

Triangle Inequality

Therefore $|L_1 - L_2| = |L_1 - f(x) - L_2 + f(x)| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon = 2\varepsilon$ so $|L_1 - L_2| < 2\varepsilon$.

$2\varepsilon = 2 \times \frac{|L_1 - L_2|}{2} = |L_1 - L_2|$

So $|L_1 - L_2| < |L_1 - L_2|$ which is a contradiction, therefore limits are unique.

One-Sided Limits of $f(x)$ as x tends to $\mathbb{R}$:

Let $f:\Omega \to \mathbb{R}$ for some $\Omega \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$ and $L \in \overline{\mathbb{R}}$

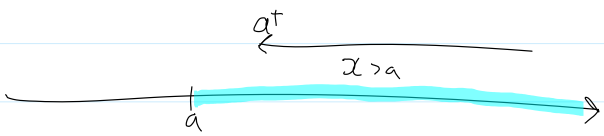
We write $\lim_{x\to a^+} f(x) = L$ whenever $\lim_{x\to a} f_{a,+}(x) = L$

where $f_{a,+}$ is the restriction $f|_{\Omega \cap (a,\infty)}$ of f to $\Omega \cap (a, \infty)$

"tends to a from above (or from the right)."

Similar definition for $\lim_{x\to a^-}$ (tending to a from below or the left).

For $\lim_{x\to a^+}$ a must be an accumulation point of $\Omega \cap (a,\infty)$. Similarly $\lim_{x\to a^-}$ of $\Omega \cap (-\infty, a)$.



Ex VI: Prove $\lim_{x\to 0^+} \frac{x}{|x|} = 1$

Pf VI: • By domain convention we see $f(x) = \frac{x}{|x|}$ has domain $\Omega = \mathbb{R} \setminus \{0\}$

• $L=1$, $a=0$. Since $\Omega \cap (0,\infty] = (0,\infty)$ we must show that $\lim_{x\to 0} f|_{(0,\infty)} = 1$.
• 0 is indeed an accumulation point of $(0,\infty)$.

Need to show that $\forall \varepsilon>0$, there exists $\delta>0$ such that for all $x\in(0,\infty)$,
$0<|x-0|<\delta \implies \left|\frac{x}{|x|}-1\right|<\varepsilon$.
Note $\frac{x}{|x|}=1 \ \forall x\in(0,\infty)$ so the above holds trivially whatever choice of $\delta>0$.
Therefore by definition this proves $\lim_{x\to 0^+} \frac{x}{|x|}=1$.

$\lim_{x\to a} f(x)=L \iff \lim_{x\to a^+} f(x)=L$ and $\lim_{x\to a^-} f(x)=L$.

Finding a limit from definition:
With the end of Ex Vi, we can now use these to verify all 15 types of limits. But what about finding one?

$\lim_{x\to a} f(x) = L$
Need all 3 of these to verify limits from the definition.
To find a limit, cannot use the definition as it is, i.e to find:
$\lim_{x\to 0} 1-e^{x}$ $\quad\lim_{x\to\infty} (1+\frac{1}{x})^2$ $\quad$ etc.

Locality of limits:
The concept of limit has a local nature (eg the limit $\lim_{x\to a} f(x)$ only depends on the behavior of $f(x)$ when x is 'close to a' (except possibly when $x=a$).
Locality - Only need to consider points in nearby neighbourhood.

$\lim_{x\to a} f(x) = \lim_{x\to a} f|_{\Omega\cap(a-\delta,\,a+\delta)}(x) \quad \forall\delta>0$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f|_{\Omega\cap(N,\infty)}(x)$ for some $N\in\mathbb{R}$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} f|_{\Omega\cap(-\infty,N)}(x)$ for some $N\in\mathbb{R}$

Proof on P38 if interested.

Ex: $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) = \begin{cases} 0 & \text{if } |x|<1 \\ \arctan x & \text{if } |x|\geq 1 \end{cases}$
By the locality of limits we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} f|_{(-1,1)}(x) = 0$

Tools to find limits

- algebra of limits
- Sandwich theorem
- continuity

Algebra of limits:

$(a, L_1, L_2 \in \mathbb{R}$ $f: \Omega \to \mathbb{R})$

$\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$

Then

Linear Rule - $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L_1 \pm L_2$

Product Rule - $\lim_{x \to a} (f(x) \times g(x)) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x) = L_1 L_2$

Quotient Rule - $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L_1}{L_2}$ Provided $L_2 \neq 0$

don't apply if one of limits don't exist.

Proof (idea) of Product Rule:

$|f(x)\, g(x) - L_1 L_2| = |f(x)\, g(x) - f(x) L_2 + f(x) L_2 - L_1 L_2|$

$\leq |f(x)| \underbrace{|g(x) - L_2|} + \underbrace{|f(x) - L_1|} |L_2|$ ← Triangle inequality

will be small as $x \to a$ by definition

$|x+y| \leq |x| + |y| \Rightarrow$

$|f(x)(g(x) - L_2) + L_2(f(x) - L_1)| \leq |f(x)(g(x) - L_2)| + |$

These 3 rules are Powerful,

Application

- $\lim_{x \to a} x = a$
- $\lim_{x \to a} b = b$

} verify by definition

$\Rightarrow \lim_{x \to a} bx = b \lim_{x \to a} x = ba$

$\lim_{x \to a} bx^2 = \lim_{x \to a} (bx) \lim_{x \to a} x = ba \times a = ba^2$

- $\lim_{x \to \infty} 1/x = 0$
- $\lim_{x \to \infty} \frac{1}{x^2} = 0$

} Verify by definition.

$2x^2 + 3x$ $\quad$ lim $\quad$ $\cancel{2x^2+3x}$ $\quad$ lim $\quad$ $2 + 3/x$

$x \to \infty \quad x$

Ex1. $\lim_{x\to\infty} \frac{2x^2+3x}{6x^2+5} = \lim_{x\to\infty} \frac{\frac{2x^2+3x}{x^2}}{\frac{6x^2+5}{x^2}} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{6+\frac{5}{x^2}}$

$\overset{\text{quotient}}{=} \frac{\lim_{x\to\infty}\left(2+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(6+\frac{5}{x^2}\right)} \underset{\text{product}}{\overset{\text{linear}}{=}} \frac{2+3\lim_{x\to\infty}\frac{1}{x}}{6+5\lim_{x\to\infty}\frac{1}{x^2}} = \frac{2+3\cdot 0}{6+5\cdot 0} = \frac{2}{6} = \frac{1}{3}$

Ex2. $\lim_{x\to a}(3x^2+2x+6) \overset{\text{linear rule}}{=} \lim_{x\to a}(3x^2) + \lim_{x\to a}(2x) + \lim_{x\to a} 6$

$= 3\left(\lim_{x\to a} x\right)\left(\lim_{x\to a} x\right) + 2\lim_{x\to a} x + 6$

Product rule + lim of constant is constant.

$= 3$

Ex3 $\lim_{x\to\infty} \frac{2x^2+6}{3x^2-2x} = \lim_{x\to\infty} \frac{2+\frac{6}{x^2}}{3-\frac{2}{x}} \overset{\text{quotient rule}}{=} \dots = \frac{2}{3}$

Ex4 $\lim_{x\to 1} \frac{x^3+2x^2+4}{x^3+6} \overset{\text{quotient rule}}{=} \frac{\lim_{x=1}(x^3+2x^2+4)}{\lim_{x=1}(x^3+6)} \overset{\text{linear}}{=} \frac{\lim_{x\to 1} x^3 + 2\lim_{x\to 1} x^2 + \lim_{x\to 1} 4}{\lim_{x\to 1} x^3 + \lim_{x\to 1} 6}$

$= \frac{1+2\cdot 1+4}{1+6} = \frac{7}{7} = 1 \quad \#$

<u>Null Limits + Absolute Value:</u>

$\lim_{x\to a} f(x) = 0 \iff \lim_{x\to a} |f(x)| = 0$

- $\big|\,|f(x)| - |c|\,\big| \le |f(x) - c|$ (Proof idea)

<u>Limits + orders:</u>

If $f(x) \le g(x) \quad \forall x \in \Omega \setminus \{a\}$

$\lim_{x\to a} f(x) = C_1 \quad \lim_{x\to a} g(x) = C_2$

Then $C_1 \le C_2$

<u>Proof:</u> Assume $a, C_1, C_2 \in \mathbb{R}$ $\quad \forall \varepsilon > 0 \ \exists \ \delta_1, \delta_2$ such that $x \in \Omega \setminus \{a\}$ and

$0 < |x-a| < \delta_1 \implies |f(x) - C_1| < \varepsilon$

$0 < |x-a| < \delta_2 \implies |g(x) - C_2| < \varepsilon$

Take $\delta = \min(\delta_1, \delta_2)$, If $0 < |x-a| < \delta$,

$\forall x \in \Omega,\ 0 < |x-a| < \delta$

$C_1 - \varepsilon < f(x)$

$\le g(x) < C_2 + \varepsilon$

$\Rightarrow L_1 - \varepsilon < L_2 + \varepsilon$

$\Rightarrow L_2 - L_2 < 2\varepsilon$

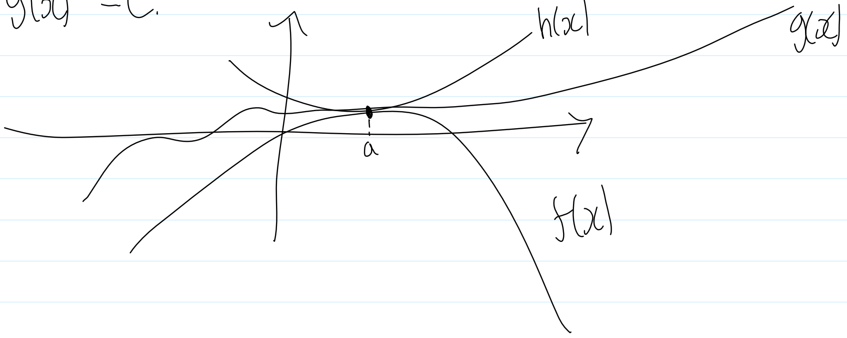
Since $\varepsilon > 0$ was arbitrary choosing, we have $L_1 - L_2 \leq 0$ as desired.

Sandwich Theorem - $f, g, h : \Omega \to \mathbb{R}$

$f(x) \leq g(x) \leq h(x) \quad \forall x \in \Omega \setminus \{a\}$

If $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

Then $\lim_{x \to a} g(x) = L$.



Proof using order convert above (see lecture notes) but from graph it is clear.

Ex $\lim_{x \to 0} x \sin\left(\frac{1}{x} + \frac{1}{x^2}\right)$

Solve using sandwich theorem.

> Since sine function is bounded ($|\sin x| \leq 1$)

$\forall x \in \mathbb{R} \setminus \{0\}$

$-|x| \leq x \sin\left(\frac{1}{x} + \frac{1}{x^2}\right) \leq |x|$

We know that $\lim_{x \to 0} x = 0$, $\Rightarrow \lim_{x \to 0} |x| = 0$ and $\lim_{x \to 0} (-|x|) = 0$

By the sandwich theorem, we get that $\lim_{x \to 0} x \sin\left(\frac{1}{x} + \frac{1}{x^2}\right) = 0$.

Ex2 $\lim_{x \to \infty} \frac{\sin x}{x}$

$|\sin x| \leq 1 \quad \forall x \in \mathbb{R}$ so $\sin x$ bounded.

Note that when $x > 0$

$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$

Also that $\lim_{x \to \infty} -\frac{1}{x} = 0$ and $\lim_{x \to \infty} \frac{1}{x} = 0$ (notable limit). hence by sandwich theorem $\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

Ex3 $\lim_{x \to -\infty} \frac{x^2(\sin x + \cos^3 x)}{(x^2+1)(x-3)}$

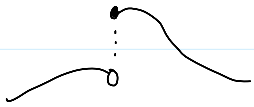
Note that $\left| \frac{\cancel{x^2}(\sin x + \cos^3 x)}{\cancel{(x^2+1)}(x-3)} \right| \leq \frac{x^2}{|(x^2+1)(x-3)|} \leq \frac{1}{|x-3|}$

$\lim_{x \to -\infty} \frac{1}{|x-3|} = \lim_{x \to -\infty} \frac{1}{x-3} = 0$ By sandwich theorem

$\lim_{x \to -\infty} \left| \frac{x^2 (\sin x) + \cos^3 x)}{(x^2+1)(x-3)} \right| = 0$ Thus $\lim_{x \to -\infty} (\dots| = 0$.

(from 'null limits + absolute value).

$f(x)$
(continuous)



$g(x)$
(discontinuous)

Continuity - $f : \Omega \to \mathbb{R}$ $\quad \Omega \subseteq \mathbb{R}$

$\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall |x-a| < \delta, x \in \Omega$. We have $|f(x) - f(a)| < \varepsilon \Rightarrow f(x)$ is continuous at $x = a$

- must be defined at $f(a)$ (unlike limits where it doesn't need to be). ie $a \in \Omega$
- a may not be an accumulation point of Ω

Ex 1 $f(x) = b \in \mathbb{R}$ constant function. is continuous proof.

$\forall \varepsilon > 0$, take $\delta = 1$ then $|x-a| < \delta$ $x \in \mathbb{R}$

$|f(x) - f(a)| = |b - b| = 0 < \cancel{\delta}\ \varepsilon$ wrote δ in lecture but should be ε?

$\Rightarrow f(x)$ is continuous at $x = a$

If $f(x)$ is continuous at every point of Ω, then f is a continuous function on Ω.

Ex 2: Prove $f(x) = x$, $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$

$\forall \varepsilon > 0$, take $\delta = \underline{\varepsilon}$ then $|x - a| < \delta\ (= \varepsilon)$ $x \in \mathbb{R}$

$|f(x) - f(a)| = |x - a| < \delta = \varepsilon$

$\Rightarrow f(x)$ is continuous at $x = a$.

$\Rightarrow f$ is continuous on $\mathbb{R}$.

Ex 3: Prove $f(x) = |x|$ $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$.

$\forall \varepsilon > 0$, take $\delta = \underline{\varepsilon}$ then $|x - a| < \delta\ (= \varepsilon)$ $x \in \mathbb{R}$

$|f(x) - f(a)| = ||x| - |a|| \leq |x - a| < \delta (= \varepsilon)$

Reverse triangle inequality: $|a - b| \geq ||a| - |b||$

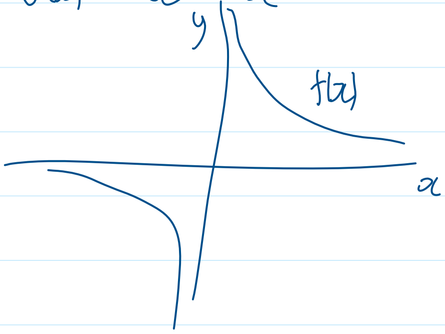
$\Rightarrow f(x)$ is continuous at $x = a$

$\Rightarrow f$ is continuous on $\mathbb{R}$.

- Polynomial functions
- Rational functions
- n-th root functions $x \mapsto \sqrt[n]{x} = x^{1/n}$
- Trigonometric functions & their inverses
  $\sin x$ $\cos x$
  $\arcsin x$ $\arccos x$
- Exponential & Logarithm
- Power Function x^a

Are all continuous on their domain (following domain convention) and their continuity does not need to be verified unless asked to.

Ex $f(x) = x^{-1} = \frac{1}{x}$



$f(x)$ is continuous on $\underbrace{\mathbb{R} \setminus \{0\}}_{\text{domain of } f}$ (not continuous on 0)

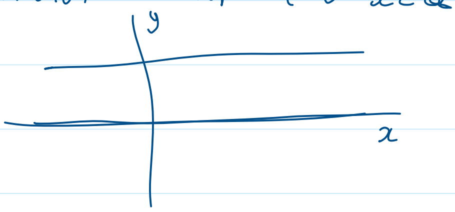
Discontinuity:

Ex: $s(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

Discontinuous at $x = 0$

continuous on $\mathbb{R} \setminus \{0\}$

Dirichlet Function $d(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$



Discontinuous on every point of $\mathbb{R}$

Continuity and limits:

Continuity and limits:

- If a is not an accumulation point of Ω, then f is continuous at a.

$\hookrightarrow \exists\, \delta$ such that $\{x : 0 < |x-a| < \delta\} \cap \Omega = \emptyset$

$\Rightarrow \{x : |x-a| < \delta\} \cap \Omega = \{a\}$

$\Rightarrow$ If $x \in \Omega$, $|x-a| < \delta$

$\Rightarrow |f(x) - f(a)| = |f(a) - f(a)| = 0$

- If a is an accumulation point of Ω, $f(x)$ is continuous at $a \iff \lim_{x \to a} f(x) = f(a)$

$\hookrightarrow$ can see by comparing definitions of limits and continuity.

Ex: Find limit of $\lim_{x \to 0} (x^{999} + 80x^3 + 100)$ ← If not using this technique would need AoL 999 times.

Polynomial functions are continuous on $\mathbb{R}$

Therefore we have $\lim_{x \to 0} (x^{999} + 80x^3 + 100) = 0^{999} + 80(0)^3 + 100 = 100.$

Ex 2: $\lim_{x \to 1} \dfrac{x^3 + 2x^2 + 4}{x^2 + 6}$

We see $(x^2 + 6)\big|_{x=1} \neq 0$ so $1 \in \Omega$ the domain of $f(x) = \dfrac{x^3 + 2x^2 + 4}{x^2 + 6}$

Therefore f is continuous at $x = 1$. Thus $\lim_{x \to 1} f(x) = f(1) = \dfrac{1^3 + 2(1)^2 + 4}{1^2 + 6} = \dfrac{7}{7} = 1.$

Algebra of continuous functions:

If f, g are continuous at a then

- $f \pm g$
- $f \cdot g$
- f/g $(g(a) \neq 0)$

are all continuous at a.

Pf: $\lim_{x \to a} (f(x) g(x)) \overset{A.O.L}{=} (\lim_{x \to a} f(x))(\lim_{x \to a} g(x)) \overset{\text{Continuity}}{=} f(a) g(a) \Rightarrow f(x) g(x)$ is cont. at $x = a$.

Ex: $\sin x + \cos^2 x$ is continuous

Pf: $\sin x$ and $\cos x$ is continuous

$\Rightarrow \cos^2 x$ is continuous by product rule

$\Rightarrow \sin x + \cos^2 x$ is continuous by linear rule.

11: Continuity II

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Composition & Limits:

$f: \Omega \to \mathbb{R}$, $g: A \subseteq \mathbb{R}$, $f(\Omega) \subseteq A$

- $\lim_{x \to a} f(x) = c$ $c \in A$
- g is continuous at c

Then: $\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right) = g(c)$

can take limit to inside in this circumstance.

Proof: Since g is continuous at c, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|y - c| < \delta \Rightarrow |g(y) - g(c)| < \varepsilon$

Since $\delta > 0$, and $\lim_{x \to a} f(x) = c$. $\exists \alpha > 0$ such that

$0 < |x - a| < \alpha \Rightarrow |f(x) - c| < \delta$

Therefore we have $|g(f(x)) - g(c)| < \varepsilon$

Provided $0 < |x - a| < \alpha$. Thus we conclude that $\lim_{x \to a} g(f(x)) = g(c)$

Prop: (1) If f is continuous at a

g is continuous at $f(a)$

$\Rightarrow g \circ f$ is continuous at a

(2) f, g are both continuous

$\Rightarrow f \circ g$ is continuous.

Ex: $\lim_{x \to 0} \arctan(\log(1 + x^2))$

Sol: $\arctan x$ and $\log(x)$ and $1 + x^2$ are continuous, therefore $\arctan(\log(1 + x^2))$ is also continuous. Also 0 is in its domain.

Thus $\lim_{x \to 0} \arctan(\log(1 + x^2)) = \arctan(\log(1 + 0^2)) = \arctan(\log 1) = \arctan 0 = 0$.

Change of Variable: $f: \Omega \to \mathbb{R}$, $g: A \to \mathbb{R}$

$\lim_{x \to a} f(x) = b$ $f(\Omega) \subseteq A$. $f(x) \neq b$, $\forall x \in \Omega \setminus \{a\}$

If $\lim_{x \to b} g(x) = c$ Then

$\lim_{x \to a} g(f(x)) = \lim_{y \to b} g(y) = c$.

Used for making more complicated limits simpler.

Pf: Set $\hat{g}(x) = \begin{cases} g(x) & x \neq b \\ c & x = b \end{cases}$ then $\hat{g}$ is continuous at b.

$$\Rightarrow \lim_{x\to a} g(f(x)) \underbrace{=}_{f(x)\neq b,\ \forall x\in\Omega\setminus\{a\}} \lim_{x\to a} \hat{g}(f(x)) \underbrace{=}_{\text{continuity of } \hat{g} \text{ at } b} \hat{g}\left(\lim_{x\to a} f(x)\right) \underbrace{=}_{\lim_{x\to a} f(x)=b} \hat{g}(b) \underbrace{=}_{\text{definition of } \hat{g}} C.$$

Ex: $\lim_{x\to\infty}\left(\sqrt{x+4}-\sqrt{x+1}\right)$

Sol: $\sqrt{x+4}-\sqrt{x+1} = \dfrac{(\sqrt{x+4}-\sqrt{x+1})(\sqrt{x+4}+\sqrt{x+1})}{\sqrt{x+4}+\sqrt{x+1}} = \dfrac{x+4-x+1}{\sqrt{x+4}+\sqrt{x+1}} = \dfrac{3}{\sqrt{x+4}+\sqrt{x+1}}$

We note that $\lim_{x\to\infty}(\sqrt{x+4}+\sqrt{x+1}) = \infty$ and therefore $\lim_{x\to\infty}(\sqrt{x+4}-\sqrt{x+1})$

$= \lim_{x\to\infty} \dfrac{3}{\sqrt{x+4}+\sqrt{x+1}} = \lim_{y\to\infty} \dfrac{3}{y} = 3\cdot 0 = 0$

coefficients not matching

Ex2: $\lim_{x\to\infty}\left(\sqrt{2x-5}-\sqrt{4x+3}\right) = \lim_{x\to\infty}\sqrt{x}\left(\sqrt{2-\frac{5}{x}}-\sqrt{4+\frac{3}{x}}\right)$

Sol: $\left(\lim_{x\to\infty}\sqrt{x}\right)\left(\lim_{x\to\infty}\sqrt{2-\frac{5}{x}} - \lim_{x\to\infty}\sqrt{4+\frac{3}{x}}\right)$

$\lim_{x\to\infty}\left(\sqrt{2-\frac{5}{x}}\right) \overset{\text{C.O.V}}{=} \lim_{y\to 0}\sqrt{2-y} = \sqrt{2}$

$\lim_{x\to\infty}\left(\sqrt{4+\frac{3}{x}}\right) \overset{\text{C.O.V}}{=} \lim_{y\to 0}\sqrt{4+y} = \sqrt{4} = 2.$

$= \infty\cdot \underbrace{(\sqrt{2}-2)}_{<0} = -\infty$

Thm: Elementary functions are continuous on their domain

Ex $\lim_{x\to 0}\left(1+\tan(\arcsin x)\right)^{(1+x)^x+(\ln(1+x))^{100}}$

As the function is an elementary function, and $x=0$ is on its domain, thus it is continuous at 0. Therefore

$= \left(1+\tan(\arcsin 0)\right)^{(1+0)^0+(\ln(1+0))^{100}} = 1$

Some notable limits: (which we can't solve with our techniques)

$\lim_{x\to 0}\dfrac{\sin x}{x} = 1$

Proof: See 10:40. of lecture recording

comes from $\sin x \leq x \leq \tan x \quad x\in(0,\frac{\pi}{2})$

$|\sin x| \leq |x| \quad x\in(0,\frac{\pi}{2})\cup(-\frac{\pi}{2},0)$

$\Rightarrow \lim_{x\to 0}\sin x = 0$ by sandwich theorem

$\cos x \leq \dfrac{\sin x}{x} \leq 1 \qquad x\in(0,\frac{\pi}{2})\cup(-\frac{\pi}{2},0)$

sandwich theorem

Ex $\lim_{x\to 0}\dfrac{1-\cos x}{x^2} = \lim_{x\to 0}\dfrac{2\sin^2(\frac{x}{2})}{x^2} = \frac{1}{2}\lim_{x\to 0}\left(\dfrac{\sin(\frac{x}{2})}{(\frac{x}{2})}\right) \overset{\text{C.O.V}}{=} \frac{1}{2}\lim_{y\to 0}\left(\dfrac{\sin y}{y}\right) \overset{\text{notable limit}}{=} \frac{1}{2}\cdot 1 = \frac{1}{2}.$

12: Notable Limits & Cont Properties

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- $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- $\lim_{x\to 0} \frac{1-\cos x}{x^2} = 1/2$

- $\lim_{x\to\infty} (1+\frac{1}{x})^x = e$ (Euler's constant)

Ex: $\lim_{x\to -\infty} (1+1/x)^x \underset{\text{change of var}}{\overset{y=-x}{=}} \lim_{y\to\infty} (1-1/y)^{-y} = \lim_{y\to\infty} \left(\frac{y}{y-1}\right)^y = \lim_{y\to\infty} \left(1+\frac{1}{y-1}\right)^y = \lim_{y\to\infty} \left[\left(1+\frac{1}{y-1}\right)^{y-1}\right]^{\frac{y}{y-1}}$

$$= e^{\lim_{y\to\infty} \frac{y}{y-1}} = e$$

Ex2: $\lim_{x\to 0^+} (1+x)^{1/x} \overset{x=1/y}{=} \lim_{y\to\infty} (1+1/y)^y = e$

$\lim_{x\to 0^-} (1+x)^{1/x} \overset{x=1/y}{=} \lim_{y\to -\infty} (1+1/y)^y = e$ Hence

- $\lim_{x\to 0} (1+x)^{1/x} = e$

- $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ ($\log_e$ or $\ln$)

$\Leftarrow = \lim_{x\to 0} \log((1+x)^{1/x}) \overset{\text{continuity of log, composition}}{=} \log\left(\lim_{x\to 0} (1+x)^{1/x}\right) = \log e = 1$

$\lim_{x\to 0} \frac{e^x - 1}{x} \overset{y=e^x}{=} \lim_{y\to 1} \frac{y-1}{\ln y} \overset{z=y-1}{=} \lim_{z=0} \frac{z}{\ln(1+z)} = \lim_{z=0} \frac{1}{\left(\frac{\ln(1+z)}{z}\right)} \underset{\text{of limit}}{\overset{\text{algebra}}{=}} \frac{1}{\lim_{z\to 0}\left(\frac{\ln(1+z)}{z}\right)} \underset{\text{limit}}{\overset{\text{Notable}}{=}} \frac{1}{1} = 1$

- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

More Notable Limits: $(b>0)$

- $\lim_{x\to\infty} x^b = \lim_{x\to\infty} e^x = \lim_{x\to\infty} \log x = \infty$
- $\lim_{x\to 0^+} x^b = \lim_{x\to -\infty} e^x = 0$
- $\lim_{x\to 0^+} \log x = -\infty$
- $\lim_{x\to\infty} \frac{e^x}{x^b} = \infty$
- $\lim_{x\to -\infty} |x|^b e^x = 0$
- $\lim_{x\to\infty} \frac{\log x}{x^b} = 0$
- $\lim_{x\to 0^+} x^b \log x = 0$

lim ... A.O.L lim ... lim ... Notable

Ex $\lim_{x\to\infty}\left(\sqrt[3]{x}+4\sqrt[4]{x}\right) \overset{A.O.L}{=} \lim_{x\to\infty}\left(x^{1/3}\right)+4\lim_{x\to\infty}\left(x^{1/4}\right) \underset{\text{limits}}{\overset{\text{notable}}{=}} \infty+3\infty=\infty.$

$\left|\lim_{x\to\infty} x^b=\infty,\ b>0\right|$

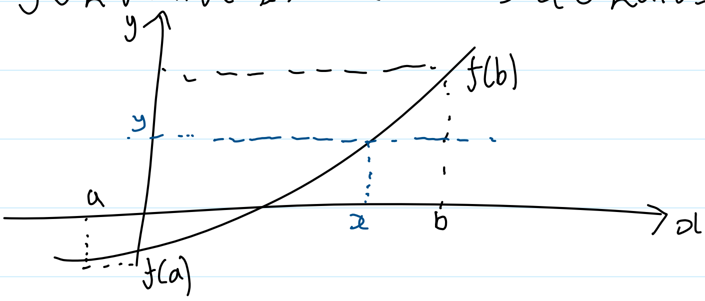
Ex $\lim_{x\to\infty} x^{-1/x} \overset{x>0}{=} \lim_{x\to\infty} e^{\ln\left(x^{-1/x}\right)} = \lim_{x\to\infty} e^{-\frac{1}{x}\ln x} \overset{\text{cont. of } e^x}{=} e^{\lim_{x\to\infty}\left(-\frac{1}{x}\ln x\right)} \overset{a.o.l}{=} e^{-\lim_{x\to\infty}\frac{\ln x}{x}} \underset{\text{limits}}{\overset{\text{notable}}{=}} e^{-0}=1$

Properties of Continuous Functions:

Thm: (Intermediate Value Theorem)

$f:[a,b]\to\mathbb{R}$ be continuous. $f(a)\le f(b)$

Then, $\forall\, y\in[f(a),f(b)]$, there exists $x\in[a,b]$ such that $f(x)=y$.



Proof: Assume $f(a)\le 0\le f(b)$ and $y=0$.

We want to find $x\in[a,b]$, such that $f(x)=0$.

- If $f(a)=0$ (or $f(b)=0$) we just take $x=a$ (or b).
- If $f(a)<0<f(b)$. Take

$\bar{x}=\sup\{x\in[a,b];\ f(x)<0\}$

Then we can show that $f(\bar{x})=0$

Ex: $f(x)=x^3+2x^2-100$ has a root on $[-10,10]$

Proof: $f(-10)=(-10)^3+2(-10)^2-100=-900$

$f(10)=(10)^3+2(10)^2-100=1100$

We note that f is a polynomial and thus continuous and $f(-10)<0<f(10)$

Therefore by the intermediate value theorem we conclude that $\exists\, x^*\in[-10,10]$ such that $f(x^*)=0$.

Boundedness Theorem: $f:[a,b]\to\mathbb{R}$ be continuous $a<b$ $a,b\in\mathbb{R}$

Boundedness Theorem: $f:[a,b]\to\mathbb{R}$ be continuous $a\leq b$. $a,b\in\mathbb{R}$

Thm ① f is bounded

② f attains its bounds i.e. $\exists\, x_m, x_M$ such that

$$f(x_m) = \min\{f(x);\ x\in[a,b]\}$$
$$f(x_M) = \max\{f(x);\ x\in[a,b]\}$$

Derivative (rate of change) is a special case of limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$f'(x)$ is a function, the Derivative Function of $f(x)$.

Def^n: $f : \Omega \to \mathbb{R}$, $a \in \Omega$. If $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L$ for some $L \in \mathbb{R}$. Then f is differentiable at point a. And L is the derivative of f at a.

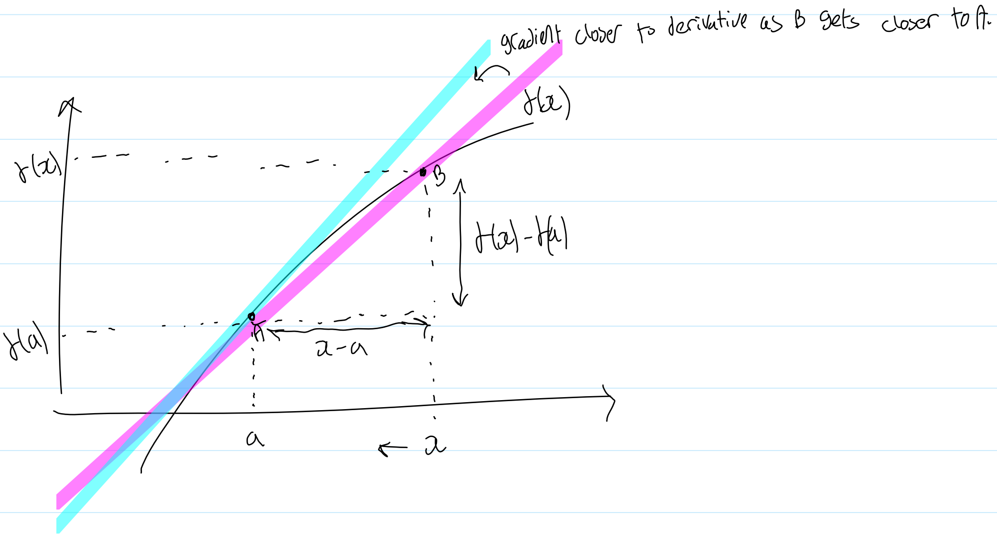
It is differentiable if the limit exists at this point.
If f is differentiable at each point $a \in \Omega$ then f is differentiable on Ω.

Ex $f(x) = x$ is differentiable on $\mathbb{R}$

Pf: $\forall x_0 \in \mathbb{R}$ we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} = \frac{x - x_0}{x - x_0} = 1 \text{ for all } x_0 \in \mathbb{R}$$

Thus f is differentiable at x_0.



Geometric Interpretation:

$$\frac{f(x) - f(a)}{x - a}$$ (Newton Quotient or Difference Quotient)

Slope/gradient of the line through $(a, f(a))$ and $(x, f(x))$

As $x \to a$ it is the gradient of the tangent at $(a, f(a))$.

Physical Interpretation:

If $f(t)$ is the position of a point moving on a line

$$\frac{f(t) - f(a)}{t - a}$$ is the average velocity on $[a, t]$

As $t \to a$ it is the instantaneous velocity at a.

Other interpretations exist in other fields.

other interpretations exist in other fields.

Notations:

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$$

$$= \left.\frac{df}{dx}\right|_{x=a}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} \qquad \Delta f(a) = f(a+\Delta x) - f(x)$$

$$= \frac{df}{dx}(x)$$

$$= \frac{d}{dx} f(x)$$

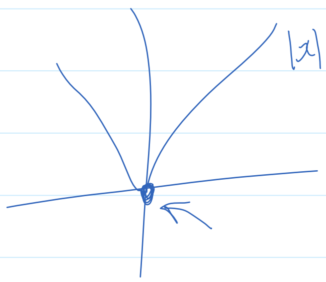
not sure is written correctly

Non-Differentiable Functions:

Ex: $f(x) = |x|$ is not differentiable at $x=0$

Pf: $\dfrac{f(x)-f(0)}{x-0} = \dfrac{|x|}{x}$

We have shown the above does not have a limit as $x \to 0$ previously. Thus by the definition, $f(x)$ is not differentiable at $x=0$.



Ex2: $f(x) = \begin{cases} x \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

f is continuous at $x=0$ $\quad \lim_{x \to 0} x \sin 1/x = 0 = f(0)$ $\quad$ Sandwich theorem?

f is continuous.

$$\frac{f(x)-f(0)}{x-0} = \frac{x \sin 1/x}{x} = \sin 1/x$$ does not have limit as $x \to 0$

how do you know?

Thus not differentiable at 0.

Differentiability and Continuity:

$f: \Omega \to \mathbb{R}$, a is an accumulation point of Ω.

If f is differentiable at a, then f is continuous at a.

↳ Differentiability $\Rightarrow$ Continuity

Pf: (rough idea)

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left[\frac{f(x)-f(a)}{x-a} \cdot (x-a) + f(a) \right]$$

Algebra of limits

$$= \left(\lim_{x \to a} \frac{f(x)-f(a)}{x-a} \right) \left(\lim_{x \to a} (x-a) \right) + f(a)$$ ← constant

why not change?

$$= f'(a) \cdot 0 + f(a) = f(a)$$

$\Rightarrow f$ is continuous at a.

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$$ ← 'right derivative'

$$f'_-(a) = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a}$$ ← 'left derivative'

Thm: f is differentiable at $a \Leftrightarrow f$ is both left and right differentiable and $f'_+(a) = f'_-(a)$.

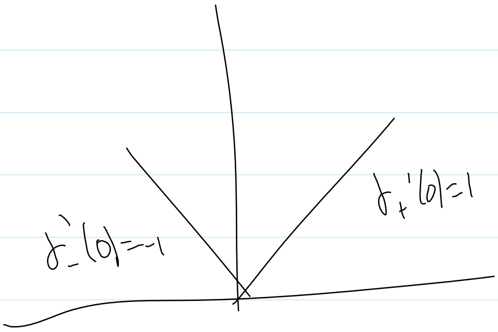
What circumstances would they be different?

Ex: $f(x) = |x|$

$$f'_+(0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^+} \frac{x-0}{x-0} = 1$$

$$f'_-(0) = \lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \frac{-x-0}{x-0} = -1$$

Since $f'_+(0) \neq f'_-(0)$ thus f is not differentiable at 0.



$$f(x) = x^2$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

14: Methods of Differentiation

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Lots of these properties overlap with limit properties. This is due to the fact derivatives are a special case of limit.

Linear Property:

$(f+g)'(x) = f'(x) + g'(x)$

Proof:

$$(f+g)'(x) \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{(f+g)(x+\Delta x) - (f+g)(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right]$$

$$\overset{\text{linear of limits}}{=} \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}$$

$$\overset{\text{definition}}{=} f'(x) + g'(x).$$

Product Property (Leibniz rule):

$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$

Product Rule from a-level $\Rightarrow uv' + vu'$

Proof:

$$(fg)'(x) \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{(fg)(x+\Delta x) - (fg)(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right) g(x+\Delta x) + f(x) \cdot \left(\frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \right]$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta f}{\Delta x} \cdot g(x+\Delta x) + f(x) \cdot \frac{\Delta g}{\Delta x} \right)$$

$$\overset{\text{algebra of limits}}{=} f'(x)g(x) + f(x)g'(x)$$

Ex $f(x) = x^2$ $\quad f'(x) = 2x$

$(fg)' = f'g + fg'$

$$(x^2)' = (x)'x + x(x)' = 2x$$

Quotient Rule/Property:

$$\left(\frac{f}{g}\right)'x = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

Quotient rule from A-level $\Rightarrow \frac{vu'-uv'}{v^2}$

Proof:

$$\frac{\left(\frac{f}{g}\right)(x+\Delta x) - \left(\frac{f}{g}\right)x}{\Delta x} = \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x}$$

$$= \frac{\big(f(x+\Delta x) - f(x)\big)\,g(x) - f(x)\big(g(x+\Delta x) - g(x)\big)}{\Delta x\, g(x+\Delta x)\,g(x)}$$

$$= \frac{\frac{\Delta f}{\Delta x} gx - f(x)\frac{\Delta g}{\Delta x}}{g(x+\Delta x)\,g(x)} \xrightarrow{\Delta x \to 0} \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Ex: $\left(\frac{1}{x}\right)'$ $\quad f = 1 \quad g = x$ $\quad f' = 0 \quad g' = 1$

$$\left(\frac{1}{x}\right)' \overset{\text{Quotient rule}}{=} \frac{(1)'x - 1(x)'}{x^2} = \frac{0\cdot x - 1\cdot 1}{x^2} = -\frac{1}{x^2}$$

Specify which rule is being used

Ex: $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$

$$\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \quad \text{Quotient rule}$$

$g = \cos x \quad g' = -\sin x$

$f = \sin x \quad f' = \cos x$

$$= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\frac{}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

Chain rule:

$$(g(f(x)))'(x) = g'(f(x)) \cdot f'(x)$$

Proof: $\dfrac{\Delta(g \circ f)}{\Delta x} = \dfrac{g(f(x+\Delta x)) - g(f(x))}{\Delta x}$

$$= \frac{g(f(x+\Delta x)) - g(f(x))}{f(x+\Delta x) - f(x)} \cdot \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} y = y_0 \quad \overset{y = f(x+\Delta x)}{\longrightarrow} \; y_0 = f(x) \qquad = \frac{g(y) - g(y_0)}{y - y_0} \cdot \frac{\Delta f}{\Delta x}$$

$$\xrightarrow{\Delta x \to 0} \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \cdot f'(x) = g'(y_0) f'(x)$$

$$= g'(f(x)) f'(x)$$

Ex2: $(\sin(x^2))' = \sin'(x^2)(x^2)'$ Chain rule

$$= 2x \cos(x^2)$$

Derivative of the inverse:

$$x = g(f(x)) \Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

Not written correctly?

At alevel f maths done with implicit differentiation

Proof: Let $g = f^{-1}$ inverse of f.

Then $x = g(f(x))$

Then $x = g(f(x))$

Take derivative on both sides or above:

$1 \overset{\text{chain}}{=} g'(f(x))f'(x)$

$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$

???

Ex: $\arcsin'(y) \overset{y=\sin x}{=} \arcsin'(\sin x) = \frac{1}{\sin'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$

$g = f^{-1}$

$g'(f(x)) = \frac{1}{f'(x)}$

Ex2: $(\arccos x)'$

$x = \cos t$

$= \arccos(\cos(t))$

$= \frac{1}{(\cos t)'} = -\frac{1}{\sin t}$

$= -\frac{1}{\sqrt{1-\cos^2 t}}$

$\overset{x=\cos t}{=} -\frac{1}{\sqrt{1-x^2}}$

Ex3: $(\arctan x)' = \frac{1}{1+x^2}$ Proof needed

Trig:

$(\sin x)' = \cos x$

Proof:

$(\sin x)' \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$

$\sin(a+b) = \sin a \cos b + \sin b \cos a$

So $= \lim_{\Delta x \to 0} \left(\sin x \frac{\cos(\Delta x)-1}{\Delta x} + \cos x \cdot \frac{\sin(\Delta x)}{\Delta x} \right)$

$\overset{\text{algebra}}{=} \sin x \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{(\Delta x)^2} \cdot \Delta x + \cos x \cdot \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$

$= \sin x \,(-1/2) \cdot 0 + \cos x \cdot 1 = \cos x$

$(\cos x)' = -\sin x$

Proof:

$$(\cos x)' = \left(\sin\left(\frac{\pi}{2} - x\right)\right)' \overset{\text{chain}}{=} \cos\left(\frac{\pi}{2} - x\right) \cdot \left(\frac{\pi}{2} - x\right)'$$

$$= -\cos\left(\pi/2 \cdot x\right)$$

$$= -\sin x$$

15: Methods of Differentiation Cont.

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<u>The exponential Function:</u>

$f(x) = e^x$

$f'(x) = e^x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

$$\overset{\text{algebra of limits}}{=} e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} \leftarrow \text{notable limit} = 1$$

$$= e^x$$

<u>Natural Log Function:</u>

$f(x) = \ln(x) = \log_e x = \log x$

$f'(x) = (\ln(x))' = \frac{1}{x}$

Proof: f is the inverse of e^x

$$f'(x) \overset{x=e^t}{=} f'(e^t) = \frac{1}{(e^t)'} = \frac{1}{e^t}$$

$$\overset{x=e^t}{=} \frac{1}{x}$$

$$\log_a x = \frac{\ln x}{\ln a}$$

<u>Exponentials:</u>

$f(x) = a^x, \quad a > 0$

$f'(x) = \ln a \cdot a^x$

Proof: $f'(x) = (a^x)' = (e^{x\ln a})'$

$$\overset{\text{chain rule}}{=} e^{x\ln a} \cdot \ln a$$

$$= \ln a \cdot a^x$$

$x = e^{\ln x}$

$\ln a^b = b\ln a$

$a^x = e^{\ln(a^x)} \leftarrow$ useful trick

$= e^{x\ln a}$

<u>Powers:</u>

$f(x) = x^\alpha \quad \alpha \neq 0$

$f'(x) = (x^\alpha)' = (e^{\ln(x^\alpha)})' = (e^{\alpha\ln x})'$

$$\overset{\text{Chain rule}}{=} e^{a\ln x} \cdot (a\ln x)' = a \cdot \frac{1}{x} \cdot e^{a\ln x}$$

$$= a \cdot \frac{1}{x} \cdot x^{a} = a \cdot x^{a-1}$$

$$(x^{-n})' = -n\,x^{-n-1}$$
$$(x^{n})' = n\,x^{n-1}$$

Ex: $f(x) = x^{x}$

Sol: $(x^{x})' = (e^{\ln x^{x}})' = (e^{x\ln x})'$

$$\overset{\text{chain}}{=} e^{x\ln x} \cdot (x\ln x)'$$

$$\overset{\text{Product rule}}{=} e^{x\ln x} \cdot ((x)' \ln x + x \cdot (\ln x)')$$

$$= x^{x} \cdot (\ln x + x \cdot \tfrac{1}{x})$$
$$= x^{x} (\ln x + 1)$$

Log Trick:

$$f(x) = u(x)^{v(x)}$$

$$f'(x) = \left(e^{\ln(u(x)^{v(x)})}\right)' = \left(e^{v(x)\ln(u(x))}\right)'$$

$$\overset{\text{chain}}{=} e^{v(x)\ln(u(x))} \cdot (v(x) \cdot \ln(u(x)))'$$

$$\overset{\text{Leibniz}}{=} u(x)^{v(x)} \cdot \left(v'(x) \cdot \ln(u(x)) + v(x) \cdot (\ln(u(x)))'\right)$$

$$\overset{\text{chain}}{=} u(x)^{v(x)} \left(v'(x)\ln(u(x)) + v(x)\frac{u'(x)}{u(x)}\right)$$

Try $\left((\cos x)^{x^{2}}\right)'$

Hyperbolic Function:

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$\tanh x = \frac{\sinh x}{\cosh x}$ $\qquad$ $\coth x = \frac{\cosh x}{\sinh x}$

$\sinh'(x) = \cosh(x)$ $\qquad$ $\cosh'(x) = \sinh x$

$\tanh'(x) = 1 - \tanh^2 x$ $\qquad$ $\coth'(x) = 1 - \coth^2(x)$

Taking ln on both sides:

Ex: $f(x) = \frac{(2x+3)^4 \sqrt{x-6}}{\sqrt[3]{x+1}}$

sol: $\ln(f(x)) = \ln\left(\frac{(2x+3)^4 (x-6)^{1/2}}{(x+1)^{1/3}}\right)$

$$= 4\ln(2x+3) + \tfrac{1}{2}\ln(x-6) - \tfrac{1}{3}\ln(x+1)$$

$\Rightarrow (\ln(f(x)))' \overset{\text{chain rule}}{=} \frac{f'(x)}{f(x)}$ $\quad$ (LHS)

$(RHS)' = \frac{8}{2x+3} + \frac{1}{2(x-6)} - \frac{1}{3(x+1)}$

$\Rightarrow \frac{f'(x)}{f(x)} = \frac{8}{2x+3} + \frac{1}{2(x-6)} - \frac{1}{3(x+1)}$

$\Rightarrow f'(x) = f(x)\left[\quad \downarrow \quad\right]$

$\Rightarrow f'(x) = \frac{(2x+3)^4 \sqrt{x-6}}{\sqrt[3]{x+1}}\left[\frac{8}{2x+3} + \frac{1}{2(x-6)} - \frac{1}{3(x+1)}\right]$

Can simplify computation

With all these techniques, can find derivatives of any elementary functions.

Ex: $\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)$

$\overset{\text{quotient}}{=} \frac{(\sin x)'(1+\cos x) - \sin x(1+\cos x)'}{(1+\cos x)^2}$

algebra $\quad \cos x(1+\cos x) + \sin^2 x$

$$\overset{\text{algebra}}{=} \frac{\cos x(1+\cos x)+\sin^2 x}{(\cos x+1)^2}$$

$$= \frac{\cos x+1}{(\cos x+1)^2} = \frac{1}{1+\cos x}$$

Higher order Derivative:

Let f be a differentiable function, then f' is also a well defined function. If f' is also differentiable, then we can define

$f'' = (f')'$

Ex $f(x) = \cos x$ then $f'(x) = -\sin x$ then $f''(x) = -\cos x$

$f''' = f''$ similarly, but the dashes are long, so can use notation:

$f^{(0)} = f \quad f^{(1)} = f' \quad f^{(2)} = f'' \quad f^{(n)} = (f^{(n-1)})' \leftarrow (n \geq 1)$ ← brackets important so not complicated with composition.

$$f^{(n)} = \frac{d^n f}{dx^n} = \frac{d^n}{dx^n} f$$

If $s(t)$ is the position function with respect to time, $s'(t)$ is velocity, $s''(t)$ is acceleration.

$f: \Omega \to \mathbb{R}$

~~n-times differentiable~~ - If $f^{(n)}(a)$ exists, then f is n times differentiable at $a \in \Omega$

If $f^{(n)} x$ is continuous, then f is n-times continuously differentiable ← written correctly

C^∞ class:

f is infinitely differentiable if f is n times differentiable for all $n \in \mathbb{N}$

Ex: $f(x) = e^x \quad f^{(n)}(x) = e^x \quad \forall n \in \mathbb{N}$

$e^x \in C^\infty$

$\cos x, \sin x, x^n \quad n \in \mathbb{N}$

Does it have to be meaningful, eg x? NO.

