REAL AND COMPLEX ANALYSIS

2RCA/2RCA3

REAL ANALYSIS

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LECTURE 0

Some assumed knowledge

We begin by recalling some notation. We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{Z}$ the set of integer numbers, and $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers.

1. Some set theory

- If an element a belongs to a set A, we write $a \in A$; and if not we write $a \notin A$.

- If A is a subset of B (perhaps equal to B), we write

$$A \subseteq B \quad (\text{or} \quad B \supseteq A).$$

- Let A and B two subsets of a set X. Then,

$$A = B \qquad \text{iff}^1 \qquad A \subseteq B \qquad \text{and} \qquad B \subseteq A.$$

- Let A and B be two sets. Then,

(1) The *union* of A and B, $A \cup B$, is the set defined by

$$A \cup B = \{x \in X \ : \ x \in A \quad \text{or} \quad x \in B\}.$$

(2) The *intersection* of A and B, $A \cap B$, is the set defined by

$$A \cap B = \{x \in X \ : \ x \in A \quad \text{and} \quad x \in B\}.$$

If $A \cap B = \emptyset$, then A and B have no points in common and A and B are said to be *disjoint*.

- Let A be a subset of a set X. Then, the *complement of A in X*, A^c (also $X - A$ or $X \backslash A$) is the set

$$A^c = \{x \in X \quad \text{such that} \quad x \notin A\}.$$

[1] Iff means "if and only if".

- Let I be any set (finite or not). For each element $i \in I$, we are given a subset A_i of X, then we denote the union and intersection of the collection of sets $\{A_i\}_{i\in I}$ by

$$\bigcup_{i\in I} A_i \qquad \text{and} \qquad \bigcap_{i\in I} A_i,$$

respectively.

The following identities, know as the *De Morgan's Laws*, are true

(1)

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I}(A_i^c)$$

(1)

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I}(A_i^c).$$

2. Intervals

An *interval* is a subset of $\mathbb{R}$ taking one of the following forms, where a and b are arbitrary real numbers and $a < b$.

$$\begin{aligned}
(a,b) &:= \{x \in \mathbb{R} : a < x < b\} && \text{(this is called an } \textit{open} \text{ interval)}\\
(a,b] &:= \{x \in \mathbb{R} : a < x \le b\}\\
[a,b) &:= \{x \in \mathbb{R} : a \le x < b\}\\
[a,b] &:= \{x \in \mathbb{R} : a \le x \le b\} && \text{(this is called a } \textit{closed} \text{ interval)}\\
(a,\infty) &:= \{x \in \mathbb{R} : x > a\}\\
[a,\infty) &:= \{x \in \mathbb{R} : x \ge a\}\\
(-\infty,b) &:= \{x \in \mathbb{R} : x < b\}\\
(-\infty,b] &:= \{x \in \mathbb{R} : x \le b\}\\
(-\infty,\infty) &:= \mathbb{R}
\end{aligned}$$

The interval $(0,\infty)$ will also be denoted by $\mathbb{R}^+$.

3. Functions

Suppose that $X \subseteq \mathbb{R}$. A *real function* $f : X \to \mathbb{R}$ is a rule which assigns to every real number $x \in X$ a unique real number $y \in \mathbb{R}$. If the number $y \in \mathbb{R}$ corresponds to the number $x \in X$, then we write $y = f(x)$.

- The *domain* of f, $Dom(f)$, is the set X.

- The *image* or the *range* of f is the set

$$f(X) = \{f(x) : x \in X\}$$

(sometimes we will also use the notation $\Im m(f)$ for $f(X)$.

- The *graph* of f is the set of points in $\mathbb{R}^2$ defined by

$$\{(x, f(x)) \in \mathbb{R}^2 : x \in X\}.$$

EXAMPLES 0.1. The first two items are standard examples of real functions. The last item shows that sequences are essentially the same as functions.

(1) $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$.
(2) $f : [0, \infty) \to \mathbb{R}$, given by $f(x) = \sqrt{x}$.
(3) If $(a_n)_{n\in\mathbb{N}}$ is a real sequence, then we may define a function $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = a_n$ for every $n \in \mathbb{N}$.

LECTURE 1

Basic topological concepts on the real line

1. Euclidean distance on the real line

DEFINITION 1.1. On $\mathbb{R}$, we define the *Euclidean distance on the real line* to be the function $d : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ defined by:

$$d(x, y) = |x - y|, \qquad \forall\, x, y \in \mathbb{R}.$$

- For any given real numbers x and $y \in \mathbb{R}$, we call $d(x, y)$ or $|x - y|$ the "distance between x and y".

- We call $\mathbb{R}$ with this distance the "Euclidean space on the real line".

- We will often write $|\cdot|$ meaning the Euclidean distance on the real line.

PROPOSITION 1.2. *[Properties] The Euclidean distance on $\mathbb{R}$ satisfies the following properties: for any $x, y, z \in \mathbb{R}$*

(1) $|x - y| \geq 0$ *and* $|x - y| = 0$ *if and only if* $x = y$.

(2) $|x - y| = |y - x|$ *(Symmetry)*

(3) $|x - y| \leq |x - z| + |z - y|$ *(Triangle inequality for real numbers)*

PROOF. Provided in the "Real Analysis and the Calculus"-module. □

REMARK 1.3. Many of the concepts introduced in the "Real Analysis and the Calculus"-module, such as the notion of convergence of sequences of real numbers and that of the continuity of a real function, are based on the idea of "distance" between real numbers and the associated idea of "closeness". For example, recall the following definition: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and $l \in \mathbb{R}$. We say that (a_n) converges to l as $n \to \infty$, and write $a_n \to l$ as $n \to \infty$, if: Given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \varepsilon \qquad \text{whenever} \qquad n \geq N.$$

In words, the sequence (a_n) converges to l as $n \to \infty$ if one can make the distance between the numbers of the sequence and l as small as we please by taking elements in the sequence associated to an index sufficiently large. What is it that makes the notion/definiton of convergence work? *Answer:* the notion of "closeness", that is the notion of "distance".

The above remark suggests that if we can find an appropriate notion of "distance" on a non-empty set X (not necessarily a subset of real numbers), we may give the definition of what it means for a sequence of elements of the set X to be a convergent sequence in X.

2. Open, closed and bounded sets

When studying the notions of continuity or differentiability for real functions, and also when studying the properties of real functions, we will often talk about open/closed intervals or closed bounded intervals. One may ask:

Q1. What do these notions (open/closed/bounded sets) mean?

Q2. What is the difference between the interval $(0, 1)$, $(0, 1]$, and $[0, 1]$?

In order to answer these two questions we need to introduce a preliminary definition.

DEFINITION 1.4. Given $x_0 \in \mathbb{R}$, an *open interval centred at* x_0 is a set of real numbers of the form

$$(x_0 - r, x_0 + r) \qquad \text{for some} \qquad r > 0$$

Notice that

$$(x_0 - r, x_0 + r) = \{x \in \mathbb{R} \ : \ |x - x_0| < r\} = \{x \in \mathbb{R} \ : \ d(x, x_0) < r\}.$$

Thus, one can describe an open interval centred at a point in terms of the euclidean distance on the real line.

DEFINITION 1.5. Let $\mathbb{R}$ be the set of real numbers.

(i) A subset $U \subseteq \mathbb{R}$ is an <u>open set</u> if:

For every $x \in U$, there exists $\varepsilon > 0$ (which may depend on x) such that

$$(x - \varepsilon, x + \varepsilon) \subseteq U.$$

In words, a set U of real numbers is an open set if it has the property that for $\overline{\text{every}}$ point x in U there is an open interval centred at the point that is contained in U. The open interval may vary with x, and it may be very small.

(ii) A subset $F \subseteq \mathbb{R}$ is a <u>closed set</u> if the complement of F in $\mathbb{R}$, F^c , is an open set.

(iii) A subset $Y \subseteq \mathbb{R}$ is a <u>bounded set</u> if there exists $K \in \mathbb{R}$ such that
$$|x| \leq K \qquad \text{for all } x \in Y.$$

We will continue to see some examples.

Examples.

1. Let $a < b$ be real numbers. The open interval (a, b) is open in $\mathbb{R}$

Indeed, given any $x \in (a, b)$, we can take $\varepsilon = \min\{|x - a|, |x - b|\}$, so that
$$(x - \varepsilon, x + \varepsilon) \subseteq (a, b).$$
Thus, (a, b) is an open set.

2. Let $a < b$ be real numbers. The closed interval $[a, b]$ is a closed set.

Indeed, notice that $[a, b]^c = (-\infty, a) \cup (b, \infty)$. We want to show that $(-\infty, a) \cup (b, \infty)$ is an open set. Now,

If $x \in (-\infty, a)$, take $\varepsilon = |x - a|/2 > 0$ (for example), then
$$(x - \varepsilon, x + \varepsilon) \subseteq (-\infty, a) \subseteq (-\infty, a) \cup (b, \infty).$$
If $x \in (b, \infty)$, take $\varepsilon = |x - b|/2 > 0$ (for example), then
$$(x - \varepsilon, x + \varepsilon) \subseteq (b, \infty) \subseteq (-\infty, a) \cup (b, \infty).$$

Thus, $(-\infty, a) \cup (b, \infty)$ is an open set, and therefore $[a, b]$ is a closed set.

3. Let $a, b \in \mathbb{R}$. Then, arguing in a similar way as above, we have that: $(-\infty, a)$ and (b, ∞) are open sets. Also, $(-\infty, a]$ and $[b, \infty)$ are closed sets.

4. Let $a < b$ be real numbers. The interval $[a, b)$ is <u>not open</u> and is <u>not closed</u>.

Indeed, to see that $[a, b)$ is not an open set, it suffices to find a point in the interval $[a, b)$ such that any open interval centred at the point contains points outside the set $[a, b)$. Consider the point a in $[a, b)$, then <u>for any</u> $\varepsilon > 0$
$$(a - \varepsilon, a + \varepsilon)$$
contains points which are <u>not</u> in the set $[a, b)$ (notice that all the points in $(a - \varepsilon, a)$ are not in $[a, b)$).

To see that $[a, b)$ is not a closed set we need to show that $[a, b)^c = (-\infty, a) \cup [b, \infty)$ is not an open set. Indeed, observe that there exists $b \in [a, b)^c$ such that any open interval centred at b has points outside the set $[a, b)^c$.

5. All of the intervals considered in **1,2** and **4** are bounded sets. For example: notice that for any $x \in (a, b)$, we have that $|x| \leq 2(|a| + |a - b|)$.

Exercise. Is the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ an open set? is it a closed set? is it a bounded set? (*Why?*)

The following theorems establish how open/closed sets behave with respect to taking union and intersections of open/closed sets.

THEOREM 1.6.

(i) *The union of an* <u>*arbitrary*</u> *family/collection of open sets is an open set.*

(ii) *The intersection of a* <u>*finite*</u> *number of open sets is an open set.*

PROOF. (*Non-examinable*)

Proof of (i). Let $\{U_i\}_{i\in I}$ be an arbitrary collection of open sets U_i. We want to show that $\bigcup_{i\in I} U_i$ is an open set. Indeed:

Given $x \in \bigcup_{i\in I}$, then there exists an index $i_o \in I$ such that $x \in U_{i_0}$ and since by hypothesis we know that U_{i_0} is an open set, then there exists $\varepsilon > 0$ such that

$$(x - r, x + r) \subseteq U_{i_0},$$

and since $U_{i_0} \subseteq \bigcup_{i\in I}$, we obtain that

$$(x - r, x + r) \subseteq U_{i_0} \subseteq \bigcup_{i\in I} U_i.$$

Thus, $\bigcup_{i\in I} U_i$ is an open set.

Proof of (ii). Let $\{U_1, \ldots U_N\}$ be a finite collection of N-open sets U_i, $i \in \{1, \ldots N\}$. We want to show that $\bigcap_{i=1}^{N} U_i$ is an open set. Indeed:

Given $x \in \bigcap_{i=1}^{N} U_i$, then for all $i \in \{1, \ldots N\}$ we have that $x \in U_i$, and since <u>each</u> U_i is an open set (by hypothesis) we have that

there exists $\varepsilon_1 > 0$ such that $(x - r_1, x + r_1) \subseteq U_1$,
$\vdots$
there exists $\varepsilon_N > 0$ such that $(x - r_N, x + r_N) \subseteq U_N$.

