4 Continuous random variables and distributions

In this section we begin to study continuous random variables. In many respects these are similar to discrete random variables, but they allow us to model quantities which are not confined to a discrete set [1] such as the position of a dart on a dartboard, or the lifetime of a mobile phone.

4.1 Continuous random variables

Definition 4.1 (Densities)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is called a *(probability) density function* if

(i) $f(x) \geq 0$ for all $x \in \mathbb{R}$, and

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$.

Definition 4.2 (Continuous random variables)**.** A function $X : \Omega \to \mathbb{R}$ is called a *continuous random variable* if there is a *(probability) density function* f_X with

$$\mathbb{P}\big(X \in [a,b]\big) = \int_a^b f_X(x)dx, \quad -\infty \leq a < b \leq \infty.$$

Remark 4.3. The density function of a continuous random variable plays a similar role to the mass function of a discrete random variable (see Definition 3.8). However the value $f_X(x)$ is not itself a probability – only integrals of f_X give probabilities. In particular, it is possible that $f_X(x) > 1$.

Definition 4.4 (Distribution function)**.** The *(cumulative) distribution function* of a continuous random variable X with density function f_X is the map $F_X : \mathbb{R} \to [0,1]$ defined for all $t \in \mathbb{R}$ by

$$F_X(t) := \mathbb{P}\left(X \leq t\right) = \int_{-\infty}^t f_X(x)dx.$$

Remark 4.5. This is almost identical to the distribution function of a discrete random variable (see Definition 3.8) and all properties from Proposition 3.14 remain true for such distribution functions. Note however that if $f_X(x) > 0$ for all $x \in [a,b]$ with $a < b$ then F_X is *strictly increasing* on $[a,b]$.

Example 4.6. Let $f(x) = 3 \cdot \sqrt{2x}/8$ on $[0,2]$ and $f(x) = 0$ for $x \notin [0,2]$. Clearly $f(x) \geq 0$ for all $x \in \mathbb{R}$ and for $t \in [0,2]$, we have

$$\int_{-\infty}^t f(x)dx = \frac{3\sqrt{2}}{8}\int_0^t \sqrt{x}dx = \frac{\sqrt{2}}{4}\left[x^{3/2}\right]_0^t = \frac{\sqrt{2}}{4}t^{3/2}.$$

As this expression equals 1 for $t = 2$ and $f(x) = 0$ for $x > 2$ we see that f is a density function. For a random variable X with this density, the distribution function F_X is

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{\sqrt{2}}{4} \cdot t^{3/2} & \text{if } 0 \leq t \leq 2, \\ 1 & \text{if } t > 2. \end{cases}$$

[1] If $X : \Omega \to \mathbb{R}$ is a discrete random variable then the image set S_X must also be discrete.

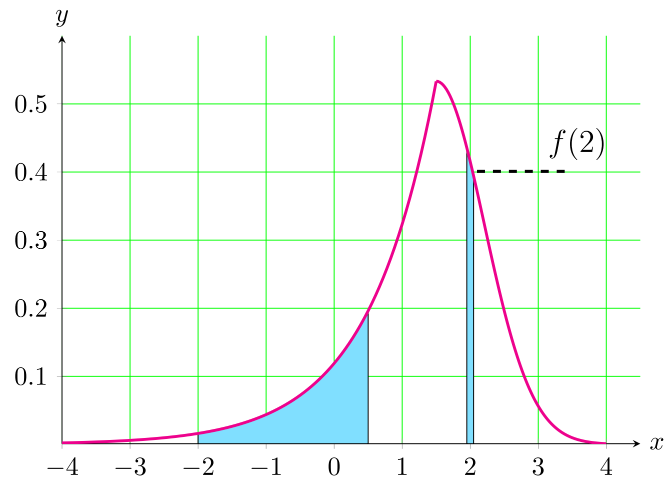

Figure 1: The curve represents the density function f_X of a continuous random variable X. The shaded area on the left indicates $\mathbb{P}\left(-2 \leq X \leq 0.5\right)$. The (almost) rectangular shaded area on the right indicates $\mathbb{P}\left(X \in I\right)$ for an interval I of length ε containing 2; for small ε this probability is approximately $\varepsilon \cdot f_X(2)$.

