Example sheet 3 – formative

1. Consider the second order linear differential equation

$$\ddot{x} + 2\dot{x} - 3x = 0.$$

   (a) Recast the problem as a system of first order differential equations
   (b) Write the system in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Show that the eigenvalues of $\mathbf{A}$ are given by $\lambda_1 = 1$, $\lambda_2 = -3$, and find the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$
   (c) Use your answer to part (b) to classify the equilibrium point $(0,0)$.

Solution:

(a) Writing $\dot{x} = y$, we have

$$\dot{x} = y,$$
$$\dot{y} = 3x - 2y.$$

(b) We can write the system as

$$\dot{x} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \mathbf{x}. \tag{1}$$

The eigenvalues of the matrix $\mathbf{A}$ are solutions of

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= 0, \\ \Rightarrow \begin{pmatrix} -\lambda & 1 \\ 3 & -2-\lambda \end{pmatrix} &= 0, \\ \Rightarrow \lambda^2 + 2\lambda - 3 &= 0, \\ \Rightarrow (\lambda+3)(\lambda-1) &= 0. \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -3$.

(c) To find the eigenvectors, write $\mathbf{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\mathbf{v}_1$ is the solution to

$$\begin{aligned} \begin{pmatrix} -\lambda_1 & 1 \\ 3 & -2-\lambda_1 \end{pmatrix} \mathbf{v}_1 &= 0, \\ \Rightarrow \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 0, \\ \Rightarrow y &= x. \end{aligned}$$

Thus, for simplicity, pick $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly, for $\mathbf{v}_2$, we have

$$\begin{pmatrix} -\lambda_2 & 1 \\ 3 & -2-\lambda_2 \end{pmatrix} \mathbf{v}_2 = 0,$$
$$\Rightarrow \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0,$$
$$\Rightarrow y = -3x.$$

Finally, again for simplicity, pick $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

(d) The eigenvalues of the matrix A are real and of opposite sign. Therefore the equilibrium point $(0,0)$ is a saddle node.

2. Consider the second order differential equation

$$\ddot{x} - \dot{x} - 2x = 0, \quad x \in \mathbb{R}.$$

(a) Write the above ODE as a system of first order differential equations.

(b) Determine the location and nature of the equilibrium points of the resulting system.

(c) Determine the horizontal and vertical isoclines and the direction of the flow along them.

(d) Hence sketch the phase plane of the system, indicating all qualitatively different solutions.

Solution:

(a) Write $\dot{x} = y$ then $\ddot{x} - \dot{x} - 2x = 0$ gives $\dot{y} - y - 2x = 0$, namely $\dot{y} = y + 2x = 0$, so we have

$$\dot{x} = y,$$
$$\dot{y} = y + 2x.$$

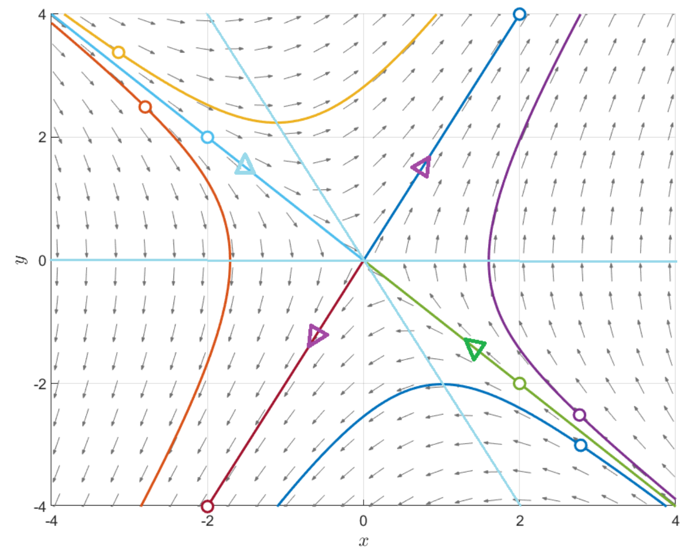
(b) There is one equilibrium point at $(0,0)$. Writing the system in matrix form,

$$\left(\begin{array}{c} \dot{x} \\ \dot{y} \end{array} \right) = A \left(\begin{array}{c} x \\ y \end{array} \right), \qquad A = \left(\begin{array}{cc} 0 & 1 \\ 2 & 1 \end{array} \right)$$

The eigenvalues of A are given by the solutions of the equation $\lambda(\lambda-1) - 2 = 0$, giving $\lambda = -1, 2$. These are real and opposite in sign, therefore the equilibrium point is a saddle.

(c) The horizontal isocline is given by $y = -2x$ and along the horizontal isocline, $\dot{x} = -2x$. The vertical isocline is given by $y = 0$ and along the vertical isocline, $\dot{y} = 2x$.

