University of Birmingham
School of Mathematics

1RA Differentiation Autumn 2022

Problem Sheet 2

Model Solutions

You have approximately 10 working days to complete and submit the $\boxed{\text{SUM}}$ questions (**Q4** and **Q9**) and you may begin working on it immediately.

<table>
<tr><td colspan="2">Assignment available from: 14 October
Submission due: 1700 on Wednesday 26 October 2022</td></tr>
<tr><td>Pre-submission</td><td>Post-submission</td></tr>
<tr><td>• Your Guided Study Support Class in Weeks 3-5.
• Tutor meetings in Weeks 3-5.
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• Office Hours: Wednesday 1300-1430 and Friday 1000-1130.</td><td>• Written feedback on your submission.
• Generic feedback (3 November).
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Instructions:

You will spend the next two weeks (including your Guided Study Support Class in weeks 4 and 5 working on the $\boxed{\text{SUM}}$ questions (**Q4** and **Q9**).

The **deadline** for submission is as follows:

- **By 1700 on Wednesday 26 October 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

Important:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

Questions:

Q1. For each of the following statements, either prove that it is true by using the definition of limit, or give a counterexample to show that it is false.

(i) Let $f, g : \mathbb{R} \to \mathbb{R}$. Suppose that $\lim\limits_{x\to\infty} f(x) = a$ and $\lim\limits_{x\to\infty} g(x) = b$ for some $a, b \in \mathbb{R}$. If $f(x) < g(x)$ for all $x \in \mathbb{R}$, then $a < b$.

(ii) Let $f, g : \mathbb{R} \to \mathbb{R}$. If $\lim\limits_{x\to\infty} f(x) = \ell$ for some $\ell \in \mathbb{R}$, and $\lim\limits_{x\to\infty} g(x) = \infty$, then $\lim\limits_{x\to\infty} f(x)g(x) = \infty$.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$. If $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0-} f(x) = \ell$ for some $\ell \in \mathbb{R}$, then $f(0) = \ell$.

(iv) Let $f, g : \mathbb{R} \to \mathbb{R}$. If $\lim\limits_{x\to a} f(x) = b$ and $\lim\limits_{x\to b} g(x) = \ell$ for some $a, b, \ell \in \mathbb{R}$, then $\lim\limits_{x\to a} g(f(x)) = \ell$.

Solution. Each of the above statements is false. Here are counterexamples.

(i). Let $f(x) = 0$ and $g(x) = 1/(1 + x^2)$ for all $x \in \mathbb{R}$. Clearly $g(x) > 0 = f(x)$ for all $x \in \mathbb{R}$. On the other hand, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 0 = 0$, while $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} 1/(1 + x^2) = 0$ (this follows by the Algebra of Limits, since $\lim_{x\to\infty}(1 + x^2) = 1 + \infty^2 = \infty$ by the Algebra of Limits). So $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x)$.

(ii). Let $f(x) = 1/(1 + x^2)$ and $g(x) = x$ for all $x \in \mathbb{R}$. Then $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1/(1+x^2) = 0 \in \mathbb{R}$ (this limit was discussed in the solution to part (i)) and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} x = \infty$. However $\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} x/(1+x^2) = 0 \neq \infty$. To justify that the last limit is zero, we note that

$$\frac{x}{1 + x^2} = \frac{1}{x + 1/x}$$

for all $x \neq 0$, and moreover $\lim_{x\to\infty} x = \infty$, so

$$\lim_{x\to\infty} 1/x = 0,$$
$$\lim_{x\to\infty} (x + 1/x) = \infty + 0 = \infty,$$
$$\lim_{x\to\infty} \frac{x}{1 + x^2} = \lim_{x\to\infty} \frac{1}{x + 1/x} = 0,$$

where at each step the Algebra of Limits was applied.

(iii). Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\lim_{x\to 0^\pm} f(x) = \lim_{x\to 0^\pm} 1 = 1$, while $f(0) = 0 \neq 1$.

(iv). Let $f : \mathbb{R} \to \mathbb{R}$ be the constant function

$$f(x) = 0$$

and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x) = \lim_{x\to 0} 0 = 0$ and $\lim_{x\to 0} g(x) = \lim_{x\to 0} 1 = 1$. However, $g(f(x)) = g(0) = 0$ for all $x \in \mathbb{R}$, so $\lim_{x\to 0} g(f(x)) = \lim_{x\to 0} 0 = 0 \neq 1$. $\square$

Q2. (i) Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^3-5x^2+7x-13}{2x^3-\pi}$.

(b) $\displaystyle\lim_{x\to 0} x\cos\frac{1}{x}$.

(ii) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt[3]{x}$ is continuous, by directly using the definition of continuous function.

Solution. (i).(a). We claim that $\displaystyle\lim_{x\to\infty} \frac{3x^3-5x^2+7x-13}{2x^3-\pi} = \frac{3}{2}$. In order to show this, we note that, for all $x \neq 0$,

$$\frac{3x^3-5x^2+7x-13}{2x^3-\pi} = \frac{3-5/x+7/x^2-13/x^3}{2-\pi/x^3}.$$

Since $\lim_{x\to\infty} x = \infty$, by the Algebra of Limits we immediately deduce that $\lim_{x\to\infty} 1/x = 0$ and consequently, again by the Algebra of Limits,

$$\lim_{x\to\infty} \frac{3-5/x+7/x^2-13/x^3}{2-\pi/x^3} = \frac{3-5\cdot 0+7\cdot 0^2-13\cdot 0^3}{2-\pi\cdot 0^3} = \frac{3}{2}.$$

(i).(b). We claim that $\displaystyle\lim_{x\to 0} x\cos\frac{1}{x} = 0$. By the Absolute Value Rule for Null Limits, proving this is the same as proving that $\displaystyle\lim_{x\to 0} \left|x\cos\frac{1}{x}\right| = 0$. Note that $\displaystyle\left|\cos\frac{1}{x}\right| \leq 1$ for all $x \in \mathbb{R}\setminus\{0\}$, whence

$$0 \leq \left|x\cos\frac{1}{x}\right| \leq |x|$$

for all $x \in \mathbb{R}$. Consequently, by the Sandwich Theorem, it is enough to prove that $\displaystyle\lim_{x\to 0} |x| = 0$. But this is the same (by the Absolute Value Rule again) as proving that $\displaystyle\lim_{x\to 0} x = 0$, and the latter is one of the basic limits discussed in lectures.

(ii). Let any $a \in \mathbb{R}$ be given. We shall prove that f is continuous at a. In other words, for any given $\epsilon > 0$, we shall prove that there exists $\delta > 0$ such that, for all $x \in \mathbb{R}$,

$$|x-a| < \delta \quad\Longrightarrow\quad |\sqrt[3]{x}-\sqrt[3]{a}| < \epsilon.$$

Let any $\epsilon > 0$ be given. Then, for all $x \in \mathbb{R}$,

$$|\sqrt[3]{x}-\sqrt[3]{a}| < \epsilon \quad\Longleftrightarrow\quad \sqrt[3]{a}-\epsilon < \sqrt[3]{x} < \sqrt[3]{a}+\epsilon \quad\Longleftrightarrow\quad (\sqrt[3]{a}-\epsilon)^3 < x < (\sqrt[3]{a}+\epsilon)^3$$

and the last inequality is equivalent to

$$-(a-(\sqrt[3]{a}-\epsilon)^3) < x-a < (\sqrt[3]{a}+\epsilon)^3 - a. \tag{1}$$

If we take $\delta = \min\{a-(\sqrt[3]{a}-\epsilon)^3, (\sqrt[3]{a}+\epsilon)^3-a\}$, then $\delta > 0$ and, for all $x \in \mathbb{R}$ such that $|x-a| < \delta$, the inequality (1) is satisfied, and consequently (by the previous chain of implications) we also have $|\sqrt[3]{x}-\sqrt[3]{a}| < \epsilon$.

