Example sheet 6 – formative

1. Locate the equilibrium points of the nonlinear dynamical system,

$$\dot{x} = \mu - xy^2,$$
$$\dot{y} = xy^2 - y,$$

where $\mu > 0$ is a constant. Classify the equilibrium points for each $\mu > 0$.

Solution:

The equilibrium points satisfy

$$\mu - xy^2 = 0, \quad xy^2 - y = 0.$$

Therefore, the nonlinear system has a unique equilibrium point at $\left(\frac{1}{\mu}, \mu\right)$. We next classify this equilibrium point via the linearization theorem. We write

$$x = \frac{1}{\mu} + \overline{x}, \quad y = \mu + \overline{y}, \tag{1}$$

with the associated linear system in the $(\overline{x}, \overline{y})$ coordinates being given by

$$\begin{aligned} \dot{\overline{x}} &= -\mu^2\overline{x} - 2\overline{y}, \\ \dot{\overline{y}} &= \mu^2\overline{x} + \overline{y}. \end{aligned} \tag{2}$$

The linear system (2) has an unique equilibrium point at $(0,0)$, and $\mathbf{A} = \begin{pmatrix} -\mu^2 & -2 \\ \mu^2 & 1 \end{pmatrix}$.

Note, the matrix $\mathbf{A}$ can also be obtained form the Jacobian

$$\mathbf{J} = \begin{pmatrix} -y^2 & -2xy \\ y^2 & 2xy - 1 \end{pmatrix}.$$

The eigenvalues, λ, of $\mathbf{A}$ are determined by evaluating $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, that is on evaluating

$$\begin{vmatrix} -\mu^2 - \lambda & -2 \\ \mu^2 & 1 - \lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^2 + \lambda(\mu^2 - 1) + \mu^2 = 0.$$

The eigenvalues of $\mathbf{A}$ are then given by

$$\lambda_{\pm} = -\frac{(\mu^2 - 1)}{2} \pm \frac{1}{2}\sqrt{(\mu^2 - 1)^2 - 4\mu^2}. \tag{3}$$

Consideration of the eigenvalues (3) indicates that we have the following three cases to consider:

(i) $\mu > 1$. $\mathrm{Re}(\lambda_\pm) < 0$ for all $\mu > 1$ with

$\lambda_\pm$ both real and distinct for $\mu^2 > 3 + 2\sqrt{2}$ or $\mu > 1 + \sqrt{2}$,
$\lambda_\pm$ both real and equal for $\mu^2 = 3 + 2\sqrt{2}$ or $\mu = 1 + \sqrt{2}$,
$\lambda_\pm$ complex conjugate for $1 < \mu^2 < 3 + 2\sqrt{2}$ or $1 < \mu < 1 + \sqrt{2}$.

Therefore, the equilibrium point $\left(\frac{1}{\mu}, \mu\right)$ of this system is, via the linearization theorem, a:

a. stable node when $\mu^2 > 3 + 2\sqrt{2}$ or $\mu > 1 + \sqrt{2}$.
b. stable degenerate node for $\mu^2 = 3 + 2\sqrt{2}$ or $\mu = 1 + \sqrt{2}$.
c. stable spiral for $1 < \mu^2 < 3 + 2\sqrt{2}$ or $1 < \mu < 1 + \sqrt{2}$.

(ii) $0 < \mu < 1$. $\mathrm{Re}(\lambda_\pm) > 0$ for all $0 < \mu < 1$ with

$\lambda_\pm$ both real and distinct for $0 < \mu^2 < 3 - 2\sqrt{2}$ or $0 < \mu < -1 + \sqrt{2}$,
$\lambda_\pm$ both real and equal for $\mu^2 = 3 - 2\sqrt{2}$ or $\mu = -1 + \sqrt{2}$,
$\lambda_\pm$ complex conjugate for $3 - 2\sqrt{2} < \mu^2 < 1$ or $-1 + \sqrt{2} < \mu < 1$.

Therefore, the equilibrium point $\left(\frac{1}{\mu}, \mu\right)$ of this system is, via the linearization theorem, an:

a. Unstable node when $0 < \mu^2 < 3 - 2\sqrt{2}$ or $0 < \mu < -1 + \sqrt{2}$.
b. Unstable degenerate node for $\mu^2 = 3 - 2\sqrt{2}$ or $\mu = -1 + \sqrt{2}$.
c. Unstable spiral for $3 - 2\sqrt{2} < \mu^2 < 1$ or $-1 + \sqrt{2} < \mu < 1$.

(iii) $\mu = 1$. In this case the eigenvalues are purely imaginary, $\lambda_\pm = \pm i$, and the linearized theory does not classify the equilibrium point in this case. The equilibrium point may be a centre or it may be a nonlinear spiral.

