University of Birmingham
School of Mathematics
Vectors, Geometry and Linear Algebra
VGLA

Problem Sheet 4

Remember that there are practise questions under the materials section for each week.

$\boxed{\text{SUM}}$ **Q1.** (i) Let $V = \mathbb{R}^n$ with $n \geq 3$ be a real vector space. Which of the following subsets of V are subspaces of V? In each case prove your assertion.

(a) $A = \{(x_1, x_2, x_3, \ldots, x_n) \mid \alpha x_1 + \beta x_2 + \gamma x_3 = 0\}$ where α, β, γ are fixed elements of $\mathbb{R}$;
(b) $B = \{(x_1, x_2, x_3, \ldots, x_n) \mid 3x_n + 4x_{n-1} + x_{n-2} = 1\}$;
(c) $C = \{(x_1, x_2, x_3, \ldots, x_n) \mid \sum_{i=1}^{n} (i^i) x_i = 0\}$;
(d) $D = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_n - x_{n-1} = x_{n-1} - x_{n-2}\}$;
(e) $E = \{(x_1, x_2, x_3, \ldots, x_n) \mid \prod_{i=1}^{n} i x_i = 0\}$.

(ii) Suppose that $V = \mathbb{C}^3$. Determine whether

$$W = \{(z_1, z_2, z_3) \in V \mid \sum_{i=1}^{3} \mathrm{Im}(z_i) = 0\}$$

is a subspace of V.

(iii) Suppose that A, B, C are subspaces of a vector space V. Set

$$W = (A \cap (B + C)) \cap (B \cap (A + C)) \cap (C \cap (B + A)).$$

Show that W is a subspace of V. Is $W = A \cap B \cap C$? Either give a counterexample which shows that they are not equal, or prove that they are equal.

$\boxed{\text{SUM}}$ **Q2.** (i) Determine the quadratic equation satisfied by the points $z = x + iy$ on the Argand diagram which satisfy the following equation

$$||z - 2i| - |z + 2i|| = 2.$$

(ii) Consider the ellipse given by the equation

$$5x^2 + 5y^2 + 6xy = 8.$$

This is obtained from the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $0 < b < a$ by rotating through some angle α. Find

(a) the angle of rotation α;
(b) the coordinates of the foci of the rotated ellipse;
(c) the length of the major and minor axes.

Q3. Suppose that V is a vector space over $\mathbb{R}$ of finite dimension $n \geq 1$. Assume that $U_1, \dots, U_k$ is a finite collection of subspaces of V with $\dim U_j \leq n-1$ for $1 \leq j \leq k$. Show that

$$\bigcup_{i=1}^{k} U_i \neq V.$$

Sketch:

(i) Use induction on $\dim V$. What is the inductive hypothesis?
(ii) Why is the result true when $n = 1$?
(iii) Assume that $n \geq 2$. Show that there are an infinite number of subspaces of V of dimension $n-1$. You could do this by fixing a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and defining subspaces

$$V_\lambda = \begin{cases} \langle \lambda \mathbf{v}_1 + \mathbf{v}_2, \dots, \mathbf{v}_n \rangle & n \geq 3 \\ \langle \lambda \mathbf{v}_1 + \mathbf{v}_2 \rangle & n = 2. \end{cases}$$

Show that for $\lambda_1, \lambda_2 \in \mathbb{R}$, $V_{\lambda_1} = V_{\lambda_2}$ if and only if $\lambda_1 = \lambda_2$.
(iv) Using (iii), let W be a subspace of dimension $n-1$ with $W \notin \{U_1, \dots U_k\}$.
(v) Show that for each $1 \leq j \leq k$, $W \cap U_j$ is a subspace of W of dimension at most $n-2$.
(vi) Suppose that $V = \bigcup_{i=1}^{k} U_i$. Show that $W = \bigcup_{i=1}^{k} (W \cap U_i)$, apply the inductive hypothesis and conclude the proof.

Is the same true if the vector space is over a finite field and finite dimensional? Either prove it, or explain why the result is not true.