Problem Sheet 3 — Model Solutions and Feedback

Question 1 (SUM)**.**

(a) If I choose a non-negative integer solution to the equation $x_1 + x_2 + x_3 + x_4 = 20$ uniformly at random, what is the probability that I have $x_1 = 6$?

(b) How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 \leq 43$ (note carefully that this is an inequality, not an equality) with $x_1 \geq 3$, $x_2 \geq 1$, $x_3 \geq 2$, $x_4 \geq 5$ and $x_5 \geq 0$?

(c) How many integer solutions are there to $x_1 + x_2 + x_3 \leq 50$ with $x_1 \geq 27$, $x_2 \geq -11$ and $x_3 \geq 35$?

Solution. (a) Let A be the set of non-negative integer solutions to $x_1+x_2+x_3+x_4 = 20$, and let $B \subseteq A$ be the set of such solutions in which $x_1 = 6$. Then we have $|A| = \binom{4+20-1}{20} = \binom{23}{20} = \binom{23}{3} = 1771$. Moreover, the solutions in B correspond bijectively[1] to the non-negative integer solutions of $x_2 + x_3 + x_4 = 14$, and so we have $|B| = \binom{3+14-1}{14} = \binom{16}{14} = \binom{16}{2} = 120$. So the desired probability is $\frac{|B|}{|A|} = \frac{120}{1771}$.

(b) The integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 \leq 43$ with $x_1 \geq 3$, $x_2 \geq 1$, $x_3 \geq 2$, $x_4 \geq 5$ and $x_5 \geq 0$ correspond bijectively to integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 43$ with with $x_1 \geq 3$, $x_2 \geq 1$, $x_3 \geq 2$, $x_4 \geq 5$, $x_5 \geq 0$ and $x_6 \geq 0$ (the bijection is given by taking $x_6 = 43 - x_1 - x_2 - x_3 - x_4 - x_5$). Similarly, by the substitution $x_1 = y_1 + 3$, $x_2 = y_2 + 1$, $x_3 = y_3 + 2$, $x_4 = y_4 + 5$ such solutions correspond bijectively to the non-negative integer solutions to

$$y_1 + y_2 + y_3 + y_4 + x_5 + x_6 = 43 - 3 - 1 - 2 - 5 = 32.$$

It follows that the number of such solutions is $\binom{6+32-1}{32} = \binom{37}{32} = \binom{37}{5} = 435897$.

(c) If $x_1 \geq 27$, $x_2 \geq -11$ and $x_3 \geq 35$ then we must have

$$x_1 + x_2 + x_3 \geq 27 - 11 + 35 = 51.$$

So there are no solutions to $x_1 + x_2 + x_3 \leq 50$ with these constraints, integer or otherwise. □

Feedback. A good answer following the approach of the model solution should do the following:

- Be clear on the sets which are being counted in (a) (explicit set notation isn't essential, though I do think it is clearer), and explain how the solutions with $x_1 = 6$ are counted (i.e. that we have the correspondence with solutions to $x_2 + x_3 + x_4 = 14$).
- For (b), explain the use of a slack variable to change the inequality to an equality, state precisely what substitution you are making, and give the restated version of the equation after this substtitution before applying the formula to this.
- Explain why there are no solutions for (c). Your answer shouldn't be 'the method doesn't work', but rather an explanation of why there are no solutions in this case. You should not refer to the $\binom{n+r-1}{r}$ formula in your answer (in particular, the values of n and r you get fall outside the range of values for which we proved this corollary) but simply show directly that the inequality and constraints of the question cannot be simultaneously satisfied.

The use of a 'dummy' or 'slack' variable (x_6 in this case) to turn an inequality into an equality is worth remembering for similar questions of this type (and you will see the same idea in other modules as you progress through your degree). Alternatively, you could count the solutions for each possible value of the sum between 11 and 43 for (b). This gives the answer as $\binom{36}{4} + \binom{35}{4} + \cdots + \binom{4}{4}$, and it is a useful exercise with multiple valid solutions to show that this sum of binomial coefficients is indeed equal to $\binom{37}{5}$ (if you want to look this up, search for the Christmas Stocking Theorem).

[1] Formally, taking C to be the set of non-negative integer solutions to $x_2 + x_3 + x_4 = 14$, the bijection is that a solution $(x_1, x_2, x_3, x_4) \in B$ corresponds to the solution $(x_2, x_3, x_4) \in C$.

Question 2 (SUM)**.** A D8 is an eight-sided die whose sides are numbered one to eight; when rolled, each side is equally likely to be selected. If I roll three D8, what is the probability that the numbers shown form an arithmetic progression (not necessarily in the order in which they are rolled, and not including the degenerate case when the numbers are all the same)?