(d) To sketch the phase portrait we need to calculate the eigenvectors. We have for $\lambda_1 = -1$, the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and for $\lambda = 2$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



3. Sketch the phase portraits of the following linear dynamical systems. This should include details of the equilibria, their type, eigenvalues and eigenvectors and horizontal and vertical isoclines. In each case state how all qualitatively different solutions $(x(t), y(t))$ behave as $t \to \infty$ for varying initial conditions (x_0, y_0).

(a)

$$\dot{x} = -x - 3y,$$
$$\dot{y} = -x + y.$$

(b)

$$\dot{x} = 2x - y,$$
$$\dot{y} = -x + 2y.$$

(c)

$$\dot{x} = -x,$$
$$\dot{y} = -3y.$$

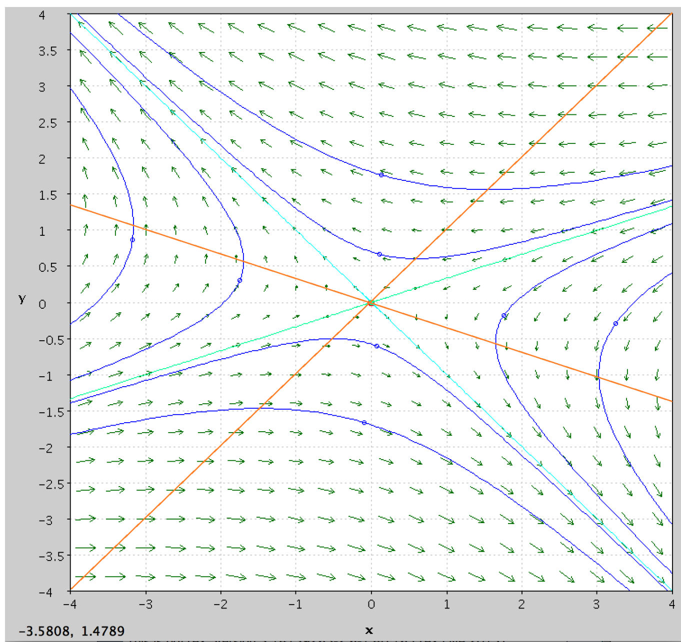
(d)

$$\dot{x} = -3x,$$
$$\dot{y} = -3y.$$

Solution:

(a) Equilibrium point is $(0,0)$, eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -2$, so equilibrium point is a saddle node. Corresponding eigenvectors are $\mathbf{v}_1 = (1,-1)^T$ and $\mathbf{v}_2 = (3,1)^T$.

The horizontal isocline is given by $y = x$ and along the horizontal isocline $\dot{x} = -4x$. The vertical isocline is given by $y = -\frac{x}{3}$ and along the vertical isocline, $\dot{y} = -\frac{4x}{3}$.

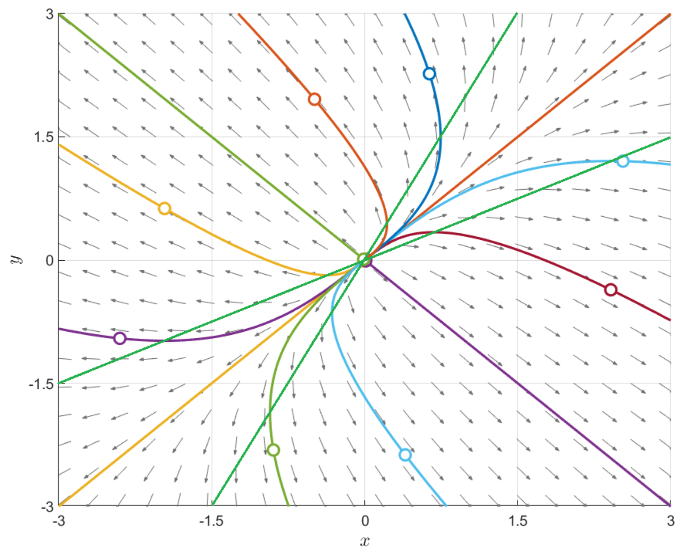


The behaviour of trajectories starting at $(x(0), y(0))$ as $t \to \infty$ is as follows:

1. If $y_0 > \frac{1}{3}x_0$ then trajectories $(x,y) \to (-\infty,\infty)$ as $t \to \infty$, parallel to $y = -x$.
2. If $y_0 < \frac{1}{3}x_0$ then trajectories $(x,y) \to (\infty,-\infty)$ as $t \to \infty$, parallel to $y = -x$.
3. if $y_0 = \frac{1}{3}x_0$ then solutions $(x,y) \to (0,0)$ as $t \to \infty$, along $y = \frac{1}{3}x$.

