2DE/2DE3 Example sheet 4: Separation of Variables and Fourier series

1. Use the method of Separation of Variables to replace the given PDE by two ODEs.
   (a) $x\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial t} = 0$;
   (b) $u_{xx} + u_{xt} + u_t = 0$;
   (c) $[a(x)u_x]_x - b(x)u_{tt} = 0$.

2. The heat equation in two space dimensions is
$$\alpha^2(u_{xx} + u_{yy}) = u_t.$$
   By looking for a solution in the form
$$u(x, y, t) = X(x)Y(y)T(t),$$
   find three ODEs that can be solved to obtain $u(x, y, t)$. (*Hint: you will need to introduce two new constants*). You do not need to solve the ODEs.

3. Show that, for $m, n \in \mathbb{N}$,
$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases}$$

4. Show that
$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\, dx = 0 \quad \forall m, n \in \mathbb{N}.$$

5. (a) Find the Fourier series of
$$f(x) = x \quad -\pi < x < \pi.$$
   (b) If $f(x)$ is defined to be periodic with period 2π, by the Fourier Convergence Theorem, to what function does this series converge?

6. (a) Find the Fourier series of
$$f(x) = x^2 \quad -\pi < x < \pi.$$
   (b) If $f(x)$ is defined to be periodic with period 2π, by the Fourier Convergence Theorem, to what function does this series converge?

7. (a) Find the Fourier series of

$$f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$$

(b) If $f(x)$ is defined to be periodic with period 2, by the Fourier Convergence Theorem, to what function does this series converge?

8. (a) Find the Fourier series of

$$f(x) = e^x \quad -\pi < x < \pi.$$

(b) If $f(x)$ is defined to be periodic with period 2π, by the Fourier Convergence Theorem, to what function does this series converge?