Problem Sheet 2 — Model Solutions and Feedback

Question 1 (SUM)**.** Let X be the set of positive integers which are factors of 63504. Calculate:

(a) the size of X;
(b) the number of elements of X which are divisible by 4;
(c) the number of elements of X which are divisible either by 4 or by 9.

Solution. Observe that $63504 = 2^4 \cdot 3^4 \cdot 7^2$. So the positive factors of 63504 are the integers of the form $2^a \cdot 3^b \cdot 5^c$ where $0 \leq a \leq 4$, $0 \leq b \leq 4$ and $0 \leq c \leq 2$. $(\dagger)$ This means that, defining $A = \{0, 1, 2, 3, 4\}$, $B = \{0, 1, 2, 3, 4\}$ and $C = \{0, 1, 2\}$, there is a bijection $f : A \times B \times C \to X$ given by $f((a, b, c)) = 2^a \cdot 3^b \cdot 5^c$. By the pairing principle it follows that

$$|X| = |A \times B \times C| = |A| \cdot |B| \cdot |C| = 5 \times 5 \times 3 = 75.$$

So there are 75 positive factors of 63504.

For (ii), observe that the factors which are divisible by 4 are those of the form $2^a \cdot 3^b \cdot 5^c$ where $2 \leq a \leq 4$, $0 \leq b \leq 4$ and $0 \leq c \leq 2$, so by a similar argument as above there are $3 \times 5 \times 3 = 45$ such outcomes.

For (iii), let X be the set of positive factors of 63504 which are divisible by 4, so $|X| = 45$ as shown in (ii). Let Y be the set of positive factors of 63504 which are divisible by 9; these have the form $2^a \cdot 3^b \cdot 5^c$ where $0 \leq a \leq 4$, $2 \leq b \leq 4$ and $0 \leq c \leq 2$, so by a similar argument as above the number of these is $|Y| = 5 \times 3 \times 3 = 45$. The set $X \cap Y$ is then the set of positive factors of 63504 which are divisible by 4 and by 9; these have the form $2^a \cdot 3^b \cdot 5^c$ where $2 \leq a \leq 4$, $2 \leq b \leq 4$ and $0 \leq c \leq 2$, so by a similar argument as above the number of these is $|X \cap Y| = 3 \times 3 \times 3 = 27$. By inclusion-exclusion it follows that the number of positive factors of 63504 which are divisible by 4 or by 9 is

$$|X \cup Y| = |X| + |Y| - |X \cap Y| = 45 + 45 - 27 = 63.$$

$\square$

Feedback. I would expect to see the following things, stated or explained clearly, in a good answer to this question.

- The prime factorisation of 63504 (I wouldn't say it's necessary to show the working for this).
- Deduced from this, the fact that factors of 63504 have the form $2^a \cdot 3^b \cdot 5^c$ for a, b and c in the appropriate ranges.
- A deduction that there are 90 such factors, for (i).
- A statement of the form of factors which are divisible by 4, and a deduction of number of these and therefore the probability for (ii).
- If you proceed by the model solution for (iii), statements of the forms of factors which are divisible by 9 and those divisible by both 4 and 9, a clear definition of the sets to which you will apply inclusion-exclusion, a statement of the sizes of $|X|, |Y|$ and $|X \cap Y|$, and therefore of $|X \cup Y|$, and from this a deduction of the probability.

There is some scope for variation from the model solution though. In particular, rather than presenting a bijection to count the factors of 63504 given the form of these, it is fine to say at the point labelled $(\dagger)$ that e.g. "there are 5 choices for a, and then 5 choices for b, and then 3 choices for c, giving $5 \times 5 \times 3 = 90$ factors in total" (this would replace the solution from $(\dagger)$ to the end of the displayed equation). However, this way of presenting the choice of a factor as a series of choices is much preferable to simply saying, e.g. "so there are $5 \times 5 \times 3 = 90$ factors" at this point.