(b) Equilibrium point is $(0,0)$, eigenvalues are $\lambda_1 = 3$ (with eigenvector $\mathbf{v} = (1,1)$) and $\lambda_2 = 1$ (with eigenvector $\mathbf{w} = (1,-1)$). Eigenvalues are real and both positive, so equilibrium point is an unstable node.

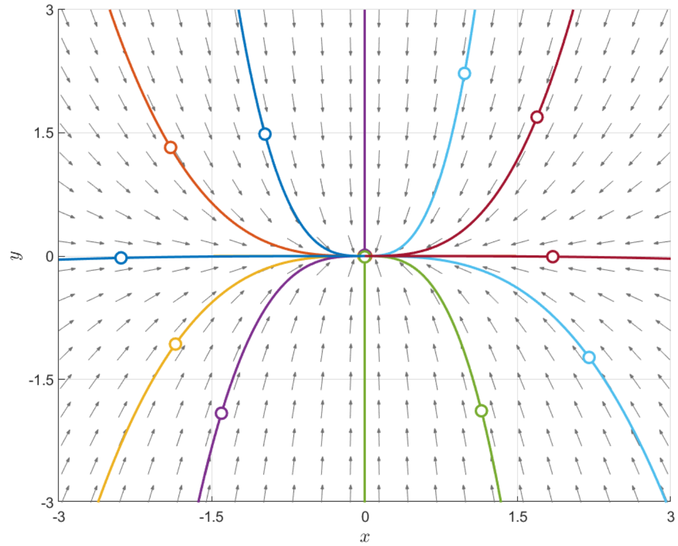
The horizontal isocline is given by $y = \frac{x}{2}$ and along the horizontal isocline $\dot{x} = -\frac{3x}{2}$. The vertical isocline is given by $y = 2x$ and along the vertical isocline, $\dot{y} = 3x$.



Trajectories above $y = x$ have $(x(t), y(t)) \to (-\infty, \infty)$ as $t \to \infty$, trajectories below $y = x$ have $(x(t), y(t)) \to (\infty, -\infty)$ as $t \to \infty$

(c) Equilibrium point is $(0, 0)$, eigenvalues are $\lambda_1 = -1$ (with eigenvector $\mathbf{v} = (1, 0)$), $\lambda_2 = -3$ (with eigenvector $\mathbf{w} = (0, 1)$). Eigenvalues are real and both negative, so equilibrium point is a stable node.

The horizontal isocline is given by $y = 0$ and along the horizontal isocline $\dot{x} = -x$. The vertical isocline is given by $x = 0$ and along the vertical isocline, $\dot{y} = -3y$. Both isoclines contain straight line solutions. They are also aligned with the eigenvectors.



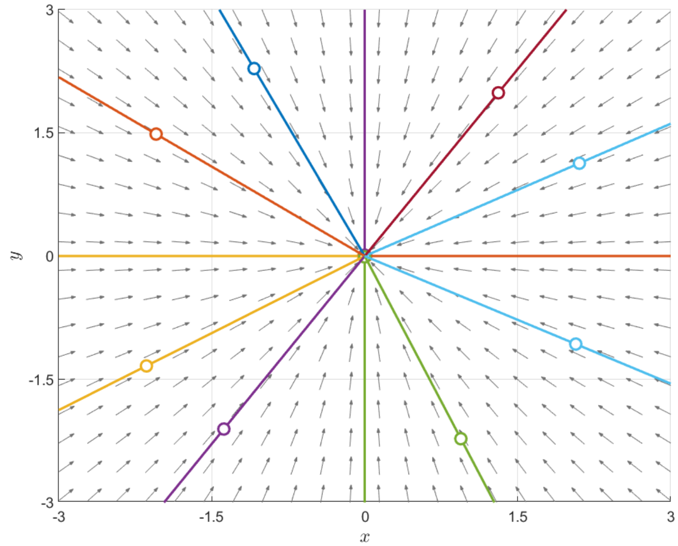
All trajectories $(x(t), y(t)) \to (0,0)$ as $t \to \infty$.

(d) Equilibrium point is $(0,0)$, single eigenvalue is $\lambda_1 = -3$ (with eigenvector $\mathbf{v} = (\alpha, \beta)$). A real repeated negative eigenvalue of a diagonal matrix, so equilibrium point is a stable star.

The horizontal isocline is given by $y = 0$ and along the horizontal isocline $\dot{x} = -3x$. The vertical isocline is given by $x = 0$ and along the vertical isocline, $\dot{y} = -3y$. Both isoclines contain straight line solutions. If you look for straight line solutions using the method shown in Chapter 2, you'll find:

$$\frac{3(mx+q)}{3x} = m \Leftrightarrow mx + q = mx \Leftrightarrow q = 0.$$

m remains undefined, i.e., every line through $(0,0)$ contains straight line solutions, as we would expect from a star.



All trajectories $(x(t), y(t)) \to (0, 0)$ as $t \to \infty$.