Problem sheet 4

Question 11. Let X and Y be continuous random variables. For some $c_1 \in \mathbb{R}$, the density function of X is given by

$$f_X(x) = \begin{cases} c_1\, x^{-3/2}, & x \in [1,9]; \\ 0, & x \notin [1,9]. \end{cases}$$

For some $c_2 \in \mathbb{R}$, the distribution function of Y is given by

$$F_Y(t) = \begin{cases} c_2\, e^{-1/t}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

(a) Find c_1 and F_X.

(b) Let $Z = \sqrt{X}$. Find the distribution function F_Z and, using this, deduce an expression for a density f_Z.

(c) Find c_2, the density function f_Y, and $t_0 \in \mathbb{R}$ with $\mathbb{P}(Y \geq t_0) = 1/4$.

Question 12. (SUM)

(a) Let $\mathcal{N} \sim N(0,1)$ and set $X = 4\mathcal{N}+1$. Also let Y be a random variable with $Y \sim \text{hyp}_{2,3,6}$ and suppose that X, Y are independent.

(i) What probability distribution does X follow?

(ii) Determine $\mathbb{P}(X \geq 6)$ and $\mathbb{P}(|X-2| \geq 5)$ to three decimal places.

(iii) Explaining your answer, find $\mathbb{P}(X = Y)$.

(iv) Find $\mathbb{P}(X \geq Y)$ to three decimal places.

Hint: It might be helpful to consider events $\{X \geq i, Y = i\}$

(b) Suppose that X and Y are independent random variables, and that both X and Y follow an exponential distribution with parameter 2.

(i) Find F_X and F_Y.

(ii) Find distribution function F_Z of the random variable $Z = \min(X, Y)$.

Hint: It is helpful to consider $1 - F_Z$.

(iii) What distribution does Z follow?

(iv) Find a value $t_0 \in \mathbb{R}$ such $\mathbb{P}(Z \geq t_0) = 1/3$.

Question 13. An urn contains 40 white balls and 2 red balls. Let $m \in \{1, \ldots, 40\}$. We select m balls without replacement from the urn and if the sample only contains white balls we win m pounds; otherwise, we win nothing. Let X denote the amount of money we win.

(a) Determine S_X, the mass function p_X and $\mathbb{E}[X]$ in terms of m.

(b) How should you choose m in order to maximize the expected payoff?

Question 14. Let $X \sim \text{bin}_{n,p}$ where $n \geq 1$ and $p \in [0, 1]$.

(a) Show that $\mathbb{E}[X] = np$ using only the definition of $\mathbb{E}[X]$ and the mass function of X.

(b) Show that $\mathbb{E}\big[X(X-1)\big] = n(n-1)p^2$.

(c) Defining[1] $\mathbb{V}\text{ar}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, deduce that $\mathbb{V}\text{ar}(X) = np(1-p)$.

Question 15. A bowl contains n strands of boiled spaghetti, with each strand having two ends. Over n rounds we proceed as follows.

- In round 1, we randomly select two ends from the bowl and tie them together.
- In round $i \geq 2$, we randomly select two ends from the remaining untied ends and tie these together.

After the n rounds there are no remaining untied ends and the bowl consists of spaghetti loops.

(a) What is the probability that we create a loop in round 1?

(b) What is the probability that we create a loop in round i?

(c) Let X denote the number of loops in the bowl. By viewing X as a sum of simpler random variables, show that [2]

$$\mathbb{E}[X] = \sum_{i=1}^{n} \frac{1}{2i-1}.$$

[1] While we have not defined variance yet (Week 9), you can still make this calculation.

[2] Bonus! By comparing $\sum_{i=1}^{n} \frac{1}{i}$ and $\int_1^n x^{-1} dx$, show that $\log(n) \leq \sum_{i=1}^{n} \frac{1}{i} \leq \log(n) + 1$ and deduce in (c) that $\left|\mathbb{E}[X] - \frac{1}{2}\log(n)\right| \leq 3$.

t	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999
3.1	0.99903	0.99906	0.9991	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.9994	0.99942	0.99944	0.99946	0.99948	0.9995
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.9996	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.9997	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.9998	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.9999	0.9999	0.9999	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998