University of Birmingham
School of Mathematics

1RA Differentiation Autumn 2022

Problem Sheet 2

issued Week 3

You have approximately 10 working days to complete and submit the $\boxed{\text{SUM}}$ questions (**Q4** and **Q9**) and you may begin working on it immediately.

<table>
<tr><td colspan="2">Assignment available from: 14 October
Submission due: 1700 on Wednesday 26 October 2022</td></tr>
<tr><td>Pre-submission</td><td>Post-submission</td></tr>
<tr><td>• Your Guided Study Support Class in Weeks 3-5.
• Tutor meetings in Weeks 3-5.
• PASS from Week 4
• Library MSC from Week 4
• Office Hours: Wednesday 1300-1430 and Friday 1000-1130.</td><td>• Written feedback on your submission.
• Generic feedback (3 November).
• Model solutions (3 November).
• Tutor meetings in Week 7.
• Office Hours: Wednesday 1300-1430 and Friday 1000-1130</td></tr>
</table>

Instructions:

You will spend the next two weeks (including your Guided Study Support Class in weeks 4 and 5 working on the $\boxed{\text{SUM}}$ questions (**Q4** and **Q9**).

The **deadline** for submission is as follows:

- **By 1700 on Wednesday 26 October 2022**

Late submissions will be penalised as per University guidelines at a rate of 5% per working day late (i.e. a mark of 63% becomes a mark of 58% if submitted one day late).

Important:

Your Problem Sheet solutions must be submitted as a single PDF file. You may upload newer versions, BUT only the most recent upload will be viewed and graded. In particular, this means that subsequent uploads will need to contain ALL of your work, not just the parts which have changed. Moreover, if you upload a new version after the deadline, then your submission will be counted as late and the late penalty will be applied, REGARDLESS of whether an older version was submitted before the deadline. In the interest of fairness to all students and staff, there will be no exceptions to these rules. All of this and more is explained in detail on the Submitting Problem Sheets: FAQs Canvas page.

Questions:

Q1. For each of the following statements, either prove that it is true by using the definition of limit, or give a counterexample to show that it is false.

(i) Let $f, g : \mathbb{R} \to \mathbb{R}$. Suppose that $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$ for some $a, b \in \mathbb{R}$. If $f(x) < g(x)$ for all $x \in \mathbb{R}$, then $a < b$.

(ii) Let $f, g : \mathbb{R} \to \mathbb{R}$. If $\lim_{x\to\infty} f(x) = \ell$ for some $\ell \in \mathbb{R}$, and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} f(x)g(x) = \infty$.

(iii) Let $f : \mathbb{R} \to \mathbb{R}$. If $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \ell$ for some $\ell \in \mathbb{R}$, then $f(0) = \ell$.

(iv) Let $f, g : \mathbb{R} \to \mathbb{R}$. If $\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} g(x) = \ell$ for some $a, b, \ell \in \mathbb{R}$, then $\lim_{x\to a} g(f(x)) = \ell$.

Q2. (i) Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^3 - 5x^2 + 7x - 13}{2x^3 - \pi}$.

(b) $\displaystyle\lim_{x\to 0} x \cos\frac{1}{x}$.

(ii) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt[3]{x}$ is continuous, by directly using the definition of continuous function.

Q3. Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$.

(i) By using the definition of limit, prove that $\lim_{x\to a} f(x) = 0$ if and only if $\lim_{x\to a} |f(x)| = 0$.

(ii) Assume that $\lim_{x\to a} |f(x)| = \ell$ for some $\ell \in (0, \infty)$. Is it necessarily true that either $\lim_{x\to a} f(x) = \ell$ or $\lim_{x\to a} f(x) = -\ell$? Justify your answer.

(SUM) **Q4**. Determine the value of the following limits. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using. **Only those materials that have been discussed in lectures can be used here. For instance, you can NOT use L'Hospital's rule here**.

(i) $\displaystyle\lim_{x\to 1} \frac{x^2 - 1}{2x^2 - x - 1}$.

(ii) $\displaystyle\lim_{x\to 0} \frac{(x-1)^3 + (1-3x)}{x^2 + 2x^3}$.

(iii) $\displaystyle\lim_{x\to 1} \frac{x^n - 1}{x^m - 1}$, where $n, m \in \mathbb{N}$.