The distribution function F_X of a continuous random variable X is obtained by integrating the density function f_X. By the fundamental theorem of calculus we can also go backwards [2]: for $t \in \mathbb{R}$ we have

$$F'_X(t) = \lim_{h \downarrow 0} \frac{F_X(t+h) - F_X(t)}{h} = \lim_{h \downarrow 0} \frac{\mathbb{P}\big(X \in [t, t+h]\big)}{h} = f_X(t).$$

This is sometimes written as $\mathbb{P}\left(X \in [x, x+dx]\right) = f_X(x)dx$ indicating that the distribution of X assigns probability $f_X(x)dx$ to a small interval $[x, x+dx]$ containing $x \in \mathbb{R}$ (see Figure 1).

Example 4.7. Let $F_X(t) = 1 - e^{-t^2}, t \geq 0$ and $F_X(t) = 0, t < 0$ be the distribution function of a continuous random variable X. A density f_X can be found by taking the derivative: we have

$$f_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ 2xe^{-x^2}, & \text{if } x \geq 0. \end{cases}$$

Remark 4.8. Let X be a continuous random variable.

- Then X take any specific value $x \in \mathbb{R}$ with probability zero, since

$$\mathbb{P}\left(X = x\right) \leq \mathbb{P}\left(X \in [x-\varepsilon, x+\varepsilon]\right) = \int_{x-\varepsilon}^{x+\varepsilon} f_X(y)dy \approx 2\varepsilon \cdot f_X(x) \to 0, \quad \varepsilon \downarrow 0.$$

  For $a < b$ this gives $\mathbb{P}\left(a \leq X \leq b\right) = \mathbb{P}\left(a < X < b\right) = \mathbb{P}\left(a \leq X < b\right) = \mathbb{P}\left(a < X \leq b\right)$.

- The distribution function F_X is continuous since $F_X(t+\varepsilon) - F_X(t) \approx \varepsilon \cdot f_X(t) \to 0$ as $\varepsilon \to 0$.

- The density function f_X might not be continuous; from Defintion 4.1 the function f_X is only required to be non-negative with an improper integral from $-\infty$ to ∞ equal to one.

[2] This requires some nice behaviour of f such as continuity on an interval covering t.

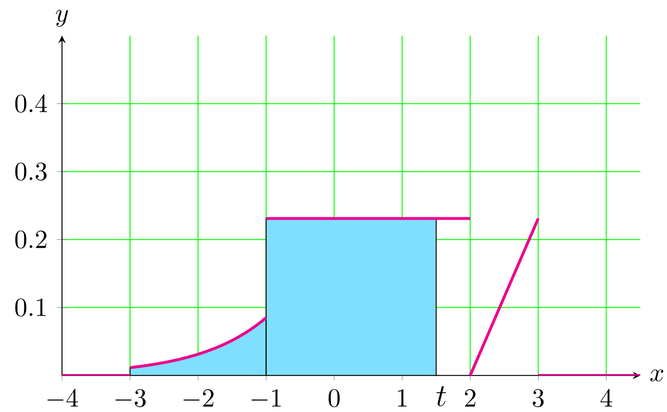

Figure 2: Plot of the density function f_X of a continuous random variable X. f_X is not continuous at points $-3, -1, 2$ and 3. The shaded area indicates $F_X(t) = \mathbb{P}(X \le t)$.

The following table illustrates some of the similarities and differences between discrete and continuous random variables.

	discrete r.v.	continuous r.v.
mass/density function	$p_X(k),\ k \in S_X$	$f_X(x),\ x \in \mathbb{R}$
distribution function F_X	$\sum_{k \in S_X, k \le t} p_X(k)$ F_X is a step function	$\int_{-\infty}^{t} f_X(x)dx$ F_X is continuous
relationship	$p_X(k) = F_X(k) - \mathbb{P}(X < k)$	$f_X(x) = F'_X(x)$

Remark 4.9 (An ignorable disclaimer)**.** You might notice that we did not check $\mathbb{P}$ in Definition 4.2 actually gives a probability distribution (in the sense of Definition 1.6). Although this seems very intuitive, as we have presented things, it is actually not true!

