1Mech — Mechanics

Mechanics exercises 3 (weeks 5 and 6)
Mechanics solutions 3

This sheet's assessed question is number 5.

1. A particle of mass m is attracted towards the origin by a force of the form mc/r^2, where r, θ are polar coordinates and $c > 0$ is a constant. The particle is initially located a distance a from the origin and moving with speed $\sqrt{c/a}$ perpendicular to the radial direction.

   (a) Show that $u = 1/r$ satisfies
   $$\frac{d^2u}{d\theta^2} + u = \frac{c}{h^2},$$
   where $h = r^2\dot{\theta}$ is constant.

   (b) Find expressions for the initial conditions for the $u(\theta)$ equation, and the value of h.

   (c) Hence solve to find the particle path. What shape is it?

Solution. (a) We know that $u = 1/r$ satisfies
$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2},$$
where $F(r)$ gives the outward pointing force. Here $F(r) = -\frac{mc}{r^2}$ and hence $F(1/u) = -mcu^2$. Thus we find
$$\begin{aligned}\frac{d^2u}{d\theta^2} + u &= \frac{mcu^2}{mh^2u^2},\\ &= \frac{c}{h^2},\end{aligned}$$
where $h = r^2\dot{\theta}$ is constant.

(b) We have the initial conditions
$$\begin{array}{lll} r = a, & \text{specifying location at} & t = 0,\\ \dot{r} = 0, \quad r\dot{\theta} = \sqrt{c/a} & \text{specifying radial and transverse velocity components at} & t = 0,\end{array}$$
and we choose $\theta = 0$ at $t = 0$ also. Rewriting these in terms of $u(\theta)$ we find

- $u = 1/r = 1/a$ at $\theta = 0$,
- $\frac{du}{d\theta} = -\frac{\dot{r}}{h} = 0$ at $\theta = 0$.

Now, since $h = r^2\dot{\theta}$ is a constant and
$$\begin{aligned}h = r^2\dot{\theta} &= r \cdot r\dot{\theta},\\ &= a \cdot \sqrt{c/a},\\ &= \sqrt{ca},\end{aligned}$$
initially, this holds for all time.

(c) We must solve

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u &= \frac{c}{h^2}, \\ &= \frac{c}{ca}, \\ &= \frac{1}{a}. \end{aligned}$$

This is a second order linear ODE with constant coefficients. We first solve the homogeneous problem

$$\frac{d^2u_c}{d\theta^2} + u_c = 0,$$

by forming the characteristic equation

$$\begin{aligned} \lambda^2 + 1 &= 0, \\ \implies \lambda &= \pm i, \\ \implies u_c &= A\cos\theta + B\sin\theta, \end{aligned}$$

where A and B are constants.
We then look for a particular integral which will satisfy the right hand side, of the form $u_p = D$, where D is a constant. Hence $D = 1/a$, and thus the general solution is

$$u = A\cos\theta + B\sin\theta + \frac{1}{a}.$$

We now use the initial conditions to find A and B. Since $u(0) = 1/a$ we have

$$\begin{aligned} \frac{1}{a} &= A + \frac{1}{a}, \\ \implies A &= 0. \end{aligned}$$

Then

$$\frac{du}{d\theta} = -A\sin\theta + B\cos\theta,$$

and $du/d\theta = 0$ at $\theta = 0$, so

$$0 = B.$$

Hence the solution is

$$u = \frac{1}{a},$$

which gives

$$r = a,$$

i.e. the particle moves in a circle with constant radius.

Feedback: *This gives some practice solving a central forces question. You don't need to derive the framework every time, but you should be clear what you're using when. The procedure for solving these problems is fairly standard; once you have the solution you should think about what shape the particle path is, and thus what you can find out from it.* ◀

2. Suppose that a particle of mass m is subject to a central force acting towards the origin of magnitude

$$\mu m\left(\frac{1}{r^2}+\frac{3a}{4r^3}\right),$$

with μ, a constant, and r the distance between the particle and the origin. The particle is initially at $r=a$ with initial velocity $\sqrt{\mu/a}$ in a direction perpendicular to the line joining the origin to the particle.

(a) Show that $u=1/r$ satisfies

$$\frac{d^2u}{d\theta^2}+\frac{u}{4} = \frac{1}{a},$$

subject to $u=1/a$, $du/d\theta=0$ at $\theta=0$.

