University of Birmingham
School of Mathematics
Vectors, Geometry and Linear Algebra
VGLA

Problem Sheet 3

Remember that there are practise questions under the materials section for each week.

$\boxed{\text{SUM}}$ **Q1.** (i) Suppose that $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{nn}(\mathbb{F})$ are invertible. Assume that $\mathbf{A}^T = \mathbf{A}^{-1}$ and $\mathbf{B}^T = \mathbf{B}^{-1}$. Show that $(\mathbf{AB})^T = (\mathbf{AB})^{-1}$.

(ii) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}.$$

(a) Use Gaussian Elimination to find the inverse of $\mathbf{A}$ is it exists.

(b) Directly from the definition of the determinant as

$$\sum_{\sigma \in S_3} (-1)^{N(\sigma)} \prod_{i=1}^{3} a_{\sigma(i),i},$$

calculate $\det(\mathbf{A})$.

(iii) Calculate the determinant of

$$\mathbf{A} = \begin{pmatrix} \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda & \lambda & 0 \\ \lambda & \lambda & \lambda & \lambda & 0 & \lambda \\ \lambda & \lambda & \lambda & 0 & \lambda & \lambda \\ \lambda & \lambda & 0 & \lambda & \lambda & \lambda \\ \lambda & 0 & \lambda & \lambda & \lambda & \lambda \end{pmatrix}$$

where $\lambda \in \mathbb{R}$. (You may use whichever method you like.)

(iv) Suppose that $n \geq 2$ and $\mathbf{A} \in \mathcal{M}_{nn}(\mathbb{R})$. Assume that $\mathbf{A}$ has at least $n^2 - n + 2$ entries which are equal to μ for some $\mu \in \mathbb{R}$. Show that $\det \mathbf{A} = 0$.

(v) Show that there exists an invertible matrix $\mathbf{B} \in \mathcal{M}_{nn}(\mathbb{R})$ with $n^2 - n + 1$ entries which are equal to μ for some fixed $\mu \in \mathbb{R}$.

SUM **Q2**. Group the following matrices into sets which have the same determinant. In this λ is a real number which is the same in each matrix. Explain your answer.

- $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & \lambda \\ 5 & 19 & 6 & 3 \\ 6 & 2 & 3 & 5 \\ 1 & -2 & 4 & 6 \end{pmatrix}.$
- $\mathbf{B} = \begin{pmatrix} 1 & -2 & 4 & 6 \\ 6 & 2 & 3 & 5 \\ 5 & 19 & 6 & 3 \\ 1 & 2 & 3 & \lambda \end{pmatrix}.$
- $\mathbf{C} = \begin{pmatrix} -1 & 2 & -4 & -6 \\ -6 & -2 & -3 & -5 \\ -5 & -19 & -6 & -3 \\ -1 & -2 & -3 & -\lambda \end{pmatrix}.$
- $\mathbf{D} = \begin{pmatrix} 6 & 21 & 9 & \lambda+3 \\ 5 & 19 & 6 & 3 \\ 6 & 2 & 3 & 5 \\ 1 & -2 & 4 & 6 \end{pmatrix}.$
- $\mathbf{E} = \begin{pmatrix} 6 & 21 & \lambda+3 & 9 \\ 5 & 19 & 3 & 6 \\ 6 & 2 & 5 & 3 \\ 1 & -2 & 6 & 4 \end{pmatrix}.$
- $\mathbf{F} = \begin{pmatrix} 6 & 21 & \lambda+3 & 9 \\ 6 & 2 & 5 & 3 \\ 5 & 19 & 3 & 6 \\ 1 & -2 & 6 & 4 \end{pmatrix}.$
- $\mathbf{G} = \begin{pmatrix} 6 & -21 & \lambda+3 & 9 \\ 6 & -2 & 5 & 3 \\ 5 & -19 & 3 & 6 \\ 1 & 2 & 6 & 4 \end{pmatrix}.$