Even with simple densities like $f(x) = 1$ for $x \in [0, 1]$ and $f(x) = 0$ for $x \notin [0, 1]$, the function $\mathbb{P}$ in Definition 4.2 cannot assign a probability to *every* set $A \subseteq \mathbb{R}$ and satisfy Definition 1.6 – it turns out that there are far too many sets $A \subseteq \mathbb{R}$ to do this consistently. This is not as bad as it might sound, and the correction in more advanced probability courses is to only assign probabilities to certain 'nice' sets $A \subseteq \mathbb{R}$. These nice sets include intervals $[a, b]$ and the nice sets are closed under complements and countable unions and intersections (e.g. $[a, b] \cup [c, d]$ or $\mathbb{R} \setminus [a, b]$, ...).

It is completely standard to ignore these subtleties in a first course on probability, and that is the approach we take here. In particular, you can happily ignore this remark. :) Practically it also doesn't make too much of a difference either, as it is very hard to find sets $A \subseteq R$ which are 'not nice'. The main purpose in mentioning this at all is to note the issue in case you continue in probability or in case you read a textbook which discusses 'σ-algebras'. If you *are* interested in hearing more then you might like to look up the terms 'non-measurable set', 'measure theory' and 'sigma algebra' in Wikipedia.

4.2 The continuous uniform distribution

Our first example is a continuous analogue of the uniform distribution.

Definition 4.10 (Continuous uniform distribution)**.** Let $a, b \in \mathbb{R}$ with $a < b$. A continuous random variable X follows the *uniform distribution on* $[a, b]$ if the density function of X is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b], \\ 0 & \text{if } x \notin [a,b]. \end{cases}$$

We then write $X \sim \text{unif}[a, b]$. In this case, the distribution function F_X is given by

$$F_X(t) = \begin{cases} 0 & \text{if } t < a, \\ \frac{t-a}{b-a} & \text{if } a \le t \le b, \\ 1 & \text{if } t > b. \end{cases}$$

Example 4.11 (Stick breaking)**.** We break a stick of length 2 at a uniformly chosen point in two pieces. What is the probability that both pieces have length at least 0.2? Let $X \sim \text{unif}[0, 2]$ be the length of the first (that is, left) piece. Then, the sought probability is $\mathbb{P}(0.2 \le X \le 1.8) = 1.6/2 = 4/5$.

Example 4.12. Let $X \sim \text{unif}[-4, 6]$. What is the probability that the absolute value of X is at least 1? We have

$$\mathbb{P}(|X| \ge 1) = \mathbb{P}(X \ge 1) + \mathbb{P}(X \le -1) = 1 - \mathbb{P}(X < 1) + \mathbb{P}(X \le -1) = 1 - 5/10 + 3/10 = 4/5.$$

Example 4.13. Let $X \sim \text{unif}[0, 1]$. What is the density function of the random variable X^2?

Clearly $F_{X^2}(t) = 0$ for $t < 0$ and $F_{X^2}(t) = 1$ for $t > 1$. For $t \in [0, 1]$, we find

$$F_{X^2}(t) = \mathbb{P}\left(X^2 \le t\right) = \mathbb{P}\left(X \le \sqrt{t}\right) = \sqrt{t}.$$

Differentiating gives

$$f_{X^2}(x) = \frac{1}{2\sqrt{x}}, \quad x \in (0, 1).$$

For $x \notin (0, 1)$, we have $f_{X^2}(x) = 0$.

4.3 The exponential distribution

The continuous analogue of the geometric distribution is the exponential distribution.

Definition 4.14 (Exponential distribution)**.** Let $\lambda > 0$. A continuous random variable X is said to follow the *exponential distribution with parameter* λ if the density function of X is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

We then write $X \sim \exp_\lambda$. In this case, the distribution function F_X is given by

$$F_X(t) = \begin{cases} 1 - e^{-\lambda t}, & t \ge 0, \\ 0, & t < 0. \end{cases}$$

A key property of the exponential distribution is that it is memoryless.