Ex2: $f(x) = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ See if differentiable

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin 1/x - 0}{x - 0} = \lim_{x \to 0} x \sin 1/x = 0 \Rightarrow f'(0) = 0$$

If $x \neq 0 \quad f'(x) = (x^2 \sin 1/x)' = (x^2)' \sin 1/x + x^2 (\sin 1/x)'$

$= 2x \sin \frac{1}{x} - \cos(\frac{1}{x})$

$$f'(x) = \begin{cases} 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Let $g(x) = 2x \sin \frac{1}{x}$ then $\lim_{x \to 0} g(x) = 0$ and $f'(x) - g(x) = \cos \frac{1}{x}$

Which is not continuous at 0 since $\lim_{x \to 0} \cos(\frac{1}{x})$ does not exist.

Therefore, $f'(x)$ is also not continuous at the point $x = 0$.

Thus $f'(x)$ is not differentiable at $x = 0$ which means $f''(0)$ does not exist.

Ex3: $f(x) = x^n$, $n \in \mathbb{N}$

$f'(x) = nx^{n-1}$

$f''(x) = n(n-1)x^{n-2}$

$f^{(n)}(x) = n(n-1)(n-2)\cdots \times x^{n-n} = n!$

17: Applications of Differentiation

25 October 2022 14:06

Stationary Points (Critical Points):

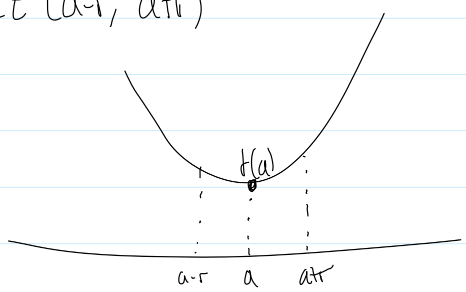
$f: \Omega \to \mathbb{R}$ for some set $\Omega \subseteq \mathbb{R}$ $a \in \Omega$

Stationary Point/Critical Point of f - If f is differentiable at a and $f'(a) = 0$.

Fermat Theorem: If a is a local maximum or a local minimum point of f and f is differentiable at the point a then a is a stationary point.

The reverse of the statement is not true (stationary points of inflection).

Pf: Assume a is a local minimum point of f. Hence there exists $r > 0$ such that $f(x) \geq f(a)$ $\forall x \in (a-r, a+r)$



For $x \in (a, a+r)$ we have $x - a > 0 \Rightarrow \dfrac{f(x) - f(a)}{x-a} \geq 0$

Since f is differentiable at a, we have

$$f'(a) = f'_+(a) = \underbrace{\lim_{x \to a^+}}_{x > a} \frac{f(x) - f(a)}{x - a} \geq 0$$

For $x \in (a-r, a)$ we have $x - a < 0 \Rightarrow \dfrac{\overset{\geq 0}{f(x) - f(a)}}{\underset{<0}{x-a}} \leq 0$

Since f is differentiable at a, we have

$$f'(a) = f'_-(a) = \underbrace{\lim_{x \to a^-}}_{x < a} \frac{f(x) - f(a)}{x - a} \leq 0$$

Finally we conclude that $f'(a) = 0$.

Rolle's Theorem:

$f: [a,b] \to \mathbb{R}$ $a, b \in \mathbb{R}$ and $a < b$ if f is:

- continuous on $[a,b]$
- differentiable on (a,b)
- $f(a) = f(b)$

Then $\exists\, c \in (a,b)$, such that $f'(c) = 0$

Proof:

By the boundedness theorem, $\exists$ global maximum $x_M \in [a,b]$ and global minimum $x_m \in [a,b]$.

If $\{x_m, x_M\} = \{a,b\}$. Therefore $\forall x \in [a,b]$
$f(x_m) = f(x) = f(x_M) \Rightarrow f$ is a constant
Thus $f'(x)=0 \quad \forall x \in (a,b)$.
Assume one or x_m or x_M is not an end point of $[a,b]$. Then one of x_m and x_M must belong to (a,b). Let c denote such point.
Then c is either a local maximum or local minimum. Thus by the Fermat theorem we conclude that $f'(c)=0$.

Mean Value Theorem:

$f:[a,b]\to\mathbb{R}$, $a<b$ f is
- continuous on $[a,b]$
- Differentiable on (a,b)

Then $\exists\, c\in(a,b)$, such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

$\hookrightarrow$ point on graph between $[a,b]$ exists that has tangent gradient same as gradient between of line AB

Proof: Let $g:[a,b]\to\mathbb{R}$ be defined by $g(x)=f(x)-\frac{f(b)-f(a)}{b-a}(x-a)$

by the algebra of continuous functions and differential rules we note that $g(x)$ is also continuous on $[a,b]$ and differentiable on (a,b). Moreover

$$g(a)=f(a)-\frac{f(b)-f(a)}{b-a}(a-a)=f(a)$$

$$g(b)=f(b)-\frac{f(b)-f(a)}{b-a}(b-a)=f(a)$$

Thus $g(a)=g(b)$. By Rolle's theorem there exists $c\in(a,b)$ such that $g'(c)=0$.

$$g'(x)=\left(f(x)-\frac{f(b)-f(a)}{b-a}(x-a)\right)'$$

$$=f'(x)-\frac{f(b)-f(a)}{b-a}$$

Then $g'(c)=0 \Rightarrow f'(c)-\frac{f(b)-f(a)}{b-a}=0$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a} \quad \#$$

Thm - $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and $f'(x)=0$ $\forall\, x\in(a,b)$, then f is constant.

Proof: $\forall x,y\in[a,b]$ with $x<y$, then f is continuous on $[x,y]$ and differentiable on (a,b), thus by the mean value theorem, $\exists\, z\in(x,y)$ such that

$$\frac{f(y)-f(x)}{y-x}=f'(z)=0$$

$\Rightarrow f(y)=f(x)$. Since x,y are arbitrary points of $[a,b]$, this proves f is a constant.

Increasing (decreasing) Properties:

not strictly like in a level

If f is differentiable on (a,b) and f is increasing (decreasing) on $(a,b) \Leftrightarrow f'(x)\geq 0$ $(f'(x)\leq 0)$

Increasing (decreasing) Properties:

If f is differentiable on (a,b), and f is increasing (decreasing) on $(a,b) \Leftrightarrow f'(x) \geq 0$ $(f'(x) \leq 0)$

not strictly like in a level

$\Rightarrow$ Proof: If f is increasing then $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$ both for $x > x_0$ and $x < x_0$.

$f(x)-f(x_0) \geq 0$, $x-x_0>0$; $f(x)-f(x_0) \leq 0$, $x-x_0<0$

Then as $x \to x_0$, we have $f'(x_0) \geq 0$

$\Leftarrow$ If $f'(x) \geq 0$ by using the MVT for any $x_1 < x_2$ $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi)$ for some $\xi \in (a,b)$

$\xRightarrow{(x_2-x_1>0)}$ $f(x_2)-f(x_1) = \underbrace{f'(\xi)(x_2-x_1)}_{\geq 0}$

$\Rightarrow f$ is increasing.

If $f'(x) > 0$ $\forall x \in (a,b)$ then f is strictly increasing on (a,b)

Pf $\forall x_1 < x_2$ By MVT $\exists \xi \in (a,b)$.

$f(x_2)-f(x_1) = \underset{>0}{f'(\xi)}\underset{>0}{(x_2-x_1)} > 0$

$\Rightarrow f$ is strictly increasing.

Thm: $f:(a-r,a+r) \to \mathbb{R}$ f is

- continuous on $(a-r,a+r)$
- differentiable on $(a-r,a+r)\setminus\{a\}$

Then: If f is decreasing on $(a-r,a)$ and
f is increasing on $(a,a+r)$

$\Rightarrow a$ is a local minimum

19: Applications of Differentiation

27 October 2022 10:02

Second Derivative Test:

$f: I \to \mathbb{R}$ where I is an open interval

$a \in I$. If f is twice continuous differentiable on I and $f'(a)=0$ Then:

① $f''(a)>0 \Rightarrow a$ is a local minimum point

② $f''(a)<0 \Rightarrow a$ is a local maximum point.

Proof: we only prove 1 ie $f''(a)>0 \Rightarrow a$ is a local min point

- $f''(x)$ is continuous, if $f''(a)>0$ then $\exists\ \delta>0$ such that $f''(x)>0\ \forall x \in (a-\delta, a+\delta)$.
- $f''(x) = (f'(x))'$ Then $f'(x)$ is strictly increasing on $(a-\delta, a+\delta)$ Therefore we have $f'(x) < f'(a) < f'(y) \quad \forall\ a-\delta < x < a < y < a+\delta$
- Since $f'(a)=0$ thus $f'(x)<0\ \forall x \in (a-\delta, a)$ and $f'(x)>0\ \forall x \in (a, a+\delta)$

$\Rightarrow a$ is a local min.

Convexity:

$f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ interval

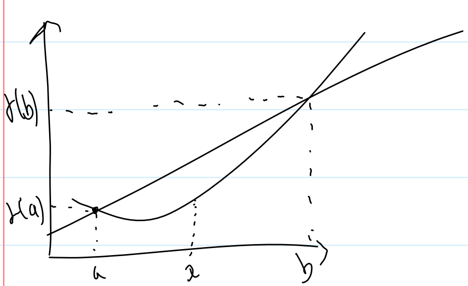
$\forall\ a,b \in I$, $t \in (0,1)$, $a<b$

Convex: $f((1-t)a+tb) \le (1-t)f(a) + tf(b)$

Concave - $f((1-t)a+tb) \ge (1-t)f(a) + tf(b)$

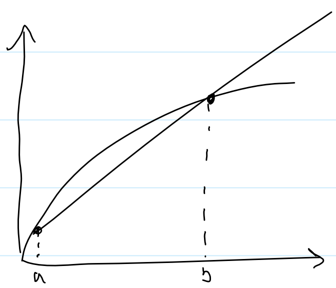
Note: $a \le \underbrace{(1-t)a + tb}_{x} \le b$ and $f(a) \le \underbrace{(1-t)f(a)+tf(b)}_{y} \le f(b)$?

Geometric Meaning:



Concave Upward / Convex

↳ curve always below line



Concave downwards / concave

↳ curve always above line

Prop: f is convex on $I \Leftrightarrow \forall a,b,c \in I$, $a<c<b$ we have $\frac{f(c)-f(a)}{c-b} \le \frac{f(b)-f(c)}{b-c}$

Pf: Let $c \in$ I such that $c=(1-t)a+tb \Rightarrow t=\frac{c-a}{b-a}$, $(1-t)=\frac{b-c}{b-a}$ thus

$f((1-t)a+tb) \leq (1-t)f(a)+tf(b)$

$\Leftrightarrow f(c) \leq \frac{b-c}{b-a}f(a)+\frac{c-a}{b-a}f(b)$

Thm: If f is differentiable then:

f is convex $\Leftrightarrow f'$ is increasing

Pf: "$\Rightarrow$" $\forall$ $a<b$ take $h>0$ such that $a+h<b-h$ since f is convex we have

$$\frac{f(a+h)-f(a)}{h} \leq \frac{f(b-h)-f(a+h)}{(b-h)-(a-h)} \leq \frac{f(b)-f(b-h)}{h}$$

Let $h \to 0^+$ we get $f_+'(a) \leq f_-'(b) \Rightarrow f'(a) \leq f'(b)$

"$\Leftarrow$" If f' increasing, then $\forall$ $a<c<b$ by the Mean value theorem we have

$$\frac{f(c)-f(a)}{c-a}=f'(\xi)$$

$\xi \in (a,c)$ $\eta \in (c,b)$

$$\frac{f(b)-f(c)}{b-c}=f'(\eta)$$

As f' is increasing, thus $f'(\xi) \leq f'(\eta)$ which gives

$$\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(c)}{b-c} \Rightarrow \text{convex.}$$

Thm: f is twice differentiable on I and f is convex $\Leftrightarrow$ $f''(x)>0$.

Pf: convex $\Rightarrow f'$ increasing

f' differentiable $\Rightarrow f''=(f')'>0$

20: Applications of Differentiation

27 October 2022 12:04

Cauchy Mean Value Theorem:

f, g as before $g(a) \neq g(b)$ $\exists$ $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

- If $g(x) = x$ then it reduces to MVT.

Proof: Let $h(x) = (f(b)-f(a))(g(x)-g(a)) - (g(b)-g(a))(f(x)-f(a))$

We note $h(a) = 0 = h(b)$. We can apply Rolle's theorem, $\exists$ $c \in (a,b)$ such that $h'(c) = 0$

i.e. $h'(c) = (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c) = 0$

This rearranges to formula.

L'Hospital Rule:

① f, g are defined on $(a-\gamma, a) \cup (a, a+\gamma)$ for some γ

② $\lim_{x\to a} f(x) = \lim_{x \to a} g(x) = 0$

③ $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists

Then: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Proof: $f_0(x) = \begin{cases} f(x) & x \in I \setminus \{a\} \\ 0 & x = a \end{cases}$, $g_0(x) = \begin{cases} g(x), & x \in I \setminus \{a\} \\ 0 & x = a \end{cases}$

Where $I = (a-\gamma, a+\gamma)$ Then f_0 and g_0 are continuous on I.

$$\frac{f(x)}{g(x)} = \frac{f_0(x) - f_0(a)}{g_0(x) - g_0(a)} \qquad x \neq a.$$

$$\overset{\text{Cauchy MVT}}{=} \frac{f_0'(\xi)}{g_0'(\xi)} \quad (\text{Where } \xi \text{ is between } x \text{ and } a)$$

$$\Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f_0'(x)}{g_0'(x)} = \lim_{\xi\to a}\frac{f'(\xi)}{g'(\xi)}$$

Ex $\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)}$ $b \neq 0$ ← Cannot go straight to A.O.L as can't have denominator of 0.

$$\underset{\text{L'Hospitals Rule}}{=} \lim_{x\to 0}\frac{(\sin(ax))'}{(\sin(bx))'} = \lim_{x\to 0}\frac{a\cos(ax)}{b\cos(bx)} \overset{\text{A.O.L}}{=} \frac{a}{b}\frac{\lim_{x\to 0}\cos(ax)}{\lim_{x\to 0}\cos(bx)} \overset{\text{continuity}}{=} \frac{a}{b}\frac{\cos 0}{\cos 0} = a/b$$

Rule $\lim_{x\to 0} \frac{(\sin(ax))'}{(\sin(bx))'} = \lim_{x\to 0} \frac{a\cos(ax)}{b\cos(bx)} = \frac{a}{b}\frac{\lim_{x\to 0}\cos(ax)}{\lim_{x\to 0}\cos(bx)} = \frac{a}{b}\frac{\cos 0}{\cos 0} = a/b$

Ex2 $\lim_{x\to 0} \frac{x-\sin x}{x^3} \overset{\text{L'Hopital}}{=} \lim_{x\to 0} \frac{(x-\sin x)'}{(x^3)'} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} \overset{\text{L'Hopital}}{=} \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6}$

$\overset{\text{continuity of cos}}{=} \frac{\cos 0}{6} = 1/6.$

could use the notable limit here

lim of top + bottom no longer 0 so can't use L'Hopital again.

Ex3: $\lim_{x\to\infty} \left(\frac{\pi}{2} - \arctan x\right)x = \lim_{x\to\infty} \frac{\pi/2 - \arctan x}{1/x} \overset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{x^2}{1+x^2}$

$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} \overset{\text{A.O.L.}}{=} \frac{1}{\lim_{x\to\infty}(1+1/x^2)} = 1/1 = 1$

trick

now both have limit of 0. (had to change)

Derivative of implicit functions:

$x^2+y^2=1 \Rightarrow \begin{cases} f_+(x) = \sqrt{1-x^2} & x\in[-1,1] \\ f_-(x) = -\sqrt{1-x^2} & x\in[-1,1] \end{cases}$

$(x^2+(y(x))^2)' = (1)'$

$2x + 2y(x)y'(x) = 0 \Rightarrow y'(x) = -\frac{x}{y(x)} \quad y(x)\neq 0.$

So assuming y is a function of x rather than a variable.

Ex2 $x^2+e^{x+y}=1$

$(x^2+e^{x+y})' = (1)' \Rightarrow 2x + e^{x+y}(x+y(x))' = 0$

$\Rightarrow 2x + e^{x+y}(1+y'(x)) = 0$

$\Rightarrow y'(x) = -2x e^{-x-y} - 1.$

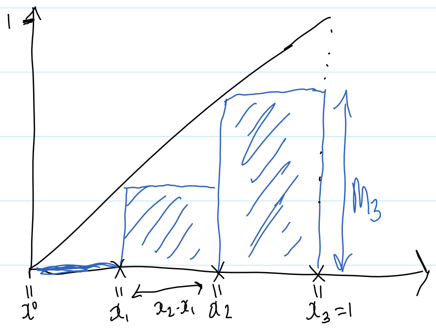
21: Defining the Integral

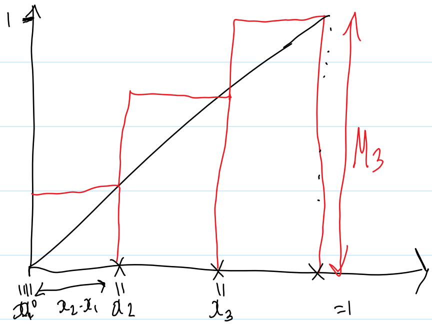
08 November 2022 13:56

Analysis is a detailed examination to determine its nature, structure or essential features whereas calculus is just a method or system of calculation.

What does it mean for $f:[a,b]\to\mathbb{R}$ to be

	Differentiable	Continuous	Integrable
Geometry	Tangent	"Connectivity"	Area
Analysis	$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists $\hookrightarrow f'(c):=$	$\lim_{x\to c} f(x)=f(c)$	??? $\hookrightarrow \int_a^b f(c)$





$m_1=\min\{f(x):x\in[x_0,x_1]\}$

$m_2=\min\{f(x):x\in[x_1,x_2]\}$

$m_3=\min\{f(x):x\in[x_2,x_3]\}$

$M_1=\max\{f(x):x\in[x_0,x_1]\}$

$M_2=\max\{f(x):x\in[x_1,x_2]\}$

$M_3=\max\{f(x):x\in[x_2,x_3]\}$

$$s_3=\begin{bmatrix} m_1(x_1-x_0)+ \\ m_2(x_2-x_1)+ \\ m_3(x_3-x_2)\end{bmatrix} \leq \text{Area} \leq S_3=\begin{bmatrix} M_1(x_1-x_0)+ \\ M_2(x_2-x_1)+ \\ M_3(x_3-x_2)\end{bmatrix}$$

underestimate $\longrightarrow$ $s_n\leq \text{Area}\leq S_n$ $\longleftarrow$ overestimate.

We use supremum and infimum instead of max and min so unbounded sets can still have areas defined.

Claim: $\sup[a,b)=b$

Proof: Observe that $x\leq b$ for all $x\in[a,b)$ so b is an upper bound for $[a,b)$

Now let $\varepsilon>0$ and observe that $b-\varepsilon<b-\varepsilon/2\in[a,b)$ $\longleftarrow$ ε tests

So $b-\varepsilon$ is not an upper bound. Hence the $\sup[a,b)=b$.

Now let $\varepsilon > 0$ and observe that $b-\varepsilon < b-\varepsilon/2 \in [a,b)$ ← ε tests

So $b-\varepsilon$ is not an upper bound, Hence the $\sup[a,b) = b$.

Partition - A Partition of $[a,b]$ is a finite set $P=\{x_0,x_1,\dots,x_n\}$ such that $a=x_0<x_1<\dots<x_n=b$ for some $n\in\mathbb{N}$. (Don't have to be equal distance apart.

Formalising lower/upper bound system:

Suppose $\begin{cases} -\infty<a<b<\infty \\ f:[a,b]\to[0,\infty] \text{ is } \underline{\text{bounded}} \end{cases}$

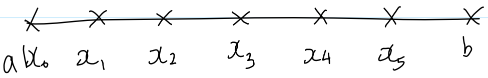
Set $m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$

$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}$

Lower Sum - $L(f,P)=\sum_{i=1}^{n} m_i(x_i-x_{i-1})$ ← n is the number of rectangles, so number of elements in the Partition-1.

Upper Sum - $U(f,P)=\sum_{i=1}^{n} M_i(x_i-x_{i-1})$

function, Partition



Example: Let $f:[0,b]\to\mathbb{R}$ be given by $f(x):=x$ $\forall x\in[0,b]$

a) Write down a formula for the Partition P_n of $[0,b]$ into n subintervals of $\underline{\text{equal width}}$.

> Let $n\in\mathbb{N}$ and $P_n=\{0,b/n,\frac{2b}{n},\dots,\frac{nb}{n}\}=\{\frac{ib}{n}: i=0,1,2,\dots,n\}$

b) Calc $L(f,P_n)$ and $U(f,P_n)$

> $L(f,P_n)=\sum_{i=1}^{n} m_i(x_i-x_{i-1})$

$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}=\inf\{x; x\in[\frac{(i-1)b}{n},\frac{ib}{n}]\}=\inf[\frac{(i-1)b}{n},\frac{ib}{n}]=(i-1)b/n$

$\Rightarrow L(f,P_n)=\sum_{i=1}^{n}(i-1)b/n\,(b/n)=b^2/n^2\sum_{i=1}^{n}(i-1)$

Arithmetic sum $a_i=a+(i-1)d$ then $\sum_{i=1}^{n}a_i=n/2(2a+(n-1)d)$ so:

$\Rightarrow L(f,P_n)=b^2/n^2\left(n/2(0+(n-1))\right)=b^2/2\left(\frac{n-1}{n}\right)$

$U(f,P_n)=\sum_{i=1}^{n}M_i(x_i-x_{i-1})=\sum_{i=1}^{n}\frac{ib}{n}\cdot b/n=b^2/2\left(\frac{n+1}{n}\right)$ ← Similar to how $L(f,P_n)$ was calculated

c) Calc i) $\sup\{L(f,P_n): n\in\mathbb{N}\}$ and ii) $\inf\{U(f,P_n): n\in\mathbb{N}\}$

> $\sup\{L(f,P_n): n\in\mathbb{N}\}$

$=\sup\{b^2/2(1-1/n): n\in\mathbb{N}\}$

$\inf\{U(f,P_n): n\in\mathbb{N}\}$

$=\inf\{b^2/2(1+1/n): n\in\mathbb{N}\}$

$= \sup\{L(f,P_n) : n\in\mathbb{N}\}$

$= \sup\{\frac{b^2}{2}(1-\frac{1}{n}) : n\in\mathbb{N}\}$

$\lim n\to\infty$

$= \frac{b^2}{2}$

$\inf\{U(f,P_n) : n\in\mathbb{N}\}$

$= \inf\{\frac{b^2}{2}(1+\frac{1}{n}) : n\in\mathbb{N}\}$

$\lim n\to\infty$

$= \frac{b^2}{2}.$

23: Partition Lemma

10 November 2022 09:58

The Partition Lemma: (andrew morris' name, not well named in literature)

P, Q, R, S Partitions $[a,b]$

~~1. $L(f,P) \leq U(f,P)$~~

2. $P \subseteq Q \Rightarrow$ a. $\{ L(f,P) \leq L(f,Q)$ b. $\{ U(f,P) \geq U(f,Q)$

3. $L(f,R) \leq U(f,S)$

Based on $\inf Q \leq \inf P \leq \sup P \leq \sup Q$

Pf 1: $L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ ← All non negative.

Recall $m_i \leq M_i$ $\leq \sum_{i=1}^{n} M_i (x_i - x_{i-1})$

$= U(f,P)$.

Pf 2:

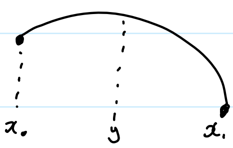
Case 1: Assume $\begin{cases} P = \{x_0, x_1, \dots, x_n\} \\ y \in (x_0, x_1) \\ Q = P \cup \{y\} \end{cases}$

$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = m_1 (x_1 - x_0) + \sum_{i=2}^{n} m_i (x_i - x_{i-1})$

$m_1 := \inf \{ f(x) : x \in [x_0, x_1] \}$

$m_1' = \inf \{ f(x) : x \in [x_0, y] \} \geq m_1$

$m_1'' = \inf \{ f(x) : x \in [y, x_1] \} \geq m_1$



$m_1 \leq m'$ $m_1 \leq m''$

$\Rightarrow L(f,P) = m_1 (y - x_0) + m_1 (x_1 - y) + \sum_{i=2}^{n} m_i (x_i - x_{i-1}) \leq m_1' (y - x_0) + m_1'' (x_1 - y) + \sum_{i=2}^{n} m_i (x_i - x_{i-1})$

was $x_1 - x_0 + y - y = x_1 - x_0$

Case 2: $y \in [a,b]$ similar

Case 3: $Q = P \cup \{y_1, y_2, \dots y_N\}$ for some $N \in \mathbb{N}$

$= P \cup \{y_1\} \cup \{y_2\} \cup \dots \cup \{y_N\}$

Iterate Case 2 N-times.

Pf 3: $L(f,R) \leq L(f, R \cup S)$ by 2a $R \subseteq R \cup S$

Pt 3: $L(\gamma, R) \leq L(\gamma, R \cup S)$ by 2a $\quad R \subseteq R \cup S$

$\leq U(\gamma, R \cup S)$ by 1

$\leq U(\gamma, S)$ by 2b $\quad S \subseteq R \cup S$

24: The Riemann-Darboux Integral

10 November 2022 11:59

Lower Integral - $\underline{\int_a^b} f := \sup\{L(f,P) : P \text{ Partition of } [a,b]\} = \sup_P L(f,P)$ ← Shorthand Notation

Upper Integral - $\overline{\int_a^b} f := \inf\{U(f,P) : P \text{ Partition of } [a,b]\} = \inf_P U(f,P)$

All Combinations

Note: $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ exist (completeness Axiom), however need not be the same.

Prop: $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

Proof: Partition Lemma: $L(f,R) \leq U(f,S) \quad \forall R, S$

$L(f,R) \leq \inf\{U(f,S) : S \text{ Partition of } [a,b]\} \quad \forall R$

upper bound

i.e $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ □

Not always equal (sometimes strict inequality). i.e:

$f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = 0$

always a point $= 0$

$U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = 1$

always a point $= 1$

$\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ is a Partition of } [a,b]\}$
$= \sup\{0\} = 0$

$\overline{\int_a^b} f = \inf\{U(f,P) : P \text{ is a Partition of } [a,b]\}$
$= \inf\{1\} = 1$

Integrable - A Property of a bounded function $f : [a,b] \to [0,\infty)$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$

Then $\int_a^b f := \underline{\int_a^b} f = \overline{\int_a^b} f$

Equal width

$\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ is Partition of } [a,b]\} \geq \sup\{L(f,P_n) : n \in \mathbb{N}\} = \frac{b^2}{2}$

$\int_a^b f = \inf\{U(f,P) : P \text{ is partition of } [a,b]\} \leq \inf\{U(f,P_n) : n \in \mathbb{N}\} = b^2/2$

"Sandwiching" $\quad b^2/2 \leq \underline{\int_a^b} f \underset{\uparrow \text{Prop.}}{\leq} \overline{\int_a^b} f \leq b^2/2 \Rightarrow \underline{\int_a^b} f = \overline{\int_a^b} f = b^2/2.$

↳ From example yesterday can now calculate the integral.

25: Equivalent Definition of Integrability

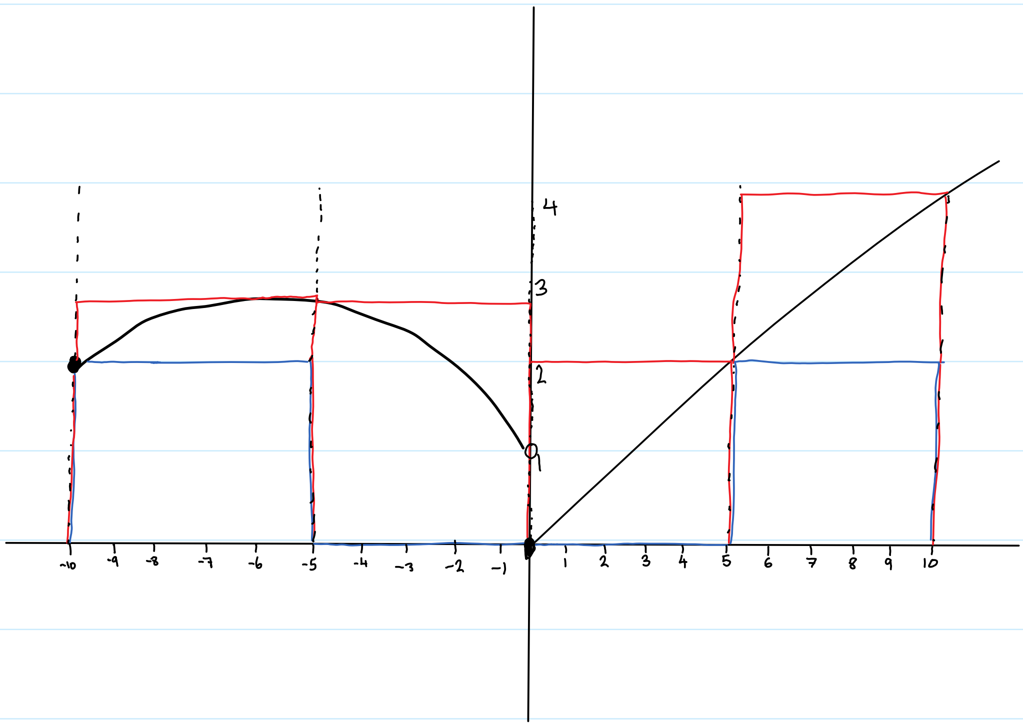
15 November 2022 14:02

Riemann's Criterion - Theorem: $f:[a,b]\to[0,\infty)$ bounded.

The following are equivalent (TFAE):

1) f is integrable

2) For each $\varepsilon>0$, there exists partition P such that $U(f,P)-L(f,P)<\varepsilon$.



Proof: (2)⇒(1): Suppose (2) holds.

Let $\varepsilon>0$, so by (2) there exists P_ε such that $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$

$\Rightarrow U(f,P_\varepsilon)<L(f,P_\varepsilon)+\varepsilon$

$\Rightarrow \overline{\int_a^b} f \leq U(f,P_\varepsilon)<L(f,P_\varepsilon)+\varepsilon \leq \underline{\int_a^b} f+\varepsilon$

$\overline{\int_a^b} f = \inf_P U(f,P_\varepsilon)$ $\qquad$ $\underline{\int_a^b} f = \sup_P L(f,P_\varepsilon)$

$\Rightarrow 0\leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq \varepsilon$ $\qquad$ $\forall \varepsilon>0$

From prop. $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f$

$\Rightarrow \underline{\int_a^b} f = \overline{\int_a^b} f$

Hence (1) holds.

<u>(1) ⇒ (2):</u> Suppose (1) holds, so $\underline{\int_a^b} f = \overline{\int_a^b} f$

Let $\varepsilon > 0$. Observe there exists P_1, P_2 such that:

$$\sup_P L(f,P) \longrightarrow \underline{\int_a^b} f - \varepsilon/2 < L(f,P_1) \le L(f, P_1 \cup P_2)$$

$$\inf_P U(f,P) \longrightarrow \overline{\int_a^b} f + \varepsilon/2 > U(f,P_2) \ge U(f, P_1 \cup P_2)$$

Subtract: $U(f,P_1 \cup P_2) - L(f,P_1 \cup P_2) < \varepsilon/2 - (-\varepsilon/2) = \varepsilon$

upper + lower integrals assumed to be equal by (1)

Hence (2) holds. □

Example: Use Riemann's criteron to prove that the following functions are integrable:

a) $f : [0,b] \to [0, +\infty)$, $f(x) := x$

> Let $n \in \mathbb{N}$ and P_n partition of $[0,b]$ into equal width.

Recall $U(f,P_n) = (1 + 1/n)\frac{b^2}{2}$, $L(f,P_n) = (1 - 1/n)\frac{b^2}{2}$. ⟵ From example last week.

Let $\varepsilon > 0$. Observe that:

$U(f,P_n) - L(f,P_n) = \frac{2}{n} \cdot \frac{b^2}{2} = \frac{b^2}{n}$ ✱

We want $\frac{b^2}{n} < \varepsilon$, i.e. $\frac{b^2}{\varepsilon} < n$

Choose $N \in \mathbb{N}$ such that $N > \frac{b^2}{\varepsilon}$

now capital as fixed

Then by ✱ $U(f,P_N) - L(f,P_N) < \varepsilon$.

Hence f integrable by Riemann's Criterion. □

b) $f:[0,2]\to[0,1]$, $f(x):=\begin{cases}1, & x=1\\ 0, & x\neq 1\end{cases}$

> Let $\delta\in(0,\frac{1}{100})$ and $P_\delta=\{0,1-\delta,1+\delta,2\}$

Recall $U(f,P_\delta)-L(f,P_\delta)=2\delta$ (*)

Let $\varepsilon>0$. We want $2\delta<\varepsilon \iff \delta<\varepsilon/2$

Now choose $\delta_0=\varepsilon/4$. So by (*)

$U(f,P_{\delta_0})-L(f,P_{\delta_0})=2\delta_0=\frac{2\varepsilon}{4}<\varepsilon$

Hence f integrable by Riemann's criterion.

Riemann Sums

Not assessed (extension)

26: Integrability of Monotonic Functions

16 November 2022 10:02

Monotonic Increasing - A function $f:[a,b]\to\mathbb{R}$ that $f(x)\le f(y)$ $\forall\, a\le x\le y\le b$.

Monotonic Decreasing - A function $f:[a,b]\to\mathbb{R}$ that $f(x)\ge f(y)$ $\forall\, a\le x\le y\le b$.

Prove that $f:[0,\pi/2]\to\mathbb{R}$ given by $f(x):=\sin x$ is integrable.

> Sketch Graph.

Let $n\in\mathbb{N}$ and $P_n=\{0,\frac{\pi}{2n},\frac{2\pi}{2n},\frac{3\pi}{2n},\frac{4\pi}{2n},\dots,\frac{n\pi}{2n}\}$ OR $P_n=\{\frac{i\pi}{2n}: i=0,1,\dots,n\}$

Calculate $U(f,P_n)-L(f,P_n)=\sum_{i=1}^{n}M_i\underbrace{(x_i-x_{i-1})}_{=\pi/2n}-\sum_{i=1}^{n}m_i(x_i-x_{i-1})$

$$=\frac{\pi}{2n}\left[\sum_{i=1}^{n}M_i-\sum_{i=1}^{n}m_i\right]=\frac{\pi}{2n}\left[\sum_{i=1}^{n}\sin\left(\frac{i\pi}{2n}\right)-\sum_{i=1}^{n}\sin\left(\frac{(i-1)\pi}{2n}\right)\right]$$

Let $j=i-1$ (change of variables)

$$=\frac{\pi}{2n}\left[\sum_{i=1}^{n}\sin\left(\frac{i\pi}{2n}\right)-\sum_{j=0}^{n-1}\sin\left(\frac{j\pi}{2n}\right)\right]$$

Just a dummy variable so can consider as i again if desired.

$$=\frac{\pi}{2n}\left[\sin\frac{n\pi}{2n}-\sin\frac{0\pi}{2n}\right]$$ The other terms all cancel

$$=\frac{\pi}{2n}[1-0]=\frac{\pi}{2n} \quad (*)$$

Let $\varepsilon>0$, We want $\frac{\pi}{2n}<\varepsilon \iff \frac{\pi}{2\varepsilon}<n$

Now choose $N\in\mathbb{N}$ such that $N>\frac{\pi}{2\varepsilon}$

(eg $N=\lceil\pi/2\varepsilon\rceil+1$)

Then by $(*)$

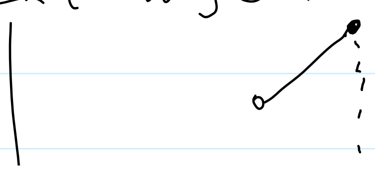
$U(f,P_n)-L(f,P_n)=\frac{\pi}{2n}<\varepsilon$ N?

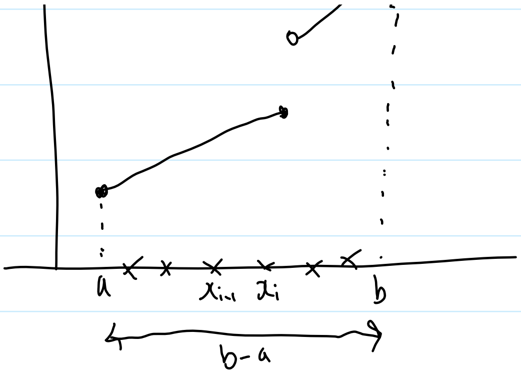
Hence f is integrable by Rienam's criterion

$\approx$ 1 Mark + 3 for Calculation

Thm: If $f:[a,b]\to\mathbb{R}$ is monotonic, then f is integrable.

Proof: (Increasing Case)





Let $n \in \mathbb{N}$ and $P_n = \{x_0, x_1, \dots x_n\}$ be equal-width Partition of $[a,b]$

$U(f,P_n) - L(f,P_n) = \sum_{i=1}^{n} (M_i - m_i)\underbrace{(x_i - x_{i-1})}_{=\frac{b-a}{n}}$

$= \frac{b-a}{n} \sum_{i=1}^{n} (M_i - m_i)$

$= \frac{b-a}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$

$= \frac{b-a}{n} [\cancel{f(x_1)} + \cancel{f(x_2)} + \dots + f(x_n) - f(x_0) - \cancel{f(x_1)} - \cancel{f(x_2)} - \dots \cancel{f(x_{n-1})}]$

$= \frac{b-a}{n} [f(x_n) - f(x_0)] = \frac{b-a}{n} [f(b) - f(a)]$ ⊛

Method of differences (a-level) is alternate method to the substitution.

Let $\varepsilon > 0$. We want $\frac{b-a}{n} [f(b) - f(a)] < \varepsilon$

$\Leftrightarrow \frac{(b-a)(f(b)-f(a))}{\varepsilon} < n$

Now choose $n \in \mathbb{N}$ such that

$n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$.

Alternative way of notating than N

Then by ⊛ $U(f,P_n) - L(f,P_n) < \varepsilon$

Hence f is integrable by Riemann's criterion. □

$< \varepsilon$ or $\leq \varepsilon$ doesn't matter.

27: Integrability of Continuous Functions

17 November 2022 10:01

Example: Define $f:[0,2]\to\mathbb{R}$ by $f(x) := \begin{cases} 2, & \text{if } x\neq 1 \\ 1, & \text{if } x=1 \end{cases}$

a) Calculate $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$ ← Doesn't matter what's happening at x, just points approaching so 2 not 1.

b) Is f continuous at $x=1$?

$\lim_{x\to 1} f(x) = 2$ but $f(1)=1$ and $1\neq 2$ hence not continuous at $x=1$.

c) Is f integrable at $x=1$?

Let $\delta \in (0, \frac{1}{100000})$ and $P_\delta = \{0, 1-\delta, 1+\delta, 2\}$ ← Delta type test

$$U(f,P_\delta) - L(f,P_\delta) = \sum_{i=1}^{3} (M_i - m_i)(x_i - x_{i-1})$$
$$= 0 + (M_2 - m_2)(2\delta) + 0$$
$$= (2-1)(2\delta) = 2\delta \quad (*)$$

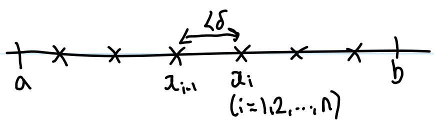
Let $\varepsilon>0$. Choose $\delta = \varepsilon/4$. Then by $(*)$

$$U(f,P_\delta) - L(f,P_\delta) = 2\delta = 2\varepsilon/4 = \varepsilon/2 < \varepsilon$$

Hence integrable by Riemann's Criterion.

Thm: If $f:[a,b]\to\mathbb{R}$ is continuous $\Rightarrow f$ is integrable.

Proof: Let $\varepsilon>0$. So $\frac{\varepsilon}{b-a}>0$. Since f is continuous there exists a $\delta>0$ such that $|x-y|<\delta \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{b-a}$ $(*)$



Let $n\in\mathbb{N}$ and P_n be equal width partition such that $|x_{i-1}-x_i|<\delta$.

$\therefore x,y\in[x_{i-1},x_i] \Rightarrow |x-y|<\delta$

$(*) \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{b-a}$.

Proof Exercise: $M_i = f(x)$ and $m_i = f(y)$ some $x,y\in[x_{i-1},x_i]$

$\Rightarrow |M_i - m_i| < \frac{\varepsilon}{b-a}$ ← → Use continuity. It is Extreme Value theorem (beyond course).

Now $U(f,P_n) - L(f,P_n) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\varepsilon(b-a)}{b-a} = \varepsilon$.

Hence f is integrable by Riemann's Criterion.

Uniformly Continuous - A function $f:[a,b] \to \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists \delta > 0$

Such that:
$$\left.\begin{array}{l} |x-y| < \delta \\ x \in [a,b] \\ y \in [a,b] \end{array}\right\} |f(x) - f(y)| < \varepsilon$$

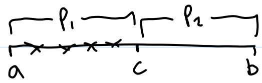
The difference is both x and y have same delta. Hence above theorem actually requires uniform continuity. BUT:

Thm: If $f:[a,b] \to \mathbb{R}$ is continuous, then f is uniformly continuous

can be continuous but not uniformly continuous if (a,b).

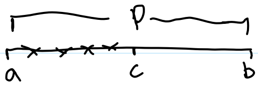
Extension/Restriction Theorem - If either side is defined, then $\int_a^b f = \int_a^c f + \int_c^b f \quad \forall a,b,c \in \mathbb{R}$
Where $\int_a^a f := 0$ and $\int_b^a f := -\int_a^b f$ when $a<b$.

Proof: (Rough) $a<c<b$
f integrable on $[a,c]$ and $[c,b]$ (using Riemann's criterion):



$P_1 \cup P_2$ is a partition for full interval

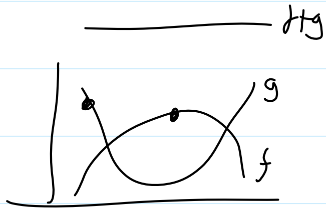
$U(f,P_1 \cup P_2) - L(f,P_1 \cup P_2) < 2\varepsilon$
Similar with f integrable on $[a,b]$:



$P \cup \{c\}$ then can split to P_1 and P_2

Recap: $(f+g)(x) := f(x)+g(x)$. $(\alpha f)(x) := \alpha f(x)$ $(fg)(x) := f(x)g(x)$
Linearity Theorem - If $\alpha \in \mathbb{R}$, $f,g:[a,b]\to\mathbb{R}$ integrable, then $f+g$ and αf integrable with
(1) $\int_a^b (f+g) = \int_a^b f + \int_a^b g$
(2) $\int_a^b (\alpha f) = \alpha \int_a^b f$

Proof (1): (Roughly)



Key Idea using R.C. $\begin{cases} U(f+g,P) \le U(f,P) + U(g,P) \\ L(f+g,P) \ge L(f,P) + L(g,P) \end{cases}$
Let $\varepsilon>0$
So there exists P_1 and P_2, $P=P_1\cup P_2$
$U(f,P)-L(f,P) \le U(f,P_1) - L(f,P_1) < \varepsilon/2$ ①
$U(g,P)-L(g,P) \le U(g,P_2) - L(g,P_2) < \varepsilon/2$ ②
①+②
$U(f,P)+U(g,P) - (L(f,P)+L(g,P)) < \varepsilon$
Using key idea
$U(f+g,P) - L(f+g,P) \le U(f,P)+U(g,P) - (L(f,P)+L(g,P)) < \varepsilon$

Hence $f+g$ integrable. $\square$

$f:[a,b]\to\mathbb{R}$ then

$$f^+(x) := \begin{cases} f(x) & \text{if } f(x)\geq 0 \\ 0 & \text{if } f(x)\leq 0 \end{cases} \qquad f^-(x) := \begin{cases} 0 & \text{if } f(x)\geq 0 \\ -f(x) & \text{if } f(x)\leq 0 \end{cases}$$

flips values

$f = f^+ - f^-$ ← Both $[0,\infty)$-valued.

We say f is integrable when both f^+ and f^- are integrable. In that case:

$$\int_a^b f := \int_a^b f^+ - \int_a^b f^-$$

Theorem: If $f,g:[a,b]\to\mathbb{R}$ integrable, then fg is integrable

Proof: (Rough) Take f integrable $\Rightarrow f^2$ integrable as fact (*)

$fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$ (can expand to see)

f,g integrable $\overset{\text{linearity}}{\Longrightarrow} f+g$ and $f-g$ are integrable.

$\overset{(*)}{\Longrightarrow} (f+g)^2$ and $(f-g)^2$ are integrable.

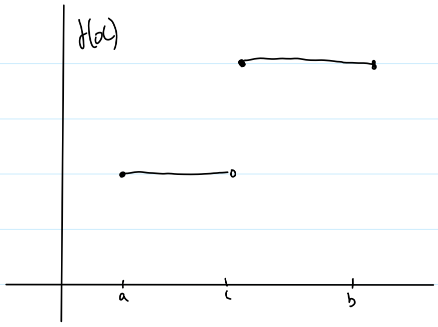
$\overset{\text{linearity}}{\Longrightarrow} \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$ is integrable. $\square$

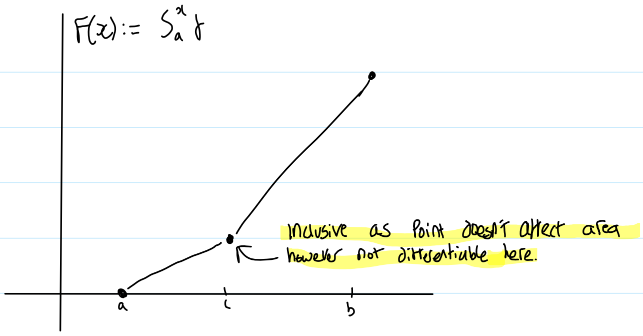
Theorem: If $f:[a,b]\to\mathbb{R}$ integrable and $m\leq f(x)\leq M \quad \forall x\in[a,b]$, $m,M\in\mathbb{R}$

$\Rightarrow m(b-a)\leq \int_a^b f \leq M(b-a)$

Proof: $\int_a^b f = \overline{\int_a^b} f = \inf_P U(f,P) \leq U(f,P_0) \leq M(b-a)$

$\int_a^b f = \underline{\int_a^b} f = \sup_P L(f,P) \geq L(f,P_0) \geq m(b-a)$





Integrals have smoothing properties (non continuous functions become continuous).

1st Fundamental Theorem of Calculus - Suppose $f:[a,b]\to\mathbb{R}$ bounded and integrable. If f is continuous at $c\in[a,b]$ then F is differentiable at c with $F'(c)=f(c)$ where $F(x):=\int_a^x f \quad \forall x\in[a,b]$

$\hookrightarrow \left[\frac{d}{dx}\left(\int_a^x f\right)\Big|_{x=c} = f(c)\right]$

Remark: $\begin{cases} F'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \\ F'(b) = \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \end{cases}$

Col: f continuous on $[a,b]$, then $F'(x)=f(x)$ for all $x\in[a,b] \Rightarrow \left[\frac{d}{dx}\left(\int_a^x f\right) = f(x)\right]$

Proof: $c\in(a,b)$ case: We want $F'(c)=\lim_{h\to 0}\frac{F(c+h)-F(c)}{h} = f(c)$

Observe: ① $F(c+h)-F(c) = \frac{1}{h}\int_a^{c+h} f - \frac{1}{h}\int_a^c f = \frac{1}{h}\int_c^{c+h} f$

② $\underbrace{\left[\inf_{x\in[c,c+h]} f(x)\right]}_{m}[c+h-c]\frac{1}{h} \le \frac{1}{h}\int_c^{c+h} f \le \underbrace{\left[\sup_{x\in[c,c+h]} f(x)\right]}_{M}[c+h-c]\frac{1}{h}$

$\lim_{h\to 0} m \to f(c)$ $\qquad \lim_{h\to 0} M \to f(c)$

Hence by Sandwich lemma / Squeeze Principle $\lim_{h\to 0}\frac{1}{h}\int_c^{c+h} f = f(c)$

Example: Prove that $F:[0,3]\to\mathbb{R}$, $F(x):=\int_0^x \frac{1}{1+\sin^2 t}\,dt$ is differentiable at $x=2$ and compute $F'(2)$.

> Let $f:[0,3]\to\mathbb{R}$ be given by $f(x):=\frac{1}{1+\sin^2 x}$ $\forall x\in[0,3]$.

By composition of continuous functions f is continuous.

So by 1st FTC F is differentiable with $F'(x)=f(x)=\frac{1}{1+\sin^2 x}$ for all $x\in[0,3]$.

