University of Birmingham
School of Mathematics

2RCA/2RCA3 Real and Complex Analysis

Part A: Real Analysis Semester 2

Problem Sheet 2

The questions indicated with $\boxed{\text{SUM}}$ below constitute the second of the summative assessments of this module, and will contribute to the overall mark of the module. Please submit your answers to the questions indicated with a $\boxed{\text{SUM}}$ below by the deadline of **17:00, Thursday 15 February 2024**, as a single pdf file into 2RCA/2RCA3 Assignment 2: Real Analysis in the Canvas page of the module.

Please note that it is the student's responsibility to make sure that their submission has been uploaded correctly into Canvas and that the uploaded file contains the submission of their assessment (eg, the uploaded file is not corrupted and contains all the pages of their answers).

Please be also aware that where assessments are submitted late without an extension being granted that has been confirmed by the Wellbeing Officer, the standard University penalty of a 5% will be imposed for each working day that the assignment is late. Any work submitted after five working days passed the deadline of submission, with no extension granted by the Wellbeing Officer/s, will be awarded a 0% mark.

In addition to the $\boxed{\text{SUM}}$-questions, Problem Sheet 2 also contains exercises that will not contribute to your module mark. You are strongly encouraged to attempt these before the relevant Guide Study sessions and/or during the course of the semester. Solutions to all exercises will be provided.

The examples/feedback classes (Guided Study) and the lecturer's office hours should be used to ask about the problem sheets.

Q1. Using the definition, show that the sequence $\{f_n\}_{n=1}^{\infty}$ defined by

$$f_n(x) = \begin{cases} 1 - nx, & \text{if} \quad 0 \le x \le \frac{1}{n}, \\ 0, & \text{if} \quad \frac{1}{n} < x \le 1, \end{cases}$$

converges pointwise to the function $f : [0,1] \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 < x \le 1, \\ 1, & \text{if} \quad x = 0. \end{cases}$$

Q2. Using the definition of uniform convergence, show that the sequence $\{f_n\}_{n=1}^{\infty}$ defined by

$$f_n(x) = \frac{x}{1+xn}, \qquad x > 0$$

converges uniformly to the function f given by

$$f(x) = 0 \qquad x > 0.$$

Q3. Find the pointwise limit (if exists) of the sequence $\{f_n\}$ of functions defined from X into $\mathbb{R}$ in each of the following cases. In each of the examples determine whether the convergence is uniform.

(i) $X = [0,1]$,

$$f_n(x) = \begin{cases} 1 - nx, & \text{if} \quad 0 \le x \le \frac{1}{n}, \\ 0, & \text{if} \quad \frac{1}{n} < x \le 1. \end{cases}$$

(ii) $X = [0,1]$, $f_n(x) = x^n$.

(iii) $X = [0,1]$, $f_n(x) = \dfrac{x^n}{n}$.

Remark: In the above exercises, if a pointwise limit f of the sequence $\{f_n\}$ exists, you are not require to justify why. You just need to find a explicit formula for the pointwise limit.

Q4. Let $(f_n)_{n=1}^{\infty}$ be the sequence of functions $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f_n(x) = \left(\sin\left(\frac{x}{\pi}\right)\right)^{2n}, \qquad \text{for all} \qquad x \in \mathbb{R}.$$

(i) Find the pointwise limit of the sequence $(f_n)_{n=1}^{\infty}$. Justify your answer.

(ii) Does the sequence $(f_n)_{n=1}^{\infty}$ converge uniformly on $\mathbb{R}$? Justify your answer.

SUM **Q5**. Let $(f_n)_{n=4}^{\infty}$ be the sequence of functions $f_n : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} \dfrac{n^4}{16}\left(x - \dfrac{4}{n^2}\right)^2, & \text{if} \quad x \in \left[0, \dfrac{4}{n^2}\right], \\ 0, & \text{if} \quad x \in \left(\dfrac{4}{n^2}, \dfrac{9}{n^2}\right), \\ \dfrac{n^2}{n^2-9}\left(x - \dfrac{9}{n^2}\right), & \text{if} \quad x \in \left[\dfrac{9}{n^2}, 1\right]. \end{cases}$$

(i) Find the pointwise limit of the sequence $(f_n)_{n=4}^{\infty}$.

(ii) Does the sequence $(f_n)_{n=4}^{\infty}$ converge uniformly on $[0,1]$ to its pointwise limit?

Justify your answers. (In particular, state any result you use in the justification of your answer and justify its application.)

Q6. Suppose that $\alpha \in \mathbb{R}$ and the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y|^{\alpha}, \qquad \text{for all} \qquad x \neq y. \tag{1}$$

(i) Show that if $\alpha > 1$ then f is differentiable at every $x_0 \in \mathbb{R}$.

(ii) Give an example of a function satisfying (1) for $\alpha = 1$, which fails to be differentiable at 0.

Q7. Let $\alpha, \beta \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} \alpha x^2 + \beta & \text{if } x \leq 1 \\ 4x & \text{if } x > 1. \end{cases}$$

For which values of α and β is g continuous at $x = 1$? For which values of α and β is g differentiable at $x = 1$? Justify any assertions that you make.