Take $\varepsilon = \min(\varepsilon_1, \cdots, \varepsilon_N)/2 > 0$, then

$$(x - r, x + r) \subseteq U_i, \qquad \text{for all } i \in \{1, 2, \ldots N\},$$

Thus

$$(x - r, x + r) \subseteq \bigcap_{i=1}^{N} U_i.$$

The above argument shows that $\bigcap_{i=1}^{N} U_i$ is an open set. $\square$

As a straightforward consequence of Theorem 1.6 and the De Morgan's Laws, we obtain the following:

THEOREM 1.7.

(i) *The intersection of an arbitrary family/collection of closed sets is a closed set.*

(ii) *The union of a finite number of closed sets is a closed set.*

Both Theorems 1.6 and 1.7 are useful tools to generate open/closed sets by taking (appropriate) unions or intersection of open/closed sets.

Example. Since for all $n \in \mathbb{N}$ the interval $(-n, n)$ is an open sets, using the above theorem we conclude that $\bigcup_{n=1}^{\infty}(-n, n) = \mathbb{R}$ is an open set.

LECTURE 2

Limits at a point

To define the limit of $f(x)$ when x tends to a, we consider values of x *in a neighbourhood of* a, that is, close to a. We distinguish between when x is to the left of a and when x is to the right of a.

Definition 2.1 (Right-hand limit). Suppose that f is defined on $(a, a+R)$, for some $R \in \mathbb{R}^+$. Then we say "$f(x)$ tends to ℓ as x tends to a from above" if, given any ϵ in $\mathbb{R}^+$, there exists δ in $\mathbb{R}^+$ such that

$$|f(x) - \ell| < \epsilon \quad \text{whenever } a < x < a + \delta.$$

Using symbols, we write $f(x) \to \ell$ as $x \to a+$, or

$$\lim_{x \to a+} f(x) = \ell.$$

The symbol $a+$ indicates that we are taking the limit *from the right*, or *from above*.

Definition 2.2 (Left-hand limit). Suppose that f is defined on $(a - R, a)$, for some $R \in \mathbb{R}^+$. Then we say "$f(x)$ tends to ℓ as x tends to a from below" if, given any ϵ in $\mathbb{R}^+$, there exists δ in $\mathbb{R}^+$ such that

$$|f(x) - \ell| < \epsilon \quad \text{whenever } a - \delta < x < a.$$

Using symbols, we write $f(x) \to \ell$ as $x \to a-$, or

$$\lim_{x \to a-} f(x) = \ell.$$

The symbol $a-$ indicates that we are taking the limit *from the left*, or *from below*.

Definition 2.3 (ϵ-δ definition of a limit at a point). Suppose that $f(x)$ is defined on $(a - R, a) \cup (a, a+R)$, for some $R \in \mathbb{R}^+$. Then we say "$f(x)$ tends to ℓ as $x \to a$" if, given any ϵ in $\mathbb{R}^+$, there exists δ in $\mathbb{R}^+$ such that

$$|f(x) - \ell| < \epsilon \quad \text{whenever } 0 < |x - a| < \delta.$$

Using symbols, we write $f(x) \to \ell$ as $x \to a$, or

$$\lim_{x \to a} f(x) = \ell.$$

Remark 2.4. (1) The assumptions on f in Definitions 2.1, 2.2 and 2.3 do not preclude the possibility that f is defined on a larger set than the set considered. However, the conditions on x, such as $x \in (a, a+R)$, $x \in (a-R, a)$ or $0 < |x - a| < \delta$, strictly exclude the possibility that $x = a$. In particular, $\lim_{x \to a} f(x)$, if it exists, is independent of $f(a)$, and *it does not matter whether $f(a)$ is defined.*

(2) Sometimes we call the set $\{x \in \mathbb{R} : 0 < |x - a| < \delta\}$ a *punctured neighbourhood of* a.

We illustrate this with an example.

EXAMPLE 2.5. We consider the following three *distinct* functions:

$$\begin{aligned} f_1 : \mathbb{R} \to \mathbb{R} \quad &\text{defined by} \quad f_1(x) = 2x \\ f_2 : \mathbb{R} \setminus \{1\} \to \mathbb{R} \quad &\text{defined by} \quad f_2(x) = 2x \\ f_3 : \mathbb{R} \to \mathbb{R} \quad &\text{defined by} \quad f_3(x) = \begin{cases} 2x & \text{if } x \in \mathbb{R} \setminus \{1\} \\ 1 & \text{if } x = 1. \end{cases} \end{aligned}$$

Then $\lim_{x\to 1} f_i(x)$ exists and is equal to 2 for each $i \in \{1, 2, 3\}$, while

$$\begin{gathered} f_1(1) = 2 = \lim_{x\to 1} f_1(x) \\ f_2(1) \quad \text{is not defined} \\ f_3(1) = 1 \neq 2 = \lim_{x\to 1} f_3(x). \end{gathered}$$

REMARK 2.6. (1) It follows from the definitions that $\lim_{x\to a} f(x) = \ell$ if and only if both $\lim_{x\to a-} f(x) = \ell$ and $\lim_{x\to a+} f(x) = \ell$.

(2) Quite often both $\lim_{x\to a+} f(x)$ and $\lim_{x\to a-} f(x)$ exist, but they are different. This is quite a common reason for $\lim_{x\to a} f(x)$ to fail to exist.

EXAMPLE 2.7. Let $f : \mathbb{R} \to \mathbb{R}$ be the floor function, written $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer n such that $n \leq x$. For instance, $\lfloor 3/2 \rfloor = 1$ and $\lfloor -\pi \rfloor = -4$. In particular, the range of the floor function is $\mathbb{Z}$. For fixed $a \in \mathbb{R}$, we consider $\lim_{x\to a} f(x)$.

There are two cases to consider.

Case 1: $a \notin \mathbb{Z}$. In this case, $n < a < n + 1$, where $n = \lfloor a \rfloor$. If $\delta = \min\{a - n, n + 1 - a\}$ and $0 < |x - a| < \delta$, then $n < x < n + 1$ and so $f(x) = \lfloor x \rfloor = n$. Thus

$$|f(x) - n| = 0 < \epsilon$$

for all $\epsilon \in \mathbb{R}^+$ and

$$\lim_{x\to a} f(x) = n.$$

Case 2: $a \in \mathbb{Z}$. If $a - 1 < x < a$, then $\lfloor x \rfloor = a - 1$, so

$$|f(x) - (a - 1)| = 0 < \epsilon$$

for all $\epsilon \in \mathbb{R}^+$ and

$$\lim_{x\to a-} f(x) = a - 1.$$

When $a < x < a + 1$, on the other hand, $\lfloor x \rfloor = a$, so that

$$|f(x) - a| = 0 < \epsilon$$

for all $\epsilon \in \mathbb{R}^+$ and

$$\lim_{x\to a+} f(x) = a.$$

Noting that

$$\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x),$$

we deduce that $\lim_{x \to a} f(x)$ does not exist when $a \in \mathbb{Z}$, from Remark 2.6.

EXERCISE 2.8. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = x^2 \sin(1/x)$. Prove that $\lim_{x \to 0} f(x)$ exists and is equal to 0.

ANSWER. Let $\epsilon \in \mathbb{R}^+$. Since $|\sin(1/x)| \leq 1$ for all $x \neq 0$,

$$\left| x^2 \sin(1/x) - 0 \right| \leq x^2,$$

and $x^2 < \epsilon$ when $|x| < \sqrt{\epsilon}$, so, on setting $\delta = \sqrt{\epsilon}$, we find that

$$\left| x^2 \sin(1/x) - 0 \right| < \epsilon \quad \text{whenever } 0 < |x - 0| < \delta.$$

Hence $\lim_{x \to 0} f(x)$ exists and is equal to 0. $\triangle$

LECTURE 3

From theorems about sequences to theorems about functions

In this lecture, we prove a theorem connecting limits of sequences and limits of functions, which can be used to prove results about functions from results about sequences, to show that certain limits do not exist, and to deduce the Algebra of Limits and the Sandwich Theorem for limits at a point.

A connection between sequences and functions

Recall that if (a_n) is a sequence of real numbers that converges to a limit $a \in \mathbb{R}$, then we write

$$\lim_{n\to\infty} a_n = a.$$

THEOREM 3.1. *Suppose that f is a real function, defined on $(a-R, a) \cup (a, a+R)$ for some R in $\mathbb{R}^+$. Then $\lim_{x\to a} f(x) = \ell$ if and only if $\lim_{n\to\infty} f(a_n) = \ell$ for all sequences (a_n) such that $\lim_{n\to\infty} a_n = a$ and $a_n \neq a$ for all $n \in \mathbb{N}$.*

We give this proof later. For now, we use it to analyse the function $x \mapsto \sin(1/x)$ whose behaviour near 0 is rather strange.

EXERCISE 3.2. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$f(x) = \sin\Big(\frac{1}{x}\Big).$$

Prove that the limit of $f(x)$ as $x \to 0$ does not exist.

ANSWER. Define two sequences (a_n) and (b_n) by

$$a_n = \frac{1}{2n\pi} \quad \text{and} \quad b_n = \frac{1}{(\pi/2) + 2n\pi}$$

for all $n \in \mathbb{N}$. Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

and $a_n, b_n \neq 0$ for all $n \in \mathbb{N}$. Now

$$f(a_n) = \sin(2n\pi) = 0 \text{ and } f(b_n) = \sin\big(2n\pi + \tfrac{\pi}{2}\big) = 1$$

for all $n \in \mathbb{N}$. Hence the (constant) sequences $(f(a_n))$ and $(f(b_n))$ have limits 0 and 1 respectively. Now by Theorem 3.1, if $\lim_{x\to 0} f(x)$ were to exist, then both $(f(a_n))$ and $(f(b_n))$

would tend to the common limit $\lim_{x\to 0} f(x)$. Since they do not converge to a common limit, $\lim_{x\to 0} f(x)$ does not exist. $\triangle$

Proof of Theorem 3.1. (*Non-examinable. Seen in 1RAC-exercise sheets*) Suppose first that $\lim_{x\to a} f(x) = \ell$; we must show that $\lim_{n\to\infty} f(a_n) = \ell$ for all sequences with the properties described above.

Let $\epsilon \in \mathbb{R}^+$. Then by definition, there exists $\delta \in \mathbb{R}^+$ such that

$$|\,f(x) - \ell| < \epsilon \quad \text{whenever } 0 < |x - a| < \delta. \tag{3.1}$$

Now let (a_n) be a sequence such that $a_n \neq a$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} a_n = a.$$

By the definition of convergence of a sequence (treated in 1RAC-"Real Analysis and the Calculus"), there exists $N \in \mathbb{N}$ such that

$$|a_n - a| < \delta \quad \text{whenever } n > N,$$

so putting $x = a_n$ in (3.1),

$$|\,f(a_n) - \ell| < \epsilon \quad \text{whenever } n > N.$$

Hence

$$\lim_{n\to\infty} f(a_n) = \ell.$$

This completes the proof of the first half of the theorem.

Conversely, suppose that $\lim_{x\to\infty} f(x) \neq \ell$; we must show that there exists a sequence (a_n), such that $a_n \neq a$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, for which the sequence $(f(a_n))$ does not tend to ℓ.

Since $\lim_{x\to a} f(x) \neq \ell$, there exists $\epsilon \in \mathbb{R}^+$ such that for every $\delta \in \mathbb{R}^+$, there exists x with the property

$$0 < |x - a| < \delta \quad \text{and} \quad |\,f(x) - \ell| > \epsilon.$$

In particular, for every positive integer n, take δ to be $1/n$. Then there exists x such that $0 < |x - a| < 1/n$ and $|\,f(x) - \ell| > \epsilon$. We now rename this x as a_n, and then $|\,f(a_n) - \ell| > \epsilon$ and

$$0 < |a_n - a| < \frac{1}{n}\,.$$

This process defines a sequence (a_n), and $\lim_{n\to\infty} a_n = a$ because of the inequality above. Further, $a_n \neq a$ for all $n \in \mathbb{N}$. Now since

$$|\,f(a_n) - \ell| > \epsilon \quad \text{for all } n \in \mathbb{N},$$

the sequence $(f(a_n))$ does not tend to ℓ. $\square$

Corollaries of Theorem 3.1

THEOREM 3.3 (Algebra of Limits). *Suppose that f and g are real functions, whose domains both include a set of the form $(a-R,a)\cup(a,a+R)$ for some positive real number R. Suppose also that*

$$\lim_{x\to a} f(x)=\ell \quad \textit{and} \quad \lim_{x\to a} g(x)=m,$$

where $\ell, m\in\mathbb{R}$. Then the following statements hold:

(1) $\lim_{x\to a}(f(x)+g(x))=\ell+m$;
(2) $\lim_{x\to a}(f(x)-g(x))=\ell-m$;
(3) $\lim_{x\to a}(f(x)\,g(x))=\ell m$.