Chain Rule:

$\frac{d}{dx}(\sin(3x)) = 3\cos(3x)$

$(f\circ g)'(x) = f'(g(x))g'(x)$ $\qquad \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$g(x)=3x \quad f(x)=\sin(x)$

Prove that $G:[0,3]\to\mathbb{R}$ given by $G(x):=\int_0^{3x}\frac{1}{1+\sin^2 t}\,dt$, $x\in[0,3]$ is differentiable and find formula for $G'(x)$ without integral symbol.

Let f be as before, then F is differentiable. Observe $G(x)=F(3x)$.

So if $g(x)=3x$, then $G(x)=(F\circ g)(x)$.

So G is differentiable by chain rule with $G'(x)=F'(g(x))g'(x) = \frac{1}{1+\sin^2(g(x))}\cdot 3$

$G'(x) = \frac{3}{1+\sin^2(3x)}$.

Prove that $H : [0,3] \to \mathbb{R}$ given by $H(x) := \int_x^3 \frac{1}{1+\sin^2 t}\,dt$, $x \in [0,3]$ is differentiable and find formula for $H'(x)$ without integral symbol.

$H(x) = \int_0^3 \frac{1}{1+\sin^2 t}\,dt - \int_0^x \frac{1}{1+\sin^2 t}\,dt$

$H(x) = C - F(x)$ some $C \in \mathbb{R}$

Hence $H'(x) = -F'(x) = -\frac{1}{1+\sin^2 x}$.

Antiderivative - Suppose $\Omega \subseteq \mathbb{R}$ and $f: \Omega \to \mathbb{R}$. An antiderivative of f is any (differentiable) function $g: \Omega \to \mathbb{R}$ such that $g' = f$. (in a-level an indefinite integral)

If $f: [a,b] \to \mathbb{R}$ is continuous write write down an antiderivative for f.
> Define $g: [a,b] \to \mathbb{R}$ by $g(x) := \int_a^x f$. Then by 1^{ST} FTC (since f is cont) we have $g'(x) = f(x)$.
Hence g is an antiderivative of f.
Remark: 1. If $g' = f$ then write $\int f := g$. $\int f = \int_a^x f + c$.
2. Not unique. This is an antiderivative (so no $+c$ needed).

2nd Fundamental Theorem of Calculus - If $f: [a,b] \to \mathbb{R}$ is bounded and integrable and $f = g'$ for some (differentiable) function g, then:

$$\int_a^b f = g(b) - g(a). \text{ OR } \int_a^b g' = g(b) - g(a)$$

How alevel Qs were solved.

Proof: (case when f is continuous):
Assume $f = g'$. Now define $F: [a,b] \to \mathbb{R}$ by $F(x) := \int_a^x f \quad \forall x \in [a,b]$.
We know $F' = f$ by example (1 FTC)
Hence

$$\begin{aligned} F' = f = g' &\Rightarrow F' - g' = 0 \\ &\Rightarrow (F - g)' = 0 \\ &\Rightarrow F(x) - g(x) = c \quad \text{some } c \in \mathbb{R} \end{aligned}$$

$F(a) = \int_a^a f = 0$

$$\begin{aligned} &\Rightarrow F(a) - g(a) = c \quad \text{so } c = -g(a): \\ &\Rightarrow F(x) = g(x) - g(a) \\ &\Rightarrow F(b) = g(b) - g(a) \\ &\Rightarrow \int_a^b f = g(b) - g(a) \end{aligned}$$

Find an antiderivative $\int f$ for each function f, $f(x) := x^n$, $x \in [a,b]$, $n \in \mathbb{N}$, $a < b$.
> Let $g: [a,b] \to \mathbb{R}$ be $g(x) := \frac{x^{n+1}}{n+1}$ for $x \in [a,b]$
Now as a polynomial, g is differentiable with $g'(x) = x^n = f(x)$.
Hence $g = \int f$.

Find an antiderivative $\int f$ for each function f, $f(x) := x^{-1}$, $x \in [a,b]$, $b < 0$
> Let $g: [a,b] \to \mathbb{R}$ be $g(x) := \log(-x)$ for $x \in [a,b]$
Then $g'(x) = \frac{-1}{-x}$ (chain rule).

$= \frac{1}{x} = f(x)$

Hence $g = \int f$

31: Theorems for integrating fg and fog

24 November 2022 10:02

Integration by Parts (Theorem): If $u,v:[a,b]\to\mathbb{R}$ are differentiable (⇒ integrable) and u' and v' are integrable then

$$\begin{cases} \int_a^b uv' = [uv]_a^b - \int_a^b u'v \\ \int uv' = uv - \int u'v \end{cases}$$

where $[uv]_a^b = u(b)v(b) - u(a)v(a)$

Proof: Product rule $(uv)' = u'v + uv'$

Observe $\int_a^b (uv)' = \int_a^b u'v + \int_a^b uv'$ (linearity)

All integrable so: $[uv]_a^b = \int_a^b u'v + \int_a^b uv'$ (2nd FTC)

Then rearrange.

Example 1: $\int_0^{\pi/2} x\sin x \, dx$

Let $u = x$ $\quad v = -\cos x$

so $u' = 1$ $\quad v' = \sin x$

Now by IBP we have

$$\int_0^{\pi/2} \underset{u}{x}\underset{v'}{\sin x}\, dx = [-x\cos x]_0^{\pi/2} - \int_0^{\pi/2} -\cos x \, dx = 0 - 0 + \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1.$$

Example 2: Find the antiderivative of $f:\mathbb{R}\to\mathbb{R}$ where $f(x) := x^2e^x$ (i.e. $\int x^2e^x\,dx$)

Trick 1 By parts twice

Let $u = x^2$ $\quad v = e^x$

$u' = 2x$ $\quad v' = e^x$

IBP: $\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$

Due to 'an' antiderivative idea don't need to worry about +c's.

$\int xe^x\,dx$

Let $u = x$ $\quad v = e^x$

$u' = 1$ $\quad v' = e^x$

IBP: $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$

Hence $\int x^2e^x\,dx = x^2e^x - 2[xe^x - e^x] = e^x(x^2 - 2x + 2)$

Example 3: $\int \log(x)\,dx$ Trick 2: $\int \log(x)\cdot 1\,dx$

Let $u = \log x$ $\quad v = x$

$u' = 1/x$ $\quad v' = 1$

Now $\int \log(x) = x\log x - \int \frac{x}{x}\,dx = x\log x - x + b_0$ constant doesn't matter

Substitution (Theorem): If $g:[a,b]\to\mathbb{R}$ differentiable with g' integrable, and $f:\mathbb{R}\to\mathbb{R}$ continuous then: $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

In a-level Let $u = g(x) \to f(u)$ $\frac{du}{dx}dx$. Still do as always done it but have appreciation for theorem.

Proof: Let F denote antiderivative of f so $F' = f$. Now chain rule

$(F\circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$

$\therefore \int_a^b f(g(x))g'(x)\,dx = \int_a^b (F\circ g)' = (F\circ g)(b) - (F\circ g)(a) = F(g(b)) - F(g(a)) \overset{\text{2nd FTC}}{=} \int_{g(a)}^{g(b)} F' \leftarrow f$

32: Integrating Rational Functions

24 November 2022 11:57

Example 4: Find $\int_1^3 e^{x^2+1} 2x \, dx$

Let $u(x) = x^2+1$

so $\frac{du}{dx} = 2x$ $\{du = 2x \, dx\}$

$\therefore \int_1^3 e^{x^2+1} 2x \, dx = \int_1^3 (\exp \circ u)(x) u'(x) \, dx$

can go straight to here $\longrightarrow = \int_2^{10} e^u \, du = [e^u]_2^{10} = e^{10} - e^2$

How to solve $\int \frac{P(x)}{Q(x)} dx$: (Polynomials)

Step 0: If $\deg P \geq \deg Q$ then do long division

Step 1: Factorize $Q(x)$ into linear $(ax+b)$ + quadratics (ax^2+bx+c) Irreducable.

Step 2: Choose correct partial fraction expansion

Step 3: Integrate

Factor	P.F. Expansion
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^N$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_N}{(ax+b)^N}$
ax^2+bx+c	$\frac{A+Bx}{ax^2+bx+c}$
$(ax^2+bx+c)^N$	See notes (but unlikely to come up)

Solve $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$

$$
\begin{array}{r|l}
 & x + 1 \\
\hline
x^3-x^2-x+1 & x^4 \quad -2x^2+4x+1 \\
 & x^4-x^3-x^2+x \quad - \\
\hline
 & 0 - x^3 - x^2 + 3x + 1 \\
 & x^3 - x^2 - x + 1 \quad - \\
\hline
 & 4x
\end{array}
$$

So

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx = \int x+1 + \frac{4x}{x^3-x^2-x+1} dx$$

$Q(x) = x^3-x^2-x+1$: $Q(1) = 0 \Rightarrow (x-1)$ factor.

Find $Q(x) = (x-1)^2(x+1)$

$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$

$x^3-x^2-x+1 \quad - \quad x+1 \quad , \quad x-1 + (x-1)^2$

$4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$

$\Rightarrow \begin{cases} A+B=0 \\ -2A+C=4 \\ A-B+C=0 \end{cases} \Rightarrow \dots \Rightarrow \begin{cases} A=-1 \\ B=1 \\ C=2 \end{cases}$

↑ comparing coefficients.

$I = \frac{x^2}{2} + x + \int \frac{-1}{x+1} + \int \frac{1}{x-1} + \int \frac{2}{(x-1)^2}$

$\quad = \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1}$

• $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$ ① $(a>0)$ • $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|$ ②

Evaluate $\int \frac{x}{x^3-1}\,dx$ Trick: Using above

$Q(x) = x^3-1 = (x-1)(x^2+x+1)$

$\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$x = A(x^2+x+1) + (Bx+C)(x-1)$

$\begin{cases} A+B=0 \\ A-B+C=1 \\ A-C=0 \end{cases} \Rightarrow \dots \Rightarrow \begin{matrix} A=C=1/3 \\ B=-1/3 \end{matrix}$

$\therefore \int \frac{x}{x^3-1}\,dx = \frac{1}{3}\log|x-1| - \frac{1}{3}\int \frac{x-1}{x^2+x+1}\,dx$

Want $\quad x-1 = \underset{\text{to get } f'/f}{a(2x+1)} + b = 2ax + a + b$

Compare coef: $1 = 2a \Rightarrow a = 1/2$

$-1 = a+b \Rightarrow b = -1 - 1/2 = -3/2$

So

$\int \frac{x-1}{x^2+x+1} = \frac{1}{2}\int \frac{2x+1}{x^2+x+1} - \frac{3}{2}\int \frac{1}{x^2+x+1} = \underset{\text{(From 2)}}{\frac{1}{2}\log|x^2+x+1|} - \frac{3}{2}\int \frac{1}{(x+1/2)^2 + (\sqrt{3}/2)^2} = \frac{1}{2}\log|x^2+x+1| - \frac{3}{2}\underset{\text{(From 1)}}{\left(\frac{1}{\sqrt{3}/2}\right)\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)}$

$\therefore \int \frac{x}{x^3-1}\,dx = \frac{1}{3}\log|x-1| - \frac{1}{6}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

33: Integrating Trigonometric Functions

29 November 2022 14:00

Integrating $\int \sin^m(x)\cos^n(x)\,dx$ Strategy:

Ex1 $\int \sin^2 x \cos^3 x \, dx$ (sin E cos O)

$= \int \sin^2(x)\cos^2 x \cdot \cos x$

Hoping for one odd power as then determines substitution.

Let $u = \sin x$ so $du = \cos x \, dx$

$= \int u^2(1-u^2)\,du$

$= \int u^2 - u^4 \, du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5$

$= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x)$

Ex2 $\int \sin^{2k+1}(x)\cos^n(x)\,dx$ (sin O cos E)

$= \int \sin^{2k}(x)(\cos^n(x)) \cdot \sin(x)\,dx$ (*)

Let $u = \cos(x)$ so $du = -\sin x \, dx$

Observe $\sin^{2k}(x) = (\sin^2(x))^k = [1-\cos^2(x)]^k$

$\therefore$ (*) $= \int [1-u^2]^k u^n (-du)$

$$\begin{cases} \sin^2(x) = \frac{1}{2}(1-\cos(2x)) \\ \cos^2(x) = \frac{1}{2}(1+\cos(2x)) \\ \sin x \cos x = \frac{1}{2}\sin(2x) \end{cases}$$

Double Angle Formulas can be used for a reduction in order.

Ex3 $\int \sin^m(x)\cos^n(x)\,dx$ where m, n are even

$\int \cos^4 dx = \int (\cos^2(x))^2 dx$

$= \int [\tfrac{1}{2}(1+\cos(2x))]^2 \; dx$

$= \tfrac{1}{4} \int 1 + 2\cos(2x) + \cos^2(2x) \; dx$

$= \tfrac{1}{4} \int 1 + 2\cos(2x) + \tfrac{1}{2}(1+\cos(4x)) \; dx$

$= \tfrac{x}{4} + \tfrac{1}{4}\sin(2x) + \tfrac{x}{8} + \tfrac{1}{32}\sin(4x)$

$= \tfrac{3x}{8} + \tfrac{1}{4}\sin(2x) + \tfrac{1}{32}\sin(4x)$

$\tan^2 x = \sec^2 x - 1$ (from $s^2 + c^2 = 1$)

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

Integrating $\int \tan^m x \sec^n x \; dx$ Strategy:

$\int \tan^m(x) \sec^{2(n-1)}(x) \cdot \sec^2(x) \; dx$ Sec E

Let $u = \tan(x)$ so $du = \sec^2 x \; dx$

$\int u^m [u^2+1]^{n-1} du$

$\int \tan^{2n+1} x \sec^n x \; dx = \int \tan^{2n} x \cdot \sec^{n-1} \cdot \tan x \cdot \sec x \; dx$ tan O

So $u = \sec(x)$ So $du = \sec(x)\tan(x) \; dx$

$\int [u^2-1]^n u^{n-1} \; du$

$\tan^2 x = \sec^2 x - 1$

$\int \tan^m(x)\sec^n(x)\,dx$ tan E sec O

↳ ??

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \cos^{-1}x \cdot [\sin x\,dx]$

Let $u = \cos x$ so $du = -\sin(x)\,dx$

$= \int \frac{1}{u}(-du) = -\log|\cos x| = \log|\sec(x)|$

So $\int \tan x = \log|\sec(x)|$

or Quotient rule from a-level.

$\int \sec(x)\,dx = \int \sec(x)\frac{\sec x + \tan x}{\sec x + \tan x}\,dx$

$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x}$

$= \log|\sec x + \tan x|$

So $\int \sec x\,dx = \log|\sec x + \tan x|$

Application with RMS Voltage: $230 = \sqrt{50}$ ...

34: Further Integration Strategies

30 November 2022 10:02

$\int \sec^3(x)\, dx$

$\int \sec^3(x)\, dx = \int \sec^2(x) \sec(x)\, dx$

By Parts: Let $u = \sec(x)$ $\quad v = \tan(x)$

$u' = \sec(x)\tan(x)$ $\quad v' = \sec^2(x)$

$\int \sec^3(x) = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\, dx = \sec(x)\tan(x) - \int \sec(x)[\sec^2(x) - 1]\, dx$

$= \sec(x)\tan(x) - \int \sec^3(x)\, dx + \int \sec(x)\, dx$

$\Rightarrow 2\int \sec^3(x)\, dx = \sec(x)\tan(x) + \log|\sec(x) + \tan(x)|$

$\Rightarrow \int \sec^3(x)\, dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\log|\sec(x) + \tan(x)|$

Recursive Formulas:

For these Qs By Parts should always be first strategy.

Let $S(n) = \int \sin^n(x)\, dx \quad (n \in \mathbb{N})$.

a) Prove $nS(n) = -\sin^{n-1}(x)\cos(x) + (n-1)S(n-2) \quad \forall (n \geq 2)$

$> S(n) = \int \sin^{n-1}(x) \cdot \sin(x)\, dx$

Let $u = \sin^{n-1}(x)$ $\quad v = -\cos(x)$

$u' = (n-1)\sin^{n-2}(x)\cos(x)$ $\quad v' = \sin(x)$

Integration by Parts gives

$S(n) = -\sin^{n-1}(x) \cdot \cos x + (n-1)\int \sin^{n-2}(x)\underbrace{\cos^2(x)}_{= 1 - \sin^2(x)}\, dx$

$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x) - (n-1)\int \sin^n(x)\, dx$

$S(n) = -\sin^{n-1}(x)\cos(x) + (n-1)S(n-2) \underbrace{- (n-1)S(n)}_{\text{Add to LHS}}$

$nS(n) = -\sin^{n-1}(x)\cos(x) + (n-1)S(n-2)$

b) Find $\int \sin^3(x)\, dx$

$\Rightarrow \int \sin^3(x)\, dx = S(3) = -\frac{1}{3}\sin^2(x)\cos(x) + \frac{2}{3}S(1)$

$\Rightarrow \int \sin^3(x)\,dx = S(3) = -\frac{1}{3}\sin^2(x)\cos(x) + \frac{2}{3} S(1)$
$= -\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x)$ $\qquad S(1) = \int \sin(x)\,dx$

eg Let $u = 3x$ $\qquad \frac{du}{dx} = 3 \quad \sim$ "$du = 3dx$"
$u = g(x)$ $\qquad \to g'(x)$

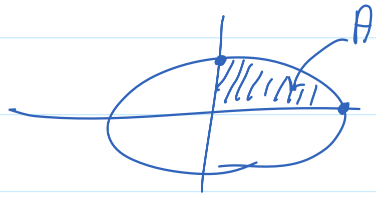
VS

eg Let $x = \sin\theta$ $\qquad \frac{dx}{d\theta} = \cos\theta \quad \sim$ "$dx = \cos\theta\, d\theta$"
$\to x = h(\theta)$
$\to \theta = h^{-1}(x)$ $\qquad$ Therefore in this case h needs to be bijective

Theorem (Bijective Substitution) - If $h: [a,b] \to [c,d]$ bijective differentiable and h' integrable, and $f: [a,b] \to \mathbb{R}$ continuous, then

$$\int_{h^{-1}(a)}^{h^{-1}(b)} f(h(\theta))h'(\theta)\,d\theta = \int_a^b f(x)\,dx$$

Eg Find area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$y = \pm b\sqrt{(1 - x^2/a^2)}$

$A = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$

Let $x = a\cos\theta$ $\quad$ so $\; dx = -a\cos\theta\, d\theta$

$x = 0 \Rightarrow 0 = \cos\theta \Rightarrow \theta = \pi/2$

$x = a \Rightarrow 1 = \cos\theta \Rightarrow \theta = 0$

$A = \frac{b}{a}\int_{\pi/2}^{0} \sqrt{a^2(1-\cos^2\theta)}\,(-a\sin\theta)\,d\theta$

keep a's and b's in place even if larger value on bottom.

$A = -ba\int_{\pi/2}^{0} \sin^2\theta\, d\theta$
$A = \frac{\pi ab}{4}$ $\qquad \sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$

So Total area of ellipse is πab.

35: Improper Integrals I

01 December 2022 10:06

Useful substitutions:

Term	Substitute
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ (OR $x = a\cos\theta$)
$\sqrt{a^2+x^2}$	$x = a\tan\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$

$\tan^2\theta + 1 = \sec^2\theta$

Eg $\int \frac{dx}{x^2\sqrt{x^2+4}}$

Let $x = 2\tan\theta$ so $dx = 2\sec^2\theta \, d\theta$

$= \int \frac{2\sec^2\theta \, d\theta}{4\tan^2\theta \sqrt{4(\tan^2\theta+1)}}$

$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} d\theta$

Let $u = \sin\theta$ so $du = \cos\theta \, du$

$= \frac{1}{4}\int \frac{du}{u^2} = \frac{-1}{4u} = \frac{-1}{4\sin\theta}$

$\tan\theta = x/2 = \frac{opp}{adj} \Rightarrow \sin\theta = \frac{opp}{hyp} = \frac{x}{\sqrt{x^2+4}}$

$\therefore \int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x}$

All methods of integration so far:

- Integration by Parts
- Substitution
- Partial Fractions
- Trig Integrals
- Trig Substitutions
- Recursive Formulas / Reduction Formulas.

Improper Integral - If $f:[a,\infty)\to\mathbb{R}$ bounded + integrable on $[a,t]$ for all $t>a$ then the improper integral is $\int_a^\infty f := \lim_{t\to\infty}\int_a^t f$ when it exists.

Remark: • $\int_{-\infty}^b f := \lim_{t\to-\infty}\int_t^b f$

- $\int_a^\infty f$ or $\int_{-\infty}^b f$ convergent means limit exists + finite
- otherwise divergent

Eg Determine if $\int_1^\infty \frac{dx}{x}$ is convergent

$$\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\left(\int_1^t \frac{dx}{x}\right)$$
$$= \lim_{t\to\infty}\Big[\log(|x|)\Big]_1^t$$
$$= \lim_{t\to\infty}\big[\log(t)-\log(1)\big]$$
$$= \lim_{t\to\infty}\log t$$
$$= +\infty$$

Two step problem, solve as normal integral then find limit.

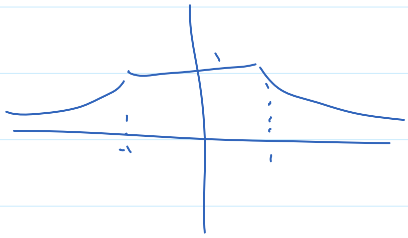
Hence divergent

If $f:\mathbb{R}\to\mathbb{R}$ and <u>both</u> $\int_a^\infty f$ and $\int_{-\infty}^a f$ convergent then $\int_{-\infty}^\infty f := \int_{-\infty}^a f + \int_a^\infty f$ (any $a\in\mathbb{R}$)

Eg $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) := \begin{cases} 1/x^2 & \text{if } |x|>1 \\ 1 & \text{if } |x|\le 1\end{cases}$

By symmetry just consider

$$\int_0^\infty f = \lim_{t\to\infty}\left[\int_0^t f\right]$$
$$= \lim_{t\to\infty}\left[\int_0^1 1\,dx + \int_1^t x^{-2}\,dx\right]$$
$$= 1 + \lim_{t\to\infty}\left[-x^{-1}\right]_1^t$$
$$= 1 + \lim_{t\to\infty}\left[\frac{-1}{t}+1\right]$$
$$= 1+0+1 = 2$$



Do separately NOT as $\lim_{t\to\infty}\left[\int_{-t}^t x\,dx\right]$

By symmetry $\int_{-\infty}^0 f = 2$

Hence $\int_{-\infty}^\infty f = 2+2 = 4.$

36: Improper Integrals II

01 December 2022 12:01

If $f:(a,b]\to\mathbb{R}$ is bounded and integrable on $[a+\varepsilon,b]$ $\forall\, \varepsilon>0$ (small), then the improper integral

$$\int_a^b f := \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^b f \quad \text{when limit exists}$$

* Convergent $\equiv$ exists + finite

Remark: $f:[a,b)\to\mathbb{R}$ $\quad \int_a^b f := \lim_{\varepsilon\to 0^+}\left(\int_a^{b-\varepsilon} f\right)$

If $f:[a,c)\cup(c,d]\to\mathbb{R}$ and both improper integrals $\int_a^c f, \int_c^b f$ convergent then $\int_a^b := \int_a^c f + \int_c^b f$

Evaluate a) $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

> $\int_2^5 \frac{1}{\sqrt{x-2}} = \lim_{\delta\to 0^+}\int_{2+\delta}^5 (x-2)^{-1/2}\,dx = \lim_{\delta\to 0^+}\left[\frac{(x-2)^{1/2}}{1/2}\right]_{2+\delta}^5 = \lim_{\delta\to 0^+}(2\sqrt{3}-2\sqrt{\delta}) = 2\sqrt{3}.$

b) $\int_0^{\pi/2}\sec(x)\,dx$

$\int_0^{\pi/2}\sec(x)\,dx = \lim_{\delta\to 0^+}\left(\int_0^{\pi/2-\delta}\sec(x)\,dx\right)$

$= \lim_{\delta\to 0^+}\Big[\log|\sec(x)+\tan(x)|\Big]_0^{\pi/2-\delta}$

$= \lim_{\delta\to 0^+}\Big[\log|\underbrace{\sec(\pi/2-\delta)}_{\to\infty}+\underbrace{\tan(\pi/2-\delta)}_{\to\infty}| - \log|\underbrace{\sec(0)}_{1}+\underbrace{\tan(0)}_{0}|\Big]$

$= +\infty$

$\therefore$ Divergent improper integral

c) $\int_0^3 \frac{dx}{x-1}\,dx$ $\qquad$ Check integrals for asymptotes

$= \lim_{\delta\to 0^+}\int_0^{1-\delta}\frac{1}{x-1}\,dx + \lim_{\delta\to 0^+}\int_{1+\delta}^3\frac{1}{x-1}\,dx$

$= \lim_{\delta\to 0^+}\Big[\log|x-1|\Big]_0^{1-\delta} + \lim_{\delta\to 0^+}\Big[\log|x-1|\Big]_{1+\delta}^3$

$= \underbrace{\lim_{\delta\to 0^+}\big(\log(\delta)-\log(1)\big)}_{=-\infty} + \ldots$

So $\int_0^1\frac{dx}{x-1}$ diverges, so $\int_0^3\frac{dx}{x-1}$ diverges

Integral Comparison Test (Theorem) - $\begin{cases} f,g : [a,\infty) \to [0,\infty) \\ 0 \le f(x) \le g(x) \ \forall x \in [a,\infty) \end{cases}$ then

1) $\int_a^\infty g$ convergent $\Rightarrow \int_a^\infty f$ convergent
2) $\int_a^\infty f$ divergent $\Rightarrow \int_a^\infty g$ divergent

Proof (idea): If $f(x) \ge g(x) \ \forall x \in [a,b]$ then $\int_a^b f \ge \int_a^b g$

$\left[m \le f(x) \le M \Rightarrow m(b-a) \le \int_a^b f \le M(b-a) \right]$

$\Rightarrow \int_a^b (f-g) \ge 0(b-a) = 0$

$\Rightarrow \int_a^b f - \int_a^b g \ge 0$

37: First-Order Separable Equations

06 December 2022 13:58

Notation: Let y denote function of x "$y = y(x)$"

$y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$

Differential Equations at a-level

Ordinary Differential Equations - With respect to only one variable (not Partial derivatives)

Suppose $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ ($\Omega_1, \Omega_2 \in \mathbb{R}$) and $I \subseteq \mathbb{R}$ an open interval. A solution of the ordinary Differential equation $y' = f(x, y)$ on I is a differentiable function $y: I \to \mathbb{R}$ such that $y'(x) = f(x, y(x))$ $\forall x \in I$.

Let $x_0, y_0 \in \mathbb{R}$. A solution of the initial Value Problem (IVP) $\begin{cases} y' = f(x,y) \text{ on } (x_0, \infty) \\ y(x_0) = y_0 \end{cases}$ is a function $y: [x_0, \infty) \to \mathbb{R}$ that is
- Continuous on $[x_0, \infty)$
- Differentiable on (x_0, ∞)
- Solution of ODE such that $y(x_0) = y_0$

Theorem: If $f: [x_0, \infty) \to \mathbb{R}$ is continuous, then IVP $\begin{cases} y' = f(x) \\ y(x_0) = y_0 \end{cases}$ has solution $y(x) := \left(\int_{x_0}^{x} f\right) + y(x_0)$ $\forall x \in [x_0, \infty)$

Pf: f is continuous $\Rightarrow$ f integrable and $y'(x) = f(x) + 0 = f(x)$ $\forall x \in [x_0, \infty)$

$\Rightarrow y$ differentiable on $[x_0, \infty) \Rightarrow y$ continuous on $[x_0, \infty)$

$y(x_0) = \int_{x_0}^{x_0} f + y(x_0) = 0 + y(x_0) = y(x_0)$.

Find solution $y: [0, \infty) \to \mathbb{R}$ of IVP

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+x+1}}$ $\quad y(0) = 3$

Use a-level calculus but now with understanding of analysis.

$> \int \frac{dy}{dx} dx = \int \frac{x}{\sqrt{x^2+x+1}} dx$

$y = \sqrt{x^2+x+1} - \frac{1}{2}\log\left|\sqrt{x^2+x+1} + x + \frac{1}{2}\right| + C$

$y(0) = 3 = 1 - \frac{1}{2}\log\frac{5}{2} + C \Rightarrow C = 2 + \frac{1}{2}\log\frac{5}{2}$

This is a solution IVP.

For others no such theorem, but formal methods (temporarily abandon rigor):

Separation of Variables:

To solve $y' = f(x) g(y)$
$\qquad\qquad\uparrow \quad \uparrow$
$\qquad\qquad x$ only $\quad y$ only

① First if $g(y_0) = 0$, then $y(x) := y_0$ is solution

② Assume $g(y) \neq 0$

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

$$\Rightarrow \int \frac{1}{g(y)} \frac{dy}{\cancel{dx}} \cancel{dx} = \int f(x)\, dx$$

$$\Rightarrow \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

③ Integrate and solve for y.

Eg Find solution $y: [0, R) \to \mathbb{R}$, for some $R > 0$, of IVP $\begin{cases} y' = (y^2 - 1)x \\ y(0) = 2 \end{cases}$

① If $y^2 - 1 = 0$, then $y_1 = +1$ or $y_2 = -1$ are two solutions to the ODE, BUT not solutions to IVP.

② Assume $y^2 - 1 \neq 0$

$$\int \frac{1}{y^2-1} \frac{dy}{\cancel{dx}} \cancel{dx} = \int x\, dx$$

$$\int \frac{1}{y^2-1}\, dy = \int x\, dx$$

$$\int \frac{-1/2}{y+1}\, dy + \int \frac{1/2}{y-1}\, dy = \tfrac{1}{2} x^2 + c \quad \text{(Using Partial fractions)}$$

$$-\tfrac{1}{2} \log|y+1| + \tfrac{1}{2} \log|y-1| = \tfrac{1}{2} x^2 + c$$

$$\log\left|\frac{y-1}{y+1}\right| = x^2 + c \qquad\qquad c \text{ can vary from line-to-line.}$$

$$\left|\frac{y-1}{y+1}\right| = c e^{x^2}$$

$$\frac{y-1}{y+1} = c e^{x^2} \quad \text{(try without modulus)}$$

$$y(c e^{x^2} - 1) = -1 - c e^{x^2}$$

$$y(x) = \frac{1 + c e^{x^2}}{1 - c e^{x^2}}$$

$$y(0) = 2 \Rightarrow \frac{1+c}{1-c} = 2 \Rightarrow c = \tfrac{1}{3}$$

Have to check it meets conditions: $y(x) = \dfrac{1 + \frac{1}{3} e^{x^2}}{1 - \frac{1}{3} e^{x^2}} \leftarrow 0$ when $x = \sqrt{\log 3}$

38: First-Order Linear Equations

07 December 2022 10:01

$y' + g(x)y = f(x)$

Linear with respect to y (not x).

Integration Factor

(From a-level F Maths).

$I(x) = e^{\int_a^x g}$

Let $F(x) := \int_a^x g$ is differentiable by FTC because g is continuous with $F'(x) = g(x)$.

$I(x) = e^{F(x)} = (\exp \circ F)(x)$

By chain rule I is differentiable with

$I'(x) = \exp(F(x))F'(x)$ $\qquad I'(x) = Ig$

$= e^{F(x)} g(x)$

$= I(x)g(x)$

Integration Factor Method:

To solve $y' + g(x)y = f(x)$ (linear first order)

① Find IF. $I(x) = e^{\int g(x)\,dx}$

② Multiply ODE by I.

$Iy' + Igy = If$

$Iy' + I'y = If$

③ Integrate

$\int \frac{d}{dx}(Iy)\,dx = \int If\,dx$ (Product rule)

$Iy = \int If + C$

In this part of course C important.

$\therefore y(x) = \frac{1}{I(x)}\left(\int If + C\right)$

Can either write method or write IF then go straight to here

Eg Find solution $y : [0,\infty) \to \mathbb{R}$ of IVP $y' - 2xy = 2x$, $y(0) = 1$.

> $I = e^{\int -2x\,dx} = e^{-x^2}$

$Iy' - I2xy = I2x$

$(Iy)' = I2x$

$e^{-x^2}y = \int e^{-x^2} 2x \, dx$

$\quad = -e^{-x^2} + C$

$\therefore y = -1 + Ce^{x^2}$

Now $y(0)=1 \Rightarrow -1+C=1$

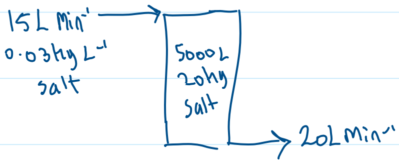
$\quad \Rightarrow C = 2$

Hence $y(x) = 2e^{x^2} - 1 \quad \forall x \in [0,\infty)$

Solves IVP as differentiable..., continuous..., initial data...

Example (Mixing Problem): A 5,000L mixing tank is filled with a salt solution containing 20kg of dissolved salt. The tank is drained at a rate of $20 L\,min^{-1}$ whilst a mixture containing $0.03\ kg L^{-1}$ of salt is pumped into the tank at a rate of $15 L\,min^{-1}$. Find the mass of salt in the tank after t minutes of mixing.

> Let $y(t)$ denote mass of salt (kg) at time t (minutes)



$y'(t) =$ Rate in $-$ Rate out, in $\frac{kg}{min}$. (Flow equation)

$$y'(t) = \left(0.03 \frac{kg}{L}\right)\left(15 \frac{L}{min}\right) - \left(\frac{y(t)}{5000-5t} \frac{kg}{L}\right)\left(20 \frac{L}{min}\right)$$

Requires more thought

$$(IVP) = \begin{cases} y'(t) = 0.45 - \dfrac{4y}{1000-t}, & 0<t<1000 \\ y(0) = 20 \end{cases}$$

As will be empty.

$$y' + \left(\frac{4}{1000-t}\right) y = 0.45$$

Let $I = e^{\int \frac{4}{1000-t} dt} = e^{-4\log|1000-t|} = e^{\log(1000-t)^{-4}}$

So $I = (1000-t)^{-4}$

Next $\frac{d}{dt}\left((1000-t)^{-4} y\right) = 0.45(1000-t)^{-4}$

$$(1000-t)^{-4} y = 0.45 \int (1000-t)^{-4}\, dt$$

$$= 0.45\left(\frac{-(1000-t)^{-3}}{-3}\right) + C$$

Crucial for I.F. method.

$$y(t) = 0.15(1000-t) + c(1000-t)^4$$

Need $y(0)=20 \Leftrightarrow 20 = 0.15(1000) + C(10^3)^4$

$\Leftrightarrow 20 - 150 = C\,10^{12}$

$\Leftrightarrow \quad C = -130 \times 10^{-12} = -1.3 \times 10^{-10}$

Hence $y(t) = \begin{cases} 0.15(1000-t) - 1.3\times10^{-10}(1000-t)^{3}, & 0 \le t < 1000 \\ 0, & t \ge 1000 \end{cases}$

Definition: $a,b,c \in \mathbb{R}$ and $I \subseteq \mathbb{R}$ open interval. A Solution of ODE $ay''+by'+cy=0$ ⊛ on I is a twice differentiable function $y: I \to \mathbb{R}$ such that $ay''(x)+by'+cy=0 \ \forall x \in I$.

Lemma: If y_1 and y_2 are solutions of ⊛ and $C_1, C_2 \in \mathbb{R}$, then $y := C_1y_1 + C_2y_2$ is also a solution of ⊛ (solutions far from being unique for homogenous equations).

Proof: observe
$$\begin{cases} ay_1''+by_1'+cy_1=0 & ① \times C_1 \\ ay_2''+by_2'+cy_2=0 & ② \times C_2 \end{cases}$$

①+② $a\underbrace{(y_1''+y_2'')}_{y''}+b\underbrace{(y_1'+y_2')}_{y'}+c\underbrace{(y_1+y_2)}_{y}=0$ $\quad y:=1y_1+1y_2$

$ay''+by'+cy=0$ where $(C_1=C_2=1)$

If $\lambda \in \mathbb{C}$ and $y(x) := e^{\lambda x} \ \forall x \in \mathbb{R}$ then compute $ay''+by'+cy$.

$> y(x)=e^{\lambda x} \to \begin{cases} y'=\lambda e^{\lambda x} \\ y''=\lambda^2 e^{\lambda x} \end{cases}$

$ay''+by'+cy=(a\lambda^2+b\lambda+c)e^{\lambda x} \overset{?}{=} 0$ if $(a\lambda^2+b\lambda+c)=0$.

From this exercise and lemma we can see to generate the infinitely many solutions we need two different solutions. Can also see where the characteristic equation comes from (Previously in alevel didn't come with explaination). Doesn't justify for other y though?

Method of Characteristic Equations:

To solve $ay''+by'+cy=0$ reduce to $a\lambda^2+b\lambda+c=0$ (characteristic equation).

Roots of $a\lambda^2+b\lambda+c$	General solution $(c_1, c_2 \in \mathbb{R})$
$\lambda_1 \neq \lambda_2$ both real.	$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$
$\lambda_1=\lambda_2$ real	$y=C_1e^{\lambda x}+C_2xe^{\lambda x}$ ← Added as 2 solutions needed to generate all solutions.
$\lambda_1=\alpha+i\beta, \lambda_2=\alpha-i\beta$	$y=e^{\alpha x}[C_1\cos(\beta x)+C_2\sin(\beta x)]$ ← From $e^{i\theta}=\cos\theta+i\sin\theta$. ↳ $e^{(\alpha+i\beta)x}=e^{\alpha x}e^{i\beta x}$

eg Find a general solution of $y''+3y'+2y=0$

Consider $\lambda^2+3\lambda+2=0$

$\Rightarrow (\lambda+1)(\lambda+2)=0 \Rightarrow \lambda_1=-1, \ \lambda_2=-2$

$\therefore y=C_1e^{-x}+C_2e^{-2x} \ \forall x \in \mathbb{R}$.

$\underline{\text{Def}^n}$: Let $x_0, x_1, y_0, y_1 \in \mathbb{R}$. A $\underline{\text{solution of}}$ IVP $\begin{cases} ay''+by'+cy=0 & \text{on } (x_0,\infty) \\ y(x_0)=y_0 \\ y'(x_0)=y_1 \end{cases}$

is solution $y:[x_0,\infty)\to\mathbb{R}$ of ODE such that:

① Continuous on $[x_0,\infty)$

② $y(x_0)=y_0$ and $y'(x_1)=y_1$

Boundary Valued Problem

A $\underline{\text{solution of}}$: BVP $\begin{cases} ay''+by'+cy=0 & \text{on } (x_0,x_1) \\ y(x_0)=y_0 \\ y(x_1)=y_1 \end{cases}$

a solution $y:[x_0,x_1]\to\mathbb{R}$ of ↑ such that

① Continuous on $[x_0, x_1]$

② $y(x_0)=y_0$ and $y(x_1)=y_1$

$\underline{\text{Eg}}$ Find solution $y:[0,\infty)\to\mathbb{R}$ of IVP $=\begin{cases} y''+36y=0 \\ y(0)=0 \\ y'(0)=3 \end{cases}$

Char eqn: $\lambda^2+36=0 \Rightarrow \lambda=0\pm 6i$

$\therefore y = C_1\cos(6x)+C_2\sin(6x)$ ← solution to ODE

When $y(0)=0 \Rightarrow C_1+0=0 \Rightarrow C_1=0$

When $y'(0)=3 \Rightarrow 0+6C_2=3 \Rightarrow C_2=1/2$

$\therefore y(x)=\frac{1}{2}\sin(6x) \quad \forall x\in[0,\infty)$ solves IVP.

$\underline{\text{Eg 2}}$ Find solution $y:[0,\frac{\pi}{2}]\to\mathbb{R}$ of BVP $\begin{cases} y''+36y=0 \\ y(0)=0 \\ y(\frac{\pi}{12})=1 \end{cases}$

$y=C_1\cos(6x)+C_2\sin(6x)$ (From eg 1)

$y(0)=0 \Rightarrow C_1=0$

$y(\frac{\pi}{12})=1 \Rightarrow C_2\sin(\frac{6\pi}{12})=1 \Rightarrow C_2=1$

$\therefore y(x)=\sin(6x) \quad \forall x\in[0,\frac{\pi}{2}]$

40: Second-Order Inhomogeneous Equations

08 December 2022 12:02

Theorem: The general solution to (*) $ay''+by'+cy=f(x)$ (inhomogenous as $f(x)\neq 0$) is given by $y=y_p+y_c$ where:

- y_p is any 'Particular solution': $ay_p''+by_p'+cy_p=f(x)$
- y_c is general homogenous solution: $ay_c''+by_c'+cy_c=0$

Proof: If y solves (*), then $ay''+by'+cy=f(x)$

So $a(y-y_p)''+b(y-y_p)'+c(y-y_p)$

$=(ay''+by'+cy)-(ay_p''+by_p'+cy_p)$

$=f(x)-f(x)=0$

ie. $y-y_p=y_c$ $\square$

Method of undetermined coefficients:

To find y_p:

Inhomogeneous $f(x)$	Trial y_p
Polynomial degree n	$y_p=A_0+A_1x+\dots A_nx^n$
exponential e^{ax}	$y_p=Ae^{ax}$
$\cos(ax)$ OR $\sin(ax)$	$y_p=A\cos(ax)+B\sin(ax)$

Remark: ① If y_p contains y_c, then multiply by x.

② Combined $f\to$ combined trial y_p ?

③ When in doubt, multiply by x in guess.

Repeated roots

Eg Find general solution of ODE $y''+4y'+4y=x^2-6x+2$

We know $y=y_p+y_c$

- y_c: Char Eqn $\lambda^2+4\lambda+4=0$

$\Leftrightarrow(\lambda+2)^2=0\Leftrightarrow\lambda=-2$

Hence $y_c=C_1e^{-2x}+C_2xe^{-2x}$ (last lecture)

Same from A-level F-Maths.

- y_p: We guess $y_p=Ax^2+Bx+C$

$\Rightarrow y_p'=2Ax+B$

$\Rightarrow y_p''=2A$

$y_p''+4y_p'+4y_p=x^2-6x+2$

$y_p'' + 4y_p' + 4y_p = x^2 - 6x + 2$

$\Rightarrow 2A + (8Ax + 4B) + (4Ax^2 + 4Bx + 4C) = x^2 - 6x + 2$

$\Rightarrow \begin{cases} 4A = 1 \\ 8A + 4B = -6 \\ 2A + 4B + 4C = 2 \end{cases} \Rightarrow \begin{cases} A = 1/4 \\ B = -2 \\ C = 19/8 \end{cases}$

Hence $y(x) = C_1 e^{-2x} + C_2 e^{-2x} + \frac{1}{4}x^2 - 2x + \frac{19}{8}$ is general solution.

<u>Eg.</u> Solve IVP = $\begin{cases} y'' + 3y' - 18y = 9e^{3x} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

$y = y_c + y_p$

- y_c: $\lambda^2 + 3\lambda - 18 = 0$

  $\Leftrightarrow (\lambda + 6)(\lambda - 3) = 0 \Leftrightarrow \lambda = -6, 3$

  $y_c = C_1 e^{-6x} + C_2 e^{3x}$

- y_p guess $y_p = Axe^{3x}$ so $y_p' = Ae^{3x} + 3Ae^{3x}$

  $y_p'' = 3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x} = 6Ae^{3x} + 9Axe^{3x}$

$(6A + 9Ax + 3A + 9Ax - 18Ax)e^{3x} = 9e^{3x}$

$9Ae^{3x} = 9e^{3x} \Rightarrow A = 1$

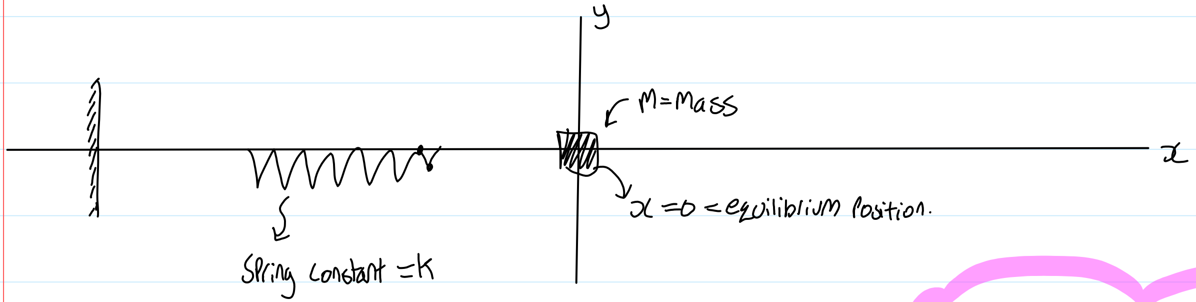
So $y = C_1 e^{-6x}\ C_2 e^{3x} + xe^{3x}$

$y(0) = 0 \Rightarrow C_1 + C_2 + 0 = 0$

$y'(0) = 0 \Rightarrow -6C_1 + 3C_2 + 1 + 0 = 0$

$\Rightarrow C_1 = 1/9,\ C_2 = -1/9$

Hence $y = \frac{1}{9}e^{-6x} - \frac{1}{9}e^{3x} + xe^{-3x} \quad \forall x \geq 0$



All in a-level F Maths too with Mrs Andrews!

Hooke's Law: $F = -kx$

Newton's Law: $F = mx''$

$mx'' = -kx \Rightarrow mx'' + kx = 0$

Which would have general solution $x(t) = A\cos(\sqrt{\frac{k}{m}}t) + B\sin(\sqrt{\frac{k}{m}}t)$

which would have general solution $x(t) = A\cos(\sqrt{k/m}\,t) + B\sin(\sqrt{k/m}\,t)$

Frictional Force: $F_f = -cx'$

$mx'' = -kx - cx' \Rightarrow mx'' + cx' + kx = 0$

$m\lambda^2 + c\lambda + k = 0$ so $\lambda = \frac{-c}{2m} \pm \sqrt{\frac{c^2-4mk}{2m}}$ Let $\Delta = c^2 - 4mk$

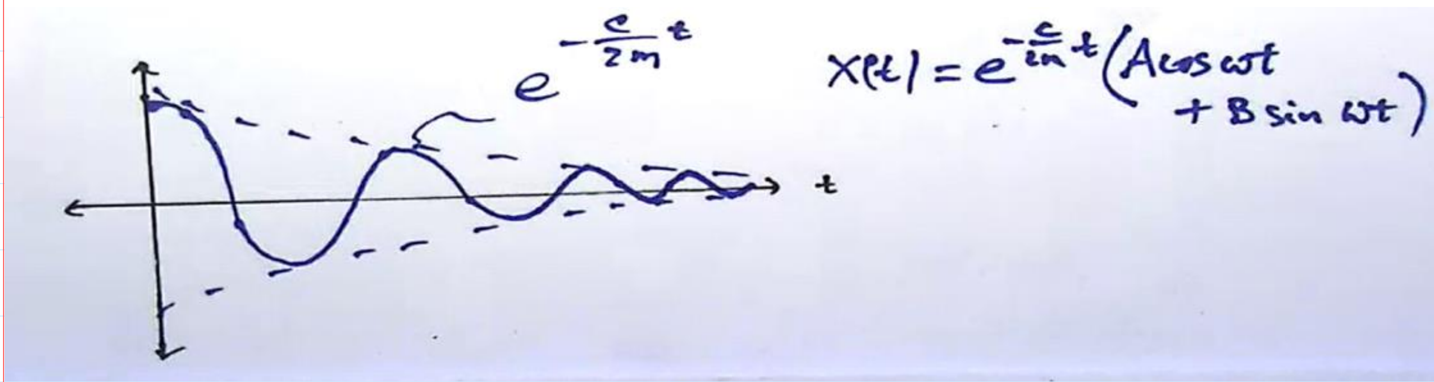
$\Delta > 0$ then 2 real $\Rightarrow$ Under damped

$\Delta = 0$ then 1 real $\Rightarrow$ Critically damped

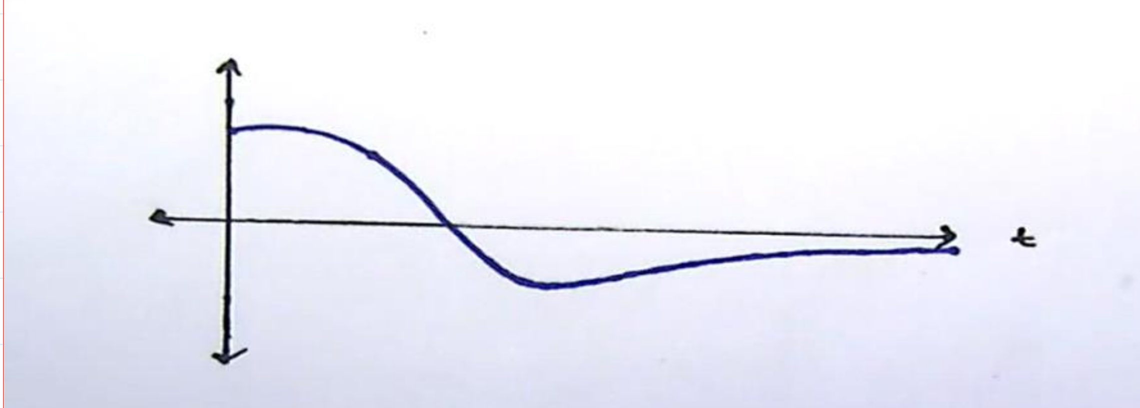
$\Delta < 0$ then 2 complex $\Rightarrow$ Over damped

Check against discriminant.