Q8. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x \sin(x) \sin\left(\frac{1}{x}\right) & \text{if } x \in (-1, 1) \text{ and } x \neq 0 \\ 0 & \text{if } x = 0 \\ x + 1 & \text{if } x \in (-\infty, -1] \cup [1, \infty). \end{cases}$$

Show that f is differentiable at 0 and give the value of $f'(0)$. Justify your answer.

Q9. Let $\alpha \in \mathbb{N}$ be fixed and let the function $f : (-\frac{\pi}{4}, \frac{\pi}{4}) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{x^{3\alpha} \cos(x)}{\sin(x)}, & \text{if } x \neq 0 \quad \text{and} \quad x \in (-\frac{\pi}{4}, \frac{\pi}{4}), \\ 0, & \text{if } x = 0. \end{cases}$$

(i) Show that f is a differentiable function for all $\alpha \in \mathbb{N}$ and give the value of the derivative $f'(x)$ for all $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$.

(ii) Is f', the derivative of f, a continuous function at 0 when $\alpha = 2$?

Justify your answers.

Q10. Let $\alpha, \beta \in \mathbb{Z}$ be fixed and let the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^{\alpha} (\sin(x))^{\beta} \cos\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(i) Show that f is differentiable at 0 if and only if $\alpha + \beta > 1$. In the case $\alpha + \beta > 1$, give the value of $f'(0)$.

(ii) In the case $\alpha + \beta > 1$, explain why f is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and give the value of $f'(x)$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(iii) For which values of α and β is f continuous at $x = 0$?

Justify your answers.

Q11. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} (x-1)^2 \cos(\frac{1}{x-1}), & \text{if } x \neq 1 \\ 0, & \text{if } x = 1. \end{cases}$$

Determine those $x_0 \in \mathbb{R}$ for which f is differentiable at x_0. Give the value of $f'(x_0)$. If $f'(x_0)$ does not exist, you must justify why it does not exists..

(ii) For the function f given in Part (i), determine whether the statement

$$f'(1) = \lim_{x \to 1} f'(x)$$

is true or false. Justify your answer.

Q12. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} (\tan x)^2 \sin\left(\frac{1}{x}\right), & \text{if } x \in (-1,1) \text{ and } x \neq 0, \\ 0, & \text{if } x = 0, \\ x+1, & \text{if } x \in (-\infty,-1] \cup [1,\infty). \end{cases}$$

Show that f is differentiable at 0 and give the value of $f'(0)$.

Q13. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \max\{x, x^3\}$. Determine those $x_0 \in \mathbb{R}$ for which f is differentiable at x_0. Give the value of $f'(x_0)$ when it exists, and prove any assertions that you make. If $f'(x_0)$ does not exist, you must prove why.

Q14. You are given that the function $f : [0,\pi] \to [-1,1]$ given by

$$f(x) = \cos(x) \qquad \text{for each } x \in [0,\pi]$$

is continuous and strictly decreasing on $[0,\pi]$, and therefore the inverse function $f^{-1} : [-1,1] \to [0,\pi]$ is well-defined.

(i) Fix $x_0 \in (0,\pi)$. Prove that f is differentiable at x_0 and

$$f'(x_0) = -\sin(x_0).$$

(ii) Fix $y_0 \in (-1,1)$. Prove that f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = -\frac{1}{\sqrt{1-y_0^2}}.$$

Q15. Using the Intermediate Value Theorem and Rolle's Theorem, show that the equation

$$e^x - x - 2 = 0$$

has exactly one positive real solution.
Hint: Use the Intermediate Value Theorem to prove that the equation has al least one positive real solution. Using proof by contradiction and Rolle's theorem show that the equation has exactly one positive real solution.

$\boxed{\text{SUM}}$ **Q16**. Using the Intermediate Value Theorem and Rolle's Theorem, show that the equation

$$2x - 1 - \sin(x) = 0$$

has exactly one real solution.
Hint: Use the Intermediate Value Theorem to prove that the equation has al least one real solution. Using proof by contradiction and Rolle's theorem show that the equation has exactly one real solution.

Q17. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable, and that $f'(x) > 1$ for all $x \in (a, b)$. Using Rolle's theorem, show that f has at most one fixed point; i.e. there exists at most one point $x_0 \in (a, b)$ such that $f(x_0) = x_0$.