Proposition 4.15. Let $X \sim \exp_\lambda$ with $\lambda > 0$. Then, for $t, s \geq 0$, we have

$$\mathbb{P}\big(X \geq t + s | X \geq t\big) = \mathbb{P}\left(X \geq s\right).$$

Proof. Given $t \geq 0$, we have $\mathbb{P}(X \geq t) = 1 - F_X(t) = e^{-\lambda t}$. Then

$$\mathbb{P}\big(X \geq t + s | X \geq t\big) = \frac{\mathbb{P}\big(X \geq t + s\big)}{\mathbb{P}\big(X \geq t\big)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathbb{P}(X \geq s),$$

as required. □

The memoryless property is natural in several contexts and can justify using the exponential distribution to model

- emission time of a radioactive particle,
- the life-time of electronic devices,
- the time between two car accidents on the same road or plane crashes.

Example 4.16 (Radioactive decay)**.** Uranium-235 is an uranium isotope used as fuel in nuclear power plants (and weapons). Its half-life is ≈ 703.8 million years, which means that the lifetime X (in years) of a single atom is (approximately) exponentially distributed [3] with $\lambda = \log 2 \cdot (7.038)^{-1} \cdot 10^{-8}$ since

$$\mathbb{P}\left(X \geq 7.038 \cdot 10^8\right) = e^{-\lambda \cdot 7.038 \cdot 10^8} = 1/2.$$

Once the half-life λ is determined other calculations follow easily. For example, the probability that the lifetime of a uranium-235 atom is at least twice its half-life is $\mathbb{P}(X \geq 2 \cdot 7.038 \cdot 10^8) = e^{-2 \cdot \lambda \cdot 7.038 \cdot 10^8} = (1/2)^2 = 1/4$.

4.4 The normal distribution

In this subsection we introduce the normal distribution, which is by far the most important distribution in statistics. Before doing this, we need the following definition.

Definition 4.17 (Error function)**.** The *(Gauss) error function* Φ is the map $\Phi : \mathbb{R} \to [0, 1]$ given by

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx, \qquad \text{for } t \in \mathbb{R}.$$

There is no simple formula for Φ, but its approximate values (which are often required) can be found in Table 1 on the last page of this handout. A useful property of this function is that, by symmetry of the integrand, the value $\Phi(t)$ for $t < 0$ can be determined using the relation [4]

$$\Phi(t) = 1 - \Phi(-t), \qquad \text{for } t \in \mathbb{R}.$$

[3] The law of large numbers, which is discussed in Section 6, will explain why in a bulk of uranium isotopes, roughly 50% have decayed after 703.8 million years.

[4] In particular, this explains why values of $\Phi(t)$ are only given in Table 1 for $t \geq 0$.

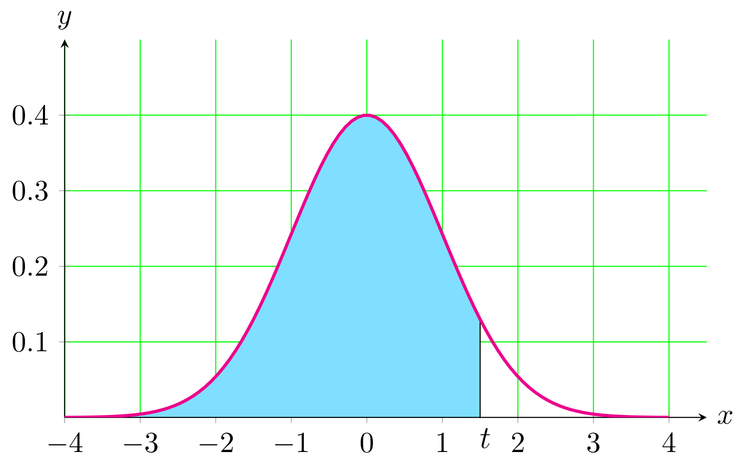

Figure 3: Plot of $f(x) := (2\pi)^{-1/2}e^{-\frac{x^2}{2}}$ for $x \in \mathbb{R}$. The value $\Phi(t)$ is equal to the area of the region between the x-axis, the graph of f and the vertical line at t. (In this diagram the area is $\Phi(1.5)$.)

Definition 4.18 (Normal distribution)**.** Let $\mu \in \mathbb{R}$ and $\sigma > 0$. A continuous random variable X is said to follow the *normal distribution with parameters* μ *and* σ^2 if the density function f_X is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad \text{for } x \in \mathbb{R}.$$

We then write $X \sim N(\mu, \sigma^2)$. With Φ as in Definition 4.17 the distribution function F_X is then

$$F_X(t) = \Phi\left(\frac{t-\mu}{\sigma}\right), \qquad \text{for } t \in \mathbb{R}.$$

When $\mu = 0$ and $\sigma = 1$, this distribution is called the *standard normal distribution.*

In the next section we obtain intuitive descriptions of the parameters μ and σ: if a random variable X follows the distribution $N(\mu, \sigma^2)$ then the *expectation* of X is μ and the *variance* of X is σ^2.