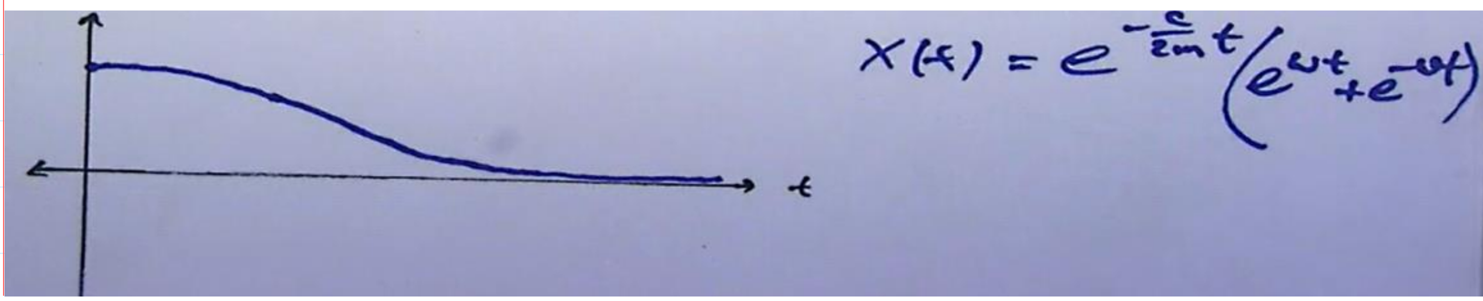
Under damped Sketch:

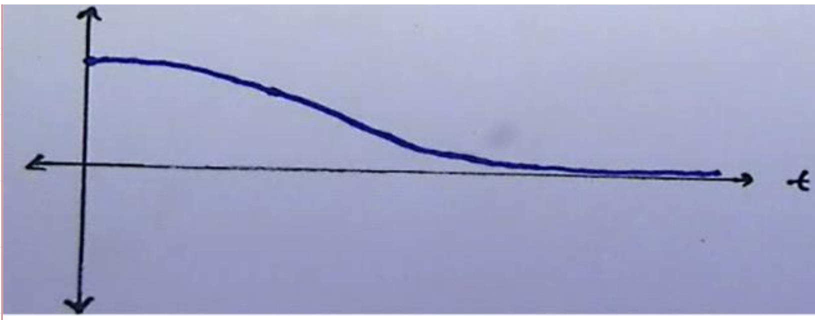


Critically Damped Sketch:



Over damped Sketch:





$$x(t) = e^{-\frac{c}{2m}t}\left(e^{\omega t} + e^{-\omega t}\right)$$

Revision

13 December 2022 14:16

$\lim_{x\to 0} \frac{\sin x}{x}$

$\lim_{x\to\infty} \left(1 \pm \frac{1}{x}\right)^x$ all notable + others

$\lim_{x\to 0} \frac{e^x - 1}{x}$

Intermediate value theorem + Boundedness theorem proofs are not required.

Rolle's theorem and Mean value theorem better to know how to proof

Important Exam Info

Func
Limits } need to accurately state these definition.
Derivatives

Slides on canvas.

Verify limit from definition ← always a Q on this and half students get wrong
Find derivates by using the definition } Most important Proofs

Most important Proofs II (less important but still a requirement)

- Algebra of limits
- Proof of notable limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$
- Leibniz rule in differentiation
- Rolle's Theorem
- Mean value Theorem

Important definitions:

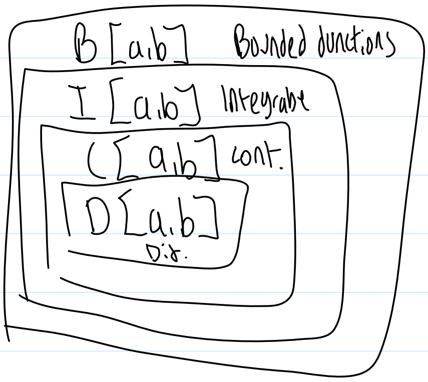
Sets, subsets, intervals

Functions, range, graph, composition, . . .

. . .

Revision #2

15 December 2022 15:18



Dif $\not\Rightarrow$ cont as $f(x)=|x|$

Cont $\not\Rightarrow$ Int $f(x) =$ 2, 1

Int $\not\Rightarrow$ Bound $f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$

Estimating integrals use $m \leq f(x) \leq M \quad \forall\, x \in [a,b]$

$\Rightarrow M(b-a) \leq \int_a^b f \leq M(b-a)$

FTC: $f:[a,b] \to \mathbb{R}$ integrable

abbreviations, in exam must use sentences.

1^{st} FTC $\left.\begin{array}{l} F(x) := \int_a^x f \\ f \text{ cont. @ } c \end{array}\right\} F'(c) = f(c)$

2^{nd} FTC $f = g' \Rightarrow \int_a^b f = g(b) - g(a)$

Prove $f(x) := \begin{cases} 1, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$ is integrable

Let $\varepsilon > 0$

Let $\delta \in \left(0, \frac{1}{1000}\right)$

Then $P_\delta := \{0, 1-\delta, 1\}$

$L(f, P_\delta) = \sum_{i=1}^{2} m_i (x_i - x_{i-1})$

$= 1(1-\delta) + 0 \cdot \delta$

$= 1 - \delta$

$U(f, P_\delta) = \sum_{i=1}^{2} M_i (x_i - x_{i-1}) = 1$

So in 'Prove' you cannot use Monotonic + continuous Proofs?

Now have $U(f, P_\delta) - L(f, P_\delta) = 1 - (1-\delta) = \delta$

We want $\delta < \varepsilon$, so choose $\delta := \varepsilon/2$

Then $U(f, P_{\varepsilon/2}) - L(f, P_{\varepsilon/2}) = \varepsilon/2 < \varepsilon$

Hence f is integrable by RC

$f(x) = \int_1^{2x} (\sin(t))^{\log(t)} \, dt \qquad x \in [2,3]$

Define $F(x) := \int_1^x (\sin(t))^{\log(t)}\,dt$

Then $g(t) := \sin(t)^{\log(t)}$ is continuous by composition of continuous functions

Hence 1^{st} FTC $\Rightarrow$ F is differentiable

With $F'(x) = (\sin(x))^{\log(x)}$

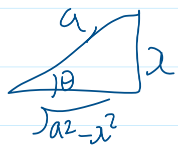
Now observe $f(x) = F(2x)$

By chain rule f differentiable

$f'(x) = F'(2x)\,(2x)' = 2(\sin(2x))^{\log(2x)}$

$x = a\sin\theta$

$\frac{x}{a} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$



So $\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$$\boxed{\begin{aligned} x &= a\sin\theta \\ \cos &= \sqrt{1-\sin^2\theta} \\ \cos\theta &= \sqrt{a^2-x^2} \end{aligned}}$$

---

linearity property for homogeneous equations

$\{(x,y): x\in\mathbb{R}, y\in\mathbb{R}\} = \mathbb{R}\times\mathbb{R}$

Closed: [Open: (Bounded: not (∞ or ∞)

$\mathbb{R}$

$\notin$ $\in$

$A = \{0,1\}$

$B = \{x\in\mathbb{R}: 0<x<1\}$

$\{x+3: x\in B$

A function takes a set A and set B and is a rule that assigns every element from set A a value in set B

$x \mapsto f(x)$

Preimage of $f(x)$. range/image

If bijective

$\Gamma f = \{(x,y)\in A\times B : y = f(x)\}$

$f: A\to B$ $id_S\ S\to \cancel{B}\, S$

$f\circ id_S = f|_S$ $f(S)\in A$

Codomain = domain

$g\circ f: A\to D$ $g\circ f := g(f(x))$ Valid if $f(A)\subseteq C$ Two sets so subset Not $\in$ as that's about elements

- $f: A\to B$ is invertible is there exists a $g: B\to A$ such that $\forall x\in A$ ~~$\exists$~~ and $y\in B$ where $f(x)=y \Leftrightarrow g(y)=x$

- Real-valued: $f: A\to B$ has a real valued inverse if f is ~~bijective~~ surjective and $\forall x\in A\ \exists y\in B$ such that $f(x)=y \Rightarrow g(y)=x$ where $g: B\to\mathbb{R}$.
If $f(A)=B$ and $g(B)=A$ AND $f(x)=y \Leftrightarrow g(y)=x$ $\forall x\in A, y\in B$.

Injective: $f: A\to B$ ~~$\exists$~~ $\forall$ $x, x'\in A$ if $f(x)=f(x')$, then $x=x'$
(1-1 function)

Surjective $f: A\to B$ f is surjective if $f(A)=B$. ✓

Bijective if f is both injective and surjective ✓

Invertibility $\Leftrightarrow$ bijective. ✓

injective $\Leftrightarrow$ real valued inverse ✗

injective $\Leftrightarrow$ real valued inverse ✗

The domain convension states that if unspecified, the domain of a function is all real values in which the function is defined (and codomain $\mathbb{R}$)

b) An elementary function is the composition ✓ of a single variable, defined with and arithmetic operations of any finite number of ~~key~~ basic functions: ....

c) $|x+y| \le |x| + |y|$ $\qquad |x-y| \ge \big||x|-|y|\big|$ ✓

d) $x^3 - 3x^2 + 2x > 0$

$x(x^2-3x+2) > 0$

$x(x-1)(x-2) > 0$ $\qquad$ $\otimes 2,\ \oplus 3 \} -2, -1$

		0	+	+	+	+	+
		−	−	0	+	+	+
		−	−	−	−	0	+
$x(x-1)(x-2)$	−	0	+	0	−	0	+

So $\{x \in \mathbb{R} : 0 < x < 1\} \cup \{x \in \mathbb{R} : x > 2\}$ or intervals $(0,1) \cup (2, \infty)$

set ↑

$f: A \to \mathbb{R}$

e) Even function if $\forall x \in A$, $-x \in A$ and $f(-x) = f(x)$

odd if $\forall x \in A$, $-x \in A$ and $f(-x) = -f(x)$.

f) Even: Symmetrical about y-axis. ✓

odd: Rotational symmetry of $180°$ ✓

a) Monotonicity if monotonic increasing or decreasing

Monotonic increasing if $f: A \to \mathbb{R}$ $\forall x, y \in A$ $\quad x < y \Rightarrow f(x) \le f(y)$

decreasing $\qquad \forall x, y \in A$ $\quad x < y \Rightarrow f(x) \ge f(y)$

Strictly inc./dec. $\Rightarrow$ injectivity.

b) Upper bound of A $\qquad \forall x \in A\ \exists y \in \mathbb{R}$ such that $x \le y$. then y is an upper bound

c) Bounded if an upper and lower bound exist for A, else unbounded ✓

d) Max A is greatest value in set A, need not exist if open interval

Min similarly. $\quad$ Max$A$: $x \in A$ such that $y \le x$ $\forall y \in A$

↑ upper bound but x in set so max.

e) Supremum is the smallest upper bound. $\min\{b \in \mathbb{R} : x \le b\ \forall x \in A\}$

"upper bound but x in set so max.

e) Supremum is the smallest upper bound. $\min\{b \in \mathbb{R} : x \leq b \ \forall x \in A\}$

Infimum is the greatest lower bound. ↑uses max and min for sup, inf

f) For functions consider the image

a) completeness axiom states that are no gaps on the number line, (because a supremum and infimum always exist)

b) For empty: $\inf A = \infty$ $\quad \sup A = -\infty$

If unbounded $\inf A = -\infty$ $\quad \sup A = \infty$

a) $\lim_{x \to \pm\infty} f(x) = L \in \mathbb{R}$ $\quad (f: \Omega \to \mathbb{R})$

$\forall \varepsilon > 0$ there exists $N > 0$ $N \in \mathbb{R}$ such that if the limit is valid:

$|f(x) - L| < \varepsilon \Rightarrow$

$\forall x \in \Omega$ and $x > N / x < N \Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to \infty} f(x) = L \in \mathbb{R}$

If the limit is valid, $\forall \varepsilon > 0 \ \exists N \in \mathbb{R}$ such that

$\forall x \in \Omega$ and $x > N \Rightarrow |f(x) - L| < \varepsilon$

b) $\lim_{x \to \infty} f(x) = \infty$

If the limit is valid, $\forall M > 0 \ \exists N \in \mathbb{R}$ such that

$x \in \Omega$ and $x > N \Rightarrow f(x) > M$

c) $\lim_{x \to a_0} f(x) = L \in \mathbb{R}$

If the limit is valid, $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$\forall x \in \Omega$ and $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Let $a_0 \in \mathbb{R}$ be an accumulation point of Ω

d) $\lim_{x \to a_0} f(x) = -\infty$

Let $a_0 \in \mathbb{R}$ be an accumulation point on Ω. If the limit is valid,

$\forall M > 0 \ \exists \delta > 0$ such that

$\forall x \in \Omega$ and $0 < |x - a| < \delta \Rightarrow f(x) < -M$.

e) $\lim_{x \to a^+} f(x) = L \in \mathbb{R}$

Let $a \in \mathbb{R}$ be an accumulation point on $\Omega \cap (a, \infty)$. If the limit exists,

$\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$\forall x \in \Omega \cap (a, \infty)$ and $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

f) Contradiction.

f) Contradiction.

a) An accumulation point $(a-\epsilon, a+\epsilon)$
a is an accumulation point of Ω if
$\forall \delta > 0\ \exists\, x \in \Omega$ such that $0 < |x-a| < \delta$

a) Yes
b) $\lim_{x\to a} f(x) = L$ if a^+ a^-

c) $\lim_{x\to 0} f(x) = \lim_{x\to 0} f|_{(-1,1)}(x) = 0$

a) Algebra of limits states if $\lim_{x\to a} f(x) = L_1$ $\lim_{x\to a} g(x) = L_2$
then $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
also product and quotient rules.
b) $\lim_{x\to a} |f(x)| = 0 \iff \lim_{x\to a} f(x) = 0$

c) If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ $\quad f,g,h : \Omega \to \mathbb{R}$

and $f(x) \le g(x) \le h(x)$ $\forall x \in \Omega \setminus \{a\}$ Sandwich theorem doesn't need a in $\forall x$
then $\lim_{x\to a} g(x) = L$.

d) Continuity: f is continuous at a is $\forall \epsilon > 0\ \exists\, \delta > 0$ such that
$\forall x \in \Omega$ and $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

b) If a not accumulation point $x=a$ is continuous
If it is an accumulation point and continuous $\Rightarrow \lim_{x\to a} f(x) = f(a)$.
d) $\lim_{x\to a} f(x) = L$
Then $\lim_{x\to a} g\circ f(x) = g\left(\lim_{x\to a} f(x)\right) = g(L)$

e) f, g continuous then $g\circ f$ continuous

a) $\lim_{x\to\infty}\left(\sqrt{2x-5} - \sqrt{4x+3}\right) = \lim_{x\to\infty}\left[\sqrt{x}\left(\sqrt{2-\frac{5}{x}} - \sqrt{4+\frac{3}{x}}\right)\right] \underset{=}{\text{algebra of limits}} \lim_{x\to\infty}\sqrt{x} \cdot \lim_{x\to\infty}\left[\sqrt{2-\frac{5}{x}} - \sqrt{4+\frac{3}{x}}\right]$
$= \infty \cdot \lim_{y\to 0^+}\left[\sqrt{2-5y} - \sqrt{4+3y}\right] = \infty \cdot \left[\sqrt{2} - 2\right] = \infty$

a) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\quad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$
$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e = \lim_{x\to-\infty} \left(1+\frac{1}{x}\right)^x$
$\lim_{x\to 0} (1+x)^{1/x} = e$
$\lim_{x\to 0} x \sin \frac{1}{x} = 0$ $\quad \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
$\lim \quad e^x - 1$

$\lim_{x\to 0} x\sin\frac{1}{x} = 0$

$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

$\lim_{x\to 0} \frac{e^x-1}{x} = 1$

$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

$\lim_{x\to 0} \frac{e^x-1}{x} = 1$

Intermediate Value Theorem: $f:[a,b]\to\mathbb{R}$ ~~bounded~~ continuous, $a<b$ $f(a)\le f(b)$, ~~$\exists\, c\in(a,b)$ such that $f(c)\in[f(a),f(b)]$~~ $\forall\, y\in[f(a),f(b)]$ $\exists\, x\in[a,b]$ such that $f(x)=y$.

Boundedness Theorem: $f:[a,b]\to\mathbb{R}$ continuous $a\le b$ $a,b\in\mathbb{R}$

① f is bounded

② (i) f attains its bounds ie $\exists\, x_m, x_M$ such that $f(x_m)=\min\{f(x): x\in[a,b]\}$

---

a) $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 1$

b) If exists then differentiable at $x=0$ for f. $C\in\mathbb{R}$ then differentiable.

c) Geometric tangent as $x\to a$, before gradient of line xA

Physical average velocity between x and A, as $x\to a$ instantaneous

---

a) Differentiable $\Rightarrow$ continuous

b) $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \frac{x}{x} = 1$

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \frac{-x}{x} = -1$

$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \ne \lim \ldots$ So not differentiable as $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist

c) $(f+g)' = f'+g'$

$(fg)' = fg'+gf'$

$\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$

Proof: $(fg)' = \lim_{x\to\Delta x} \frac{fg(x+\Delta x)-fg(x)}{\Delta x} = \lim_{x\to\Delta x}\left[\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}\right] =$

$\lim_{x\to\Delta x}\left[\frac{f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)+f(x+\Delta x)g(x)-f(x)g(x)}{\Delta x}\right]$

$= \lim_{x\to\Delta x}\left[\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)g(x) + f(x+\Delta x)\left(\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)\right] = \lim_{x\to\Delta x}\left[f'(x)g(x)+f(x+\Delta x)g'(x)\right]$

$= f'(x)g(x)+fg'(x)$

---

a) $(f(g(x)))' = g'(x)f'(g(x))$

b) $(g(f(x)))' = \frac{1}{f'(x)}$ where $g=f^{-1}$

$\arcsin(y) \underset{\cos}{\overset{y=\sin x}{=}} \arcsin'(\sin x) = \frac{1}{\cos x} \Rightarrow \frac{1}{\sqrt{1-\sin^2 x}} \underset{\cos}{\overset{y=\sin x}{=}} \frac{1}{\sqrt{1-y^2}}$

$(e^x)' = \lim_{\Delta x\to 0} \frac{e^{x+\Delta x}-e^x}{\Delta x} \overset{\text{and}}{=} e^x\left(\frac{e^{\Delta x}-1}{\Delta x}\right) \underset{\text{limit}}{\overset{\text{n take}}{=}} = e^x$

$$(e^x)' = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \overset{\text{AOL}}{=} e^x \left(\frac{e^{\Delta x} - 1}{\Delta x} \right) \overset{\text{notable limit}}{=} e^x$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \qquad \lim_{x \to 0} \frac{\log(x+1)}{x} = 1$$

$$(\ln(x))' \overset{x=e^y}{=} (\ln(e^y))' = \frac{1}{e^y} \overset{x=e^y}{=} 1/x$$

b) $\log_a x = \frac{\ln x}{\ln a}$

c) $(a^x)' = (e^{\ln a^x})' = (e^{x \ln a})' \overset{\text{chain rule}}{=} \ln a \cdot e^{x \ln a} = \ln a \cdot a^x$

d) $(x^\alpha) = (e^{\ln x^\alpha})' = (e^{\alpha \ln x})' \overset{\text{chain rule}}{=} (\alpha \ln x)' e^{\alpha \ln x} = \alpha \cdot 1/x \cdot x^\alpha = \alpha \cdot x^{\alpha - 1}$

e) e trick

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$(\tanh)' = \operatorname{sech}^2 x = 1 - \tanh^2 x$$

$f(x) = \sim$

$\ln f(x) = \_\_ - \_\_ \_\_$ ← partial fractions

$\frac{f'(x)}{f(x)} = \_\_$

$f'(x) = f(x)[\ldots]$

$f'' \quad f^{(2)} \quad f^{(n)}$

A function is n-times differentiable if $f^{(n)}$ is ~~defined $\forall n \in \mathbb{N}$~~ if $f^{(n)}(a)$ exists then n-times differentiable at $a \in \Omega$

d) C^∞ class are functions that are n-times differentiable (infinitely)

C^∞ class when f is infinitely differentiable, which is if f is n-times differentiable $\forall n \in \mathbb{N}$ $\quad f \in C^\infty$

A stationary point

If f is differentiable at a and $f'(a) = 0$ then stationary

b) Fermat's Theorem states that ~~stationary point $\Rightarrow f'(x) = 0$.~~

If a is local min or a local max of $f \Rightarrow$ differentiable at a + stationary point $\Leftarrow$ ~~$\not\Leftarrow$ e.g. inflection points~~ stationary

c) $f : [a,b] \to \mathbb{R}$ if: $a, b \in \mathbb{R}$ and $a < b$.

- f continuous on $[a,b]$
- f differentiable on (a,b)
- $f(a) = f(b)$

then $\exists\, c \in (a,b)$ such that $f'(c) = 0$

d) MVT: $f : [a,b] \to \mathbb{R}$ $\quad a, b \in \mathbb{R}$, $a < b$. If f:

- continuous on $[a,b]$
- differentiable on (a,b)

- continuous on $[a,b]$
- differentiable on (a,b)

Then $\exists\, c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

b) $f:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and $f'(x) = 0 \quad \forall x \in (a,b)$

$\forall x,y \in (a,b)$ $x<y$ We know continuous on $[x,y]$ and differentiable on (x,y).

By MVT $\dfrac{f(y)-f(x)}{y-x} = f'(c) = 0 \qquad c \in (x,y)$

$\therefore$ $f(y)=f(x)$, x,y were arbitrary points on interval $[a,b]$, hence constant function

f is differentiable + increasing on (a,b)

a) If f is an increasing function $\Leftrightarrow f'(x) > 0 \quad \forall x \in (a,b)$

Proven with MVT

both ways

Strictly just $\Rightarrow$ ex

b) Local min $\Leftarrow f''(a) > 0$

Local max $\Leftarrow f''(a) < 0$

from MVT $\quad$ f twice continuously differentiable on I & $f'(a)=0$ $\quad f: I \to \mathbb{R} \quad a \in I$.

THEN

c) $f: I \to \mathbb{R} \quad I \subseteq \mathbb{R} \quad \forall a,b \in I \quad t \in (0,1) \quad a<b$

Convex/concave upward $= f((1-t)a + tb) \le (1-t)f(a) + tf(b)$

$\hookrightarrow$ or always below tangent or line AB

Concave/concave downwards $= f((1-t)a + tb) \ge (1-t)f(a) + tf(b)$

$\hookrightarrow$ or always above tangent or line AB

---

a) From MVT:

① f convex $\Leftrightarrow f'$ increasing

f concave $\Leftrightarrow f'$ decreasing

② f convex $\Leftrightarrow f'' > 0$ if f is twice differentiable on I.

If about second derivative will require this

b) Cauchy MVT

$f,g: [a,b] \to \mathbb{R} \qquad a<b \quad a,b \in \mathbb{R}$ If f,g:

- continuous on $[a,b]$
- differentiable on (a,b)

Then $\exists\, c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

---

f,g defined on $(a-\gamma, a) \cup (a, a+\gamma)$ for some γ

L'Hospital Rule: $\quad f,g$ ~~differentiable~~

If $\lim_{x\to a} f = \lim_{x\to a} g = 0$ and $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists

Then $\lim_{x\to a} \dfrac{f}{g} = \lim_{x\to a} \dfrac{f'}{g'}$

b) $\lim_{x\to 0} \dfrac{\sin(ax)}{\sin(bx)} \underset{\text{Rule}}{\overset{\text{L'Hospital}}{=}} \lim_{x\to 0} \dfrac{a\cos(ax)}{b\cos(bx)} = \dfrac{a}{b} \dfrac{\lim_{x\to 0}\cos(ax)}{\lim_{x\to 0}\cos(bx)} \overset{\text{continuity}}{=} \dfrac{a}{b}\dfrac{\cos 0}{\cos 0} = a/b$.

$\lim_{x\to 0} \sin ax = \lim_{x\to 0} \sin bx = 0$

b) $\lim_{x \to 0} \frac{\sin(ax)}{\sin(bx)} \underset{\text{Rule}}{\overset{\text{L'Hôpital}}{=}} \frac{\lim_{x\to 0} a\cos(ax)}{\lim_{x \to 0} b\cos(bx)} = \frac{a}{b} \frac{\lim_{x\to 0}\cos(ax)}{\lim_{x\to 0}\cos(bx)} \overset{\text{continuity}}{=} \frac{a}{b}\frac{\cos 0}{\cos 0} = a/b.$

$\lim_{x\to 0} \sin ax = \lim_{x \to 0} \sin bx = 0$ ↑

c) When conditions met.

d) Common trick $x = \frac{1}{(1/x)}$ ↖ if $\lim_{x\to\infty}$ now in correct form

---

a) $\lim_{x\to\infty} \frac{x^2}{1+x^2} = \frac{1}{\frac{1}{x^2}+1} \overset{\text{AOL}}{=} \frac{1}{\lim_{x\to\infty}(\frac{1}{x^2}+1)} = \frac{1}{0+1} = 1/1 = 1$

b) $x^2+y^2=1$

$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$

$\Rightarrow 2x + 2y\frac{dy}{dx} -$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -x/y$

$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$

$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$

↑ so works for unbounded functions

d) A partition is a finite set of points of $[a,b]$ $P = \{x_0, x_1, \dots x_n\}$ such that $a = x_0 < x_1 < \dots < x_n = b$ for some $n \in \mathbb{N}$ ($x_0 = a$, $x_n = b$)

e) $L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$

$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ ✓

---

a) $\sup[a,b) = b$, $x \leq b$ $\forall x \in [a,b)$ so b is an upper bound of $[a,b)$.
Let $\varepsilon > 0$. Then $b - \varepsilon < b - \varepsilon/2 \in [a,b)$ hence $b-\varepsilon$ is not an upper bound. Hence b is the lowest upper bound $\Rightarrow$ supremum. ✓

b) $P_n = \{a + \frac{ib}{n} : i = 0, 1, \dots, n\}$ where $n \in \mathbb{N}$

c) $S_n = n/2\,(2a + (n-1)d)$ ✓

d) $S_n = \frac{a(1-r^n)}{1-r}$ ✓

---

a) Partition Lemma: P, Q, R, S are partitions

① $L(f,P) \leq U(f,P)$

② $P \subseteq Q$ i) $L(f,P) \leq L(f,Q)$ ii) $U(f,P) \geq U(f,Q)$

③ $L(f,R) \leq U(f,S)$

Proofs: ① back to m_i, M_i from inf and sup. $\Rightarrow m_i \leq M_i$ and all other parts non negative.

② Consider $P_n \cup \{y\}$ where $n \in \mathbb{N}$ $P = \{x_0, x_1, \dots x_n\}$ and $y \in (x_0, x_1)$ case.

Prove as necessary $Q = P \cup \{y\}$

Similarly for $y \in [a,b]$ between any two elements in P_n. × $Q = P \cup \{y_1, y_2, y_3 \dots y_N\}$ for $N \in \mathbb{N}$

Prove as necessary $Q = P \cup \{y\}$

Similarly for $y \in [a,b]$ between any two elements in P_n. $\times$ $Q = P \cup \{y_1, y_2, y_3 \dots y_N\}$ for $N \in \mathbb{N}$

for $P_n \cup Q = P_n \cup \{q_0\} \cup \{q_1\} \cup \dots \cup \{q_n\}$ and iterate step 2.

③ $\underset{2i}{L(f,R) \leq} L(f, R\cup S) \underset{1}{\leq} U(f, R\cup S) \underset{2ii.}{\leq} U(f,S)$

---

a) $\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$

$\overline{\int_a^b} f = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$

b) $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

c) $f : [a,b] \to \mathbb{R}$

If equal then f is integrable

$\int_a^b f := \underline{\int_a^b} f = \overline{\int_a^b} f$

---

a) $f : [a,b] \to \mathbb{R}$. Bounded $\varepsilon > 0$, f is integrable there exists a partition P such that

$U(f,P) - L(f,P) < \varepsilon$ ✓ Riemann's criterion requires to be bounded

b) $f : [0,b] \to [0,\infty)$ $f(x) := x$

$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1}) = x_{i-1} = \frac{(i-1)b}{n}$

$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i) = x_i = \frac{ib}{n}$

Let $n \in \mathbb{N}$, P_n be an equal width partition.

$U(f,P_n) - L(f,P_n) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1})^2 = \frac{b^2}{\cancel{n^2}\, n}$ ⊛

$x_i - x_{i-1} = \frac{b}{n}$

$\sum_{i=1}^{n} \frac{b^2}{n^2}$ ← n times

$= \frac{b^2}{n^2} \times n = \frac{b^2}{n}$.

Let $\varepsilon > 0$. We want $\frac{b^2}{n} < \varepsilon \Rightarrow \frac{b^2}{\varepsilon} < n$.

Choose $N \in \mathbb{N}$ where $N > \frac{b^2}{\varepsilon}$, then by ⊛

$U(f,P_N) - L(f,P_N) < \frac{b^2}{b^2/\varepsilon} = \varepsilon$

Therefore by Riemann's Criterion f is integrable.

$f : [0,2] \to [0,1]$ $\qquad f(x) := \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$

Let P_δ be a partition of $[0,2]$, $P_\delta = \{0, 1-\delta, 1+\delta, 2\}$ where $\delta > 0$.

$\times$ $\delta \in (0, \frac{1}{100})$

$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$

$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$

$U(f,P_\delta) = \sum_{i=1}^{3} M_i (x_i - x_{i-1}) = (1-\delta)\cdot 1 + (1+\delta - (1-\delta))\cdot 1 + (2-(1+\delta))\cdot 1 = 2\delta + 2 - 2\delta = 2$

$L(f,P_\delta) = \sum_{i=1}^{3} m_i (x_i - x_{i-1}) = (1-\delta)\cdot 1 + (1+\delta - (1-\delta))\cdot 0 + (2-(1+\delta))\cdot 1 = 2 - 2\delta$

$U(f,P_\delta) - L(f,P_\delta) = 2 - (2-2\delta) = 2\delta$ ⊛

Let $\varepsilon > 0$. We want $2\delta < \varepsilon \Rightarrow \delta < \varepsilon/2$

$U(f,P_\delta) - L(f,P_\delta) = 2 - (2-2\delta) = 2\delta$ Ⓐ

Let $\varepsilon > 0$. We want $2\delta < \varepsilon \Rightarrow \delta < \varepsilon/2$

Now choose $\delta_0 < \varepsilon/2$. Then by Ⓐ $U(f,P_{\delta_0}) - L(f,P_{\delta_0}) < \varepsilon$.

Hence f is integrable by Riemann's criterion.

a) Extension/restriction theorem states if $\int_a^b f$ exists then $\int_a^b f = \int_a^c f + \int_c^b f$ where $c \in (a,b)$.

b) Linearity: $f,g:[a,b] \to \mathbb{R}$ integrable then $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$ and $\int_a^b (\alpha f) = \alpha \int_a^b f$ $\alpha \in \mathbb{R}$.

c) ~~If~~ $f^+ := \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) \leq 0 \end{cases}$ ~~If~~ $f^- := \begin{cases} 0 & f(x) \geq 0 \\ -f(x) & f(x) \leq 0 \end{cases}$

Then if both exist, $\int f := \int f^+ - \int f^-$

d) fg integrable if f,g integrable. (just integrable, does not have a result $\int_a^b fg = \ldots$

a) If f bounded then ~~$\int_a^b f$~~ $f:[a,b] \to \mathbb{R}$ integrable and $m \leq f(x) \leq M$ $\forall x \in [a,b]$

$\Rightarrow m(b-a) \leq \int_a^b f \leq M(b-a)$

b) Smoothing Properties

c) $f:[a,b] \to \mathbb{R}$ integrable, and bounded. If f continuous at $c \in [a,b]$ then F is differentiable at c with $F'(c) = f(c)$ where $F(x) := \int_a^x f$ $\forall x \in [a,b]$

$\Leftrightarrow \left[\frac{d}{dx} \left(\int_a^x f \, dx \right) \Big|_{x=c} = f(c) \right]$

d) $F'(a) = \lim_{h \to 0^+} \cancel{f(a+h)}$ $\frac{f(a+h) - f(a)}{h}$

$F'(b) = \lim_{h \to 0^+} \cancel{f(b-h)}$ "

e) If f continuous on $[a,b]$ then $F'(x) = f(x)$ $\forall x \in [a,b]$

$\Leftrightarrow \frac{d}{dx} \left(\int_a^x f \, dx \right) = f(x)$

Try again:

If $f:[a,b] \to \mathbb{R}$ integrable and bounded. If f is continuous at $c \in [a,b]$ then F is differentiable with $F'(c) = f(c)$ where $F(x) := \int f$. $\forall x \in [a,b]$.

$\Leftrightarrow \left[\frac{d}{dx} \left(\int_a^x f \right) \Big|_{x=c} = f(c) \right]$

If continuous on $[a,b]$ then $F'(x) = f(x)$ $\forall x \in [a,b]$ ✓

$G:[0,3] \to \mathbb{R}$ given by $G(x) := \int_0^{x} \frac{1}{1+\sin^2 t} dt$ $x \in [0,3]$ is differentiable and find formula for $G'(x)$ without integral symbol.

Let $f:[0,3] \to \mathbb{R}$ $f(x) := \frac{1}{1+\sin^2 x}$ $\forall x \in [0,3]$

By composition of continuous functions $\frac{1}{1+\sin^2 x}$ is continuous

Let $f:[0,3]\to\mathbb{R}$ $f(x):=\frac{1}{1+\sin^2 x}$ $\forall x\in[0,3]$

By composition of continuous functions $\frac{1}{1+\sin^2 x}$ is continuous

Therefore, by the 1FTC $F'(x)$ is differentiable on $[0,3]$ with $F'(x)=f(x)=\frac{1}{1+\sin^2 x}$

Where $F:[0,3]\to\mathbb{R}$ $F(x):=\int_0^x \frac{1}{1+\sin^2 t}\,dt$.

$\Rightarrow$ if $g(x)=3x$ then $G(x)=(F\circ g)(x)$

Observe that $G(x)=F(3x)$, hence G is also differentiable. By chain rule,

$G'(x)=(3x)'F'(3x)=3\cdot\frac{1}{1+\sin^2 x}=\frac{3}{1+\sin^2 x}$.

---

a) An antiderivative is any Suppose $\Omega\subseteq\mathbb{R}$ and $f:\Omega\to\mathbb{R}$. An antiderivative of f is any differentiable function $g:\Omega\to\mathbb{R}$ such that $g'=f$.

b) $\int f := \int_a^x f + C$ ✓

c) Recall 1FTC: $f:[a,b]\to\mathbb{R}$ bounded, integrable. If f is continuous at $c\in[a,b]$ then F is differentiable with $F'(c)=f(c)$ where $F:[a,b]\to\mathbb{R}$ $F(x):=\int_a^x f$ $\forall x\in[a,b]$

$\Leftrightarrow \left[\frac{d}{dx}\left(\int_a^x f\right)\Big|_{x=c}=f(c)\right]$

If continuous on $[a,b]$ then $F'(x)=f(x)$ $\forall x\in[a,b]$

$F'(a)=\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$

$F'(b)=\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}$

2FTC: $f,g:[a,b]\to\mathbb{R}$ bounded, integrable. $g=\int f$ ($f=g'$) for some (differentiable) function g then

$$\int_a^b f = g(b)-g(a) \quad \text{OR} \quad \int_a^b g' = g(b)-g(a).$$

$$\int uv' = uv - \int vu'$$

LATE

log alg trig exp

$$\int 16x(x^2+5)^3\,dx$$

recog: $\frac{d}{dx}(x^2+5)^4 = 4(x^2+5)^3(2x) = 8x(x^2+5)^3$

Hence $2\int 8x(x^2+5)^3\,dx = 2(x^2+5)^4+C$.

Subst: Let $u=x^2+5$ $\quad du=2x\,dx$

Then $\int 8u^3\,du = \frac{8u^4}{4}+C = 2u^4+C = 2(x^2+5)^4+C$.

---

a) $u,v:[a,b]\to\mathbb{R}$ differentiable functions, and u' and v' are integrable. $\int_a^b uv'=[uv]_a^b-\int_a^b vu'$

OR without limits

b) ~~$f:[a,b]\to\mathbb{R}$ bounded, integrable. Let $f=g'$ where g is a (differentiable) function~~

~~f: [a,b] → ℝ bounded, integrable. Let f = g' where g is a (differentiable) function~~

If $g:[a,b]\to\mathbb{R}$ differentiable with g' integrable and $f:\mathbb{R}\to\mathbb{R}$ continuous then $\int_a^b f(g(x))\underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du$

↳ $(u=g(x)\quad du=g'(x)\,dx)$

Try again:

$f:[a,b]\to\mathbb{R}$ differentiable function with f' integrable. $g:\mathbb{R}\to\mathbb{R}$ a continuous function then $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

1FTC: $f:[a,b]$ bounded, integrable. If f is continuous at $c\in[a,b]$ then F is differentiable with $F'(c)=f(c)$ where $F:[a,b]\to\mathbb{R}$ $F(x):=\int_a^x f$.

2FTC: $f:[a,b]$ bounded, integrable. $f=g'$ where g is a (differentiable) function. then $\int_a^b f = g(b)-g(a)$.

An antiderivative of $f:[a,b]\to\mathbb{R}$ ~~integrable~~ is a (differentiable) function g such that $f(x)=g'(x)\quad\forall x\in[a,b]$.

$\Omega\subseteq\mathbb{R}$

$g:\Omega\to\mathbb{R}$

$$\frac{2x+7}{x^2+5x+6} = \frac{A}{x+3}+\frac{B}{x+2}$$

$(x+3)(x+2)$ $\quad (\times 6)\ (+5)\ 3,2$

$2x+7 = A(x+2)+B(x+3)$

i) $\cos 2x = \tfrac{1}{2}(1-\sin^2 x)$ $\quad \sin^2 x = \tfrac{1}{2}(1-\cos 2x)$

$= \tfrac{1}{2}(\cos^2 x - 1)$ $\quad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$

$\tan^2 x = \sec^2 x - 1$

$\sin x\cos x = \tfrac{1}{2}\sin 2x$

$$x^2-3x+2\,\overline{)\,3x^2-3x-2}$$

quotient 3

$3x^2-9x+6$ $-$

$\overline{(x-1)(x-2)}$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$\sqrt{a^2-x^2} \Rightarrow a\sin\theta$

$\sqrt{a^2+x^2} \Rightarrow a\tan\theta$

$\sqrt{x^2-a^2} \Rightarrow a\sec^2\theta$

$\sin 2\alpha = 2\sin\cos$
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
$= 2\cos^2\alpha - 1$
$= 1 - 2\sin^2\alpha$

$\int \sin^m x \cos^n x \; dx$
odd one comes out eg
$\int \sin^2 x \cos^2 x \cdot \cos x \; dx$
Let $u = \sin x$ $\quad du = \cos dx$
$\int u^2 (1-u^2) \; du$

...

~~If both even~~ $\int \sin^6 x \cos^8 x \; dx$
$\cancel{\sin^2 x = \tfrac{1}{2}(1-\cos 2x)}$ $\quad (\cos 2x = 1 - 2\sin^2 x)$
$\cancel{\cos^2 x = \tfrac{1}{2}(1+\cos 2x)}$
$\Rightarrow \int (\tfrac{1}{2}[1-\cos 2x])^3 (\tfrac{1}{2}[1+\cos 2x])^4 \; dx$

b) $\int \tan^m(x) \sec^n(x) \; dx$
If tan O sec E eg $\int \tan^5 x \sec^4 x \; dx$
Let $u = \tan x$ $\quad du = \sec^2 x \, dx$
$\Rightarrow \int u^5 (1-u^2) \; du$

If tan E sec O eg $\int \tan^8 x \sec^3 x \; dx$
Let $u = \sec x$ $\quad du = \sec x \tan x \; dx$
So $\int \tan^7(x) \sec^2(x) \cdot \sec x \tan x \; dx$
$\int \tan^7(x) \, u^2 \; du$

$\int \sec^3 x \; dx = \int \sec x \sec^2 x$
~~Let $u = \sec x$ $\quad du = \sec x \tan x \; dx$~~
By Parts $u = \sec x$ $\quad v' = \tan x$
$u' = \sec x \tan x$ $\quad v' = \sec^2 x$

b) $s(n) = \int \sin^n(x) \; dx = \int \sin^{n-1}(x) \sin x \; dx$
For recursive formulas always start with by parts

Let $h : [a,b] \to [c,d]$ bijective, differentiable function and h' integrable and $f : [a,b] \to \mathbb{R}$ continuous
$\int_a^b f'(g(x))\, g'(x) \; dx = \int$

$g:[a,b] \to \mathbb{R}$ continuous

$$\int_a^b f'(g(x))\, g'(x)\, dx = \int$$

$$\boxed{\int_{h^{-1}(a)}^{h^{-1}(b)} f(h(\theta))\, h'(\theta)\, d\theta = \int_a^b f(x)\, dx}$$

---

a) $\int_a^\infty f := \lim_{t\to\infty} \int_a^t f$ if $f:[a,\infty) \to \mathbb{R}$ bounded and integrable on $[a,t]$ $\forall t > a$

b) $\int_{-\infty}^\infty f := \lim_{b\to\infty} \int_0^b f + \lim_{b\to\infty} \int_{-b}^0 f$

$\int_{-\infty}^a f + \int_a^\infty f$ (any $a \in \mathbb{R}$)

c) Use symmetry / find seperately

d) $\int_a^b f = \lim_{t\to 0^+} \int_{a+t}^b f$ or $\lim_{t\to 0^+} \int_a^{b-t} f$

on $(a,b]$

$f,g:[a,b] \to \mathbb{R}$

$[a,\infty) \to [0,\infty)$

If $0 \le f(x) \le g(x)$ $\forall x \in [a, \cancel{b}\ \infty)$ then

① If $g(x)$ convergent $\Rightarrow f(x)$ convergent $\int_a^\infty g$ and $\int_a^\infty f$ not $g(x), f(x)$.

② If $f(x)$ divergent $\Rightarrow g(x)$ divergent.

$$x^3 \frac{dy}{dx} + 3x^2 y = \sin x$$

$$(x^3 y)' = \sin x$$

$$x^3 y = \int \sin x \, dx$$

$$x^3 y = -\cos x + C$$

$$y = \frac{-\cos x + C}{x^3}$$

$$\frac{dy}{dx} - 4y = e^x$$

I.F: $e^{\int -4\, dx} = e^{-4x}$

So $\times$ I.F:

$$e^{-4x} \frac{dy}{dx} - 4e^{-4x} y = e^x e^{-4x}$$

$$(e^{-4x} y)' = e^{-3x}$$

$$e^{-4x} y = \int e^{-3x}$$

$$e^{-4x} y = -\tfrac{1}{3} e^{-3x} + C$$

$$y = \frac{-\frac{1}{3} e^{-3x} + C}{e^{-4x}} = -\tfrac{1}{3} e^x + Ce^{4x}$$

$$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0$$

$$2\lambda^2 + 5\lambda + 3 = 0$$

b^2-4ac

$25-4(2)(3) = 25-24=1$ so 2 real roots

$(2\lambda+3)(\lambda+1)=0$

so $\lambda = -3/2$ and $\lambda=-1$

so $y = Ae^{-x} + Be^{-3/2x}$

2 real: $y = Ae^{\alpha x} + Be^{\beta x}$ where α, β are roots

1 repeated real: $y = (A+Bx)e^{\alpha x}$ α = root

complex $p \pm iq$ $y = e^{px}(A\cos qx + B\sin qx)$

b) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3$

complementary function: $\lambda^2 - 5\lambda + 6 = 0$

Particular solution: $y = \lambda$

$\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} = 0$

$6\lambda = 3$

$\lambda = 1/2$

So general solution: $CF + 1/2$

Particular solutions:

c) Constant: λ

Linear $\lambda x + \mu$

Polynomial $\cancel{A\lambda^2 + B\lambda + C + \dots}$ $Ax^2 + Bx + C$

Exp $e^{\alpha x}$ $\lambda e^{\alpha x}$

sin/cos $\lambda \sin \alpha x + \mu \cos \alpha x$

a) $\frac{d^2x}{dt^2} = v\frac{dv}{ds} =$ 3

Problem Sheets

RA1)

1) i) $f, g: \mathbb{R} \to \mathbb{R}$. $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$ $a, b \in \mathbb{R}$

$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ and $x > N \Rightarrow |f(x) - a| < \varepsilon$.

~~True but~~ how to prove?

False. Counterexample $f(x):=0$ $g(x):=\frac{1}{x}$ $\frac{1}{x}>0$ $\forall x\in\mathbb{R}$ however $\lim_{x\to\infty}\frac{1}{x}=\lim_{x\to\infty}0$.

~~ii) $f,g:\mathbb{R}\to\mathbb{R}$. If $\lim_{x\to\infty} f(x)=L$ and $\lim_{x\to\infty} g(x)=\infty$, $L\in\mathbb{R}$.~~
~~$\Leftrightarrow \forall\varepsilon>0\ \exists N\in\mathbb{R}$ such that $\forall x\in\mathbb{R}$ and $x>N \Rightarrow |f(x)-L|<\varepsilon$ ①~~
~~$\Leftrightarrow \forall M>0\ \exists N\in\mathbb{R}$ such that $\forall x\in\mathbb{R}$ and $x>N \Rightarrow g(x)>M$ ②~~
~~If $\lim_{x\to\infty} f(x)g(x)=\infty$ then $\forall M_0>0\ \exists N\in\mathbb{R}$ such that $\forall x\in\mathbb{R}$ and $x>N \Rightarrow f(x)g(x)>M_0$~~
~~From ② we know $g(x)>M$ and ① we know $|f(x)-L|<\varepsilon$~~

False. Counterexample: eg $f(x):=0$, $g(x):=x$

$\lim_{x\to\infty} 0=0$ and $\lim_{x\to\infty} x=\infty$

However, $\lim_{x\to\infty} 0\cdot x=\lim_{x\to\infty} 0=0$ (hence false). ✓

iii). Consider $f(x):=\frac{1}{x^2}$ $\lim_{x\to0^+} f(x)=\infty$ and $\lim_{x\to0^-} f(x)=\infty$ however $f(0)=\infty$
hence $f(0)\notin\mathbb{R}$ ✓

iv) ? Doesn't work for $g(x)=\begin{cases}1 & \text{if } x\neq0\\ 0 & \text{if } x=0\end{cases}$
as $\lim_{x\to0} g(x)\neq g(0)$ (only works for continuous functions).

RA1)

① i) ✓ $f:\mathbb{R}\to\mathbb{R}$ $f(x)=x^4$

Consider $x=1$, $x'=-1$. $f(1)=1$, $f(-1)=1$ hence not injective. ✓

Range $f(\mathbb{R})=[0,\infty)$ hence not surjective as codomain and image different.

Therefore not bijective either. As $x^4\geq0$ $\forall x\in\mathbb{R}$ Hence cannot have -1 for example

ii) ✓ $f:\mathbb{R}\to[0,\infty)$ $f(x)=x^4$

From i) still not injective

However codomain and image/range both $[0,\infty)$ hence surjective.

Not injective so not bijective

iii) ✓ $f:(-\infty,0]\to[0,\infty)$ $f(x)=x^4$ Could also show its invertible $\Rightarrow$ bijective $\Rightarrow$ injective and surjective

Assume $f(x)=f(y)$ $\forall x,y\in(-\infty,0]$,

$x^4=y^4$

$\sqrt[4]{x^4}=\sqrt[4]{y^4}$

$x=y$ (as only taking solutions in range $(-\infty,0]$.

Hence injective.

Codomain $[0,\infty)$ and range $f((-\infty,0])=[0,\infty)$ so surjective

Therefore as both injective and surjective it is bijective.