Solution. We first count the number of possible arithmetic progressions (APs) which could result. The possible APs with common difference 1 are

$$\{1,2,3\},\{2,3,4\},\{3,4,5\},\{4,5,6\},\{5,6,7\},\{6,7,8\}.$$

Similarly, the possible APs with common difference 2 are

$$\{1,3,5\},\{2,4,6\},\{3,5,7\},\{4,6,8\},$$

and the possible APs with common difference 3 are

$$\{1,4,7\},\{2,5,8\}.$$

There are no other possible APs, as if the common difference is 4 or more then the highest and lowest elements must differ by at least 8, but any two D8 rolls differ by at most 7. So in total there are 12 possible APs which could result from our roll.

Now imagine that we roll the three dice in order, so the outcome is an ordered sequence of three integers between 1 and 8. There are $8^3 = 512$ such sequences, and moreover each outcome is equally likely to occur. Since each of the 12 APs we identified above has $3! = 6$ permutations (that is, there are 6 possible orders in which the specified three numbers could be rolled), precisely $12 \times 6 = 72$ of the possible outcomes have the property that the numbers shown form an arithmetic progression. So the probability of this event is

$$\frac{72}{512} = \frac{9}{64} = 0.140625.$$

□

Feedback. The key things I would look for in an answer are as follows:

- A clear statement that there are 12 APs which can be formed from the numbers 1 to 8 (excluding degenerate cases where you have the same number 3 times). This should be justified, for example by writing them all down, using a sensible order to make clear that you have got them all (in the model solution this is done by looking at each possible difference in turn).
- A statement that we consider the dice to be rolled *in order*, meaning that the outcome is an ordered sequence of three elements of $\{1,2,3,4,5,6,7,8\}$.
- Consequently to this, the fact that there are $8^3 = 512$ possible outcomes in total, and the dice roll selects one of these uniformly at random (i.e. every outcome is equally likely).
- A statement that each AP has $3! = 6$ permutations, i.e. 6 outcomes of the dice rolls yield that particular AP.
- Successfully combining the above facts to find that there are $12 \times 6 = 72$ outcomes which give an AP, and therefore the probability is $72/512$.

It's easy to fall into the trap of the following incorrect solution: since the order of the rolls is unimportant to whether they form an AP, consider the dice rolls as being three choices from $\{1,2,3,4,5,6,7,8\}$ allowing repetition but without order. There are $\binom{8+3-1}{3} = \binom{10}{3} = 120$ possibilities for this choice, of which 12 yield APs (i.e. the 12 sets presented in the model solution), so the probability is $1/10$. Everything in this argument is correct except the final calculation of probability, which fails because the selection from the 120 possible outcomes made by rolling the dice is *not* uniformly random, so we can't just divide the number of outcomes in which the event occurs by the total number of outcomes. Instead, we need to consider the dice being rolled in order, as in the model solution, so that we do indeed get a uniformly-random selection of outcome from the set of possible outcomes (and this applies in all similar situations where repetition can occur, which in this instance means that multiple dice rolls can give the same outcome).

Once you take the approach of considering the rolls in order, the challenge is then to count the outcomes which give APs, given that the question says these don't necessarily have to be in the order rolled. The approach used for this – to identify the APs ignoring order, then count the ways they can be ordered – is typical for this kind of situation, so look out for it.

Because the numbers involved in this question are quite small the easiest way to count the APs is just to write them all down, as in the model solution. However, a more general method which would work for larger sets (e.g. more dice with more sides) is first to count the number of choices for the middle digit of the sequence (here 8) and then to count the number of choices for the common difference (in this case, if the middle digit is i, then the number of possibilities for the common difference is $\min(i-1, 8-i)$. You can then sum over these possibilities to get the total number of APs, in this case $\sum_{i=1}^{8} \min(i-1, 8-i) = 0+1+2+3+3+2+1+0 = 12$.

You may find it useful to compare this question and solution to the example we saw in the lectures of rolling four standard dice and calculating the probability that the numbers rolled are consecutive.

Question 3. A card player is dealt a hand of 13 cards from a standard 52 card deck.

(a) What is the probability that the hand contains no hearts?
(b) What is the probability that the hand contains 5 spades, 3 diamonds and 5 clubs?
(c) What is the probability that the hand contains at most 3 hearts?

Solution. The number of possible 13-card hands is the number of ways of choosing a subset of size 13 (the hand) from a set of size 52 (the deck), that is, $\binom{52}{13}$.

