University of Birmingham
School of Mathematics
Vectors, Geometry and Linear Algebra
VGLA

Problem Sheet 3

Model Solutions

Remember that there are practise questions under the materials section for each week.

$\boxed{\text{SUM}}$ **Q1.** (i) Suppose that $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{nn}(\mathbb{F})$ are invertible. Assume that $\mathbf{A}^T = \mathbf{A}^{-1}$ and $\mathbf{B}^T = \mathbf{B}^{-1}$. Show that $(\mathbf{AB})^T = (\mathbf{AB})^{-1}$.

(ii) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}.$$

(a) Use Gaussian Elimination to find the inverse of $\mathbf{A}$ is it exists.

(b) Directly from the definition of the determinant as

$$\sum_{\sigma \in S_3} (-1)^{N(\sigma)} \prod_{i=1}^{3} a_{\sigma(i),i},$$

calculate $\det(\mathbf{A})$.

(iii) Calculate the determinant of

$$\mathbf{A} = \begin{pmatrix} \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & 0 \\ \lambda & \lambda & \lambda & \lambda & 0 & \lambda \\ \lambda & \lambda & \lambda & 0 & \lambda & \lambda \\ \lambda & \lambda & 0 & \lambda & \lambda & \lambda \\ \lambda & 0 & \lambda & \lambda & \lambda & \lambda \end{pmatrix}$$

where $\lambda \in \mathbb{R}$. (You may use whichever method you like.)

(iv) Suppose that $n \geq 2$ and $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$. Assume that $\mathbf{A}$ has at least $n^2 - n + 2$ entries which are equal to μ for some $\mu \in \mathbb{R}$. Show that $\det \mathbf{A} = 0$.

(v) Show that there exists an invertible matrix $\mathbf{B} \in \mathcal{M}_{nn}(\mathbb{R})$ with $n^2 - n + 1$ entries which are equal to μ for some fixed $\mu \in \mathbb{R}$.

Solution. (i) We have

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \mathbf{B}^T\mathbf{A}^T = (\mathbf{AB})^T.$$

(ii) (a) The augmented matrix is

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

We now implement the Gaussian elimination algorithm.

$$\boxed{r_2 \mapsto r_2 - r_1} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\boxed{r_3 \mapsto r_2 - 2r_1} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\boxed{r_2 \circlearrowleft, r_3} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \\ 0 & -2 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\boxed{r_2 \mapsto -r_2/3} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2/3 & 0 & -1/3 \\ 0 & -2 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\boxed{r_3 \mapsto r_3 + 2r_2} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2/3 & 0 & -1/3 \\ 0 & 0 & 3 & 1/3 & 1 & -2/3 \end{array}\right)$$

$$\boxed{r_3 \mapsto r_3/3} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2/3 & 0 & -1/3 \\ 0 & 0 & 1 & 1/9 & 1/3 & -2/9 \end{array}\right)$$

$$\boxed{r_2 \mapsto r_2 - r_3} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5/9 & -1/3 & -1/9 \\ 0 & 0 & 1 & 1/9 & 1/3 & -2/9 \end{array}\right)$$

$$\boxed{r_1 \mapsto r_1 - 2r_3} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 7/9 & -2/3 & 4/9 \\ 0 & 1 & 0 & 5/9 & -1/3 & -1/9 \\ 0 & 0 & 1 & 1/9 & 1/3 & -2/9 \end{array}\right)$$

$$\boxed{r_1 \mapsto r_1 - 2r_2} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/9 & 0 & 6/9 \\ 0 & 1 & 0 & 5/9 & -1/3 & -1/9 \\ 0 & 0 & 1 & 1/9 & 1/3 & -2/9 \end{array}\right)$$

Hence

$$\mathbf{A}^{-1} = \frac{1}{9}\begin{pmatrix} -3 & 0 & 6 \\ 5 & -3 & -1 \\ 1 & 3 & -2 \end{pmatrix}.$$

(b) The elements of S_3 are, using bottom row notation, given by

$$\sigma_1 = [1,2,3],\ \sigma_2 = [1,3,2],\ \sigma_3 = [2,1,3],\ \sigma_4 = [2,3,1],\ \sigma_5 = [3,1,2] \text{ and } \sigma_6 = [3,2,1],$$

with number of inversions given by

$$N(\sigma_1) = 0,\ N(\sigma_2) = 1,\ N(\sigma_3) = 1,\ N(\sigma_4) = 2,\ N(\sigma_5) = 2 \text{ and } N(\sigma_6) = 3,$$

We obtain

$$\begin{aligned} \det(\mathbf{A}) &= (-1)^{N(\sigma_1)}(1\cdot 0\cdot 1) \\ &\quad +(-1)^{N(\sigma_2)}(1\cdot 1\cdot 3) \\ &\quad +(-1)^{N(\sigma_3)}(1\cdot 2\cdot 1) \\ &\quad +(-1)^{N(\sigma_4)}(1\cdot 1\cdot 2) \\ &\quad +(-1)^{N(\sigma_5)}(2\cdot 2\cdot 3) \\ &\quad +(-1)^{N(\sigma_6)}(2\cdot 0\cdot 2) \\ &= (-1)^0(1\cdot 0\cdot 1) \\ &\quad +(-1)^1(1\cdot 1\cdot 3) \\ &\quad +(-1)^1(1\cdot 2\cdot 1) \\ &\quad +(-1)^2(1\cdot 1\cdot 2) \\ &\quad +(-1)^2(2\cdot 2\cdot 3) \\ &\quad +(-1)^3(2\cdot 0\cdot 2) \\ &= 9. \end{aligned}$$