2. Consider the nonlinear dynamical system

$$\dot{x} = 2x - ay + y^2,$$
$$\dot{y} = 4bx + y - \frac{1}{2}x^2,$$

where $(x, y) \in \mathbb{R}^2$, and a, b are constants.

(a) Determine the nature of the equilibrium point $(0, 0)$ for all a, $b \geqslant 0$. Sketch the (a, b)-plane.

(b) For the case $a = 5$, $b = \frac{1}{8}$, classify each of the equilibrium points and sketch the global phase portrait of the system. You should make sure to note the following on your sketch:

- Location of equilibrium points;
- Straight line paths and which eigenvector they correspond to (if applicable);
- Horizontal and vertical isoclines, and the direction of flow along them.

Make sure to give yourself enough room to sketch the phase portrait clearly, with clear labelling.

Solution:

(a) The Jacobian of the system is given by

$$\mathbf{J} = \begin{pmatrix} 2 & -a + 2y \\ 4b - x & 1 \end{pmatrix},$$

which at $(0, 0)$ reduces to

$$\mathbf{J} = \begin{pmatrix} 2 & -a \\ 4b & 1 \end{pmatrix},$$

with characteristic equation $\lambda^2 - 3\lambda + 2 + 4ab = 0$. This gives eigenvalues

$$\lambda_{\pm} = \frac{3}{2} \pm \frac{1}{2}\sqrt{1 - 16ab}.$$

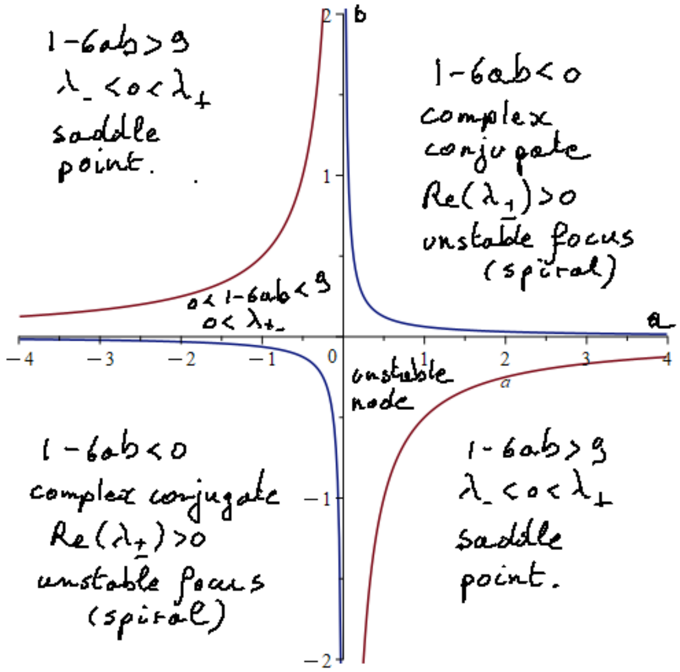
$1-16ab = 0$ forms the dividing line between the area in the (a, b)-plane where the eigenvalues are real or complex conjugate. This curve is given by

$$b = \frac{1}{16a}.$$

Outside of these curves (larger values for a and b), the eigenvalues will be a complex conjugate pair, and, between the two curves, the eigenvalues will be real. λ_+ will always be positive, but λ_- could be positive or negative, depending on whether $1 - 16ab < 9$ or $1 - 16ab > 9$ respectively. The dividing curve for these two options is given by

$$b = -\frac{1}{2a}.$$

This is summarised in the plot below.



(b) For $a = 5, b = \frac{1}{8}$, we get

$$\dot{x} = 2x - 5y + y^2,$$
$$\dot{y} = \frac{x}{2} + y - \frac{1}{2}x^2,$$

Putting $\dot{y} = 0$ yields an expression for y,

$$y = \frac{1}{2}x\,(x-1)\,,$$

which when substituted into the right hand side of the $\dot{x}$ equation, gives the expression

$$2x - 5\frac{1}{2}x\,(x-1) + \frac{1}{4}x^2\,(x-1)^2 = \frac{x}{4}\left(x^3 - 2x^2 - 9x + 18\right) = \frac{x}{4}\,(x-2)\,(x-3)\,(x+3)\,.$$

This yields four equilibreium points at $(0,0)\,,(2,1)\,,(3,3)$ and $(-3,6)$. The Jacobian is given by

$$\mathbf{J} = \begin{pmatrix} 2 & -5+2y \\ \frac{1}{2} - x & 1 \end{pmatrix},$$

1. Equilibrium $(0,0)$:

$$\mathbf{J}\,(0,0) = \begin{pmatrix} 2 & -5 \\ \frac{1}{2} & 1 \end{pmatrix},$$

with eigenvalues

$$\lambda_{\pm} = \frac{3}{2} \pm \frac{3}{2}i.$$

Because the real part of the complex eigenvalues is positive, we have an unstable spiral.

