University of Birmingham
School of Mathematics

1RA Integration Autumn 2022

Problem Sheet 4
issued Week 8

You have approximately 10 working days from the release of this problem sheet to complete and submit your answers to the $\boxed{\text{SUM}}$ questions (**Q2** and **Q5**) via the Assignments tab on the 1RA Canvas page. You are strongly encouraged to attempt all of the remaining formative questions, and as many of the extra questions as you can, to prepare for the final exam. But only your solutions to the $\boxed{\text{SUM}}$ questions should be submitted to Canvas.

<table>
<tr><td colspan="2">Assignment available from: 18 November
Submission due: 1700 on Wednesday 30 November 2022</td></tr>
<tr><td>Pre-submission</td><td>Post-submission</td></tr>
<tr><td>• Your Guided Study Support Class in Weeks 9-10.
• Tutor meetings in Weeks 8-9.
• PASS from Week 8
• Library MSC from Week 8
• Office Hours (Watson 208):
Friday 1000-1130.</td><td>• Written feedback on your submission (8 December).
• Generic feedback (8 December).
• Model solutions (8 December).
• Tutor meetings in Weeks 10-11.
• Office Hours (Watson 208):
Friday 1000-1130.</td></tr>
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Instructions:

The **deadline** for submission of the two $\boxed{\text{SUM}}$ questions (**Q2** and **Q5**) is as follows:

- **By 1700 on Wednesday 30 November 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

Important:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

Q1. (a) Sketch the graph of $f:[0,4]\to\mathbb{R}$, $f(x)=\sqrt{x}$ and highlight the area covered by the difference $U(f,P)-L(f,P)$ for the partition $P=\{0,1,2,3,4\}$.

(b) Use Riemann's Criterion to prove each of the functions below are integrable:

(i) $f:[0,3]\to[0,\infty)$, $f(x)=x^2$

(ii) $f:[2,4]\to[0,100]$, $f(x)=\begin{cases}5, & x<3;\\ 100, & x=3;\\ 3, & x>3.\end{cases}$

You may use the results stated in **Q8**(b) on Problem Sheet 3 to answer (i).

SUM **Q2**. (a) State Riemann's Criterion for integrability.

(b) For each function defined below, use Riemann's Criterion to prove that it is integrable:

(i) $f:[0,\frac{\pi}{2}]\to\mathbb{R}$, $f(x)=4\cos^2(x)$

(ii) $g:[1,10]\to\mathbb{R}$, $g(x)=\begin{cases}7, & x\in[1,2);\\ 5, & x\in[2,10)\\ 3, & x=10.\end{cases}$

(c) For each function in part (b), use properties of the function and results from Lectures to prove that it is integrable (without using Riemann's Criterion directly).

Q3. Suppose that $f:[a,b]\to[0,\infty)$ is bounded and integrable, where $-\infty<a<b<\infty$:

(a) Prove that if $\alpha\geq 0$ and P is a partition of $[a,b]$, then

$$U(\alpha f,P)=\alpha U(f,P) \quad\text{and}\quad L(\alpha f,P)=\alpha L(f,P).$$

(b) Use Riemann's Criterion and (a) to prove that αf is integrable.

(c) Explain, without working hard, why the integrability of αf implies that

$$\int_a^b(\alpha f)=\sup\{L(\alpha f,P): P \text{ is a partition of } [a,b]\}.$$

(d) Use (a) and (c) to conclude that $\int_a^b(\alpha f)=\alpha(\int_a^b f)$.

Q4. Suppose that $f:[a,b]\to[0,\infty)$ is bounded and integrable, where $-\infty<a<b<\infty$, and that $0\leq f(x)\leq M$ for all $x\in[a,b]$ and some $M>0$:

(a) Prove that $0\leq \underline{\int_a^b} f$ and that $\overline{\int_a^b} f\leq M(b-a)$.

(b) Use (a) and the definition of the integral to prove that $0\leq\int_a^b f\leq M(b-a)$.

(c) Use (b) to prove that $\lim_{h\to 0^+}\left(\int_a^{a+h} f\right)=0$ and $\lim_{h\to 0^+}\left(\int_{b-h}^b f\right)=0$.

SUM **Q5**. (a) State the First Fundamental Theorem of Calculus.

(b) Use the First Fundamental Theorem of Calculus to prove that each of the following functions is differentiable and find a formula for its derivative function that does not contain the integral symbol:

(i) $F:[2,4]\to\mathbb{R},\ F(x):=\displaystyle\int_2^x\frac{1}{\log(t)}\,\mathrm{d}t,\ x\in[2,4]$

(ii) $G:[-1,1]\to\mathbb{R},\ G(x):=\displaystyle\int_{-5}^{5x^4+3x^2+1}e^{-t^2}\,\mathrm{d}t,\ x\in[-1,1]$

(iii) $H:[\pi,2\pi]\to\mathbb{R},\ H(x):=\displaystyle\int_x^{2\pi}\frac{\sin(t)}{t}\,\mathrm{d}t,\ x\in[\pi,2\pi]$

Q6. (a) State the Second Fundamental Theorem of Calculus.

(b) Suppose that $-\infty < a < b < \infty$. Use the Second Fundamental Theorem of Calculus to prove that

$$\int_a^b |x| \, dx = \begin{cases} \frac{1}{2}(b^2 - a^2), & a \geq 0; \\ \frac{1}{2}(a^2 + b^2), & a < 0 \leq b; \\ \frac{1}{2}(a^2 - b^2), & b < 0. \end{cases}$$

You must verify all hypotheses required to apply the Second Fundamental Theorem of Calculus. In particular, if you use the fact that a certain function is an antiderivative of the absolute value function, then you must prove this fact (be careful proving differentiability at the origin).

