0 Background

Notation

We write:

- $\mathbb{N}$ for the set of natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \ldots, \}$.
- $\mathbb{Z}$ for the set of integers, i.e. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- $\mathbb{R}$ for the set of real numbers, i.e. numbers that can be written using a decimal expansion.

Given $a, b \in \mathbb{R}$ with $a < b$, we write $[a, b]$ for the closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ and (a, b) for the open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

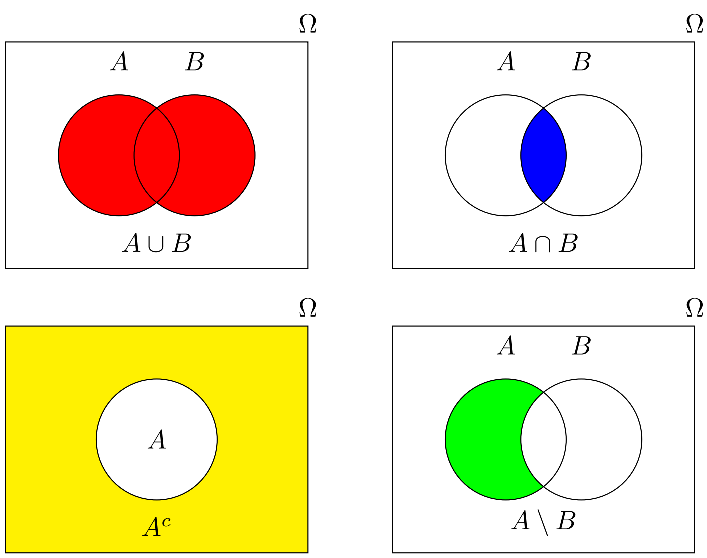
Set operations

Given a set Ω and subsets $A, B \subseteq \Omega$, we define:

- $A \cup B := \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}$, (union)
- $A \cap B := \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}$, (intersection)
- $A^c := \{\omega \in \Omega : \omega \notin A\}$, (complementary set)
- $A \setminus B := \{\omega \in \Omega : \omega \in A \text{ and } \omega \notin B\}$, (set difference)

We say that A and B are *disjoint* if $A \cap B = \emptyset$.

Here are the corresponding Venn diagrams:



For a finite or infinite family $A_1, A_2, \ldots$ of subsets of Ω:

- $\bigcup_i A_i := \{\omega \in \Omega : \omega \in A_i \text{ for some } i\}$,
- $\bigcap_i A_i := \{\omega \in \Omega : \omega \in A_i \text{ for all } i\}$,

We say that the sets $A_1, A_2, \ldots$ are *pairwise disjoint*, if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Given sets X_1, X_2 their *(Cartesian) product* is the set $X_1 \times X_2 := \{(x_1, x_2) : x_i \in X_i \text{ for } i = 1, 2\}$. More generally, given sets $X_1, X_2, \ldots, X_k$, the set $X_1 \times \cdots \times X_k := \{(x_1, \ldots x_k) : x_i \in X_i \text{ for all } i\}$; when $X_1 = \ldots = X_k = X$ this is often abbreviated to X^k.

Set cardinalities

If A is a finite set then $|A|$ will denote the *size* of A.

An infinite set A is said to be *countable* if we can write $A = \{a_1, a_2, a_3, \ldots\} = \{a_i : i \in \mathbb{N}\}$. The natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ are clearly countable, and for the purpose of this course we only really use this example. It is a fact that any subset of a countable set is either finite or countable.

We will not need this, but it may be useful to know that there are infinite sets which are not countable. For example, the real numbers $\mathbb{R}$ or the interval $[0, 1]$ are not countable (proven in 1AC).

Sums and products

We use the following notation for sums and products. Given real numbers $p_1, p_2, \ldots, p_N$, let

$$\sum_{i=1}^{N} p_i := p_1 + \ldots + p_N, \qquad \text{and} \qquad \prod_{i=1}^{N} p_i := p_1 \times \cdots \times p_N.$$

Given real numbers $p_1, p_2, p_3, \ldots$ we write

$$\sum_{i=1}^{\infty} p_i = \lim_{N \to \infty} \sum_{i=1}^{N} p_i,$$

provided this limit exists [1].

Lastly, given a finite or countable set Ω and real numbers p_ω for each $\omega \in \Omega$, we will sometimes be interested in a sum $\sum_{\omega \in \Omega} p_\omega$. This is defined as follows:

- If Ω is a finite set, with $\Omega = \{\omega_1, \ldots, \omega_k\}$ where all ω_i distinct, then $\sum_{\omega \in \Omega} p_\omega := \sum_{i=1}^{k} p_{\omega_i}$.
- If Ω is a countable set, with $\Omega = \{\omega_i : i \in \mathbb{N}\}$ with all ω_i distinct, then $\sum_{\omega \in \Omega} p_\omega := \sum_{i=1}^{\infty} p_{\omega_i}$.

For example, if $\Omega = \mathbb{N}$ and $A = \{2, 4, 6, \ldots.\} \subseteq \Omega$ then

$$\sum_{\omega \in \Omega} p_\omega = \sum_{i=1}^{\infty} p_i \qquad \text{and} \qquad \sum_{\omega \in A} p_\omega = \sum_{i=1}^{\infty} p_{2i}.$$

There are some subtleties with the countable case here, but they don't appear in our applications, so we can ignore them [2].

Two well-known elegant identities are particularly useful for evaluating sums and will appear quite often in this module. The first evaluates the sum of an arithmetic series: if $n \in \mathbb{N}$ then

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{(n+1)n}{2}.$$

The second evaluates sums of finite or infinite geometric series: for $x \in \mathbb{R}$ with $|x| < 1$ and $n \in \mathbb{N}$

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x} \qquad \text{and} \qquad \sum_{i=0}^{\infty} x^i = \frac{1}{1 - x}.$$

[1] We will not need a similar definition for products, although you might meet it in other courses.

[2] If you are interested, the issue here is that we are summing with an order on the elements of Ω, given by $\omega_1, \omega_2, \ldots$ above, and different orders sometimes give different answers. This is never a problem if Ω is finite. If Ω is countable the order of summation doesn't matter if $\sum_{i=1}^{\infty} |p_{\omega_i}| < \infty$, i.e. the series $\sum_{i=1}^{\infty} p_{\omega_i}$ is absolutely convergent (you may have seen this if you've taken 1SAS). As this is always the case for us, we ignore the technicality.