(iv) $\displaystyle\lim_{x\to 4} \frac{\sqrt{1+2x} - 3}{\sqrt{x} - 2}$.

(v) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$.

(vi) $\displaystyle\lim_{x\to\infty} x \sin\frac{1}{x}$.

Q5. Does $\displaystyle\lim_{x\to 0} \sin\frac{1}{x}$ exist? Justify your answer.

Q6. This is a guided proof of some statements in Section 4.5 of the lecture notes.

(i) By using the definition of limit, prove that
$$\lim_{x\to\infty} \log x = \infty,$$
$$\lim_{x\to\infty} \exp(x) = \infty.$$

(ii) Deduce from part (i) that
$$\lim_{x\to 0^+} \log x = -\infty,$$
$$\lim_{x\to -\infty} \exp(x) = 0,$$
and that, for all $b \in (0,\infty)$,
$$\lim_{x\to\infty} x^b = \infty,$$
$$\lim_{x\to 0} x^b = 0.$$
[Hint: change of variable in limits, $x^b = \exp(b \log x)$.]

Q7. Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $\displaystyle\lim_{x\to 0} \frac{7x}{\sin(4x)}$.

(ii) $\displaystyle\lim_{x\to 2} \frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{2x-1}-\sqrt{x+1}}$.

(iii) $\displaystyle\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$.

(iv) $\displaystyle\lim_{x\to 0} \frac{e^{2x}-1}{\log(1-5x)}$.

(v) $\displaystyle\lim_{x\to 0^+} x^x$.

Q8. Prove the following statements by directly using the definition of derivative.

(i) If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x^2 + 5$ for all $x \in \mathbb{R}$, then $f'(x) = 6x$.

(ii) If $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by $g(x) = 3/(1-x)$ for all $x \in \mathbb{R} \setminus \{1\}$, then $g'(x) = 3/(1-x)^2$.

(iii) If $k : \mathbb{R} \to \mathbb{R}$ is given by $k(x) = \sin(2x)$ for all $x \in \mathbb{R}$, then $k'(0) = 2$.

(SUM) **Q9**. Let a_1, a_2, a_3 be positive real numbers, and $\lambda_1 < \lambda_2 < \lambda_3$. Prove the following equation
$$\frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \frac{a_3}{x-\lambda_3} = 0$$
has solutions on both intervals (λ_1, λ_2) and (λ_2, λ_3).

Q10. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) If $f : (0,1) \to \mathbb{R}$ is continuous, then f is bounded.

(ii) If $g : (0,1) \to \mathbb{R}$ is continuous, then g is differentiable.

(iii) If $k : [0,1] \to \mathbb{R}$ is differentiable, then k is bounded.

Extra Questions

[Questions marked with a ✻ may be more challenging than others.]

EQ1. Prove that the following limits exist and determine their value. You can use any of the definitions and results discussed in lectures, provided you clearly state what you are using.

(i) $\lim\limits_{x\to-\infty} 2x^2 - 3x + \arctan x$.

(ii) $\lim\limits_{x\to 2} \dfrac{1}{1-x}$.

(iii) $\lim\limits_{x\to 1/2} \dfrac{4x^2-1}{2x-1}$.

EQ2. Let $A \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ be continuous. Let $a \in A$. Prove that, if $f(a) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in A \cap (a-\delta, a+\delta)$.
[This is sometimes called the "sign-preserving property" of continuous functions: informally, if a continuous function is positive at a certain point, then it is also positive at nearby points.]

✻ **EQ3**. Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $f : (a,b) \to \mathbb{R}$ be increasing. Let $\ell = \sup f$, and assume first that $\ell \in \mathbb{R}$.

(i) Prove that, for all $\epsilon > 0$, there exists $c \in (a,b)$ such that $f(c) > \ell - \epsilon$. [Hint: $\sup f$ is the minimum of the upper bounds of the range of f; can $\ell - \epsilon$ be an upper bound as well?]

(ii) Prove that, for all $\epsilon > 0$, there exists $c \in (a,b)$ such that
$$\ell - \epsilon < f(x) \leq \ell$$
for all $x \in (a,b)$ such that $x > c$. [Hint: f is monotone.]

