Problem Sheet 5 (Combinatorics part)

Assignment available: Friday 24 November 2023 (Week 9).
Submission deadline: 1700 on Wednesday 6 December 2022 (Week 11).
Required content: All necessary content will be covered by the end of Week 10.

About problem sheet questions: The comments made about the first problem sheet apply equally to this one. In particular, please don't be reluctant to seek help if you are unsure how to proceed towards a solution, or how to express your ideas, as (unlike exam questions) the questions are set on the basis that you have access to this support.

Please note that due to the length of the question, there is only one SUM question for submission this week; this is not an error.

Question 1 (SUM)**.** Let G be a graph with $n \geq 3$ vertices and $\delta(G) \geq n/2$.

(a) Prove that G is connected.

(b) Using (a), or otherwise, prove that for each $k < n$, if G contains a copy of C_k then G contains a copy of P_k.

(c) Prove that for each k, if G contains a copy of P_k then G contains either a copy of P_{k+1} or a copy of C_{k+1}.

(d) Using (b) and (c), or otherwise, prove that G contains a copy of C_n (i.e. a cycle including every vertex of G).

The purpose of this question is to lead you step-by-step through the proof of one of the founding results of extremal graph theory, namely Dirac's theorem (1952), which states that every graph G on $n \geq 3$ vertices with $\delta(G) \geq n/2$ contains a Hamilton cycle, that is, a cycle of length n (in other words, a cycle that includes every vertex of G. Please note that each part of the problem uses a different argument, so if you are struggling with an earlier part, please do go on to consider later parts of the question despite this.

There will be a Canvas announcement early in Week 10 to provide further support as to how you might try to begin each part of the question. I have kept this separate from the problem sheet as I would strongly encourage you first to attempt the questions before you look at this additional content, and to complete as much as you can without it.

The definition of a connected graph will be presented in the lecture on the Monday of week 10; all the remaining graph theory concepts you need were covered in week 9.

Question 2. Let G be a bipartite graph with vertex classes A and B. Prove that

$$\sum_{v \in A} \deg(v) = |E(G)|.$$

Question 3. Prove that any tree with at least two vertices has at least two leaves.

Question 4.

(a) For which $n \in \mathbb{N}$ does there exist a graph G with n vertices such that $|E(G)| = |E(\overline{G})|$?

(b) Find a graph G with at least two vertices for which G and $\overline{G}$ are isomorphic.

(c) Up to isomorphism, how many 2-regular graphs are there with 8 vertices?