If in addition, $m\neq 0$, then

(4) $\lim_{x\to a}(f(x)/\,g(x))=\ell/m$.

PROOF. (*Non-examinable*) By Theorem 3.1, for all sequences $\{a_n\}$ such that $a_n\neq a$ and $\lim_{n\to\infty} a_n=a$,

$$\lim_{n\to\infty} f(a_n)=\ell \quad \text{and} \quad \lim_{n\to\infty} g(a_n)=m.$$

By the analogous results for sequences (proved in 1RAC-"Real Analysis and the Calculus"),

(1) $\lim_{n\to\infty}(f(a_n)+g(a_n))=\ell+m$;
(2) $\lim_{n\to\infty}(f(a_n)-g(a_n))=\ell-m$;
(3) $\lim_{n\to\infty}(f(a_n)\,g(a_n))=\ell m$; and if in addition, $m\neq 0$, then
(4) $\lim_{n\to\infty} f(a_n)/\,g(a_n)=\ell/m$.

Since these hold for all such sequences, by Theorem 3.1,

(1) $\lim_{x\to a}(f(x)+g(x))=\ell+m$;
(2) $\lim_{x\to a}(f(x)-g(x))=\ell-m$;
(3) $\lim_{x\to a}(f(x)\,g(x))=\ell m$; and if in addition, $m\neq 0$, then
(4) $\lim_{x\to a} f(x)/\,g(x)=\ell/m$. □

THEOREM 3.4 (Sandwich Theorem). *Suppose that f, g and h are real functions, whose domains all include a set of the form $(a-R,a)\cup(a,a+R)$, for some positive real number R. Suppose also that*

$$f(x)\le g(x)\le h(x)$$

for all $x \in (a-R,a) \cup (a,a+R)$ *and*

$$\lim_{x\to a} f(x) = \ell \quad \textit{and} \quad \lim_{x\to a} h(x) = \ell,$$

where $\ell \in \mathbb{R}$. *Then*

$$\lim_{x\to a} g(x) = \ell.$$

PROOF. (*Non-examinable*) We leave this as an exercise. It follows from Theorem 3.1 combined with the Sandwich Theorem for sequences. $\square$

We remark that it is possible to prove the Algebra of Limits and the Sandwich Theorem for limits at a point using the $\varepsilon - \delta$ definition. The proofs of these results using the definition of limit of a function at a point were given in the lecture notes of 1RAC.

The proposition below shows how to "change variables" in a limit.

PROPOSITION 3.5. *Let* $a, b, c \in \mathbb{R}$ *and* $b \neq 0$. *Suppose* f *is a real function whose domain includes at least a punctured neighbourhood of* ba *and* $a+c$.

(1) *If* $\lim_{x\to ba} f(x) = \ell$ *for some* $\ell \in \mathbb{R}$, *then*

$$\lim_{x\to a} f(bx) = \ell.$$

(2) *If* $\lim_{x\to a+c} f(x) = m$ *for some* $m \in \mathbb{R}$, *then*

$$\lim_{x\to a} f(x+c) = m.$$

One should think of the claims in Proposition 3.5 as "changing variables" in the limit.

PROOF. In these notes, we prove (1) and leave (2) as an exercise.

Suppose $\lim_{x\to ba} f(x) = \ell$ and let $\epsilon \in \mathbb{R}^+$. Then we know that there exists $\delta \in \mathbb{R}^+$ such that $|f(x) - \ell| < \epsilon$ whenever $0 < |x - ba| < \delta$. Note that $|bx - ba| = |b||x-a|$. Therefore, whenever $0 < |x-a| < \delta/|b|$ we have

$$0 < |bx - ba| < \delta$$

and for such x we know that $|f(bx) - \ell| < \epsilon$. It follows that $\lim_{x\to a} f(bx) = \ell$. $\square$

Fundamental trigonometric limits. Assumed knowledge. Non-examinable

We will use Figure 3.1 below to argue for some trigonometric inequalities.

LEMMA 3.6. *We have*

$$\lim_{x\to 0} \cos x = 1.$$

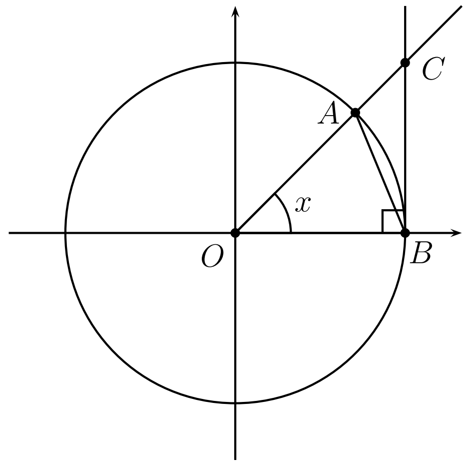

FIGURE 3.1. The unit circle

PROOF. Consider Figure 3.1, in which $0 < x < \pi/2$. From the cosine rule, and the fact that the length of the line segment AB is less than that of the arc AB,

$$2OA.OB.\cos x = OA^2 + OB^2 - AB^2 > OA^2 + OB^2 - x^2,$$

that is, $\cos x \geq 1 - x^2/2$. Since $\cos(-x) = \cos x$ for all x in $\mathbb{R}$, the same inequality holds when $-\pi/2 < x < 0$. Combining this with the obvious inequality $\cos x \leq 1$, we see that

$$1 - \frac{x^2}{2} \leq \cos x \leq 1\,.$$

The lemma now follows from the Sandwich Theorem, and the limit $\lim_{x\to 0} x^2/2 = 0$, which can be established using the Algebra of Limits. □

LEMMA 3.7. *We have*

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

PROOF. Suppose that $0 < x < \pi/2$. From Figure 3.1, we see that the area of the triangle OAB is less than the area of the sector OAB, which in turn is less than the area of the triangle OBC. Thus

$$\frac{1}{2}OA.OB.\sin x < \frac{1}{2}x < \frac{1}{2}OB.BC,$$

that is,

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x,$$

which simplifies to

$$\sin x < x < \tan x.$$

Since $\sin x > 0$ for x in the given range, we may divide by $\sin x$, and

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}\,.$$

Taking reciprocals gives

$$\cos x < \frac{\sin x}{x} < 1. \tag{3.2}$$

Now $\cos(-x) = \cos x$ and $\sin(-x) = -\sin(x)$, and so inequality (3.2) also holds when $-\pi/2 < x < 0$; that is, it holds when $0 < |x| < \pi/2$. By applying the Sandwich Theorem (Theorem 3.4) and recalling that $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} \cos x = 1$, we deduce that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

as required. $\square$

LECTURE 4

Examples of evaluating limits at a point

In this lecture we give several examples of how to evaluate various limits at a point using the tools we have developed.

EXERCISE 4.1. (i) Show that

$$\lim_{x \to 0} \frac{\sin(x/2)}{x/2} = 1.$$

(ii) Find

$$\lim_{x \to 0} \frac{\sin(x/2)}{x/3}$$

if it exists.

(iii) Show that

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

(iv) Find

$$\lim_{x \to 1} \frac{1 - x}{1 - x^2}$$

if it exists.

(v) Find $\lim_{x \to 0} f(x)$, if it exists, where

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(vi) Find $\lim_{x \to 1} f(x)$, if it exists, where

$$f(x) = \begin{cases} 2 & \text{if } x \in \mathbb{Q} \\ -2 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Answer. (i) Solution 1 We have shown (3.2) that

$$\cos x < \frac{\sin x}{x} < 1,$$

provided that $0 < |x| < \pi/2$. It follows that

$$\cos(x/2) < \frac{\sin(x/2)}{x/2} < 1,$$

provided that $0 < |x| < \pi$.

By the arguments to prove Lemma 3.6 above, for such x,

$$1 - x^2/8 \leq \cos(x/2) \leq 1,$$

and hence

$$1 - \frac{x^2}{8} \leq \frac{\sin(x/2)}{x/2} \leq 1;$$

the Sandwich Theorem implies the required result.

Solution 2 Setting $z = x/2$ we have $z \to 0$ as $x \to 0$. Also

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

so (using Proposition 3.5)

$$\lim_{x \to 0} \frac{\sin(x/2)}{x/2} = \lim_{z \to 0} \frac{\sin z}{z} = 1.$$

(ii) Note that for each $x \neq 0$ we have

$$\frac{\sin(x/2)}{x/3} = \frac{x/2}{x/3} \times \frac{\sin(x/2)}{x/2} = \frac{3}{2} \frac{\sin(x/2)}{x/2}.$$

From (i) above we know that

$$\lim_{x \to 0} \frac{\sin(x/2)}{x/2} = 1$$

and therefore (by the Algebra of Limits) we get

$$\lim_{x \to 0} \frac{\sin(x/2)}{x/3} = \frac{3}{2}.$$

(iii) Solution 1 Note that, for $x \neq 0$,

$$\frac{1 - \cos(x)}{x^2} = \frac{(1 - \cos(x))(1 + \cos(x))}{x^2(1 + \cos(x))} = \frac{1 - \cos^2(x)}{x^2(1 + \cos(x))} = \frac{\sin^2(x)}{x^2(1 + \cos(x))}.$$

Therefore

$$\frac{1-\cos(x)}{x^2} = \left(\frac{\sin(x)}{x}\right)^2 \frac{1}{1+\cos(x)}.$$

Using the limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \cos(x) = 1$$

and the Algebra of Limits, it follows that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}.$$

<u>Solution 2</u> Noting that

$$\cos x = 1 - 2\sin^2(x/2),$$

we may write

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2(x/2)}{x^2} = \frac{1}{2}\left(\frac{\sin(x/2)}{x/2}\right)^2,$$

for $x \neq 0$, and so by the Algebra of Limits,

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \left(\lim_{x\to 0} \frac{1}{2}\right)\left(\lim_{x\to 0} \frac{\sin(x/2)}{x/2}\right)^2 = \frac{1}{2}\left(\lim_{z\to 0} \frac{\sin z}{z}\right)^2 = \frac{1}{2}.$$

(iv) For $x \neq 1$ we have

$$\frac{1-x}{1-x^2} = \frac{1-x}{(1-x)(1+x)} = \frac{1}{1+x}$$

and so, by the Algebra of Limits,

$$\lim_{x\to 1} \frac{1-x}{1-x^2} = \lim_{x\to 1} \frac{1}{1+x} = \frac{1}{2}.$$

(v) For each $x \in \mathbb{R}$ we have $|f(x)| = |x|$ which means

$$-|x| \leq f(x) \leq |x|.$$

Now we claim that $|x| \to 0$ as $x \to 0$. To see this, let $\epsilon \in \mathbb{R}^+$. Then choose $\delta = \epsilon$. For x such that $0 < |x| < \delta$ we have

$$||x| - 0| = |x| < \delta = \epsilon,$$

and, by definition, this shows that $|x| \to 0$ as $x \to 0$.

By the Sandwich Theorem, it now follows that $f(x) \to 0$ as $x \to 0$.

(vi) In this case, the limit does not exist. To see this, define sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ by

$$a_n = 1 + \frac{1}{n} \in \mathbb{Q} \qquad \text{and} \qquad b_n = 1 + \frac{\pi}{n} \notin \mathbb{Q}.$$

Then $a_n, b_n \neq 1$ for each n, $a_n \to 1$ and $b_n \to 1$ as $n \to \infty$. However, for *each* $n \geq 1$ we have

$$f(a_n) = 2 \qquad \text{and} \qquad f(b_n) = -2$$

which means the sequences $(f(a_n))_{n\geq 1}$ and $(f(b_n))_{n\geq 1}$ converge to different limits (2 and -2 respectively). It follows (from Theorem 3.1) that $\lim_{x\to 1} f(x)$ does not exist. $\triangle$

Standard limits

In future, unless a question uses words like "first principles", or "using the definition" or "$\epsilon - \delta$ argument", you may assume the following results, where $a, c \in \mathbb{R}$, $b \neq 0$, and $n \in \mathbb{N}$:

(1) $\lim_{x\to a} x^n = a^n$;

(2) $\lim_{x\to a} f(bx + c) = \lim_{x\to ab+c} f(x)$; (change of variable in limits at a point)

(3) $\lim_{x\to 0} \sin x = 0$;

(4) $\lim_{x\to 0} \cos x = 1$;

(5) $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

LECTURE 5

Continuity

In this lecture we give a formal definition of continuity and begin to develop the theory of continuity for real functions.