(iii) Notice that if $\lambda = 0$, then $\det \mathbf{A} = 0$. It's not important to do this for the solution below, but if you divided by λ in your solution, you better notice this and assume that $\lambda \neq 0$ for the rest of your calculation. Hopefully you did some row or column operations. Subtract row one from all the other rows to obtain

$$\det \mathbf{A} = \det \begin{pmatrix} \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ 0 & 0 & 0 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now swapping rows 2 and 6 and then rows 3 and 5 is two swaps and so doesn't change the determinant. Hence

$$\det \mathbf{A} = \det \begin{pmatrix} \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ 0 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda \end{pmatrix}.$$

This is an upper triangular matric and so the determinant is the product of the diagonal entries. So

$$\det \mathbf{A} = -\lambda^6$$

for all $\lambda \in \mathbb{R}$.

(iv) Any such matrix will have two columns consisting of just of λs. So there are two equal columns and hence the determinant is zero.

(v) Consider the $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$ with $a_{2,n} = a_{3,n-1} = a_{4,n-2} = \ldots a_{n,2} = 0$ and all other entries equal to λ. Then, just as in part (vi) we get $\det \mathbf{A} = -\lambda^n$ (notice that when n is odd, you have to make an odd number of row swaps to get to the upper triangular form and so this introduces a negative into the determinant of $\mathbf{A}$).

□

SUM **Q2**. Group the following matrices into sets which have the same determinant. In this λ is a real number which is the same in each matrix. Explain your answer.

- $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & \lambda \\ 5 & 19 & 6 & 3 \\ 6 & 2 & 3 & 5 \\ 1 & -2 & 4 & 6 \end{pmatrix}$.
- $\mathbf{B} = \begin{pmatrix} 1 & -2 & 4 & 6 \\ 6 & 2 & 3 & 5 \\ 5 & 19 & 6 & 3 \\ 1 & 2 & 3 & \lambda \end{pmatrix}$.
- $\mathbf{C} = \begin{pmatrix} -1 & 2 & -4 & -6 \\ -6 & -2 & -3 & -5 \\ -5 & -19 & -6 & -3 \\ -1 & -2 & -3 & -\lambda \end{pmatrix}$.
- $\mathbf{D} = \begin{pmatrix} 6 & 21 & 9 & \lambda+3 \\ 5 & 19 & 6 & 3 \\ 6 & 2 & 3 & 5 \\ 1 & -2 & 4 & 6 \end{pmatrix}$.

- $\mathbf{E} = \begin{pmatrix} 6 & 21 & \lambda+3 & 9 \\ 5 & 19 & 3 & 6 \\ 6 & 2 & 5 & 3 \\ 1 & -2 & 6 & 4 \end{pmatrix}$.
- $\mathbf{F} = \begin{pmatrix} 6 & 21 & \lambda+3 & 9 \\ 6 & 2 & 5 & 3 \\ 5 & 19 & 3 & 6 \\ 1 & -2 & 6 & 4 \end{pmatrix}$.
- $\mathbf{G} = \begin{pmatrix} 6 & -21 & \lambda+3 & 9 \\ 6 & -2 & 5 & 3 \\ 5 & -19 & 3 & 6 \\ 1 & 2 & 6 & 4 \end{pmatrix}$.

Solution. If λ is such that $\det(\mathbf{A}) \neq 0$, we have two sets:

$$\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{F}\}$$

$$\{\mathbf{E}, \mathbf{G}\}.$$

If $\det(\mathbf{A}) = 0$, then all of the matrices listed have determinant 0.

To see this note that $\det(\mathbf{A}) = \det(\mathbf{B})$ as $\mathbf{B}$ is obtained from $\mathbf{A}$ by swapping rows 1 and 4 and rows 2 and 3. This multiplies the determinant by $(-1)^2$. Matrix $\mathbf{C}$ is $-\mathbf{B}$ and so, as $\mathbf{B}$ is a 4×4 matrix, $\det(\mathbf{C}) = (-1)^4 \det(\mathbf{B}) = \det(\mathbf{B})$. Matrix D is obtained from $\mathbf{A}$ by adding row 2 to row 1 and so $\det(\mathbf{A}) = \det(\mathbf{D})$. So far we have that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ all have the same determinant. Matrix $\mathbf{E}$ is obtained from $\mathbf{D}$ by swapping columns 3 and 4 and so $\det(\mathbf{E}) = -\det(\mathbf{D})$. We get $\mathbf{F}$ from $\mathbf{E}$ by swapping rows 2 and 3 and so $\det(\mathbf{F}) = \det(\mathbf{D})$. Finally $\mathbf{G}$ is obtained from $\mathbf{F}$ by multiplying column 2 by -1 and so $\det G = -\det(\mathbf{F}) = -\det(\mathbf{A})$. If $\det(\mathbf{A}) \neq 0$, this gives that the elements in $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{F}\}$ and have determinant $\det(\mathbf{A})$ and those in $\{\mathbf{E}, \mathbf{G}\}$ have determinant $-\det(\mathbf{A})$. Furthermore, if $\det(\mathbf{A}) = 0$, then all the matrices have determinant 0. $\square$