We have seen 'normal-like' curves appear many times through Section 3. One of the motivations in defining the normal distribution is given by the famous De Moivre-Laplace theorem which we discuss in Section 6: it says that given $p \in (0, 1)$, if $X \sim \text{bin}_{n,p}$ then for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X - np}{\sqrt{np(1-p)}} \leq t\right) = \Phi(t).$$

More generally, many quantities are described very accurately by the normal distribution [5], including:

- intelligence quotient ($\mu = 100, \sigma = 15$)
- adult height ($\mu = 175.3, \sigma = 7.42$ (cm) for men, $\mu = 161.9, \sigma = 7.11$ (cm) for women in England)
- baby weight at birth ($\mu = 3.40, \sigma = 1.25$ (kg) in the US)
- the amount of annual rainfall in a city

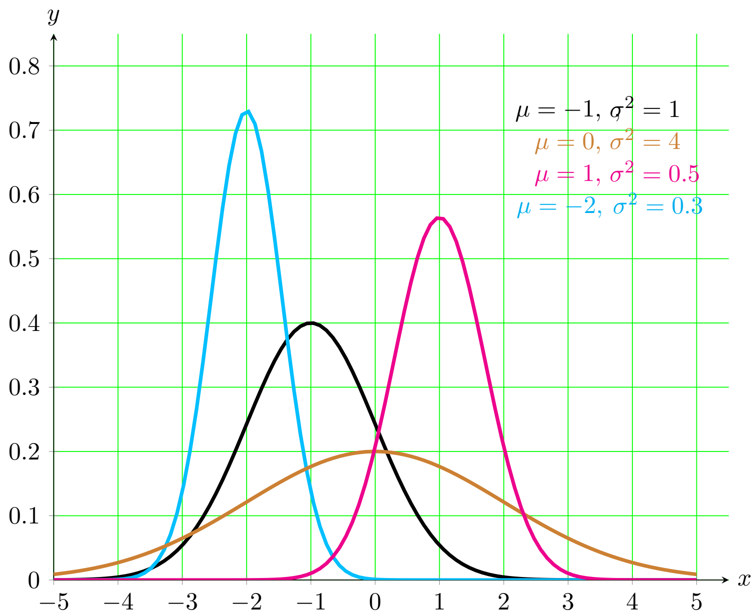

Figure 4: Densities of the normal distribution with different parameters μ and σ^2.

In order to calculate using the normal distribution, it is often useful to reduce calculations for $N(\mu, \sigma^2)$ to the case $\mu = 0, \sigma = 1$. The following proposition is very helpful in this context.

Proposition 4.19. Let $\mathcal{N} \sim N(0,1)$ and let $\mu \in \mathbb{R}$ and $\sigma > 0$. Then $\mu + \sigma\mathcal{N} \sim N(\mu, \sigma^2)$.

Proof. This follows by a change of variables. Given $t \in \mathbb{R}$, we have

$$\mathbb{P}\left(\mu + \sigma\mathcal{N} \leq t\right) = \mathbb{P}\left(\mathcal{N} \leq \frac{t-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{t-\mu}{\sigma}} e^{-\frac{x^2}{2}}\,dx \overset{y=\sigma x+\mu}{=} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy,$$

as required. $\square$

Example 4.20 (Gestation)**.** Gestation periods follow the normal distribution with $\mu = 266$ and $\sigma^2 = 256$ (so $\sigma = 16$). What percentage of births are between 10 days early and 20 days overdue?

Let X be the length of a gestation period. Then, $X \sim N(266, 256)$. We can write $X = 266 + 16\mathcal{N}$ for a random variable $\mathcal{N} \sim N(0,1)$ by Proposition 4.19. It follows that

$$\begin{aligned} \mathbb{P}\left(256 \leq X \leq 286\right) = \mathbb{P}\left(-0.675 \leq \mathcal{N} \leq 1.25\right) = \Phi(1.25) - \Phi(-0.675) &= \Phi(1.25) + \Phi(0.675) - 1 \\ &= 0.645 \text{ [3dp]}, \end{aligned}$$

where we used $\Phi(t) = 1 - \Phi(-t)$, for $t \in \mathbb{R}$.