Since $\epsilon > 0$ was arbitrary, this proves that f is continuous at a. Since $a \in \mathbb{R}$ was arbitrary, this proves that f is continuous.

<u>Alternative solution to part (ii).</u> Let any $a \in \mathbb{R}$ be given. We shall prove that f is continuous at a. In other words, we must prove that

$$\forall\epsilon > 0 : \exists\delta > 0 : \forall x \in \mathbb{R} : \left[|x-a| < \delta \Rightarrow |\sqrt[3]{x}-\sqrt[3]{a}| < \epsilon\right].$$

We consider two cases. If $a = 0$, then the condition to prove becomes

$$\forall\epsilon > 0 : \exists\delta > 0 : \forall x \in \mathbb{R} : \left[|x| < \delta \Rightarrow |\sqrt[3]{x}| < \epsilon\right].$$

Let $\epsilon > 0$. Note that, for all $x \in \mathbb{R}$, $|\sqrt[3]{x}| = \sqrt[3]{|x|}$, so

$$|\sqrt[3]{x}| < \epsilon \iff \sqrt[3]{|x|} < \epsilon \iff |x| < \epsilon^3.$$

Hence, if we choose $\delta = \epsilon^3$, we indeed have that $\delta > 0$ and, for all $x \in \mathbb{R}$, if $|x| < \delta$, then $|\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \epsilon$. Since $\epsilon > 0$ was arbitrary, this proves the above condition, that is, f is continuous at 0.

Assume now that $a \neq 0$, and let $\epsilon > 0$. For all $x \in \mathbb{R}$, if we set $X = \sqrt[3]{x}$ and $A = \sqrt[3]{a}$, then, by using the identity $X^3 - A^3 = (X - A)(X^2 + AX + A^2)$, we have

$$\sqrt[3]{x} - \sqrt[3]{a} = X - A = \frac{X^3 - A^3}{X^2 + AX + A^2}.$$

The denominator in the last fraction is a quadratic polynomial, for which a lower bound can be found by "completing the square":

$$X^2 + AX + A^2 = X^2 + AX + \frac{A^2}{4} + \frac{3A^2}{4} = \left(X + \frac{A}{2}\right)^2 + \frac{3A^2}{4} \geq \frac{3A^2}{4}$$

(here we use that $\left(X + \frac{A}{2}\right)^2 \geq 0$ because it is a square). Since $A = \sqrt[3]{a} \neq 0$, this proves that the denominator is strictly positive, and moreover

$$|\sqrt[3]{x} - \sqrt[3]{a}| = \frac{|X^3 - A^3|}{X^2 + AX + A^2} \leq \frac{|X^3 - A^3|}{3A^2/4} = \frac{|x - a|}{3\sqrt[3]{a^2}/4}.$$

In addition

$$\frac{|x - a|}{3\sqrt[3]{a^2}/4} < \epsilon \iff |x - a| < \frac{3}{4}\sqrt[3]{a^2}\epsilon.$$

In light of this discussion, if we choose $\delta = \frac{3}{4}\sqrt[3]{a^2}\epsilon$, then $\delta > 0$ and, for all $x \in \mathbb{R}$, if $|x - a| < \delta$, then

$$|\sqrt[3]{x} - \sqrt[3]{a}| \leq \frac{|x - a|}{3\sqrt[3]{a^2}/4} < \frac{\delta}{3\sqrt[3]{a^2}/4} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves the above condition, i.e., f is continuous at a. Finally, since $a \in \mathbb{R}$ was arbitrary, this proves that f is continuous. $\square$

Q3. Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$.

(i) By using the definition of limit, prove that $\lim_{x \to a} f(x) = 0$ if and only if $\lim_{x \to a} |f(x)| = 0$.

(ii) Assume that $\lim_{x \to a} |f(x)| = \ell$ for some $\ell \in (0, \infty)$. Is it necessarily true that either $\lim_{x \to a} f(x) = \ell$ or $\lim_{x \to a} f(x) = -\ell$? Justify your answer.

Solution. (i). Note that $|f(x) - 0| = |f(x)| = ||f(x)| - 0|$. Hence, if we write the definitions of $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} |f(x)| = 0$, we obtain exactly the same statement in both cases, namely:

for all $\epsilon > 0$, there exists $\delta > 0$ such that,
for all $x \in \mathbb{R}$, if $0 < |x - a| < \delta$, then $|f(x)| < \epsilon$.

This shows that the statements $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} |f(x)| = 0$ are equivalent, that is, the first one holds if and only if the second one holds.

(ii). No. For example, let $a = 0$, $\ell = 1$, and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then $|f(x)| = 1$ for all $x \in \mathbb{R}$, whence $\lim_{x \to 0} |f(x)| = 1$. On the other hand, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$, while $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$, so the limit $\lim_{x \to 0} f(x)$ does not exist (since the two one-sided limits are different). $\square$

(SUM) **Q4**. Determine the value of the following limits. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using. **Only those materials that have been discussed in lectures can be used here. For instance, you can NOT use L'Hospital's rule here**.

(i) $\lim\limits_{x\to 1} \dfrac{x^2-1}{2x^2-x-1}$.

(ii) $\lim\limits_{x\to 0} \dfrac{(x-1)^3+(1-3x)}{x^2+2x^3}$.

(iii) $\lim\limits_{x\to 1} \dfrac{x^n-1}{x^m-1}$, where $n, m \in \mathbb{N}$.

(iv) $\lim\limits_{x\to 4} \dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$.

(v) $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$.

(vi) $\lim\limits_{x\to \infty} x \sin \dfrac{1}{x}$.

Solution. (i). Observe that

$$\frac{x^2-1}{2x^2-x-1} = \frac{(x-1)(x+1)}{(x-1)(2x+1)} = \frac{x+1}{2x+1},$$

then by the Algebra of Limits, we have

$$\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \frac{x+1}{2x+1} = \frac{\lim_{x\to 1} x+1}{2\lim_{x\to 1} x+1} = \frac{2}{3}.$$

(ii). Note that

$$\frac{(x-1)^3+(1-3x)}{x^2+2x^3} = \frac{x^3-3x^2}{x^2+2x^3} = \frac{x^2(x-3)}{x^2(1+2x)} = \frac{x-3}{1+2x}.$$

By using the Algebra of Continuous Functions, so

$$\lim_{x\to 0} \frac{(x-1)^3+(1-3x)}{x^2+2x^3} \lim_{x\to 0} \frac{x-3}{1+2x} = -3.$$

(iii). Note that

$$\frac{x^n-1}{x^m-1} = \frac{(x-1)(x^{n-1}+x^{n-2}+\cdots+1)}{(x-1)(x^{m-1}+x^{m-2}+\cdots+1)} = \frac{x^{n-1}+x^{n-2}+\cdots+1}{x^{m-1}+x^{m-2}+\cdots+1},$$

hence, by the Algebra of Limits,

$$\lim_{x\to 1} \frac{x^n-1}{x^m-1} = \lim_{x\to 1} \frac{x^{n-1}+x^{n-2}+\cdots+1}{x^{m-1}+x^{m-2}+\cdots+1} = \frac{n}{m}.$$

