Problem Sheet 4 — Model Solutions and Feedback

Question 1 (SUM). (a) Prove that for every integer $n \geq 0$ we have

$$3^n = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^n\binom{n}{n}.$$

(b) Prove that for all integers $a, b \geq n \geq 0$ we have

$$\binom{a+b}{n} = \binom{a}{n} + \binom{a}{n-1}\binom{b}{1} + \binom{a}{n-2}\binom{b}{2} + \cdots + \binom{a}{1}\binom{b}{n-1} + \binom{b}{n}.$$

Solution. (a)

$$3^n = (2+1)^n = \sum_{i=0}^{n} \binom{n}{i} 2^i 1^{n-i} = \binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots + 2^n\binom{n}{n},$$

where the second equality holds by the Binomial Theorem.

(b) Let A and B be disjoint sets of sizes a and b respectively. Then $|A \cup B| = a + b$, so there are $\binom{a+b}{n}$ subsets of $A \cup B$ of size n. Each such subset must include j elements of A and $n - j$ elements of B for some $0 \leq j \leq n$, and the number of subsets of $A \cup B$ of size n which include j elements from A and $n - j$ elements of b is $\binom{a}{j}\binom{b}{n-j}$. Summing over each j between 0 and n, we conclude that the total number of subsets of $A \cup B$ of size n is equal to $\sum_{j=0}^{n} \binom{a}{j}\binom{b}{n-j}$. Since we already found that this total is equal to $\binom{a+b}{n}$, this completes the proof. □

Feedback. If you follow the approach of the model solutions, for (a) the key thing is to be clear about how you are applying the Binomial Theorem, whilst for (b) you want to be clear that you are counting the same quantity in two different ways, and how each way gives the claimed total. There are many possible reformulations for (b), for example, you could count the ways to make n choices from $A \cup B$ without order or repetition, or you could imagine a collection of $a + b$ distinct cars, of which a are red and b are blue, and count the number of possibilities for a selection of n of them.

Both of these questions admit alternative solutions, mostly using induction on n, but all such solutions that I am aware of are significantly longer than the solutions given here. In particular, the solution for (b) shows the power of combinatorial arguments in which you focus on what binomial coefficients represent, rather than simply the definition $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Question 2 (SUM). (a) Prove that $\mathbb{Q}$ is countably infinite.

(b) Prove that $\mathbb{N} \times \mathbb{Q}$ is countably infinite.

Solution. We give several proofs of (a) to illustrate the diversity of approaches which could be used.

Solution 1: Observe that the identity function $i : \mathbb{N} \to \mathbb{Q}$ with $i(n) = n$ for each $n \in \mathbb{N}$ is an injection, so $|\mathbb{N}| \leq |\mathbb{Q}|$. For the other direction, consider the function $f : \mathbb{Q} \to \mathbb{N}$ defined as follows: given $r \in \mathbb{Q}$, write $r = p/q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ where p and q are coprime (this defines p and q uniquely), and set

$$f(r) = \begin{cases} 2^p \cdot 3^q & \text{if } r > 0 \\ 1 & \text{if } r = 0 \\ 2^{-p} \cdot 3^q \cdot 5 & \text{if } r < 0. \end{cases}$$

By uniqueness of prime factorisation f is an injection, so $|\mathbb{Q}| \leq |\mathbb{N}|$, and so $|\mathbb{Q}| = |\mathbb{N}|$, that is, $\mathbb{Q}$ is countably infinite.

Solution 2: Just as in Solution 1 we have $|\mathbb{N}| \leq |\mathbb{Q}|$. Recall from lectures that $\mathbb{Z}$ is countably infinite, so there exists a bijection $f : \mathbb{N} \to \mathbb{Z}$. Consider the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ given by $g((m,n)) = \frac{f(m)}{n}$. Since $f(m) \in \mathbb{Z}$ and $n \in \mathbb{N}$ the function g is well-defined. Furthermore g is surjective since for any $r \in \mathbb{Q}$ we may write $r = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, whereupon taking $m = f^{-1}(p)$ and $n = q$ gives a pair $(m,n) \in \mathbb{N} \times \mathbb{N}$ with $g((m,n)) = r$. The existence of the surjection g implies that $|\mathbb{Q}| \leq |\mathbb{N} \times \mathbb{N}|$. We recall from lectures that $\mathbb{N} \times \mathbb{N}$ is countably infinite, meaning that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. Combining these statements we obtain $|\mathbb{Q}| \leq |\mathbb{N}|$, so $|\mathbb{Q}| = |\mathbb{N}|$, that is, $\mathbb{Q}$ is countably infinite.

