Chapter 3 – Power series solutions of second order linear equations

(for when we can't easily obtain a closed-form analytical solution)

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Given the 2nd order linear equation

$$a(x)y'' + b(x)y' + c(x)y = d(x) \tag{1}$$

and its corresponding homogeneous equation

$$a(x)y'' + b(x)y' + c(x)y = 0, \tag{2}$$

using the material of the course so far, we should be able to

- identify when (and where) a solution exists to (1) or (2) (using an Existence and Uniqueness Theorem);
- identify when a set of solutions $\{u_1(x), u_2(x)\}$ of (2) form a fundamental set of solutions (using the Wronskian);
- obtain a second linearly independent solution, u_2, to (2) if we already know one solution, u_1 (using the Reduction of Order method);
- obtain a particular solution to (1) given a fundamental set of solutions to (2) (using the Variation of Parameters method), to form a general solution.

Thus, our ability to solve the inhomogeneous equation (1) in general rests entirely on our ability to obtain a solution to the corresponding homogeneous equation (2). As we have discussed, there is no systematic way to do this for all functions $a(x), b(x), c(x)$ in terms of elementary functions. Therefore we must turn to a new tool: power series expansions (we first saw a hint of this in our discussion of the Existence and Uniqueness Theorem for nth order linear ODEs).

First, we do a review of power series.

1 Recap of power series

A **power series** about the point $x = x_0$ is an expression of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ... \tag{3}$$

where x is a variable and the a_n are constants.

1.1 Convergence

A power series is said to **converge** at a point x if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n (x - x_0)^n$$

exists for that x. The series will always converge at $x = x_0$ (to a_0). It may also converge for all x or just certain values/ranges of x. When this limit does not exist, the series is said to **diverge**. The power series (3) converges for all x in some interval centred at x_0.

The power series (3) is said to **converge absolutely** at a point x if the series

$$\sum_{n=0}^{\infty} \left| a_n (x - x_0)^n \right|$$

converges. If a series converges absolutely then it can be shown that the series converges (the reverse is not necessarily true).

We can use the **ratio test** to determine if a series converges absolutely:

$$\text{if} \quad L = \lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

then the power series (3) converges absolutely if $L < 1$ and diverges if $L > 1$. If $L = 1$ the ratio test is inconclusive and other methods must be used. Note that with the ratio test, we are considering the ratio of two consecutive terms – these are not always a_{n+1} and a_n, as we shall see in later examples.

Example: For which values of x does the following series converge:

$$f(x) = \sum_{n=0}^{\infty} \frac{(-2)^n}{n+1} (x - 3)^n \quad ?$$

Answer:

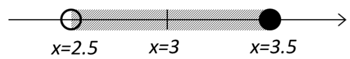

Figure 1: The interval of convergence of $\sum_{n=0}^{\infty} \dfrac{(-2)^n}{n+1}(x-3)^n$ is $(2.5 \quad 3.5]$. On the number line in the sketch we use an open circle to indicate that $x = 2.5$ is not included in the interval, and a closed circle at $x = 3.5$ to indicate that it is included in the interval.

The **radius of convergence**, ρ, is a nonnegative number such that the power series (3) converges absolutely for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$. If a series converges only at x_0, its radius of convergence is $\rho = 0$. If a series converges for all x, it has an infinite radius of convergence. The interval $|x - x_0| < \rho$ $(\rho \neq 0)$ is called the **interval of convergence**.

Example: In the previous example, what is the radius of convergence? Sketch the interval of convergence.

Answer:

A sketch of the interval of convergence is given in Figure 1.

1.2 Combining power series

Power series can be added or subtracted termwise, e.g. if $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n$ then

$$f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x - x_0)^n.$$

Power series can be multiplied:

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n(x - x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x - x_0)^n\right) = \sum_{n=0}^{\infty} c_n(x - x_0)^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

(to see this, try working out the appropriate terms for $n = 0, 1, 2, 3$). The quotient of two power series can also be obtained provided $g(x_0) \neq 0$ but there is no nice formula for this (and it may also have a smaller radius of convergence than either $f(x)$ or $g(x)$).

1.3 Differentiating and integrating power series

The power series $f(x)$ given by (3) is continuous and has derivatives of all orders for $|x - x_0| < \rho$, e.g.

$$\begin{aligned}
f'(x) &= \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}, \\
&= \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \\
f''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \\
&= \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},
\end{aligned}$$

and so on (you can see this if you consider each term in the sums separately). Each of the derivatives will converge absolutely for $|x - x_0| < \rho$. The same principle applies to integration of $f(x)$.