(a) There are 39 cards in the deck which are not hearts, so the number of ways to choose a hand of 13 such cards is $\binom{39}{13}$. So the probability that the hand contains no hearts is

$$\frac{\binom{39}{13}}{\binom{52}{13}} = \frac{\left(\frac{39\times38\times\cdots\times28\times27}{13!}\right)}{\left(\frac{52\times51\times\cdots\times41\times40}{13!}\right)} = \frac{39\times38\times\cdots\times28\times27}{52\times51\times\cdots\times41\times40} \approx 0.0128.$$

(b) A hand as described consists of any 5 of the 13 spades, any 3 of the 13 diamonds, any 5 of the 13 clubs and none of the 13 hearts. So the number of such hands is $\binom{13}{5}\binom{13}{3}\binom{13}{5}\binom{13}{0} = \binom{13}{5}\binom{13}{3}\binom{13}{5}$. The probability is therefore

$$\frac{\binom{13}{5}\binom{13}{3}\binom{13}{5}}{\binom{52}{13}} \approx 0.000746.$$

(c) For any $0 \leq i \leq 13$, a hand which contains exactly i hearts consists of i of the 13 hearts and $13-i$ of the 39 other cards. So the number of such hands is $\binom{13}{i}\binom{39}{13-i}$. By summing this value for $0 \leq i \leq 3$ we find that the number of hands containing at most 3 hearts is $\binom{13}{0}\binom{39}{13} + \binom{13}{1}\binom{39}{12} + \binom{13}{2}\binom{39}{11} + \binom{13}{3}\binom{39}{10} = \binom{39}{13} + 13\binom{39}{12} + \binom{13}{2}\binom{39}{11} + \binom{13}{3}\binom{39}{10}$. The probability is therefore

$$\frac{\binom{39}{13} + 13\binom{39}{12} + \binom{13}{2}\binom{39}{11} + \binom{13}{3}\binom{39}{10}}{\binom{52}{13}} \approx 0.585.$$

□

Feedback. I would look for the following key points in an answer:

- First, somewhere in your answer you need to state clearly that there are $\binom{52}{13}$ possible outcomes for the hand that is dealt.
- For (a), it's fine simply to say that there are $\binom{39}{13}$ outcomes with no hearts, and deduce that the probability is $\binom{39}{13}/\binom{52}{13}$.
- For (b), you should make clear that you are forming a hand by a series of choices, e.g. first choosing the spades, then diamonds, then hearts. In this way you should link each term of the expression for the number of outcomes to one of these choices (possibly implicitly, as in the model solution).

- For (c), you need to break the inequality into cases for each possible value, and should be clear about doing this. For each value you can then count the outcomes as a sequence of two choices, as per the model solution.

It's also possible to consider the cards being dealt in order. This allows you answer (a) by considering the cards one by one, to get a probability of

$$\frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50} \cdot \dots \cdot \frac{27}{40},$$

but complicates the answers to (b) and (c), as now you have to consider the possible orders in which the cards from each suit could arise. Generally I would encourage you to try to move away from thinking about questions like this on a card-by-card approach, and instead try to count outcomes in blocks (e.g. for (b) we count the possibilities for the 5 spades, then the 3 diamonds, then the 5 clubs).

Question 4. Say that integers n and m are of the same *type* if you you can form n by rearranging the digits of m (so for example 134366 and 663314 are of the same type, but 1255 and 1225 are not). How many different types of six-digit integers are there?

Solution. Given a six-digit integer n, we identify the type of n by ignoring the order of the digits of n, and simply counting how many of each digit there are. In other words, we are making six choices from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (i.e. choosing the digits), ignoring order (because we ignore the order the digits are in) but allowing repetition (because you can have the same digit more than once), and there are $\binom{10+6-1}{6} = \binom{15}{6} = 5005$ possible ways to do this. Each way of making these choices corresponds to a type of six digit integer, except for the case where we pick zero six times (since there is no six-digit number all of whose digits are zero). So there are 5004 different types of six-digit integer. $\square$

Feedback. This question is as good test of your understanding of working with order and without order, as it is easy to get tripped up by this. A six-digit integer consists of a choice of six elements of $\{0, 1, 2, \dots, 9\}$, with order and allowing repetition, and not starting with 0. For a type we only care how many of each digit we have, but not the order they are in. So a type consists of a choice of six elements of $\{0, 1, 2, \dots, 9\}$ without order but still allowing repetition (e.g. the type of the first two numbers of the question is '1 once, 3 twice, 4 once, 6 twice'). We know how to count these – the $\binom{n+r-1}{n}$ formula – except the final trap is that the type of choosing zero six times doesn't arise from any six-digit number, so we need to ignore this (check for yourself that every other type does arise from some number). I would strongly recommend you go over this question and solution if your submitted solution was incorrect or if anything is unclear to you, and please do ask for help if you still don't understand.