DEFINITION 5.1 (Continuity at a point). Let $f : X \to \mathbb{R}$ be a real function, and suppose that $a \in X$. Then f *is continuous at* a if, given any ϵ in $\mathbb{R}^+$, there exists δ in $\mathbb{R}^+$ such that

$$|\,f(x) - f(a)| < \epsilon \quad \text{whenever } x \in X \text{ and } |x - a| < \delta.$$

REMARK 5.2. First, if $f(x)$ is defined for all x in a neighbourhood of a, that is, $(a-R, a+R) \subseteq X$ for some R in $\mathbb{R}^+$, then f is continuous at a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Second, if f is defined on an interval $[a, b]$, where $a, b \in \mathbb{R}$, then f is continuous at the endpoint a if and only if

$$\lim_{x \to a+} f(x) = f(a);$$

similarly, f is continuous at b if and only if

$$\lim_{x \to b-} f(x) = f(b).$$

EXERCISE 5.3. Show, using the definition, that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous at $a = 3$.

ANSWER. Let $\epsilon \in \mathbb{R}^+$. We first note that

$$x^2 - 9 = (x-3)(x+3)$$

so that

$$|x^2 - 9| = |x-3||x+3|.$$

Now consider x such that $|x - 3| < \delta$, where δ will be chosen later. Then

$$|x+3| \leq |x-3| + 6 < \delta + 6.$$

So, in particular, if $\delta \leq 4$ then $|x+3| < 10$. Hence

$$|x^2 - 9| < 10\delta$$

for $|x - 3| < \delta$ and when $\delta \leq 4$.

Now choose $\delta = \min\{\epsilon/10, 4\}$. Then it follows that $|x^2 - 9| < \epsilon$ whenever $|x - 3| < \delta$. By definition, f is continuous at $a = 3$. $\triangle$

Algebra of continuous functions

THEOREM 5.4. *Suppose that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous at a point $a \in X$. Then*

(1) *$f + g$ is continuous at a;*
(2) *$f - g$ is continuous at a;*
(3) *$f\,g$ is continuous at a; and if in addition, $g(a) \neq 0$, then*
(4) *f/g is continuous at a.*

PROOF. (*Non-examinable*) If $f(x)$ and $g(x)$ are defined whenever $|x - a| < R$ for some R in $\mathbb{R}^+$, that is, if $(a - R, a + R) \subseteq X$, then Theorem 5.4 follows from the Algebra of Limits and Remark 5.2. We omit the proof in the general case. □

DEFINITION 5.5. Let $f : X \to \mathbb{R}$ be a real function. Then f is *continuous* if f is continuous at every point of X.

COROLLARY 5.6. *Suppose that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous. Then*

(1) *$f + g$ is continuous;*
(2) *$f - g$ is continuous;*
(3) *$f\,g$ is continuous; and if in addition, $g(x) \neq 0$ for all $x \in X$, then*
(4) *f/g is continuous.*

PROOF. (*Non-examinable*) We apply the previous theorem for all points in X. □

DEFINITION 5.7. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. Then the *composed function* or *composition* $g \circ f : X \to Z$ is defined by

$$g \circ f(x) = g(f(x)) \quad \text{for all } x \in X.$$

THEOREM 5.8 (Functions composed of continuous functions are continuous). *Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions, that f is continuous at a point $a \in X$, and that g is continuous at the point $f(a) \in Y$. Then the composed function $g \circ f : X \to Z$ is continuous at a.*

PROOF. Suppose that ϵ in $\mathbb{R}^+$ is given. Since g is continuous at $f(a)$, there exists η in $\mathbb{R}^+$ such that

$$|\,g(y) - g(f(a))| < \epsilon \quad \text{whenever } y \in Y \text{ and } |y - f(a)| < \eta. \tag{5.3}$$

Here we have used the ϵ-δ definition of continuity (Definition 5.5) of g at $f(a)$, with η in place of δ. Since f is continuous at a, there exists δ in $\mathbb{R}^+$ such that

$$|\, f(x) - f(a)| < \eta \quad \text{whenever } x \in X \text{ and } |x - a| < \delta. \tag{5.4}$$

Here we have used the ϵ-δ definition of continuity (Definition 5.5) of f at a with η in place of ϵ. Substituting (5.4) into (5.3), we conclude that

$$|\, g(f(x)) - g(f(a))| < \epsilon \quad \text{whenever } x \in X \text{ and } |x - a| < \delta.$$

□

LECTURE 6

More on continuity

In this lecture we develop our knowledge of continuous functions and use it to justify why certain familiar functions (like polynomials and the sine and cosine functions) are continuous.

COROLLARY 6.1. *Suppose that X and Y are intervals, and that $f : X \to Y$ and $g : Y \to \mathbb{R}$ are continuous functions. Then the composed function $g \circ f : X \to \mathbb{R}$ is continuous.*

PROOF. We apply Theorem 5.8 at all points of X. □

EXAMPLE 6.2. (1) A real polynomial is a function of the form $P(x) = a_0 + a_1x + \cdots + a_nx^n$ for all $x \in \mathbb{R}$, where $a_0, \ldots, a_n \in \mathbb{R}$. Real polynomials are continuous at all points in $\mathbb{R}$. To show this, it is enough to show that the functions $x \mapsto c$ and $x \mapsto x$ are continuous, and then apply Theorem 5.4 repeatedly. We leave this verification as an exercise in using the definition of continuity.

(2) If P and Q are real polynomials, then the quotient P/Q is continuous at all points $a \in \mathbb{R}$ for which $Q(a) \neq 0$. There can only be a finite number of such values of a, by the Fundamental Theorem of Algebra. The continuity of P/Q is an immediate consequence of the previous example and Theorem 5.4. Functions of the form P/Q, where P and Q are polynomials, are called *rational functions*.

(3) The functions $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$ are continuous. In fact, we already know (see the list of "standard limits") that $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} \cos x = 1$, and it follows that sin and cos are continuous at 0. Using this, the trigonometric identity

$$\cos x = \cos((x - a) + a) = \cos(a)\cos(x - a) - \sin(a)\sin(x - a)$$

and the algebra of limits, it follows that

$$\lim_{x\to a} \cos x = \cos a \quad \text{for all } a \in \mathbb{R}$$

because

$$\begin{aligned}
\lim_{x\to a} \cos x &= \lim_{x\to a}\big(\cos(a)\cos(x - a) - \sin(a)\sin(x - a)\big) \\
&= \lim_{x\to a} \cos(a) \lim_{x\to a} \cos(x - a) - \lim_{x\to a} \sin(a) \lim_{x\to a} \sin(x - a) \\
&= \cos(a) \lim_{x\to 0} \cos(x) - \sin(a) \lim_{x\to 0} \sin(x) \\
&= \cos a.
\end{aligned}$$

Hence, $\cos : \mathbb{R} \to \mathbb{R}$ is continuous. Finally, since

$$\sin x = \cos(x - \tfrac{\pi}{2}) \quad \text{for all } x \in \mathbb{R}$$

it follows from Corollary 6.1 that $\sin : \mathbb{R} \to \mathbb{R}$ is continuous.

Exercise 6.3. Define the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Determine whether these functions are continuous at 0.

Answer. From Exercise 2.8, we know that

$$\lim_{x \to 0} (x^2 \sin(1/x)) = 0.$$

Since $\lim_{x \to 0} f(x) = 0$ and $f(0) = 0$, we see that f is continuous at 0. However, g is not continuous at 0, since

$$\lim_{x \to 0} g(x) = 0 \neq 1 = g(0).$$

△

Exercise 6.4. Let f be the modulus function $f : \mathbb{R} \to [0, \infty)$ given by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

Prove that f is continuous on $\mathbb{R}$.

Answer. Let $x_0 \in \mathbb{R}$ and let $\epsilon \in \mathbb{R}^+$. Choose $\delta = \epsilon$. Then $\delta \in \mathbb{R}^+$ and whenever $|x - x_0| < \delta$ we have

$$|f(x) - f(x_0)| = ||x| - |x_0|| \leq |x - x_0| < \delta = \epsilon.$$

It follows that f is continuous at x_0. Since $x_0 \in \mathbb{R}$ was arbitrary, f is continuous on $\mathbb{R}$.

△

Before coming to the Intermediate Value Theorem, we prove two results concerning continuous functions on intervals (which go some way to explain the intuitive idea that a continuous function "cannot instantaneously jump").

Below, a, b belong to $\mathbb{R}$ and $a < b$.

Lemma 6.5. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $\lambda \in \mathbb{R}$. If $f(w) > \lambda$ for some $w \in [a, b]$, then there exists δ in $\mathbb{R}^+$ such that*

$$f(x) > \lambda \quad \textit{whenever } x \in [a, b] \textit{ and } |x - w| < \delta.$$

Proof. Take ϵ to be $f(w) - \lambda$; this is positive by hypothesis. By the definition of continuity at w, there exists δ in $\mathbb{R}^+$ such that

$$|\, f(x) - f(w)| < \epsilon \quad \text{whenever } x \in [a,b] \text{ and } |x - w| < \delta.$$

For these x, we observe that $-\epsilon < f(x) - f(w) < \epsilon$, that is,

$$-(f(w) - \lambda) < f(x) - f(w) < f(w) - \lambda$$

and the left hand inequality, which boils down to $\lambda - f(w) < f(x) - f(w)$, implies that

$$f(x) > \lambda.$$

as required. $\square$

Lemma 6.6. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and $\lambda \in \mathbb{R}$. If $f(x) \geq \lambda$ for all $x \in (a,b)$, then*

$$f(a) \geq \lambda \quad \textit{and} \quad f(b) \geq \lambda.$$

The same holds if the three inequality signs $\geq$ are all replaced by $\leq$.

Proof. We treat the case where $f(x) \geq \lambda$ for all $x \in (a,b)$ only; the case where $f(x) \leq \lambda$ can be handled similarly.

Suppose, towards a contradiction, that $f(a) < \lambda$. By Lemma 6.5, there exists δ in $\mathbb{R}^+$ such that

$$f(x) < \lambda \quad \text{whenever } x \in [a,b] \text{ and } |x - a| < \delta.$$

Since $a < b$, there exists w in (a,b) such that $f(w) < \lambda$. But $f(x) \geq \lambda$ for all x in (a,b) by hypothesis. This contradiction shows that the supposition that $f(a) < \lambda$ is impossible. Hence $f(a) \geq \lambda$.

The proof that $f(b) \geq \lambda$ is similar. $\square$

LECTURE 7

Intermediate Value Theorem

In this lecture, we establish the Intermediate Value Theorem, a fundamental result concerning continuous functions on closed and bounded intervals. We highlight its importance with some applications.

The Intermediate Value Theorem

THEOREM 7.1 (Intermediate Value Theorem). *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, that $f(a) \neq f(b)$, and that λ lies strictly between $f(a)$ and $f(b)$. Then there exists c in (a, b) such that*

$$f(c) = \lambda.$$

PROOF. If $f(a) < \lambda < f(b)$, then write $g(x) = f(x) - \lambda$; if $f(a) > \lambda > f(b)$, then write $g(x) = \lambda - f(x)$. We see that $g(x) = 0$ exactly when $f(x) = \lambda$, and that

$$g(a) < 0 < g(b). \tag{7.5}$$

It will suffice to show that there exists c in (a, b) such that $g(c) = 0$.

Define $c := (a + b)/2$. If $g(c) = 0$, then we are done. Otherwise, if $g(c) < 0$, then we define $a_1 := c$ and $b_1 := b$, while if $g(c) > 0$, then we define $a_1 := a$ and $b_1 := c$. In either case, we now have a smaller interval $[a_1, b_1]$, half as long as $[a, b]$, such that $g(a_1) < 0 < g(b_1)$.

