1Mech — Mechanics

Mechanics solutions 1

This sheet's assessed question is number 6.

1. In this question ρ is density, t is time, $\mathbf{v}$ is velocity, x is position, v is speed, m is mass, a is acceleration, V is volume, A is area and g is acceleration due to gravity.

   (a) What are the dimensions of the following expressions?

   i. $\frac{d\rho}{dt}$

   ii. $\dfrac{d^2\mathbf{v}}{dx^2}$

   (b) Are these equations dimensionally correct? Make sure you justify your answer.

   i. $vma = \frac{dV}{dt}$

   ii. $\int \rho \, \mathrm{d}t = \dfrac{m}{Av} + \dfrac{\sqrt{2}mt}{V}$

   iii. $A^{1/2}g = v^2 + \dfrac{mg}{\rho A}$

   Solution. The given quantities each have dimensions:

   - density ρ $[\mathrm{ML}^{-3}]$ (mass per unit volume)
   - time t $[\mathrm{T}]$
   - velocity $\mathbf{v}$ $[\mathrm{LT}^{-1}]$ (length per unit time)
   - position x $[\mathrm{L}]$
   - speed v $[\mathrm{LT}^{-1}]$ (length per unit time)
   - mass m $[\mathrm{M}]$
   - acceleration a $[\mathrm{LT}^{-2}]$ (length per time squared)
   - volume V $[\mathrm{L}^3]$
   - area A $[\mathrm{L}^2]$
   - acceleration due to gravity g $[\mathrm{LT}^{-2}]$.

   Then

   (a) $[\frac{d\rho}{dt}] = [\rho]/[t] = [\mathrm{ML}^{-3}]/[\mathrm{T}] = [\mathrm{ML}^{-3}\mathrm{T}^{-1}]$.

   (b) $$\left[\frac{d^2\mathbf{v}}{dx^2}\right] = \left[\frac{d}{dx}\left(\frac{d\mathbf{v}}{dx}\right)\right] = \left[\frac{1}{\mathrm{L}}\right]\frac{\left[\mathrm{LT}^{-1}\right]}{[\mathrm{L}]} = [\mathrm{L}^{-1}\mathrm{T}^{-1}].$$

   The dimensions of each additive term must be the same for the equation to be dimensionally homogeneous - they are not then the model is definitely wrong!

   (a) $[vma] = [\mathrm{LT}^{-1}\mathrm{MLT}^{-2}] = [\mathrm{L}^2\mathrm{MT}^{-3}]$, $\frac{dV}{dt} = [\mathrm{L}^3\mathrm{T}^{-1}]$. Therefore the expression cannot be correct.

(b) $[\int \rho \, \mathrm{d}t] = [\rho][t] = [\mathrm{ML}^{-3}\ \mathrm{T}]$, $\left[\dfrac{m}{Av}\right] = [\mathrm{M\ L}^{-3}\ \mathrm{T}]$, $\left[\dfrac{\sqrt{2}mt}{V}\right] = \dfrac{\left[\sqrt{2}mt\right]}{[V]} =$ $[\mathrm{ML}^{-3}\mathrm{T}]$, noting that $\sqrt{2}$ here is dimensionless. The expression is therefore dimensionally consistent and **may** be correct.

(c) $\left[A^{1/2}g\right] = [\left(\mathrm{L}^2\right)^{1/2}\ \mathrm{L\ T}^{-2}] = [\mathrm{L}^2\ \mathrm{T}^{-2}]$, $[v^2] = [\left(\mathrm{LT}^{-1}\right)^2] = [\mathrm{L}^2\ \mathrm{T}^{-2}]$, $\left[\dfrac{mg}{\rho A}\right] =$ $\left[\dfrac{\mathrm{MLT}^{-2}}{\mathrm{ML}^{-3}\mathrm{L}^2}\right] = [\mathrm{L}^2\mathrm{T}^{-2}]$. The expression is therefore dimensionally consistent and **may** be correct.

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Feedback: *These questions hopefully shouldn't cause too many problems, but aim to get you thinking about variables having physical meaning and dimensions. The principle of dimensional homogeneity gives a useful tool for checking whether you have made any obvious mistakes in your model/analysis. Remember that your models and your solutions should be dimensionally consistent!*

2. If a particle moves along a path $\mathbf{r} = Vt\mathbf{i} + (h - gt^2/2)\,\mathbf{j}$, what is its velocity and acceleration? By eliminating t, what path does the particle travel along in space? What shape is this?