✓ iv) $f:\mathbb{R}\to\mathbb{R}$ $f(x)=x^3+1$. ✓ ‖

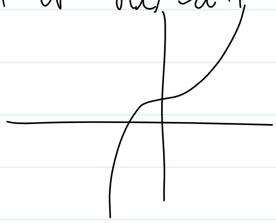
Assume $f(x)=f(y)$ $\forall x,y\in\mathbb{R}$

$x^3+1=y^3+1$

$x^3=y^3$

$x = y$ Hence f is injective

Graph of $f(x) = x^3+1$



Hence image / $f(\mathbb{R}) = \mathbb{R}$ and codomain is $\mathbb{R}$ hence surjective

injective and surjective $\Rightarrow$ bijective.

RA3

① i) If $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x^2+5 \quad \forall x \in \mathbb{R}$.

By definition of derivative, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2+5-[3x^2+5]}{h}$

$= \lim_{h\to 0} \frac{3(x^2+2xh+h^2)+5-3x^2-5}{h} = \lim_{h\to 0} \frac{3x^2+6xh+3h^2-3x^2}{h} = \lim_{h\to 0} \frac{6xh+3h^2}{\cancel{h}} = \lim_{h\to 0} 6x+3h$

$= 6x$ Hence $f'(x) = 6x$. ✓

ii) $g: \mathbb{R}\setminus\{1\} \to \mathbb{R}$ $\quad g(x) = \frac{3}{1-x} \quad \forall x \in \mathbb{R}\setminus\{1\}$ Newton Quotient

By definition, $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\frac{3}{1-(x+h)} - \frac{3}{1-x}}{h} = \lim_{h\to 0} \frac{\frac{3(1-x)}{(1-x-h)(1-x)} - \frac{3(1-x-h)}{(1-x-h)(1-x)}}{h}$

$= \lim_{h\to 0} \frac{3(1-x)-3(1-x-h)}{h(1-x-h)(1-x)} = \lim_{h\to 0} \frac{\cancel{3}-\cancel{3x}-\cancel{3}+\cancel{3x}+3h}{h(1-x-h)(1-x)} = \lim_{h\to 0} \frac{3}{(1-x-h)(1-x)} = \frac{3}{(1-x)^2}$.

iii) $K: \mathbb{R} \to \mathbb{R}$ $\quad H(x) = \sin(2x) \quad \forall x \in \mathbb{R}$

By definition $H'(x) = \lim_{h\to 0} \frac{H(x+h)-H(x)}{h} = \lim_{h\to 0} \frac{\sin(2x+2h)-\sin(2x)}{h}$ $\quad \lim_{h\to 0} \frac{\sin(2h)-\sin 0}{h}$

$\sin(2x+2h) = \sin 2x\cos 2h + \sin 2h \cos 2x$ (double angle formulae). $\quad \lim_{h\to 0} \frac{\sin 2h}{h}$

$H'(x) = \lim_{h\to 0} \frac{(\cos 2h-1)\sin 2x + \sin 2h \cos 2x}{h}$ $\quad = 2 \lim_{h\to 0} \frac{\sin 2h}{2h}$

notable limit = 1

For small angles $\sin\theta \approx \theta$ and $\cos\theta \approx 1-\frac{\theta^2}{2}$

so $\sin 2h \approx 2h \quad \cos 2h \approx 1-\frac{(2h)^2}{2} = 1-2h^2$ $\quad = 2$.

Sub in:

$H'(x) = \lim_{h\to 0} \frac{(1-2h^2-1)\sin 2x + 2h\cos 2x}{h} = \lim_{h\to 0} -2h\sin 2x + 2\cos 2x$

$= 2\cos 2x$ $\quad$ $h'(0)$ immediately

Hence $H'(0) = 2\cos(2\cdot 0) = 2\cos 0 = 2$. Same ans, dif method, then used

iv) $r: \mathbb{R} \to \mathbb{R}$ $\quad r(x) := \begin{cases} x^3 \sin\frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$\lim_{x\to 0} x^3 \sin\frac{1}{x^2}$

$\sin\frac{1}{x^2}$ is bounded so $|\sin\frac{1}{x^2}| < 1$

Hence $0 < |x^3 \sin\frac{1}{x^2}| < |x^3|$ $\quad$ Absolute rule for null limits

$\lim_{x\to 0} 0 = 0 \quad \lim_{x\to 0} |x^3| = 0$ Hence by Sandwich theorem $\lim_{x\to 0} |x^3 \sin\frac{1}{x^2}| = 0 \Rightarrow \lim_{x\to 0} x^3 \sin\frac{1}{x^2} = 0$

algebra of limits.

Therefore r is continuous.

By definition, $r'(x) = \lim_{h\to 0} \frac{r(x+h)-r(x)}{h} = \frac{(x+h)^3 \sin\frac{1}{(x+h)^2} - x^3\sin\frac{1}{x^2}}{h} = \frac{(x^3+3x^2h+3xh^2+h^3)\sin\frac{1}{(x+h)^2} - x^3\sin\frac{1}{x^2}}{h}$

$\Rightarrow$ aol $\lim_{h\to 0} \left[\frac{x^3\left(\frac{1}{\sin(x+h)^2} - \frac{1}{\sin x^2}\right)}{h}\right] + \lim_{h\to 0}\left[\frac{1}{\sin(x+h)^2}\left(3x^2+3xh+h^2\right)\right]$

$\Rightarrow \frac{3x^2}{\sin x^2}$

$r'(0) = \frac{3(0)^2}{\sin 0} = $ undefined

Try again

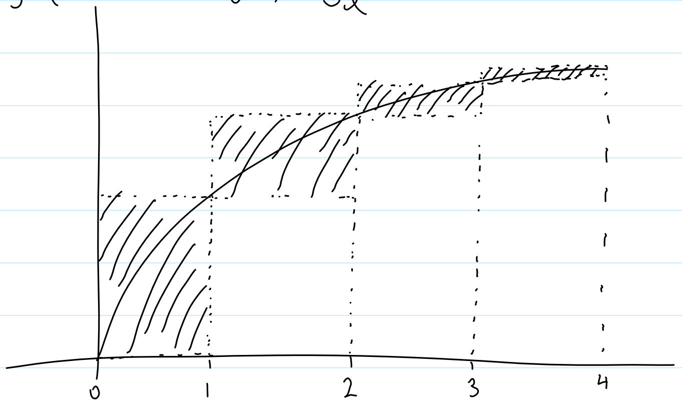
$r'(0) = \lim_{h\to 0} \frac{r(0+h)-r(0)}{h} = \lim_{h\to 0}\frac{r(h)-r(0)}{h} = \lim_{h\to 0}\frac{h^3\sin\frac{1}{h^2} - 0}{h} \overset{\text{dd}}{=} \lim_{h\to 0} h^2 \sin\frac{1}{h^2} = 0$

(Previous method.

RA4)

① $f:[0,4]\to\mathbb{R}$ $\quad f(x) := \sqrt{x}$

a)



b) i) $f:[0,3]\to[0,\infty)$, $\quad f(x) = x^2$

Let $n\in\mathbb{N}$ with P_n an equal width partition of $[0,3]$ of n sub intervals. or equal width

$U(f,P_n) - L(f,P_n) = \sum_{i=1}^n M_i(x_i - x_{i-1}) - \sum_{i=1}^n m_i(x_i - x_{i-1})$

$m_i = \inf\{f(x) : x\in[x_{i-1},x_i]\} = (x_{i-1})^2$

$M_i = \sup\{f(x) : x\in[x_{i-1},x_i]\} = (x_i)^2$ as f is an increasing function ($f'(x)>0\ \forall x>0$)

$= \sum_{i=1}^n (x_i - x_{i-1})^2(x_i + x_{i-1})$

$\Rightarrow U(f,P_n) - L(f,P_n) = \sum_{i=1}^n (x_i^2 - x_{i-1}^2)(x_i - x_{i-1}) = \sum_{i=1}^n (x_i - x_{i-1})^2 = (\frac{3}{n})^2 \times n = \frac{9}{n}$

$= \frac{b-a}{n} = \frac{3-0}{n}$ $\quad (x_i^2 - x_{i-1}^2) = (x_i + x_{i-1})(x_i - x_{i-1})$

$= \frac{ib}{n} + \frac{(i-1)b}{n}$

$= \frac{(2i-1)b}{n}$

Let $\varepsilon>0$. We want $U(f,P_n) - L(f,P_n) = \frac{9}{n} < \varepsilon$. ⊛

Choose N $n > \frac{9}{\varepsilon}$. Then by ⊛ $U(f,P_N) - L(f,P_N) < \varepsilon$.

Hence f is integrable by Riemann's criterion.

$\frac{9}{n^2}\sum_{i=1}^n \frac{6}{n}i - \frac{3}{n}$

$\frac{54}{n^3}\sum_{i=1}^n i - \frac{27}{n^3}\sum_{i=1}^n 1$

$\sum_{i=1}^n i = \frac{n}{2}(n+1)$

$\frac{54}{n^3}(\frac{n}{2})(n+1) - \frac{27}{n^2} = \frac{27}{n^2}(n+1-1) = \frac{27}{n}$.

ii) $f:[2,4]\to[0,100]$ $\quad f(x) = \begin{cases} 5, & x<3 \\ 100, & x=3 \\ 3, & x>3 \end{cases}$

Let P be the partition of $[2,4]$, $P_\delta = \{2, 3-\delta, 3+\delta, 4\}$ where $0<\delta<1$.
OR $\delta\in(0,\frac{1}{10})$.

$U(f,P_\delta) - L(f,P_\delta) = \sum_{i=1}^3 M_i(x_i - x_{i-1}) - \sum_{i=1}^3 m_i(x_i - x_{i-1})$

$m_i = \inf\{f(x) : x\in[x_{i-1},x_i]\}$ $\quad m_1 = 5 \quad m_2 = 3 \quad m_3 = 3$

$M_i = \sup\{f(x) : x\in[x_{i-1},x_i]\}$ Whilst $M_1 = 5 \quad M_2 = 100 \quad M_3 = 3$

$\Rightarrow U(f,P_n) - L(f,P_n) = (\cancel{5\cdot(3-\delta-2)} + 100\cdot(3+\delta-(3-\delta)) + \cancel{3\cdot(4-(3+\delta))}) - (\cancel{5\cdot(3-\delta-2)} + 3(3+\delta-(3-\delta)) + \cancel{3\cdot(4-(3+\delta))})$

$= 100\cdot(2\delta) - 3(2\delta) = 200\delta - 6\delta = 194\delta$ ✓

Let $\varepsilon>0$. We want $U(f,P_\delta) - L(f,P_\delta) = 194\delta < \varepsilon$. ⊛

Choose $\delta_0 = \frac{\varepsilon}{200}$, then by ⊛ $U(f,P_{\delta_0}) - L(f,P_{\delta_0}) < \varepsilon$. $\quad (\delta < \frac{\varepsilon}{194})$

Hence f integrable by Riemann's criterion. ✓

or just $\delta_0 < \frac{\varepsilon}{194}$ with $\delta_0 \in (0,\frac{1}{10})$

RA5)

$\int \sin^3(3x)\cos^5(3x)\,dx$

$= \int \sin^3(3x)\cos^4(3x)\cdot\cos(3x)\,dx$

Let $u=\sin(3x)$ so $du = 3\cos 3x\,dx$

$\frac{1}{3}du = \cos 3x\,dx$

then $\int \sin^3(3x)[1-\sin^2\theta]^2\cdot\cos(3x)\,dx$

$= \int u^3[1-u^2]^2\cdot\frac{1}{3}du.$

$= \frac{1}{3}\int u^3[1-2u^2+u^4]\,du = \frac{1}{3}\int u^3-2u^5+u^7\,du$

$= \frac{1}{3}\left[\frac{u^4}{4}-2\frac{u^6}{6}+\frac{u^8}{8}\right]+C$

$= \frac{u^4}{12}-\frac{u^6}{9}+\frac{u^8}{24}+C$

$\overset{u=\sin 3x}{=} \frac{\sin^4(3x)}{12}-\frac{\sin^6(3x)}{9}+\frac{\sin^8(3x)}{24}+C$ ✓

b) $\int\frac{\sin^7(x)}{\cos^4(x)}\,dx = \int \sin^7(x)\cos^{-4}(x)\,dx = \int\sin^6(x)\cos^{-4}(x)\cdot\sin(x)\,dx$

Let $u=\cos x$ so $du=-\sin x\,dx$

$\{-du=\sin x\,dx\}$

$\Rightarrow \int[1-\cos^2x]^3\cos^{-4}(x)\cdot\sin(x)\,dx$

$\overset{\text{change of variable}}{=} -\int[1-u^2]^3u^{-4}\,du = -\int[1^3+3(1)^2(-u^2)+3(1)(-u^2)^2+(-u^2)^3]u^{-4}\,du$

$=-\int[1-3u^2+3u^4-u^6]u^{-4}\,du = -\int u^{-4}-3u^{-2}+3-u^2\,du$

$=-\left[\frac{-u^{-3}}{3}+3u^{-1}+3u-\frac{u^3}{3}+C\right]$

$= \frac{u^{-3}}{3}-3u^{-1}-3u+\frac{u^3}{3}+C.$

$\overset{\text{cov}}{=} \frac{1}{3\cos^3x}-\frac{3}{\cos x}-3\cos x+\frac{1}{3}\cos^3x+C.$ ✓ an Antiderivative so no +c needed.

c) $\int_0^{\pi/4}(\tan(x)\sec(x))^8\,dx = \int_0^{\pi/4}\tan^8(x)\sec^8(x)\,dx$

$=\int_0^{\pi/4}\tan^8(x)\sec^6(x)\cdot\sec^2x\,dx$

Let $u=\tan x$ so $du=\sec^2x\,dx$

$u=\tan\frac{\pi}{4}=1$ $\quad u=\tan 0 = 0$

✗ $1+3u^2+3u^4+u^6$ b. expansion wrong. ⇒

$\overset{\text{cov}}{\Rightarrow}\int_0^1 u^8[u^2+1]^3\,du = \int_0^1 u^8[u^2+3u^4+3u^6+u^8]\,du$

$=\int_0^1 u^{10}+3u^{12}+3u^{14}+u^{16}\,du = \left[\frac{u^{11}}{11}+\frac{3u^{13}}{13}+\frac{3u^{15}}{15}+\frac{u^{17}}{17}\right]_0^1$

$=\left(\frac{1}{11}+\frac{3}{13}+\frac{3}{15}+\frac{1}{17}\right)-(0)$ Correct method though.

RA1)

② i) $\mathbb{N}$ $\quad A=\{x\in\mathbb{R}: \frac{1}{x}\in\mathbb{Q}\}$ ie $x\neq 0, x\in\mathbb{Q}$ ie $x=2.2$ $\frac{1}{2.2}=\frac{10}{22}\in\mathbb{Q}$

So $\mathbb{N}\subseteq A$ ✓

ii) $B=\{x\in\mathbb{R}: 3x\in\mathbb{Q}\}$ $\quad\mathbb{Q}$

If $3x\in\mathbb{Q}$ then $x\in\mathbb{Q}$

So $B=\mathbb{Q}$ ✓ Explain as $B\subseteq\mathbb{Q}$ and $\mathbb{Q}\subseteq B$.

iii) $C=\{x\in\mathbb{R}: 4x\in\mathbb{Z}\}$ $\quad D=\{x\in\mathbb{R}: x-3\in\mathbb{Z}\}$

If $4x\in\mathbb{Z}$ then $x\in\mathbb{Q}$ $\quad$ If $x-3\in\mathbb{Z}$ then $x\in\mathbb{Z}$.

Hence $D \subseteq C$. ✓ explain why not $C \subseteq D$

iv) $\mathbb{Q}$ $\quad E = \{x \in \mathbb{R} : \frac{1}{2-x} \in \mathbb{Q}\}$
$x \in \mathbb{Q}\ x \neq 2$

so $E \subseteq \mathbb{Q}$. ✓

RA 2)

② i) $\lim_{x\to\infty} \frac{3x^3-5x^2+7x-13}{2x^3-\pi} \begin{matrix}\div x^3\\ \div x^3\end{matrix} = \lim_{x\to\infty} \frac{3-5/x+7/x^2-13/x^3}{2-\frac{\pi}{x^3}}$

now as all limits exist we can apply algebra of limits:

$= \frac{\lim_{x\to\infty}(3-5/x+7/x^2-13/x^3)}{\lim_{x\to\infty} 2-\frac{\pi}{x^3}} \overset{\text{algebra of limits linear rule}}{=} \frac{3-0+0-0}{2-0} = 3/2.$ ✓

b) $\lim_{x\to 0} x\cos 1/x$

Note that cos is bounded such that $0<|\cos \frac{1}{x}|<1$

Therefore $0 < |x\cos 1/x| < x$.

$\lim_{x\to 0} 0 = 0$ and $\lim_{x\to 0} x = 0$, hence by the sandwich theorem $\lim_{x\to 0}|x\cos 1/x| = 0$.

From the absolute value with null limits rule, $\lim_{x\to 0}|x\cos 1/x| = 0 \Rightarrow \lim_{x\to 0} x\cos 1/x = 0$

Hence $\lim_{x\to 0} x\cos 1/x = 0$. ✓

ii) $f: \mathbb{R}\to\mathbb{R}$ $\quad f(x) = \sqrt[3]{x}$.

f is continuous at $x=a$ if $\forall \varepsilon>0, \exists \delta>0$ such that

$\forall x \in \mathbb{R}$ and $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$. ✓

Note that $\sqrt[3]{x}-\sqrt[3]{a} = x^{1/3}-a^{1/3} = (x-a)Q(x)$ where $Q(x) := \frac{x^{1/3}-a^{1/3}}{x-a}$

factor theorem. $x^{1/3}-a^{1/3} = \frac{x-a}{(x^{2/3}+a^{1/3}x^{1/3}+a^{2/3})}$

We want $|(x-a)Q(x)| = |x-a||Q(x)| < \varepsilon$. Let $\delta = \frac{\varepsilon}{|Q(x)|}$

Then $|x-a|<\delta \Rightarrow |x-a||Q(x)|<\varepsilon \Rightarrow |f(x)-f(a)|<\varepsilon$.

As $a\in\mathbb{R}$ therefore f is continuous on $\mathbb{R}$.

open up modulus on RHS $|\sqrt[3]{x}-\sqrt[3]{a}|<\varepsilon \Rightarrow -\varepsilon < \sqrt[3]{x}-\sqrt[3]{a} < \varepsilon$

$\Rightarrow \sqrt[3]{a}-\varepsilon < \sqrt[3]{x} < \sqrt[3]{a}+\varepsilon$. $\Rightarrow (\sqrt[3]{a}-\varepsilon)^3 < x < (\sqrt[3]{a}+\varepsilon)^3$

$\Rightarrow -(a-(\sqrt[3]{a}-\varepsilon)^3) < x-a < -a+(\sqrt[3]{a}+\varepsilon)^3$. ⊛

$\Rightarrow |x-a| < a-(\sqrt[3]{a}-\varepsilon)^3$ and $|x-a| < (\sqrt[3]{a}+\varepsilon)^3-a$

If we take $\delta = \min\{a-(\sqrt[3]{a}-\varepsilon)^3, (\sqrt[3]{a}+\varepsilon)^3-a\}$ then $\delta>0$ and $\forall x\in\mathbb{R}$

and $|x-a|<\delta$ ⊛ is satisfied and consequently $|\sqrt[3]{x}-\sqrt[3]{a}|<\varepsilon$.

Since $\varepsilon>0$ was arbitrary, proves f continuous at a. Since $a\in\mathbb{R}$ was arbitrary proves f is continuous on $\mathbb{R}$

RA 3)

② $\lim_{x\to 0} \frac{17^x-1}{2x} = \frac{e^{x\ln 17}-1}{2x} = \frac{\ln 17}{2}\cdot\frac{e^{x\ln 17}-1}{x\ln 17}$. As $\lim_{x\to 0} x\ln 17 = 0$ and $x\ln 17 \neq 0$ when $x\neq 0$

RA 3)

② $\lim_{x\to 0} \frac{17^x - 1}{2x} = \frac{e^{x\ln 17}-1}{2x} = \frac{\ln 17}{2} \cdot \frac{e^{x\ln 17}-1}{x\ln 17}$. As $\lim_{x\to 0} x\ln 17 = 0$ and $x\ln 17 \neq 0$ when $x \neq 0$

So change of variable $\overset{y = x\ln 17}{=} \lim_{y\to 0}\left(\frac{\ln 17}{2}\cdot\frac{e^y-1}{y}\right) \underset{\text{of limits}}{\overset{\text{algebra}}{=}} \frac{\ln 17}{2}\cdot\lim_{y\to 0}\frac{e^y-1}{y}$. Notable limit $\lim_{y\to 0}\frac{e^y-1}{y}=1$

hence $\Rightarrow \frac{\ln 17}{2}\cdot\lim_{y\to 0}\frac{e^y-1}{y} = \frac{\ln 17}{2}\cdot 1 = \frac{\ln 17}{2}$.

Could do all limits one at a time (with own COV)

ii) $\lim_{x\to 2} \frac{\log(2x-3)}{\tan(x-2)} \underset{x=\frac{1}{2}y+2}{\overset{\text{Change of variable}}{=}} \lim_{y\to 0} \frac{\log(2(\frac{1}{2}y+2)-3)}{\tan(\frac{1}{2}y+2-2)} = \lim_{y\to 0}\left(\frac{\log(y+1)}{\tan(\frac{1}{2}y)}\right) = \lim_{y\to 0}\left(\frac{\log(y+1)}{y}\cdot\frac{y}{\left(\frac{\sin(\frac{1}{2}y)}{\cos(\frac{1}{2}y)}\right)}\right)$

$= \lim_{y\to 0} \underbrace{\frac{\log(y+1)}{y}}_{=1} \cdot \underbrace{\frac{\frac{1}{2}y}{\sin(\frac{1}{2}y)}}_{=1} \cdot \frac{\cos(\frac{1}{2}y)}{\frac{1}{2}}$

$= \lim_{y\to 0} 2\cos(\frac{1}{2}y) = 2\cos 0 = 2$.

iii) $\lim_{x\to 0} \frac{x^2}{e^x - x - 1}$

Note that $f,g:\mathbb{R}\to\mathbb{R}$ given by $f(x) := x^2$ $g(x) := e^x - x - 1$ are both differentiable, hence continuous and $\lim_{x\to 0} f(x) = f(0) = 0$ and $\lim_{x\to 0} g(x) = g(0) = e^0 - 0 - 1 = 0$.

While $f'(x) = 2x$ $g'(x) = e^x - 1$

hence by the algebra of limits $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} \Rightarrow \lim_{x\to 0}\frac{x^2}{e^x-x-1} = \lim_{x\to 0}\frac{2x}{e^x-1} \overset{\text{aol}}{=} 2\cdot\lim_{x\to 0}\frac{x}{e^x-1}$

$\lim_{x\to 0}\frac{e^x-1}{x} = 1$ is a notable limit, hence, $2\cdot\left(\lim_{x\to 0}\frac{e^x-1}{x}\right)^{-1} = 2\cdot 1 = 2$.

Methods that can be used: aol, cov, notable limits, continuity, sandwich theorem and L'Hospital rule.

iv) $\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} = \lim_{x\to 0}\left(\frac{\cos x - 1}{x^4} + \frac{1}{2x^2}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\cdot\frac{-1}{x^2} + \frac{1}{2x^2}\right)$

Observe that $f,g:\mathbb{R}\to\mathbb{R}$ $f(x) := \cos x - 1 + \frac{1}{2}x^2$ $g(x) := x^4$ then f,g differentiable $\Rightarrow$ continuous

$\lim_{x\to 0} f(x) = f(0) = 0$, $\lim_{x\to 0} g(x) = g(0) = 0$ and $f'(x) = -\sin x + x$ and $g'(x) = 4x^3$.

Note that f', g' are differentiable too hence continuous and

$\lim_{x\to 0} f'(x) = f'(0) = 0$, $\lim_{x\to 0} g'(x) = g'(0) = 0$ while $f''(x) = -\cos x + 1$ $g''(x) = 12x^2$

Therefore by algebra of limits $\lim_{x\to 0}\frac{f''(x)}{g''(x)} = \frac{1}{2}$ ...

$\overset{\text{L'Hospital}}{\Rightarrow} \lim_{x\to 0}\frac{-\sin x + x}{4x^3} \overset{\text{L'Hospital}}{\Rightarrow} \lim_{x\to 0}\frac{-\cos x + 1}{12x^2} \overset{\text{aol}}{=} \frac{1}{12}\cdot\lim_{x\to 0}\frac{1-\cos x}{x^2} \overset{\text{Notable limit}}{=} \frac{1}{12}\cdot\frac{1}{2} = \frac{1}{24}$. ✓

RA 4

② Riemann's criterion states that for $f:[a,b]\to\mathbb{R}$ bounded to be integrable, that is the same to say that $\forall \varepsilon > 0$ there exists a partition P from $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

RA 5

② a) $T_n(x) = \int \tan^n(4x)\,dx$ $n \in \mathbb{N}\cup\{0\}$

$T_n(x) = \tan^{n-1}(4x)\tan(4x)\,dx$

$U = \tan^{n-1}(4x)$ $V = \frac{1}{4}\ln|\sec(4x)|$

$U' = (n-1)\tan^{n-2}(4x)(4\sec^2 4x)$ $V' = \tan(4x)$

Using integration by parts $\int UV' = UV - \int VU'$

$$\Rightarrow \int \tan^{n-1}(4x)\tan(4x)\ dx = \tan^{n-1}(4x)\cdot \tfrac{1}{4}\ln|\sec(4x)| - \int \tfrac{1}{4}\ln|\sec(4x)|\cdot(n-1)\tan^{n-2}(4x)(4\sec^2(4x))\,dx$$

$$= \tan^{n-1}(4x)\cdot\tfrac{1}{4}\ln|\sec(4x)| - (n-1)\int \underset{u}{\ln|\sec(4x)|}\cdot\underset{v'}{\tan^{n-2}(4x)}\ dx$$

$u = \ln|\sec(4x)|$ $\qquad v = \int \tan^{n-2}(4x) = T_{n-2}(x)$

$u' = 4\tan(4x)$ $\qquad v' = \tan^{n-2}(4x)$

so $\int \ln|\sec(4x)|\cdot\tan^{n-2}(4x)\,dx = \ln|\sec(4x)|T_{n-2}(x) - \int T_{n-2}(x)\cdot 4\tan 4x$

$T_n = \int \tan^n(4x)\ dx$

$$\int \tan^{n-2}(4x)\cdot\tan^2(4x)\ dx = \int \tan^{n-2}(4x)\left[\sec^2(4x)-1\right]dx$$

$$= \int \tan^{n-2}(4x)\sec^2(4x) - \tan^{n-2}(4x)\ dx$$

$$= \int \tan^{n-2}(4x)\sec^2(4x)\ dx - T_{n-2}(x)$$

Let $u = \tan(4x)$ so $du = 4\sec^2(4x)\,dx$

$\tfrac{1}{4}du = \sec^2(4x)\,dx$

Sub in: $= \int u^{n-2}\cdot\tfrac{1}{4}du = \left[\frac{u^{n-1}}{4(n-1)}\right] = \frac{\tan^{n-1}(4x)}{4(n-1)}$

Hence $T_n = \dfrac{\tan^{n-1}(4x)}{4(n-1)} - T_{n-2}(x)$

ii) $\int \tan^3(4x) = T_3 = \dfrac{\tan^2(4x)}{4(2)} - T_1(x)$

$T_1(x) = \int \tan 4x\ dx = \tfrac{1}{4}\ln|\sec 4x|$

$$\int \tan^3(4x) = \frac{\tan^2(4x)}{8} - \tfrac{1}{4}\ln|\sec 4x|$$

b) $\int \tan^5(x)\sec^6(x)\ dx = \int \tan^5(x)\sec^4(x)\cdot\sec^2(x)\,dx$

Let $u = \tan x$ so $du = \sec^2 x\ dx$

$s^2 + c^2 = 1$ $\qquad t^2 + 1 = \sec^2 x$

so $\int \tan^5(x)\sec^6(x)\,dx = \int u^5[u^2+1]^2\ du = \int u^5[u^4+2u^2+1]\ du$

$= \int u^9 + 2u^7 + u^5\ du = \frac{u^{10}}{10} + \frac{u^8}{4} + \frac{u^6}{6} + C$

$= \tfrac{1}{10}\tan^{10}x + \tfrac{1}{4}\tan^8 x + \tfrac{1}{6}\tan^6 x + C$ ✓

ii) $I = \int_0^{\pi/4} \cos^2(6x)\sin^2(6x)\ dx = \int_0^{\pi/4} \cos(6x)\sin^2(6x)\cdot\cos(6x)\,dx$

~~Let $u = \sin 6x$ so $du = 6\cos 6x\ dx$~~

~~so $\tfrac{1}{6}du = \cos 6x\ dx$~~

~~$u = \sin\pi/4 = 1/2$ $\qquad u = \sin 0 = 0$~~

$(\sin(12x))^2 = (2\cos(6x)\sin(6x))^2$ $\qquad \sin 2x = 2\sin x\cos x$

~~$u=\sin\pi/4=1/\sqrt{2}$~~ ~~$u=\sin 0=0$~~

$(\sin(2x))^2=(2\cos(6x)\sin(6x))^2$ $\quad$ $\sin 2x = 2\sin x\cos x$

$I=\frac{1}{4}\int_0^{\pi/4}\sin^2 12x\,dx$

$\cos 24x = 2\sin^2 12x - 1 \Rightarrow \sin^2 12x = \frac{1}{2}+\frac{1}{2}\cos 24x$ $\quad x$

$I=\frac{1}{4}\int_0^{\pi/4}\frac{1}{2}+\frac{1}{2}\cos 24x\,dx = \frac{1}{4}\left[\frac{1}{2}x+\frac{1}{48}\sin 24x\right]_0^{\pi/4} = \left[\frac{1}{2}\left(\frac{\pi}{4}\right)+\frac{1}{48}\sin\left(24\cdot\frac{\pi}{4}\right)-(0)\right]\times\frac{1}{4}$

$=\frac{1}{4}\left[\frac{\pi}{8}+\frac{1}{48}\sin(8\pi)\right]=\frac{\pi}{8}\times\frac{1}{4}=\frac{\pi}{32}$

ii) $\frac{x^2+5x-4}{x^3-x}$ $\quad$ $\frac{x^2+5x-4}{x(x+1)(x-1)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$

$x^2+5x-4 = A(x+1)(x-1)+B(x-1)x+C(x+1)x$

when $x=1$,

$1+5-4=2C \Rightarrow 2=2C \Rightarrow C=1$

When $x=-1$,

$1-5-4=2B \Rightarrow -8=2B \Rightarrow B=-4$.

When $x=0$,

$-4=-A \Rightarrow A=4$.

$\therefore \int\frac{x^2+5x-4}{x^3-x}\,dx = 4\int\frac{1}{x}\,dx - 4\int\frac{1}{x+1}\,dx + \int\frac{1}{x-1}\,dx = 4\ln|x|-4\ln|x+1|+\ln|x-1|+C$.

$=\ln|x^4|-\ln|(x+1)^4|+\ln|x-1|+C$

$=\ln\left|\frac{x^4(x-1)}{(x+1)^4}\right|+C$.

RA 1)

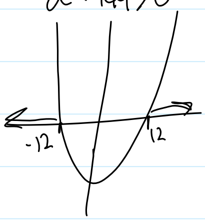
3) $[2,5]\cap(3,9]=(3,5]$ ✓

ii) $(-\infty,5)\cup(3,7]=(-\infty,7]$ ✓

iii) $(-4,1)\cup(-1,2)\cup\{\pi\}=(-4,2)\cup\{\pi\}$ ✓

iv) $\{x\in\mathbb{R}: x^2>144\}\cup\{12\}=(-\infty,-12)\cup[12,\infty)$ ✓

$x^2-144>0$



RA2)

2) Let $f:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$

i) If $\lim_{x\to a} f(x)=0$ then a is an accumulation point of $\mathbb{R}$. $\forall\varepsilon>0, \exists\delta>0$ such that $\forall x\in\mathbb{R}$ and $0<|x-a|<\delta \Rightarrow |f(x)-0|<\varepsilon$ ⊛

If $\lim_{x\to a}|f(x)|=0$ then a is an accumulation point of $\mathbb{R}$. $\forall\varepsilon>0\ \exists\delta>0$ such that $\forall x\in\mathbb{R}$ and $0<|x-a|<\delta \Rightarrow ||f(x)|-0|<\varepsilon$ ⊛⊛

Observe for definition ⊛ $|f(x)-0| = |f(x)|$ and for ⊛⊛ $||f(x)|-0| = ||f(x)|| = |f(x)|$.
Therefore both $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} |f(x)| = 0$ have the same definition. ✓

ii) Assume $\lim_{x\to a} |f(x)| = L$. Therefore a is an accumulation point of $\mathbb{R}$. $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in \mathbb{R}$ and $0 < |x-a| < \delta \Rightarrow ||f(x)|-L| < \varepsilon$ ⊛
If $\lim_{x\to a} f(x) = L$, then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x \in \mathbb{R}$ and $0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$
If $L > 0$ $||f(x)|-L| = ||f(x)|-|L|| \leq |f(x)-L|$. Hence not necessarily.
If $L < 0$ $||f(x)|-L| = ||f(x)|+|L|| \geq |f(x)-(-L)|$. Hence $|f(x)-(-L)| \leq ||f(x)|+|L|| < \varepsilon$ ✓
$\lim_{x\to a} |f(x)| = L \Rightarrow \lim_{x\to a} f(x) = -L$ however $\lim_{x\to a} |f(x)| = L \not\Rightarrow \lim_{x\to a} f(x) = L$
Try use intuition and think of basic counterexample when asked if statement is true.

RA3)

3) i) $f(x) = \dfrac{\sin(x^2)\sin^2(x)}{2+\sin x}$ /Leibniz rule

$(\sin(x^2)\sin^2(x))'$ requires product rule:

$u = \sin(x^2)$, $u' = 2x\cos(x^2)$ chain rule
$v = \sin^2(x) = (\sin x)^2$, $v' = 2(\sin x)\cos x$ (chain rule)

$= uv' + vu' = 2\sin x \cos x \sin(x^2) + 2x\cos(x^2)\sin^2 x$

Using quotient rule let $U = \sin(x^2)\sin^2(x)$ $\quad V = 2+\sin x$
$U' = 2\sin x\cos x \sin(x^2) + 2x\cos(x^2)\sin^2 x$ $\quad V' = \cos x$

then $$f'(x) = \frac{VU' - UV'}{V^2} = \frac{(2+\sin x)(2\sin x\cos x\sin(x^2) + 2x\cos(x^2)\sin^2 x) - \sin(x^2)\sin^2 x\cos x}{(2+\sin x)^2}$$ ✓

ii) $g(x) = e^{\sin x} + \cos(x + \sin x)$ sum rule
$g'(x) = \cos x\, e^{\sin x} - (1+\cos x)\sin(x+\sin x)$
chain rule for each statement.

iii) $h(x) = (5+\sin x)^{\log x} = e^{\log[(5+\sin x)^{\log x}]} = e^{\log x \log(5+\sin x)}$
$\log(h(x)) = \log(e^{\log x \log(5+\sin x)})$
$(\log(h(x)))' = (\log x \log(5+\sin x))'$

$u = \log x$, $u' = 1/x$ $\quad v = \log(5+\sin x)$, $v' = \dfrac{\cos x}{5+\sin x}$ chain rule

$\dfrac{h'(x)}{h(x)} = uv' + vu' = \dfrac{\log x \cos x}{5+\sin x} + \dfrac{\log(5+\sin x)}{x}$

log differentiation $\quad$ product rule

so $h'(x) = h(x)[\ \swarrow\ ]$ ✓

RA4

3) $f: [a,b] \to [0,\infty)$ bounded integrable. $-\infty < a < b < \infty$.
a) $\alpha \geq 0$.
$M_i := \sup\{\alpha f(x) : x \in [x_{i-1}, x_i]\} = \alpha \sup\{f(x) : x \in [x_{i-1}, x_i]\}$
$U(\alpha f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} \alpha \cdot \sup\{f(x) : x \in [x_{i-1}, x_i]\} \cdot (x_i - x_{i-1}) = \alpha \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\} \cdot (x_i - x_{i-1})$
$= \alpha U(f, P)$ $\quad$ Wanted to try $M_{i,\alpha}$ and M_i terminology but didn't ⇒ mark scheme did.
$L(\alpha f, P) = \alpha L(f, P)$ similarly

b)

f is integrable, hence $\forall \cancel{\varepsilon}\,\delta > 0$ $\exists$ a partition of $[a,b]$ P such that
$U(f,P_\delta) - L(f,P_\delta) < \cancel{\varepsilon}\,\delta$
Hence $\alpha[U(f,P_\delta) - L(f,P_\delta)] < \alpha \cancel{\varepsilon}\,\delta$
Now let $\varepsilon > 0$ and choose $\delta_0 > 0$ $\delta_0 := \varepsilon/\alpha$ so that
$\Rightarrow \alpha[U(f,P_\delta) - L(f,P_\delta)] = U(\alpha f, P_\delta) - L(\alpha f, P_\delta) < \varepsilon$

c) We can create a function $g(x) := \alpha f$ which is integrable (part b).
Then for integrable functions $\int_a^b g := \underline{\int_a^b} g = \sup\{L(g,P) : P \text{ is a partition of } [a,b]\}$
$\Rightarrow \int_a^b \alpha f = \sup\{L(\alpha f, P) : P \text{ is a partition of } [a,b]\}$ ✓

d) $\int_a^b \alpha f \underset{\text{Pt c.}}{=} \sup\{L(\alpha f, P) : P \text{ is a partition of } [a,b]\} \underset{\text{Pt a.}}{=} \sup\{\alpha L(f,P) : P \ldots\}$ can also use $\sup_P L(f,P)$
$= \alpha \sup\{L(f,P) : P \ldots\} = \alpha \int_a^b f$ ✓

RA5

③ a) $\int \frac{6x+1}{x^2+3x+5}\,dx = \int \frac{3(2x+3)-8}{x^2+3x+5}\,dx = 3\int \frac{2x+3}{x^2+3x+5}\,dx - \int \frac{8}{x^2+3x+5}\,dx$
$(x+\frac{3}{2})^2 - \frac{9}{4} + 5 = (x+\frac{3}{2})^2 + \frac{20}{4} - \frac{9}{4} = (x+\frac{3}{2})^2 + \frac{11}{4}$
$\int \frac{8}{(x+\frac{3}{2})^2 + \frac{11}{4}}$ $\quad \underline{u = x + \frac{3}{2}}$ $(du = dx)$ $\quad \int \frac{8}{u^2 + \frac{11}{4}}\,du = \frac{8}{\sqrt{11/4}} \arctan\left(\frac{u}{\sqrt{11/4}}\right) = \frac{8}{\sqrt{11}/2} \arctan(u)$
Not SQRT (this or tanh)
or instead of memorising could've done $\tan\theta = x + \frac{3}{2}$ ...
$= \frac{16\sqrt{11}}{11} \arctan(u) = \frac{16\sqrt{11}}{11} \arctan(x + \frac{3}{2})$
So $I = 3\ln|x^2+3x+5| + \frac{16\sqrt{11}}{11} \arctan(\cancel{x+\frac{3}{2}}) + C$
$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
only tan has $\frac{1}{a}$ in front as well

b) $I = \int \sec^4(3x)\tan^4(3x)\,dx = \int \sec^2(3x)\tan^4(3x) \cdot \sec^2(3x)\,dx$
Let $u = \tan(3x)$ $\quad$ so $du = 3\sec^2(3x)$
$\frac{1}{3}du = \sec^2(3x)$
Also $s^2 + c^2 = 1 \Rightarrow t^2 + 1 = \sec^2$
So $I = \int [u^2+1]u^4 \cdot \frac{1}{3}du = \frac{1}{3}\int u^6 + u^4\,du = \frac{1}{3}\left[\frac{u^7}{7} + \frac{u^5}{5}\right] + C$
$= \frac{u^7}{21} + \frac{u^5}{15} + C \overset{u=\tan 3x}{=} \frac{1}{21}\tan^7(3x) + \frac{1}{15}\tan^5(3x) + C$ ✓

c) Asymptote is at $x = \frac{3}{2} \notin [2,\infty)$.
$\int_2^\infty \frac{1}{\sqrt{2x-3}} = \lim_{t\to\infty} \int_2^t \frac{1}{\sqrt{2x-3}}\,dx = \lim_{t\to\infty} \int_2^t (2x-3)^{-1/2}\,dx$
$\frac{d}{dx}\left[(2x-3)^{1/2}\right] = \frac{1}{2}(2x-3)^{-1/2}(2) = (2x-3)^{-1/2}$ hence by recognition:
$\Rightarrow \lim_{t\to\infty}\left[\sqrt{2x-3}\right]_2^t = \lim_{t\to\infty}\left[\sqrt{2t-3} - \sqrt{2(2)-3}\right] = \lim_{t\to\infty}\left[\sqrt{2t-3} - 1\right] \underset{u=2t-3}{\overset{\text{change of variable}}{=}} \lim_{u\to\infty}\left[\sqrt{u} - 1\right] = \infty$ ✓
Hence divergent.

d) $I = \int_5^6 \frac{1}{\sqrt{x^2-25}}\,dx$
Let $x = 5\sec\theta$ $\quad$ so $dx = 5\sec\theta\tan\theta\,d\theta$

$\theta = \arccos(5/x)$ so $\theta_1 = \arccos(5/5) = 0$ $\quad \theta_2 = \arccos(5/6)$

So $I = \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\arccos(5/6)} \frac{1}{\sqrt{25\sec^2\theta - 25}} \cdot 5\sec\theta\tan\theta \, d\theta$

$\Rightarrow I = \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\arccos 5/6} \frac{\cancel{5}\sec\theta\tan\theta}{\cancel{5}\sqrt{\sec^2\theta - 1}} \, d\theta = \lim_{\varepsilon\to 0^+} \int_\varepsilon^{\arccos 5/6} \frac{\sec\theta \cancel{\tan\theta}}{\cancel{\tan\theta}} \, d\theta$

From above

$\hookrightarrow \tan^2\theta + 1 = \sec^2\theta \Rightarrow \tan^2\theta = \sec^2\theta - 1$

$\Rightarrow I = \lim_{\varepsilon\to 0} \int_\varepsilon^{\arccos 5/6} \sec\theta$ $\quad$ ~~$d\theta = \lim_{\varepsilon\to 0} \int_\varepsilon^{\arccos 5/6} \frac{1}{\sin\theta} d\theta = \lim_{\varepsilon\to 0} \int_\varepsilon^{\arccos 5/6} \operatorname{cosec}\theta \, d\theta$~~

~~$= \lim_{\varepsilon\to 0} [\ln(\tan(\theta/2))$~~

$= I = \lim_{\varepsilon\to 0^+} \left[\ln|\sec\theta + \tan\theta|\right]_\varepsilon^{\arccos 5/6} = \lim_{\varepsilon\to 0^+} \left[\ln|\sec(\arccos 5/6) + \tan(\arccos 5/6)| - \ln|\sec\varepsilon + \tan\varepsilon|\right]$

2ND FTC. $\quad = 6/5 \quad = \frac{\sqrt{x^2-25}}{5}$ $\quad$??

Find $\int \frac{1}{\sqrt{x^2-25}}$ first $= \log(x + \sqrt{x^2-25}) - \log 5$ then can use original limits to avoid

RA 1 Q4 is summative.

RA2 Q4

i) $\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{(2x+1)\cancel{(x-1)}} \underset{\text{or limits}}{\overset{\text{algebra}}{=}} \frac{\lim_{x\to 1} x+1}{\lim_{x\to 1} 2x+1} = \frac{1+1}{2(1)+1} = \frac{2}{3}.$ ✓

ii) $\lim_{x\to 0} \frac{(x-1)^3 + (1-3x)}{x^2+2x^3} = \lim_{x\to 0} \frac{x^3 + 3x^2(-1) + 3x(-1)^2 + (-1)^3 + 1 - 3x}{x^2(1+2x)} = \lim_{x\to 0} \frac{x^3 - 3x^2 \cancel{+3x} \cancel{-1+1} \cancel{-3x}}{x^2(1+2x)}$

$= \lim_{x\to 0} \frac{\cancel{x^2}(x-3)}{\cancel{x^2}(1+2x)} \underset{\text{or limits (quotient rule)}}{\overset{\text{algebra}}{=}} \frac{\lim_{x\to 0} x-3}{\lim_{x\to 0} 1+2x} = \frac{0-3}{1+2(0)} = \frac{-3}{1} = -3.$ ✓

algebra or continuous functions

iii) $\lim_{x\to 1} \frac{x^n-1}{x^m-1}$. As $x=1$ is a factor of the numerator and denominator, it is a factor of both.

$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x^{n-1} + x^{n-2} + \dots + x + 1)}{\cancel{(x-1)}(x^{m-1} + x^{m-2} + \dots + x + 1)} \underset{\text{or limits}}{\overset{\text{algebra}}{=}} \frac{\lim_{x\to 1} x^{n-1} + x^{n-2} + \dots + x + 1}{\lim_{x\to 1} x^{m-1} + x^{m-2} + \dots + x + 1} = \frac{1^{n-1} + 1^{n-2} + \dots + 1 + 1}{1^{m-1} + 1^{m-2} + \dots + 1 + 1} = \frac{n}{m}.$ ✓

iv) $\lim_{x\to 4} \frac{\sqrt{1+2x} - 3}{\sqrt{x} - 2} \frac{\times(\sqrt{x}+2)}{\times(\sqrt{x}+2)} = \lim_{x\to 4} \frac{(\sqrt{1+2x}-3)(\sqrt{x}+2)}{x-4} \times \frac{\sqrt{1+2x}+3}{\sqrt{1+2x}+3}$ $\quad \lim_{x\to 4} \frac{(1+2x-9)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} = \lim_{x\to 4} \frac{2\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{(x-4)}(\sqrt{1+2x}+3)}$

$= \lim_{x\to 4} \frac{2(\sqrt{x}+2)}{(\sqrt{1+2x}+3)} \underset{\text{or limits}}{\overset{\text{algebra}}{=}} \frac{2(\sqrt{4}+2)}{(\sqrt{1+2(4)}+3)} = \frac{2(4)}{6} = 4/3.$ ✓

composition of continuous functions and continuity of $\sqrt{x}$.

v) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{\frac{1}{\cos x} - 1}{x^2}\right] = \lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{1-\cos x}{x^2 \cos x}\right] = \lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1-\cos x}{x^2}\right]$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ is a notable limit, as is $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$. From composition of continuous functions

$\frac{1}{\cos x}$ is continuous hence $\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = 1/1 = 1$.

Hence from algebra of limits $\lim_{x\to 0} \left[\frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1-\cos x}{x^2}\right] = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} \cdot \lim_{x\to 0} \frac{1-\cos x}{x^2}$

Product rule.

$= 1 \cdot 1 \cdot 1/2 = 1/2.$ ✓

Notable limit

vi) $\lim_{x\to\infty} x\sin 1/x \underset{\text{variable, } y=1/x}{\overset{\text{change of}}{=}} \lim_{y\to 0^+} \cancel{1/y} \frac{\sin y}{y}$. Observe $\sin y$ is continuous hence $\lim_{y\to 0^+} \sin y = \sin 0 = 0$.

However, $\lim_{y\to 0^+} 1/y = \infty$, ~~hence $\lim_{y\to 0^+} 1/y \sin y = \infty = \lim_{x\to\infty} x\sin 1/x$.~~ This logic does not apply as using algebra of limits is not valid.

$x\sin 1/x = \frac{\sin 1/x}{1/x}$

RA3) Q4 is summative.

RA4)

④ $f: [a,b] \to [0,\infty)$ bounded and integrable where $-\infty < a < b < \infty$

and $0 \le f(x) \le M$ $\forall x \in [a,b]$ and some $M > 0$.

a) $\int_a^b f = \sup\{L(f,P) : P$ is a partition of $[a,b]\}$

and $0 \leq f(x) \leq M$ $\forall x \in [a,b]$ and some $M > 0$.

a) $\underline{\int_a^b} f = \sup\{L(f,P) : P \text{ is a Partition of } [a,b]\}$
where $L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$, $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \geq 0$ Keep to inequalities.
Considering the case where $\underline{\int_a^b} f \geq 0$ is smallest, is when $f(x) = 0$ $[0 \leq f(x) \leq M]$
$\forall x \in [a,b]$, then $m_i = 0$, $L(f,P) = 0$ for any Partition P. Hence $0 \leq \underline{\int_a^b} f$.