1.4 Taylor series

The **Taylor series** (or **Maclaurin series** if $x_0 = 0$) is given by

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \\
&= f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots
\end{aligned}$$

A function $f(x)$ that has a Taylor series expansion about $x = x_0$ with a radius of convergence $\rho > 0$ is said to be **analytic** at $x = x_0$. If f and g are both analytic at x_0 then $f \pm g, fg$ and f/g (provided $g(x_0) \neq 0$) are also analytic at $x = x_0$.

Note that a function that is analytic in a neighbourhood of x_0 is continuous and possesses derivatives of all orders in that neighbourhood.

If $f(x) = g(x)$ for all x, then $a_n = b_n$ for all n, so that if we have obtained a power series expansion for an analytic function, then this power series must be its Taylor series. We can generally use the Taylor series formula to obtain a power series expansion of a function.

If $f(x) = 0$ for all x then $a_n = 0$ for all n.

1.5 The index of summation

Sometimes it is useful to change our index of summation (it does not matter what we call this index as long as we are still expressing the same sum).

Example: Write

$$\sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + ...$$

as a series whose first term corresponds to $n = 0$ rather than $n = 2$.

Answer:

Shifting the index is useful when we want to combine sums (so that the start point is the same for both sums, or so that the powers of $(x - x_0)$ are the same for both sums). Make sure you can follow this, it will be crucial in later sections.

2 Ordinary and singular points

Before we can proceed to obtain series solutions of the homogeneous equation (2), we must first establish some definitions.

We can rewrite (2) in the standard form

$$y'' + p(x)y' + q(x)y = 0 \tag{4}$$

where $p(x) = b(x)/a(x)$ and $q(x) = c(x)/a(x)$. A point x_0 is called an **ordinary point** of (4) if both $p(x)$ and $q(x)$ are analytic at x_0 (i.e. $a(x_0) \neq 0$). Otherwise x_0 is a **singular point**. A singular point x_0 of (4) is said to be a **regular singular point** if both $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ are analytic at x_0. Otherwise x_0 is called an **irregular singular point**.

Example: What are the singular points of

$$(x^2-1)^2 y'' + (x+1)y' - y = 0 \quad ?$$

Are they regular or irregular?

Answer:

How we approach finding series solutions to differential equations depends on whether we are looking for solutions within an interval of an ordinary point, a regular singular point or an irregular singular point.

3 Series solutions near an ordinary point

We look for solutions in the form of a power series to equation (4) around a value x_0 where $p(x)$ and $q(x)$ are analytic, i.e. x_0 is an ordinary point. Since $p(x)$ and $q(x)$ are analytic, they are continuous around x_0 and the Existence and Uniqueness theorem guarantees that a solution does exist (so we're not wasting out time).

We approach this by substituting the power series into the differential equation and matching the resulting coefficients of powers of $(x - x_0)$ to obtain the a_n. We can assume that the resulting power series converges in some interval of convergence around x_0 and is therefore a solution to (4).

The best way to understand this is by doing an example.

Example: Find a series solution of Airy's equation[1]

$$y'' - xy = 0,$$

and show that the solution converges.

Answer:

[1] Airy's equation arises in the solution of various problems in optics and astronomy, for example.

If we have two initial conditions associated with the original equation we could use these

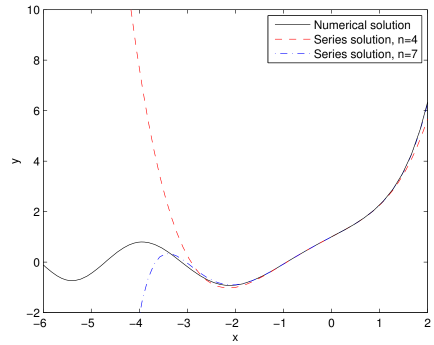

Figure 2: Solid black line: the full solution to Airy's equation ($y'' - xy = 0$) subject to $y(0) = y'(0) = 1$ (note that this has been computed numerically); dashed red line: the series solution for $n = 4$; dot-dash blue line: the series solution for $n = 7$. Note that the series solution is accurate around $x = 0$ and becomes more accurate the more terms that are included in the series solution.

to calculate a_0 and a_1 (just as we would with determining unknown constants in a solution) and to generate the unique solution satisfying those conditions.

Example cont. Impose the initial conditions: $y(0) = 1, y'(0) = 1$.

Answer:

See Figure 2 for a graphical representation of this solution.

Note that the solution will be most accurate in a region around $x = 0$ and this solution will become more accurate the more terms that are included. If we are looking for a solution at a different value of $x = x_0$, then we would derive a series solution in terms of powers of $(x - x_0)$.