(c) Let $f : \mathbb{R} \setminus [-3, -2] \to \mathbb{R}$ be defined by $f(x) := |x|$ for all $x \in \mathbb{R} \setminus [-3, -2]$. Find two antiderivatives F_1 and F_2 of f such that

$$F_1(x) - F_2(x) = \begin{cases} 9, & x > -2; \\ 3, & x < -3. \end{cases}$$

You must prove that your choices for F_1 and F_2 are antiderivatives of f.

Q7. Find the following antiderivatives and integrals (henceforth $\log(x) := \log_e(x)$):

(a) $\displaystyle\int_1^e x^2 \log(x) \, dx$

(b) $\displaystyle\int (\log(x))^2 \, dx$

(c) $\displaystyle\int_0^\pi e^x \cos(x) \, dx$

Q8. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{x - 4}{x^2 - 5x + 6} \, dx$

(b) $\displaystyle\int_1^2 \frac{x^5 + x - 1}{x^3 + 1} \, dx$

(c) $\displaystyle\int \frac{x^2 + 2x - 1}{x^3 - x} \, dx$

Extra Questions

EQ1. Suppose that $f : [a, b] \to [0, \infty)$ is bounded and integrable, where $-\infty < a < b < \infty$. A *tagged partition* (P, T) of $[a, b]$ consists of a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and a collection $T = \{t_1, \ldots, t_n\}$ of *tags* satisfying $t_1 \in [x_0, x_1], \ldots, t_n \in [x_{n-1}, x_n]$. The corresponding *Riemann Sum* is defined by

$$R(f, P, T) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

(a) Prove that $L(f, P) \leq R(f, P, T) \leq U(f, P)$ for any tagged partition (P, T).

(b) Prove that $L(f, P) \leq \int_a^b f \leq U(f, P)$ for any partition P.

(c) Use Riemann's Criterion to prove that for each $\epsilon > 0$, there exists a partition P such that $|R(f, P, T) - \int_a^b f| < \epsilon$ whenever T is a collection of tags for P.

EQ2. (a) For each function defined below, use the properties of the function and results from Lectures/Lectures Notes to prove that it is integrable:

(i) $f : [0, 10] \to \mathbb{R}$, $f(x) = 3x^2 + 5x + 9$

(ii) $g : [1, 100] \to \mathbb{R}$, $g(x) = \lfloor x \rfloor := \max\{n \in \mathbb{N} : n \leq x\}$

(b) Use Riemann's Criterion to prove that each function in part (a) is integrable. You may use the results stated in **Q8**(b) on Problem Sheet 3 (but you are not required to do so).

EQ3. (a) Prove that if $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are both uniformly continuous, then $f + g$ is uniformly continuous.

(b) Suppose that $f : [a, \infty) \to \mathbb{R}$ is a continuous function, where $-\infty < a < b < \infty$. Prove that if f is uniformly continuous on both $[a, b]$ and $[b, \infty)$, then f is uniformly continuous on $[a, \infty)$.

EQ4. Suppose that $f : [1, 5] \to \mathbb{R}$ and $g : [2, 6] \to [-2, 10]$ are both integrable functions, whilst $10 \leq f(x) \leq 1000$ for all $x \in [2, 4]$. For each of the integrals below, use the properties of integrable functions to prove that the integral exists, and then find an upper bound and a lower bound for the value of the integral:

(a) $\int_2^4 f$

(b) $\int_2^5 (f - g)$

(c) $\int_3^4 6fg$

EQ5. Use the First Fundamental Theorem of Calculus and apply it to prove that each of the following functions are differentiable and find a formula for their derivatives:

(a) $F : [0, 2] \to \mathbb{R},\ F(x) := \displaystyle\int_0^x \sin(t^2)\ \mathrm{d}t,\ x \in [0, 2]$

(b) $G : [1, 2] \to \mathbb{R},\ G(x) := \displaystyle\int_1^x \sin(t^2)\ \mathrm{d}t,\ x \in [1, 2]$

(c) $H : [0, 1] \to \mathbb{R},\ H(x) := \displaystyle\int_x^1 \sin(t^2)\ \mathrm{d}t,\ x \in [0, 1]$

(d) $I : [0, 1] \to \mathbb{R},\ I(x) := \displaystyle\int_0^{2x^3} \sin(t^2)\ \mathrm{d}t,\ x \in [0, 1]$

(e) $J : [0, 2] \to \mathbb{R},\ J(x) := \left(\displaystyle\int_0^x \sin(t^2)\ \mathrm{d}t\right)^2,\ x \in [0, 2]$

The formula for the derivative $J'(x)$ may contain an integral expression.

EQ6. Suppose that $f : [1, 3] \to \mathbb{R}$ is differentiable and that its derivative f' is continuous:

(a) If $f(1) = 10$ and $\int_1^3 f' = 16$, then calculate $f(3)$.

(b) Explain why the continuity of f' allowed for the application of the Fundamental Theorem of Calculus in part (a).

(c) State a weaker condition on f' that would suffice to apply the Fundamental Theorem of Calculus in part (a).

EQ7. Find the following antiderivatives and integrals:

(a) $\displaystyle\int x\sin(5x)\ \mathrm{d}x$

(b) $\displaystyle\int_1^2 \frac{(\log(x))^2}{x^3}\ \mathrm{d}x$

(c) $\displaystyle\int e^{2x}\sin(3x)\ \mathrm{d}x$

EQ8. Find the following antiderivatives and integrals:

(a) $\displaystyle\int \frac{x^2+1}{x+4}\ \mathrm{d}x$

(b) $\displaystyle\int \frac{10}{(x-1)(x^2+4)}\ \mathrm{d}x$

(c) $\displaystyle\int_3^4 \frac{x^2+1}{x^2-4x+4}\ \mathrm{d}x$