(b) Solve the system to find the particle path. What is the distance of closest approach to the origin?

Solution. For an outward pointing central force $F(r)$, the particle path satisfies

$$\frac{d^2u}{d\theta^2}+u = -\frac{F(1/u)}{mh^2u^2},$$

where $u=1/r$ and $h=r^2\dot{\theta}$ is a constant. Here

$$\begin{aligned} F(r) &= -\mu m\left(\frac{1}{r^2}+\frac{3a}{4r^3}\right), \\ \implies F(1/u) &= -\mu m\left(u^2+\frac{3au^3}{4}\right). \end{aligned}$$

Hence

$$\begin{aligned} \frac{d^2u}{d\theta^2}+u &= \frac{\mu m\left(u^2+\frac{3au^3}{4}\right)}{mh^2u^2}, \\ &= \frac{\mu\left(1+\frac{3au}{4}\right)}{h^2}. \end{aligned}$$

We also have initial conditions

$$\begin{array}{llll} r=a, & \theta=0, & \text{specifying location at} & t=0, \\ \dot{r}=0, & r\dot{\theta}=\sqrt{\mu a} & \text{specifying radial and transverse velocity components at} & t=0, \end{array}$$

and we choose $\theta=0$ initially. Rewriting these in terms of $u(\theta)$ we find

- $u=1/r=1/a$ at $\theta=0$,
- $\frac{du}{d\theta}=-\frac{\dot{r}}{h}=0$ at $\theta=0$.

Now, since $h=r^2\dot{\theta}$ is a constant and

$$\begin{aligned} h=r^2\dot{\theta} &= r\cdot r\dot{\theta}, \\ &= a\cdot\sqrt{\mu/a}, \\ &= \sqrt{\mu a}, \end{aligned}$$

initially, this holds for all time.

We must solve

$$
\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= \frac{\mu\left(1 + \frac{3au}{4}\right)}{h^2}, \\
&= \frac{\frac{\mu\left(1 + \frac{3au}{4}\right)}{h^2}}{\mu a}, \\
&= \frac{1}{a} + \frac{3u}{4}, \\
\implies \frac{d^2u}{d\theta^2} + \frac{u}{4} &= \frac{1}{a}.
\end{aligned}
$$

This is a second order linear ODE with constant coefficients. We first solve the homogeneous problem

$$
\frac{d^2u_c}{d\theta^2} + \frac{u_c}{4} = 0,
$$

by forming the characteristic equation

$$
\begin{aligned}
\lambda^2 + \frac{1}{4} &= 0, \\
\implies \lambda &= \pm\frac{i}{2}, \\
\implies u_c &= A\cos\frac{\theta}{2} + B\sin\frac{\theta}{2},
\end{aligned}
$$

where A and B are constants.

We then look for a particular integral which will satisfy the right hand side, of the form $u_p = D$, where D is a constant. Hence $D/4 = 1/a$, and thus the general solution is

$$
u = A\cos\frac{\theta}{2} + B\sin\frac{\theta}{2} + \frac{4}{a}.
$$

We now use the initial conditions to find A and B. Since $u(0) = 1/a$ we have

$$
\begin{aligned}
\frac{1}{a} &= A + \frac{4}{a}, \\
\implies A &= -\frac{3}{a}.
\end{aligned}
$$

Then

$$
\frac{du}{d\theta} = -\frac{A}{2}\sin\frac{\theta}{2} + \frac{B}{2}\cos\frac{\theta}{2},
$$

