## LH: Homework sheet 1 – Solutions – Linear Algebra

**1.** Consider the Gram-Schmidt procedure applied to $B$: define $\mathbf{u}_1 := \mathbf{v}_1$ and compute

$$\mathbf{u}_{k+1} = \mathbf{v}_{k+1} - \sum_{j=1}^{k} \frac{\langle \mathbf{v}_{k+1}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \mathbf{u}_j, \qquad k = 1, 2, \ldots, n-1. \tag{1}$$

**(a)** Let us re-write the general form of the Gram-Schmidt algorithm by placing $\mathbf{v}_{k+1}$ on the left:

$$\mathbf{v}_{k+1} = \mathbf{u}_{k+1} + \sum_{j=1}^{k} \frac{\langle \mathbf{v}_{k+1}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \mathbf{u}_j =: \sum_{j=1}^{k+1} a_j \mathbf{u}_j,$$

which is a linear combination of $\mathbf{u}_j$. Hence $\mathbf{v}_{k+1} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_{k+1}\}$. **[2]**

**(b)** Consider the general statement

$$P(k) : U_k := \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} =: V_k. \tag{2}$$

With this notation, we observe that we trivially have $U_k \subset U_{k+1}$ and $V_k \subset V_{k+1}$.
We show that $P(k)$ holds for $k = 1, \ldots, n$. We proceed as follows.

- **Base case**: $P(k = 1)$ holds trivially, as $\operatorname{span}\{\mathbf{u}_1\} = \operatorname{span}\{\mathbf{v}_1\}$ since $\mathbf{u}_1 = \mathbf{v}_1$. **[2]**
- **Inductive hypothesis**: Assume the general case $k = m$ holds: $P(k = m)$ is true:

$$U_m := \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} =: V_m.$$

**[2]**

- **Inductive step:** we show that $P(k = m) \implies P(k = m+1)$. More precisely, we need to show that

$$U_{m+1} := \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{m+1}\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+1}\} =: V_{m+1}.$$

This a set equality, so we need to show a double inclusion: $U_{m+1} \subseteq V_{m+1}$ and $V_{m+1} \subseteq U_{m+1}$.
We first show $V_{m+1} \subseteq U_{m+1}$. Let $\mathbf{v} \in V_{m+1}$ be arbitrary. Then

$$\mathbf{v} = \sum_{j=1}^{m+1} a_j \mathbf{v}_j = \sum_{j=1}^{m} a_j \mathbf{v}_j + a_{m+1} \mathbf{v}_{m+1}.$$

The first term on the right is in $V_m$ and $V_m = U_m \subset U_{m+1}$. The second term is also in $U_{m+1}$ by part **(a)**. Hence, $\mathbf{v} \in U_{m+1}$ and therefore $V_{m+1} \subseteq U_{m+1}$. **[3]**
We now show $U_{m+1} \subseteq V_{m+1}$. Let $\mathbf{u} \in U_{m+1}$ be arbitrary. Then

$$\mathbf{u} = \sum_{j=1}^{m+1} b_j \mathbf{u}_j = \sum_{j=1}^{m} b_j \mathbf{u}_j + b_{m+1} \mathbf{u}_{m+1}.$$

The first term on the right is in $U_m$ and $U_m = V_m \subset V_{m+1}$. Observe now that, by its definition in the Gram-Schmidt algorithm,

$$\mathbf{u}_{m+1} = \mathbf{v}_{m+1} - \sum_{j=1}^{m} \frac{\langle \mathbf{v}_{k+1}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \mathbf{u}_j,$$

which is a linear combination of elements in $V_{m+1}$ (first term) and $U_m$ (second term). But $U_m = V_m \subset V_{m+1}$. Hence, $b_{m+1}\mathbf{u}_{m+1} \in V_{m+1}$ and therefore $\mathbf{u} \in V_{m+1}$, i.e., $U_{m+1} \subseteq V_{m+1}$.
Therefore $U_{m+1} = V_{m+1}$ and the inductive step is proved. **[3]**
By the Principle of Mathematical Induction, the general statement (2) holds.

**(c)** Taking the inner product of the general Gram-Schmidt step (1) with $\mathbf{u}_{k+1}$ we find

$$\langle \mathbf{u}_{k+1}, \mathbf{u}_{k+1} \rangle = \left\langle \mathbf{u}_{k+1}, \mathbf{v}_{k+1} - \sum_{j=1}^{k} \frac{\langle \mathbf{v}_{k+1}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \mathbf{u}_j \right\rangle = \langle \mathbf{u}_{k+1}, \mathbf{v}_{k+1} \rangle - \sum_{j=1}^{k} \frac{\langle \mathbf{v}_{k+1}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \overbrace{\langle \mathbf{u}_j, \mathbf{u}_{k+1} \rangle}^{=0} = \langle \mathbf{v}_{k+1}, \mathbf{u}_{k+1} \rangle ,$$

since by construction the vector $\mathbf{u}_{k+1}$ is orthogonal to the previous ones, i.e., to $\mathbf{u}_1, \dots, \mathbf{u}_k$. **[2]**
Finally, using the above identity together with the Cauchy-Schwarz inequality we find

$$\|\mathbf{u}_{k+1}\|^2 = \langle \mathbf{u}_{k+1}, \mathbf{u}_{k+1} \rangle = \langle \mathbf{v}_{k+1}, \mathbf{u}_{k+1} \rangle \leq \|\mathbf{v}_{k+1}\| \|\mathbf{u}_{k+1}\| \Longrightarrow \|\mathbf{u}_{k+1}\| \leq \|\mathbf{v}_{k+1}\| .$$ **[2]**

