University of Birmingham
School of Mathematics
Vectors, Geometry and Linear Algebra
VGLA

Problem Sheet 5

Model Solutions

Remember that there are practise questions under the materials section for each week. Note the question 2 in on page 2.

$\boxed{\text{SUM}}$ **Q1.** (i) Explain why a system of equations $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ of m equations in n unknowns has a non-zero solution if and only if n is greater than the rank of $\mathbf{A}$.

(ii) Using (i), determine if the following system of equations has a non-zero solution.

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 2x_4 &= 0. \end{aligned}$$

(iii) Using (i), determine if the following system of equations has a non-zero solution.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0. \end{aligned}$$

(iv) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

where $x_1, x_2, x_3 \in \mathbb{R}$. Calculate the determinant of $\mathbf{A}$ (remember Vandermonde) and calculate the rank of A for all possible x_1, x_2 and x_3.

Solution. (i) We know that a homogeneous equation always has at least one solution, namely the trivial solution. We also know that if $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ has rank n, then $\mathbf{A}$ has a unique solution by Theorem 7.48. Since the rank of $\mathbf{A}$ is at most n (the number of columns of $\mathbf{A}$), we have proved that, if $\mathbf{A}$ has more than 1 solution, then the rank of $\mathbf{A}$ is less than n.

Now suppose that the rank is less than n. We need to see that there is a non-zero solution to the equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$. Since the transposes of the columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ span the column space col($\mathbf{A}$) and since the dimension of the column space is the rank of $\mathbf{A}$, we have that the vectors $\mathbf{v}_1^T, \dots, \mathbf{v}_n^T$ are linearly dependent. Hence there exists $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ not all zero such that

$$\lambda_1 \mathbf{v}_1^T + \cdots + \lambda_n \mathbf{v}_n^T = 0.$$

It follows that $\mathbf{x} = (\lambda_1, \dots, \lambda_n)$ is a non-zero solution to the system of equations. We have shown that if the rank of $\mathbf{A}$ is less than n, then the equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ has a non-zero solution. This explains the statement in the question.

(ii) We have

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & 1 & 2 \end{pmatrix}.$$

Clearly the rank of A is at most 3 and $3 < 4 = n$ and so there are non-zero solutions.

(iii) We have

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{pmatrix}.$$

Subtracting row 1 from row 2 and 2 lots of row 1 from row 3 we get

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & -5 & -1 \end{pmatrix}.$$

Dividing row 2 by -3, we obtain

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2/3 \\ 0 & -5 & -1 \end{pmatrix}.$$

Adding 5 lots of row 2 to row 3 gives

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2/3 \\ 0 & 0 & -13/3 \end{pmatrix}.$$

Finally scaling the third row reveals

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix}$$

and so we see that $\mathbf{A}$ has rank 3 and so the only solution is the zero solution.

(iv)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

is a Vandermonde matrix and so the determinant is $(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$ as we saw in an examples class.

If $x_3 \neq x_2$, $x_3 \neq x_1$ and $x_2 \neq x_1$ then $\mathbf{A}$ has non-zero determinant and so is invertible and has rank 3.

If $x_1 = x_2$ and $x_1 \neq x_3$, then the column rank is visibly 2. The same holds if $x_1 = x_3$ and $x_1 \neq x_2$ or $x_2 = x_3$ and $x_1 \neq x_2$. Hence the rank is 2 in these cases.

If $x_1 = x_2 = x_3$ then the rank is 1.

□

SUM **Q2**. Suppose that V and W are vector spaces over $\mathbb{R}$ with $\dim V = 4$ and $\dim W = 3$. Assume that $T : V \to W$ is a linear transformation.

