

Problem sheet 5

Question 16. Let $X \sim \text{Poi}_\lambda$ for $\lambda > 0$. Show that

$$\mathbb{E} \left[\frac{1}{1+X} \right] = \frac{1}{\lambda} (1 - e^{-\lambda}).$$

Question 17. (SUM)

- (a) Let X be a continuous random variable with distribution function

$$F_X(t) = \begin{cases} 0 & t < -1, \\ c_1(t+1)^2 & t \in [-1, 1], \\ 1 & t > 1. \end{cases}$$

Find c_1 , a density f_X , $\mathbb{E}[X]$, $\mathbb{E}[X^2]$, $\mathbb{V}\text{ar}(X)$ and a median of X .

- (b) Let Y be a continuous random variable with density

$$f_Y(x) = \begin{cases} c_2 x^{-4}, & x \in [1, 2], \\ 0, & \text{otherwise.} \end{cases}$$

Find c_2 , F_Y , $\mathbb{E}[Y]$, $\mathbb{V}\text{ar}(Y)$ and a median of Y .

- (c) Find $\mathbb{E}[3X - 7Y]$. Under the assumption that X, Y are independent, find $\mathbb{V}\text{ar}(3X - 7Y)$, $\mathbb{E}[XY]$ and $\mathbb{V}\text{ar}(XY)$.

Question 18.

- (a) Suppose that X is a non-negative random variable with $\mathbb{P}(X \geq a) \geq b$, where $a > 0$. Deduce that $\mathbb{E}[X] \geq ab$.
- (b) Bound $\mathbb{P}(|X - \mathbb{E}[X]| \geq 5)$ using Chebyshev's inequality when
- (i) X follows the Poisson distribution, with $\lambda = 5$.
 - (ii) X follows the normal distribution with $\mu = 5, \sigma^2 = 6.25$.

In case (i) and (ii) also find $\mathbb{P}(|X - \mathbb{E}[X]| \geq 5)$ exactly. Do you think that Chebyshev's inequality is effective in these cases?

Question 19. Let $n \in \mathbb{N}$ be an even number.

- (a) Using Chebyshev's inequality for a binomial random variable $X \sim \text{bin}_{n,1/2}$, show that $\mathbb{P}(|X - n/2| < \sqrt{n}) \geq 3/4$.
- (b) From (a) deduce that $\binom{n}{n/2} \geq \frac{2^n}{4\sqrt{n}}$.

Hint: Recall that in Question 4(a) on Problem Sheet 1 you showed that $\binom{n}{n/2}$ is maximal among the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$.

Question 20. A fair 6-sided dice has the numbers $\{1, 2, \dots, 6\}$ appearing on the faces. The dice is rolled 30000 times. Let X denote the number of rolls on which a multiple of 3 appears.

- (a) What distribution does X follow?
- (b) Use the De Moivre–Laplace theorem to estimate the probability that X lies between 9900 and 10150.

t	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999
3.1	0.99903	0.99906	0.9991	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.9994	0.99942	0.99944	0.99946	0.99948	0.9995
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.9996	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.9997	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.9998	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.9999	0.9999	0.9999	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998

Table 1: Table for $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx, t \geq 0$.