Note the use of "by a similar argument" in (ii) and (iii). More precisely, this means it's the same argument but with the numbers involved changed slightly. It's fine for you to use this in your solutions, but carefully

– it's easy to miss subtle differences between cases, and moreover you are liable to lose marks if you try to use this in cases where the arguments are not sufficiently similar. Whenever in doubt, write the case in full.

Question 2 (SUM)**.** How many integers n are there with $1,000 \leq n < 10,000$ which do not have 8 as a digit?

You can solve this either by using the 4-set inclusion-exclusion formula or by using the product rule; the latter is simpler. Both methods are given here, with the inclusion-exclusion formula first:

Solution. Note that all integers in the given range have four digits. Define sets A, B, C and D as follows.

$$\begin{aligned}
A &= \{n \in \mathbb{Z} : 1,000 \leq n < 10,000 \text{ and the first digit of } n \text{ is 8}\}\\
B &= \{n \in \mathbb{Z} : 1,000 \leq n < 10,000 \text{ and the second digit of } n \text{ is 8}\}\\
C &= \{n \in \mathbb{Z} : 1,000 \leq n < 10,000 \text{ and the third digit of } n \text{ is 8}\}\\
D &= \{n \in \mathbb{Z} : 1,000 \leq n < 10,000 \text{ and the fourth digit of } n \text{ is 8}\}
\end{aligned}$$

So any integer in $A \cap B \cap C \cap D$ has all four digits equal to 8, that is, is 8888. So $|A \cap B \cap C \cap D| = 1$. Similarly, an integer in $B \cap C$ must have the form "x88y", where y can be any integer between 0 and 9, and x can be any integer between 1 and 9 (because the first digit cannot be zero!). So $|B \cap C| = 90$. Similarly we obtain:

- $|A \cap B \cap C| = |A \cap B \cap D| = |A \cap C \cap D| = 10$ and $|B \cap C \cap D| = 9$,
- $|A \cap B| = |A \cap C| = |A \cap D| = 100$ and $|B \cap C| = |B \cap D| = |C \cap D| = 90$,
- $|A| = 1000$ and $|B| = |C| = |D| = 900$.

So by the inclusion-exclusion formula for four sets, we have

$$\begin{aligned}
|A \cup B \cup C \cup D| =& 1000 + 900 + 900 + 900\\
&- 100 - 100 - 100 - 90 - 90 - 90\\
&+ 10 + 10 + 10 + 9\\
&- 1\\
=& 3168.
\end{aligned}$$

Since $A \cup B \cup C \cup D$ consists of all the integers n with $1000 \leq n \leq 10000$ which do have 8 as a digit, we conclude that there are 3168 such integers, and therefore $9000 - 3168 = 5832$ integers in the given range which do not have 8 as a digit. So the solution is 5832. □

Now the product rule method, presented in terms of counting choices:

Solution. We count the ways to form an integer n as in the question. To do this, there are 8 choices for the first digit (i.e. $1, 2, 3, 4, 5, 6, 7, 9$), then 9 choices for the second digit (i.e. $0, 1, 2, 3, 4, 5, 6, 7, 9$), then 9 choices for the third digit (i.e. $0, 1, 2, 3, 4, 5, 6, 7, 9$) and finally 9 choices for the fourth digit (i.e. $0, 1, 2, 3, 4, 5, 6, 7, 9$). Each combination of these choices gives a different integer n with $1000 \leq n < 10000$ which does not have 8 as a digit. Also, it is clear that every integer n with $1000 \leq n < 10000$ which does not have 8 as a digit will be formed by some combination of choices. So the number of such integers is $8 \cdot 9 \cdot 9 \cdot 9 = 5832$. □

The use of the product rule in the argument above was implicit, in the step where we multiply the number of choices for each digit to obtain the total number of possibilities. This usage could be made explicit as follows:

Solution. The integers we want to count are the integers "$abcd$" with $(a, b, c, d) \in A \times B \times C \times D$, where

$$\begin{aligned} A &= \{1, 2, 3, 4, 5, 6, 7, 9\} \\ B &= \{0, 1, 2, 3, 4, 5, 6, 7, 9\} \\ C &= \{0, 1, 2, 3, 4, 5, 6, 7, 9\} \\ D &= \{0, 1, 2, 3, 4, 5, 6, 7, 9\} \end{aligned}$$

By the product rule, the number of such integers is

$$|A \times B \times C \times D| = |A||B||C||D| = 8 \cdot 9 \cdot 9 \cdot 9 = 5832.$$

□

Feedback. If you used the inclusion-exclusion approach, or the formal application of the product rule, you should clearly define the sets to which you are applying the result. For inclusion-exclusion you should then explain your claimed sizes for each of the intersections, and that the union is the set you are trying to count, whilst for the product rule you should explain how the product of the sets corresponds to the numbers you want to count. Similarly, for the counting choices approach that you make clear how you are forming the objects you want to count via a sequence of choices (here this means choosing digit by digit).

In general, once you have got used to the idea you will probably find it easier and simpler to proceed by counting choices rather than formally applying the product rule, but whenever you are in doubt go back to thinking about the set product which is underpinning your counting choices approach.

All of the solutions presented above are fully valid in their own right – it is simply a matter of personal preference which you prefer to use. In fact it is quite common for counting problems to have admit a range of different possible solutions.

Question 3. Let $X = \{1, 2, 3\}$ and $Y = \{1, 2, \ldots, 10\}$.

(a) How many functions are there from X to Y?
(b) How many of these functions are injective?
(c) How many injections from X to Y have 1 as a fixed point?
(d) How many injections from X to Y have no fixed points?

(A fixed point of a function $f : X \to Y$ is an element $x \in X$ with $f(x) = x$, i.e. which maps to itself.)

Solution. (a) Each function $f : X \to Y$ corresponds to an ordered sequence $(f(1), f(2), f(3))$ of three elements of Y, and likewise each such sequence (a, b, c) corresponds to a function $f : X \to Y$ with $f(1) = a$, $f(2) = b$, $f(3) = c$. This gives a bijection between the set of functions $f : X \to Y$ and the set of ordered sequences of three elements of Y (allowing repetition), so we conclude that the number of functions $f : X \to Y$ is the same as the number of such sequences, i.e. $10^3 = 1000$.

Alternatively, to form a function $f : X \to Y$ we need to choose an image for 1, an image for 2 and then an image for 3. There are 10 options for each of these choices, giving $10 \times 10 \times 10 = 1000$ possible functions overall.

(b) The same correspondence between functions $f : X \to Y$ and ordered sequences of three elements of Y as in (a) gives a bijection between the set of injective functions $f : X \to Y$ and the set of ordered sequences of three elements of Y with no repeated element (because f being injective implies that different elements of X have different images, which is equivalent to saying each element of Y appears at most once in the sequence). So these sets have the same size, i.e. $10!/(10-3)! = 10 \times 9 \times 8 = 720$.

Again one can alternatively say that there are 10 choices for the image of 1, and having fixed this there are then 9 possibilities for the image of 2 (since $f(2)$ must be different from $f(1)$ for the function to be injective), following which there are 8 possibilities for the image of 3, giving $10 \times 9 \times 8 = 720$ functions overall.

(c) The injective functions $f : X \to Y$ with 1 as a fixed point correspond to the sequences $(1, b, c)$ with $b, c \in \{2, \ldots, 10\}$ and $b \neq c$. By omitting the first coordinate these correspond to ordered pairs (b, c) of elements of $\{2, \ldots, 10\}$ without repetition, of which there are $9 \times 8 = 72$.

Alternatively, the image of 1 is already fixed, after which there are 9 choices for the image of 2, and then 8 choices for the image of 3, giving $9 \times 8 = 72$ choices overall.

A third option is to argue by symmetry that the number of injections $f : X \to Y$ with $f(1) = i$ is the same for each $i \in Y$, and so using (b) this shared number is $720/10 = 72$.