(iv). Note that

$$\begin{aligned}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2} &= \frac{(\sqrt{1+2x}-3)(\sqrt{1+2x}+3)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)} \\ &= \frac{2(x-4)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)} = \frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3},\end{aligned}$$

by the Algebra of continuous function, continuity of $\sqrt{x}$, and continuity of composition of continuous functions, we know that

$$\lim_{x\to 4} \frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3} = \frac{2(\lim_{x\to 4}\sqrt{x}+2)}{\lim_{x\to 4}\sqrt{1+2x}+3} = \frac{4}{3}.$$

Hence, we have,

$$\lim_{x\to 4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}=\lim_{x\to 4}\frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3}=\frac{4}{3}.$$

(v). Note that

$$1-\cos x = 2\sin^2\frac{x}{2}.$$

Therefore, we have

$$\frac{\tan x-\sin x}{x^3}=\frac{(1-\cos x)\sin x}{x^3\cos x}=\frac{2\sin x\sin^2\frac{x}{2}}{x^3\cos x}=\frac{\sin x}{x}\cdot\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2}\cdot\frac{1}{2\cos x}$$

By results from lectures ("notable limits") and algebra of limits, we know that

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\quad \lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2}=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=1.$$

From algebra of limits and the continuous of $\cos x$, we see that

$$\lim_{x\to 0}\frac{1}{2\cos x}=\frac{1}{2\lim_{x\to 0}\cos x}=\frac{1}{2\cos 0}=\frac{1}{2}.$$

Therefore, we have from the product rule that

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}&=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2}\cdot\frac{1}{2\cos x}\\&=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2}\cdot\lim_{x\to 0}\frac{1}{2\cos x}=\exp(0)=\frac{1}{2},\end{aligned}$$

(vi). Note that

$$x\sin\frac{1}{x}=\frac{\sin\frac{1}{x}}{\frac{1}{x}}.$$

By setting $t=\frac{1}{x}$ ("change of variable"), we know that

$$t\to 0,$$

as $x\to\infty$. Therefore, we have

$$\lim_{x\to\infty}x\sin\frac{1}{x}=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=\lim_{t\to 0}\frac{\sin t}{t}=1,$$

where we used the "notable limit" in the last step. □

Q5. Does $\lim_{x\to 0}\sin\frac{1}{x}$ exist? Justify your answer.

Solution. No, the limit does not exist.

For a contradiction, assume that $\lim_{x\to 0}\sin\frac{1}{x}=\ell$ for some $\ell\in\overline{\mathbb{R}}$. Note that $-1\le\sin(1/x)\le 1$ for all $x\in\mathbb{R}\setminus\{0\}$. Since, by a result from lectures, inequalities of the form "$\le$" pass to the limit, we deduce that $-1\le\ell\le 1$, and in particular ℓ is a real number.

Since the range of the function $\sin:\mathbb{R}\to\mathbb{R}$ is the whole interval $[-1,1]$, we can find $a\in\mathbb{R}$ such that $\sin a\neq\ell$. Consequently, for all $n\in\mathbb{N}$, if $x_n=1/(a+2\pi n)$, then $\sin(1/x_n)=\sin(a+2\pi n)=\sin a$, and therefore

$$|\sin(1/x_n)-\ell|=|\sin a-\ell|>0.$$

Now, let $\epsilon=|\sin a-\ell|$. Since $\lim_{x\to 0}\sin\frac{1}{x}=\ell$, by the definition of limit we deduce that there exists $\delta>0$ such that, for all $x\in\mathbb{R}\setminus\{0\}$,

$$0<|x|<\delta\quad\Longrightarrow\quad|\sin(1/x)-\ell|<\epsilon. \tag{2}$$

On the other hand, if we choose $n \in \mathbb{N}$ sufficently large, then $0 < x_n < \delta$ (indeed, it is enough to take n such that $a + 2\pi n > 1/\delta$, that is, $n > (1/\delta - a)/(2\pi)$), but, at the same time, $|\sin(1/x_n) - \ell| = |\sin a - \ell| = \epsilon$; so, by taking $x = x_n$, we reach a contradiction to (2). $\square$

Q6. This is a guided proof of some statements in Section 4.5 of the lecture notes.

(i) By using the definition of limit, prove that

$$\lim_{x\to\infty} \log x = \infty,$$
$$\lim_{x\to\infty} \exp(x) = \infty.$$

(ii) Deduce from part (i) that

$$\lim_{x\to 0^+} \log x = -\infty,$$
$$\lim_{x\to -\infty} \exp(x) = 0,$$

and that, for all $b \in (0,\infty)$,

$$\lim_{x\to\infty} x^b = \infty,$$
$$\lim_{x\to 0} x^b = 0.$$

[Hint: change of variable in limits, $x^b = \exp(b \log x)$.]

Solution. (i). Note that the domain of the function log is $(0,\infty)$. To prove that $\lim_{x\to\infty} \log x = \infty$, by the definition of limit we must prove that

$$\forall M > 0 : \exists N \in \mathbb{R} : \forall x \in (0,\infty) : (x > N \Rightarrow \log x > M)\,. \tag{3}$$

Let $M > 0$, and note that, for all $x \in (0,\infty)$, we have $\log x > M$ if and only if $x > e^M$. Hence, if we take $N = e^M$, then the implication $x > N \Rightarrow \log x > M$ holds true. Since $M > 0$ was arbitrary, this proves (3).

Similarly, by the definition of limit, proving that $\lim_{x\to\infty} \exp(x) = \infty$ is the same as proving that

$$\forall M > 0 : \exists N \in \mathbb{R} : \forall x \in \mathbb{R} : (x > N \Rightarrow \exp(x) > M)\,. \tag{4}$$

Let $M > 0$, and note that, for all $x \in \mathbb{R}$, we have $\exp(x) > M$ if and only if $x > \log M$. Hence, if we take $N = \log M$, then the implication $x > N \Rightarrow \exp(x) > M$ holds true. Since $M > 0$ was arbitrary, this proves (4).

(ii). Note that, for all $x \in (0,\infty)$, we have $\log x = -\log(1/x)$, and that $\lim_{x\to 0^+} 1/x = \infty$ by the Algebra of Limits. Hence, by a change of variables and the Algebra of Limits, we conclude that

$$\lim_{x\to 0^+} \log x = -\lim_{x\to 0^+} \log \frac{1}{x} = -\lim_{y\to\infty} \log y = -\infty, \tag{5}$$

where the last equality is justified in part (i).

Similarly, note that $\exp(x) = 1/\exp(-x)$, and $\lim_{x\to\infty}(-x) = -\infty$ by the Algebra of Limits. Hence, by a change of variables and the Algebra of Limits,

$$\lim_{x\to -\infty} \exp(x) = \lim_{x\to -\infty} \frac{1}{\exp(-x)} = \lim_{y\to\infty} \frac{1}{\exp(y)} = 0, \tag{6}$$

where we used that $\lim_{y\to\infty} \exp(y) = \infty$ by part (i).

Let $b \in (0,\infty)$. We now observe that $x^b = \exp(b \log x)$, for all $x \in (0,\infty)$. From part (i) and the Algebra of Limits we deduce that

$$\lim_{x\to\infty} (b \log x) = b \lim_{x\to\infty} \log x = b \cdot \infty = \infty.$$

Consequently, by a change of variable,

$$\lim_{x\to\infty} x^b = \lim_{x\to\infty} \exp(b\log x) = \lim_{y\to\infty} \exp(y) = \infty,$$

where the last equality was proved in part (i).