Solution 3: Just as in Solution 1 we have $|\mathbb{N}| \leq |\mathbb{Q}|$. Since every rational number can be written as p/q for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, each member of $\mathbb{Q}$ appears somewhere in the infinite arrangement shown on the left hand side of the diagram below.

$$
\begin{array}{ccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & -\frac{4}{6} & -\frac{3}{6} & -\frac{2}{6} & -\frac{1}{6} & \frac{0}{6} & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \cdots \\
\cdots & -\frac{4}{5} & -\frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} & \frac{0}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \cdots \\
\cdots & -\frac{4}{4} & -\frac{3}{4} & -\frac{2}{4} & -\frac{1}{4} & \frac{0}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\
\cdots & -\frac{4}{3} & -\frac{3}{3} & -\frac{2}{3} & -\frac{1}{3} & \frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\cdots & -\frac{4}{2} & -\frac{3}{2} & -\frac{2}{2} & -\frac{1}{2} & \frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\
\cdots & -\frac{4}{1} & -\frac{3}{1} & -\frac{2}{1} & -\frac{1}{1} & \frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots
\end{array}
\qquad
\begin{array}{ccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & -\frac{4}{6} & -\frac{3}{6} & -\frac{2}{6} & -\frac{1}{6} & \frac{0}{6} & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \cdots \\
\cdots & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & \cdots \\
\cdots & 32 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 42 & \cdots \\
\cdots & 31 & 26 & 9 & 10 & 11 & 12 & 13 & 18 & 43 & \cdots \\
\cdots & 30 & 27 & 8 & 5 & 4 & 3 & 14 & 17 & 44 & \cdots \\
\cdots & 29 & 28 & 7 & 6 & 1 & 2 & 15 & 16 & 45 & \cdots
\end{array}
$$

The right hand side of the diagram shows how the elements of the arrangement may be labelled with elements of $\mathbb{N}$, by proceeding outwards from the centre. This creates a function $f : \mathbb{N} \to \mathbb{N}$ in which for each $n \in \mathbb{N}$ in the right hand arrangement we map n to the fraction in the corresponding position in the left hand arrangement, so, for example, $f(1) = 0/1$, $f(14) = 2/2$ and $f(9) = -2/3$. This function f is not a bijection since many elements of $\mathbb{N}$ may map to the same fraction (for example $f(3) = f(21) = 1/2$), but it is surjective since every element of $\mathbb{Q}$ appears somewhere in the left hand arrangement so is mapped to by the element of $\mathbb{N}$ in the corresponding position in the right hand arrangement. The existence of the surjection f implies that $|\mathbb{Q}| \leq |\mathbb{N}|$, so $|\mathbb{Q}| = |\mathbb{N}|$, that is, $\mathbb{Q}$ is countably infinite.

Solution 4: We can adapt Solution 3 by first deleting all the 'surplus' rationals (those whose numerator and denominator share a common factor) from the left hand table. We then labelling the remaining rationals with $\mathbb{N}$ as before, but simply leave a blank space corresponding to each deleted rational.

$$
\begin{array}{ccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & -\frac{4}{5} & -\frac{3}{5} & -\frac{2}{5} & -\frac{1}{5} & \xcancel{\frac{0}{5}} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \cdots \\
\cdots & \xcancel{-\frac{4}{4}} & -\frac{3}{4} & \xcancel{-\frac{2}{4}} & -\frac{1}{4} & \xcancel{\frac{0}{4}} & \frac{1}{4} & \xcancel{\frac{2}{4}} & \frac{3}{4} & \xcancel{\frac{4}{4}} & \cdots \\
\cdots & -\frac{4}{3} & \xcancel{-\frac{3}{3}} & -\frac{2}{3} & -\frac{1}{3} & \xcancel{\frac{0}{3}} & \frac{1}{3} & \frac{2}{3} & \xcancel{\frac{3}{3}} & \frac{4}{3} & \cdots \\
\cdots & \xcancel{-\frac{4}{2}} & -\frac{3}{2} & \xcancel{-\frac{2}{2}} & -\frac{1}{2} & \xcancel{\frac{0}{2}} & \frac{1}{2} & \xcancel{\frac{2}{2}} & \frac{3}{2} & \xcancel{\frac{4}{2}} & \cdots \\
\cdots & -\frac{4}{1} & -\frac{3}{1} & -\frac{2}{1} & -\frac{1}{1} & \frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots
\end{array}
\qquad
\begin{array}{ccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 22 & 23 & 24 & 25 & & 26 & 27 & 28 & 29 & \cdots \\
\cdots & & 17 & & 16 & & 15 & & 14 & & \cdots \\
\cdots & 21 & & 7 & 8 & & 9 & 10 & & 30 & \cdots \\
\cdots & & 18 & & 4 & & 3 & & 13 & & \cdots \\
\cdots & 20 & 19 & 6 & 5 & 1 & 2 & 11 & 12 & 31 & \cdots
\end{array}
$$