$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$, $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$
Considering case where $\overline{\int_a^b} f$ is greatest, is when $f(x) = M$ $[0 \leq f(x) \leq M]$
$\forall x \in [a,b]$, then $M_i = M$, $U(f,P) = \sum_{i=1}^{n} M(x_i - x_{i-1}) = \sum_{i=1}^{n} M\left(\frac{b-a}{n}\right) = M\left(\frac{b-a}{n}\right) \cdot n = M(b-a)$
for any Partition P. Hence $\overline{\int_a^b} f \leq M(b-a)$.

b) If f is integrable, $\int_a^b f := \underline{\int_a^b} f = \overline{\int_a^b} f$. From Part a,
$0 \leq \underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f \leq M(b-a)$. Hence $0 \leq \int_a^b f \leq M(b-a)$ ✓

c) $\lim_{h \to 0^+} 0 = 0$ and $\lim_{h \to 0^+} M((a+h) - a) = \lim_{h \to 0^+} M \cdot h \overset{\text{algebra of limits}}{=} \lim_{h \to 0^+} M \cdot 0 = 0$
Hence by Sandwich theorem, as $0 \leq \int_a^b f \leq M(b-a)$, $0 \leq \lim_{h \to 0^+} \int_a^{a+h} f \leq 0$
$\lim_{h \to 0^+} \int_a^{a+h} f = 0$. ✓

RA5

4) a) $I := \int_0^1 x \log x = \lim_{t \to 0^+} \int_t^1 x \log x$

$u = \log x$ $\quad v = \frac{1}{2}x^2$
$u' = \frac{1}{x}$ $\quad v' = x$

Applying integration by Parts, $\int_a^b uv' = [uv]_a^b - \int_a^b vu'$:
$I = \lim_{t \to 0^+} \left[\frac{1}{2}x^2 \log x\right]_t^1 - \lim_{t \to 0^+} \int_t^1 \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx = \lim_{t \to 0^+} \left[\frac{1}{2}(1)^2 \log(1) - \frac{1}{2}(t)^2 \log(t)\right] - \lim_{t \to 0^+} \left[\frac{x^2}{4}\right]_t^1$
$\frac{x}{2}$

$= \lim_{t \to 0^+} \left[0 - \frac{1}{2}t^2 \log t - \frac{1^2}{4} + \frac{t^2}{4}\right] \overset{\text{aol}}{=} -\frac{1}{4} + \frac{1}{2} \lim_{t \to 0^+} \left(\frac{-\log(t)}{\left(\frac{1}{t^2}\right)}\right) + 0$
$\lim_{t \to 0} t^2 \log t = 0$ but how to Prove?

Can apply L'Hospital's rule if $\frac{\infty}{\infty}$ as well as $\frac{0}{0}$.

b) $\int \frac{x}{x-5} = \int \frac{(x-5)+5}{x-5}\, dx = \int 1 + \frac{5}{x-5}\, dx = x + 5\ln|x-5| + C$

$\int_0^{10} \frac{x}{x-5}\, dx = \lim_{t \to 0^-} \int_0^{5+t} \frac{x}{x-5} + \lim_{t \to 0^+} \int_{5+t}^{10} \frac{x}{x-5} = \lim_{t \to 0^+} [x + 5\ln|x-5|]_0^{5+t \;\; 5-t} + \lim_{t \to 0^+} [x + 5\ln|x-5|]_{5+t}^{10}$
$= \lim_{t \to 0^-} [5+t+5\ln(t) - (0+5\ln(-5))] + \lim_{t \to 0^+} [10 + 5\ln 5 - (5+t+5\ln(t))$
$\lim_{t \to 0^+} \ln t = -\infty$, hence $\lim_{t \to 0^+} \int_0^{5-t} \frac{x}{x-5} = -\infty$, $\int_0^{10} \frac{x}{x-5}\, dx$ divergent
No need to calculate other integral if first divergent.

c) $\int_{-\pi/2}^{0} \sec(x)\, dx = \lim_{t \to -\frac{\pi}{2}} \int_t^0 \sec(x)\, dx = \lim_{t \to -\frac{\pi}{2}} [\ln|\sec x + \tan x|]_t^0$
$\overset{\text{2FTC}}{=} \lim_{t \to -\frac{\pi}{2}} [\ln|\sec 0 + \tan 0| - \ln|\sec(t) + \tan(t)|] = \lim_{t \to 0} [\ln|1+0| - \ln|\sec t + \tan t|]$
$\lim \ln|\sec t + \tan t|$ is undefined. Hence divergent.

$\overset{\text{2FTC}}{=} \lim_{t\to\frac{\pi}{2}} \left[\ln|\sec 0+\tan 0| - \ln|\sec(t)+\tan(t)|\right] = \lim_{t\to 0}\left[\ln|1+0| - \ln|\sec t+\tan t|\right]$

$\lim_{t\to\frac{\pi}{2}} \ln|\sec t+\tan t|$ is Undefined, Hence divergent

$t\to 0^+$ always (instead $\lim_{t\to 0^+}\int_{-\pi/2+\delta}^{0}$)

as t $\frac{\sin t}{\cos t} = \frac{1+\sin t}{\cos t}$

$\lim_{t\to 0^+}\left[\ln\left|\frac{1+\sin(t-\pi/2)}{\cos(t-\pi/2)}\right|\right] = \infty$ because $\lim_{t\to 0^+}\left|\frac{1+\sin(t-\pi/2)}{\cos(t-\pi/2)}\right| = \lim_{t\to 0^+}\frac{1+\sin(t-\pi/2)}{\cos(t-\pi/2)}$

$\underset{\text{L'Hospital}}{=} \lim_{t\to 0^+}\frac{\cos(t-\pi/2)}{-\sin(t-\pi/2)} \underset{\text{continuity}}{=} \frac{\cos(-\pi/2)}{-\sin(-\pi/2)} = 0$. and $\lim_{x\to 0^+}\log(x) = -\infty$.

(Full justification needed)

RA1)

(5) $\frac{x^3(x-3)^2(x+4)}{(x-1)(x+1)} \geq 0$ ← always positive

	$x<-4$	$x=-4$	$-4<x<-1$	$x=-1$	$-1<x<0$	$x=0$	$0<x<1$	$x=1$	$1<x<3$	$x=3$	$x>3$
x^3	−	−	−	−	−	0	+	+	+	+	+
$x+4$	0	+	+	+	+	+	+	+	+	+	
$(x-3)^2$	+	+	+	+	+	+	+	+	+	0	+
$x-1$	−	−	−	−	−	−	−	0	+	+	+
$x+1$	−	−	−	0	+	+	+	+	+	+	+
$x^3(x-3)^2(x+4)$	+	0	−	−	−	0	+	+	+	0	+
$(x-1)(x+1)$	+	+	+	0	−	−	−	0	+	+	+
$\frac{x^3(x-3)^2(x+4)}{x^2-1}$	+	0	−	under.	+	0	−	under.	+	0	+

Hence solutions in the interval $x\in A$ where $A=(-\infty,-4)\cup(-1,0)\cup(1,3),\cup(3,\infty)$

≥ 0 was in question Hence $(-\infty,4]\cup(-1,0]\cup(1,\infty)$

RA2)

(5) $\lim_{x\to 0}\sin 1/x$.

$\sin 1/x$ is bounded such that $0<|\sin 1/x|<1$ ↙ $\forall x\in\mathbb{R}\setminus\{0\}$. We cannot apply Sandwich theorem here as the LHS and RHS limits are not equal. The function is also not continuous at $x=0$. Hence as the function is oscillating, between $[0,1]$, it will not have a limit.

They did contradiction to assuming $\lim_{x\to 0}\sin 1/x = C$ $C\in\mathbb{R}$

$-1\leq\sin(1/x)\leq 1$ $\forall x\in\mathbb{R}\setminus\{0\}$ and therefore $-1\leq C\leq 1$ so $C\in\mathbb{R}$.

If $\lim_{x\to 0}\sin 1/x = C$ is valid, then $x=0$ is an accumulation point of $x\in\mathbb{R}\setminus\{0\}$ and $\forall\varepsilon>0, \exists\delta>0$ such that

$\forall x\in\mathbb{R}\setminus\{0\}$ and $0<|x-a|<\delta \Rightarrow |f(x)-C|<\varepsilon$. ⊛

We can find ∴ when $x=x_n=\frac{1}{a+2\pi n}$ we find contradiction to ⊛ as $0<x$ (if n sufficiently large), $|f(x)-C| = \left|\sin\frac{1}{\left(\frac{1}{a+2\pi n}\right)} - C\right| = |\sin(a+2\pi n) - C| = |\sin a - C|$

Does ... exist? Q8 should use definition of limit

RA3

(5) a) $f:[0,4]\to\mathbb{R}$ $f(x):=x^2+7$ $\forall x\in[0,4]$

RA3

(5) a) $f:[0,4]\to\mathbb{R}$ $f(x):=x^2+2$ $\forall x\in[0,4]$.

$A:=\{f(x): x\in[1,2]\}\cup\{x: x\in[1,2]\} = \{x: x\in[3,6]\}\cup\{x: x\in[1,2]\}$

$\sup A = 6$. Proof:

$x\leq 6$ $\forall x\in A$ hence 6 is an upper bound. Observe $\forall\varepsilon>0$, $6-\varepsilon < 6-\frac{\varepsilon}{2}\in A$ hence $6-\varepsilon$ is not an upper bound. Hence 6 is the lowest upper bound $\Rightarrow$ supremum.

Did Q on problem sheet.

RA4

(5) a) The first Fundamental Theorem of Calculus states that $f:[a,b]\to\infty$ f is integrable and bounded, there exists a (differentiable) function F such that $c\in(a,b)$ $F'(c)=f(c)$ where $F(x):=\int_a^x f$. $F:[a,b]\to\mathbb{R}$

If continuous at c

$\leadsto \left[\frac{d}{dx}\left(\int_a^x f\right)=f(x)\right]$

i) $F:[2,4]\to\mathbb{R}$, $F(x):=\int_2^x \frac{1}{\log(t)}\,dt$, $x\in[2,4]$

Consider $f:[2,4]\to\mathbb{R}$ $f(x):=\frac{1}{\log(x)}$. $\forall x\in[2,4]$

From the composition of continuous functions $(\log x, \frac{1}{x})$, f is continuous $\Rightarrow$ integrable. By the first fundamental theorem of calculus, F is therefore differentiable with $F'(x)=f(x)=\frac{1}{\log x}$ ✓

ii) $G:[-1,1]\to\mathbb{R}$ $G(x):=\int_{-5}^{5x^4+3x^2+1} e^{-t^2}\,dt$, $x\in[-1,1]$

Consider $f:[-1,1]\to\mathbb{R}$ $f(x):=e^{-x^2}$ $x\in[-1,1]$.

From the composition of continuous functions $(e^x, -x^2)$ f is continuous $\Rightarrow$ integrable. From the 1st Fundamental theorem of calculus we therefore have differentiable function $F:[-1,1]\to\mathbb{R}$ $F(x):=\int_{-1}^x e^{-t^2}\,dt$ with $F'(x)=f(x)=e^{-x^2}$.

Observe that $G(x)=\int_{-5}^{-1}e^{-t^2}\,dt+\int_{-1}^{5x^4+3x^2+1}e^{-t^2}\,dt=\int_{-5}^{-1}e^{-t^2}+F(5x^4+3x^2+1)$

constant

Hence $G'(x)=\frac{d}{dx}\left[F(5x^4+3x^2+1)\right]$

so by chain rule $G'(x)=(20x^3+6x)F'(5x^4+3x^2+1)=(20x^3+6x)\left(e^{-(5x^4+3x^2+1)^2}\right)$

$G'(x)=\frac{20x^3+6x}{e^{(5x^4+3x^2+1)^2}}$ ✓

iii) $H:[\pi,2\pi]\to\mathbb{R}$, $H(x):=\int_x^{2\pi}\frac{\sin t}{t}\,dt$ $x\in[\pi,2\pi]$

$\int_\pi^{2\pi}\frac{\sin t}{t}\,dt=\int_\pi^x\frac{\sin t}{t}\,dt+\int_x^{2\pi}\frac{\sin t}{t}$ $\forall x\in[\pi,2\pi]\Rightarrow\int_x^{2\pi}\frac{\sin t}{t}=\int_\pi^{2\pi}\frac{\sin t}{t}\,dt-\int_\pi^x\frac{\sin t}{t}\,dt$.

$H(x)=\int_\pi^{2\pi}\frac{\sin t}{t}\,dt-\int_\pi^x\frac{\sin t}{t}\,dt$

$f:[\pi,2\pi]\to\mathbb{R}$ $f(x):=\frac{\sin x}{x}$.

Note from the composition of continuous functions ($\sin x$, $\frac{1}{x}$) f is continuous $\Rightarrow$ integrable. From the 1st Fundamental Theorem of Calculus F is differentiable with $F'(x) = f(x)$ where $F(x) := \int_a^x \frac{\sin t}{t}\, dt$.

Observe $H(x) = \int_{\pi}^{2\pi} \frac{\sin t}{t}\, dt - F(x)$, so $H'(x) \overset{\text{(differentiation rules)}}{=} -F'(x) = -\frac{\sin x}{x}$. ✓

RA5

(5) a) $f: [0,4] \to [0,1]$ $\qquad f(x) := \begin{cases} x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \le 4 \end{cases}$

$y: [0,4] \to \mathbb{R}$ $\quad y' = f(x) \Rightarrow y = \int f(x)\, dx$

when $0 \le x \le 1$, $\int f(x)\, dx = \int x\, dx = \frac{1}{2}x^2 + C_1$

when $1 < x \le 4$ $\int f(x)\, dx = \int 1\, dx = x + C_2$

Hence $y(x) := \begin{cases} \frac{1}{2}x^2 + C_1 & \text{if } 0 \le x \le 1 \\ x + C_2 & \text{if } 1 \le x \le 4 \end{cases}$

$y(0) = 1 \Rightarrow y(0) = \frac{1}{2}(0)^2 + C_1 = 1 \Rightarrow C_1 = 1$.

Must be continuous at $[0,4)$ so $\frac{3}{2} = \lim_{x \to 1^-} (\frac{1}{2}x^2 + 1) = \lim_{x \to 1^+} (x + D) = 1 + D$

Integrals have smoothing properties hence at $x = 1$, $\frac{1}{2}(1)^2 + 1 = 1 + C_2 \Rightarrow C_2 = \frac{1}{2}$

so $y(x) = \begin{cases} \frac{1}{2}x^2 + 1 & \text{if } 0 \le x \le 1 \\ x + \frac{1}{2} & \text{if } 1 \le x \le 4 \end{cases}$ $\qquad$ use limits when referring to continuity

Prove differentiability: From the first fundamental theorem of calculus f is continuous $\Rightarrow$ integrable and bounded, and $y: [0,4] \to \mathbb{R}$ $y(x) := \int f(x)$. $\cancel{y}$ f is continuous $\forall x \in [0,4]$ as polynomials are continuous and ~~integrals~~ and $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} = f(x)$ ~~have smoothing properties~~, hence y is differentiable. They didn't do this. They said continous and differentiable on $(0,1) \cup (1,4)$ then did derivative from first principles to check limit at $x = 1$ using $\lim_{h \to 0^+}$ and $\lim_{h \to 0^-}$. My proof was the same I believe but made one mistake

b) $y: [0, \infty) \to \mathbb{R}$ $\qquad y y' = \log x \qquad y(0) = 2$.

$y \frac{dy}{dx} = \log x$

$\int y \frac{dy}{\cancel{dx}} \cancel{dx} = \int \log x\, dx$

$\Rightarrow \int y\, dy = \int \log x\, dx$

$\int \log x \cdot 1\, dx$ $\quad$ Let $u = \log x$ $\quad v = x$

$u' = \frac{1}{x}$ $\quad v' = 1$

Then by integration by parts $\int \log x\, dx = uv - \int v u'\, dx = x \log x - \int x \cdot \frac{1}{x}\, dx$

$= x \log x - \int 1\, dx$

$= x \log x - x + C$

Hence $\frac{1}{2}y^2 = x \log x - x + C$.

$y(x) = \sqrt{2x \log x - 2x + C}$

$y(0) = 2 \Rightarrow \lim_{x \to 0} (y(x)) = \lim_{x \to 0} (\sqrt{2x \log x - 2x + C})$ $\qquad$ $\lim_{x \to 0^+}$ as not defined below 0.

~~y composition of continuous functions hence continuous so,~~ y must be cont on $[0,\infty)$

$(y(0))^2 = \lim_{x \to 0} (2x \log x - 2x + C)$

$\lim_{x \to 0^+} x \log x = \lim_{x \to 0} \frac{\log x}{(\frac{1}{x})}$. $\lim_{x \to 0} \log x = \lim_{x \to 0} \frac{1}{x} = \infty$ hence L'Hospital's rule can be applied:

$\lim_{x \to 0^+} \frac{\log x}{(\frac{1}{x})} = \lim_{x \to 0} \frac{(\log x)'}{(\frac{1}{x})'} = \lim_{x \to 0} \frac{(\frac{1}{x})}{-x^{-2}} = \lim_{x \to 0} \frac{-x^2}{x} = \lim_{x \to 0} -x \underset{\text{continuity}}{=} 0$

$\lim_{x\to0^+} x\log x = \lim_{x\to0} \frac{\log x}{(1/x)}$. $\lim_{x\to0}\log x = \lim_{x\to0} 1/x = \infty$ hence L'Hospital's rule can be applied.

$\lim_{x\to0^+}\frac{\log x}{(1/x)} \underset{\text{L'Hospital's rule.}}{=} \lim_{x\to0}\frac{(\log x)'}{(1/x)'} = \lim_{x\to0}\frac{(1/x)}{-x^{-2}} = \lim_{x\to0}\frac{-x^2}{x} = \lim_{x\to0} -x \underset{\text{continuity}}{=} 0$

Therefore going back to $(y(0))^2 = \lim_{x\to0}(2x\log x - 2x + c)$ by algebra of limits

$2^2 = 2\lim_{x\to0} x\log x - 2\lim_{x\to0} x + \lim_{x\to0} c \Rightarrow 4 = 2\cdot0 - 2\cdot0 + c \Rightarrow 4 = c$.

If this was case, Precewise.

Hence $y^2 = 2x\log x - 2x + 4 \Rightarrow y = +\sqrt{2x\log x - 2x + 4}$ ~~and $y = -\sqrt{2x\log x - 2x + 4}$~~

they just ignored negative

$y(x): \begin{cases} 2 & \forall x=0 \\ \sqrt{2x\log x - 2x + 4} & \text{if } x>0 \end{cases}$ as not defined at $x=0$. as assumed y is solution on $(0,\infty)$

Confusing Parts: ① Assumed y is solution on $[0,\infty)$ without any merit?

② Why must y be continuous at $x=0$? → Smoothing Properties comp of cont. func?

RA1

⑥ i) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) := x^2 - x - 1$

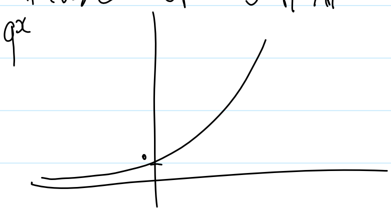
$f(-1) = (-1)^2 - (-1) - 1 = 1 + 1 - 1 = 1.$

$f\left(\frac{1+\sqrt5}{2}\right) = \left(\frac{1+\sqrt5}{2}\right)^2 - \left(\frac{1+\sqrt5}{2}\right) - 1 = \frac{1+2\sqrt5+5}{4} - \frac12 - \frac{\sqrt5}{2} - 1 = \frac64 + \frac{\sqrt5}{2} \ldots$

$\ldots$

ii)

d) image of $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x) := 9^x$



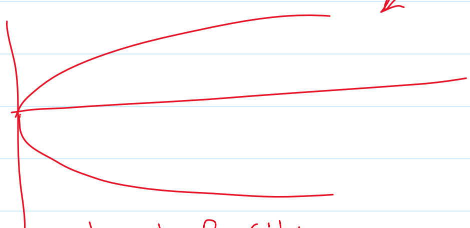
so $f(\mathbb{R}) = (0,\infty)$.

iii) $y^2 = x+1 \Rightarrow y = \sqrt{x+1}$ or $y = -\sqrt{x+1}$ ✗ No. $y^2 = x+1$ so

a) so $y:\mathbb{R}\to\mathbb{R}$ not possible as if $x<-1$ then then complex $\mathbb{C}\notin\mathbb{R}$.

b) $y: [-1,\infty)\to\mathbb{R}$ is possible as y defined on $[-1,\infty)$ and $f([-1,\infty)) = [0,\infty)$ or $(-\infty, 0]$.

c) also possible. ✓



so b not possible as on $[-1,\infty)\to\mathbb{R}$ it is not a function by definition.

RA2)

⑥ i) $\lim_{x\to\infty}\log x = \infty$

If the limit is valid, $\forall M>0, \exists N \in ~~(0,\infty)~~ \mathbb{R}$ such that $\forall x\in ~~\mathbb{R}~~ (0,\infty)$ and $x>N \Rightarrow f(x)>M$. ⊛

Observe $f(x)>M \Rightarrow \log x > M. \Rightarrow x > e^M$

If the limit is valid, $\forall M>0$, $\exists N\in\mathbb{R}$ such that $\forall x\in\mathbb{R}$ and $x>N \Rightarrow f(x)>M$. ⊕

Observe $f(x)>M \Rightarrow \log x > M. \Rightarrow x > e^M$

Choose $N=e^M$, then ⊕ is satisfied such that $x>e^M \Rightarrow f(x)>M$.

Therefore $\lim_{x\to\infty}\log x = \infty$.

!!

$\lim_{x\to\infty}\exp(x)=\infty$

If limit is valid, $\forall M>0$ $\exists N\in\mathbb{R}$ such that $\forall x\in\mathbb{R}$ and $x>N\Rightarrow f(x)>M$. ⊕⊕

Observe $f(x)>M \Rightarrow e^x > M \Rightarrow \log e^x > \log M \Rightarrow x>\log M$.

Choose $N=\log M$, then ⊕⊕ is satisfied such that $x>\log M \Rightarrow f(x)>M$.

!!

c.i) $\lim_{x\to 0^+}\log x \underset{y=1/x}{\overset{\text{change of variable}}{=}} \lim_{y\to\infty}\log 1/y = \lim_{y\to\infty}\log(y^{-1}) = \lim_{y\to\infty}-\log y \overset{\text{algebra of limits}}{=} -\lim_{y\to\infty}\log y = -\infty$.

$\lim_{x\to-\infty}\exp(x) \underset{y=-x}{\overset{\text{change of variable}}{=}} \lim_{y\to\infty}\exp(-y) = \lim_{y\to\infty}e^{-y} = \lim_{y\to\infty}\frac{1}{e^y} \overset{\text{algebra of limits}}{=} \frac{1}{\lim_{y\to\infty}e^y} = \frac{1}{\infty}=0$.

$\lim_{x\to\infty}x^b = \lim_{x\to\infty}e^{\log x^b} = \lim_{x\to\infty}e^{b\log x} = \lim_{x\to\infty}(e^b)^{\log x} \overset{\text{limits and composition of cont. functions}}{=} \lim_{y\to\infty}(e^b)^y = \infty$

only exp must be continuous at the limit
↳ in this case ∞ so not relevant?

$\lim_{x\to 0}x^b = \lim_{x\to 0}e^{b\log x} = \lim_{x\to 0}(e^b)^{\log x} = \lim_{y\to-\infty}(e^b)^y = 0$.

Answers don't mention?

Calling it $\exp(by)$ may be easier to understand how composed.

RA3)

⑥ a) i) $f:[0,10]\to\mathbb{R}$ $f(x):=x^2$ $P=\{0,2,7,10\}$

$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$ so as x^2 is an increasing function $m_1=0, m_2=4, m_3=49$

$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}$ and $M_1=4, M_2=49, M_3=100$

$L(f,P)=\sum_{i=1}^{3}m_i(x_i-x_{i-1}) = 0\cdot(2-0)+4\cdot(7-2)+49\cdot(10-7) = 0+4(5)+49(3) = 20+147=167$.

$U(f,P)=\sum_{i=1}^{3}M_i(x_i-x_{i-1}) = 4\cdot(2-0)+49\cdot(7-2)+100\cdot(10-7) = \overset{8}{0}+49(5)+100(3) = 245+300 = \overset{553}{545}$.

ii) $f:[0,10]\to\mathbb{R}$ $f(x):=e^{-x}$ $P=\{0,1,5,8,9,10\}$

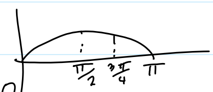
$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$ e^{-x} is a decreasing function hence $m_1=e^{-1}, m_2=e^{-5}, m_3=e^{-8}, m_4=e^{-9}, m_5=e^{-10}$

$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}$ so $M_1=e^0, M_2=e^{-1}, M_3=e^{-5}, M_4=e^{-8}, M_5=e^{-9}$.

$L(f,P)=\sum_{i=1}^{5}m_i(x_i-x_{i-1}) = e^{-1}(1-0)+e^{-5}(5-1)+e^{-8}(8-5)+e^{-9}(9-8)+e^{-10}(10-9) = e^{-1}+4e^{-5}+3e^{-8}+e^{-9}+e^{-10}$.

$U(f,P)=\sum_{i=1}^{5}M_i(x_i-x_{i-1}) = e^0(1-0)+e^{-1}(5-1)+e^{-5}(8-5)+e^{-8}(9-8)+e^{-9}(10-9) = 1+4e^{-1}+3e^{-5}+e^{-8}+e^{-9}$

iii) $f:[0,\pi]\to\mathbb{R}$ $f(x):=\sin(x)$ $P=\{0,\frac{\pi}{2},\frac{3\pi}{4},\pi\}$



$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$ so $m_1=\sin(0)=0$, $m_2=\sin\frac{3\pi}{4}=\frac{1}{2}$ $m_3=\sin\pi=0$

$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}$ and $M_1=\sin\frac{\pi}{2}=1$, $M_2=\sin\frac{\pi}{2}=1$, $M_3=\sin\frac{3\pi}{4}=\frac{1}{2}$.

$L(f,P)=\sum_{i=1}^{3}m_i(x_i-x_{i-1}) = 0\cdot(\frac{\pi}{2}-0)+\frac{1}{2}\cdot(\frac{3\pi}{4}-\frac{\pi}{2})+0\cdot(\pi-\frac{3\pi}{4}) = 0+\frac{1}{2}(\frac{\pi}{4})+0=\frac{\pi}{8}$

$U(f,P)=\sum_{i=1}^{3}M_i(x_i-x_{i-1}) = 1\cdot(\frac{\pi}{2}-0)+1\cdot(\frac{3\pi}{4}-\frac{\pi}{2})+\frac{1}{2}\cdot(\pi-\frac{3\pi}{4}) = \frac{\pi}{2}+\frac{\pi}{4}+\frac{1}{2}(\frac{\pi}{4}) = \frac{4\pi}{8}+\frac{2\pi}{8}+\frac{\pi}{8}=\frac{7\pi}{8}$

d) $f:[-2,2]\to\mathbb{R}$ $f(x)=\begin{cases}5 & x\in\mathbb{N}\\ 3 & x\notin\mathbb{N}\end{cases}$ $P_\delta=\{-2,1-\delta,1+\delta,2-\delta,2\}$ $\delta\in(0,\frac{1}{10})$

$m_i=\inf\{f(x): x\in[x_{i-1},x_i]\}$ so $m_1=3, m_2=3, m_3=3, m_4=3$

$M_i=\sup\{f(x): x\in[x_{i-1},x_i]\}$ so $M_1=3, M_2=5, M_3=3, M_4=5$.

$L(f,P_\delta)=\sum_{i=1}^{4}m_i(x_i-x_{i-1})=3(1-\delta-(-2))+3(1+\delta-(1-\delta))+3(2-\delta-(1+\delta))+3(2-(2-\delta))$

$=3(3-\delta)+3(2\delta)+3(1-2\delta)+3(\delta)$

$=9-\cancel{3\delta}+\cancel{6\delta}+3-\cancel{6\delta}+\cancel{3\delta}=12.$ ✓

$U(f,P_\delta)=\sum_{i=1}^{4}M_i(x_i-x_{i-1})=3(3-\delta)+5(2\delta)+3(1-2\delta)+5(\delta)$

$=9-3\delta+10\delta+3-6\delta+5\delta$

$=12+6\delta$ ✓

RA4

a) $f:[a,b]\to\mathbb{R}$ is integrable and bounded, and (differentiable) function g such that $g'=f$. Then $\int_a^b f=g(b)-g(a)$. ✓

b) $|x|:[a,b]\to\mathbb{R}$ $|x|:=\begin{cases}x & \text{if } x\geq 0\\ -x & \text{if } x\leq 0\end{cases}$

$|x|$ is continuous if $\forall\varepsilon>0, \exists\delta>0$ such that $\forall x\in\mathbb{R}$ and $|x-a|<\delta\Rightarrow|f(x)-f(a)|<\varepsilon$.

Observe $|f(x)-f(a)|=||x|-|a||\leq|x-a|<\delta$ so $|f(x)-f(a)|<\delta$ They didn't do by definition - just stated

(Reverse Triangle Inequality states $|x-y|\geq||x|-|y||$)

so choose $\delta=\varepsilon$ then $|x-a|<\delta\Rightarrow|f(x)-f(a)|<\varepsilon$ hence $|x|$ is continuous on $\mathbb{R}$

$|x|$ continuous $\Rightarrow|x|$ integrable. By boundedness theorem $|x|$ continuous $\Rightarrow|x|$ is bounded.

Consider case where $x\geq 0$, $|x|=x$.

$g_1:[0,\infty)\to\mathbb{R}$ $g_1(x):=\frac{1}{2}x^2$.

Observe $g_1'(x)=\lim_{h\to0}\frac{g_1(x+h)-g_1(x)}{h}=\lim_{h\to0}\frac{\frac{1}{2}(x+h)^2-\frac{1}{2}x^2}{h}=\lim_{h\to0}\frac{\frac{1}{2}(x^2+2xh+h^2)-\frac{1}{2}x^2}{h}=\lim_{h\to0}\frac{xh+\frac{1}{2}h^2}{h}\overset{a.o.l}{=}x$

hence $g_1'(x)=f$ (and differentiable).

Consider case where $x\leq 0$, $|x|=-x$.

$g_2:(-\infty,0]\to\mathbb{R}$ $g_2(x):=-\frac{1}{2}x^2$

Observe $g_2'(x)=\lim_{h\to0}\frac{g_2(x+h)-g_2(x)}{h}=\lim_{h\to0}\frac{-\frac{1}{2}(x^2+2xh+h^2)-(-\frac{1}{2}x^2)}{h}=\lim_{h\to0}\frac{-xh-\frac{1}{2}h^2}{h}\overset{a.o.l}{=}\lim_{h\to0}-x-\frac{1}{2}h=-x.$

hence $g_2'(x)=f$ (and differentiable).

Therefore by 2nd Fundamental Theorem of Calculus:

In the case of $a\geq0$ $\int_a^b|x|=g_1(b)-g_1(a)=\frac{1}{2}b^2-\frac{1}{2}a^2=\frac{1}{2}(b^2-a^2)$

In the case of $a<0\leq b$ $\int_a^b|x|=g_1(b)-g_2(a)=\frac{1}{2}b^2-(-\frac{1}{2}a^2)=\frac{1}{2}(b^2+a^2)$

In the case of $b<0$ $\int_a^b|x|=g_2(b)-g_2(a)=-\frac{1}{2}b^2-(-\frac{1}{2}a^2)=\frac{1}{2}(a^2-b^2)$

so $\int_a^b|x|\,dx=\{$ as shown

Correct I believe but instead of defining 2 functions they defined

$g(x):=\begin{cases}\frac{1}{2}x^2 & \text{if } x\geq0\\ -\frac{1}{2}x^2 & \text{if } x<0\end{cases}$ and proved this was differentiable with $g'(x)=f(x)=|x|$.

Only need to check differentiability with limits on $x=0$ as g not given by a single open interval containing the origin $\Rightarrow$ If contained by one polynomial expression can just state it is differentiable for that reason.

Use Piecewise More

c) $f: \mathbb{R}\setminus[-3,-2]\to\mathbb{R}$ $\quad f(x):=|x| \ \forall x\in\mathbb{R}\setminus[-3,-2]$.

~~From, $\int_a^b |x|\,dx = \begin{cases} \frac{1}{2}(b^2-a^2) & \text{if } a\geq 0 \\ \frac{1}{2}(b^2+a^2) & \text{if } a<0\leq b \\ \frac{1}{2}(a^2-b^2) & \text{if } b<0\end{cases}$~~

From b, $\quad \int |x|\,dx = \begin{cases} \frac{1}{2}x^2+C_1 & \text{if } x\in[0,\infty) \\ -\frac{1}{2}x^2+C_2 & \text{if } x\in(-\infty,-3)\cup(-2,0)\end{cases}$

For example $F_1,F_2: \mathbb{R}\setminus[-3,-2]\to\mathbb{R}$ $\quad F_1(x):=\begin{cases} \frac{1}{2}x^2+9 & \text{if } x\in[0,\infty) \\ -\frac{1}{2}x^2+3 & \text{if } x\in(-\infty,-3)\cup(-2,0)\end{cases}$

$\begin{cases} \frac{1}{2}x^2+9 & \text{if } x\in[0,\infty) \\ -\frac{1}{2}x^2+9 & \text{if } x\in(-2,0) \\ -\frac{1}{2}x^2+3 & \text{if } x\in(-\infty,-3)\end{cases}$

Extremely obvious now I've seen the answer...

$F_2(x):=\begin{cases} \frac{1}{2}x^2 & \text{if } x\in[0,\infty) \\ -\frac{1}{2}x^2 & \text{if } x\in(-\infty,-3)\cup(-2,0)\end{cases}$

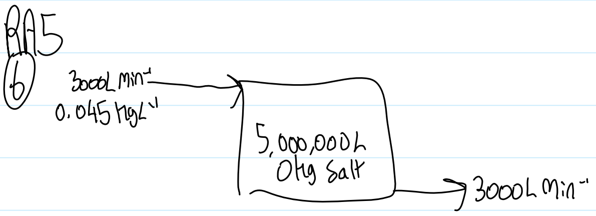
$F_1(x)-F_2(x) = \begin{cases} \cancel{9}\,\frac{1}{2}x^2 & \text{if } x\in[0,\infty) \\ \cancel{3}\,-\frac{1}{2}x^2 & \text{if } x\in(-\infty,-3)\cup(-2,0)\end{cases}$

How to move this to

$\begin{cases} \frac{1}{2}x^2 & \text{if } x\in[0,\infty) \\ -\frac{1}{2}x^2 & \text{if } x\in(-2,0) \\ -\frac{1}{2}x^2 & \text{if } x\in(-\infty,-3)\end{cases}$

RA5

6)



a) $y(t)$ denotes mass of salt (kg) in pool at t minutes

$y'(t)=$ Salt in $-$ Salt out,

Salt in: By chain rule $\frac{dm}{dt}$ in $=\frac{dV}{dt}$ in $\times \frac{dm}{dV}$ in $=3000\times 0.045=135\ \text{kg}\,\text{min}^{-1}$ in.

Salt out: $\frac{\text{total mass}}{\text{total volume}}\times 3000\,(\text{L}\,\text{min}^{-1}) = \frac{y(t)}{5{,}000{,}000}\times 3000 = 0.0006\,y(t)\ \text{kg}\,\text{min}^{-1}$ out.

So $\quad y'(t)=135-0.0006\,y(t) \qquad y(0)=0.$

Could've just $y'(t)=135-\frac{3y}{5000}$

$y'(t)=\frac{675000-3y}{5000}$

then separation of variables

$\frac{1}{675000-3y}\,y'(t)=\frac{1}{5000}$

b) $\quad 0.0006\,y(t)+y'(t)=135$

I.F: $\quad e^{\int 0.0006\,dt}=e^{0.0006t}$

so $\quad 0.0006e^{0.0006t}y(t)+e^{0.0006t}y'(t)=135e^{0.0006t}$

$(e^{0.0006t}y(t))'=135e^{0.0006t}$

$e^{0.0006t}y(t)=\int 135e^{0.0006t}\,dt$

$e^{0.0006t}y(t)=135\cdot\frac{e^{0.0006t}}{0.0006}+C$

$y(t)=\frac{225{,}000e^{0.0006t}+C}{e^{0.0006t}}=225000+Ce^{-0.0006t}$

$y(0)=0 \Rightarrow 0=225{,}000+Ce^{0}$

$C=-225000$

Hence $y(t)=225000\left(1-e^{-0.0006t}\right)$

Concentration $=\frac{\text{mass}}{\text{volume}} \Rightarrow y(t)=0.0035\times 5{,}000{,}000=17500$

$17500=225000\left(1-e^{-0.0006t}\right)$

$\frac{17500}{225000}=1-e^{-0.0006t}$

$e^{-0.0006t}=1-\frac{17500}{225000}$

$t=\frac{\ln\left(1-\frac{17500}{225000}\right)}{-0.0006}=134.9\text{ min (4sf)}.$ ✓

c) salt in: $1000\times 0.045=45$ kg min^{-1} in

Salt out: $\frac{\text{total mass}}{\text{total volume}}\times 3000\,(\text{L min}^{-1})=\frac{y(t)}{5{,}000{,}000}\times 3000=0.0006\,y(t)$ kg min^{-1} out.

So $y'(t)=45-0.0006\,y(t)$ $\quad y(0)=0.$

no longer constant as pool size decreases
should be $5{,}000{,}000-2000t$
Now we would've had to do IF.

$0.0006\,y(t)+y'(t)=45$

I.F: $e^{\int 0.0006\,dt}=e^{0.0006t}$

so $0.0006e^{0.0006t}y(t)+e^{0.0006t}y'(t)=45e^{0.0006t}$

$\left(e^{0.0006t}y(t)\right)'=45e^{0.0006t}$

$e^{0.0006t}y(t)=\int 45e^{0.0006t}\,dt$

$e^{0.0006t}y(t)=45\cdot\frac{e^{0.0006t}}{0.0006}+C$

$y(t)=\frac{75{,}000e^{0.0006t}+C}{e^{0.0006t}}=75{,}000+Ce^{-0.0006t}$

$y(0)=0 \Rightarrow 0=75{,}000+Ce^{0}$

$C=-75{,}000$

Hence $y(t)=75{,}000\left(1-e^{-0.0006t}\right)$

As wanted $[0,\infty)\to\mathbb{R}$ then define $\tilde{y}(t)$ piecewise function which is 0 when pool empty (to be valid).

RA1

⑦ a) $f(x):=2x^2+1$ $\quad f:\mathbb{R}\to\mathbb{R}$ $\quad f(\mathbb{R})=~~[0,\infty)~~$ $[1,\infty)$

b) $g(x)=\frac{2-x}{3+x}$ $\quad g:\mathbb{R}\setminus\{-3\}\to\mathbb{R}$ $\quad g(\mathbb{R})=\mathbb{R}\setminus\{-1\}$

	$x<-3$	$x=-3$	$-3<x<2$	$x=2$	$x>2$
$2-x$	+	+	+	0	−
$3+x$	−	0	+	+	+
$\frac{2-x}{3+x}$	−	undef.	+	0	−

f not one to one as $f(1)=f(-1)=3.$

g assume it is 1-1 and has a real valued inverse:

$y = \frac{2-x}{3+x} \Rightarrow 3y + xy = 2-x$

$\Rightarrow x(y+1) = 2-3y$

$x = \frac{2-3y}{y+1}$

Here we've found a real valued inverse $g^{-1}: \mathbb{R}\setminus\{-1\} \to \mathbb{R}$ $g^{-1}(x) = \frac{2-3x}{x+1}$ ✓

RA2

(7) i) $\lim_{x\to 0} \frac{7x}{\sin(4x)}$

Observe if $f, g: \mathbb{R}\to\mathbb{R}$ $f(x) := 7x$ and $g(x) := \sin(4x)$ that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$
(continuity)

Also note that $f'(x) = 7$ and $g'(x) = 4\cos(4x)$. (chain rule).

From L'Hospital's Rule we can therefore say that

$\lim_{x\to 0} \frac{7x}{\sin(4x)} = \lim_{x\to 0} \frac{7}{4\cos 4x} \underset{\text{algebra of limits}}{=} \frac{7}{4 \lim_{x\to 0} \cos(4\cdot 0)} = \frac{7}{4\cdot 1} = 7/4.$ ✓

or can pull a $\frac{\sin x}{x}$ out of expression: $\frac{7}{4} \cdot \frac{1}{\left(\frac{\sin 4x}{4x}\right)}$

ii) $\lim_{x\to 2} \frac{\sqrt{3x-2} - \sqrt{5x-6}}{\sqrt{2x-1} - \sqrt{x+1}} \times \frac{(\sqrt{2x-1} + \sqrt{x+1})}{(\sqrt{2x-1} + \sqrt{x+1})} = \lim_{x\to 2} \frac{(\sqrt{3x-2} - \sqrt{5x-6})(\sqrt{2x-1} + \sqrt{x+1})}{2x-1 - (x+1)} = \lim_{x\to 2} \frac{(\sqrt{3x-2} - \sqrt{5x-6})(\sqrt{2x-1} + \sqrt{x+1})}{x-2}$

algebra of limits $\Rightarrow \frac{\lim_{x\to 2}(\sqrt{3x-2} - \sqrt{5x-6}) \cdot \lim_{x\to 2}(\sqrt{2x-1} + \sqrt{x+1})}{\lim_{x\to 2} x} = \frac{(\sqrt{3\cdot 2-2} - \sqrt{5\cdot 2-6}) \cdot (\sqrt{2\cdot 2-1} + \sqrt{2+1})}{2} = \frac{(\sqrt{4} - \sqrt{4})(\sqrt{3}+\sqrt{3})}{2}$

$= \sqrt{3}$. ✗ (basic math error) (need to multiply by top conjugate too) ↳ end up with $-\sqrt{3}$

iii) $\lim_{x\to 0} \frac{x \sin x}{1-\cos x} = \lim_{x\to 0} \left(\frac{x^2}{1-\cos x} \cdot \frac{\sin x}{x} \right)$

Observe the notable limits $\lim_{x\to 0} \frac{1-\cos x}{x^2} = 1/2$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$ so

algebra of limits $\Rightarrow \lim_{x\to 0}\left(\frac{x^2}{1-\cos x}\right) \cdot \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \left(\lim_{x\to 0} \frac{1-\cos x}{x^2}\right)^{-1} \cdot \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \left(\left(\tfrac{1}{2}\right)^{-1} \cdot 1\right) = 2$ ✓

iv) $\lim_{x\to 0} \left(\frac{e^{2x}-1}{\log(1-5x)} \right) = \lim_{x\to 0} \left(\frac{(e^{2x}-1)(-5x)(2x)}{2x \log(1-5x)(-5x)} \right) = \lim_{x\to 0} \left(\frac{e^{2x}-1}{2x} \cdot \frac{-5x}{\log(1-5x)} \cdot \frac{2x}{-5x} \right)$

$\underset{\text{algebra of limits}}{=} \left(\lim_{x\to 0} \frac{e^{2x}-1}{2x}\right) \cdot \left(\lim_{x\to 0} \frac{-5x}{\log(1-5x)}\right) \cdot \left(\lim_{x\to 0} \frac{-2}{5}\right)$.

Observe $\lim_{x\to 0} \frac{e^{2x}-1}{2x} \underset{\substack{\text{change of variable} \\ y=2x}}{=} \lim_{y\to 0} \frac{e^y-1}{y} \underset{\text{notable limit}}{=} 1$ and $\lim_{x\to 0} \frac{\log(1-5x)}{-5x} \underset{\substack{\text{change of variable} \\ y=-5x}}{=} \lim_{y\to 0} \frac{\log(1+y)}{y} \underset{\text{notable limit}}{=} 1$

hence $\Rightarrow \left(\lim_{x\to 0} \frac{e^{2x}-1}{2x}\right) \cdot \left(\lim_{x\to 0} \frac{-5x}{\log(1-5x)}\right) \cdot \left(\lim_{x\to 0} \frac{-2}{5}\right) = 1 \cdot (1)^{-1} \cdot -2/5 = -2/5.$ ✓

v) $f: \mathbb{R}\to\mathbb{R}$ $f(x) := x^x$

f is continuous due to the composition of continuous functions.

Hence $\lim_{x\to 0^+} f(x) = f(0) = 0^0 = 1$.

$x^x = e^{\log x^x} = e^{x \log x}$

$\lim_{x\to 0^+} x\log x = 0$ is notable limit.

exp is continuous at 0 $\Rightarrow \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} \exp(x\log x) = \exp(0) = 1$.

RA3

(7) $f: [a,b] \to [0,\infty)$ bounded. P, Q Partitions of $[a,b]$ $P \le Q$

Prove $U(f,P) \ge U(f,Q)$

(7) $f: [a,b] \to [0,\infty)$ bounded. P, Q Partitions of $[a,b]$ $P \leq Q$
Prove $U(f,P) \geq U(f,Q)$
↳ Will go over Partition Lemma when doing flashcards

RA4
(7)

Scrap Paper 2

08 January 2023 20:10

RA4

7 a) $\int_1^e x^2 \log(x)\ dx$

Let $U = \log(x)$ $V = \frac{1}{3}x^3$

$U' = \frac{1}{x}$ $V' = x^2$

Then using integration by parts: $\int x^2 \log(x)\, dx = UV - \int U'V\, dx = \frac{1}{3}x^3 \log x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\, dx$

$= \frac{1}{3}x^3 \log x - \int \frac{1}{3}x^2\, dx = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 = \frac{1}{3}x^3(\log x - \frac{1}{3})$

Hence from 2nd Fundamental theorem of calculus:

$\int_1^e x^2 \log(x)\, dx = [\frac{1}{3}x^3(\log x - \frac{1}{3})]_1^e = \frac{1}{3}e^3(\log e - \frac{1}{3}) - \frac{1}{3}(1)^3(\log 1 - \frac{1}{3}) = \frac{1}{3}e^3(1-\frac{1}{3}) - \frac{1}{3}(0-\frac{1}{3})$

$= \frac{2}{9}e^3 + \frac{1}{9}$ ✓

b) $\int (\log(x))^2\ dx$

Let $U = (\log(x))^2$ $V = x$

$U' = \frac{2\log(x)}{x}$ $V' = 1$

Then using integration by parts $\int (\log x)^2 = UV - \int U'V\, dx$

$= x(\log x)^2 - \int \frac{2\log(x)}{x} \cdot x\ dx = x(\log x)^2 - 2\int \log x$ ⊛

Let $U = \log x$ $V = x$

$U' = \frac{1}{x}$ $V' = 1$

Then do integration by parts one more on $\int \log x = UV - \int U'V\, dx$

$\int \log x = x\log x - \int \frac{1}{x} \cdot x\ dx = x\log x - x$

Sub into ⊛:

$\int (\log x)^2 = x(\log x)^2 - 2(x\log x - x) = x(\log x)^2 - 2x\log x + 2x$ ✓

c) $\int e^x \cos(x)\ dx$

Let $U = \cos(x)$ $V = e^x$

$U' = -\sin x$ $V' = e^x$

Then using integration by parts $(\int UV' = UV - \int U'V)$:

$\int e^x \cos(x)\, dx = e^x \cos x + \int e^x \sin x\ dx$ ⊛

$\int e^x \sin x\ dx$

Let $U = \sin x$ $V = e^x$

$U' = \cos x$ $V' = e^x$

then using integration by parts:

$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\ dx$

Sub into ⊛:

$\int e^x \cos(x)\, dx = e^x \cos x + (e^x \sin x - \int e^x \cos x\, dx)$

$2\int e^x \cos x\ dx = e^x(\sin x + \cos x)$

$\int e^x \cos x\ dx = \frac{1}{2}e^x(\sin x + \cos x)$

If do without limits initially, remember to calculate at the end

$\int_0^\pi e^x \cos(x)\ dx = [\frac{1}{2}e^x(\sin x + \cos x)]_0^\pi = ..$ ✓

RA5

7 $y'' - 7y' + 12y = 0$ Characteristic EQN

Auxiliary EQN: $\lambda^2 - 7\lambda + 12 = 0$

$(\lambda - 3)(\lambda - 4) = 0$ $\begin{matrix} \times\ 12 \\ +\ -7 \end{matrix}\Big\} -3, -4$

hence two real roots, $\lambda = 3, \lambda = 4$

So the ~~ODE Solution~~ is $y(x) = Ae^{3x} + Be^{4x}$ ✓

call it general solution still

b) $y'' + 64y = 0$

~~Auxiliary~~ equation: $\lambda^2 + 64 = 0$

$\lambda = \pm 8i$: hence complex roots.