We continue this process recursively. At the nth step, we have an interval $[a_n, b_n]$, such that $b_n - a_n = 2^{-n}(b - a)$ and $g(a_n) < 0 < g(b_n)$; we define $c_n := (a_n + b_n)/2$. If $g(c_n) = 0$, then we are done, and we stop the recursive process. Otherwise, if $g(c_n) < 0$, then we define $a_{n+1} := c_n$ and $b_{n+1} := b_n$, while if $g(c_n) > 0$, then we define $a_{n+1} := a_n$ and $b_{n+1} := c_n$. In either case, we now have a smaller interval $[a_{n+1}, b_{n+1}]$, such that $b_{n+1} - a_{n+1} = 2^{-n-1}(b - a)$ and $g(a_{n+1}) < 0 < g(b_{n+1})$.

If this process stops at some point, we have found c such that $g(c) = 0$. Otherwise, at each recursive step, we took an interval $[a_{n+1}, b_{n+1}]$ that is either the left half or the right half of $[a_n, b_n]$, and it follows that

$$a \leq a_1 \leq a_2 \leq a_3 \leq \dots \quad \text{and} \quad b \geq b_1 \geq b_2 \geq b_3 \geq \dots.$$

The sequence $\{a_n\}$ is monotone increasing and bounded above by b, while the sequence $\{b_n\}$ is monotone decreasing and bounded below by a, so there exist c and d in $[a, b]$ such that

$$\lim_{n\to\infty} a_n = c \quad \text{and} \quad \lim_{n\to\infty} b_n = d.$$

Now $d - c = \lim_{n\to\infty} b_n - a_n = \lim_{n\to\infty} 2^{-n}(b-a) = 0$, that is, $c = d$. Since g is continuous in $[a, b]$, it is continuous at c, and by Theorem 3.1,

$$g(c) = \lim_{x\to c} g(x) = \lim_{n\to\infty} g(a_n) \leq 0$$

because $g(a_n) < 0$ for all n. Similarly,

$$g(c) = g(d) = \lim_{x\to d} g(x) = \lim_{n\to\infty} g(b_n) \geq 0$$

because $g(b_n) > 0$ for all n. Finally, since $g(c) \geq 0$ and $g(c) \leq 0$, it follows that $g(c) = 0$. Notice also that since $g(c) = 0$, $c \neq a$ and $c \neq b$ and therefore $c \in (a, b)$ (indeed, if $c = a$, since $g(c) = 0$, we have that $g(a) = 0$ which contradicts (7.5), and we can argue similarly if we assume that $c = b$). This concludes the proof. $\square$

The recursive process described in this proof is the basis of a commonly used algorithm for finding roots of equations. For an alternative proof using Lemma 6.5, see Problem Sheet 3.

We remark that the continuity hypothesis in the statement of the Intermediate Value Theorem is *essential*. For example, the floor function takes the values 0 and 1, but does not take the intermediate value 1/2.

Finding roots of equations

The Intermediate Value Theorem is often helpful for locating solutions to equations, as the following examples show.

EXERCISE 7.2. Show that the equation

$$5 + x\sin(x^3) - x^4 = 0$$

has at least one real solution.

ANSWER. Let $f(x) = 5 + x\sin(x^3) - x^4$. We know that polynomials and the sin function are continuous on $\mathbb{R}$. By the algebra of continuous functions, and since the composition of continuous functions is continuous, it follows that the function f is continuous on $\mathbb{R}$.

Consider f on $[0, 3]$. Note that

$$f(0) = 5 > 0 \quad \text{and} \quad f(3) = 5 + 3\sin(27) - 81 < 5 + 3 - 81 = -73 < 0.$$

Hence $f(3) < 0 < f(0)$. By the Intermediate Value Theorem, there exists a point $c \in (0, 3)$ such that $f(c) = 0$, that is, $5 + c\sin(c^3) - c^4 = 0$. $\triangle$

A fixed point theorem

COROLLARY 7.3 (Fixed Point Theorem). *If* $f : [a, b] \to [a, b]$ *is continuous, then there exists* $c \in [a, b]$ *such that* $f(c) = c$.

PROOF. If $f(a) = a$ or $f(b) = b$, then we are done, and so we may suppose that $f(a) \neq a$ and $f(b) \neq b$. Since $f(a), f(b) \in [a, b]$, it follows that $f(a) > a$ and $f(b) < b$.

We now define a function $g : [a, b] \to \mathbb{R}$ by

$$g(x) = x - f(x).$$

By the algebra of continuous functions, g is continuous, and moreover

$$g(a) < 0 < g(b).$$

From the Intermediate Value Theorem, there exists $c \in (a, b)$ such that $g(c) = 0$; that is, $f(c) = c$, as claimed. □

The corollary above is an example of a *fixed point theorem.* There are many fixed point theorems in mathematics, which have powerful consequences such as the existence of solutions to complicated differential equations.

LECTURE 8

The Boundedness Theorem

In this lecture, we state and prove the Boundedness Theorem for continuous functions on closed bounded intervals, which tells us that such functions really do have maxima and minima.

The Completeness Axiom

In 1RAC-"Real Analysis and the Calculus", several properties of the real numbers are stated.

THEOREM (Least Upper Bound Axiom). *Any bounded set X of real numbers has a least upper bound.*

THEOREM (Monotone Convergence Theorem). *Any monotone bounded sequence (x_n) of real numbers is convergent.*

THEOREM (Bolzano–Weierstrass). *Any bounded sequence (x_n) of real numbers has a convergent subsequence.*

The Least Upper Bound Axiom also has a Greatest Lower Bound version. These statements are logically equivalent, that is, each implies the others, so any one of them can be taken to be the basic property (which is usually called an axiom). In the last lecture, we assumed the Monotone Convergence Theorem. In this lecture, we will need to assume the other versions of this property.

The Boundedness Theorem

Recall that a subset X of $\mathbb{R}$ is *bounded* if there exists R in $\mathbb{R}^+$ such that $X \subseteq [-R, R]$. In particular, a *closed bounded interval* is an interval of the form

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$

where $a, b \in \mathbb{R}$ and $a \leq b$. We assume that $a < b$ throughout this lecture.

DEFINITION 8.1. A function $f : X \to \mathbb{R}$ is *bounded* if there is a number R in $\mathbb{R}^+$ such that

$$|f(x)| \leq R \quad \text{for all } x \in X.$$

The condition of this definition is equivalent to the requirement that the range of f, which we denote by $f(X)$, is a bounded subset of $\mathbb{R}$.

By the Completeness Axiom, if f is bounded, then $\sup f(X)$ and $\inf f(X)$ both exist.

Definition 8.2. A bounded function $f : X \to \mathbb{R}$ *attains its bounds* if there exist points c and d in X such that

$$f(c) = \sup f(X) \quad \text{and} \quad f(d) = \inf f(X).$$

Theorem 8.3 (Boundedness Theorem). *A continuous function on a closed bounded interval is bounded and attains its bounds.*

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function.

Assume, with a view to a contradiction, that f is *not* bounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Since each $x_n \in [a, b]$, the sequence (x_n) is bounded. By the Bolzano–Weierstrass Theorem, the sequence (x_n) has a subsequence, (x_{n_k}) say, that converges, to c, say. Since each $x_{n_k} \in [a, b]$, the limit $c \in [a, b]$. From Theorem 3.1 and the continuity of f at c it follows that

$$\lim_{k\to\infty} f(x_{n_k}) = f(c).$$

However,

$$|f(x_{n_k})| > n_k \geq k \quad \text{for all } k \in \mathbb{N},$$

and so the sequence $(f(x_{n_k}))$ is not bounded and hence not convergent. This contradiction shows that our assumption that f is not bounded must be incorrect, that is, f is bounded.

Now define $M = \sup f([a, b])$. By definition of the least upper bound, $M - 1/n$ is not an upper bound for $f([a, b])$ for any $n \in \mathbb{N}$. It follows that for all $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $f(x_n) \geq M - 1/n$. Again we have a sequence of real numbers (x_n), all of which lie in the bounded set $[a, b]$. By the Bolzano–Weierstrass Theorem, this sequence also has a convergent subsequence, (x_{n_k}) say. Define c to be $\lim_{k\to\infty} x_{n_k}$. Then $c \in [a, b]$, since each $x_{n_k} \in [a, b]$. Because f is continuous and $x_{n_k} \to c$ as $k \to \infty$, Theorem 3.1 implies that

$$f(c) = \lim_{k\to\infty} f(x_{n_k}) \geq \lim_{k\to\infty} M - 1/n_k = M;$$

on the other hand $f(c) \leq M$ since M is an upper bound for $f([a, b])$. We conclude that $f(c) = M$, as required.

To show that the lower bound m is also attained, we proceed in the same way, exchanging least upper bounds with greatest lower bounds, changing minus signs to plus signs, and changing the directions of all the inequalities. □

Remark 8.4. (1) It is essential that, in the statement of Theorem 8.3, the domain of f is *closed*. In general, a continuous function on an interval that is not closed need not be bounded.

For example, consider the functions $f : (a, b] \to \mathbb{R}$ and $g : [a, b) \to \mathbb{R}$ given by

$$f(x) = \frac{1}{x-a} \quad \text{and} \quad g(x) = \frac{1}{x-b}.$$

These functions are continuous, but *not* bounded.

(2) It is also essential that the domain of f be *bounded*; for example, consider the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x$.

(3) Even if a function is bounded, if the interval is not closed, then the infimum or supremum need not be attained. For example, consider the function $f : (0, 1) \to (0, 1)$ given by $f(x) = x$.

The proof of the Boundedness Theorem is not constructive, that is, it does not provide an effective algorithm for finding the maximum and minimum of a continuous function. When we consider differentiation, we will see that finding maxima and minima of differentiable functions is tied to the question of finding roots of equations, so "smooth optimisation" is a problem that can be tackled. But for general continuous functions, "optimisation" is a much harder problem.

LECTURE 9

The Inverse Function Theorem for continuous functions

In this lecture we see that the image of a closed bounded interval under a continuous map is a closed bounded interval, and we present the Inverse Function Theorem applicable to continuous functions.

The image of an interval

By putting together the Boundedness Theorem and the Intermediate Value Theorem, we deduce the following description of the image of a closed bounded interval under a continuous function.

THEOREM 9.1. *Suppose that* $f : [a, b] \to \mathbb{R}$ *is continuous. Then* $f([a, b])$ *is a closed bounded interval. More precisely,*

$$f([a, b]) = [m, M]$$

where $M = \sup f([a, b])$ *and* $m = \inf f([a, b])$.

PROOF. We have $f([a, b]) \subseteq [m, M]$ and furthermore there exist $x_1, x_2 \in [a, b]$ such that $f(x_1) = M$ and $f(x_2) = m$, by the Boundedness Theorem. It thus suffices to show that for every $\lambda \in (m, M)$ there exists $c \in (a, b)$ such that $f(c) = \lambda$. To this end let $\lambda \in (m, M)$. Since

$$f(x_2) < \lambda < f(x_1),$$

the existence of c follows from the Intermediate Value Theorem. □

Bijections, monotone functions and inverse functions

DEFINITION 9.2. (1) A real function $f : X \to Y$ is *injective* if

$$f(x_1) \neq f(x_2)$$

whenever $x_1, x_2 \in X$ and $x_1 \neq x_2$. Equivalently, f is injective if, for each $y \in Y$, there exists *at most one* $x \in X$ such that

$$f(x) = y.$$

(2) The function f is *surjective* if, for each $y \in Y$, there exists *at least one* $x \in X$ such that

$$f(x) = y.$$

(3) The function f is *bijective* if it is both injective and surjective. Thus f is bijective if, for each $y \in Y$, there exists *exactly one* $x \in X$ such that

$$f(x) = y.$$

DEFINITION 9.3. Let $f : X \to Y$ be a bijection. The *inverse function* $f^{-1} : Y \to X$ is defined by

$$f^{-1}(y) = x$$

for each $y \in Y$, where x is the unique element of X such that $f(x) = y$.

It is natural to ask under what circumstances the inverse of a continuous bijective function is continuous. It turns out that, for function on intervals, the answer to this question involves monotone functions.

DEFINITION 9.4. A real function $f : X \to Y$ is *strictly increasing* if

$$f(x_1) < f(x_2)$$

whenever $x_1, x_2 \in X$ and $x_1 < x_2$. A real function $f : X \to Y$ is *strictly decreasing* if

$$f(x_1) > f(x_2)$$

whenever $x_1, x_2 \in X$ and $x_1 < x_2$. A real function is *strictly monotone* if it is either strictly increasing or strictly decreasing.

