University of Birmingham
School of Mathematics

Vectors, Geometry and Linear Algebra
VGLA

Problem Sheet 1

Model Solutions

SUM **Q1**. Suppose that $\mathbf{a} = (2,1,5)$, $\mathbf{b} = (1,2,3)$ and $\mathbf{c} = (1,1,1)$ are vectors. Throughout your answers be careful to distinguish points, vectors and scalars.

(i) Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.
(ii) Calculate $(\mathbf{a} \cdot \mathbf{c})\mathbf{a} - (\mathbf{b} \cdot \mathbf{c})\mathbf{b}$.
(iii) Determine $\mathrm{proj}_{\mathbf{a}}(\mathbf{c})$.
(iv) Find $\lambda \in \mathbb{R}$ such that $\mathbf{b} + \lambda\mathbf{c}$ is perpendicular to $\mathbf{a}$.
(v) Write down the set of points on the line which has direction vector parallel to $\mathbf{a}$ and passes through the point $\mathbf{d} = (2,3,4)$.
(vi) Describe the points of the plane Π perpendicular to $\mathbf{a}$ containing the point $P = (1,3,3)$.

Solution. (i) [Note we will study determinants in Chapter 5, Week 7.] We have

$$\mathbf{a} \times \mathbf{b} = \left| \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 5 \\ 1 & 2 & 3 \end{pmatrix} \right| = -7\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

[Check that the output vector is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.] Hence

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \left| \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & -1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \right| = -4\mathbf{i} + 10\mathbf{j} - 6\mathbf{k} = (-4, 10, -6).$$

[Check that the output vector is perpendicular to $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$.]

(ii) This time

$$\mathbf{a} \cdot \mathbf{c} = 2 \cdot 1 + 1 \cdot 1 + 5 \cdot 1 = 2 + 1 + 5 = 8$$

and

$$\mathbf{b} \cdot \mathbf{c} = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 = 1 + 2 + 3 = 6.$$

Hence

$$(\mathbf{a} \cdot \mathbf{c})\mathbf{a} - (\mathbf{b} \cdot \mathbf{c})\mathbf{b} = 8\mathbf{a} - 6\mathbf{b} = (16, 8, 40) - (6, 12, 18) = (10, -4, 22).$$

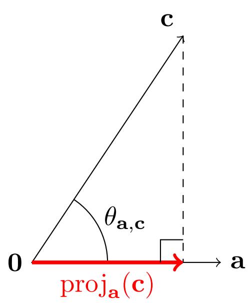

FIGURE 1. Geometric representation of the projection of $\mathbf{c}$ onto $\mathbf{a}$.

(iii) By definition (or using Figure 1 which is what I recommend), we have

$$\mathrm{proj}_{\mathbf{a}}(\mathbf{c}) = |\mathbf{c}| \cos(\theta_{\mathbf{a},\mathbf{c}}) \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}|^2} \right) \mathbf{a} = \frac{8}{30}(2,1,5) = \frac{4}{15}(2,1,5)$$

as $|\mathbf{a}|^2 = 2^2 + 1 + 5^2 = 30$ and $\mathbf{a} \cdot \mathbf{c} = 8$.

(iv) We require

$$(1,2,3) + \lambda(1,1,1) = (\lambda+1, \lambda+2, \lambda+3)$$

to be perpendicular to $\mathbf{a} = (2,1,5)$. Hence we want to find all $\lambda \in \mathbb{R}$ such that

$$(\lambda+1, \lambda+2, \lambda+3) \cdot (2,1,5) = 0$$

which means

$$2\lambda + 2 + \lambda + 2 + 5\lambda + 15 = 8\lambda + 19 = 0.$$

Hence $\lambda = -\frac{19}{8}$.

(v) We have $\{\lambda(2,1,5) \mid \lambda \in \mathbb{R}\}$ is the line parallel to $\mathbf{a}$ through $(0,0,0)$ and so

$$L = \{(2,3,4) + \lambda(2,1,5) \mid \lambda \in \mathbb{R}\}$$

is the line parallel to $\mathbf{a}$ through $\mathbf{d} = (2,3,4)$.

