2RCA Problem Sheet 4

Questions marked (SUM) are assessed. Solutions to these questions should be submitted by the deadline of 17:00 Thursday 21st March.

1) Determine the centre and radius of convergence of the following power series:
   a) $\sum_{n=0}^{\infty} n3^{-n}(z-2i)^n$
   b) $\sum_{n=0}^{\infty} \frac{1}{n}(z+5+i)^{3n}$
   c) $\sum_{n=0}^{\infty}(3+(-1)^n)^{-n}(z+i)^n$.
2) Find all the complex solutions of the following equations:
   a) $e^{iz}=1-i$.
   b) $\sinh z = i$.
   c) $\cos z = 5$.
3) Provide a parameterisation γ of the line segment with endpoints 0 and $2-i$ oriented from 0 towards $2-i$. Compute the contour integral $\int_\gamma z^2+5\overline{z}\,dz$.
4) Show that

$$\sin(x+iy)=\cosh y\sin x + i\sinh y\cos x$$

   for all $x, y \in \mathbb{R}$. Derive a similar identity for $\cos(x+iy)$.

5) Let $\Gamma \subseteq \mathbb{C}$ be the set formed by the union of the line segment with endpoints 1 and i, the line segment with endpoints i and -1, and the set

$$\{z \in \mathbb{C} :\ |z| = 1, Im(z) \leq 0\}.$$

   Sketch Γ and show that Γ is a piecewise smooth, simple, closed contour. Suppose Γ is given the anticlockwise orientation. Provide a simple, closed contour $\gamma: [a, b] \to \mathbb{C}$ which parameterises Γ and respects the orientation on Γ.
6) (SUM)
   a) Compute

$$\int_\Gamma z^2+2\overline{z}\,dz$$

   where Γ is the straight line segment connecting i to $1+i$, oriented from i towards $1+i$.

   b) Compute

$$\int_\Gamma \overline{z}^3\,dz$$

   where $\Gamma = \{z \in \mathbb{C} :\ |z-i| = 1, Im(z) \geq 1\}$, oriented from right to left.

7) (SUM) Let Γ be the circular arc given by

$$\Gamma = \{z \in \mathbb{C} : \ |z| = 5, 0 \leq Arg(z) \leq \frac{\pi}{3}\},$$

oriented in the anticlockwise direction. Use the ML-Lemma to show that

$$\left|\int_{\Gamma} \frac{e^z}{z^2+1}\, dz\right| \leq \frac{5\pi e^5}{72}.$$

8) (SUM) Evaluate the following integrals:

a) $\int_{\Gamma} \frac{\sin z}{z^2+2iz-2}\, dz,$

where Γ is the unit circle with the anticlockwise orientation.

b) $\int_{\gamma} \cosh 3z \ dz,$

where $\gamma: [0, \pi] \to \mathbb{C}$ is given by the formula $\gamma(t) = i + e^{it}$.

9) Let $f: \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = (z^3 - 3iz^2 + 5z - 2)e^z(\cosh z)^7$. Suppose that $g: \mathbb{C} \to \mathbb{C}$ is a holomorphic function for which $g(z) = f(z)$ whenever $z \in \mathbb{C}$ and $|z| = 1$. Show that $f(z) = g(z)$ for all $z \in \mathbb{C}$.

10) (SUM) Express the function $\frac{1}{(z+2)(z+5)}$ in the following ways:

a) As a Taylor series with centre 1 for $z \in B(1,1)$.
b) As a Laurent series with centre 0 for $|z| > 5$.
c) As a Laurent series with centre -2 for $0 < |z+2| < 3$.