(d) Let A be the set of injections $f : X \to Y$ with 1 as a fixed point, let B be the set of injections $f : X \to Y$ with 2 as a fixed point, and let C be the set of injections $f : X \to Y$ with 3 as a fixed point. Then $|A| = |B| = |C| = 72$ by (c) (the same argument holds with 2 or 3 as a fixed point rather than 1). Furthermore, $A \cap B$ is the set of injections $f : X \to Y$ with 1 and 2 as fixed points, and there are 8 such functions (since we have fixed the image of 1 and 2 and there are then 8 possibilities for the image of c), so $|A \cap B| = 8$. Likewise $A \cap C$ is the set of injections $f : X \to Y$ with 1 and 3 as fixed points, and $A \cap C$ is the set of injections $f : X \to Y$ with 2 and 3 as fixed points, so we have $|B \cap C| = |A \cap C| = 8$. Finally, $A \cap B \cap C$ is the set of injections $f : X \to Y$ with 1, 2 and 3 as fixed points, and there is just one such function, namely the identity function from X to Y, so $|A \cap B \cap C| = 1$. By inclusion-exclusion we conclude that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 72 + 72 + 72 - 8 - 8 - 8 + 1 = 193.$$

Since $|A \cup B \cup C|$ is the number of injections $f : X \to Y$ with at least one fixed point, and we know from (b) that there are 720 injections $f : X \to Y$ in total, we conclude that the number of injections $f : X \to Y$ with no fixed points is $720 - 193 = 527$. $\square$

Feedback. For parts (a), (b) and (c) of this question the model solution includes both a 'counting choices' argument, in which we form a function by a sequence of choices and count up the possibilities, and a more formal argument in which we are explicit about the bijection involved. Either approach to present your answer is fine – the 'counting choices' argument gives shorter solutions which are often easier to understand, but it's important to understand how this is underpinned by the more formal argument (which is the reason why you can count choices in this way). See the Week 4 Guided Study Session for more discussion of this point.

A common source of confusion is the difference between a function $f : X \to Y$ and the mapping of a single element of X. To form a single function you need to specify where each of the three elements maps to, so it's important to maintain this distinction. Similarly, the presentation of your answer for (a), (b) and (c) will depend on which approach you use, but for the 'counting choices' approach the key is to set out clearly that the sequence of three choices of where each element maps to forms a single function, and so we count the functions by counting the number of ways to make these choices.

Part (d) is a typical inclusion-exclusion argument, similar to the problem of binary sequences discussed in the Week 4 Guided Study Session. The two key elements to highlight are firstly, that instead of counting the functions with no fixed points we took the complement and counted the functions with at least one fixed point, and secondly that the problem in counting these is that we don't know which element is the fixed point, so we consider each possibility for this and use inclusion-exclusion to handle the overlaps (i.e. functions with more than one fixed point). In terms of presenting your answer, it's important that you set out clearly what each of the sets are (A, B and C in the model solution above), explain why the intersections have the claimed sizes, apply inclusion-exclusion and explain how and why the conclusion you get (the size of $|A \cup B \cup C|$) leads you to the solution.

Question 4. Prove the general form of the inclusion-exclusion formula (this is Theorem 2.7 from the Week 2 lecture notes; you can use either form of the theorem).

An outline of how to proceed was given in Lecture 4, and also appears in the Week 2 lecture notes. As noted there the main difficulty is notational: you need to handle a large number of terms in a clear and understandable way.

Solution. You can work with either way of writing the theorem, but this model solution will prove the theorem in the following form: for any $r \in \mathbb{N}$ and any finite sets $A_1, \ldots, A_r$ we have

$$\left| \bigcup_{i=1}^{r} A_i \right| = \sum_{\substack{J \subseteq \{1,2,\ldots,r\} \\ J \neq \emptyset}} (-1)^{|J|+1} \left| \bigcap_{i \in J} A_i \right|.$$

We prove this statement by induction on r. Our base case is $r = 1$, for which the statement says that for any finite set A_1 we have $|A_1| = |A_1|$, which is a tautology.