Finally, from (5) and the Algebra of Limits we deduce that

$$\lim_{x\to 0^+} (b\log x) = b \lim_{x\to 0^+} \log x = b\cdot(-\infty) = -\infty$$

(note that $b > 0$). A change of variables then yields

$$\lim_{x\to 0^+} x^b = \lim_{x\to 0^+} \exp(b\log x) = \lim_{y\to -\infty} \exp(y) = 0,$$

where the last equality was proved in (6). □

Q7. Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $\displaystyle\lim_{x\to 0} \frac{7x}{\sin(4x)}$.

(ii) $\displaystyle\lim_{x\to 2} \frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{2x-1}-\sqrt{x+1}}$.

(iii) $\displaystyle\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$.

(iv) $\displaystyle\lim_{x\to 0} \frac{e^{2x}-1}{\log(1-5x)}$.

(v) $\displaystyle\lim_{x\to 0^+} x^x$.

Solution. (i). Observe that

$$\frac{7x}{\sin(4x)} = \frac{7}{4}\frac{4x}{\sin(4x)} = \frac{7}{4}\frac{1}{f(4x)},$$

where $f(y) = \frac{\sin y}{y}$ for $y \neq 0$. Note that $\lim_{x\to 0} 4x = 4\cdot 0 = 0$ (by the Algebra of Limits), and moreover $4x \neq 0$ whenever $x \neq 0$. From lectures ("notable limits") we know that

$$\lim_{y\to 0} f(y) = 1,$$

hence, by a change of variables,

$$\lim_{x\to 0} f(4x) = 1$$

as well, and finally, by the Algebra of Limits,

$$\lim_{x\to 0} \frac{7x}{\sin(4x)} = \lim_{x\to 0}\left(\frac{7}{4}\frac{1}{f(4x)}\right) = \frac{7}{4}\cdot\frac{1}{1} = \frac{7}{4}.$$

(ii). Note that

$$\sqrt{3x-2}-\sqrt{5x-6} = \frac{(3x-2)-(5x-6)}{\sqrt{3x-2}+\sqrt{5x-6}} = \frac{-2x+4}{\sqrt{3x-2}+\sqrt{5x-6}},$$
$$\sqrt{2x-1}-\sqrt{x+1} = \frac{(2x-1)-(x+1)}{\sqrt{2x-1}+\sqrt{x+1}} = \frac{x-2}{\sqrt{2x-1}+\sqrt{x+1}},$$

so

$$\frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{2x-1}-\sqrt{x+1}} = \frac{-2x+4}{\sqrt{3x-2}+\sqrt{5x-6}} \cdot \frac{\sqrt{2x-1}+\sqrt{x+1}}{x-2}$$
$$= -2\frac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{3x-2}+\sqrt{5x-6}}.$$

By using the Algebra of Continuous Functions, the fact that composition of continuous functions is continuous, and the continuity of the function $y \mapsto \sqrt{y}$, we deduce that $x \mapsto -2\frac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{3x-2}+\sqrt{5x-6}}$ is continuous at 2, so

$$\lim_{x\to 2} \frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{2x-1}-\sqrt{x+1}} = \lim_{x\to 2}\left(-2\frac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{3x-2}+\sqrt{5x-6}}\right) = -2\frac{\sqrt{3}+\sqrt{3}}{\sqrt{4}+\sqrt{4}} = -\sqrt{3}.$$

(iii). Note that

$$\frac{x\sin x}{1-\cos x} = \frac{x^2}{1-\cos x}\frac{\sin x}{x} = \frac{f(x)}{g(x)},$$

where $f(x) = \frac{\sin x}{x}$ and $g(x) = \frac{1-\cos x}{x^2}$. From lectures (“notable limits”) we know that

$$\lim_{x\to 0} f(x) = 1, \qquad \lim_{x\to 0} g(x) = 1/2,$$

hence, by the Algebra of Limits,

$$\lim_{x\to 0}\frac{x\sin x}{1-\cos x} = \lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{1}{1/2} = 2.$$

(iv). Note that

$$\frac{e^{2x}-1}{\log(1-5x)} = -\frac{2}{5}\frac{e^{2x}-1}{2x}\frac{-5x}{\log(1-5x)} = -\frac{2}{5}\frac{f(2x)}{g(-5x)},$$

where $f(y) = (e^y-1)/y$ and $g(y) = \log(1+y)/y$. Note that both $2x$ and $-5x$ tend to 0 as $x \to 0$ (by the Algebra of Limits), and moreover $2x \neq 0$ and $-5x \neq 0$ whenever $x \neq 0$. Further, from lectures (“notable limits”) we know that

$$\lim_{y\to 0} f(y) = 1, \qquad \lim_{y\to 0} g(y) = 1.$$

Hence, by two changes of variables,

$$\lim_{x\to 0} f(2x) = 1, \qquad \lim_{x\to 0} g(-5x) = 1$$

as well, and finally, by the Algebra of Limits,

$$\lim_{x\to 0}\frac{e^{2x}-1}{\log(1-5x)} = \lim_{x\to 0}\left(-\frac{2}{5}\frac{f(2x)}{g(-5x)}\right) = -\frac{2}{5}\cdot\frac{1}{1} = -\frac{2}{5}.$$

(v). Note that

$$x^x = \exp(x\log x).$$

By a result from lectures (“notable limits”), we know that

$$\lim_{x\to 0^+} x\log x = 0.$$

Since exp is continuous at 0, by another result from lectures we conclude that

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+}\exp(x\log x) = \exp(0) = 1,$$

□

Q8. Prove the following statements by directly using the definition of derivative.

(i) If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x^2 + 5$ for all $x \in \mathbb{R}$, then $f'(x) = 6x$.

(ii) If $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by $g(x) = 3/(1-x)$ for all $x \in \mathbb{R} \setminus \{1\}$, then $g'(x) = 3/(1-x)^2$.

(iii) If $k : \mathbb{R} \to \mathbb{R}$ is given by $k(x) = \sin(2x)$ for all $x \in \mathbb{R}$, then $k'(0) = 2$.

Solution. (i). Let $x, h \in \mathbb{R}$, and observe that (provided $h \neq 0$)

$$\frac{f(x+h) - f(x)}{h} = \frac{(3(x+h)^2 + 5) - (3x^2 + 5)}{h} = 6x + 3h.$$

Hence, by the Algebra of Limits,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h) = 6x.$$

(ii). Let $x, h \in \mathbb{R}$, and observe that (provided $x \neq 1$, $h \neq 0$, and $x + h \neq 1$)

$$\frac{g(x+h) - g(x)}{h} = \frac{3/(1-(x+h)) - 3/(1-x)}{h} = \frac{3}{(1-x)(1-(x+h))}.$$

Hence, by the Algebra of Limits,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{3}{(1-x)(1-(x+h))} = \frac{3}{(1-x)^2}.$$

(iii). Let $h \in \mathbb{R} \setminus \{0\}$, and observe that

$$\frac{k(h) - k(0)}{h} = \frac{\sin(2h)}{h} = 2r(2h),$$

where $r(y) = \frac{\sin(y)}{y}$. Note that $2h$ tends to 0 as $h \to 0$ (by the Algebra of Limits), and $2h \neq 0$ whenever $h \neq 0$. So, by a change of variables and a result from lectures ("notable limits"),

$$\lim_{h \to 0} r(2h) = \lim_{y \to 0} r(y) = 1,$$

and consequently, by the Algebra of Limits,

$$k'(0) = \lim_{h \to 0} \frac{k(h) - k(0)}{h} = \lim_{h \to 0} (2r(2h)) = 2 \cdot 1 = 2.$$

□

(SUM) **Q9**. Let a_1, a_2, a_3 be positive real numbers, and $\lambda_1 < \lambda_2 < \lambda_3$. Prove the following equation

$$\frac{a_1}{x - \lambda_1} + \frac{a_2}{x - \lambda_2} + \frac{a_3}{x - \lambda_3} = 0$$

has solutions on both intervals (λ_1, λ_2) and (λ_2, λ_3).