THEOREM 9.5 (Inverse Function Theorem). *Suppose that $f : [a, b] \to [c, d]$ is continuous and strictly increasing, that $f(a) = c$ and that $f(b) = d$. Then the inverse function $f^{-1} : [c, d] \to [a, b]$ exists, is continuous, strictly increasing and surjective.*

PROOF. Observe that by the Intermediate Value Theorem, f is surjective. Moreover, f is injective because if $x_1, x_2 \in [a, b]$ and $x_1 \neq x_2$, then either $x_1 < x_2$ and $f(x_1) < f(x_2)$ or $x_2 < x_1$ and $f(x_2) < f(x_1)$; in both cases, $f(x_1) \neq f(x_2)$. It follows that the inverse function $f^{-1} : [c, d] \to [a, b]$ *exists.*

Next we show that f^{-1} is strictly increasing. Suppose that $f^{-1}(y_1) \geq f^{-1}(y_2)$. Then since f is strictly increasing, $f \circ f^{-1}(y_1) \geq f \circ f^{-1}(y_2)$, which simplifies to $y_1 \geq y_2$. It follows that the negation of $y_1 \geq y_2$, that is, $y_1 < y_2$, implies the negation of $f^{-1}(y_1) \geq f^{-1}(y_2)$, that is, $f^{-1}(y_1) < f^{-1}(y_2)$. Thus f^{-1} is strictly increasing.

It remains to prove that f^{-1} is continuous; that is, f^{-1} is continuous at every point $y_0 \in [c, d]$. Let $x_0 = f^{-1}(y_0)$. It is convenient to consider two cases.

Case 1: $y_0 \in (c, d)$. Let ϵ in $\mathbb{R}^+$ be arbitrary. We wish to show that there exists δ in $\mathbb{R}^+$ such that $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon$ whenever $y \in [c, d]$ and $|y - y_0| < \delta$. To this end, we let

$$\epsilon' = \min\{\epsilon, x_0 - a, b - x_0\}.$$

Observe that, by construction, $0 < \epsilon' \leq \epsilon$ and

$$a \leq x_0 - \epsilon' < x_0 < x_0 + \epsilon' \leq b,$$

and so since f is strictly increasing,

$$f(a) \leq f(x_0 - \epsilon') < f(x_0) < f(x_0 + \epsilon') \leq f(b).$$

Now if we choose

$$\delta = \min\{f(x_0 + \epsilon') - y_0, y_0 - f(x_0 - \epsilon')\},$$

then whenever $|y - y_0| < \delta$,

$$f(x_0 - \epsilon') \leq y_0 - \delta < y < y_0 + \delta \leq f(x_0 + \epsilon').$$

Since f^{-1} is strictly increasing,

$$x_0 - \epsilon' < f^{-1}(y) < x_0 + \epsilon'$$

whenever $|y - y_0| < \delta$. Since $x_0 = f^{-1}(y_0)$, this becomes

$$|f^{-1}(y) - f^{-1}(y_0)| < \epsilon' \leq \epsilon \quad \text{whenever } |y - y_0| < \delta,$$

as required.

Case 2: $y_0 = c$ (or $y_0 = d$). Again we let ϵ in $\mathbb{R}^+$ be arbitrary and set $\epsilon' = \min\{\epsilon, b - a\}$. Choosing $\delta = f(a + \epsilon') - c$ (or $\delta = d - f(b - \epsilon')$) and arguing as in Case 1, leads to the desired conclusion. The details are left as an exercise. $\square$

Theorem 9.5 holds for strictly decreasing continuous functions too. The appropriate modifications to the statement and proof are left as an exercise.

LECTURE 10

Uniform convergence

Uniform convergence and pointwise convergence

DEFINITION 10.1. Let (f_n) be a sequence of functions $f_n : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, and $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say that the sequence (f_n) converges pointwise to f on X if for each $x \in X$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$, that is, given any $x \in X$, and $\epsilon > 0$ there exists $N \in \mathbb{N}$ (depending on x and ϵ) such that

$$|f_n(x) - f(x)| < \epsilon, \qquad \text{whenever} \qquad n \geq N.$$

In this case, we call the function f the pointwise limit of the sequence (f_n) and write

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{for} \qquad x \in X, \qquad \text{or} \qquad f_n(x) \longrightarrow f(x) \qquad \text{as} \qquad n \to \infty \quad \text{for} \qquad x \in X.$$

Examples.

(1) Consider the sequence (f_n) with $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f_n(x) = \frac{x}{n}, \qquad n \in \mathbb{N}.$$

For fixed $x \in \mathbb{R}$, using the algebra of limits of sequences of real numbers, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = \left(\lim_{n\to\infty} x\right)\left(\lim_{n\to\infty} \frac{1}{n}\right) = 0.$$

Therefore the sequence (f_n) converges pointwise to the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = 0, \qquad \forall\, x \in \mathbb{R}.$$

(2) For $n \in \mathbb{N}$, let $f_n : [0,1] \longrightarrow \mathbb{R}$ be the function defined by

$$f_n(x) = x^n, \qquad \forall\, x \in [0,1].$$

Let $x \in [0,1]$ be given. Then
- If $x = 0$, $f_n(0) = 0$, $\forall n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0.$$

- If $x = 1$, $f_n(1) = 1$, $\forall n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1.$$

- If $0 < x < 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n \underbrace{=}_{x\in(0,1)} 0.$$

Then the sequence (f_n) converges pointwise to the function $f : [0,1] \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 \le x < 1 \\ 1, & \text{if} \quad x = 1. \end{cases}$$

(In other words, the function f defined above is the pointwise limit of the sequence (f_n).)

(3) For $n \in \mathbb{N}$, let $f_n : [0,\infty) \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} nx, & \text{if} \quad 0 \le x \le \frac{1}{n} \\ \dfrac{1}{nx}, & \text{if} \quad x > \frac{1}{n}. \end{cases}$$

Let $x \in [0,\infty)$ be given. Then,

- If $x = 0$, $f_n(0) = 0$, $\forall n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0.$$

- If $0 < x < \infty$, since $\frac{1}{n} \to 0$ as $n \to \infty$, then there exists $N \in \mathbb{N}$ such that

$$\frac{1}{n} < x, \qquad \forall\, n \ge N.$$

Therefore,

$$f_n(x) = \frac{1}{nx}, \qquad \forall\, n \ge N,$$

and consequently, using the algebra of limits of sequences of real numbers,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{nx} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence, the sequence (f_n) converges pointwise to the function $f \equiv 0$ on $[0,\infty)$.

(4) For $n \in \mathbb{N}$, let $f_n : [0,\infty) \longrightarrow \mathbb{R}$ be the function defined by

$$f_n(x) = \frac{1}{n}\sin(nx+n), \qquad \forall\, x \in \mathbb{R}.$$

Let $x \in \mathbb{R}$ be fixed. We have that

$$|f_n(x)| = \left|\frac{1}{n}\sin(nx+n)\right| \le \frac{1}{n}, \qquad \forall\, n \in \mathbb{N},$$

so that

$$-\frac{1}{n} \le f_n(x) \le \frac{1}{n}, \qquad \forall\, n \in \mathbb{N}.$$

Since $\pm\frac{1}{n} \to 0$ as $n \to \infty$, from the above inequality and the Sandwich Theorem we get that

$$f_n(x) \longrightarrow 0 \qquad \text{as} \qquad n \to \infty.$$

Therefore, the sequence (f_n) converges pointwise to the function $f \equiv 0$ on $\mathbb{R}$.

Uniform convergence

Definition 10.2. Let (f_n) be a sequence of functions, $f_n : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, and $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say that the sequence (f_n) *converges uniformly* to f on X if for each $\epsilon > 0$ there is a natural number $N \in \mathbb{N}$ (depending on ϵ but not on $x \in X$) such that

$$|f_n(x) - f(x)| < \epsilon, \qquad \text{whenever} \qquad n \geq N \qquad \text{and} \qquad \underline{x \in X}.$$

If such a function f exists we say that the sequence (f_n) converges uniformly on X to f.

Remark. From the definitions, it is immediate to see that if a sequence (f_n) converges uniformly to a function f on X, then the sequence (f_n) converge pointwise to f on X.

The converse of this statement is not true in general, that is there are sequences of functions (f_n) that converge pointwise to a function f on X, but does not converge uniformly to f (see Example (5) below).

It is sometimes useful to have the following necessary and sufficient condition for a sequence (f_n) to *fail* to converge uniformly on X to f.

Criteria for non-uniform convergence: A sequence of functions (f_n) with $f_n : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *does not* converge uniformly to a function $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ if for some $\epsilon_0 > 0$ there exists a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in X such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0 \qquad \text{for all} \qquad k \in \mathbb{N}.$$

Example. (5) Consider the sequence (f_n) defined by

$$f_n(x) = \begin{cases} nx, & \text{if} \quad 0 \leq x \leq \frac{1}{n}, \\ \dfrac{1}{nx}, & \text{if} \quad x > \frac{1}{n}, \end{cases} \qquad \forall\, x \in [0, \infty).$$

We have seen that (see Example (3)) that (f_n) converges pointwise to the function $f \equiv 0$ on $[0, \infty)$. We continue to show that the sequence (f_n) does not converge uniformly to $f \equiv 0$. To this end, for $k \in \mathbb{N}$, consider

$$n_k = k \qquad \text{and} \qquad x_k = \frac{1}{k},$$

and notice that

$$|f_{n_k}(x_k) - f(x_k)| = |f_k\left(\frac{1}{k}\right) - 0| = |k\left(\frac{1}{k}\right) - 0| = 1.$$

Therefore the sequence (f_n) does not converge uniformly to $f \equiv 0$ on $[0, \infty)$.

The above example (Example (5)) provides an example of a sequence of continuous functions that converges pointwise to a function, but the pointwise limit is *not* a continuous function. It will now be seen that the additional hypothesis of uniform convergence is sufficient to guarantee that the limit of a sequence of continuous functions be continuous.

THEOREM 10.3. *Let (f_n) be a sequence of* <u>*continuous*</u> *functions $f_n : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, and $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$.*

If (f_n) converges uniformly to f on X, then f is a continuous function.

PROOF. Let (f_n) be a sequence of continuous functions converging uniformly to f. We want to show that f is continuous at any $c \in X$.

Let $c \in X$ and $\epsilon > 0$ be given. Since (f_n) converges uniformly to f, then there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \qquad \forall\, n \geq N, \quad \text{and} \quad x \in X,$$

and in particular

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3}, \qquad \forall\, x \in X. \tag{10.6}$$

On the other hand, since f_N is continuous at c, there exists $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \frac{\varepsilon}{3}, \qquad \text{whenever} \quad |x - c| < \delta, \qquad \text{and} \qquad x \in X. \tag{10.7}$$

Using the triangle inequality, (10.6) and (10.7), we get that

$$\begin{aligned} |f(x) - f(c)| &= |(f(x) - f_N(x)) + (f_N(x) - f_N(c)) + (f_N(c) - f(c))| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c))| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

whenever $|x - c| < \delta$ and $x \in X$. Therefore f is continuous at c. $\square$

LECTURE 11

Uniform convergence (continuation)

Example. (6) Let (f_n) be the sequence defined by

$$f_n(x) = \frac{1}{(1+x)^n}, \qquad x \in [0,1].$$

Show that (f_n) does not converge uniformly on $[0,1]$.

Solution. We argue by contradiction. Assume that the sequence (f_n) converges uniformly on $[0,1]$ to some function f. Then, in particular, (f_n) converges pointwise to f. Now, notice that

- If $x = 0$, then $f_n(0) = 1$ for all $n \in \mathbb{N}$, and so $\lim_{n\to\infty} f_n(0) = 1$.
- If $0 < x \leq 1$, then $\frac{1}{1+x} < 1$, and consequently

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{(1+x)^n} = \lim_{n\to\infty} \left(\frac{1}{1+x}\right)^n = 0.$$

Therefore the pointwise limit of the sequence (f_n) is the function $f : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if} \quad x = 0, \\ 0, & \text{if} \quad 0 < x \leq 1. \end{cases}$$

Now, since we are assuming that the sequence (f_n) converges uniformly to some f, then in particular the sequence (f_n) converges pointwise to f and, from the uniqueness of the limit of convergent sequences, it follows that f is the function given above. But, notice that f is not continuous on $[0,1]$ (indeed the function f is not continuous at 1, since $\lim_{x\to 0^+} f(x) = 0$ but $f(0) = 1 \neq 0$). This gives a contradiction because we know (see Theorem 10.3 above) that if a sequence of continuous functions converges uniformly to f, then f is a continuous function.