Again we have a function $f : \mathbb{N} \to \mathbb{Q}$ in which each $n \in \mathbb{N}$ in the right hand arrangement maps to the element of $\mathbb{Q}$ in the corresponding position in the left hand arrangement, so, for example, $f(1) = 0$, $f(9) = \frac{1}{3}$, $f(18) = -\frac{3}{2}$ and $f(20) = -4$. The function f is surjective just as in Solution 3, but now is also injective since no element of $\mathbb{Q}$ appears more than once in the left hand arrangement, so if $f(n) = f(m)$ for $n, m \in \mathbb{N}$ then n and m must appear in the same place in the right hand arrangement (namely the position corresponding to $f(n)$), which implies that $n = m$. So f is a bijection, from which it follows that $\mathbb{Q}$ is countably infinite.

For (b), note that by (a) there exists a bijection $f : \mathbb{N} \to \mathbb{Q}$. Also recall from lectures that $\mathbb{N} \times \mathbb{N}$ is countably infinite, so there exists a bijection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. So we may define a function $h : \mathbb{N} \to \mathbb{N} \times \mathbb{Q}$ by, for each $n \in \mathbb{N}$, setting $(a,b) = g(n)$ and $h(n) = (a, f(b))$. To see that h is surjective, observe that for any $(m,r) \in \mathbb{N} \times \mathbb{Q}$ we have $h(g^{-1}((m, f^{-1}(r))) = (m,r)$. To see that h is injective, suppose that $h(m) = h(n)$

for some $m, n \in \mathbb{N}$. Then, writing $(a,b) = g(n)$ and $(c,d) = g(m)$, we have $(a, f(b)) = (c, f(d))$, which, since f is a bijection, implies that $(a,b) = (c,d)$, so $g(n) = g(m)$. Since g is a bijection it follows that $m = n$, so h is indeed injective. We conclude that h is a bijection, and the existence of this bijection shows that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$. $\square$

Feedback. Multiple solutions are given for part (a) to show a selection of the diverse range of possible approaches for this kind of question. In the same way there are many possible approaches for (b), but the approach given is chosen because it follows nicely from (a) rather than 'starting again' with the new set.

The reason for there being so many possible approaches to showing that $\mathbb{Q}$ is countably infinite is that, once we've observed that $\mathbb{Q}$ is infinite and therefore $|\mathbb{N}| \leq |\mathbb{Q}|$, it remains to prove that $|\mathbb{Q}| \leq |\mathbb{N}|$. This can be done by (i) proving the existence of an injection $f : \mathbb{Q} \to \mathbb{N}$ or (ii) proving the existence of a surjection $f : \mathbb{N} \to \mathbb{Q}$. In either case this can be done by giving a function explicitly, or composing other functions, or using a labelling argument. Furthermore, we don't need to compare directly to $\mathbb{N}$; for any set X which we already know to be countably infinite, we have $|X| \leq |\mathbb{N}|$, so it suffices to show that $|\mathbb{Q}| \leq |X|$. We could do this by (i) proving the existence of an injection $f : \mathbb{Q} \to X$ or (ii) proving the existence of a surjection $f : X \to \mathbb{Q}$. It's not hard then to see how you can end up with a great number of possible ways to make the argument!