(i) Explain why $\ker(T) \neq \{\mathbf{0}\}$.
(ii) List all possibilities for the dimension on the image of T.
(iii) Suppose that

$$B_V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

is a basis for V and

$$B_W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

is a basis for W. Assume that

$$\begin{aligned} T(\mathbf{v}_1) &= \mathbf{w}_1 - \mathbf{w}_2 \\ T(\mathbf{v}_2) &= \mathbf{w}_2 - \mathbf{w}_3 \end{aligned}$$

$$\begin{aligned} T(\mathbf{v}_3) &= \mathbf{w}_3 - \mathbf{w}_1 \\ T(\mathbf{v}_4) &= \mathbf{w}_1 + \mathbf{w}_2 - 2\mathbf{w}_3 \end{aligned}$$

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(a) Suppose that $\mathbf{v} \in V$ has coordinate vector $(1, 2, 3, 4)$ with respect to B_V of V. Calculate $T(\mathbf{v})$ and write down its coordinate vector with respect to the basis B_W.
(b) Write down the matrix representing T with respect to B_V and B_W.
(c) Calculate the dimension of the image of T and find a basis for the image.
(d) Calculate $\ker(T)$.

Solution. (i) Since the image of T is a subspace of W, we have that the rank of T is at most 3. Hence the nullity of T is at least 1 by the rank-nullity Theorem. Hence $\ker(T) \neq \{0\}$.

(ii) The image could be any possible dimension. Hence the dimension of the image is one of 0, 1, 2, $3 = \dim(W)$.

(iii) As T is a linear transformation we have

$$\begin{aligned} T(\mathbf{v}) &= T(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 + 4\mathbf{v}_4) \\ &= T(\mathbf{v}_1) + 2T(\mathbf{v}_2) + 3T(\mathbf{v}_3) + 4T(\mathbf{v}_4) \\ &= \mathbf{w}_1 - \mathbf{w}_2 + 2(\mathbf{w}_2 - \mathbf{w}_3) + 3(\mathbf{w}_3 - \mathbf{w}_1) + 4(\mathbf{w}_1 + \mathbf{w}_2 - 2\mathbf{w}_3) \\ &= 2\mathbf{w}_1 + 5\mathbf{w}_2 - 7\mathbf{w}_3. \end{aligned}$$

Thus the coordinate vector of $T(\mathbf{v})$ with respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is $(2, 5, -7)$.

(iv) We are looking for a 3×4 matrix which we'll call $\mathbf{A}$. The ith column of $\mathbf{A}$ is the transpose of coordinate vector of $T(\mathbf{v}_i)$ with respect to the basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Hence the matrix representing T is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{pmatrix}.$$

We can check

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix}.$$

and so

$$T(\mathbf{v}) = 2\mathbf{w}_1 + 5\mathbf{w}_2 - 7\mathbf{w}_3$$

just as above.

(v) The image of T is spanned by the the images of the vectors in any basis of V. Hence the image of T is

$$\langle T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), T(\mathbf{v}_4) \rangle = \langle \mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_2 - \mathbf{w}_3, \mathbf{w}_3 - \mathbf{w}_1, \mathbf{w}_1 + \mathbf{w}_2 - 2\mathbf{w}_3 \rangle$$

To find the dimension of the image, we just calculate the column rank of the matrix $\mathbf{A}$. Since I'm more comfortable with row operations, I'll transpose the matrix.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Add row 1 to row 3 and subtract row 1 from row 4.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix}$$

Add row 2 to row 3 and subtract 2 lots of row 2 from row 4.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the image of T has basis $\{\mathbf{w}_1 - \mathbf{w}_2, \mathbf{w}_2 - \mathbf{w}_3\}$ and the dimension of the image is 2.

(vi) To find the kernel of T we solve the equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$. Thus we row reduce A to echelon form. (Or you can augment the matrix with a column of zeros if you wish.)

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{pmatrix}.$$

Add row 1 to row 2.

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{pmatrix}.$$

Add row 2 to row 3

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence the solution set to the homogeneous equation is

$$\{(\lambda - \mu, \lambda - 2\mu, \lambda, \mu) \mid \lambda, \mu \in \mathbb{R}\}$$

and this is the kernel. Taking $\lambda = 1$ and $\mu = 0$ and $\lambda = 0$ and $\mu = 1$. Gives us a basis for $\ker(T)$:

$$\{(1, 1, 1, 0), (-1, -2, 0, 1)\}.$$

$\square$