2. Equilibrium $(2,1)$:
$$\mathbf{J}(2,1) = \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & 1 \end{pmatrix},$$
with eigenvalues
$$\lambda_{\pm} = \frac{3}{2} \pm \frac{\sqrt{19}}{2}.$$
As $\lambda_- < 0$ this is a saddle point. The eigenvectors are given by
$$\lambda_+ : \quad \mathbf{v}_+ = \left(1, \frac{1-\sqrt{19}}{6}\right)^{\mathsf{T}} \approx (1, -0.56)^{\mathsf{T}},$$
$$\lambda_- : \quad \mathbf{v}_- = \left(1, \frac{1+\sqrt{19}}{6}\right)^{\mathsf{T}} \approx (1, 0.89)^{\mathsf{T}}.$$

3. Equilibrium $(3,3)$:
$$\mathbf{J}(3,3) = \begin{pmatrix} 2 & 1 \\ -\frac{5}{2} & 1 \end{pmatrix},$$
with eigenvalues
$$\lambda_{\pm} = \frac{3}{2} \pm \frac{3}{2}i.$$
As the real part of the eigenvalues is positive, this is an unstable spiral.

4. Equilibrium $(-3,6)$:
$$\mathbf{J}(-3,6) = \begin{pmatrix} 2 & 7 \\ \frac{7}{2} & 1 \end{pmatrix},$$
with eigenvalues
$$\lambda_{\pm} = \frac{3}{2} \pm \frac{3}{2}\sqrt{11}.$$
As $\lambda_- < 0$ this is a saddle point. The eigenvectors are given by
$$\lambda_+ : \quad \mathbf{v}_+ = \left(1, -\frac{1-3\sqrt{11}}{14}\right)^{\mathsf{T}} \approx (1, 0.64)^{\mathsf{T}},$$
$$\lambda_- : \quad \mathbf{v}_- = \left(1, -\frac{1+3\sqrt{11}}{14}\right)^{\mathsf{T}} \approx (1, -0.78)^{\mathsf{T}}.$$

For the horizontal isocline, we need
$$y = \frac{1}{2}x(x-1),$$

which is an upwards parabola with zero's at $x = 0$ and $x = 1$. The sign of $\dot{x}$ is determined by

$$\dot{x} = \frac{x}{4}\left(x-2\right)\left(x-3\right)\left(x+3\right).$$

so the flow is to the right for $x < -3$, $0 < x < 2$ or $3 < x$, whilst it is to the left for $-3 < x < 0$ or $2 < x < 3$.

For the vertical isocline, we need

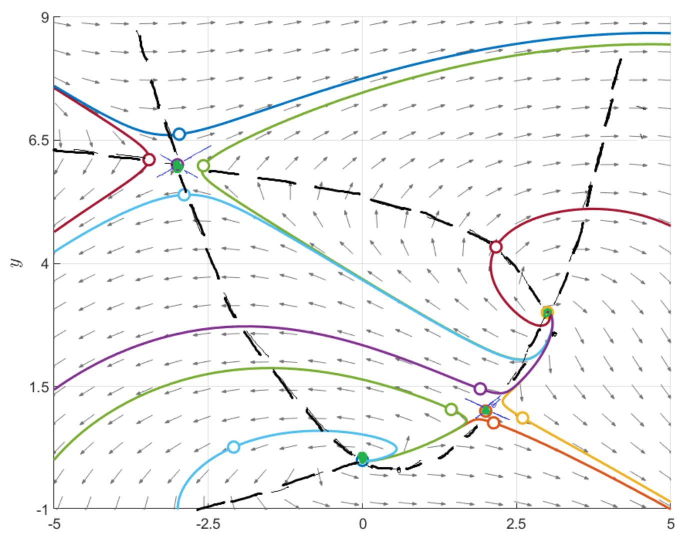
$$x = -\frac{1}{2}y\left(y-5\right),$$

which is a parabola opening op downwards with zero's at $y = 0$ and $y = 5$ in the (y, x)-plane, when flipped into the (x, y)-plane, it is a parabola opening to the left and through the points $(0, 0)$ and $(0, 5)$. The most right point is at $\left(\frac{25}{8}, \frac{5}{2}\right)$. The sign of $\dot{y}$ is determined by

$$\dot{y} = -\frac{y}{8}\left(y^3 - 10y^2 + 27y - 18\right) = -\frac{y}{8}\left(y-1\right)\left(y-3\right)\left(y-6\right).$$

Hence the flow is upwards for $y < 0, 1 < y < 3$ and $y > 6$, and downwards elsewhere.

The phase portrait is given in the Figure below.



An heteroclinic orbit from $(3, 3)$ towards the saddle at $(-3, 6)$ separates those trajectories going upwards and to the right from those going to the left. Another heterocliic orbit connects $(3, 3)$ and $(2, 1)$ and the last heteroclinic orbit goes from $(0, 0)$ to $(2, 1)$