Example. (7) Let (f_n) be the sequence of functions defined by

$$f_n(x) = \frac{x^n}{1+x^n}, \qquad 0 \leq x \leq 1.$$

Notice that (f_n) is a sequence of continuous functions. Also

- If $x = 1$, then $f_n(x) = \frac{1}{2}$ for all $n \in \mathbb{N}$, and consequently

$$\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}.$$

- If $0 \le x < 1$, then $\lim_{n\to\infty} x^n = 0$, and using the algebra of limits of sequences of real numbers we conclude that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0.$$

Therefore the pointwise limit of the sequence (f_n) is given by the function $f : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 \le x < 1, \\ \frac{1}{2}, & \text{if} \quad x = 1. \end{cases} \tag{11.8}$$

We will continue to show that the sequence (f_n) does not converge uniformly on $[0,1]$. We will see two different arguments.

(1) If we argue by contradiction and assume that the sequence (f_n) converges uniformly to some f on $[0,1]$, then in particular the sequence (f_n) converges pointwise to f and, from the uniqueness of the limit of convergent sequences, it follows that f is the function given in (11.8). But, notice that f is not continuous on $[0,1]$ (indeed the function f is not continuous at 1, since $\lim_{x\to 1^-} f(x) = 0$ but $f(1) = \frac{1}{2} \neq 0$). This gives a contradiction because we know (see Theorem 10.3 above) that if sequence of continuous functions converge uniformly to f, then f is a continuous function.

(2) Alternatively, if one assumes (arguing again by contradiction) that the sequence (f_n) converges uniformly to some f, then in particular the sequence (f_n) converges pointwise to f and, from the uniqueness of the limit of convergent sequences, it follows that f is the function given in (11.8). However, (f_n) does not converge uniformly to f. Indeed, for $k \in \mathbb{N}$ define

$$n_k = k \qquad \text{and} \qquad x_k = 1 - \frac{1}{k}.$$

Then, observe that

$$|f_{n_k}(x_k) - f(x_k)| \;=\; |f_k\left(1 - \frac{1}{k}\right) - f\left(1 - \frac{1}{k}\right)| = |\frac{\left(1-\frac{1}{k}\right)^k}{1+\left(1-\frac{1}{k}\right)^k} - 0| = \frac{\left(1-\frac{1}{k}\right)^k}{1+\left(1-\frac{1}{k}\right)^k},$$

and since $\lim_{k\to\infty}\left(1-\frac{1}{k}\right)^k = \frac{1}{e}$, then by the Algebra of limits we have that

$$\lim_{k\to\infty} \frac{\left(1-\frac{1}{k}\right)^k}{1+\left(1-\frac{1}{k}\right)^{1/k}} = \frac{\frac{1}{e}}{1+\frac{1}{e}} = \frac{1}{1+e}$$

and consequently there exists $K \in \mathbb{N}$ such that

$$\begin{aligned}
|f_{n_k}(x_k) - f(x_k)| &= \left| f_k\left(1 - \frac{1}{k}\right) - f\left(1 - \frac{1}{k}\right) \right| = \left| \frac{\left(1 - \frac{1}{k}\right)^k}{1 + \left(1 - \frac{1}{k}\right)^k} - 0 \right| = \frac{\left(1 - \frac{1}{k}\right)^k}{1 + \left(1 - \frac{1}{k}\right)^k} \\
&= \left(\frac{\left(1 - \frac{1}{k}\right)^k}{1 + \left(1 - \frac{1}{k}\right)^k} - \left(\frac{1}{1+e} \right) \right) + \left(\frac{1}{1+e} \right) \\
&\geq \frac{1}{2(1+e)}, \qquad \text{for} \qquad k \geq K.
\end{aligned}$$

Example. (8) Let $(f_n)_{n=2}^{\infty}$ be the sequence of (continuous) functions defined as follows: For $n \geq 2$, let $f_n : [0,1] \longrightarrow \mathbb{R}$ be the function given by

$$f_n(x) = \begin{cases} n^2 x, & \text{if} \quad 0 \leq x \leq \dfrac{1}{n}, \\ -n^2(x - \frac{2}{n}), & \text{if} \quad \dfrac{1}{n} \leq x \leq \dfrac{2}{n}, \\ 0, & \text{if} \quad \dfrac{2}{n} \leq x \leq 1. \end{cases}$$

It is easy to see that the pointwise limit of the sequence (f_n) exists and it is given by the function $f : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f(x) = 0, \qquad \forall x \in [0,1].$$

Indeed, let $x \in [0,1]$. We observe that

- If $x = 0$, then by definition $f_n(0) = 0$ for all $n \geq 2$. Therefore

$$f_n(0) = 0 \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty.$$

- If $0 < x \leq 1$, since $\frac{2}{n} \longrightarrow 0$ as $n \longrightarrow \infty$, there exists $N \in \mathbb{N}$ such that

$$\frac{2}{n} < x, \qquad \text{whenever} \qquad n \geq N.$$

Therefore, by definition of f_n, we have that

$$f_n(x) = 0 \qquad \text{for all} \qquad n \geq N,$$

and thus

$$f_n(x) = 0 \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty.$$

Notice that in this case the pointwise limit of the sequence of function (f_n) is a continuous function. However, the sequence (f_n) does not converge uniformly on $[0,1]$ to its pointwise limit $f \equiv 0$. Indeed, for $k \in \mathbb{N}$, consider

$$n_k = k \qquad \text{and} \qquad x_k = \frac{1}{k^2} \in [0,1],$$

and observe that, since $\frac{1}{k^2} \leq \frac{1}{k}$ for all $k \geq 1$, then

$$|f_{n_k}(x_k) - f(x_k)| = |f_k\left(\frac{1}{k^2}\right) - 0| = |k^2\left(\frac{1}{k^2}\right)| = 1.$$

Finally, in this example it is also worth mentioning that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

Indeed, a straightforward calculation, or using the interpretation of the integral in terms of area under a curve, shows that

$$\int_0^1 f_n(x)\,dx = 1, \qquad \text{for all} \qquad n \geq 2,$$

therefore

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty} 1 = 1. \tag{11.9}$$

However, since form part (a) the pointwise limit of the sequence (f_n) is the function $f = 0$, we have that

$$\int_0^1 f(x)\,dx = 0. \tag{11.10}$$

From (11.9) and (11.10), it follows that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

Interchange of limit and integral

This last example (Example (8)) shows that it is not always true that we can interchange the limit with integration, i.e the limit of the integral of a sequence of functions does not need to be the integral of the (pointwise limit) of the sequence of functions. However, the following theorem is true:

THEOREM 11.1. *Let (f_n) be a sequence of functions $f_n : [a,b] \longrightarrow \mathbb{R}$ that are integrable on $[a,b]$, and suppose that (f_n) converges uniformly to a function $f : [a,b] \longrightarrow \mathbb{R}$. Then f is integrable and*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

PROOF. (Not examinable). A proof of this result can be found in Bartle and Sherbert's book. $\square$

A couple of references for the topic of uniform convergence and pointwise convergence of a sequence of functions are the following:

- "Introduction to Real Analysis" by Robert G. Bartle by Donal R. Sherbert. Second Edition, John Wiley & Sons, Inc., 1982.
- "Real Analysis: An Introduction"' by A. J. White, Addison-Wesley Publishing Company, Inc., 1968.

This section has been prepared using the above references.

LECTURE 12

Differentiability

In this lecture, we begin with some examples of continuous inverse functions given by the Inverse Function Theorem. Then we introduce the definition of differentiability and meet some examples of differentiable/non-differentiable functions. We prove the important fact that differentiability at a point implies continuity at a point.

Examples

EXAMPLE 12.1. (1) Suppose that $f(x) = x^n$ for all $x \in \mathbb{R}$, where n is an odd positive integer. Then f is continuous and strictly increasing, and its range is $\mathbb{R}$. By Theorem 9.5, there is a continuous inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$, the *nth root*, written $f^{-1}(y) = \sqrt[n]{y}$ (or $y^{1/n}$).

(2) Suppose that $f(x) = x^n$ for all $x \in [0, \infty)$, where n is an even positive integer. Then f is continuous and strictly increasing, and its range is $[0, \infty)$. So by Theorem 9.5, there is a continuous inverse function $f^{-1} : [0, \infty) \to [0, \infty)$, the *nth root*, written $f^{-1}(y) = \sqrt[n]{y}$ (or $y^{1/n}$).

EXAMPLE 12.2. (1) Suppose that $f(x) = \sin(x)$. This is a continuous function, strictly increasing on $[-\pi/2, \pi/2]$. Further, $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$. So by Theorem 9.5, there is an inverse function

$$f^{-1} : [-1, 1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

that is continuous and is strictly increasing. The function f^{-1} is called the *arcsin* function and written as

$$f^{-1}(y) = \sin^{-1}(y).$$

(2) Suppose that $f(x) = \cos(x)$. This is a continuous function, strictly decreasing on $[0, \pi]$. Further, $\cos(0) = 1$ and $\cos(\pi) = -1$. So by Theorem 9.5, there is an inverse function

$$f^{-1} : [-1, 1] \to [0, \pi]$$

that is continuous and is strictly decreasing. The function f^{-1} is called the *arccos* function and written as

$$f^{-1}(y) = \cos^{-1}(y).$$

Differentiable Functions

We now change topic somewhat, and start to look at differentiable functions. We begin with differentiability at a point, and then we look at some examples.

Differentiability at a point

DEFINITION 12.3. A real function $f : (a, b) \to \mathbb{R}$ is *differentiable at a point* $x_0 \in (a, b)$ if the limit

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If this limit exists, it is called *the derivative of* f *at* x_0, and denoted $f'(x_0)$.

Often we write $y = f(x)$, and then an alternative notation for $f'(x_0)$ is

$$\left(\frac{dy}{dx}\right)_{x=x_0} \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=x_0}.$$

EXAMPLE 12.4. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$. Then f is differentiable at all points in $\mathbb{R} \setminus \{0\}$ and f is not differentiable at 0.

ANSWER. When $x_0 > 0$ and h is small, that is, when $|h| < x_0$, we see that $x_0 + h > 0$. So

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{x_0 + h - x_0}{h} = \frac{h}{h} = 1.$$

Therefore

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = 1.$$

When $x_0 < 0$ and h is small, that is, when $|h| < |x_0|$, we see that $x_0 + h < 0$. So

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{-(x_0 + h) - (-x_0)}{h} = \frac{-h}{h} = -1.$$

Therefore

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = -1.$$

We have now shown that f is differentiable at all points in $\mathbb{R} \setminus \{0\}$.

Finally, when $x_0 = 0$, we see that

$$\frac{f(h) - f(0)}{h} = \frac{|h| - 0}{h} = \frac{|h|}{h},$$

and so

$$\lim_{h\to 0+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0+} 1 = 1,$$

while

$$\lim_{h\to 0-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0-}(-1)=-1.$$

Since the left and right hand limits are different,

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

does not exist (see Remark 2.6). Thus f is not differentiable at 0, as claimed. △

EXERCISE 12.5. Define the function $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^{1/3}$. Is f is differentiable at 0? If so, find $f'(0)$, and if not, explain why not.

ANSWER. Observe that

$$\frac{f(0+h)-f(0)}{h}=\frac{f(h)}{h}=\frac{h^{1/3}}{h}=\frac{1}{h^{2/3}}$$

and $1/h^{2/3}$ increases without bound as $h\to 0$. Thus $f'(0)$ does not exist. △

The following is an important result which says that differentiability at a point implies continuity at a point.

THEOREM 12.6. *If a function is differentiable at a point $x_0\in\mathbb{R}$, then it is continuous at x_0.*

PROOF. Since f is differentiable at x_0,

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

exists. By the algebra of limits,

$$\begin{aligned}\lim_{h\to 0}(f(x_0+h)-f(x_0)) &= \lim_{h\to 0}\Big(\frac{f(x_0+h)-f(x_0)}{h}\cdot h\Big)\\ &= \Big(\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\Big)\cdot\Big(\lim_{h\to 0}h\Big)=f'(x_0)\cdot 0=0.\end{aligned}$$

This means

$$\lim_{h\to 0}f(x_0+h)=f(x_0)$$

and therefore

$$\lim_{x\to x_0}f(x)=f(x_0).$$

It follows that f is continuous at x_0. □

COROLLARY 12.7. *If f is not continuous at a, then it is not differentiable at a.*

REMARK 12.8. The converse of Theorem 12.6 is *not true*. Indeed, consider the absolute value function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=|x|$. It is true that f is continuous on $\mathbb{R}$ (see Exercise 6.4). However, by Example 12.4 we know that f is not differentiable at 0.