Solution. Let

$$f(x) = a_1(x - \lambda_2)(x - \lambda_3) + a_2(x - \lambda_1)(x - \lambda_3) + a_3(x - \lambda_1)(x - \lambda_2)$$

which is a fundamental function, and thus continuous. Since $a_1, a_2, a_3 > 0$ and $\lambda_1 < \lambda_2 < \lambda_3$, we note that ("sign analysis")

$$\begin{aligned} f(\lambda_1) &= a_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) > 0, \\ f(\lambda_2) &= a_2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3) < 0, \\ f(\lambda_3) &= a_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) < 0, \end{aligned}$$

By using the intermediate value theorem, there exist $\xi_1 \in (\lambda_1, \lambda_2)$ and $\xi_2 \in (\lambda_2, \lambda_3)$ such that

$$f(\xi_1) = f(\xi_2) = 0.$$

We also note that

$$\begin{aligned} g(x) &= \frac{a_1}{x - \lambda_1} + \frac{a_2}{x - \lambda_2} + \frac{a_3}{x - \lambda_3} \\ &= \frac{f(x)}{(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)} \end{aligned}$$

and

$$g(\xi_1) = \frac{f(\xi_1)}{(\xi_1 - \lambda_1)(\xi_1 - \lambda_2)(\xi_1 - \lambda_3)} = 0$$
$$g(\xi_2) = \frac{f(\xi_2)}{(\xi_2 - \lambda_1)(\xi_2 - \lambda_2)(\xi_2 - \lambda_3)} = 0$$

which finishes the proof.

Another proof (ideas), for instance we see that

$$\lim_{x \to \lambda_1^+} g(x) = \infty$$
$$\lim_{x \to \lambda_2^-} g(x) = -\infty$$

which implies (by using the definition of limits) the existence of $x_1, x_2 \in (\lambda_1, \lambda_2)$ and $x_1 < x_2$ such that $f(x_1) > 0$ and $f(x_2) < 0$. Then we apply the intermediate value theorem to conclude there exists $\xi_1 \in (\lambda_1, \lambda_2)$ such that $g(\xi) = 0$. Similar argument for the interval (λ_2, λ_3).

Marking scheme: 40 marks for this question. Either proofs are fine. Full marks for those of complete (with some detailed computation) and correct works. They also need to clearly state what they are using in their computation, such as the definition of limits, intermediate value theorem, etc. □

Q10. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) If $f : (0, 1) \to \mathbb{R}$ is continuous, then f is bounded.
(ii) If $g : (0, 1) \to \mathbb{R}$ is continuous, then g is differentiable.
(iii) If $k : [0, 1] \to \mathbb{R}$ is differentiable, then k is bounded.

Solution. (i). The statement is false. For example, if $f : (0, 1) \to \mathbb{R}$ is defined by $f(x) = 1/x$ for all $x \in (0, 1)$, then f is continuous (by the Algebra of Continuous Functions), but f is unbounded (indeed $f((0, 1)) = (1, \infty)$, and therefore $\sup f = \infty$).

(ii). The statement is false. For example, if $g : (0, 1) \to \mathbb{R}$ is defined by $g(x) = |x - 1/2|$, then g is continuous (since $x \mapsto |x|$ is continuous and composition of continuous functions is continuous); however g is not differentiable at $1/2$, because, for all $h \in \mathbb{R} \setminus \{0\}$,

$$\frac{f(1/2 + h) - f(1/2)}{h} = \frac{|h|}{h}$$

and the latter expression has no limit as $h \to 0$ (the one-sided limits are ± 1).

(iii). The statement is true. Indeed, by a result from lectures, if $k : [0, 1] \to \mathbb{R}$ is differentiable, then k is continuous; moreover, by the Boundedness Theorem, if $k : [0, 1] \to \mathbb{R}$ is continuous, then it is bounded. □

Extra Questions

EQ1. Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $\lim_{x \to -\infty} 2x^2 - 3x + \arctan x$.

(ii) $\lim_{x \to 2} \dfrac{1}{1 - x}$.

(iii) $\lim_{x\to 1/2} \frac{4x^2-1}{2x-1}$.

Solution. (i). We claim that $\lim_{x\to-\infty} 2x^2-3x+\arctan x = \infty$. In order to show this, we first observe that $\arctan x \le \pi/2$ for all $x \in \mathbb{R}$, whence

$$2x^2 - 3x + \arctan x \ge 2x^2 - 3x - \pi/2.$$

By the Sandwich Theorem for Infinite Limits, it is then enough to prove that $\lim_{x\to-\infty} 2x^2 - 3x - \pi/2 = \infty$. On the other hand,

$$2x^2 - 3x - \pi/2 = x^2(2 - 3/x - \pi/(2x^2)).$$

Since $\lim_{x\to-\infty} x = -\infty$, by repeatedly applying the Algebra of Limits we obtain that $\lim_{x\to-\infty} 1/x = 0$, whence

$$\lim_{x\to-\infty} x^2 = (-\infty)\cdot(-\infty) = \infty, \qquad \lim_{x\to-\infty} 2-3/x-\pi/(2x^2) = 2-3\cdot 0-(\pi/2)\cdot 0^2 = 2$$

and finally

$$\lim_{x\to-\infty} 2x^2 - 3x - \pi/2 = \lim_{x\to-\infty} x^2(2 - 3/x - \pi/(2x^2)) = \infty \cdot 2 = \infty.$$

(ii) We claim that $\lim_{x\to 2} \frac{1}{1-x} = -1$. This is an immediate consequence of the Algebra of Limits, since $\lim_{x\to 2} x = 2$, whence

$$\lim_{x\to 2} \frac{1}{1-x} = \frac{1}{1-2} = -1.$$

(iii) We claim that $\lim_{x\to 1/2} \frac{4x^2-1}{2x-1} = 2$. Indeed, for all $x \neq 1/2$,

$$\frac{4x^2-1}{2x-1} = \frac{(2x-1)(2x+1)}{2x-1} = 2x+1;$$

since moreover $\lim_{x\to 1/2} x = 1/2$, by the Algebra of Limits we deduce that

$$\lim_{x\to 1/2} \frac{4x^2-1}{2x-1} = \lim_{x\to 1/2} 2x+1 = 2(1/2)+1 = 2.$$

□

EQ2. Let $A \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ be continuous. Let $a \in A$. Prove that, if $f(a) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in A \cap (a-\delta, a+\delta)$.
[This is sometimes called the “sign-preserving property” of continuous functions: informally, if a continuous function is positive at a certain point, then it is also positive at nearby points.]