So the ~~ODE Solution~~ is $y(x) = A\cos(8x) + B\sin(8x)$

$y'(x) = -8A\sin(8x) + 8B\cos(8x)$
$y(0) = 0 \Rightarrow 0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$
$y'(0) = 3 \Rightarrow 3 = -8\sin(0) + 8B\cos(0) \Rightarrow 3/8 = B$
Hence Solution to IVP $y(x) = 3/8 \sin(8x)$ ✓
a

c) $y'' - 2y' + y = 0$
Auxillary EQN: $\lambda^2 - 2\lambda + 1 = 0$
$(\lambda - 1)^2 = 0$ $\quad \begin{matrix} (\times) 1 \\ (+) -2 \end{matrix} \Big\} -1, -1$
Hence repeated root of $\lambda = 1$.
Solution to ODE is $y(x) = (A + Bx)e^x$
$y(0) = 1 \Rightarrow 1 = (A + 0)e^0 \Rightarrow A = 1$
$y(1) = 2 \Rightarrow 2 = (1 + B)e^1 \Rightarrow 2/e - 1 = B$
Hence Solution BVP is $y(x) = (1 + (2/e - 1)x)e^x$ ✓ $\forall x \in [0,1]$.
emphasise continuous function

RA1
⑧ max $\{f(x), g(x)\}$ must simplify.
$M(x) = \max\{f(x), g(x)\} = \frac{1}{2}\left[f(x) + g(x) + \sqrt{(f(x) - g(x))^2}\right]$
$m(x) = \min\{f(x), g(x)\} = \frac{1}{2}\left[f(x) + g(x) - \sqrt{(f(x) - g(x))^2}\right]$

RA2
⑧ If $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 3x^2 + 5 \ \forall x \in \mathbb{R}$ then $f'(x) = 6x$:
$f'(x) = \lim_{h \to 0} \frac{f(x+h)}{} \ldots$ ✓

b) $g: \mathbb{R} \setminus \{1\}$ $\quad g(x) = \frac{3}{1-x}$
$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\frac{3}{(1-x-h)} - \frac{3}{(1-x)}}{h} = \lim_{h\to 0} \frac{\frac{3(1-x)}{(1-x-h)(1-x)} - \frac{3(1-x-h)}{(1-x-h)(1-x)}}{h} = \lim_{h\to 0} \frac{3 - 3x - (3 - 3x + 3h)}{h(1-x-h)(1-x)}$$
$$= \lim_{h\to 0} \frac{-3}{(1-x-h)(1-x)} = \frac{-3}{(1-x)^2}$$ ✓

c) $h(x) = \sin(2x)$
$$h'(0) = \lim_{h\to 0} \frac{h(0+h) - h(0)}{h} = \lim_{h\to 0} \frac{\sin(2(0+h)) - \sin(2\cdot 0)}{h} = \lim_{h\to 0} \frac{\sin(2h)}{h}.$$
Small angle approximation $\sin a \approx a$ for very small a: Dont use small angle use notable limit
$\lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2.$
$$\frac{\sin(2h)}{h} = 2 \cdot \frac{\sin(2h)}{2h}$$

RA3
⑧ $f: [0,b] \to [0,\infty)$ $\quad f(x) = x^2$ $\quad b \in (0,\infty)$
a) $x_i = \frac{ib}{n}$ ✓

b) $\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$

$L(f, P_n) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$
$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ $\quad f(x)$ is increasing on $[0,b]$ hence $m_i = (x_{i-1})^2$
$$L(f, P_n) = \sum_{i=1}^{n} (x_{i-1})^2 \underbrace{(x_i - x_{i-1})}_{b/n} = \frac{b}{n} \sum_{i=1}^{n} (x_{i-1})^2 = \frac{b}{n} \cdot \sum_{i=0}^{n-1} \left(\frac{ib}{n}\right)^2 = \frac{b^3}{n^3} \sum_{i=0}^{n-1} i^2 = \frac{b^3}{n^3} \cdot \frac{1}{6}(n-1)((n-1)+1)(2(n-1)+1)$$
$$= \frac{b^3}{6n^3}(n-1)n(2n-1)$$
$U(f, P_n) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$
$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ $f(x)$ is increasing on $[0,b]$ hence $M_i = (x_i)^2$
$$U(f, P_n) = \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \cdot \frac{b}{n} = \frac{b^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{b^3}{6n^3} n(n+1)(2n+1)$$ ✓

c) $L(f, P_n) = \frac{b^3}{6} \cdot \frac{1}{n^2}(n-1)(2n-1) = \frac{b^3}{6} \cdot \frac{(n-1)}{n} \cdot \frac{2n-1}{n} = \underset{\text{constant}}{\frac{b^3}{6}} \cdot (1 - \frac{1}{n}) \cdot (2 - \frac{1}{n})$ or $\frac{b^3}{6} \cdot \left(\frac{2n^2 - 3n + 1}{n^2}\right)$
maximised when $n \to \infty$
$\Rightarrow \frac{b^3}{6} \cdot \left(2 - \frac{3}{n} + \frac{1}{n^2}\right)$
$\sup\{L(f, P_n) : n \in \mathbb{N}\} = \frac{b^3}{6} \cdot 1 \cdot 2 = \frac{b^3}{3}.$
$a_n := \frac{b^3}{6} \cdot \left(2 - \frac{3}{n} + \frac{1}{n^2}\right)$
Increasing sequence
$\lim_{n\to\infty} a_n = \frac{b^3}{3}$

$U(f, P_n) = \frac{b^3}{6} \cdot \frac{1}{n^2}(n+1)(2n+1) = \frac{b^3}{6} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} = \underset{\text{constant}}{\frac{b^3}{6}} \cdot (1 + \frac{1}{n}) \cdot (2 + \frac{1}{n})$
$\inf\{U(f, P_n) : n \in \mathbb{N}\} = \frac{b^3}{6} \cdot 1 \cdot 2 = \frac{b^3}{3}$ ✓
Minimised when $n \to \infty$
(equal width partitions subset of all partitions).

d) $\underline{\int_0^b} f = \sup\{L(f,P_n) : n \in \mathbb{N}\} = b^3/3$ $\times$ $\underline{\int_0^b} f := \sup_P L(f,P) \geq \sup_{n\in\mathbb{N}} L(f,P_n) = \frac{b^3}{n}$

$\overline{\int_0^b} f = \inf\{U(f,P_n) : n \in \mathbb{N}\} = b^3/3$ $\overline{\int_0^b} f := \ldots\ldots = \frac{b^3}{n}$

Then $\frac{b^3}{3} \leq \underline{\int_0^b} f \leq \overline{\int_0^b} f \leq \frac{b^3}{3}$

so $\underline{\int_0^b} f = \frac{b^3}{3}$ $\overline{\int_0^b} f = \frac{b^3}{3}$.

e) f is a Polynomial and so Continuous By the boundedness theorem f is therefore bounded. Further, f continuous $\Rightarrow$ f integrable.

Therefore $\int_0^b f := \underline{\int_0^b} f = \overline{\int_0^b} f = \frac{b^3}{3}$. ✓

RA 4)

8) $\frac{x-4}{x^2-5x+6}$ $\left.\begin{matrix}\times 6\\ + -5\end{matrix}\right\} -3, -2,$

$= \frac{x-4}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$

$\Rightarrow x-4 = A(x-2) + B(x-3)$ ✓

b) $\int_1^2 \frac{x^5+x-1}{x^3+1} dx$

$$\begin{array}{r|l} & x^2 \\ x^3+1 & x^5 + 0x^4 + 0x^3 + 0x^2 + x - 1 \\ & x^5 \qquad\qquad x^2 \qquad - \\ \hline & 0 \qquad\qquad -x^2 + x - 1 \end{array}$$

So $\frac{x^5+x-1}{x^3+1} = x^2 - \frac{x^2-x+1}{x^3+1} = x^2 - \frac{x^2-x+1}{(x+1)(x^2-x+1)}$

$\frac{x^2-x+1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$\Rightarrow x^2-x+1 = A(x^2-x+1) + (Bx+C)(x+1)$

$\Rightarrow x^2-x+1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$.

Compare Coefficients:

$1 = A+B$ ①

$-1 = -A+B+C$ ②

$1 = A+C$

Solve then sub into $\int_1^2 \frac{x^5+x-1}{x^3+1} dx = \int_1^2 x^2 - \frac{A}{x+1} - \frac{Bx+C}{x^2-x+1}$. ✓

c) $\frac{x^2+2x-1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$

$x^2+2x-1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$

Solve then integrate. ✓

RA5)

8 a) $y'' - 2y' + 10y = e^x$ ⊛

Complementary function: $\lambda^2 - 2\lambda + 10 = 0$

$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-4(10)}}{2} = 1 \pm \frac{\sqrt{4-40}}{2} = 1 \pm \frac{\sqrt{-36}}{2} = 1 \pm 3i$

So complex roots. Hence the Complementary function $y_c = e^x(A\cos 3x + B\sin 3x)$

Choose Particular Solution of $y = \lambda e^x$, then $y' = \lambda e^x$ and $y'' = \lambda e^x$

Sub into ⊛ $\lambda e^x - 2\lambda e^x + 10\lambda e^x = e^x$

$\lambda - 2\lambda + 10\lambda = 1$

$9\lambda = 1$

$\lambda = 1/9$ $y_p = 1/9\, e^x$

Hence the general Solution is C.F + P.S: $y(x) = e^x(A\cos 3x + B\sin 3x) + 1/9\, e^x$. ✓

$y(x) = y_p + y_c \Rightarrow$

b) $y'' + 5y' + 4y = 3 - 2x$ ⊛

C.F: $\lambda^2 + 5\lambda + 4 = 0$ $\left.\begin{matrix}\times 4\\ + 5\end{matrix}\right\} 4, 1$

$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-5 \pm \sqrt{25-4(4)}}{2} = \frac{-5 \pm \sqrt{9}}{2} = \frac{-5 \pm 3}{2} \Rightarrow \lambda = -1$ or $\lambda = -4$.

Hence C.F, $y(x) = Ae^{-x} + Be^{-4x}$.

P.S: Choose $y(x) = \lambda + \mu x$, then $y'(x) = \mu$ and $y''(x) = 0$ sub into ⊛:

$0 + 5\mu + 4(\lambda + \mu x) = 3 - 2x$

Compare Coefficients:

$5\mu + 4\lambda = 3$ ①

$4\mu = -2$ ② $\Rightarrow \mu = -\frac{1}{2}$

then ①: $5(-\frac{1}{2}) + 4\lambda = 3 \Rightarrow 4\lambda = 3 + \frac{5}{2} \Rightarrow \lambda = \frac{11}{8}$

so $y(x) = \frac{11}{8} - \frac{1}{2}x$.

General Solution is P.S + C.F.: $y(x) = Ae^{-x} + Be^{-4x} + \frac{11}{8} - \frac{1}{2}x$

$y'(x) = -Ae^{-x} - 4Be^{-4x} - \frac{1}{2}$

$y(0) = 0 \Rightarrow 0 = A + B + \frac{11}{8} \Rightarrow A = -B - \frac{11}{8}$ ①

$y'(0) = 0 \Rightarrow 0 = -A - 4B - \frac{1}{2}$ ②

Sub ① into ②:

$0 = -(-B - \frac{11}{8}) - 4B - \frac{1}{2}$

$0 = -3B - \frac{1}{2} + \frac{11}{8}$

$3B = \frac{11}{8} - \frac{4}{8} = \frac{7}{8}$

$B = \frac{7}{24}$

$A = -\frac{7}{24} - \frac{11}{8} = -\frac{7}{24} - \frac{33}{24} = -\frac{40}{24} = -\frac{20}{12} = -\frac{10}{6} = -\frac{5}{3}$.

Hence a solution to IVP is $y(x) = -\frac{5}{3}e^{-x} + \frac{7}{24}e^{-4x} + \frac{11}{8} - \frac{1}{2}x$.

c) $y'' + 9y = x\cos x$ ⊛

Complementary function: $\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$

Imaginary roots, hence $y_c := A\sin 3x + B\cos 3x$.

linear so $Ax+b$

$x\cos x$ ← need cos and sin each with own coefficients!

Particular solution choose $y_p = \lambda x\cos x + \mu x\sin x$ ← needed $(Ax+B)\cos(x) + (Cx+D)\sin x$!!

then let $u = \lambda x$, $v = \cos x$, $u' = \lambda$, $v' = -\sin x$ and by product rule $y_p' = uv' + vu' = \lambda\cos x - \lambda x\sin x + \mu\sin x + \mu x\cos x$

and again for y_p'' $(x\sin x)'$ requires product rule $\Rightarrow (x\sin x)' = x\cos x + \sin x$

$u = x$, $u' = 1$, $v = \sin x$, $v' = \cos x$

hence $y_p'' = -\lambda\sin x - \lambda(x\cos x + \sin x) + \mu\cos x + \mu(\cos x - x\sin x)$

$= -2\lambda\sin x - \lambda x\cos x + 2\mu\cos x - \mu x\sin x$

Sub y_p, y_p', y_p'' into ⊛

$-2\lambda\sin x + 2\mu\cos x - \lambda x\cos x - \mu x\sin x + 9(\lambda x\cos x + \mu x\sin x) = x\cos x$

Compare Coefficients: equate $\sin x$, $\cos x$, $x\cos x$ and $x\sin x$.

$\sin x, \cos x$: $-2\lambda + 2\mu = 0 \Rightarrow \lambda = \mu = 0$

$x\cos x$: $-\lambda + 9\lambda = 1 \Rightarrow 8\lambda = 1 \Rightarrow \lambda = \frac{1}{8}$

Flashcard runthrough

$f: A \to B$

$\Gamma_f = \{(x,y) \in A \times B : y = f(x)\}$

$f|_S = f \circ id_S$ where $id_S: S \to S$, $id_S(x) = x$

$g \circ f = g(f(x))$ to be valid $\cancel{B} \in C$ $\quad g \circ f: A \to D$

$f(A) \subseteq C$ ← Range NOT codomain.

A function $f: A \to B$ is invertible if there exists $g: B \to A$ such that $\forall x \in A, y \in B$ $f(x) = y \Leftrightarrow g(y) = x$.

Real valued inverses ignores codomain (only has to be injective not surjective)
⇔ If $f(A)=B$ and $g(B)=A$ AND .... (old condition)

Injective - $f: A \to B$ $\forall x, x' \in A$ $f(x)=f(x') \Rightarrow x=x'$ (1-1 function).
Surjective - $f: A \to B$ $f(A)=B$
Bijective both
Bijective ⇔ invertibility
Injective ⇔ real valued inverse

$|x+y| \leq |x|+|y|$ $|x-y| \geq ||x|-|y||$

symmetrical about vertical axis
Even if $\forall x \in A$ then $-x \in A$ And $f(-x)=f(x)$
Odd " " $f(-x)=-f(x)$
Rotational symmetry of 180°

a) Monotonic increasing: $\forall x, y \in A$ such that $x \leq y \Rightarrow f(x) \leq f(y)$
decreasing: $x \leq y$ $f(x) \geq f(y)$
If strictly inc / dec. ⇒ injective

b) $x \in A$, $x \leq y$ then y is upper bound

Completeness Axiom: if a set is bounded above its supremum exists
if a set is bounded below its infimum exists
⇔ There are no gaps on the number line.
A is non empty

Unbounded: $\sup A = \infty$
empty $\sup A = -\infty$

a) $\lim_{x \to \pm\infty} f(x) = C \in \mathbb{R}$
If limit is valid, $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ and $x > N / x < N \Rightarrow |f(x) - C| < \varepsilon$.

b) $\lim_{x \to \infty} f(x) = \pm\infty$
If lim valid, $\forall M > 0$, $\exists N \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ and $x > N \Rightarrow f(x) > M / f(x) < -M$.

c) $\lim_{x \to a} f(x) = L \in \mathbb{R}$
If lim valid, a is an accumulation point on $\mathbb{R}$, Also $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
$\forall x \in \mathbb{R}$ and $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

d) –
e) $\lim_{x \to a^+} f(x) = L \in \mathbb{R}$
If lim valid a is an accum. point on $\mathbb{R} \cap (a, \infty)$. Also $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
$\forall x \in \mathbb{R} \cap (a, \infty)$ ...

f) Assume it does and find contradiction. (approx if ∞ or $L \in \mathbb{R}$)

a is an accumulation point if $\forall \varepsilon > 0$ $\exists x \in \mathbb{R}$ such that $0 < |x-a| < \varepsilon$.
⇔ Used δ but still correct.
a accumulation point of Ω ⇔ $a = \inf\{x : x \in \Omega \cap (a, \infty)\}$
$\inf\{\Omega \cap (a, \infty)\}$

$f(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ \arctan x & \text{if } |x| \geq 1 \end{cases}$

$\lim_{x \to 0} f(x)$ $\arctan 1 = \pi/4$ so not continuous

If .. $= L_1$ - - . . $= L_2$
then ~ . $- L_1 + L_2$

$\lim_{x\to a} f(x)g(x) = L_1L_2$
If lim valid, $\forall \varepsilon>0, \exists \delta>0$ such that $\forall x \in \mathbb{R}$ and $0<|x-a|<\delta \Rightarrow |f(x)g(x)-L_1L_2|<\varepsilon$.
Observe $|f(x)g(x)-L_1L_2| = |f(x)g(x)+f(x)L_2-f(x)L_2-L_1L_2| = |f(x)[g(x)-L_2]+L_2[f(x)-L_1]|$
know $|g(x)-L_2|<\varepsilon$ $|f(x)-L_1|<\varepsilon$

so $|f(x)g(x)-L_1L_2|$ is

$f,g,h:\Omega\to\mathbb{R}$ If $f(x)\le g(x)\le h(x)$ $\forall x\in\Omega\setminus\{a\}$
and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L_1$ then $\lim_{x\to a} g(x) = L_1$

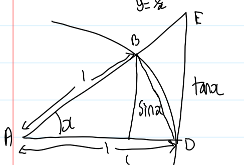
Continuity: f is continuous at $x=a$ if $\forall \varepsilon>0, \exists\delta>0$ such that $\forall x\in\Omega$ and $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$.

Let $\varepsilon>0$. Then let $\varepsilon' = \min\{1, \frac{\varepsilon}{(1+|L_1|+|L_2|)}\}$ and observe $\varepsilon'>0, \varepsilon'\le 1$, and $(1+|L_1|+|L_2|)\varepsilon' \le \varepsilon$. Then $\exists \delta_1,\delta_2>0$ such that $\forall x\in\Omega$
$0<|x-a|<\delta_1 \Rightarrow |f(x)-L_1|<\varepsilon'$
$0<|x-a|<\delta_2 \Rightarrow |g(x)-L_2|<\varepsilon'$
Let ~~$\delta>0$~~ $\delta = \min\{\delta_1,\delta_2\}$ (then $\delta>0$) and for the lim to be valid we require $\forall x\in\Omega$ and $0<|x-a|<\delta$
$\Rightarrow |f(x)g(x)-L_1L_2|<\varepsilon$.
Observe $|f(x)g(x)-L_1L_2| = |f(x)g(x)-L_2f(x)+L_2f(x)-L_1L_2| = |f(x)(g(x)-L_2)+L_2(f(x)-L_1)|$
$\le |f(x)(g(x)-L_2)|+|L_2(f(x)-L_1)| = |f(x)||g(x)-L_2|+|L_2||f(x)-L_1| = |f(x)-L_1+L_1||g(x)-L_2|+|L_2||f(x)-L_1|$
$\le (|f(x)-L_1|+|L_1|)(|g(x)-L_2|)+|L_2||f(x)-L_1| = |f(x)-L_1||g(x)-L_2|+|L_1||g(x)-L_2|+|L_2||f(x)-L_1|$
$<(\varepsilon')^2+|L_1|\varepsilon'+|L_2|\varepsilon' = (\cancel{\varepsilon'}1+|L_1|+|L_2|)\varepsilon' \le \varepsilon$. Hence valid.
$\le \varepsilon'+|L_1|\varepsilon'+|L_2|\varepsilon'$
as $0<\varepsilon'\le 1$.

$\lim_{x\to\infty}(\sqrt{2x-5}-\sqrt{4x+3}) = \lim_{x\to\infty}(\sqrt{x(2-\frac{5}{x})}-\sqrt{x(4+\frac{3}{x})}) = \lim_{x\to\infty}\sqrt{x}(\sqrt{2-\frac{5}{x}}-\sqrt{4+\frac{3}{x}})$
$\lim_{x\to\infty}\sqrt{x}=\infty$ $\lim_{x\to\infty}(\sqrt{2-\frac{5}{x}}-\sqrt{4+\frac{3}{x}}) = \sqrt{2-0}-\sqrt{4+0} = \sqrt{2}-2$ ← could change of variable $y\to 0$ $\frac{1}{x}\to y$
algebra of limits $\lim_{x\to\infty}\sqrt{x}\cdot\lim_{x\to\infty}(\sqrt{2-\frac{5}{x}}-\sqrt{4+\frac{3}{x}}) = \infty\cdot(\underbrace{\sqrt{2}-2}_{<0}) = -\infty$

Keep an eye on signs of reals.

change of variable $y=\frac{1}{x}$ $= \lim_{y\to 0^+}(\sqrt{2/y-5}-\sqrt{4/y+3})$



- $BC = \sin x$, $ED = \tan x$ (trig).
- Area $A\hat{B}D = \frac{1}{2}bh = \frac{1}{2}\times 1\times\sin x = \frac{1}{2}\sin x$
Area of sector $ABC = \frac{1}{2}r\theta = \frac{1}{2}\times 1\times x = \frac{1}{2}x$
Area of AED triange $= \frac{1}{2}\times 1\times\tan x = \frac{1}{2}\tan x$

Therefore $\frac{1}{2}\sin x \le \frac{1}{2}x \le \frac{1}{2}\tan x$
$\sin x \le x \le \tan x$
$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$
$\cos x \le \frac{\sin x}{x} \le 1$
$\lim_{x\to 0}\cos x \underset{\text{continuity}}{=} \cos 0 = 1$ $\lim_{x\to 0} 1 = 1$
Hence from sandwich theorem $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

a) $\lim_{x\to 0}\frac{\sin x}{x} = 1$ $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$ (from 1st) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to\infty}(1+\frac{1}{x})^x = e$ ($e^{\log}$ trick? ?) And $\lim_{x\to-\infty}$
$\lim_{x\to 0}\frac{(1+\log x)}{x} = 1$ $\lim_{x\to 0}\frac{e^x-1}{x} = 1$ $\lim_{x\to 0} x\log x = 0$ (L'Hospital) $\lim_{x\to 0} x\sin\frac{1}{x} = 0$ (sandwich)
Inside log $\frac{\log(x+1)}{x} = \log((x+1)^{\frac{1}{x}})$ so from prev.
from reciprocal of $\lim_{x\to 0}\frac{(1+\log x)}{x}$ (change of variable $y = e^x - 1$ $\Rightarrow y+1 = e^x$, $x = \log(y+1)$)
$\lim_{x\to 0}\frac{e^x-1}{x} \underset{y=e^x-1}{=} \frac{y}{\log(y+1)}$

Intermediate Value Theorem: $f:[a,b]\to\mathbb{R}$ continuous on $[a,b]$
where $a\le b$ and $f(a)\le f(b) \Rightarrow \forall y\in[f(a),f(b)]\ \exists\, c\in(a,b)$ such that $f(c)=y$.

Boundedness Theorem: $f:[a,b]\to\mathbb{R}$ ($a\le b$, $a,b\in\mathbb{R}$) continuous then f is bounded and attains its bounds (ie closed interval).

$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x} = \lim_{d\to 0} \frac{f(a+d)-f(a)}{d}$? $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

Prove that $(uv)' = u'v + vu'$ =

$(uv)' = \lim_{\Delta x\to 0} \frac{uv(x+\Delta x) - uv(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{u(x+\Delta x)v(x+\Delta x) - v(x+\Delta x)u(x) + v(x+\Delta x)u(x) - u(x)v(x)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{[u(x+\Delta x)-u(x)]\,v(x+\Delta x) + u(x)[v(x+\Delta x)-v(x)]}{\Delta x} = \lim_{\Delta x\to 0} u'(x)v(x+\Delta x) + u(x)v'(x) = u'v + v'u$

cosh sinh tanh

$\frac{e^x + e^{-x}}{2}$

Chain rule: $f, g: \Omega \to \mathbb{R}$ differentiable, then $(f(g(x)))' = g'(x) f'(g(x))$

$\arcsin'(y) \underset{\substack{\text{or limits} \\ y=\sin x}}{\overset{\text{change}}{=}} (\arcsin(\sin x))' = \frac{1}{\cos x} = \frac{1}{\sqrt{1-s^2}} = \frac{1}{\sqrt{1-y^2}}$.

inverse derivative rule.

not small angle ⇒ uses notable limits $\sin(\frac{\pi}{2}-x)$

$\cos x$ uses translation $\cos x = \sin(x + \frac{\pi}{2})$

$(e^x)' = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$ notable lim

$(\ln(x))' = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0} \frac{\ln(\frac{x+h}{x})}{h} = \lim_{h\to 0} \frac{\ln(1+\frac{h}{x})}{h} = \frac{1}{x} \lim_{h\to 0} \frac{\ln(1+h/x)}{(h/x)} = \frac{1}{x} \lim_{y\to 0} \frac{\ln(1+y)}{y} = \frac{1}{x}\cdot 1$

$y = h/x$ notable limit

OR inverse method: $\lim (\ln x)' \underset{x\to a}{\overset{\lim}{=}} (\ln e^y)'$

$\log_a x = \frac{\ln x}{\ln a}$.

$(a^x)' = (e^{\log a^x})' = (e^{x\log a})' = (\exp(x\log a))' \overset{\text{chain}}{\underset{\text{rule}}{=}} (x\log a)' \exp'(x\log a) = \log a \cdot e^{x\log a} = \log a \cdot a^x$

$1 - t^2 = s^2$ sech

Stationary point of f when $f'(x) = 0$. (and differentiable)

b) Fermat's Theorem states that local min or local max ⇒ stationary point and differentiable at that point (~~*~~)

c) Rolle's Theorem: $f: [a,b] \to \mathbb{R}$ $a<b$: $a, b \in \mathbb{R}$. f is:

- continuous on $[a,b]$
- differentiable on (a,b)
- $f(a) = f(b)$

Then ~~$\frac{f(b)-f(a)}{b-a} = 0$~~. $\exists\, c \in (a,b)$ such that $f'(c) = 0$

(Proof is boundedness theorem the Fermat's theorem).

MVT: $f: [a,b] \to \mathbb{R}$ ~~$a \le b$~~ $a<b$: $a, b \in \mathbb{R}$

- cont.
- ~

Then $\exists\, c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Proof Let $g: [a,b] \to \mathbb{R}$ $g(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$

observe $g(a) = f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = f(a)$

and $g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(b) - f(b) + f(a) = f(a)$

Hence $g(a) = g(b)$ g also continuous $[a,b]$ comp of cont func. and differentiable on (a,b)

Then by Rolle's theorem $\exists\, c \in (a,b)$ such that $g'(c) = 0$

$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

$\Rightarrow g'(c) = 0 \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$ as required.

f increasing $\Leftrightarrow f'(x) \geq 0$

f strictly increasing ~~⇒~~ $f'(x) > 0$

and differentiable on (a,b) (~~*~~)

f increasing $\Longleftrightarrow f'(x) \geq 0$
f Strictly increasing $\Rightarrow \cancel{\Leftarrow} f'(x) > 0$
and differentiable on (a,b) $(\cancel{\Leftarrow})$

Convex/Concave UP (always ~~above~~ below the line AB $0<t<1$
Mathematically $f((1-t)a+tb) \leq (1-t)f(a) + tf(b)$
Convex: $f''(x) > 0$ (twice differentiable) Also f' is increasing.
f differentiable when using a differential. f differentiable when theorem has differential.

Cauchy MVT $f,g: [a,b] \to \mathbb{R}$ $a<b$ if f,g:
· Cont on $[a,b]$
· dif. on (a,b) and $g(a) \neq g(b)$
Then $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ where $\exists$ $c \in (a,b)$.

$\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \left(\frac{1}{x^2}\right) \cdot \frac{\left(\frac{1}{1+x^2}\right)}{1} = \lim_{x\to\infty} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{x^2}\right)}$
observe $\lim_{x\to\infty}\left(\frac{1}{1+x^2}\right) = \lim_{y\to\infty}\left(\frac{1}{y}\right) = 0$ and $\lim_{x\to\infty}\left(\frac{1}{x^2}\right) = 0$.
change of variable $y = x^2+1$

and $\left(\frac{1}{1+x^2}\right)' = ((1+x^2)^{-1})' = (-(1+x^2)^{-2}) = \frac{-1}{(1+x^2)^2}$ and $\left(\frac{1}{x^2}\right)' = (x^{-2})' = -x^{-3} = \frac{-1}{x^3}$
both limits exist hence we can apply L'Hospital's Rule:
$\lim_{x\to\infty}\left(\frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{x^2}\right)}\right) = \lim_{x\to\infty} \frac{\left(\frac{1}{1+x^2}\right)'}{\left(\frac{1}{x^2}\right)'} = \frac{\left(\frac{-1}{(1+x^2)^2}\right)}{\left(-\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{x^3}{(1+x^2)^2}$
DO $\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{2x}{2x} = \lim_{x\to\infty} 1$ because L'Hospital works when both ∞ too.

OR divide top and bottom by x^2 $\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^2}+1}$ then algebra of limits.

$S_n = \frac{n}{2}(2a + (\overset{n-1}{1-n})d)$
$S_n = \frac{a(1-r^n)}{1-r}$

Partition Lemma: For Partitions P,Q,R,S:
① $L(f,P) \leq U(f,P)$
② $P \subseteq Q$: i) $L(f,P) \leq L(f,Q)$
ii) $U(f,P) \geq U(f,Q)$
③ $L(f,R) \leq U(f,S)$.
Proofs ① $L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ always Positive non negative $U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$
$m_i \leq M_i$ hence $L(f,P) \leq U(f,P)$
② Consider case where $P = \{x_0, x_1, \cdots, x_n\}$ and $Q = \{x_0'\} \cup P$ call it y not x_0'
such that $x_0' \in (x_0, x_1)$.
then $L(f,Q) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ all terms in the series would remain same
except when $i=1$. Previously the term was $m_1(x_1 - x_0)$ and now it is
$m_1'(x_0' - x_0) + m_1''(x_1 - x_0')$ m_1' and $m_1'' \geq m_1$ hence $L(f,Q) \geq L(f,P)$.
Similarly can show that $M_1'(x_0' - x_0)$ and $M_1''(x_1 - x_0)$ where M_1' and $M_1'' \leq M_1$
hence $U(f,Q) \leq U(f,P)$.
case when $Q = P \cup \{x_i'\}$ where x_i' is between any (x_i, x_{i-1}) follow
the same logic as previous
case when $Q = P \cup \{q_0, q_1, q_2 \cdots q_n\}$ then $Q = P \cup \{q_0\} \cup \{q_1\} \cdots \cup \{q_n\}$
and repeat step 2 iteratively
③ $L(f$ . . — —

$\underline{\int_a^b} f = \sup_P \{L(f,P)\} = \sup\{L(f,P): P \text{ is a partition of } [a,b]\}$

$\overline{\int_a^b} f$

$f: [0,b] \to [0,\infty)$ $f(x) := x$
Let $n \in \mathbb{N}$ and P_n be a partition of $[\overset{0}{a},b]$ with equal width subintervals.
$L(f,P_n) = \sum_{i=1}^{n} m_i \underbrace{(x_i - x_{i-1})}_{b/n}$
where $m_i = \inf\{f(x): x \in [x_{i-1}, x_i]\} = x_{i-1} = \frac{(i-1)b}{n}$ (as x increasing func.
so $L(f,P_n) = \sum_{i=1}^{n} \frac{(i-1)b}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2}\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) = \frac{b^2}{n^2}\left(\frac{n}{2}(n+1) - n\right) = \frac{b^2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2} - n\right) = \frac{b^2}{n^2}\left(\frac{n^2}{2} - \frac{n}{2}\right) = \frac{b^2}{2}\left(1 - \frac{1}{n}\right)$

$M_i = \sup\{f(x): x \in [x_{i-1}, x_i]\} = x_i = \frac{ib}{n}$

so $U(f,P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{ib}{n}\cdot\frac{b}{n} = \frac{b^2}{n^2}\sum_{i=1}^{n} i = \frac{b^2}{n^2}\left(\frac{n}{2}(n+1)\right) = \frac{b^2}{2n^2}(n^2+n) = \frac{b^2}{2}\left(1+\frac{1}{n}\right)$

Let $\varepsilon > 0$. Then $U(f,P_n) - L(f,P_n) = \frac{b^2}{2}\left(1+\frac{1}{n}\right) - \frac{b^2}{2}\left(1-\frac{1}{n}\right) = \frac{b^2}{2}\left(1+\frac{1}{n} - 1 + \frac{1}{n}\right) = \frac{b^2}{2}\left(\frac{2}{n}\right) = \frac{b^2}{n}$ (*)
We want $\frac{b^2}{n} < \varepsilon$. (or $\frac{b^2}{\varepsilon} < n$). choose $N > \frac{b^2}{\varepsilon}$ then $N \in \mathbb{R}$ by (*)
$U(f,P_N) - L(f,P_N) < \varepsilon$
Hence by Riemann's Criterion f is integrable.

Monotonic $\Rightarrow$ integrable
Continuous $\Rightarrow$ integrable
Uniformly Continuous definition uses $x,y \in [a,b]$ and $|x-y|$ instead of $y \in [a,b]$ and $\forall x \in \Omega$ $|x-y|$

$\int_a^b f = \int_a^c f + \int_c^b f \quad c \in [a,b]$.

If f,g integrable $\quad \alpha \in \mathbb{R}$
Then $\int f+g = \int f + \int g$ and $\int \alpha f = \alpha \int f$
REMEMBER TO CALL This <u>linearity</u>

$f_- \quad f_+$
$f = f_+ - f_-$
f,g integrable
fg integrable $\Rightarrow \int fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$

$m(b-a)$ ~~0~~ $< \int_a^b f < M(b-a)$ where $m \le f(x) \le M$ $\forall x \in \mathbb{R}$

1FTC: $f:[a,b] \to \mathbb{R}$ bounded, integrable and differentiable function F, then if f cont on $c \in [a,b]$ is continuous on f then $F'(c) = f(c)$ where $F(x) := \int_a^x f$.
$\hookrightarrow \frac{d}{dx}\left(\int_a^x f\right) = f(x)$
$F'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$
$F'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$

Then $F'(x) = f(x)$ if cont on $x \in [a,b]$.

$f(x) = \frac{1}{1+\sin^2 x}$ ... —
cont - —
$\Rightarrow$ integrable and bounded and by 1FTC $F(x) = \int_0^x \frac{1}{1+\sin^2 t}\,dt$ and $F'(x) = f(x)$

observe: $G(x) = F(3x)$ hence by chain rule also differentiable with
$G'(x) = 3F'(3x) = \frac{3}{1+\sin^2 3x}$.

An antiderivative of $f:[a,b] \to \mathbb{R}$ integrable is a (differentiable) function g,
$g' = f$. ie $g = \int f$
$\int f = g + c \qquad \int f := \int_a^x f + c$

2FTC: $f:[a,b]$ integrable, bounded and g is a (differentiable) function such that
$g' = f$. Then $\int_a^b f = g(b) - g(a)$.

$\int \sec^2 x\,dx = \tan x$
$\int \tan x\,dx = \ln|\sec x|$
$\int \sec x\,dx = \ln|\sec x + \tan x|$
$\int \sec x \tan x\,dx = \sec x$.

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a) Let $u,v:[a,b] \to \mathbb{R}$ integrable, bounded, and differentiable on (a,b). (included in differentiable.)
Then $\int uv' = uv - \int u'v$ and u' and v' are integrable.

b) Let $\cancel{f}, g : [a,b] \Rightarrow \mathbb{R}$ integrable and differentiable. and g' integrable

Then let $u = g(x)$ so $du = g'(x)\,dx$ — f continuous so works

Then $\int g'(x) f(g(x))\,dx = \int f(u)\,du$

and $\int_a^b g'(x) f(g(x))\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\tan^2 x = \sec^2 x - 1$

$\sin x \cos x = \frac{1}{2}\sin 2x$

$\cos 2x = 2\cos^2 x - 1$ $\Rightarrow \cos^2 x = \frac{\cos 2x + 1}{2}$

$1 - 2\sin^2 x$ $\quad \sin^2 x = 1 - \cos 2x$

$\frac{1}{\sqrt{a^2 - x^2}} \Rightarrow x = a\sin\theta$

$\frac{1}{\sqrt{a^2 + x^2}} \Rightarrow x = a\tan\theta$

$\frac{1}{\sqrt{x^2 - a^2}} \Rightarrow x = a\sec\theta$

$\int \sin^m \cos^n\,dx$

take odd one and separate

$\int \tan^m \sec^n\,dx$ $\quad \tan^2 = \sec^2 - 1$

If tan O then use $u = \sec\theta \Rightarrow du = \sec\theta\tan\theta$ then

If sec E then use $u = \tan\theta \Rightarrow du = \sec^2\theta$ and take $\sec^2\theta$ out.

$t^2 + 1 = \sec^2$

$\int \sec^3 x\,dx = \int \sec^2 x \sec x\,dx = \int (\tan^2 x + 1)\sec x\,\cancel{dx}$

$= \int \tan^2 x \sec x + \int \sec x\,dx$ — Then by Parts $u = \sec x$ $\underline{v' = \sec^2 x}$

$= \int \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos x}\,dx + \int \sec x\,dx$ — Last resort, int. by Parts.

$S(n) = \int \sin^n(x)\,dx \quad n \in \mathbb{N}$

Prove $nS(n) = -\sin^{n-1}(x)\cos x + (n-1)S(n-2) \quad \forall n \geq 2.$

$S(n) = \int \sin^{n-1} x \sin x\,dx$

Let $u = \sin^{n-1} x \quad v = -\cos x$

$u' = (n-1)\sin^{n-2} x \cos x \quad v' = \sin x$

Then using integration by parts: $\int uv' = uv - \int u'v\,dx$

$\int \sin^{n-1} x \sin x\,dx = -\sin^{n-1}(x)\cos x + \int (n-1)\sin^{n-2} x \cos^2 x$

$\Rightarrow S(n) = -\sin^{n-1}(x)\cos x + (n-1)\,?$ $\quad (n-1)\int \sin^{n-2} x(1 - \sin^2 x)$

$= (n-1)\left[\int \sin^{n-2} x - \int \sin^n x\right]$

$= (n-1)S(n-2) - (n-1)S(n)$

So $S(n) + (n-1)S(n) = -\sin^{n-1}(x)\cos x + (n-1)S(n-2)$

~~Remember if LHS is more than just S(n).~~

If $f(x) \leq g(x) \quad \forall x \in [a,b]$, then $\int_a^x f(x) \leq \int_a^x g(x)$ — Probably true though.

X $\quad 0 \leq f(x) \leq g(x) \;\forall x \in [a,\infty)$ then

If $\int_a^\infty g$ convergent $\Rightarrow \int_a^\infty f$ Convergent.

If $\int_a^\infty f$ divergent $\Rightarrow \int_a^\infty g$ divergent.

For sep. of variables remember to also have solution for it $= 0$ (can't divide it)

$b^2 - 4ac > 0$ the over

$= 0$ critical

< 0 under

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1) a) $f : \mathbb{R} \Rightarrow \mathbb{R}$ $\quad \lim_{x \to -\infty} f(x) = \infty$ means that

$\forall M>0, \exists N\in\mathbb{R}$ such that $\forall x\in\mathbb{R}$ and $x<N \Rightarrow f(x)<-M$.

b) $\lim_{x\to-\infty} \sqrt[5]{x^2} = \infty$

$\forall M>0, \exists N\in\mathbb{R}$ such that, $\forall x\in\mathbb{R}$ and $x<N \Rightarrow \sqrt[5]{x^2}<-M$ to be valid.
Observe that $\sqrt[5]{x^2} = (x^2)^{1/5} = x^{2/5}$, choose $N=M^{5/2}$
such that $x<M^{5/2} \Rightarrow (x)^{2/5}<M$ therefore the limit is valid.

c) i) $\lim_{x\to0}\left(\frac{(e^{5x}-1)\tan x}{\cos(2x)-1}\right) = \lim_{x\to0}\left(\frac{e^{5x}-1}{5x}\cdot\frac{2x}{\cos(2x)-1}\cdot\tan x\cdot\frac{5}{2}\right)$

Note that $\lim_{x\to0}\frac{e^{5x}-1}{5x} \underset{\substack{\text{change}\\\text{of variable}\\y=5x}}{=} \lim_{y\to0}\frac{e^y-1}{y} \underset{\substack{\text{notable}\\\text{limit}}}{=} 1$

and that $\lim_{x\to0}\frac{\cos(2x)-1}{2x} \underset{\substack{\text{change}\\\text{of variable}\\y=2x}}{=} \lim_{y\to0}\frac{\cos y-1}{y} \underset{\substack{\text{notable}\\\text{limit}}}{=} 1/2$

and due to the continuity of $\tan x$, $\lim_{x\to0}\tan x = \tan 0 = 0$.
Hence the initial limit, using the algebra of limits:
$\Rightarrow \lim_{x\to0}\frac{e^{5x}-1}{5x}\cdot\lim_{x\to0}\frac{2x}{\cos(2x)-1}\cdot\lim_{x\to0}\tan x\cdot\lim_{x\to0}\frac{5}{2} = 1\cdot(1/2)^{-1}\cdot0\cdot5/2 = 0.$
$\lim_{x\to0}\frac{(e^{5x}-1)\tan x}{\cos(2x)-1} = 0.$

ii) $\lim_{x\to\infty}\frac{3x^3+x^2+\sin(e^x)}{5x-8x^3+\arctan(\log x)} \begin{matrix}\div x^3\\\div x^3\end{matrix} = \lim_{x\to\infty}\frac{3+\frac{1}{x}+\frac{\sin(e^x)}{x^3}}{\frac{5}{x^2}-8+\frac{\arctan(\log x)}{x^3}}$

$\lim_{x\to\infty}\frac{\sin(e^x)}{x^3} = \lim_{x\to\infty}\left(\frac{\sin(e^x)}{e^x}\cdot e^x\cdot\frac{1}{x^3}\right)$ which with algebra of limits + notable limits:
such that $\lim_{x\to\infty}\left(\frac{\sin(e^x)}{e^x}\right) \underset{\substack{\text{change}\\\text{of variable}\\y=e^x}}{=} \lim_{y\to\infty}\frac{\sin y}{y} \underset{\substack{\text{notable}\\\text{limit}}}{=} 1$

$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} 1/x^3 = 0$. Hence $\lim_{x\to\infty}\left(\frac{\sin(e^x)}{e^x}\right)$ is undefined as $\lim_{x\to\infty}\frac{\sin(e^x)}{x^3} = 1\cdot\infty\cdot0 =$ undefined.
$\Rightarrow$ The whole limit is undefined.

~~Sandwich Theorem or L'Hospital need~~ to be attempted if algebra + notable limits make it appear to look undefined. In this case sandwich theorem.

$\lim_{x\to\infty}\frac{\sin(e^x)}{x^3}$ Note that $|\sin(e^x)|\le1 \quad \forall x\in\mathbb{R}$
Hence $0\le|\sin(e^x)|\le1 \Rightarrow 0\le\left|\frac{\sin(e^x)}{x^3}\right|\le\frac{1}{x^3} \quad \forall x\in\mathbb{R}$.
$\lim_{x\to\infty}0=0$ $\lim_{x\to\infty}1/x^3=0$ hence by sandwich theorem $\lim_{x\to\infty}\left|\frac{\sin(e^x)}{x^3}\right|=0$. By null limits and absolute values $\Rightarrow \lim_{x\to\infty}\frac{\sin(e^x)}{x^3}=0$.

$\lim_{x\to\infty}\frac{\arctan(\log x)}{x^3}$, Note $0\le|\arctan(\log x)|\le\frac{\pi}{2} \quad \forall x\in\mathbb{R}$ as $|\arctan y|\le\frac{\pi}{2}\ \forall y\in\mathbb{R}$
Hence $0\le\left|\frac{\arctan(\log x)}{x^3}\right|\le\frac{\pi}{2x^3}$ as $\lim_{x\to\infty}0=0$ $\lim_{x\to\infty}\frac{\pi}{2x^3} \underset{\substack{\text{algebra}\\\text{of limits}}}{=} \frac{\pi}{2}\lim_{x\to\infty}\frac{1}{x^3}=0$
Then by sandwich theorem $\lim_{x\to\infty}\left|\frac{\arctan(\log x)}{x^3}\right|=0 \Rightarrow \lim_{x\to\infty}\frac{\arctan(\log x)}{x^3}=0$.

or $\Rightarrow \arctan(\lim_{x\to\infty}\log x)\cdot\lim_{x\to\infty}\frac{1}{x^3} = \arctan(\infty)\cdot0 = \frac{\pi}{2}\cdot0=0.$

$\lim_{x\to\infty}\arctan(\log x) = \lim_{y\to\infty}\arctan(y) = \frac{\pi}{2}$

Can do like this without sandwich theorem (it was always defined).

So back to initial limit $\lim_{x\to\infty}\frac{3+\frac{1}{x}+\frac{\sin(e^x)}{x^3}}{\frac{5}{x^2}-8+\frac{\arctan(\log x)}{x^3}}$ by algebra of limits:

$\frac{3+\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{\sin(e^x)}{x^3}}{5\lim_{x\to\infty}(x^{-2})-8+\lim_{x\to\infty}\frac{\arctan(\log x)}{x^3}} = \frac{3+0+0}{0-8+0} = -3/8.$

d) $g:(0,\infty)\to\mathbb{R}$ $\quad g(x)=x^{\sin(x^2)} \quad \forall x\in(0,\infty)$
$g(x) = x^{\sin(x^2)} = e^{\log x^{\sin(x^2)}} = e^{\sin(x^2)\log x}$ (log laws)
log both sides:
$\log g(x) = \log e^{\sin(x^2)\log x} = \sin(x^2)\log x$.
Take derivative:
$(\log(g(x)))' = (\sin(x^2)\log x)'$
using chain rule $(\log(g(x)))' = \frac{g'(x)}{g(x)}$

and let $u=\sin(x^2) \quad v=\log x$
$u'=2x\cos(x^2) \quad v'=\frac{1}{x}$

So using Product/Leibniz Rule (chain rule) $(uv)' = uv' + vu'$:

$$(\sin(x^2)\log x)' = \frac{\sin(x^2)}{x} + 2x\cos(x^2)\log x$$

Therefore $(\log(g(x)))' = (\sin(x^2)\log x)'$

$$\Rightarrow g'(x) = g(x)\left[\frac{\sin(x^2)}{x} + 2x\cos(x^2)\log x\right.$$

$$g'(x) = x^{\sin(x^2)}\left(\frac{\sin(x^2)}{x} + 2x\cos(x^2)\log x\right).$$

$$\int_0^4 \frac{x^3-2}{|x-1|}\,dx = \lim_{t\to 1^-}\int_0^t \frac{x^3-2}{|x-1|}\,dx + \lim_{t\to 1^+}\int_t^4 \frac{x^3-2}{|x-1|}\,dx$$ (extension/restriction theorem).

Note $|x-1| := \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x \leq 1 \end{cases}$ So the above integrals can be simplified to:

$$\Rightarrow \lim_{t\to 1^-}\int_0^t \frac{x^3-2}{1-x}\,dx + \lim_{t\to 1^+}\int_t^4 \frac{x^3-2}{x-1}\,dx.$$

$$\begin{array}{rl} & -x^2 - x - 1 \\ -x+1 \,) & x^3 \quad\quad -2 \\ & x^3 - x^2 \quad\quad - \\ \hline & 0 + x^2 \quad -2 \\ & x^2 - x \quad\quad - \\ \hline & 0 \quad x - 2 \\ & x - 1 \quad\quad - \\ \hline & 0 \quad -1 \end{array}$$

hence $\frac{x^3-2}{1-x} = -x^2 - x - 1 - \frac{1}{1-x}$

and $\frac{x^3-2}{x-1} = -\left(-x^2 - x - 1 - \frac{1}{1-x}\right) = x^2 + x + 1 + \frac{1}{1-x}$

$$\text{So } \lim_{t\to 1^-}\int_0^t \frac{x^3-2}{1-x}\,dx + \lim_{t\to 1^+}\int_t^4 \frac{x^3-2}{x-1}\,dx = -\lim_{t\to 1^-}\int_0^t x^2 + x + 1 + \frac{1}{1-x}\,dx + \lim_{t\to 1^+}\int_t^4 x^2 + x + 1 + \frac{1}{1-x}\,dx$$

$$\Rightarrow -\lim_{t\to 1^-}\left[\frac{x^3}{3} + \frac{x^2}{2} + x - \log|1-x|\right]_0^t + \lim_{t\to 1^+}\left[\frac{x^3}{3} + \frac{x^2}{2} + x - \log|1-x|\right]_t^4$$

$$\Rightarrow -\lim_{t\to 1^-}\left[\frac{t^3}{3} + \frac{t^2}{2} + t - \log|1-t| - (0)\right]$$

$\lim_{t\to 1^-}(\log|1-t|) = -\infty$ and hence $\lim_{t\to 1^-}\int_0^t \frac{x^3-2}{|x-1|}\,dx$ is divergent hence $\int_0^4 \frac{x^3-2}{|x-1|}$ is divergent.