Solution. If the particle path is

$$\mathbf{r} = Vt\mathbf{i} + \left(h - gt^2/2\right)\mathbf{j},$$

then the velocity is given by differentiating with respect to time, hence

$$\begin{aligned} \mathbf{v} &= \dot{\mathbf{r}}, \\ &= V\mathbf{i} + (-gt)\,\mathbf{j}, \end{aligned}$$

and acceleration is given by differentiating a second time with respect to time, so that

$$\begin{aligned} \mathbf{a} &= \ddot{\mathbf{r}}, \\ &= \dot{\mathbf{v}}, \\ &= -g\mathbf{j}. \end{aligned}$$

The components of $\mathbf{r}$ give the x and y coordinates of the particle path, so $x = Vt$ (the component in the $\mathbf{i}$ direction) and $y = h - gt^2/2$ (in the $\mathbf{j}$ direction). This can be rearranged to give

$$\begin{aligned} t &= \frac{x}{V}, \\ \Longrightarrow y &= h - \frac{g}{2}\left(\frac{x}{V}\right)^2, \\ &= h - \frac{gx^2}{2V^2}, \end{aligned}$$

which gives a parabola (try to plot it if you don't know why!).

This is the particle motion when it is projected at velocity V in the x direction, from a height h, under the effect of gravity g acting in the negative y direction. ◀

3. Suppose a charged particle moving under the influence of a magnetic field follows a helical path, at constant rate $\dot{\theta} = \omega$. The position vector is given by

$$\mathbf{r} = a\cos(\theta(t))\,\mathbf{i} + a\sin(\theta(t))\,\mathbf{j} + b\theta(t)\,\mathbf{k},$$

where a and b are constants. Calculate the velocity and acceleration of the particle. Hence determine the speed which the particle moves along the helical path, and the magnitude of the acceleration. Which direction does the acceleration point in (in terms of the helix geometry)?

Solution. The first thing to do in a question like this is get your head around the geometry! The **i** and **j** components here give a circle, with an additional component in the **k** direction stretching it out to form a helix. See `https://www.geogebra.org/m/m4we3vsz` for an interactive version.

Velocity is the first derivative of position vector with respect to time, hence

$$\mathbf{v} = \dot{\mathbf{r}} = -a\dot{\theta}\sin\theta\mathbf{i} + a\dot{\theta}\cos\theta\mathbf{j} + b\dot{\theta}\mathbf{k},$$

using the chain rule such that (e.g.)

$$\begin{aligned} \frac{d}{dt}(\sin(\theta(t))) &= \frac{d\theta}{dt}\frac{d}{d\theta}(\sin\theta), \\ &= \dot{\theta}\cos\theta. \end{aligned}$$

Now, since $\dot{\theta} = \omega$, this gives

$$\mathbf{v} = \dot{\mathbf{r}} = -a\omega\sin\theta\mathbf{i} + a\omega\cos\theta\mathbf{j} + b\omega\mathbf{k}.$$

Similarly, acceleration is given by

$$\begin{aligned} \mathbf{a} &= \ddot{\mathbf{r}} \\ &= \dot{\mathbf{v}}, \\ &= -a\omega\dot{\theta}\cos\theta\mathbf{i} - a\omega\dot{\theta}\sin\theta\mathbf{j}, \\ &= -a\omega^2\cos\theta\mathbf{i} - a\omega^2\sin\theta\mathbf{j}. \end{aligned}$$

Speed is then given by the magnitude of velocity, so

$$\begin{aligned} |\mathbf{v}| &= \sqrt{(-a\omega\sin\theta)^2 + (a\omega\cos\theta)^2 + (b\omega)^2}, \\ &= \sqrt{(a^2+b^2)\,\omega^2}, \end{aligned}$$

acting along the helix. The magnitude of acceleration is given by

$$\begin{aligned} |\mathbf{a}| &= \sqrt{(-a\omega^2\cos\theta)^2 + (-a\omega^2\sin\theta)^2}, \\ &= \sqrt{a^2\omega^4}, \\ &= a\omega^2. \end{aligned}$$

The accceleration vector only contains components in the **i** and **j** directions, such that $\mathbf{a} = -a\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$. Therefore it acts towards the centreline of the helix. ◀

