Example sheet 2: Analytical solutions to ODEs

1. Show that $u_1 = e^{-x}$ is one solution of
$$xy'' + (x-1)y' - y = 0, \quad x > 0,$$
and use the Reduction of Order method to obtain the general solution.

2. Show that $u_1 = x$ is one solution of
$$y'' + xy' - y = 0, \quad x > 0,$$
and use the Reduction of Order method to obtain the general solution in terms of an integral (note that you do not need to evaluate the final integral).

3. Find a solution to the following equation in the form of x^r for some constant r and use this solution to obtain the general solution using the Reduction of Order method.
$$x^2y'' - 3xy' + 4y = 0, \quad x > 0,$$

4. You have seen in previous modules that the solution to a 2nd order linear ODE with constant coefficients whose characteristic equation has a repeated root, a say, is of the form $y = \alpha_1 x e^{ax} + \alpha_2 e^{ax}$. By looking for a solution of the form $u_1 = e^{rx}$ to
$$y'' - 2ay' + a^2y = 0,$$
and then using the Reduction of Order method with this first solution, verify that this is the case.

5. Given that $u_1 = 1 + x$ and $u_2 = e^x$ form a fundamental set of solutions to the homogeneous version of
$$xy'' - (1+x)y' + y = x^2e^{2x}, \quad x > 0,$$
use the Variation of Parameters method to obtain the general solution to the inhomogeneous equation. What is the particular solution of this equation?

6. Find the general solution of
$$y'' + 4y' + 4y = x^{-2}e^{-2x}, \quad x > 0$$
using the Variation of Parameters method. What is the particular solution of this equation?

7. Given that $u_1(x) = x$ is a solution to
$$x^2y'' - 2xy' + 2y = 0, \quad x > 0,$$

   (a) find the general solution to this equation;

(b) and find the particular solution to

$$x^2y'' - 2xy' + 2y = 4x^2, \quad x > 0.$$

8. Given that $u_1(x) = e^x$ is a solution to

$$(1-x)y'' + xy' - y = 0, \quad 0 < x < 1,$$

(a) find the general solution to this equation;

(b) and find the particular solution to

$$(1-x)y'' + xy' - y = 2(x-1)^2e^{-x}, \quad 0 < x < 1.$$

9. (a) By looking for solutions to

$$y''' - y'' = 0,$$

in the form $y = e^{rx}$, find a fundamental set of solutions to the above equation.

(b) Use the Variation of Parameters method to find the particular solution to

$$y''' - y'' = e^x.$$

10. Use the Reduction of Order method for **nonlinear** ODEs to obtain solutions to the following ODE for $y = y(x)$:

$$y'' + y(y')^3 = 0.$$

You do not need to express the solution explicitly for $y = y(x)$.

11. Use the Reduction of Order method for nonlinear ODEs to obtain solutions to the following ODE for $y = y(x)$:

$$yy'' + (y')^2 = yy', \quad y > 0.$$

12. If we have the 2nd order linear homogeneous ODE

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{1}$$

where

$$a''(x) - b'(x) + c(x) = 0 \tag{2}$$

then the equation is called **exact** and can be transformed into the 1st order (inhomogeneous) linear equation in y:

$$a(x)y' + (b(x) - a'(x))y = \alpha_1 \tag{3}$$

where α_1 is a constant. By differentiating (3) prove that the two equations are equivalent.

13. Solve $x^2 y'' + xy' - y = 0$ on $x > 0$.

14. Check that the following equation is an exact 2nd order linear ODE and solve it in terms of an integral (note that you do not need to evaluate the integral).

$$y'' + xy' + y = 0$$

15. If a 2nd order linear ODE is not exact, it can be made exact by multiplying through by an appropriate integrating factor $\mu(x)$:

$$\mu(x)a(x)y'' + \mu(x)b(x)y' + \mu(x)c(x)y = 0$$

where $\mu(x)$ is chosen to satisfy the condition

$$[\mu(x)a(x)]'' - [\mu(x)b(x)]' + \mu(x)c(x) = 0, \quad \text{(taken directly from (2))}.$$

Derive a 2nd order linear ODE that must be solved to obtain $\mu(x)$. This equation is called the **adjoint** of the original equation. [However, solving the adjoint equation to find $\mu(x)$ is usually just as hard as solving the original equation, so this approach is not often much use...]

16. A 2nd order linear equation is called **self-adjoint** if its adjoint is the same as the original equation. Show that Airy's equation (use in optics and astronomy, amongst other applications):

$$y'' - xy = 0$$

is self-adjoint.

17. Show that a necessary condition for the general 2nd order linear homogeneous ODE

$$a(x)y'' + b(x)y' + c(x)y = 0$$

to be self-adjoint is that

$$a'(x) = b(x).$$