(iii) Prove that $\lim\limits_{x\to b} f(x) = \ell$.

Assume now instead that $\ell = \infty$.

(iv) Is it still true that $\lim\limits_{x\to b} f(x) = \ell$ in this case? Justify your answer.

EQ4. This question is aimed at proving that trigonometric functions are continuous.

(i) Recall from Lemma 4.20 the inequality
$$\sin\theta \leq \theta \qquad \text{for all } \theta \in [0, \pi/2).$$
Prove that
$$|\sin\theta| \leq |\theta| \qquad \text{for all } \theta \in (-\pi/2, \pi/2).$$

(ii) Deduce that the function sin is continuous at 0. [Hint: Sandwich Theorem.]

(iii) Deduce that the function cos is continuous at 0. [Hint: $\cos x = 1 - 2\sin^2(x/2)$.]

(iv) Prove that, for all $a \in \mathbb{R}$,
$$\lim_{h\to 0} \sin(a+h) = \sin a.$$
[Hint: Angle Sum Formula.]

(v) Deduce that the function sin is continuous at every $a \in \mathbb{R}$.

(vi) Conclude that the functions cos, tan, cot are continuous too.

EQ5. This is a guided proof of Proposition 4.23 of the lecture notes.

(i) Recall from Proposition 1.24 the inequality
$$2^n \geq n+1 \qquad \text{for all } n \in \mathbb{N}_0.$$
From this inequality, or otherwise, deduce that
$$\exp(x) \geq x$$
for all $x \geq 0$. [Hint: $e^x \geq 2^x \geq 2^{\lfloor x \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer smaller than or equal to x.]

(ii) Deduce that, for all $b > 0$, there exists $C > 0$ (depending on b) such that
$$\exp(x) \geq Cx^b$$
for all $x \geq 0$. [Hint: $\exp(x) = (\exp(x/b))^b$.]

(iii) Using the previous inequality, or otherwise, prove that, for all $b > 0$,
$$\lim_{x\to\infty} \frac{\exp(x)}{x^b} = \infty.$$
[Hint: "Sandwich Theorem for Infinite Limits".]

(iv) Using the previous result, or otherwise, prove that, for all $b > 0$,
$$\lim_{x\to-\infty} |x|^b \exp(x) = 0, \qquad \lim_{x\to\infty} \frac{\log x}{x^b} = 0, \qquad \lim_{x\to 0^+} x^b \log x = 0.$$
[Hint: change of variables in limits ($x = \exp(\log x)$, $\log x = -\log \frac{1}{x}$).]

EQ6. Let $a, b, c \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial
$$f(x) = x^3 + ax^2 + bx + c.$$

(i) Prove that
$$\lim_{x\to-\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty.$$

(ii) Prove that there exist $x_1, x_2 \in \mathbb{R}$ such that
$$f(x_1) < 0 \qquad \text{and} \qquad f(x_2) > 0.$$
[Hint: definition of limit.]

(iii) Prove that there exists $x_0 \in \mathbb{R}$ such that
$$f(x_0) = 0.$$
[Hint: Intermediate Value Theorem.]

[Note: This shows that every polynomial of degree 3 has a zero in $\mathbb{R}$. A similar argument proves that every polynomial of odd degree has a zero in $\mathbb{R}$; in turn this fact can be used as a starting point for a proof of the Fundamental Theorem of Algebra.]

∗ **EQ7**. Let $A \subseteq \mathbb{R}$. Recall that, by the Leibniz rule, if $f, g : A \to \mathbb{R}$ are differentiable, then their product fg is differentiable too, and the formula
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
holds for all $x \in \mathbb{R}$.

(i) Let $f_1, f_2, f_3 : A \to \mathbb{R}$ be differentiable functions. Prove that their product $f_1 f_2 f_3$ is differentiable too, and find a formula for its derivative $(f_1 f_2 f_3)'$. [Hint: $f_1 f_2 f_3 = (f_1 f_2) f_3$.]

(ii) Let $n \in \mathbb{N}$, and let $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$ be differentiable functions. Prove that their product $f_1 f_2 \cdots f_n$ is differentiable. [Hint: induction on n.]

(iii) Can you write a formula for the derivative $(f_1 f_2 \cdots f_n)'$?