Feedback: *Questions 2 and 3 look at differentiating to find velocity and acceleration from position, and thinking about these in two and three dimensions respectively. The aim is to connect the mathematics with what the particle is physically doing. Question 3 demonstrates the importance of knowing the dependencies in your expressions - here θ is a function of t, so e.g. differentiating requires the chain rule. You do not need to always write these dependencies out explicitly, but you do need to be very confident you know them! If in doubt, write it out!*

4. A particle of mass m is moving on a straight line under the action of an exponentially decreasing force $F = F_0 e^{-\lambda t}$, where F_0 and λ are positive constants. The particle passes the point x_0 with velocity v_0 at time $t = 0$. Find the displacement of the particle at time t, and sketch a graph of displacement versus time for the special case when $v_0 = -F_0/m\lambda$. What happens for large time?

Solution. The mass is subject to a force $F_0 e^{-\lambda t}$ so using Newton's second law we have

$$m\ddot{x} \;=\; F_0 e^{-\lambda t},$$

where $x(t)$ gives the displacement at time t. Hence

$$\begin{aligned} \ddot{x} &= \frac{F_0}{m} e^{-\lambda t}, \\ \implies \dot{x} &= -\frac{F_0}{\lambda m} e^{-\lambda t} + c_1, \\ \implies x &= \frac{F_0}{\lambda^2 m} e^{-\lambda t} + c_1 t + c_2, \end{aligned}$$

where c_1, c_2 are constants, and we have integrated twice.

We now use the initial conditions to find the constants. These are the location (x) and velocity ($\dot{x}$) of the particle initially i.e. $x = x_0$, $\dot{x} = v_0$ at $t = 0$. Hence

$$\begin{aligned} \dot{x}\,(0) = v_0 &= -\frac{F_0}{m\lambda} + c_1, \\ \implies c_1 &= v_0 + \frac{F_0}{m\lambda}, \end{aligned}$$

and

$$\begin{aligned} x\,(0) = x_0 &= \frac{F_0}{m\lambda^2} + c_2, \\ \implies c_2 &= x_0 - \frac{F_0}{m\lambda^2}. \end{aligned}$$

Hence the displacement at time t is given by

$$x \;=\; \frac{F_0}{\lambda^2 m} e^{-\lambda t} + \left(v_0 + \frac{F_0}{m\lambda} \right) t + x_0 - \frac{F_0}{m\lambda^2}.$$

When $v_0 = -\dfrac{F_0}{m\lambda}$, then

$$\begin{aligned} x &= \frac{F_0}{\lambda^2 m} e^{-\lambda t} + x_0 - \frac{F_0}{m\lambda^2}, \\ &= x_0 + \frac{F_0}{\lambda^2 m} \left(e^{-\lambda t} - 1 \right). \end{aligned}$$

For large time ($t \to \infty$), $x \to x_0 - \dfrac{F_0}{\lambda^2 m}$, i.e. a fixed value, and velocity $\dot{x} \to 0$, i.e. the particle will eventually come to rest at a position dependent on where it started, the mass of the particle and the magnitude of the force applied to it.

Figure 1: Displacement time graph, see the interactive version at `https://www.geogebra.org/m/rt8hugbn`.

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Feedback: *This question finds the position for a one-dimensional time dependent force. Once you have your solution, you should think about what it actually says about what the particle is doing.*

5. A particle of mass m is attached to a spring with spring constant k, which is fixed at the opposite end. The mass is subject to an additional force $F_0 \cos \Omega t$ directed away from the fixed end of the spring, for constant Ω.

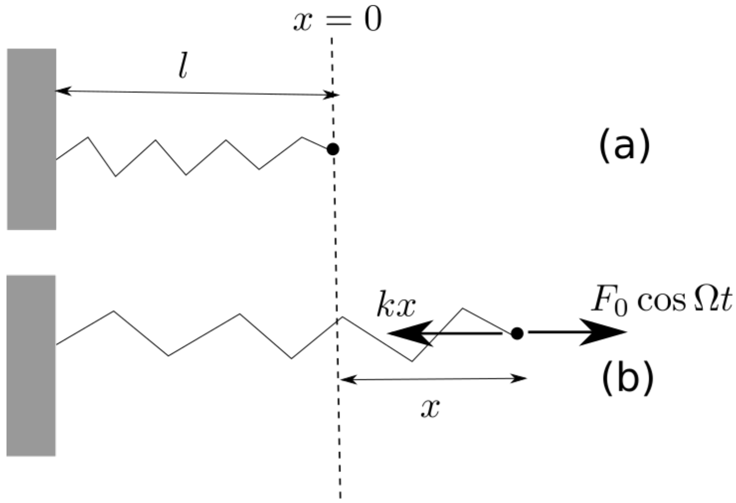
   (a) Show that the equation of motion is

   $$\ddot{x} + \omega^2 x = \frac{F_0}{m} \cos \Omega t,$$

   where x gives the displacement from the equilibrium position (when the spring is neither stretched nor compressed), and $\omega^2 = k/m$.