(vi) Let Π be the plane that we would like to describe. We have that $\mathbf{a}$ is a normal vector to the plane Π. Then, as $P \in \Pi$, the points R of Π satisfy $\vec{PR}$ is perpendicular to $\mathbf{a}$. Let R have position vector $\mathbf{r}$ with respect to the same origin as $\mathbf{a}$. Then

$$\mathbf{a} \cdot \vec{PR} = \mathbf{a} \cdot (\mathbf{r} - \mathbf{p}) = 0.$$

Write $\mathbf{r} = (x_1, x_2, x_3)$. Then

$$(2,1,5) \cdot (x_1, x_2, x_3) = (2,1,5) \cdot (1,3,3) = 20.$$

Thus

$$\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 + x_2 + 5x_3 = 20\}.$$

□

Q2. Suppose that $\triangle ABC$ is a triangle in $\mathbb{R}^2$ with vertices A, B and C. Show that the perpendicular bisectors of the sides go through a common point. You may like to use the following plan of a proof:

- Choose an origin to be the intersection of the perpendicular bisector of two of the sides of $\triangle ABC$ say AB and BC.
- With respect to this origin write down position vectors for all the vertices of $\triangle ABC$ and midpoints of sides of the triangle.
- Use the fact that the position vector through of the midpoint of line segment AB is perpendicular to $\vec{AB}$ and the position vector of the midpoint of the line segment BC is perpendicular to $\vec{BC}$ to deduce that the origin is the same distance from each vertex of $\triangle ABC$.
- Solve the problem.

Solution. Let O be the intersection of the perpendicular bisectors of the lines segments AB and BC. Assume that A has position vector $\mathbf{a}$, B position vector $\mathbf{b}$ and C has position vector $\mathbf{c}$ relative to the origin O.

The midpoint of line segment AB has position vector $\frac{\mathbf{b}+\mathbf{a}}{2}$, the midpoint of line segment BC has position vector $\frac{\mathbf{c}+\mathbf{b}}{2}$ and the midpoint of line segment CA has position vector $\frac{\mathbf{a}+\mathbf{c}}{2}$ with respect to the origin O.

Furthermore, as O is the origin and by definition the intersection of the perpendicular bisectors of the line segments of AB and BC , $\frac{\mathbf{b}+\mathbf{a}}{2}$ is perpendicular to $\vec{AB} = \mathbf{a} - \mathbf{b}$ and $\frac{\mathbf{c}+\mathbf{b}}{2}$ is perpendicular to $\vec{BC} = \mathbf{c} - \mathbf{b}$.

Hence using the scalar product we know:

$$(\mathbf{a} - \mathbf{b}) \cdot \left(\frac{\mathbf{b}+\mathbf{a}}{2}\right) = 0$$

which yields

$$\mathbf{a}\cdot\mathbf{a}+(-\mathbf{a}\cdot\mathbf{b})+\mathbf{a}\cdot\mathbf{b}+(-\mathbf{b}\cdot\mathbf{b})=0.$$

So

(1) $$|a|^2-|b|^2=0.$$

Similarly

$$(\mathbf{c}-\mathbf{b})\cdot\left(\frac{\mathbf{c}+\mathbf{b}}{2}\right)=0$$

and therefore

(2) $$|c|^2-|b|^2=0.$$

Subtracting (2) from (1) gives

(3) $$|a|^2-|c|^2=0.$$

This shows that all the vertices of $\triangle ABC$ have the same distance from O. Now using (3), the same calculation as above shows that

$$\left(\frac{\mathbf{a}+\mathbf{c}}{2}\right)\cdot(\mathbf{c}-\mathbf{a})=|c|^2-|a|^2=0.$$

Hence the position vector $\frac{\mathbf{a}+\mathbf{c}}{2}$ with respect to O is perpendicular to $\vec{AC}$. Thus the perpendicular bisector $\vec{AC}$ passes through 0. This proves the claim. □

$\boxed{\text{SUM}}$ **Q3**. Find the line of intersection of the planes

$$\{(x,y,z)\in\mathbb{R}^3 : x+2y+z=3\}$$

and

$$\{(x,y,z)\in\mathbb{R}^3 : x+y+2z=4\}.$$

Solution. The first plane has vector equation

$$(1,2,1)\cdot\mathbf{r}=3$$

and the second has vector equation

$$(1,1,2)\cdot\mathbf{r}=4.$$

A vector $\mathbf{u}$ which is parallel to the line of intersection is given by

$$\mathbf{u}=\left|\begin{pmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\1&1&2\end{pmatrix}\right|=(3,-1,-1).$$

We now find the coordinate of some point on the line of intersection. For this we can assume that the x-coordinate is zero. That is $x=0$. This gives the pair of linear equations

$$2y+z=3$$

and

$$y+2z=4.$$

Hence $y=\frac{2}{3}$ and $z=\frac{5}{3}$.

The line of intersection is now seen to be

$$L=\{(0,\frac{2}{3},\frac{5}{3})+\alpha(3,-1,-1)\mid\alpha\in\mathbb{R}\}.$$

□