and $du/d\theta = 0$ at $\theta = 0$, so

$$
0 = \frac{B}{2}.
$$

Hence the solution is

$$
u = \frac{-3\cos\frac{\theta}{2} + 4}{a},
$$

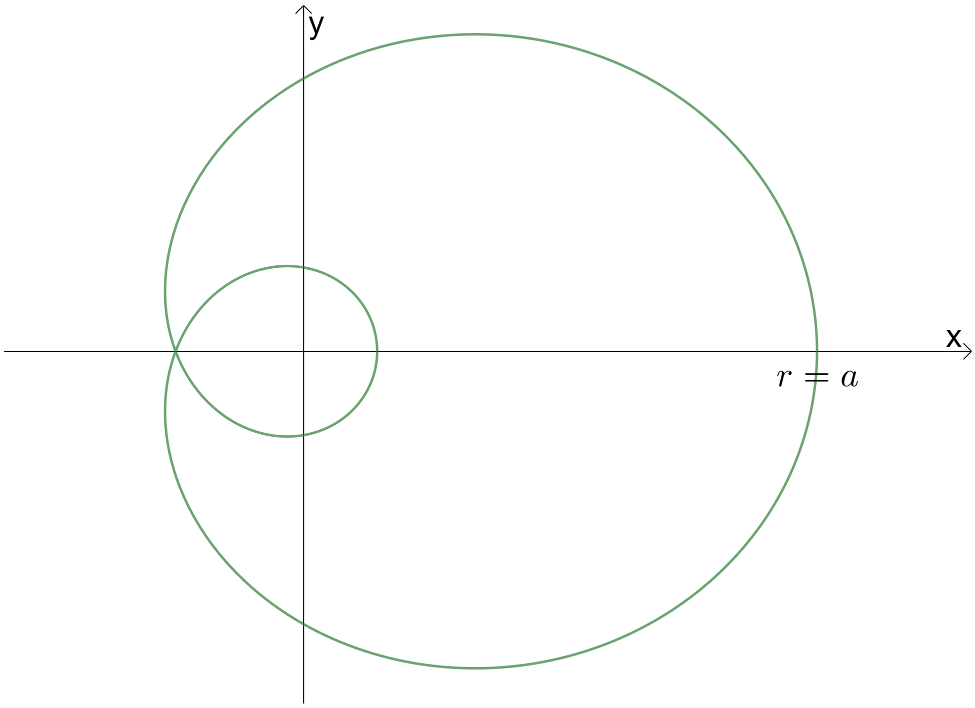

Figure 1: Particle path for question 2, showing a closed orbit, with an extra "loop" around the origin.

giving

$$r = \frac{a}{4 - 3\cos\frac{\theta}{2}}.$$

This gives a closed orbit with an extra "loop" around the origin (see Figure 1 and `https://www.geogebra.org/m/jvczvgud`). The point of closest approach is when r is smallest, or equivalently u is biggest. This is when $\cos\dfrac{\theta}{2} = -1$ and hence

$$\begin{aligned} r &= \frac{a}{4+3}, \\ &= \frac{a}{7} \end{aligned}$$

Feedback: *This gives another central forces problem to solve, so the above feedback holds again. Here, the shape of the particle path leads to natural questions such as what are the closest/furthest distances between the particle and the centre of the force? Pay particular attention to how you present your argument, it should include everything you need to solve the problem, and "flow" nicely. Try reading what you have written out loud - does it make sense?*

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3. For the following question you may find it advantageous to convert into $u(\theta)$ where $u = 1/r$. Note the question does not ask about stability of these paths.

   (a) If a particle under a central force $F(r)$ moves along the spiral $r = e^{-k\theta}$ where k is a constant, show that $F(r) = -C/r^3$ where C is a constant.

(b) If a particle under a central force $F(r)$ moves along a circular arc terminating at $r = 0$, show that $r = a\cos\theta$ gives the particle path, where a gives the diameter of the circle, and hence show that $F(r) = -D/r^5$ where D is a constant. [Hint: Sketch the semi circular part of the particle path and draw a line joining $r = 0$ to any point on the particle path to find a right angled triangle.]

Solution. (a) If a particle moves under a central force F such that $r = e^{-k\theta}$, then $u = 1/r = e^{k\theta}$ must satisfy

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2},$$

and hence

$$\begin{aligned} F(1/u) &= -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right), \\ &= -mh^2e^{2k\theta}\left(k^2e^{k\theta} + e^{k\theta}\right), \\ &= -mh^2\left(k^2+1\right)e^{3k\theta}, \\ &= -mh^2\left(k^2+1\right)u^3. \end{aligned}$$