   (b) Find the displacement as a function of time of a particle subject to the forcing described, given that initially the particle is stationary at $x = 0$, and assuming that $\omega \neq \Omega$.

   (c) **Optional extension:** What happens when $\omega = \Omega$?



Solution. (a) *Figure description for accessibility: (a) Shows the spring/mass system in the unstretched state, indicating the natural length of the spring, with the particle at the point $x = 0$. (b) shows the system when stretched, including the balance of forces.*

Let $x = 0$ give the location of the mass when the spring is at its natural length; x then denotes the extension in the spring as the particle moves, with the positive x direction pointing away from the fixed end of the spring. The tension in the spring acting on the particle is therefore $-kx$, using Hooke's law. The additional force applied to the particle is $F_0 \cos \Omega t$; hence the net force acting on the particle is $F_0 \cos \Omega t - kx$. Newton's second law therefore gives

$$\begin{aligned} m\ddot{x} &= F_0 \cos \Omega t - kx, \\ \implies \ddot{x} + \omega^2 x &= \frac{F_0}{m} \cos \Omega t, \end{aligned}$$

where $\omega^2 = k/m$ to ensure that k/m remains positive.

(b) This is a second order linear ODE with constant coefficients. First let's write down the initial conditions for the system. At $t = 0$ we have $x = 0$ (zero displacement) and $\dot{x} = 0$ (zero velocity). We then solve the homogeneous problem, that is

$$\ddot{x}_c + \omega^2 x_c = 0.$$

The characteristic equation is given by

$$\begin{aligned} \lambda^2 + \omega^2 &= 0, \\ \implies \lambda = \pm i\omega, & \end{aligned}$$

so the solution is of the form

$$x_c = A \cos \omega t + B \sin \omega t.$$

Next we find the particular solution. When $\omega \neq \Omega$ the homogeneous solution and the right hand side are not of the same form, so we can guess the particular solution will be of the form

$$x_p = C \cos \Omega t + D \sin \Omega t.$$

This gives

$$\begin{aligned} \dot{x}_p &= -C\Omega \sin \Omega t + D\Omega \cos \Omega t, \\ \ddot{x}_p &= -C\Omega^2 \cos \Omega t - D\Omega^2 \sin \Omega t. \end{aligned}$$

Then

$$\begin{aligned} \ddot{x}_p + \omega^2 x_p &= \frac{F_0}{m} \cos \Omega t, \\ \implies -C\Omega^2 \cos \Omega t - D\Omega^2 \sin \Omega t + C\omega^2 \cos \Omega t + D\omega^2 \sin \Omega t &= \frac{F_0}{m} \cos \Omega t. \end{aligned}$$

We equate coefficients of $\sin \Omega t$ and $\cos \Omega t$ to find:

$$\begin{aligned} D\left(\omega^2 - \Omega^2\right) &= 0, \\ \implies D &= 0, \end{aligned}$$

since $\omega^2 \neq \Omega^2$, and

$$\begin{aligned} C\left(\omega^2 - \Omega^2\right) &= \frac{F_0}{m}, \\ \Longrightarrow C &= \frac{F_0}{m\left(\omega^2 - \Omega^2\right)}. \end{aligned}$$

Hence the full solution is given by

$$x = A\cos\omega t + B\sin\omega t + \frac{F_0}{m\left(\omega^2 - \Omega^2\right)}\cos\Omega t.$$

We now use the initial conditions to find the constants A and B. At $t = 0$, $x = 0$, hence

$$\begin{aligned} 0 &= A + \frac{F_0}{m\left(\omega^2 - \Omega^2\right)}, \\ \Longrightarrow A &= -\frac{F_0}{m\left(\omega^2 - \Omega^2\right)}, \end{aligned}$$

Now, since

$$\dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t - \frac{F_0\Omega}{m\left(\omega^2 - \Omega^2\right)}\sin\Omega t.$$

and $\dot{x} = 0$ at $t = 0$, we find $B = 0$. Thus

$$x = \frac{F_0}{m\left(\omega^2 - \Omega^2\right)}\left(\cos\Omega t - \cos\omega t\right),$$

gives the position of the particle at time t. This will give oscillatory behaviour (as it's trig functions).