Q18. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \cos^2(3x) \qquad \text{for each } x \in \mathbb{R}.$$

Use the Mean Value Theorem to prove that for all $a, b \in [-\frac{\pi}{24}, \frac{\pi}{24}]$ we have

$$|f(a) - f(b)| \leq \frac{3}{\sqrt{2}}|a - b|.$$

Hint: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Q19. Use the Mean Value Theorem to prove that for all $a, b \in (\frac{\pi}{8}, \frac{\pi}{4})$ we have

$$|\tan(2a) - \tan(2b)| \geq 4|a - b|.$$

Hint: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Q20. Using the Mean Value Theorem, prove that

$$\arctan(x) < x \quad \text{for all} \qquad x > 0.$$

Q21. Using the Mean Value Theorem, show that

$$|\arcsin(x)| \geq |x| \quad \text{for all} \qquad x \in (-1, 1).$$

Q22. Let $f : (-\frac{\pi}{4}, \frac{\pi}{4}) \to (-1, 1)$ be given by

$$f(x) = \sin(2x) \qquad \text{for each } x \in (-\tfrac{\pi}{4}, \tfrac{\pi}{4}).$$

Use the Inverse Function Theorem for differentiable functions and the Mean Value Theorem to prove that for all $a, b \in [-\frac{1}{10}, \frac{1}{10}]$ we have

$$|f^{-1}(a) - f^{-1}(b)| \leq \frac{5}{3\sqrt{11}}|a - b|.$$

[Here, $f^{-1} : (-1, 1) \to (-\frac{\pi}{4}, \frac{\pi}{4})$ denotes the inverse function of f.]

SUM **Q23**. Using the Mean Value Theorem show that

$$\arccos(x) - \frac{\pi}{2} \leq -x \qquad \text{for all} \qquad x \in [0, 1].$$

Here, $\arccos : [-1, 1] \longrightarrow [0, \pi]$ is the inverse of the cosine function.

Q24. Suppose that $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions on $\mathbb{R}$ and that

$$0 < e^x f'(x) \leq x g'(x) \qquad \text{for all} \quad x \in [1, 10]$$

Show that for any $a, b \in [1, 10]$ with $a < b$

$$\frac{f(b) - f(a)}{g(b) - g(a)} < 10.$$

L'Hôpital's Rule

(i) From lectures, we have the following case of L'Hôpital's Rule:

0/0 form of L'Hôpital's Rule. Suppose that f and g are differentiable on (a,b). Suppose further that $x_0 \in (a,b)$ and that $g'(x) \neq 0$ for all $x \in (a,b) \setminus \{x_0\}$. If $f(x_0) = g(x_0) = 0$ and

$$\frac{f'(x)}{g'(x)} \to A \quad \text{as } x \to x_0,$$

then

$$\frac{f(x)}{g(x)} \to A \quad \text{as } x \to x_0.$$

(ii) For verifying the hypothesis that $g'(x) \neq 0$ for x in a *punctured* interval around the point x_0, the following result from lectures is very useful.

Lemma from lecture notes. Suppose that g, g', g'', …, $g^{(n)}$ exist and are continuous on (a,b), and that for some point $x_0 \in (a,b)$,

$$g^{(k)}(x_0) = 0 \text{ whenever } 0 \leq k \leq n-1 \qquad \text{and} \qquad g^{(n)}(x_0) \neq 0.$$

Then there exists $\delta \in \mathbb{R}^+$ such that

$$g^{(k)}(x) \neq 0 \quad \text{whenever } 0 \leq k \leq n \text{ and } 0 < |x - x_0| < \delta.$$

SUM **Q25**. Using L'Hôpital's Rule, show that the function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{1}{x^2} - \dfrac{1}{(\tan(x))^2}, & \text{if } x \neq 0, \\[2ex] \dfrac{2}{3}, & \text{if } x = 0 \end{cases}$$

is differentiable at $x_0 = 0$ and give he value of $f'(0)$.

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Q26. Using L'Hôpital's Rule, show that the function $f : (-\frac{\pi}{3}, \frac{\pi}{3}) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{\sin x}, & \text{if } x \neq 0, \\[2ex] 0, & \text{if } x = 0 \end{cases}$$

is differentiable at $x_0 = 0$ and give the value of $f'(0)$.

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Q27. Use l'Hopital's Rule to evaluate the following limits:

(i)
$$\lim_{x\to 0} \frac{\sin^2 x}{1-\cos x}.$$

(ii)
$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{\tan(x)}.$$

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Q28. Using L'Hôpital's Rule evaluate the following limit
$$\lim_{x\to 0} \frac{\cos(x) - \cos(2x)}{\sin^2(x) + x\sin^2(x)}.$$

Note: As always, if you are going to use a theorem, be sure to justify each of the hypotheses of the theorem before applying it.

Q29. Using Taylor's Theorem, show that
$$\sin x \geq x - \frac{x^3}{6}, \quad \text{for all} \quad x \geq 0.$$

Q30. Using Taylor's Theorem, show that for each $t \in [0, 1)$,
$$\sqrt{1+t} = 1 + \sum_{k=1}^{\infty} c_k t^k,$$
where
$$c_k = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(k-1)\right)}{k!}.$$

Hint: Consider the behaviour of the remainder term $R_n(t)$ given by Taylor's Theorem as $n \to \infty$.

Q31. Use Taylor's Theorem to prove that for each $t \in \mathbb{R}$, the series
$$\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$
converges to $\cos(t)$. [You may use without proof the fact that for any $t \in \mathbb{R}$, $\frac{t^n}{n!} \to 0$ as $n \to \infty$.]