Therefore

$$F(r) = -\frac{mh^2\left(k^2+1\right)}{r^3}.$$

(b) A circular arc going through $r = 0$is represented by $r = a\cos\theta$ since the angle inscribed in a semi circle is a right angle. Hence

$$\begin{aligned} u = \frac{1}{r} &= \frac{1}{a\cos\theta}, \\ &= \frac{\sec\theta}{a}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{du}{d\theta} &= \frac{\sec\theta\tan\theta}{a}, \\ \frac{d^2u}{d\theta^2} &= \frac{\sec\theta\tan^2\theta + \sec^3\theta}{a}, \end{aligned}$$

and therefore

$$\begin{aligned} F(1/u) &= -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right), \\ &= -mh^2\frac{\sec^2\theta}{a^2}\left(\frac{\sec\theta\tan^2\theta + \sec^3\theta}{a} + \frac{\sec\theta}{a}\right), \\ &= -mh^2\frac{\sec^3\theta}{a^3}\left(\tan^2\theta + \sec^2\theta + 1\right), \\ &= -mh^2\frac{\sec^3\theta}{a^3}\left(2\sec^2\theta\right), \\ &= -2mh^2\frac{\sec^5\theta}{a^3}, \\ &= -2mh^2u^5a^2. \end{aligned}$$

Hence

$$F(r) = -\frac{2mh^2a^2.}{r^5}.$$

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Feedback: *These questions go the other way around - given the solution and particle path, what was the force applied. Part (b) is more technically challenging and require some geometry/trig knowledge.*

4. A comet moves under the inverse square law attraction of the Sun. The force is given by

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r = -\frac{GMm}{r^3}\mathbf{r},$$

where $\mathbf{r}$ is the position vector of the comet relative to the Sun, r gives the distance between the Sun and the comet such that $\mathbf{r} = r\mathbf{e}_r$, G is the gravitational constant and M is the mass of the sun. Starting from Newton's second law show that

(a) the moment of momentum of the comet with respect to the Sun is constant.

(b) the orbit of the comet lies in a plane containing the Sun.

(c) $r^2\dot{\theta}$ is constant.

Solution. (a) If

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r = -\frac{GMm}{r^3}\mathbf{r},$$

then

$$\begin{aligned} \mathbf{F} &= m\ddot{\mathbf{r}}, \\ \implies m\ddot{\mathbf{r}} &= -\frac{GMm}{r^3}\mathbf{r}. \end{aligned}$$

We now take the cross product with $\mathbf{r}$ to find the moment of momentum:

$$\begin{aligned} \mathbf{r} \times m\ddot{\mathbf{r}} &= \mathbf{r} \times \left(-\frac{GMm}{r^3}\mathbf{r}\right), \\ &= 0, \end{aligned}$$

since $\mathbf{r} \times \mathbf{r} = 0$. Hence

$$\begin{aligned} 0 = \mathbf{r} \times m\ddot{\mathbf{r}} &= \mathbf{r} \times m\ddot{\mathbf{r}} + \dot{\mathbf{r}} \times (m\dot{\mathbf{r}}), \\ &= \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}}), \end{aligned}$$

as $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$, and therefore

$$\begin{aligned} \mathbf{r} \times m\dot{\mathbf{r}} &= \text{constant}, \\ &= m\mathbf{h} \end{aligned}$$

where $\mathbf{h}$, a constant, is the moment of momentum.

(b) Since $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector, then $\mathbf{r} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{h} \cdot \mathbf{r} = 0$, as $\mathbf{r} \times \dot{\mathbf{r}}$ is perpendicular to $\mathbf{r}$. This gives the equation of a plane.

(c) Now

$$\begin{aligned}\mathbf{h} &= \mathbf{r} \times \dot{\mathbf{r}}, \\ &= r\mathbf{e}_r \times \left(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta\right), \\ &= (r\mathbf{e}_r \times \dot{r}\mathbf{e}_r) + \left(r\mathbf{e}_r \times r\dot{\theta}\mathbf{e}_\theta\right), \\ &= r^2\dot{\theta}\left(\mathbf{e}_r \times \mathbf{e}_\theta\right), \\ &= r^2\dot{\theta}\mathbf{k},\end{aligned}$$

where $\mathbf{k}$ is a vector coming out of the plane, perpendicular to both $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Hence $r^2\dot{\theta}$ is conserved.