(c) **Optional** If $\omega = \Omega$ then the complementary function remains the same, however the form of the particular solution you would normally guess is now part of the homogeneous solution. We therefore instead guess that

$$x_p = Ct\cos\omega t + Dt\sin\omega t,$$

and hence

$$\begin{aligned} \dot{x}_p &= C\cos\omega t + D\sin\omega t - Ct\omega\sin\omega t + Dt\omega\cos\omega t, \\ \ddot{x}_p &= -C\omega\sin\omega t - C\omega\sin\omega t - Ct\omega^2\cos\omega t \\ &\quad + D\omega\cos\omega t + D\omega\cos\omega t - Dt\omega^2\sin\omega t. \end{aligned}$$

Hence

$$\begin{aligned} -\left(2C\omega - Dt\omega^2\right)\sin\omega t + \left(-Ct\omega^2 + 2D\omega\right)\cos\omega t & \\ +\omega^2 Ct\cos\omega t + \omega^2 Dt\sin\omega t &= \frac{F_0}{m}\cos\omega t. \end{aligned}$$

Equating coefficients gives $-2\omega C = 0$ from the $\sin\omega t$ terms and $2\omega D = F_0/m$ from the $\cos\omega t$ terms. Note that the terms with t in them cancel out. Finally

$$x = A\cos\omega t + B\sin\omega t + \frac{F_0}{2\omega m}t\sin\omega t.$$

We again use the initial conditions to find A and B, where $x = 0$ at $t = 0$ gives $A = 0$, and since

$$\dot{x} \;=\; -A\omega \sin \omega t + B\omega \cos \omega t + \frac{F_0}{2\omega m}\left(\sin \omega t + \omega t \cos \omega t\right),$$

$\dot{x} = 0$ at $t = 0$ gives $B\omega = 0$ and hence $B = 0$. Thus the complete solution is given by

$$x \;=\; \frac{F_0}{2\omega m} t \sin \omega t.$$

This grows for large t rather than oscillates - when the forcing frequency matches the natural frequency (i.e. $\omega = \Omega$) we see resonance.

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Feedback: *This question gives practice forming and solving a model. Pay particular attention to how the model is presented - you need to explain your model very thoroughly. An equation without connection to the real world isn't a model! Your model should stand on its own, defining everything that you need. We then solve the resulting ODE, including defining initial conditions - it's a good idea to give these before you start solving things. Finally, this question shows the importance of making any assumptions about parameter values explicit; the answer has very different behaviour when* $\omega = \Omega$.

6. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

   Electric charge is a derived quantity with dimensions [electric current × time]. When two metallic plates, one positively charged and the other negatively charged, are placed parallel to each other, an electric field is created between them. Let two such plates each have area α, with charges $+Q > 0$ and $-Q < 0$ respectively. The electric field between the plates exerts a force on any charged particle. If a particle has mass m, charge q which may be positive or negative, and position x which increases towards the $+Q$ plate, then the equation of motion for the particle is

$$\ddot{x} = -\frac{qQ}{m\alpha\varepsilon},$$

   where ε is a constant known as the *permittivity* of the material between the plates, such as air.

   (a) Derive the dimensions of the permittivity ε.

   (b) Let the particle be an electron, with negative charge $q < 0$. For convenience, let

$$\beta = -\frac{q}{m\alpha\varepsilon} > 0.$$

   i. Let $Q(t) = Q_0(1 + \omega t)$ where $Q_0 > 0$ and $\omega > 0$ are constants. Given that the electron has initial position $x(0) = 0$ and initial velocity

$$v(0) = -\frac{\beta Q_0}{2\omega} < 0,$$

   find the electron's displacement $x(t)$.