Note that it is not straightforward to demonstrate an explicit bijection between $\mathbb{N}$ and $\mathbb{Q}$, so the indirect approaches in which we argue that such a bijection exists give simpler answers to the question. However, the following function is indeed a bijection $f : \mathbb{Q} \to \mathbb{N}$ (the proof that f is a bijection is left as an exercise). Given $r \in \mathbb{Q}$, write $r = p/q$ where $p \in \mathbb{Z}, q \in \mathbb{N}$ and p and q are coprime (this specifies p and q uniquely). Let $C = \{x \in \{1, 2, \ldots, |p|\} : \gcd(x, q) = 1\}$, and set

$$m = \begin{cases} 2|C| + 1 & \text{if } p \leq 0, \\ 2|C| & \text{if } p > 0. \end{cases}$$

Finally set $f(r) = \binom{q+m}{2} - m + 1$.

Question 3. Let $\Upsilon := \{(a_1, a_2, a_3, \ldots) : a_i \in \{0, 1\} \text{ for every } i \in \mathbb{N}\}$, so in other words Υ is the set of all infinite sequences of zeroes and ones indexed by the natural numbers. Prove that Υ is uncountable.

Hint: Mimic the proof from lectures that $\mathbb{R}$ is uncountable, by considering an arbitrary function $f : \mathbb{N} \to \Upsilon$ and proving that f is not surjective by constructing an element of Υ which is not in the image of f.

Solution. We follow the approach outlined in the hint. Consider an arbitrary function $f : \mathbb{N} \to \Upsilon$, so for each $n \in \mathbb{N}$ the image $f(n)$ is a zero-one sequence. For each $n \in \mathbb{N}$ write $f(n) = (b_1^n, b_2^n, b_3^n, \ldots)$, so that b_j^i denotes the j-th term in the sequence $f(i)$. Then, for each $n \in \mathbb{N}$, define

$$c_n = \begin{cases} 1 & \text{if } b_n^n = 0, \\ 0 & \text{if } b_n^n = 1. \end{cases}$$

Let c be the sequence $(c_1, c_2, c_3, \ldots)$. Then for each $n \in \mathbb{N}$ we have $c \neq f(n)$, since the n-th element of c is c_n, the n-th element of $f(n)$ is b_n^n, and $c_n \neq b_n^n$ by definition. In other words c is not in the image of f. Since $c \in \Upsilon$ (because c is a zero-one sequence indexed by natural numbers), this shows that f is not surjective, and since f was arbitrary we conclude that there is no surjection $f : \mathbb{N} \to \Upsilon$.

Recall from lectures that $|X| \leq |Y|$ if and only if either X is empty or there exists a surjection $f : Y \to X$. Since $|\Upsilon|$ is non-empty it follows that $|\Upsilon| \not\leq |\mathbb{N}|$, that is, Υ is uncountable. $\square$

Feedback. As indicated by the hint, conceptually this proof is very similar to the proof from lectures that $\mathbb{R}$ is uncountable; the main challenge is to work out exactly where the changes in this argument are required.

Question 4. Let x_1, x_2, x_3, x_4 and x_5 each be chosen uniformly at random in turn from the set $\{1, 2, \ldots, 20\}$, independently of each other (in other words, each is chosen uniformly at random regardless of the values of previously-chosen x_i). Find the probabilities of each of the following events.

(a) That there exist $i, j \in \{0, 1, 2, 3, 4\}$ with $i \neq j$ such that three of the integers x_r are congruent to i modulo 5 and the other two integers x_r are congruent to j modulo 5.

(b) That exactly three of the integers x_1, x_2, x_3, x_4 and x_5 are divisible by 2 and exactly three are divisible by 3.

Solution. Observe that there are $20^5 = 2^{10} \cdot 5^5$ possibilities for the ordered sequence $(x_1, x_2, x_3, x_4, x_5)$ formed by the random choices. For (a), observe that we can form a sequence in which three of the integers x_r are congruent to i modulo 5 and the other two integers x_r are congruent to j modulo 5 by first choosing the value of i (for which there are five possibilities), then choosing the value of j (for which there are then four possibilities), then choosing which of the integers x_1, x_2, x_3, x_4 and x_5 will be congruent to i (for which there are $\binom{5}{3} = 10$ possibilities), then choosing the values for each of these integers (there are 4 possibilities for each, giving 4^3 possibilities overall), then choosing the values of the remaining two integers, which must be congruent to j modulo 5 (so there are 4 possibilities for each, giving 4^2 possibilities overall). So in total the number of outcomes where the described event occurs is $5 \cdot 4 \cdot 10 \cdot 4^3 \cdot 4^2 = 2^{13} \cdot 5^2$. So overall the probability of this event is $\frac{2^{13} \cdot 5^2}{2^{10} \cdot 5^5} = \frac{8}{125}$.