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Feedback: *This is mainly bookwork, but good practice to think in particular about what dot and cross products must be zero due to vectors being perpendicular or parallel.*

5. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

Let (r, θ) be the standard polar coordinates in 2D and let the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ be defined in the usual way. A particle of mass m is attracted towards the origin by a force of magnitude

$$m\omega^2\left(\frac{a^4}{r^3} + r\right),$$

where ω and a are positive constants. The particle is initially at distance a from the origin and moving with initial velocity $a\omega\mathbf{e}_\theta$.

(a) Write down Newton's Second Law for the particle in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Hence, show that the quantity $h = r^2\dot{\theta}$ is constant throughout the particle's motion, and find the value of h.

(b) Starting from Newton's Second Law, derive the following equation satisfied by $u = 1/r$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = \frac{1}{a^4u^3}. \tag{1}$$

Determine the initial conditions for u and $\frac{\mathrm{d}u}{\mathrm{d}\theta}$ at $\theta = 0$.

(c) Verify that the function

$$u(\theta) = \frac{\sqrt{\theta^2 + c^4}}{ac},$$

where c is any constant, solves (1) and satisfies the initial condition for $\frac{\mathrm{d}u}{\mathrm{d}\theta}$. Use the initial condition for u to determine the value of c.

(d) Show that

$$\dot{\theta} = \omega(\theta^2 + 1). \tag{2}$$

Verify that the function

$$\theta(t) = \tan(\omega t)$$

solves (2) and satisfies the initial condition for θ at $t = 0$. How much time does it take the particle to reach the origin? Justify your answer.

Solution. (a) Newton's Second Law:

$$m\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta = -m\omega^2\left(\frac{a^4}{r^3} + r\right)\mathbf{e}_r. \tag{3}$$

Equating the $\mathbf{e}_\theta$ component gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0,$$

and hence $h = r^2\dot{\theta}$ is a constant. Since it is a constant, its value can be found by evaluating $r^2\dot{\theta}$ at $t = 0$:

$$h = [r \cdot r\dot{\theta}]_{t=0} = a \cdot a\omega = a^2\omega, \tag{4}$$

where we have used the fact that $r\dot{\theta}$ is the $\mathbf{e}_\theta$ component of velocity.

(b) "Starting from Newton's Second Law" means you are not allowed to simply put the force F into the formula $\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{F(1/u)}{mh^2u^2}$; you must derive it. Writing $\dot{\theta} = h/r^2$, the $\mathbf{e}_r$ component of (3) becomes

$$m\left(\ddot{r} - h^2/r^3\right) = -m\omega^2\left(\frac{a^4}{r^3} + r\right), \tag{5}$$

or equivalently,

$$m\left(\ddot{r} - h^2u^3\right) = -m\omega^2\left(a^4u^3 + \frac{1}{u}\right). \tag{6}$$

By the chain rule,

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{h}{r^2} = -h\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right) = -h\frac{\mathrm{d}u}{\mathrm{d}\theta}, \tag{7}$$

and

$$\ddot{r} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}\theta}\dot{\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)\dot{\theta} = -h\dot{\theta}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -\frac{h^2}{r^2}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}. \tag{8}$$

Putting (8) into (5) or (6) and diving both sides by $-mh^2u^2$ yields

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\omega^2}{h^2u^2}\left(a^4u^3 + \frac{1}{u}\right). \tag{9}$$

And since $h = a^2\omega$, we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = u + \frac{1}{a^4 u^3},$$

and the desired equation follows. For the initial conditions, we have $\theta = 0$ at $t = 0$ by choice of coordinate system, and so

$$[u]_{\theta=0} = [u]_{t=0} = \frac{1}{[r]_{t=0}} = \frac{1}{a},$$

and

$$\left[\frac{\mathrm{d}u}{\mathrm{d}\theta}\right]_{\theta=0} = -\frac{[\dot{r}]_{\theta=0}}{h} = -\frac{[\dot{r}]_{t=0}}{h} = 0,$$

where we have used (7) and the fact that $\dot{r}$ is the $\mathbf{e}_r$ component of velocity.