Could've done $\lim_{y\to 0^+}(\log|t|)$ to emphasise.

$y = 1-t$

$x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}.$

Real Analysis 1 (Don't Submit)

03 October 2022 15:09

(1) i) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^4$

* Consider $x = 1$, $f(1) = 1$

$x = -1$, $f(-1) = (-1)^4 = 1$

Therefore f is not injective. Proof by counterexample.

* Range of f is $x \geq 0$, codomain is $\mathbb{R}$ hence not surjective
* Not injective or surjective, so not bijective.

ii) $f: \mathbb{R} \to [0, \infty)$

$f(x) = x^4$

* By part i, not injective
* Codomain is $[0, \infty)$, Range is $[0, \infty]$ hence surjective.
* Surjective but not injective, hence not bijective.

Let $y \in [0, \infty]$ Note $x \in \mathbb{R}$

Let $x = \sqrt[4]{y}$

We see that $x^4 = y$, so $f(x) = y$

Therefore $f(x) \in [0, \infty)$

and codomain $\in [0, \infty)$

iii) $f: (-\infty, 0] \to [0, \infty)$

$f(x) = x^4$

* By part i, not injective. ~~X~~
* Codomain is $[0, \infty)$, Range is $[0, \infty)$ so surjective
* Surjective but not injective, so not bijective.

Assume $x \neq y$, $x, y \in (-\infty, 0]$

then if $x^4 = y^4$

then $\sqrt[4]{x^4} = \sqrt[4]{y^4}$

$|x| = |y|$

$-x = -y$ ← as $x, y \leq 0$

so $-x = -y$, so $x = y$. Contradiction so bijective.

iv) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^3 + 1$

* Let $x, y \in \mathbb{R}$, where $f(x) = f(y)$.

$f(x) = f(y)$

* Let $x, y \in \mathbb{R}$, where $f(x) = f(y)$.

$f(x) = f(y)$

$x^3 + 1 = y^3 + 1$

$x^3 = y^3$

$x = y$

Hence f is injective.

Proving injectivity

$A \Rightarrow B$ the same as

not $B \Rightarrow$ not A

* Codomain is $\mathbb{R}$, Range is $\mathbb{R}$ so surjective
* Injective and surjective so bijective

Find inverse, let $y = x^3 + 1$

Swap outputs and inputs,

$x = y^3 + 1$

$y = \sqrt[3]{x - 1}$

Hence $f^{-1}(x) = \sqrt[3]{x-1}$

(2) i) $A = \{x \in \mathbb{R} : \frac{1}{x} \in \mathbb{Q}\}$, $\mathbb{N}$

For $\frac{1}{x} \in \mathbb{Q}$, $x \in \mathbb{Q}$

hence $\mathbb{N} \subseteq A$

ii) $B = \{x \in \mathbb{R} : 3x \in \mathbb{Q}\}$, $\mathbb{Q}$

If $3x \in \mathbb{Q}$, then $x \in \mathbb{Q}$

$B = \mathbb{Q}$

iii) $C = \{x \in \mathbb{R} : 4x \in \mathbb{Z}\}$ and $D = \{x \in \mathbb{R} : x - 3 \in \mathbb{Z}\}$

Real Analysis 1 (Submissions)

10 October 2022 14:59

4) $f(x) = \frac{1}{1+x}$ $\quad g(x) = e^{-x}$

i) $f(2+x) = \frac{1}{1+(2+x)} = \frac{1}{3+x}$

ii) $f(2x) = \frac{1}{1+(2x)} = \frac{1}{1+2x}$

iii) $f(x^2) = \frac{1}{1+x^2}$

iv) $f \circ f(x) = f\left(\frac{1}{1+x}\right) = \frac{1}{1+\left(\frac{1}{1+x}\right)} = \frac{1}{\left(\frac{1+x}{1+x}\right)+\left(\frac{1}{1+x}\right)} = \frac{1}{\left(\frac{2+x}{1+x}\right)} = \frac{1+x}{2+x}$

v) $f\left(\frac{1}{f(x)}\right) = f\left(\frac{1}{\left(\frac{1}{1+x}\right)}\right) = f(1+x) = \frac{1}{1+(1+x)} = \frac{1}{2+x}$

vi) $f \circ g(x) = f(e^{-x}) = \frac{1}{1+e^{-x}} = \frac{1}{1+\frac{1}{e^x}} = \frac{1}{\left(\frac{e^x+1}{e^x}\right)} = \frac{e^x}{e^x+1}$

11)

i) a) $g(x) = x \cdot \frac{2^x-1}{2^x+1}$

$$g(-x) = (-x) \cdot \left(\frac{2^{(-x)}-1}{2^{(-x)}+1}\right) = (-x) \cdot \left(\frac{\frac{1}{2^x} - \frac{2^x}{2^x}}{\frac{1}{2^x} + \frac{2^x}{2^x}}\right) = (-x) \cdot \left(\frac{\left(\frac{1-2^x}{2^x}\right)}{\left(\frac{1+2^x}{2^x}\right)}\right)$$

$$= (-x) \cdot \left(\frac{1-2^x}{1+2^x}\right) = x \cdot \left(\frac{2^x-1}{2^x+1}\right) = g(x)$$

× top and bottom by 2^x

$g(-x) = g(x)$ ∴ g is an even function (and not odd as $g(-x) \neq -g(x)$)

b) $h(x) = x + \sin x$

$h(-x) = (-x) + \sin(-x)$

$\sin(-x) \equiv -\sin(x)$, sub in:

$h(-x) = -x - \sin(x) = -(x + \sin x) = -h(x)$

$h(-x) = -h(x)$ ∴ h is an odd function (and not even as $h(-x) \neq h(x)$).

c) $h(x) = x^3 + \cos(\pi x)$

$h(-x) = (-x)^3 + \cos(-\pi x)$

$\cos(x) \equiv \cos(-x)$, and $(-x)^3 = -x^3$

$h(-x) = -(x)^3 + \cos(\pi x)$

$h(x) \neq h(-x)$ therefore not even.

$-h(x) = -x^3 - \cos(\pi x)$

$h(x) \neq h(-x)$ Therefore not odd.

ii) $f: \mathbb{R} \to \mathbb{R}$

$f(x) = a\sin x + b\cos x \qquad a, b \in \mathbb{R}$

For an odd function, $-f(x) = f(-x)$

$-(a\sin x + b\cos x) = a\sin(-x) + b\cos(-x)$

$-a\sin x - b\cos x = -a\sin(x) + b\cos(x)$

$0 = 2b\cos(x)$

Therefore to be odd, $b = 0$ and $a \in \mathbb{R}$.

For an even function, $f(x) = f(-x)$

$a\sin x + b\cos x = a\sin(-x) + b\cos(-x)$

$a\sin x + b\cos x = -a\sin(x) + b\cos(x)$

$0 = 2a\sin x$

Therefore to be even, $a = 0$ and $b \in \mathbb{R}$

For both an odd and even function, both conditions must be met so

- $b = 0$ and $a \in \mathbb{R}$
- $a = 0$ and $b \in \mathbb{R}$

$0 \in \mathbb{R}$ and so when $a = 0, b = 0$ the function is both odd and even.

Real Analysis 2 (Submissions)

23 October 2022 19:45

4) i) $\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(2x+1)} = \lim_{x\to 1} \frac{x+1}{2x+1} = \frac{1+1}{2(1)+1} = 2/3$

Algebra of limits

ii) $\lim_{x\to 0} \frac{(x-1)^3+(1-3x)}{x^2+2x^3} = \lim_{x\to 0} \frac{x^3+3x^2(-1)+3x(-1)^2+(-1)^3+(1-3x)}{x^2(1+2x)}$

$= \lim_{x\to 0} \frac{x^3-3x^2+3x-3x-1+1}{x^2(1+2x)}$

$= \lim_{x\to 0} \frac{x^3-3x^2}{x^2(1+2x)}$

$= \lim_{x\to 0} \frac{\cancel{x^2}(x-3)}{\cancel{x^2}(1+2x)}$ Algebra of limits

$= \lim_{x\to 0} \frac{x-3}{1+2x}$

$= \frac{0-3}{1+0} = -3$

iii) $\lim_{x\to 1} \frac{x^n-1}{x^m-1}$ $n, m \in \mathbb{N}$

If a limit exists top and bottom must have a factor of $(x-1)$ that cancels (factor theorem).

$x^n - 1 = \underbrace{x^{n-1}(x-1)}_{-x^{n-1}} + \underbrace{x^{n-2}(x-1)}_{-x^{n-2}} + x^{n-3}(x-1) + \dots + \underbrace{x(x-1)}_{-x^2} + \underbrace{1}_{-x}(x-1)$

Hence $\frac{x^n-1}{x-1} = x^{n-1}+x^{n-2}+x^{n-3}+\dots+x+1 = \sum_{r=1}^{n} x^{r-1}$. Similarly for $\frac{x^m-1}{x-1}$.

So $\lim_{x\to 1} \frac{x^n-1}{x^m-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}\sum_{r=1}^n x^{r-1}}{\cancel{(x-1)}\sum_{r=1}^m x^{r-1}}$ Algebra of limits

$= \lim_{x\to 1} \frac{\sum_{r=1}^n x^{r-1}}{\sum_{r=1}^m x^{r-1}} = \frac{\sum_{r=1}^n 1^{r-1}}{\sum_{r=1}^m 1^{r-1}} = \frac{\sum_{r=1}^n 1}{\sum_{r=1}^m 1} = n/m.$

iv) $\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2} \times \frac{(\sqrt{1+2x}+3)(\sqrt{x}+2)}{(\sqrt{1+2x}+3)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{(\sqrt{1+2x}-3)(\sqrt{1+2x}+3)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)(\sqrt{1+2x}+3)}$

$= \lim_{x\to 4} \frac{[(1+2x)-9](\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)}$

$= \lim_{x\to 4} \frac{2\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{(x-4)}(\sqrt{1+2x}+3)}$ Algebra of limits

$= \lim_{x\to 4} \frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3}$

$= \frac{2(\sqrt{4}+2)}{\sqrt{1+2(4)}+3} = \frac{2(4)}{3+3} = 8/6 = 4/3$

✓ v) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\left(\frac{\sin x}{\cos x}\right)-\sin x}{x^3} = \lim_{x\to 0} \frac{\sin x\left(\frac{1}{\cos x}-1\right)}{x^3} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\left(\frac{1}{\cos x}-1\right)}{x^2}\right]$

$= \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2\cos x}\right] = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2}\cdot\frac{1}{\cos x}\right] \underset{\text{limits}}{\overset{\text{algebra of}}{=}} \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0}\frac{1-\cos x}{x^2}\right)\cdot\left(\lim_{x\to 0}\frac{1}{\cos x}\right)$

v) $\lim_{x\to 0} \frac{[illegible]}{x^3} = \lim_{x\to 0} \frac{[illegible]}{x^3} = \lim_{x\to 0} \frac{[illegible]}{x^3} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{[illegible]}{x^2}\right]$

$= \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2\cos x}\right] = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2}\cdot\frac{1}{\cos x}\right] \xrightarrow[\text{limits}]{\text{algebra of}} \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0}\frac{1-\cos x}{x^2}\right)\cdot\left(\lim_{x\to 0}\frac{1}{\cos x}\right)$

We have the notable limits $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} = \frac{1}{2}$ ① and $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ ②, which can be substituted in:

$\Rightarrow 1\cdot\frac{1}{2}\cdot\left(\lim_{x\to 0}\frac{1}{\cos x}\right) = 1\cdot\frac{1}{2}\cdot\frac{1}{\cos 0} = 1\cdot\frac{1}{2}\cdot 1 = \frac{1}{2}.$

vi) $\lim_{x\to\infty} x\sin\frac{1}{x}$

Let $\theta = \frac{1}{x}$. Then $x\sin\frac{1}{x} = \frac{\sin\theta}{\theta}$ and if $x\to\infty$ then $\theta\to 0$ so:

$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ Using the substitution

$\hookrightarrow$ We know the notable limit $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

Hence $\lim_{x\to\infty} x\sin\frac{1}{x} = 1$.

⑨ $a_1, a_2, a_3 \in \mathbb{R}^+$ and $\lambda_1 < \lambda_2 < \lambda_3$.

$$\frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \frac{a_3}{x-\lambda_3} = \frac{a_1(x-\lambda_2)(x-\lambda_3)}{(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)} + \frac{a_2(x-\lambda_1)(x-\lambda_3)}{(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)} + \frac{a_3(x-\lambda_1)(x-\lambda_2)}{(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)}$$

$$= \frac{a_1(x-\lambda_2)(x-\lambda_3) + a_2(x-\lambda_1)(x-\lambda_3) + a_3(x-\lambda_1)(x-\lambda_2)}{(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)} = 0$$

Let $f(x) = a_1(x-\lambda_2)(x-\lambda_3) + a_2(x-\lambda_1)(x-\lambda_3) + a_3(x-\lambda_1)(x-\lambda_2)$ $\quad f:\mathbb{R}\to\mathbb{R}$.

and $g(x) = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ $\quad g:\mathbb{R}\to\mathbb{R}$

Such that $\frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \frac{a_3}{x-\lambda_3} = f(x)\cdot\frac{1}{g(x)}$ $\quad x \neq \lambda_1, \lambda_2, \lambda_3$.

A polynomial is just a sum of monomial functions. These are the product of a constant and the n-fold product of the identity function. The constant and identity functions are both continuous, so from the algebra of continuous functions as a polynomial is just the sum and product of continuous functions, all polynomial functions must be continuous. Hence f and g are continuous.
As $\frac{1}{g(x)}$ is continuous when $g(x)\neq 0$, the given expression $f(x)\cdot\frac{1}{g(x)}$ is continuous when $x \neq \lambda_1, \lambda_2, \lambda_3$ as it is the product of 2 continuous functions (algebra of cont. functions).

Sign Analysis at $f(\lambda_1)$, $f(\lambda_2)$ and $f(\lambda_3)$:

$f(\lambda_1) = a_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3) + 0 + 0 = a_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)$

Expression	Sign	Reason
a_1	$+$	$a_1 \in \mathbb{R}^+$
$(\lambda_1-\lambda_2)$	$-$	$\lambda_1 < \lambda_2$
[illegible]		[illegible]

$(\lambda_1-\lambda_2)$	$-$	$\lambda_1<\lambda_2$
$(\lambda_1-\lambda_3)$	$-$	$\lambda_1<\lambda_3$
$a_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)$	$+$	Follows from individual expression values

$$f(\lambda_2)=0+a_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)+0=a_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)$$

Expression	Sign	Reason
a_2	$+$	$a_2\in\mathbb{R}^+$
$(\lambda_2-\lambda_1)$	$+$	$\lambda_1<\lambda_2$
$(\lambda_2-\lambda_3)$	$-$	$\lambda_2<\lambda_3$
$a_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)$	$-$	Follows from individual expression values

$$f(\lambda_3)=0+0+a_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)=a_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$$

Expression	Sign	Reason
a_3	$+$	$a_3\in\mathbb{R}^+$
$(\lambda_3-\lambda_1)$	$+$	$\lambda_1<\lambda_3$
$(\lambda_3-\lambda_2)$	$+$	$\lambda_2<\lambda_3$
$a_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$	$+$	Follows from individual expression values

For the expression to be equal to 0, the numerator must be 0. Hence solutions are when $f(x)=0$ (in other words the roots of $f(x)$). As f is continuous across the whole domain, the intermediate value theorem applies such that as $f(\lambda_1)>0$ and $f(\lambda_2)<0$, an $f(a)=0$ $a\in(\lambda_1,\lambda_2)$ must exist. Similarly $f(\lambda_2)<0$ and $f(\lambda_3)>0$, so $f(b)=0$ $b\in(\lambda_2,\lambda_3)$ must exist. As previously shown, $\frac{f(x)}{g(x)}$ is continuous and non-zero when $x\neq\lambda_1,\lambda_2,\lambda_3$ [so continuous (λ_1,λ_2) and (λ_2,λ_3)]. So these solutions for f are the same solutions for $\frac{f(x)}{g(x)}$.

Therefore $\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}+\frac{a_3}{x-\lambda_3}=0$ has solutions in both intervals (λ_1,λ_2) and (λ_2,λ_3).

4) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 3x^4 - 26x^3 + 60x^2 - 11$

i) Stationary Points when $f'(x) = 0$ and differentiable at x.

$f'(x) = 12x^3 - 78x^2 + 120x$ (Differentiable $\forall x \in \mathbb{R}$ as f is a Polynomial).

$12x^3 - 78x^2 + 120x = 0$

$\Rightarrow 6x(2x^2 - 13x + 20) = 0$

$\Rightarrow 6x(2x-5)(x-4) = 0$

Hence the Stationary Points when $x=0, x=5/2, x=4$:

When $x = 0$, $f(x) = 3(0)^4 - 26(0)^3 + 60(0)^2 - 11 = -11$ so $(0,-11)$

When $x = 5/2$, $f(5/2) = 3(5/2)^4 - 26(5/2)^3 + 60(5/2)^2 - 11 = \frac{1199}{16}$ so $(5/2, \frac{1199}{16})$

when $x = 4$, $f(4) = 3(4)^4 - 26(4)^3 + 60(4)^2 - 11 = 53$ so $(4, 53)$

f is twice differentiable on $\mathbb{R} \Rightarrow$ second derivative test can determine the stationary point's nature:

$f''(x) = 36x^2 - 156x + 120$ (Differentiable $\forall x \in \mathbb{R}$ as f' is a Polynomial).

As $f''(0) = 36(0)^2 - 156(0) + 120 = 120 > 0$, the stationary Point $(0,-11)$ is a local Minimum.

As $f''(5/2) = 36(5/2)^2 - 156(5/2) + 120 = -45 < 0$, the stationary Point $(5/2, \frac{1199}{16})$ is a local Maximum.

As $f''(4) = 36(4)^2 - 156(4) + 120 = 72 > 0$, the stationary Point $(4, 53)$ is a local Minimum.

ii) $f'(x) = 12x^3 - 78x^2 + 120x = 6x(2x-5)(x-4)$

	$x<0$	$x=0$	$0<x<5/2$	$x=5/2$	$5/2<x<4$	$x=4$	$x>4$
$6x$	$-$	0	$+$	$+$	$+$	$+$	$+$
$2x-5$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	0	$+$
$6x(2x-5)(x-4)$	$-$	0	$+$	0	$-$	0	$+$

f is increasing when $f'(x) \geq 0$, therefore when $\{x : 0 \leq x \leq 5/2\} \cup \{x : x \geq 4\}$

f is decreasing when $f'(x) \leq 0$, therefore when $\{x : x \leq 0\} \cup \{x : 5/2 \leq x \leq 4\}$

iii) $f''(x) = 36x^2 - 156x + 120 = 12(3x^2 - 13x + 10) = 12(3x-10)(x-1)$

	$x<10/3$	$x=10/3$	$10/3<x<1$	$x=1$	$x>1$
12	$+$	$+$	$+$	$+$	$+$
$3x-10$	$-$	0	$+$	$+$	$+$

12	+	+	+	+	+
$3x-10$	−	0	+	+	+
$x-1$	−	−	−	0	+
$12(3x-10)(x-1)$	+	0	−	0	+

f is convex when $f''(x)>0$, therefore when $\{x: x<10/3\} \cup \{x: x>1\}$.
f is concave when $f''(x)<0$, therefore when $\{x: 10/3<x<1\}$.

iv) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 3x^4-26x^3+60x^2-11 = \lim_{x\to-\infty} x^4\left(3-\frac{26}{x}+\frac{60}{x^2}-\frac{11}{x^4}\right)=+\infty$ (as $a^4\geq 0 \ \forall a\in\mathbb{R}$)
$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 3x^4-26x^3+60x^2-11 = \lim_{x\to\infty} x^4\underbrace{\left(3-\frac{26}{x}+\frac{60}{x^2}-\frac{11}{x^4}\right)}_{g(x)}=+\infty$ (Similarly).
$\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = 3.$

v) By Part iv $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = +\infty$ hence f is unbounded above but bounded below.
If f is unbounded above, by definition $\sup f = \infty$ (and by consequence a global max cannot exist).

However the function is bounded below as the function is continuous on $\mathbb{R}$ and does not tend to $-\infty$.
The global minimum must therefore be the smallest local minimum, so $(0,-11)$.
Claim: $\inf f = -11$.
Proof: Let $A=\{f(x): x\in\mathbb{R}\}$. $a\in f(\mathbb{R})$
$a\geq -11$ for $a\in A$ (as global minimum) so -11 is a lower bound of f.
Now let $\varepsilon>0$, if we consider a potential lower bound $-11+\varepsilon$, we observe that $-11+\varepsilon > -11+\varepsilon/2 \in A$
Hence $-11+\varepsilon$ is not a lower bound for any $\varepsilon>0$, therefore -11 is the greatest lower bound of A, $\inf f = -11$.

5) a) $f:[0,4]\to\mathbb{R}$ $f(x) := x^2+2$
$A := \{f(x): x\in[1,2]\} \cup \{x: x\in[1,2]\}$
Claim: $\sup A = 6$
Proof: $f(1)=1^2+2=3$, $f(2)=2^2+2=6$
Hence $A=\{x: 3\leq x\leq 6\} \cup \{x: 1\leq x\leq 2\}$
$x\leq 6$ for $x\in A$ so 6 is an upper bound of A.
Now let $\varepsilon>0$, if we consider a potential upper bound $6-\varepsilon$, we observe that $6-\varepsilon<6-\varepsilon/2\in A$
Hence $6-\varepsilon$ is not an upper bound for any $\varepsilon>0$, therefore 6 is the lowest upper bound of A, $\sup A = 6$.

b) $B := \left\{\frac{4n^3+3n+1}{n^3} : n\in\mathbb{N}\right\} = \left\{4+\frac{3}{n^2}+\frac{1}{n^3} : n\in\mathbb{N}\right\}$

b) $B := \left\{ \frac{4n^3+3n+1}{n^3} : n \in \mathbb{N} \right\} = \{ 4 + \frac{3}{n^2} + \frac{1}{n^3} : n \in \mathbb{N} \}$

Claim: $\inf B = 4$.

Proof: Let $f: \mathbb{N} \to \mathbb{R}$ for $f(n) = 4 + \frac{3}{n^2} + \frac{1}{n^3}$

$f'(n) = -\frac{6}{n^3} - \frac{3}{n^4}$ (as f is differentiable $\forall\, n \in \mathbb{N}$).

$\frac{6}{n^3} > 0$ and $\frac{3}{n^4} > 0$ hence $f'(n) \leq 0$ $\forall n \in \mathbb{N}$, so f is a decreasing function.

$\lim_{n\to\infty} \left(4 + \frac{3}{n^2} + \frac{1}{n^3}\right) \overset{\text{algebra of limits}}{=} \lim_{n\to\infty} 4 + \lim_{n\to\infty} \frac{3}{n^2} + \lim_{n\to\infty} \frac{1}{n^3} = 4+0+0 = 4$ is a lower bound for B.

Now let $\varepsilon > 0$, if we consider a potential lower bound $4+\varepsilon$, we observe that $4+\varepsilon > 4 + \varepsilon/2 \in B$

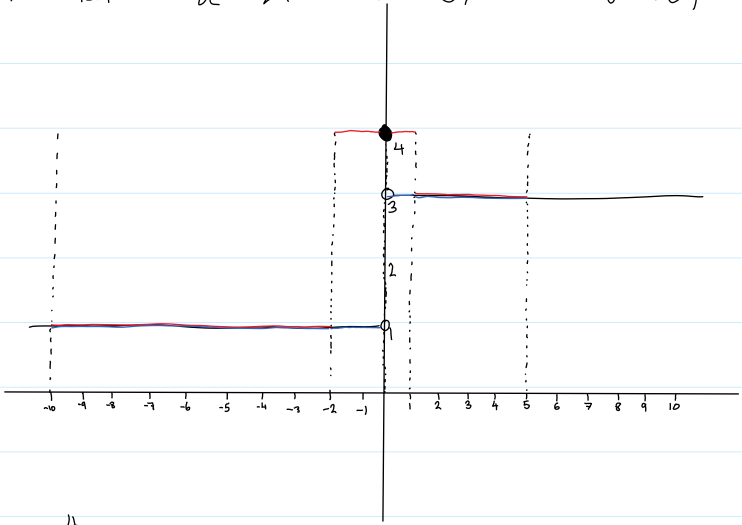
Hence $4+\varepsilon$ is not a lower bound for any $\varepsilon > 0$, therefore 4 is the greatest lower bound of B, $\inf B = 4$.

c) $P = \{-10, -2, 0, 1, 5\}$

$g: [-10, 5] \to \mathbb{R}$ $\quad g(x) := \begin{cases} \frac{x+2|x|}{|x|} & \text{if } x \neq 0; \\ 4 & \text{if } x = 0. \end{cases} = \begin{cases} 3 & \text{if } x > 0; \\ 4 & \text{if } x = 0; \\ 1 & \text{if } x < 0. \end{cases}$

$(x>0)$ $\frac{x+2|x|}{|x|} = \frac{x}{x} + 2 = 1+2 = 3,$ $\qquad (x<0)$ $\frac{x+2|x|}{|x|} = \frac{x}{-x} + 2 = -1+2 = 1,$

i)



$L(g,P) = \sum_{i=1}^{4} m_i (x_i - x_{i-1})$ $\qquad U(f,P) = \sum_{i=1}^{4} M_i (x_i - x_{i-1})$

Where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ $\qquad$ Where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$

i	1	2	3	4
x_i	-2	0	1	5
x_{i-1}	-10	-2	0	1
$x_i - x_{i-1}$	8	2	1	4
m_i	1	1	3	3
M_i	1	4	4	3

So $L(g,P) = 1(8) + 1(2) + 3(1) + 3(4) = 25$ $\qquad U(g,P) = 1(8) + 4(2) + 4(1) + 3(4) = 32$

ii) When 0 is in Partition, $L(g,Q)$ will always be 25.
If there are Partition elements closer to 0, $U(g,Q)$ Will approach 25.
For example, take $Q=\{-10,-0.0001,0,0.0001,5\}$. Then:

i	1	2	3	4
x_i	-0.0001	0	0.0001	5
x_{i-1}	-10	-0.0001	0	0.0001
x_i-x_{i-1}	9.9999	0.0001	0.0001	4.9999
m_i	1	1	3	3
M_i	1	4	4	3

So $L(g,Q)=L(g,P)=\sum_{i=1}^{4} m_i(x_i-x_{i-1}) = 1(9.9999)+1(0.0001)+3(0.0001)+3(4.9999)=25$
and $U(g,Q)=U(g,P)=\sum_{i=1}^{4} M_i(x_i-x_{i-1}) = 1(9.9999)+4(0.0001)+4(0.0001)+3(4.9999)=25.0004$.
$U(g,Q)-L(g,Q)=25.0004-25=0.0004<0.001$.

Contribution to:

$L(Q,g)$ $1\times\varepsilon+3\times\varepsilon=4\varepsilon$

$U(Q,g)$ $4\times\varepsilon+4\times\varepsilon=8\varepsilon$

$U(Q,g)-L(Q,g)=4\varepsilon<\frac{1}{1000}$

$\varepsilon<\frac{1}{4000}$

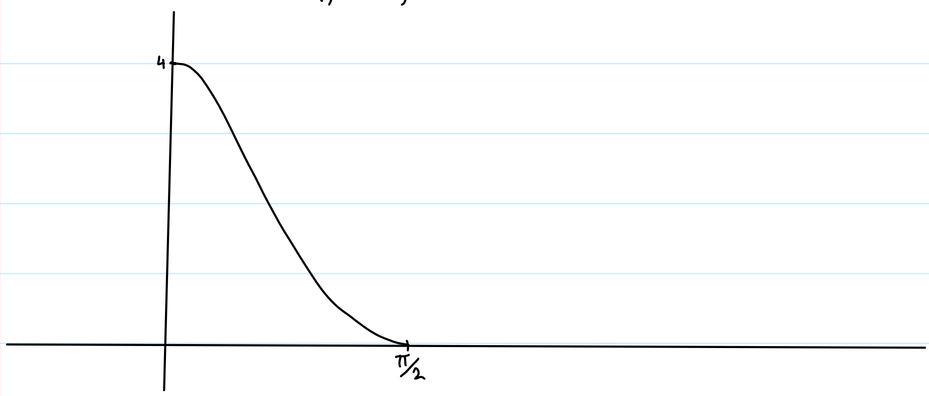
Eg choose $\varepsilon=\frac{1}{5000}$

Problem Sheet 4 (Submission)

21 November 2022 14:56

② a) For $f:[a,b]\to\mathbb{R}$ where f is bounded, Riemann's criterion states that f is integrable then if for each $\varepsilon>0$, there exists partition P such that $U(f,P)-L(f,P)<\varepsilon$.

b) i) $f:[0,\pi/2]\to\mathbb{R}$, $f(x)=4\cos^2(x)$.



For each $n\in\mathbb{N}$ using the partition P_n of $[0,\frac{\pi}{2}]$ into n subintervals of equal width we have:

$M_i=\sup\{f(x):x\in[x_{i-1},x_i]\}=f(x_{i-1})$, $m_i=\inf\{f(x):x\in[x_{i-1},x_i]\}=f(x_i)$ as decreasing on interval.

$U(f,P_n)=\sum_{i=1}^{n}M_i(x_i-x_{i-1})=\sum_{i=1}^{n}4\cos(x_{i-1})^2(x_i-x_{i-1})=\frac{4\pi}{2n}[\cos^2(x_0)+\cos^2(x_1)+\dots+\cos^2(x_{n-1})]$

$L(f,P_n)=\sum_{i=1}^{n}m_i(x_i-x_{i-1})=\sum_{i=1}^{n}4\cos(x_i)^2\underline{(x_i-x_{i-1})}=\frac{4\pi}{2n}[\cos^2(x_1)+\cos^2(x_2)+\dots+\cos^2(x_n)]$

As equal width partitions $=\frac{\pi}{2n}$

$$\begin{aligned}U(f,P_n)-L(f,P_n)&=\frac{2\pi}{n}[\cos^2(x_0)+\cos^2(x_1)+\dots+\cos^2(x_{n-1})]-\frac{2\pi}{n}[\cos^2(x_1)+\cos^2(x_2)+\dots+\cos^2(x_n)]\\&=\frac{2\pi}{n}[\cos^2(x_0)-\cos^2(x_n)]\\&=\frac{2\pi}{n}[\cos^2(0)-\cos^2(\tfrac{\pi}{2})]=\frac{2\pi}{n}[1-0]\\&=\frac{2\pi}{n}\quad (*)\end{aligned}$$

Let $\varepsilon>0$, we want $\frac{2\pi}{n}<\varepsilon$

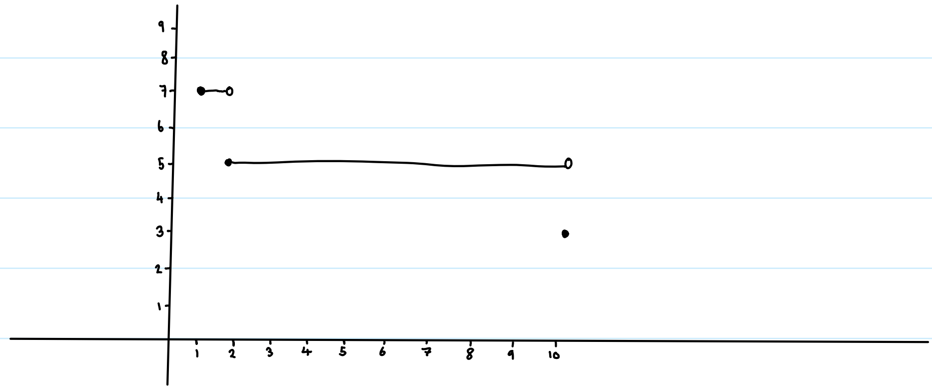
Now choose $N\in\mathbb{N}$ such that $N>\frac{2\pi}{\varepsilon}$ eg $\lceil\frac{2\pi}{\varepsilon}\rceil+1$.

Then by $(*)$ $U(f,P_n)-L(f,P_n)=\frac{2\pi}{N}<\varepsilon$.

Hence f is integrable by Riemann's criterion.

ii) $g:[1,10]\to\mathbb{R}$, $g(x)=\begin{cases}7, & x\in[1,2);\\5, & x\in[2,10);\\3, & x=10.\end{cases}$

ii) $g:[1,10]\to\mathbb{R},\ g(x) = \begin{cases} 7, & x\in[1,2); \\ 5, & x\in[2,10); \\ 3, & x=10. \end{cases}$



Let $n\in\mathbb{N}$ and P_n be an equal width partition in the interval $[1,10]$.

$M_i = \sup\{f(x): x\in[x_{i-1},x_i]\} = g(x_{i-1})$, $m_i = \inf\{f(x): x\in[x_{i-1},x_i]\} = g(x_i)$ as decreasing on interval.

$U(g,P_n) = \sum_{i=1}^{n} M_i(x_i-x_{i-1}) = \frac{9}{n}\left[g(x_0)+g(x_1)+\dots g(x_{n-2})+g(x_{n-1})\right]$

$L(g,P_n) = \sum_{i=1}^{n} m_i\underbrace{(x_i-x_{i-1})}_{} = \frac{9}{n}\left[g(x_1)+g(x_2)+\dots g(x_{n-1})+g(x_n)\right]$

As it is an equal width partition, $=\frac{10-1}{n}=\frac{9}{n}$

Therefore, $U(g,P_n)-L(g,P_n) = \frac{9}{n}\left[g(x_0)+g(x_1)+\dots+g(x_{n-2})+g(x_{n-1})\right] - \frac{9}{n}\left[g(x_1)+g(x_2)\dots+g(x_n)\right]$

$= \frac{9}{n}\left[g(x_0)-g(x_n)\right] = \frac{9}{n}\left[g(1)-g(10)\right]$

$= \frac{9}{n}[7-3] = \frac{9}{n}\cdot 4$

$= \frac{36}{n}$ (*)

Let $\varepsilon>0$. We want $\frac{36}{n}<\varepsilon$.

Choose $N\in\mathbb{N}$, such that $N>\frac{36}{\varepsilon}$. Eg $N=\lceil\frac{36}{\varepsilon}\rceil+1$.

Then by (*), Observe that $U(g,P_N)-L(g,P_n) = \frac{36}{N}<\varepsilon$

Hence g is integrable by Riemann's criterion.

c) (For f) $f:[0,\frac{\pi}{2}]\to\mathbb{R},\ f(x)=4\cos^2(x)$.

Theorem: If $f:[a,b]\to\mathbb{R}$ is continuous $\Rightarrow f$ is integrable.

cos and polynomials $(4u^2)$ are continuous on $\mathbb{R}$ so by composition of continuous functions f is continuous therefore by the theorem f is integrable.

(For g): $g:[1,10]\to\mathbb{R},\ g(x) = \begin{cases} 7, & x\in[1,2); \\ 5, & x\in[2,10); \\ 3, & x=10. \end{cases}$

Theorem: If $g:[a,b]\to\mathbb{R}$ is monotonic, then g is integrable.

Need to prove g is monotonic decreasing $[g(x)\geq g(y)\ \forall\ 1\leq x\leq y\leq 10]$:

Choose $y \in [1,10]$. Then $g(y) = \begin{cases} 7, & y \in [1,2); \\ 5, & y \in [2,10); \\ 3, & y = 10. \end{cases}$ As $x \leq y$:

If $g(y)=3$, $g(x) \in \{3,5,7\}$

If $g(y)=5$, $g(x) \in \{5,7\}$

If $g(y)=7$, $g(x) \in \{7\}$.

In all cases $g(x) \geq g(y)$ hence g is monotonic (decreasing) so therefore integrable.

(5) a) The First Fundamental Theorem of Calculus states that suppose $f:[a,b] \to \mathbb{R}$ bounded and integrable. If f is continuous at $c \in [a,b]$ then F is differentiable at c with $F'(c) = f(c)$ where $F(x) := \int_a^x f \quad \forall x \in [a,b]$.

b) i) $F:[2,4] \to \mathbb{R}$, $F(x) := \int_2^x \frac{1}{\log(t)}\, dt$, $x \in [2,4]$

Let $f:[2,4] \to \mathbb{R}$ be given by $f(x) := \frac{1}{\log(x)} \quad \forall x \in [2,4]$.

By composition of continuous functions f is continuous.

↳ (log followed by x^{-1} which are both continuous in the interval $[2,4]$).

so by 1^{st} FTC F is differentiable with $F'(x) = f(x) = \frac{1}{\log(x)}$ for all $x \in [2,4]$.

ii) $G:[-1,1] \to \mathbb{R}$, $G(x) := \int_{-5}^{5x^4+3x^2+1} e^{-t^2}\, dt$, $x \in [-1,1]$

Observe $G(x) = \int_{-5}^{-1} e^{-t^2} dt + \int_{-1}^{5x^4+3x^2+1} e^{-t^2}\, dt$ (extension/restriction)

Let $A(x) := \int_{-5}^{-1} e^{-t^2} dt + \int_{-1}^{x} e^{-t^2}\, dt$

Let $a:[-1,1] \to \mathbb{R}$ be given by $a(x) := e^{-x^2} \quad \forall x \in [-1,1]$. (because $\int_5^{-1} e^{-t^2} dt$ is a constant).

By composition of continuous functions a is continuous.

↳ ($-x^2$ followed by exp which are both continuous in the interval $[-1,1]$).

so by 1^{st} FTC A is differentiable with $A'(x) = a(x) = e^{-x^2} \quad \forall x \in [-1,1]$.

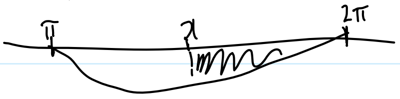
Observe $G(x) = A(5x^4+3x^2+1)$

So if $g(x) := 5x^4+3x^2+1$, then $G(x) = (A \circ g)(x)$

So G is differentiable with chain rule with $G'(x) = A'(g(x))\, g'(x) = e^{-(5x^4+3x^2+1)^2} \cdot (20x^3+6x)$

$G'(x) = \frac{20x^3+6x}{e^{(5x^4+3x^2+1)^2}}$

iii) $H:[\pi,2\pi]\to\mathbb{R}$, $H(x):=\int_x^{2\pi}\frac{\sin t}{t}\,dt$, $\quad x\in[\pi,2\pi]$



Observe $H(x)=\int_\pi^{2\pi}\frac{\sin t}{t}\,dt-\int_\pi^{x}\frac{\sin t}{t}\,dt=\int_\pi^{2\pi}\frac{\sin t}{t}\,dt+\int_\pi^{x}\frac{-\sin t}{t}\,dt$

Let $h:[\pi,2\pi]\to\mathbb{R}$ given by $h(x)=\frac{-\sin x}{x}$ $\quad\forall x\in[\pi,2\pi]$

↳ Because $\int_\pi^{2\pi}\frac{\sin t}{t}\,dt$ is a constant it can be disregarded.

By composition of continuous functions h is continuous

↳ ($\sin$ continuous on $\mathbb{R}$ and division of t continuous $\mathbb{R}/\{0\}$)

so by 1^{st} FTC H is differentiable with $H'(x)=h(x)=\frac{-\sin x}{x}$ $\quad\forall x\in[\pi,2\pi]$

3 a) $I = \int \frac{6x+1}{x^2+3x+5}\,dx = \int \frac{3(2x+3)-8}{x^2+3x+5}\,dx$

$= 3\int \frac{2x+3}{x^2+3x+5}\,dx - \int \frac{8}{x^2+3x+5}\,dx$

$= 3\ln|x^2+3x+5| + C - \int \frac{8}{(x+3/2)^2 - 9/4 + 5}\,dx$

$= 3\ln|x^2+3x+5| + C - 8\int \frac{1}{(x+3/2)^2 + 11/4}\,dx$

Let $u = x + 3/2$, so $du = dx$ evaluate $\int \frac{1}{(x+3/2)^2+11/4}\,dx$:

$\int \frac{1}{u^2+11/4}\,du = \frac{1}{(\sqrt{11}/2)}\arctan\left(u/(\sqrt{11}/2)\right) + c$ $\left[\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)\right]$

$\Rightarrow \frac{2}{\sqrt{11}}\arctan\left(\frac{2u}{\sqrt{11}}\right)$

$\Rightarrow \frac{2\sqrt{11}}{11}\arctan\left(\frac{2\sqrt{11}u}{11}\right) = \frac{2\sqrt{11}}{11}\arctan\left(\frac{2\sqrt{11}x+3\sqrt{11}}{11}\right)$

So $I = 3\ln|x^2+3x+5| - \frac{16\sqrt{11}}{11}\arctan\left(\frac{2\sqrt{11}x+3\sqrt{11}}{11}\right)$

b) $\int \sec^4(3x)\tan^4(3x)\,dx = \int \tan^4(3x)\sec^2(3x)\cdot\sec^2(3x)\,dx$

Let $u = \tan 3x$ so $du = 3\sec^2(3x)\,dx$

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \tan^2\theta + 1 = \sec^2\theta \Rightarrow \cancel{\sec^2\theta = \tan^2\theta - 1}$

Substitute in:

$\int u^4(u^2+1)\cdot\frac{1}{3}\,du$

$= \int \frac{u^6}{3} + \frac{u^4}{3}\,du$

$= \frac{u^7}{21} + \frac{u^5}{15} + C$

$= \frac{1}{21}\tan^7 3x + \frac{1}{15}\tan^5 3x + C$

c) Undefined at $x = 3/2$, which is below the lower band 2 hence continuous on $[2,\infty)$.

$\lim_{a\to\infty}\int_2^a \frac{1}{\sqrt{2x-3}}\,dx = \lim_{a\to\infty}\int_2^a (2x-3)^{-1/2}\,dx = \lim_{a\to\infty}\left[(2x-3)^{1/2}\right]_2^a$

$\frac{d}{dx}(2x-3)^{1/2} = \frac{1}{2}(2x-3)^{-1/2}(2) = (2x-3)^{-1/2}$

$= \lim_{a\to\infty}\left[\sqrt{2a-3} - \sqrt{2(2)-3}\right] = \lim_{a\to\infty}\left[\sqrt{2a-3} - 1\right] = \infty$

Therefore the integral is divergent.

d) $\int_5^6 \frac{1}{\sqrt{x^2-25}}\,dx$

Let $x = 5\sec\theta$ So $dx = 5\sec\theta\tan\theta$

$\theta = \sec^{-1}(x/5)$

$\lim_{t\to 0^+}\left[\int_{\operatorname{arcsec}(5+t)}^{\operatorname{arcsec} 6} \frac{5\sec\theta\tan\theta}{\sqrt{25\sec^2\theta - 25}}\right.$

$\Rightarrow \lim_{t\to 0^+}\left[\int_{\operatorname{arcsec}(5+t)}^{\operatorname{arcsec} 6} \frac{\cancel{5}\sec\theta\tan\theta}{\cancel{5}\sqrt{\sec^2\theta - 1}}\,d\theta\right]$

$\tan^2\theta = \sec^2\theta - 1$ so:

$\Rightarrow \lim_{t\to 0^+}\left[\int^{\operatorname{arcsec} 6} \frac{\sec\theta\tan\theta}{\ldots}\,d\theta\right]$

$\tan^2\theta = \sec^2\theta - 1$ so:

$\Rightarrow \lim_{t\to 0^+}\left[\int_{\operatorname{arcsec}(5+t)}^{\operatorname{arcsec}6} \frac{\sec\theta\cancel{\tan\theta}}{\cancel{\tan\theta}}\, d\theta\right]$

$\Rightarrow \lim_{t\to 0^+}\left[\int_{\operatorname{arcsec}(5+t)}^{\operatorname{arcsec}6} \sec\theta\, d\theta\right]$

$\Rightarrow \lim_{t\to 0^+}\left[\log|\sec\theta + \tan\theta|\right]_{\operatorname{arcsec}(5+t)}^{\operatorname{arcsec}6}$

$\Rightarrow \lim_{t\to 0^+}\left[\log|\sec\theta + \sqrt{1-\sec^2\theta}|\right]_{\operatorname{arcsec}(5+t)}^{\operatorname{arcsec}6}$

$\Rightarrow \lim_{t\to 0^+}\left[\log|6 + \underbrace{\sqrt{1-6^2}}_{\text{Does not exist}}| - \log|5+t+\sqrt{1+(5+t)^2}|\right]$

Therefore as the limit does not exist, the improper integral is diverging.

⑤ a) $f:[0,4]\to[0,1]$ where $f(x) := \begin{cases} x, & \text{if } 0\le x\le 1; \\ 1, & \text{if } 1<x\le 4. \end{cases}$ and $y:[0,4)\to\mathbb{R}$

$\frac{dy}{dx} = f(x)$

$\int \frac{dy}{\cancel{dx}}\cancel{dx} = \int f(x)\,dx$ As $f(x)$ is continuous $\Rightarrow$ integrable.

$\Rightarrow y = \int f(x)$

$\Rightarrow y = \begin{cases} \frac{1}{2}x^2 + C_1 & \text{if } 0\le x\le 1. \\ x + C_2 & \text{if } 1<x\le 4. \end{cases}$

$y(0) = 1 \Rightarrow \frac{1}{2}(0)^2 + C_1 = 1 \Rightarrow C_1 = 1$

$y = \begin{cases} \frac{1}{2}x^2 + 1 & \text{if } 0\le x\le 1. \\ x + C_2 & \text{if } 1<x\le 4. \end{cases}$

Need to prove y is differentiable.

$\forall x_0 \in [0,1]$ $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{\frac{1}{2}x^2+1-\frac{1}{2}x_0^2-1}{x-x_0} = \lim_{x\to x_0}\frac{x^2-x_0^2}{2(x-x_0)}$

$= \lim_{x\to x_0}\frac{(x+x_0)\cancel{(x-x_0)}}{2\cancel{(x-x_0)}} = \frac{2x_0}{2} = x_0$ Thus differentiable at x_0.

$\forall x_1 \in (1,4]$ $\lim_{x\to x_1}\frac{f(x)-f(x_1)}{x-x_1} = \lim_{x\to x_1}\frac{x+C_2-(x_1+C_2)}{x-x_1} = \frac{\cancel{x-x_1}}{\cancel{x-x_1}} = 1$ Thus differentiable at x_1.

Therefore y is indeed differentiable.

Differentiability $\Rightarrow$ continuity of $y \Rightarrow y(1)$ continuous $\Rightarrow \lim_{\delta\to 0} y(1+\delta) \Rightarrow \lim_{\delta\to 0^+} y(1+\delta) = \lim_{\delta\to 0^-} y(1+\delta)$

$\Rightarrow 1 + C_2 = \frac{1}{2}(1)^2 + 1 \Rightarrow C_2 = \frac{1}{2}+1-1 = \frac{1}{2}$

Therefore $y = \begin{cases} \frac{1}{2}x^2 + 1 & \text{if } 0\le x\le 1. \\ x + \frac{1}{2} & \text{if } 1<x\le 4. \end{cases}$

b) $y:[0,\infty)\to\mathbb{R}$

$yy' = \log x$ $y(0) = 2.$

$y\frac{dy}{dx} = \log x$

$y = \log x \frac{dx}{dy}$

$\int y\, dy = \int \log x \frac{dx}{\cancel{dy}}\cancel{dy}$

$\int y\,dy = \int \log x\,dx$

Evaluate $\int \underset{v'}{1} \cdot \underset{u}{\log x}\,dx$

$u = \log x \qquad v = x$

$u' = 1/x \qquad v' = 1$

Using Integration by Parts: $\int \log x\,dx = x\log x - \int \frac{1}{x} \cdot x\,dx = x\log x - x + C$

$\int y\,dy = \int \log x\,dx$

$\Rightarrow \frac{1}{2}y^2 = x\log x - x + C$

$\Rightarrow y^2 = 2x\log x - 2x + C$ ⊛

$\Rightarrow y = \begin{cases} \sqrt{2x\log x - 2x + C} & \text{if } x \geq 0 \\ -\sqrt{2x\log x - 2x + C} & \text{if } x < 0 \end{cases}$

However $y: [0,\infty)$ hence $y = \sqrt{2x\log x - 2x + C}$

$y(0) = 2 \Rightarrow 2 = \sqrt{2(0)\log 0 - 2(0) + C}$

Which is undefined as $\log 0$ is undefined. So must instead consider the differential equation.

$yy' = \log x$ ① sub into ⊛

$y^2 = 2xyy' - 2x + C$

$y(0) = 2 \Rightarrow (2)^2 = 2 \cdot 0 \cdot 2 \cdot y' - 2 \cdot 0 + C$

$\Rightarrow C = 4$

Therefore $y = \sqrt{2x\log x - 2x + 4}$