For the inductive step, fix $k \in \mathbb{N}$ and suppose that the statement holds for $r = k$. Let $A_1, \dots, A_{k+1}$ be finite sets, and write $Y = \bigcup_{i=1}^{k} A_i$. Then

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = |Y \cup A_{k+1}| = |Y| + |A_{k+1}| - |Y \cap A_{k+1}|, \tag{1}$$

where the second equality uses the inclusion-exclusion formula for two sets[1]. We now consider each term of the right hand side separately. First observe that by our inductive hypothesis applied to the sets $A_1, \dots, A_k$ we have

$$|Y| = \left|\bigcup_{i=1}^{k} A_i\right| = \sum_{\substack{J\subseteq\{1,2,\dots,k\}\\ J\neq\emptyset}} (-1)^{|J|+1} \left|\bigcap_{i\in J} A_i\right| = \sum_{\substack{J\subseteq\{1,2,\dots,k,k+1\}\\ J\neq\emptyset,\ k+1\notin J}} (-1)^{|J|+1} \left|\bigcap_{i\in J} A_i\right|. \tag{2}$$

(To see this, observe that the sums in the final two terms are taken over exactly the same subsets). For the second term we have simply

$$|A_{k+1}| = \sum_{J=\{k+1\}} (-1)^{|J|+1} \left|\bigcap_{i\in J} A_i\right| \tag{3}$$

(Note that there is only one term in the sum on the right hand side). For the third term, note that by distributivity we have

$$Y \cap A_{k+1} = (A_1 \cup \dots \cup A_k) \cap A_{k+1} = (A_1 \cap A_{k+1}) \cup (A_2 \cap A_{k+1}) \cup \dots \cup (A_k \cap A_{k+1}) = \bigcup_{i=1}^{k} (A_i \cap A_{k+1}).$$

So by our inductive hypothesis applied to the sets $(A_1 \cap A_{k+1}), (A_2 \cap A_{k+1}), \dots, (A_k \cap A_{k+1})$, we have

$$\begin{aligned}|Y \cap A_{k+1}| = \left|\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right| &= \sum_{\substack{I\subseteq\{1,2,\dots,k\}\\ I\neq\emptyset}} (-1)^{|I|+1} \left|\bigcap_{i\in I} (A_i \cap A_{k+1})\right| \\ &= (-1) \sum_{\substack{J\subseteq\{1,2,\dots,k,k+1\}\\ J\neq\{k+1\},\ k+1\in J}} (-1)^{|J|+1} \left|\bigcap_{i\in J} A_i\right|. \end{aligned} \tag{4}$$

To see this, note that the summand corresponding to I in the penultimate term of the equation is equal to the summand corresponding to $J = I \cup \{k+1\}$ in the final term of the equation, and so the two sums are equal on a term-by-term basis. Combining equations (1), (2), (3) and (4) we have

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{\substack{J\subseteq\{1,2,\dots,k,k+1\}\\ J\neq\emptyset}} (-1)^{|J|+1} \left|\bigcap_{i\in J} A_i\right|.$$

Since $A_1, \dots, A_{k+1}$ were arbitrary finite sets, this proves that the statement holds for $r = k + 1$, and so completes the inductive step. We conclude that the statement holds for any $r \in \mathbb{N}$. $\square$

Feedback. The main difficulty with this question is not working out the steps needed for the proof – for the inductive step these are essentially the same as in the proof of 3-set inclusion-exclusion seen in lectures – but instead in handling the notation. I prefer the set notation used in the model solution as it is clearer and more concise, but indexing sums by subsets of a set can take some getting used to (but it is well-worth the effort, as this kind of indexing occurs commonly in higher-level combinatorics). You can also proceed by using a large number of ellipses (i.e. ... symbols), but then the challenge is to make sure it's always clear what these mean, since for example you will require nested ellipses for the proof.

[1] I am allowing use of this because we already proved this earlier in this section of the course. Without that we would need our induction argument to have two base cases, namely $r = 1$ and $r = 2$.