Example 4.21 (IQ)**.** The intelligence quotient of a randomly chosen person (roughly) follows the normal distribution with parameters $\mu = 100$ and $\sigma^2 = 225$ (so $\sigma = 15$). One common criteria for a

[5] Gauss originally studied the normal distribution in 1809 to to model measurement errors in physical experiments.

'genius' is that a person has an IQ of at least 130. The probability that a randomly chosen person falls into this category is

$$\mathbb{P}\left(100 + 15\mathcal{N} \geq 130\right) = \mathbb{P}\left(\mathcal{N} \geq 30/15\right) = 1 - \mathbb{P}\left(\mathcal{N} \leq 30/15\right) = 1 - \Phi(2) = 0.023 \text{ [3dp]}$$

So, according to this criteria, 2.3% of the population falls into this category.

Example 4.22. Let $X \sim N(3, 25)$. What is $\mathbb{P}\left(|X-3| \geq 1\right)$? We can set $X = 3 + 5\mathcal{N}$, where $\mathcal{N} \sim N(0,1)$ and compute

$$\begin{aligned}\mathbb{P}\left(|X-3| \geq 1\right) = \mathbb{P}\left(\{X \geq 4\} \cup \{X \leq 2\}\right) &= \mathbb{P}\left(X \geq 4\right) + \mathbb{P}\left(X \leq 2\right) \\ &= \mathbb{P}\left(\mathcal{N} \geq 1/5\right) + \mathbb{P}\left(\mathcal{N} \leq -1/5\right) \\ &= 1 - \mathbb{P}\left(\mathcal{N} \leq 1/5\right) + \Phi(-1/5) \\ &= 2 - 2\Phi(1/5) = 0.841 \text{ [3dp]}\end{aligned}$$

Example 4.23. Let $X \sim N(0, \sigma^2)$. What is the distribution of X^2? For $\mathcal{N} \sim N(0,1)$ and $t \geq 0$

$$F_{X^2}(t) = \mathbb{P}\left(X^2 \leq t\right) = \mathbb{P}\left(\sigma^2\mathcal{N}^2 \leq t\right) = \mathbb{P}\left(-\frac{\sqrt{t}}{\sigma} \leq \mathcal{N} \leq \frac{\sqrt{t}}{\sigma}\right) = \Phi\Big(\frac{\sqrt{t}}{\sigma}\Big) - \Phi\Big(\frac{-\sqrt{t}}{\sigma}\Big) = 2\Phi\Big(\frac{\sqrt{t}}{\sigma}\Big) - 1.$$

Noting that $f_{X^2}(t) = F'_{X^2}(t)$ and that $\Phi'(x) = e^{-x^2/2}/\sqrt{2\pi}$, by the chain rule we find

$$f_{X^2}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{\sqrt{2\pi\sigma^2 x}} e^{-\frac{x}{2\sigma^2}} & \text{for } x \geq 0. \end{cases}$$

For $\sigma^2 = 1$, the distribution of X^2 is called the χ^2-distribution and has an important role in statistics[6].

4.5 Independence of random variables

Lastly, we require a notion of independence for both discrete and continuous random variables [7].

Definition 4.24 (Independence of random variables)**.** A collection of discrete or continuous random variables $X_1, \ldots, X_n$ are called *independent* if, for all sets $A_1, \ldots, A_n \subseteq \mathbb{R}$, we have

$$\mathbb{P}\big(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\big) = \prod_{i=1}^{n} \mathbb{P}\big(X_i \in A_i\big).$$

Most important takeaways in this chapter. You should

- understand the concept of densities and distribution functions of continuous random variables and their connection,
- be able to solve standard problems involving the uniform, exponential and normal distributions.

[6] You will see much more about this distribution in the module 2S in Year 2 .

[7] Note: By Proposition 3.37 this agrees with our previous definition for discrete random variable (Definition 3.33).

Table 1: Table for $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} dx, t \geq 0$.

t	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999