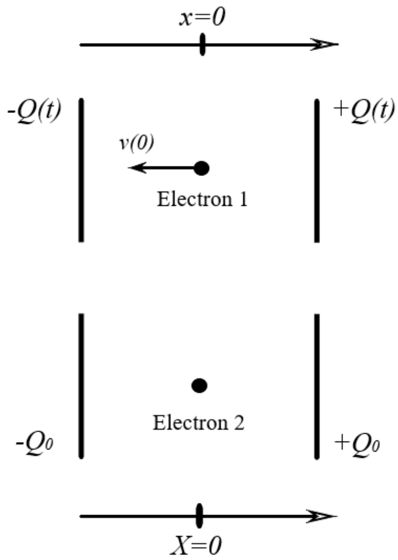

Figure 2: Parallel-plate capacitors and charged particles.

ii. Let a second electron be between two parallel plates with constant charges, $\pm Q_0$, and let this electron's position be X, increasing towards the $+Q_0$ plate, with initial position $X(0) = 0$ and initial velocity $V(0) = 0$ (see Figure 2). Let $t_* > 0$ be the time at which electron 1 "catches up" with electron 2, meaning $x(t_*) = X(t_*)$. Show that

$$t_* = \frac{\sqrt{3}}{\omega}.$$

Interpret this result (i.e. explain what it means physically) in the limit $\omega \to 0$.

Solution. (a) The principle of dimensional homogeneity says that the dimensions of both sides of any equation must match. In particular, for the equation of motion here,

$$[\ddot{x}] = \frac{[\mathrm{iT}][\mathrm{iT}]}{[\mathrm{M}][\mathrm{L}^2][\varepsilon]}.$$

Rearranging gives

$$[\varepsilon] = \frac{[\mathrm{i}^2\mathrm{T}^2]}{[\mathrm{ML}^2][\ddot{x}]} = \frac{[\mathrm{i}^2\mathrm{T}^2]}{[\mathrm{ML}^2][\mathrm{L\ T}^{-2}]} = [\mathrm{i}^2\mathrm{T}^4\mathrm{M}^{-1}\mathrm{L}^{-3}].$$

(b) i. Integrating the equation of motion $\ddot{x} = \beta Q_0(1 + \omega t)$ gives

$$v(t) = \frac{\beta Q_0}{2\omega}(1 + \omega t)^2 + c_1,$$

and the initial condition $v(0) = -\frac{\beta Q_0}{2\omega}$ gives

$$c_1 = v(0) - \frac{\beta Q_0}{2\omega} = -\frac{\beta Q_0}{\omega},$$

hence

$$v(t) = \frac{\beta Q_0}{2\omega}\Big((1+\omega t)^2 - 2\Big) = \frac{\beta Q_0}{2\omega}\Big(-1 + 2\omega t + \omega^2 t^2\Big).$$

Integrating again, we find

$$x(t) = \frac{\beta Q_0}{2\omega}\left(-t + \omega t^2 + \frac{\omega^2 t^3}{3}\right) + c_2,$$

and the initial condition $x(0) = 0$ gives $c_2 = 0$.

ii. Integrating the equation of motion $\ddot{X} = \beta Q_0$ for the second electron, we find

$$V(t) = \beta Q_0 t + d_1,$$

and the initial condition $V(0) = 0$ gives $d_1 = 0$. Integrating again gives

$$X(t) = \frac{\beta Q_0 t^2}{2} + d_2,$$

and the initial condition $X(0) = 0$ gives $d_2 = 0$. Setting $x(t_*) = X(t_*)$ gives

$$\frac{\beta Q_0}{2\omega}\left(-t_* + \omega t_*^2 + \frac{\omega^2 t_*^3}{3}\right) = \frac{\beta Q_0 t_*^2}{2}.$$

Multiplying by $2\omega/(\beta Q_0)$ gives

$$-t_* + \omega t_*^2 + \frac{\omega^2 t_*^3}{3} = \omega t_*^2,$$

or,

$$t_*\left(-1 + \frac{\omega^2 t_*^2}{3}\right) = 0.$$

Solving the quadratic and keeping the positive solution gives

$$t_* = \frac{\sqrt{3}}{\omega}.$$

As $\omega \to 0$, $t_* \to \infty$, meaning electron 1 never catches up with electron 2. *The following is nice but not required for credit: ω is the rate at which the first capacitor increases its charge. As the charging rate decreases, the electron's acceleration becomes more and more similar to that of electron 2. In the limit, their accelerations become identical, so electron 1 can never recover the "head-start" that electron 2 makes towards the right.*

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