Summary: to find a power series solution around an ordinary point at $x_0 = 0$,

(a) check that $x = 0$ is an ordinary point,

(b) look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

(c) substitute this series and its derivatives into the equation,

(d) shift the indices so that all sums are in terms of x^n (or a different power if you'd prefer),

(e) equate the coefficients of x^n across both sides of the equation to obtain a recurrence relation for the a_n,

(f) the general solution is a linear combination of two resulting power series.

4 Frobenius method for regular singular points

4.1 Frobenius method for $r_1 \neq r_2$ and $r_1 - r_2$ not an integer

We cannot use the above method in the interval of a regular singular point because the solution will not be analytic there and we cannot therefore represent it with a Taylor series. This means that the method will break down.

If $x = 0$ is a regular singular point of $(4)^2$, we instead seek solutions of the form

$$\begin{aligned} y &= x^r \sum_{n=0}^{\infty} a_n x^n, \\ &= \sum_{n=0}^{\infty} a_n x^{n+r}. \end{aligned}$$

We then must find values of r for which y is a solution of the ODE in addition to finding the coefficients a_n. In general, there will be two possible values of r which will give rise to the two linearly independent solutions. We then calculate the coefficients a_n for each value of r and ensure that the series converge. We shall assume that all series presented here do converge on a relevant interval.

It should be easiest to understand the method by following an example.

Example: Find a series solution of

$$2x^2 y'' - xy' + (1+x)y = 0$$

in an interval centred on $x = 0$.

Wrong answer:

[2] We just consider $x_0 = 0$ for simplicity, all analysis extends to general x_0 by defining a new variable $x^* = x - x_0$ so that when $x = x_0$, $x^* = 0$

Correct answer:

In this case, we have been able to find two linearly independent solutions to the equation. This may not always be the case: extra work must be done if the roots of the indicial equation are equal or if they differ by an integer. We shall consider this in §4.2.

Summary: to use the Frobenius method to find a power series solution around a regular singular point at $x_0 = 0$,

(a) check that $x = 0$ is a regular singular point,

(b) look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

(c) substitute this series and its derivatives into the equation,

(d) shift the indices so that all sums are in terms of x^{n+r} (or a different power if you'd prefer),

(e) equate the coefficients of the lowest power of x (e.g. take the lowest value of n) to obtain the indicial equation in r (using arbitrary $a_0 \neq 0$)

(f) solve for r to obtain the roots r_1 and r_2. If $r_1 \neq r_2$ and r_1 and r_2 do not differ by an integer quantity, proceed,

(g) equate the coefficients of x^{n+r} across both sides of the equation to obtain a recurrence relation for the a_n in terms of r,

(h) determine the recurrence relation for each r,

(i) the general solution is a linear combination of the two resulting power series

$$y = \alpha_1 \sum_{n=0}^{\infty} a_n x^{n+r_1} + \alpha_2 \sum_{n=0}^{\infty} b_n x^{n+r_2},$$

where α_1, α_2 are arbitrary constants.

Note that we have only considered the regular singular point $x_0 = 0$ in lectures for simplicity. The above procedure works in the same way about a general regular singular point x_0 by looking for a solution in the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}.$$

4.2 Frobenius method for $r_1 = r_2$ and $r_1 - r_2$ an integer

Let x_0 be a regular singular point of $a(x)y'' + b(x)y' + c(x)y = 0$ and r_1, r_2 be the roots of the associated indicial equation with $r_1 \geq r_2$.

- If $r_1 - r_2$ is not an integer then $\exists$ two linearly independent solutions of the form

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \quad a_0 \neq 0,$$
$$y_2 = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}, \quad b_0 \neq 0,$$

(as we saw in §4.1).

- If $r_1 = r_2$ then $\exists$ two linearly independent solutions of the form

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \quad a_0 \neq 0,$$
$$y_2 = y_1(x) \ln (x - x_0) + \sum_{n=1}^{\infty} b_n (x - x_0)^{n+r_1}.$$

- If $r_1 - r_2$ is a positive integer then $\exists$ two linearly independent solutions of the form

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \quad a_0 \neq 0,$$
$$y_2 = \alpha y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2},$$

  where α is a constant that could be zero.

The details of why the above holds are rather complicated and we omit them here for brevity.

5 Power series solutions of inhomogeneous linear ODEs

So far we have only looked for power series solutions to homogeneous ODEs. What happens if we want to find a solution to the following inhomogeneous ODE:

$$a(x)y'' + b(x)y' + c(x)y = d(x) \quad ? \tag{5}$$

All of the above method applies, but we must ensure that we match coefficients of powers of x (or of $(x - x_0)$ if $x_0 \neq 0$) to the correct coefficients on the right-hand side. If any of the functions are not written in terms of powers of x (or $(x - x_0)$), we can use their Taylor series expansion to write them in a form where we can match powers of x (or $(x - x_0)$) across the whole equation.