**2. (a)** First, we check that $S$ is an orthogonal set in $U$ with respect to the given inner product. We have

$$\langle 1, x \rangle = \int_{-1}^{1} \frac{1 \cdot x}{\sqrt{1-x^2}} \mathrm{d}x = 0,$$

since the integrand is an odd function. [Note that when the integrand is not an odd function, the evaluation of the integral is via substitution: a suitable one is $x = \sin\theta$.] **[2]**
To find another polynomial in $U$ that is orthogonal to $S$, we use the Gram-Schmidt process. First, we choose a basis for $U$: the simplest is the power basis $B = \{1, x, x^2\}$. Note that this basis is not orthogonal since 1 and $x^2$ are not orthogonal. Note that we only need to compute a third basis element orthogonal to 1 and $x$ to replace $x^2$. **[2]**
Let us denote the resulting orthogonal basis by $B_3 := \{q_1, q_2, q_3\}$, with $q_1(x) = 1, q_2(x) = x$. We find

$$q_3(x) = x^2 - \frac{\langle x^2, q_1(x) \rangle}{\langle q_1(x), q_1(x) \rangle} q_1(x) - \frac{\langle x^2, q_2(x) \rangle}{\langle q_2(x), q_2(x) \rangle} q_2(x).$$

We first evaluate the numerators, then the denominators, if necessary.

$$\left\langle x^2, q_1(x) \right\rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} \mathrm{d}x = \frac{\pi}{2}, \quad \left\langle x^2, q_2(x) \right\rangle = \int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}} \mathrm{d}x = 0$$

$$\langle q_1(x), q_1(x) \rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \pi.$$ **[6]**

We find

$$q_3(x) = x^2 - \frac{\pi/2}{\pi} \cdot 1 - 0 = x^2 - \frac{1}{2},$$

which happens to be a monic polynomial. **[2]**

**(b)** This part can be solved by using orthogonal projections (see recording of Lecture 8: Orthogonal projections: an application, see also Q12, Q13 on Examples sheet 2). We start with the statement: let $U < V$ and let $\mathbf{v} \in V$. Then by Theorem 6.7, there exists a unique decomposition $\mathbf{v} = \mathbf{v}_U^{\perp} + \mathbf{v}_U^{\parallel}$, where $\mathbf{v}_U^{\parallel} \in U$ is the orthogonal projection of $\mathbf{v}$ onto $U$ (see Definition 6.7). By Q12 or Lecture 8,

$$\left\| \mathbf{v} - \mathbf{v}_U^{\parallel} \right\| \leq \|\mathbf{v} - \mathbf{u}\| , \qquad \text{for all } \mathbf{u} \in U.$$

We identify the following quantities:

- $V = \mathcal{P}_3([-1, 1])$

- $U = \mathcal{P}_2([-1,1])$
- $\mathbf{v} \longleftrightarrow q(x) = x^3$
- $\mathbf{v}_U^{\|} \longleftrightarrow p^*$
- $\mathbf{u} \in U \longleftrightarrow p \in U$

Since $p^*$ was identified as the orthogonal projection of $q$ onto $U$, which has an orthogonal basis computed in part **(a)**, we can use the formula for $\mathbf{v}_U^{\|}$ given in Definition 6.7 using the elements of $B_3$. We find

$$p^* = \sum_{i=1}^{3} \frac{\langle q, q_i \rangle}{\langle q_i, q_i \rangle} q_i.$$

The above description justifies the approach in the computation of $p^*$. **[4]**

We first evaluate the numerators:

$$\langle q, q_1 \rangle = \int_{-1}^{1} \frac{x^3 \cdot 1}{\sqrt{1-x^2}} \mathrm{d}x = 0, \quad \langle q, q_2 \rangle = \int_{-1}^{1} \frac{x^3 \cdot x}{\sqrt{1-x^2}} \mathrm{d}x = \frac{3\pi}{8}, \quad \langle q, q_3 \rangle = \int_{-1}^{1} \frac{x^3 \left(x^2 - \frac{1}{2}\right)}{\sqrt{1-x^2}} \mathrm{d}x = 0.$$

**[6]**

Hence, there is only one non-zero term in the sum. Since $\langle q_2, q_2 \rangle = \langle x^2, q_1 \rangle = \pi/2$, we find

$$p^*(x) = \frac{3\pi/8}{\pi/2} x = \frac{3}{4} x.$$

**[2]**