Solution. By our assumptions, we know that f is continuous at a, which means, by definition, that

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in A : (|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)\,. \tag{7}$$

Since we know that $f(a) > 0$, we can apply (7) with $\epsilon = f(a)$ and obtain the corresponding $\delta > 0$. We now notice that, if $x \in A \cap (a-\delta, a+\delta)$, then $x \in A$ and $|x-a| < \delta$, and therefore from (7) we deduce that

$$\begin{aligned} f(a) - f(x) &\le |f(x) - f(a)| && \text{(by properties of the absolute value)} \\ &< \epsilon && \text{(by (7))} \\ &= f(a), && \text{(by our choice of } \epsilon) \end{aligned}$$

that is, $f(a) - f(x) < f(a)$, which, rearranged, gives $f(x) > 0$, as desired. □

* **EQ3**. Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $f : (a, b) \to \mathbb{R}$ be increasing. Let $\ell = \sup f$, and assume first that $\ell \in \mathbb{R}$.

(i) Prove that, for all $\epsilon > 0$, there exists $c \in (a, b)$ such that $f(c) > \ell - \epsilon$. [Hint: $\sup f$ is the minimum of the upper bounds of the range of f; can $\ell - \epsilon$ be an upper bound as well?]

(ii) Prove that, for all $\epsilon > 0$, there exists $c \in (a, b)$ such that

$$\ell - \epsilon < f(x) \leq \ell$$

for all $x \in (a, b)$ such that $x > c$. [Hint: f is monotone.]

(iii) Prove that $\lim_{x \to b} f(x) = \ell$.

Assume now instead that $\ell = \infty$.

(iv) Is it still true that $\lim_{x \to b} f(x) = \ell$ in this case? Justify your answer.

Solution. (i). Let $\epsilon > 0$. By definition, $\ell = \sup f$ is the least upper bound of the range of f. Since $\ell - \epsilon < \ell$, we conclude that $\ell - \epsilon$ is not an upper bound of the range of f, that is, there exists an element $y \in f((a, b))$ such that $y > \ell - \epsilon$. By definition of $f((a, b))$, there exists $c \in (a, b)$ such that $y = f(c)$, and the desired inequality $f(c) > \ell - \epsilon$ follows.

(ii). Let $\epsilon > 0$, and let $c \in (a, b)$ be constructed as in part (i), so that $f(c) > \ell - \epsilon$. Since f is increasing, we conclude that, for all $x \in (a, b)$ such that $x > c$, we also have $f(x) \geq f(c) > \ell - \epsilon$. On the other hand, for any such x, the value $f(x)$ belongs to the range of f, and ℓ is an upper bound of the range, so $f(x) \leq \ell$ as well.

(iii). According to the definition of limit, we must prove that

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in (a, b) : (|x - b| < \delta \Rightarrow |f(x) - \ell| < \epsilon)\,. \tag{8}$$

Let $\epsilon > 0$. From part (ii), we know that there exists $c \in (a, b)$ such that $\ell - \epsilon < f(x) \leq \ell$ for all $x \in (a, b)$ such that $x > c$; in particular, for those x, we have $f(x) - \ell \leq 0$ and

$$|f(x) - \ell| = \ell - f(x) < \epsilon. \tag{9}$$

Let $\delta = b - c$. Then $\delta > 0$; moreover, for all $x \in (a, b)$, if $|x - b| < \delta$, then $b - x < \delta$ and $x > b - \delta = c$; we can then apply the inequality (9) to such x and obtain that $|f(x) - \ell| < \epsilon$, as desired. Since $\epsilon > 0$ was arbitrary, this proves (8).

(iv). Yes, the identity $\lim_{x \to b} f(x) = \ell$ holds also in the case where $\ell = \infty$. To prove this, we need to show that

$$\forall M > 0 : \exists \delta > 0 : \forall x \in (a, b) : (|x - b| < \delta \Rightarrow f(x) > M)\,.$$

Let $M > 0$. Since $\sup f = \infty$, that is, f is unbounded above, the number M cannot be an upper bound of the range of f, hence there exists (as in part (i)) a point $c \in (a, b)$ such that $f(c) > M$. Since f is increasing, we then deduce (as in part (ii)) that $f(x) > M$ for all $x > c$. If we take $\delta = b - c$, then (arguing as in part (iii)) we can show that, for all $x \in (a, b)$ such that $|x - b| < \delta$, we also have $x > b - \delta = c$, and therefore $f(x) > M$. Since $M > 0$ was arbitrary, this proves the desired statement. □

EQ4. This question is aimed at proving that trigonometric functions are continuous.

(i) Recall from Lemma 4.20 the inequality

$$\sin\theta \leq \theta \qquad \text{for all } \theta \in [0, \pi/2).$$

Prove that

$$|\sin\theta| \leq |\theta| \qquad \text{for all } \theta \in (-\pi/2, \pi/2).$$

(ii) Deduce that the function sin is continuous at 0. [Hint: Sandwich Theorem.]

(iii) Deduce that the function cos is continuous at 0. [Hint: $\cos x = 1-2\sin^2(x/2)$.]

(iv) Prove that, for all $a \in \mathbb{R}$,

$$\lim_{h\to 0} \sin(a+h) = \sin a.$$

[Hint: Angle Sum Formula.]

(v) Deduce that the function sin is continuous at every $a \in \mathbb{R}$.

(vi) Conclude that the functions cos, tan, cot are continuous too.

Solution. (i). Let $\theta \in (-\pi/2, \pi/2)$. If $\theta \in [0, \pi/2)$, then $\sin\theta \geq 0$, so

$$|\sin\theta| = \sin\theta \leq \theta = |\theta|, \tag{10}$$

where the inequality from Lemma 4.20 was used. If instead $\theta \in (-\pi/2, 0)$, then we use the fact that the function sin is odd (that is, $\sin(-\theta) = -\sin\theta$) and moreover $-\theta \in (0, \pi/2)$ to deduce that

$$\begin{aligned}
|\sin\theta| &= |-\sin(-\theta)| && \text{(sin is odd)} \\
&= |\sin(-\theta)| \\
&\leq |-\theta| && \text{(inequality (10) applied to } -\theta) \\
&= |\theta|,
\end{aligned}$$

as desired.

(ii). By the characterisation of continuous functions in terms of limits, we need to prove that $\lim_{x\to 0} \sin x = 0$. By the Absolute Value Rule for Null Limits, this is the same as proving that $\lim_{x\to 0} |\sin x| = 0$. In view of the inequality in part (i), we have

$$0 \leq |\sin x| \leq |x|$$

for all $x \in (-\pi/2, \pi/2)$, and therefore, by the Sandwich Theorem, it is enough to prove that $\lim_{x\to 0} |x| = 0$; this however is a consequence of the fact that the modulus function $x \mapsto |x|$ is continuous and $|0| = 0$.

(iii). By the Double Angle Formulas,

$$\cos x = 1 - 2\sin^2(x/2) \tag{11}$$

for all $x \in \mathbb{R}$. Since $x \mapsto x/2$ is continuous (it is a polynomial, so it is continuous the Algebra of Continuous Functions) and maps 0 to 0, and moreover the function sin is continuous at 0 by part (ii), by the Composition Rule for continuous functions we deduce that $x \mapsto \sin(x/2)$ is continuous at 0 too. The Algebra of Continuous Functions and the identity (11) then allow us to conclude that $x \mapsto \cos x$ is continuous at 0 as well.

(iv). By the Angle Sum Formulas,

$$\sin(a+x) = \sin a \cos x + \cos a \sin x$$

for all $x \in \mathbb{R}$. Note that $\sin a$ and $\cos a$ are constants (that is, they do not depend on x), while $x \mapsto \sin x$ and $x \mapsto \cos x$ are both continuous at 0 by parts (ii) and (iii). Hence, by the Algebra of Continuous Functions, the function $x \mapsto \sin(a+x)$ is continuous at 0 too, and therefore

$$\lim_{x\to 0} \sin(a+x) = \sin(a+0) = \sin a,$$

as desired.

