1Mech — Mechanics
Mechanics exercises 1 (weeks 1 and 2)

This sheet's assessed question is number 6.

1. In this question ρ is density, t is time, $\mathbf{v}$ is velocity, x is position, v is speed, m is mass, a is acceleration, V is volume, A is area and g is acceleration due to gravity.

   (a) What are the dimensions of the following expressions?

      i. $\frac{d\rho}{dt}$

      ii. $\dfrac{d^2\mathbf{v}}{dx^2}$

   (b) Are these equations dimensionally correct? Make sure you justify your answer.

      i. $vma = \frac{dV}{dt}$

      ii. $\int \rho \, dt = \dfrac{m}{Av} + \dfrac{\sqrt{2}mt}{V}$

      iii. $A^{1/2}g = v^2 + \dfrac{mg}{\rho A}$

2. If a particle moves along a path $\mathbf{r} = Vt\mathbf{i} + (h - gt^2/2)\mathbf{j}$, what is its velocity and acceleration? By eliminating t, what path does the particle travel along in space? What shape is this?

3. Suppose a charged particle moving under the influence of a magnetic field follows a helical path, at constant rate $\dot{\theta} = \omega$. The position vector is given by

$$\mathbf{r} = a\cos(\theta(t))\,\mathbf{i} + a\sin(\theta(t))\,\mathbf{j} + b\theta(t)\,\mathbf{k},$$

   where a and b are constants. Calculate the velocity and acceleration of the particle. Hence determine the speed which the particle moves along the helical path, and the magnitude of the acceleration. Which direction does the acceleration point in (in terms of the helix geometry)?

4. A particle of mass m is moving on a straight line under the action of an exponentially decreasing force $F = F_0 e^{-\lambda t}$, where F_0 and λ are positive constants. The particle passes the point x_0 with velocity v_0 at time $t = 0$. Find the displacement of the particle at time t, and sketch a graph of displacement versus time for the special case when $v_0 = -F_0/m\lambda$. What happens for large time?

5. A particle of mass m is attached to a spring with spring constant k, which is fixed at the opposite end. The mass is subject to an additional force $F_0 \cos \Omega t$ directed away from the fixed end of the spring, for constant Ω.

(a) Show that the equation of motion is

$$\ddot{x} + \omega^2 x = \frac{F_0}{m} \cos \Omega t,$$

where x gives the displacement from the equilibrium position (when the spring is neither stretched nor compressed), and $\omega^2 = k/m$.

(b) Find the displacement as a function of time of a particle subject to the forcing described, given that initially the particle is stationary at $x = 0$, and assuming that $\omega \neq \Omega$.

(c) **Optional extension:** What happens when $\omega = \Omega$?

6. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

Electric charge is a derived quantity with dimensions [electric current × time]. When two metallic plates, one positively charged and the other negatively charged, are placed parallel to each other, an electric field is created between them. Let two such plates each have area α, with charges $+Q > 0$ and $-Q < 0$ respectively. The electric field between the plates exerts a force on any charged particle. If a particle has mass m, charge q which may be positive or negative, and position x which increases towards the $+Q$ plate, then the equation of motion for the particle is

$$\ddot{x} = -\frac{qQ}{m\alpha\varepsilon},$$

where ε is a constant known as the *permittivity* of the material between the plates, such as air.
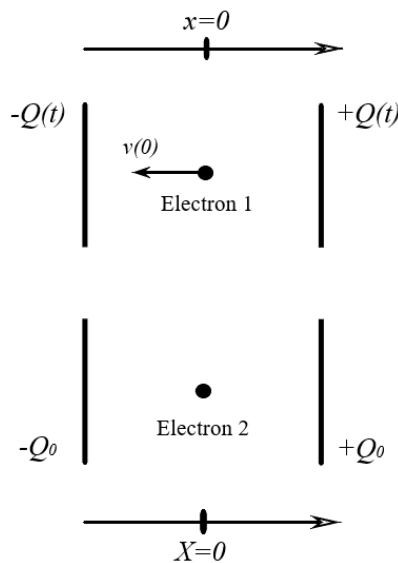

Figure 1: Parallel-plate capacitors and charged particles.

(a) Derive the dimensions of the permittivity ε.

(b) Let the particle be an electron, with negative charge $q < 0$. For convenience, let

$$\beta = -\frac{q}{m\alpha\varepsilon} > 0.$$

i. Let $Q(t) = Q_0(1 + \omega t)$ where $Q_0 > 0$ and $\omega > 0$ are constants. Given that the electron has initial position $x(0) = 0$ and initial velocity

$$v(0) = -\frac{\beta Q_0}{2\omega} < 0,$$

find the electron's displacement $x(t)$.

ii. Let a second electron be between two parallel plates with constant charges, $\pm Q_0$, and let this electron's position be X, increasing towards the $+Q_0$ plate, with initial position $X(0) = 0$ and initial velocity $V(0) = 0$ (see Figure 1). Let $t_* > 0$ be the time at which electron 1 "catches up" with electron 2, meaning $x(t_*) = X(t_*)$. Show that

$$t_* = \frac{\sqrt{3}}{\omega}.$$

Interpret this result (i.e. explain what it means physically) in the limit $\omega \to 0$.