(c) If $u = \frac{\sqrt{\theta^2+c^4}}{ac}$, then

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \frac{\theta}{ac\sqrt{\theta^2 + c^4}}, \tag{10}$$

and

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} = \frac{1}{ac\sqrt{\theta^2 + c^4}} - \frac{\theta^2}{ac(\theta^2 + c^4)^{3/2}} = \frac{\theta^2 + c^4 - \theta^2}{ac(\theta^2 + c^4)^{3/2}} = \frac{c^3}{a(\theta^2 + c^4)^{3/2}}. \tag{11}$$

Meanwhile,

$$\frac{1}{a^4 u^3} = \frac{1}{a^4}\frac{(ac)^3}{\sqrt{\theta^2 + c^4}^3} = \frac{c^3}{a(\theta^2 + c^4)^{3/2}}, \tag{12}$$

and so the two sides of (1) are equal, as required. Moreover, (10) implies

$$\left[\frac{\mathrm{d}u}{\mathrm{d}\theta}\right]_{\theta=0} = 0,$$

which is exactly the required initial condition for $\frac{\mathrm{d}u}{\mathrm{d}\theta}$. To fix c, we use the requirement that $[u]_{\theta=0} = 1/a$, which yields

$$\frac{\sqrt{0 + c^4}}{ac} = 1/a \quad \Longrightarrow \quad \frac{c^2}{ac} = \frac{1}{a} \quad \Longrightarrow \quad c = 1.$$

(d) From $h = r^2\dot{\theta} = a^2\omega$ and $c = 1$, we find

$$\dot{\theta} = h/r^2 = hu^2 = a^2\omega u^2 = a^2\omega\frac{\theta^2 + 1}{a^2} = \omega(\theta^2 + 1), \tag{13}$$

as required. If $\theta(t) = \tan(\omega t)$, then $\dot{\theta} = \omega\sec^2(\omega t)$, and $\omega(\theta^2 + 1) = \omega\sec^2(\omega t)$ using a trigonometric identity. So, (2) is satisfied. Moreover, $\theta(0) = \tan(0) = 0$, which is the required initial condition for θ. As $\omega t \to \pi/2$, i.e. $t \to \pi/(2\omega)$, we have $\theta \to \infty$ and so $u = \frac{\sqrt{\theta^2+1}}{a} \to \infty$, which means $r = 1/u \to 0$. Thus, the particle reaches the origin in time $\pi/(2\omega)$.

There is a trap that one might easily fall into for part (d). If you put $\theta = \tan(\omega t)$ into $u = \sqrt{\theta^2 + 1}/a$, and use the trig identity "$\tan^2 X + 1 = \sec^2 X$", you obtain $u = \sec(\omega t)/a$ (taking the positive square root), and hence $r = 1/u = a\cos(\omega t)$. So it seems that r becomes 0 periodically: whenever $t = \pi/(2\omega), 3\pi/(2\omega), 5\pi/(2\omega), \ldots$. Mathematically, this all seemingly checks out. But physically it makes no sense. The force is attractive: it draws the particle towards the origin; and its magnitude becomes infinite at $r = 0$. So, once the particle reaches the origin, it certainly does not leave the origin again. (There is a name in physics for a place where the force becomes infinite: it is called a "singularity".) How do we reconcile this physical reality with the mathematical fact that the particle path seems to be $r = a\cos(\omega t)$? The answer is, the actual particle path is just a part of $r = a\cos(\omega t)$, not all of it. This is true in general: the function $r(\theta) = 1/u(\theta)$ found by solving the central force equation generally only provides the curve that *contains* the particle path; the particle may trace out all of the curve, or, in the case of this problem, some of the curve. How do we know whether the particle will trace out all of the curve or not? We need some physical insight, like the fact that the particle cannot possibly escape an attractive singularity. Or, if you insist on a mathematical argument, consider the fact that in this problem, $\theta = \tan(\omega t)$ is discontinuous at $t = \pi/(2\omega)$. A particle path must be continuous. So the particle cannot move beyond $t = \pi/(2\omega)$.

Feedback: *This is the style of a typical exam question. Parts (a) and (b) are mostly bookwork, with a small amount of "typical application". Parts (c) and (d) crank up the difficulty. Gaining full marks on a question of this type requires not only high familiarity with bookwork and typical applications, but the ability to think outside the box. The most common mistake was forgetting to put a minus sign in the force in part (a), which messes up a lot of what follows. The force is towards the origin, which means it is in the* $-\mathbf{e_r}$ *direction.* ◀