(v). This is an immediate consequence of part (iv) and the characterisation of continuous functions in terms of limits.

(vi). By repeating the argument in part (iii), since the function sin is continuous (by part (v)) and $x \mapsto x/2$ is continuous too, the Composition Rule allows to deduce that $x \mapsto \sin(x/2)$ is continuous; hence, by the formula (11) and the Algebra of Continuous Functions, we deduce that the function cos is continuous.

Finally, the function tan is the quotient of sin and cos, while the function cot is the quotient of cos and sin; since the quotient of two continuous functions is continuous (by the Algebra of Continuous Functions), from the continuity of sin and cos we deduce that of tan and cot. □

EQ5. This is a guided proof of Proposition 4.23 of the lecture notes.

(i) Recall from Proposition 1.24 the inequality

$$2^n \geq n+1 \qquad \text{for all } n \in \mathbb{N}_0. \tag{12}$$

From this inequality, or otherwise, deduce that

$$\exp(x) \geq x$$

for all $x \geq 0$. [Hint: $e^x \geq 2^x \geq 2^{\lfloor x \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer smaller than or equal to x.]

(ii) Deduce that, for all $b > 0$, there exists $C > 0$ (depending on b) such that

$$\exp(x) \geq Cx^b$$

for all $x \geq 0$. [Hint: $\exp(x) = (\exp(x/b))^b$.]

(iii) Using the previous inequality, or otherwise, prove that, for all $b > 0$,

$$\lim_{x\to\infty} \frac{\exp(x)}{x^b} = \infty.$$

[Hint: "Sandwich Theorem for Infinite Limits".]

(iv) Using the previous result, or otherwise, prove that, for all $b > 0$,

$$\lim_{x\to-\infty} |x|^b \exp(x) = 0, \qquad \lim_{x\to\infty} \frac{\log x}{x^b} = 0, \qquad \lim_{x\to 0^+} x^b \log x = 0.$$

[Hint: change of variables in limits ($x = \exp(\log x)$, $\log x = -\log \frac{1}{x}$).]

Solution. (i). Since $e \geq 2$, we have that $e^x \geq 2^x \geq 2^{\lfloor x \rfloor}$; so, by applying (12) with $n = \lfloor x \rfloor$ we deduce that

$$\exp(x) \geq 2^{\lfloor x \rfloor} \geq \lfloor x \rfloor + 1 \geq x.$$

(ii). Note that $e^x = (e^{x/b})^b$. So, by part (i) applied with x/b in place of x, we deduce that, for all $x \geq 0$,

$$\exp(x) = (\exp(x/b))^b \geq (x/b)^b = b^{-b}x^b,$$

and the desired inequality follows by taking $C = b^{-b}$.

(iii). Let $b > 0$. Note that, by part (ii), applied with $b+1$ in place of b, there exists $C > 0$ such that

$$\exp(x) \geq Cx^{b+1}$$

for all $x \geq 0$, whence

$$\frac{\exp(x)}{x^b} \geq Cx. \tag{13}$$

From the Algebra of Limits it is clear that the right-hand side of (13) tends to ∞ as $x \to \infty$, whence (by the "Sandwich Theorem for Infinite Limits")

$$\lim_{x\to\infty} \frac{\exp(x)}{x^b} = \infty$$

as well.

(iv). Note that, for all $x < 0$,

$$|x|^b \exp(x) = \frac{(-x)^b}{\exp(-x)} = \frac{1}{f(-x)},$$

where $f(y) = \frac{\exp(y)}{y^b}$. Since $\lim_{x\to-\infty}(-x) = -(-\infty) = \infty$ (by the Algebra of Limits), and we know that $\lim_{y\to\infty} f(y) = \infty$ from part (iii), by a change of variables we deduce that

$$\lim_{x\to-\infty} f(-x) = \infty,$$

and therefore, by the Algebra of Limits,

$$\lim_{x\to-\infty} |x|^b \exp(x) = \lim_{x\to-\infty} \frac{1}{f(-x)} = 0.$$

Note now that, for all $x > 0$,

$$\frac{\log x}{x^b} = \frac{\log x}{(\exp(\log x))^b} = \frac{1}{b}\frac{b\log x}{\exp(b\log x)} = \frac{1}{b}\frac{1}{f(b\log x)},$$

where $f(y) = \frac{e^y}{y}$ as before. Moreover

$$\lim_{x\to\infty} \log x = \infty$$

(see **Q6**), and therefore

$$\lim_{x\to\infty} (b\log x) = b\cdot\infty = \infty$$

by the Algebra of Limits, because $b > 0$. Since $\lim_{y\to\infty} f(y) = \infty$ (by part (iii)), by a change of variables we conclude that

$$\lim_{x\to\infty} f(b\log x) = \infty$$

as well, and consequently

$$\lim_{x\to\infty} \frac{\log x}{x^b} = \lim_{x\to\infty} \left(\frac{1}{b}\frac{1}{f(b\log x)}\right) = \frac{1}{b}\cdot 0 = 0,$$

by the Algebra of Limits.

Finally, note that, for all $x > 0$,

$$x^b \log x = -\frac{\log(1/x)}{(1/x)^b} = -h(1/x),$$

where $h(y) = \frac{\log y}{y^b}$. Note also that $\lim_{x\to 0^+}(1/x) = \infty$ by the Algebra of Limits, and that $\lim_{y\to\infty} h(y) = 0$ (as we have just proved). Therefore, by a change of variables, $\lim_{x\to 0^+} h(1/x) = 0$ as well, and

$$\lim_{x\to 0^+} x^b \log x = \lim_{x\to 0^+} -h(1/x) = -0 = 0$$

by the Algebra of Limits. $\square$

EQ6. Let $a, b, c \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial

$$f(x) = x^3 + ax^2 + bx + c.$$

(i) Prove that

$$\lim_{x\to-\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty.$$

(ii) Prove that there exist $x_1, x_2 \in \mathbb{R}$ such that

$$f(x_1) < 0 \qquad \text{and} \qquad f(x_2) > 0.$$

[Hint: definition of limit.]

(iii) Prove that there exists $x_0 \in \mathbb{R}$ such that

$$f(x_0) = 0.$$

[Hint: Intermediate Value Theorem.]

[Note: This shows that every polynomial of degree 3 has a zero in $\mathbb{R}$. A similar argument proves that every polynomial of odd degree has a zero in $\mathbb{R}$; in turn this fact can be used as a starting point for a proof of the Fundamental Theorem of Algebra.]

Solution. (i). We can write, for all $x \in \mathbb{R} \setminus \{0\}$,

$$f(x) = x^3(1 + a/x + b/x^2 + c/x^3).$$

We know that $\lim_{x\to\pm\infty} x = \pm\infty$, and therefore $\lim_{x\to\pm\infty} 1/x = 0$ by the Algebra of Limits. Again by the Algebra of Limits we deduce that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^3(1 + a/x + b/x^2 + c/x^3) = \infty^3 \cdot (1 + a \cdot 0 + b \cdot 0^2 + c \cdot 0^3) = \infty$$

and similarly

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3(1+a/x+b/x^2+c/x^3) = (-\infty)^3 \cdot (1+a\cdot 0+b\cdot 0^2+c\cdot 0^3) = -\infty.$$

(ii). Since $\lim_{x\to\infty} f(x) = \infty$, by the definition of limit we know that

$$\forall M > 0 : \exists N \in \mathbb{R} : \forall x \in \mathbb{R} : (x > N \Rightarrow f(x) > M)\,.$$

In particular, if we take $M = 1$, we deduce that there exists $N \in \mathbb{R}$ such that $f(x) > 1$ for all $x > N$. In particular, if $x_2 = N + 1$, we obtain $f(x_2) > 1 > 0$. Similarly, from $\lim_{x\to-\infty} f(x) = -\infty$ and the definition of limit we know that

$$\forall M' > 0 : \exists N' \in \mathbb{R} : \forall x \in \mathbb{R} : (x < N' \Rightarrow f(x) < -M')\,;$$

hence, if we take $M' = 1$, then we deduce that there exists $N' \in \mathbb{R}$ such that $f(x) < -1$ for all $x < N$, and by taking $x_1 = N' - 1$ we obtain that $f(x_1) < -1 < 0$.