For (b), let $X = \{1, 2, \ldots, 20\}$, and let

$$\begin{aligned}
A &= \{n \in X : 2 \mid n \text{ and } 3 \mid n\} = \{6, 12, 18\} \\
B &= \{n \in X : 2 \mid n \text{ and } 3 \nmid n\} = \{2, 4, 8, 10, 14, 16, 20\} \\
C &= \{n \in X : 2 \nmid n \text{ and } 3 \mid n\} = \{3, 9, 15\} \\
D &= \{n \in X : 2 \nmid n \text{ and } 3 \nmid n\} = \{1, 5, 7, 11, 13, 17, 19\}
\end{aligned}$$

Then sequences $(x_1, x_2, x_3, x_4, x_5)$ for which the described outcome occurs could comprise:

- 3 integers from A and 2 from D. Such sequences can be formed by deciding which three of the integers x_i will be chosen from A (there are $\binom{5}{3}$ possibilities for this), and then choosing the values of the three integers from A (three choices for each, so 3^3 possibilities overall) and the two integers from D (seven choices for each, so 7^2 possibilities overall). In total this gives $\binom{5}{3} \cdot 3^3 \cdot 7^2$ outcomes of this form.

- 2 integers from A and 1 from each of B, C, D. Such sequences can be formed by deciding which two of the integers x_i will be chosen from A (there are $\binom{5}{2}$ possibilities for this), then which of the three remaining integers will be chosen from B (3 possibilities), then which of the two remaining integers will be chosen from C (2 possibilities). We then need to choose the values of each of the integers, for which there are 3^2 possibitilies for the two from A, 7 possibilities for the one from B, 3 possibilities for the one from C and 7 possibilities for the one from D. In total this gives $\binom{5}{2} \cdot 3 \cdot 2 \cdot 3^3 \cdot 7^2$ outcomes of this form.

- 1 integer from A and 2 from each of B and C. Such sequences can be formed by deciding which one of the integers x_i will be chosen from A (there are 5 possibilities for this), then which two of the four remaining integers will be chosen from B ($\binom{4}{2}$ possibilities). We then need to choose the values of each of the integers, for which there are 3 possibitilies for the one from A, 7^2 possibilities for the two from B, and 3^2 possibilities for the two from C. In total this gives $5 \cdot \binom{4}{2} \cdot 3^3 \cdot 7^2$ outcomes of this form.

Summing over all cases, we conclude that there are

$$\left(\binom{5}{3} + 3 \cdot 2 \cdot \binom{5}{2} + 5 \cdot \binom{4}{2}\right) 3^3 \cdot 7^2 = 100 \cdot 27 \cdot 49$$

outcomes where this event occurs, and so the probability is $\frac{100 \cdot 27 \cdot 49}{20^5} = \frac{27 \cdot 49}{4 \cdot 20^3} = \frac{1323}{32000}$. $\square$

Feedback. Both parts of this question are excellent examples of how sets can be counted by visualising them as a sequence of choices. For example, in (a) we form outcomes by first choosing i, then j, then which integers are congruent to i (note this term is easily omitted), and then the actual values of each integer, and multiplying the number of possibilities at each step. You can of course make these choices in a different order, or there are alternative sequences of choices you could make which lead to the same answer.

Part (b) illustrates another key idea for this kind of question, as seen in an example in the lectures, where we have to split into cases to count the choices. This is necessary because if we were just to choose first the three integers which are divisible by 2, then without knowing which of these are divisible by 3 we do not know how many more integers divisible by 3 we require. So we consider separately the different possibilities for the number of integers which are divisible by both 2 and 3; once you specify this number then you can form the corresponding outcomes by a sequence of choices similarly as in (a).

The solution to (a) can actually be simplified by observing that the actual values of x_1, x_2, x_3, x_4 and x_5 do not matter, only their congruence modulo 5, and that this also is uniformly random. So if we choose $y_1, y_2, \ldots, y_5 \in \{0, 1, 2, 3, 4\}$ with $y_i \equiv x_i \pmod 5$, then each y_i is chosen uniformly at random and independently of previous choices. There are then 5^5 possible outcomes for the sequence $(y_1, y_2, y_3, y_4, y_5)$, and the number of outcomes in which the described event occurs is $\binom{5}{3} \cdot 5 \cdot 4 = 8 \cdot 5^2$ by similar arguments as in the model solution.