(iii). Let $x_1, x_2 \in \mathbb{R}$ be as in part (ii). Clearly $x_1 \neq x_2$, since $f(x_1)$ and $f(x_2)$ have opposite signs. If $x_1 < x_2$, then the function $g = f|_{[x_1,x_2]}$ is continuous (note that, by the Algebra of Continuous Functions, every polynomial is continuous) and $g(x_1) < 0 < g(x_2)$; so by the Intermediate Value Theorem we deduce that there exists $x_0 \in [x_1, x_2]$ such that $g(x_0) = 0$, and consequently $f(x_0) = 0$ too. If instead $x_1 > x_2$, then we can argue similarly by applying the Intermediate Value Theorem to $f|_{[x_2,x_1]}$. □

* **EQ7**. Let $A \subseteq \mathbb{R}$. Recall that, by the Leibniz rule, if $f, g : A \to \mathbb{R}$ are differentiable, then their product fg is differentiable too, and the formula

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

holds for all $x \in \mathbb{R}$.

(i) Let $f_1, f_2, f_3 : A \to \mathbb{R}$ be differentiable functions. Prove that their product $f_1f_2f_3$ is differentiable too, and find a formula for its derivative $(f_1f_2f_3)'$. [Hint: $f_1f_2f_3 = (f_1f_2)f_3$.]

(ii) Let $n \in \mathbb{N}$, and let $f_1, f_2, \dots, f_n : A \to \mathbb{R}$ be differentiable functions. Prove that their product $f_1f_2\cdots f_n$ is differentiable. [Hint: induction on n.]

(iii) Can you write a formula for the derivative $(f_1f_2\cdots f_n)'$?

Solution. (i). Let $h = f_1f_2$ be the product of f_1 and f_2. Since both f_1 and f_2 are differentiable, by the Leibniz rule their product h is differentiable too, and

$$h' = f_1'f_2 + f_1f_2'.$$

Note now that $f_1f_2f_3 = (f_1f_2)f_3 = hf_3$ is the product of f and f_3. Since both h and f_3 are differentiable, again by the Leibniz rule we conclude that their product hf_3 is differentiable, and

$$\begin{aligned}(f_1f_2f_3)' &= (hf_3)' = h'f_3 + hf_3' \\ &= (f_1'f_2 + f_1f_2')f_3 + (f_1f_2)f_3' = f_1'f_2f_3 + f_1f_2'f_3 + f_1f_2f_3'.\end{aligned}$$

In other words, for all $x \in A$,

$$(f_1 f_2 f_3)'(x) = f_1'(x) f_2(x) f_3(x) + f_1(x) f_2'(x) f_3(x) + f_1(x) f_2(x) f_3'(x).$$

(ii). Let $P(n)$ be the statement "for any differentiable functions $f_1, \dots, f_n : A \to \mathbb{R}$, their product $f_1 \cdots f_n$ is differentiable". We prove that $P(n)$ is true for all $n \in \mathbb{N}$, by induction on n.

Base case: $P(1)$. In the case $n = 1$, the product $f_1 \cdots f_n$ reduces to the single function f_1, and the statement $P(1)$ becomes "if f_1 is differentiable, then f_1 is differentiable", which is trivially true.

Induction step: $\forall k \in \mathbb{N}\, (P(k) \Rightarrow P(k+1))$. Let $k \in \mathbb{N}$, and assume that $P(k)$ is true. To prove $P(k+1)$, let us consider any $k+1$ differentiable functions $f_1, \dots, f_{k+1} : A \to \mathbb{R}$, and note that we can write their product as $f_1 f_2 \cdots f_k f_{k+1} = (f_1 f_2 \cdots f_k) f_{k+1} = h f_{k+1}$, where $h = f_1 \cdots f_k$ is the product of the first k functions. Since $f_1, \dots, f_k$ are differentiable, by the induction hypothesis $P(k)$ we deduce that their product $h = f_1 \cdots f_k$ is differentiable too. Now we can apply the Leibniz rule to the product $h f_{k+1}$ and conclude that, since both factors h and f_{k+1} are differentiable, their product is too. Since $f_1 \cdots f_{k+1} = h f_{k+1}$, this proves that $f_1 \cdots f_{k+1}$ is differentiable. Since $f_1, \dots, f_{k+1} : A \to \mathbb{R}$ were arbitrary differentiable functions, this proves $P(k+1)$. Since $k \in \mathbb{N}$ was arbitrary, this proves the induction step.

Conclusion. From the base case and the induction step, by the induction principle we deduce that $P(n)$ is true for all n, that is, the product of any n differentiable functions is differentiable, as desired.

(iii). By proceeding as in part (i), if we apply the Leibniz rule iteratively we obtain, for example, that

$$\begin{aligned}
(f_1 f_2)' &= f_1' f_2 + f_1 f_2', \\
(f_1 f_2 f_3)' &= f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3', \\
(f_1 f_2 f_3 f_4)' &= f_1' f_2 f_3 f_4 + f_1 f_2' f_3 f_4 + f_1 f_2 f_3' f_4 + f_1 f_2 f_3 f_4'
\end{aligned}$$

in the cases $n = 2, 3, 4$. From this one can recognise a pattern: we may expect that the derivative of the n-fold product $f_1 \cdots f_n$ is a sum of n terms, each of which is a product of the form $f_1 \cdots f_{k-1} f_k' f_{k+1} \cdots f_n$, where the kth factor is differentiated, but the others are not, and k ranges from 1 to n. In other words,

$$\begin{aligned}
(f_1 \cdots f_n)' = {} & f_1' f_2 f_3 \cdots f_{n-2} f_{n-1} f_n \\
& + f_1 f_2' f_3 \cdots f_{n-2} f_{n-1} f_n \\
& + \dots \\
& + f_1 f_2 f_3 \cdots f_{n-2} f_{n-1}' f_n \\
& + f_1 f_2 f_3 \cdots f_{n-2} f_{n-1} f_n'.
\end{aligned}$$

In order to avoid so many "..." and write this in a more compact form, we could use the summation symbol:

$$(f_1 \cdots f_n)' = \sum_{k=1}^{n} f_1 \cdots f_{k-1} f_k' f_{k+1} \cdots f_n.$$

We could also get rid of the "$\cdots$" in the products by using the iterated product notation:

$$\begin{aligned}
(f_1 \cdots f_n)' &= \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} f_j \right) f_k' \left(\prod_{j=k+1}^{n} f_j \right) \\
&= \sum_{k=1}^{n} f_k' \prod_{\substack{1 \le j \le n \\ j \ne k}} f_j.
\end{aligned}$$

This formula (however written) can be proved by induction on n, by suitably adapting the arguments in parts (i) and (ii) above. $\square$