LECTURE 13

Rules for derivatives and an alternative criterion for differentiability

In this lecture we establish some fundamental rules for differentiation (including the "product" and "quotient" rules), and follow this with some examples of differentiable functions including polynomials and the sine function. We also state and prove an alternative criterion for differentiability which will be useful, for example, to prove the chain rule.

The following theorem, which is analogous to Theorem 5.4 for continuous functions, allows us to dramatically increase our armory of differentiable functions. Parts (3) and (4) are often referred to as the "product rule" and "quotient rule" for differentiation.

THEOREM 13.1 (Rules for Differentiation). *Suppose that f and g are both differentiable at a point x_0. Then*

(1) *the sum $f+g$ is differentiable at x_0, and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$;*
(2) *the difference $f-g$ is differentiable at x_0, and $(f-g)'(x_0) = f'(x_0) - g'(x_0)$;*
(3) *the product fg is differentiable at x_0, and $(fg)'(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$; and if in addition, $g(x_0) \neq 0$, then*
(4) *the quotient f/g is differentiable at x_0 and $(f/g)'(x_0) = (f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0))/(g(x_0))^2$.*

The proof we give below uses only the Algebra of Limits and the fact that differentiability implies continuity (Theorem 12.6).

PROOF. For $h \neq 0$,

$$\frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} = \frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h}$$

so, by the Algebra of Limits it follows that

$$\begin{aligned}\lim_{h\to 0} \frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\ &= f'(x_0) + g'(x_0).\end{aligned}$$

Hence, $f+g$ is differentiable at x_0 and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$. A similar argument works for the difference.

For the product,

$$\begin{aligned}
(fg)'(x_0) &= \lim_{h\to 0} \frac{(fg)(x_0+h)-(fg)(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)\,g(x_0+h)-f(x_0)\,g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)\,g(x_0+h)-f(x_0+h)\,g(x_0)+f(x_0+h)\,g(x_0)-f(x_0)\,g(x_0)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x_0+h)\,g(x_0+h)-f(x_0+h)\,g(x_0)}{h}+\frac{f(x_0+h)\,g(x_0)-f(x_0)\,g(x_0)}{h}\right] \\
&= \lim_{h\to 0} \left[f(x_0+h)\,\frac{g(x_0+h)-g(x_0)}{h}\right]+\lim_{h\to 0}\left[g(x_0)\,\frac{f(x_0+h)-f(x_0)}{h}\right] \\
&= \lim_{h\to 0} f(x_0+h)\lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}+g(x_0)\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \\
&= f(x_0)\,g'(x_0)+g(x_0)\,f'(x_0),
\end{aligned}$$

since f is continuous at x_0.

For the quotient,

$$\begin{aligned}
(f/g)'(x_0) &= \lim_{h\to 0} \frac{(f/g)(x_0+h)-(f/g)(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)/\,g(x_0+h)-f(x_0)/\,g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)\,g(x_0)-f(x_0)\,g(x_0+h)}{g(x_0)\,g(x_0+h)h} \\
&= \lim_{h\to 0} \frac{1}{g(x_0)\,g(x_0+h)} \\
&\quad \times \lim_{h\to 0}\left[\frac{f(x_0+h)\,g(x_0)-f(x_0)\,g(x_0)+f(x_0)\,g(x_0)-f(x_0)\,g(x_0+h)}{h}\right] \\
&= \frac{1}{(g(x_0))^2}\lim_{h\to 0}\left[\frac{f(x_0+h)\,g(x_0)-f(x_0)\,g(x_0)}{h}-\frac{f(x_0)\,g(x_0+h)-f(x_0)\,g(x_0)}{h}\right] \\
&= \frac{1}{(g(x_0))^2}\left[g(x_0)\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}-f(x_0)\lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}\right] \\
&= \frac{f'(x_0)\,g(x_0)-f(x_0)\,g'(x_0)}{(g(x_0))^2},
\end{aligned}$$

since g is continuous at x_0. □

Example 13.2. (1) We recall that a real polynomial is an expression of the form

$$a_0 + a_1 x + \cdots + a_n x^n,$$

for some real numbers $a_0, \ldots, a_n$. Polynomials are differentiable at all points of $\mathbb{R}$. This follows from Theorem 13.1 (applied repeatedly), and the differentiability of constant functions and the identity function $x \mapsto x$ at all points of $\mathbb{R}$.

(2) Recall also that a rational function is a quotient of real polynomials. By Theorem 13.1, any rational function p/q is differentiable at all points x_0 for which $q(x_0) \neq 0$.

Exercise 13.3. Show that the function $\sin : \mathbb{R} \to \mathbb{R}$ is differentiable at all points in $\mathbb{R}$, and $\sin'(x) = \cos(x)$.

Answer. From the trigonometric identity $\sin(x+h) = \cos h \sin x + \sin h \cos x$,

$$\begin{aligned}
\frac{\sin(x+h) - \sin(x)}{h} &= \frac{\cos h \sin x + \sin h \cos x - \sin x}{h} \\
&= \frac{\sin h}{h} \cos x - \frac{1 - \cos h}{h} \sin x \\
&= \frac{\sin h}{h} \cos x - \frac{1 - \cos^2 h}{(1 + \cos h) h} \sin x \\
&= \frac{\sin h}{h} \cos x - \frac{\sin h}{1 + \cos h} \frac{\sin h}{h} \sin x \,.
\end{aligned}$$

The sine and cosine functions are continuous, and $\lim_{h \to 0} \sin h = 0$ and $\lim_{h \to 0} \cos h = 1$. We also know that $\lim_{h \to 0} (\sin h)/h = 1$ (see Lemma 3.7), so by the algebra of limits we conclude that

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \cos x,$$

as required. △

Now we come to the analogue of Theorem 5.8 (compositions of continuous functions are continuous) for functions that are differentiable at a point. Recall that the composed function $g \circ f$ is defined by $g \circ f(x) = g(f(x))$ for all x (for which this makes sense). It is helpful to have a preliminary lemma.

Lemma 13.4 (Alternative criterion for differentiability at a point). *A function $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$ if and only if there exists a function $F : (a,b) \to \mathbb{R}$ that satisfies*

$$f(x) = f(x_0) + (x - x_0)\, F(x) \quad \text{for all } x \in (a,b), \tag{13.11}$$

and is continuous at x_0. Furthermore, if f is differentiable at x_0 then the function F is given by

$$F(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x = x_0. \end{cases}$$

Proof. A function F satisfies (13.11) if and only if

$$F(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

for all $x \in (a, b) \setminus \{x_0\}$. If this holds, then F is continuous at x_0 if and only if $\lim_{x \to x_0} F(x)$ exists and is equal to $F(x_0)$. However, this is true if and only if f is differentiable at x_0 and $f'(x_0) = F(x_0)$. $\square$

Theorem 13.5 (Chain Rule). *Suppose that $f : (a, b) \to (c, d)$ is differentiable at $x_0 \in (a, b)$ and that $g : (c, d) \to \mathbb{R}$ is differentiable at $f(x_0)$. Then the composed function $g \circ f$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0).$$

Proof. By the alternative criterion for differentiability, applied to the function g and the point $f(x_0)$, there is a function $G : (c, d) \to \mathbb{R}$, continuous at $f(x_0)$, such that

$$g(y) = g(f(x_0)) + (y - f(x_0))\, G(y) \quad \text{for all } y \in (c, d)$$

and $G(f(x_0)) = g'(f(x_0))$. By substituting $f(x_0 + h)$ for y, we deduce that

$$g(f(x_0 + h)) - g(f(x_0)) = (f(x_0 + h) - f(x_0))\, G(f(x_0 + h)).$$

Hence, for all $h \neq 0$,

$$\begin{aligned} \frac{g \circ f(x_0 + h) - g \circ f(x_0)}{h} &= \frac{g(f(x_0 + h)) - g(f(x_0))}{h} \\ &= G(f(x_0 + h))\, \frac{f(x_0 + h) - f(x_0)}{h}. \end{aligned}$$

Since f is differentiable at x_0,

$$\frac{f(x_0 + h) - f(x_0)}{h} \to f'(x_0)$$

as $h \to 0$, and by Theorem 5.8 (composition of continuous functions),

$$G(f(x_0 + h)) \to G(f(x_0)) = g'(f(x_0))$$

as $h \to 0$. Here we have also used the fact that f is continuous at x_0, which follows from Theorem 12.6.

Hence by the algebra of limits,

$$\lim_{h \to 0} \frac{g \circ f(x_0 + h) - g \circ f(x_0)}{h} = g'(f(x_0))\, f'(x_0).$$

$\square$

LECTURE 14

More on differentiation

In this lecture, we find the derivative of an inverse function, and show some applications of this.

Inverse functions and differentiability

We recall the Inverse Function Theorem (Theorem 9.5).

THEOREM. *Suppose that $f : [a,b] \to [c,d]$ is continuous and strictly increasing, that $f(a) = c$ and that $f(b) = d$. Then the inverse function $f^{-1} : [c,d] \to [a,b]$ exists, is continuous, strictly increasing and surjective. The same holds if both occurrences of increasing are replaced by decreasing.*

We may extend this to deal with differentiation.

THEOREM 14.1. *Suppose that f satisfies the conditions of the Inverse Function Theorem, and that $x_0 \in (a,b)$ and $y_0 = f(x_0)$. If in addition f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

PROOF. Take x close to, but different from x_0, and write $y = f(x)$ as well as $y_0 = f(x_0)$; equivalently, $x = f^{-1}(y)$ and $x_0 = f^{-1}(y_0)$.

Since f is differentiable at x_0,

$$\frac{f(x) - f(x_0)}{x - x_0} \to f'(x_0) \quad \text{as } x \to x_0.$$

Hence

$$\frac{y - y_0}{f^{-1}(y) - f^{-1}(y_0)} \to f'(x_0) \quad \text{as } x \to x_0.$$

Because $f'(x_0) \neq 0$, when x is close enough to x_0, the quotient on the left hand side is nonzero, and by the algebra of limits,

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \to \frac{1}{f'(x_0)} \quad \text{as } x \to x_0.$$

Since f is continuous, $y \to y_0$ when $x \to x_0$; since f^{-1} is also continuous, $x \to x_0$ when $y \to y_0$. Thus $y \to y_0$ exactly when $x \to x_0$, and it follows that

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \to \frac{1}{f'(x_0)} \quad \text{as } y \to y_0.$$

This means that f^{-1} is indeed differentiable at y_0 and that its derivative is $1/f'(x_0)$, which is what we were required to show. $\square$

Note that we may also write

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

An easy way to remember this is to write $y = f(x)$, and then $x = f^{-1}(y)$, so

$$(f^{-1})'(y) = \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{f'(x)}\,.$$

The derivatives of rational powers

Example 14.2. Suppose that $r \in \mathbb{Q}$ and $f(x) = x^r$. Then

$$f'(x) = rx^{r-1}.$$

For brevity, the above claim has been stated in a concise way, and we now make a few clarification remarks about the values of x under consideration. In order for $f(x) = x^r$ to define a *real function*, it may be necessary to restrict the domain of f to, say, $[0, \infty)$. For example, this is the case if $r = \frac{1}{2}$, but if, say, $r \in \mathbb{N}$ then one may take the domain to be $\mathbb{R}$. We also remark that if $r < 0$ then f is not defined when $x = 0$, and if $r < 1$ then rx^{r-1} is not defined when $x = 0$. In such cases, f is not differentiable at $x = 0$.

We outline below how to prove the above claim is a series of four steps, using tools we have developed to this point. The details are left as an exercise.

Step 1. One can show that the claim holds when r is a nonnegative integer by the Product Rule (Theorem 13.1(3)) and induction.

Step 2. One can show that the claim holds when r is a negative integer by the Quotient Rule (Theorem 13.1(4)) and the conclusion of the previous step.

Step 3. One can show that the claim holds when $r = \frac{1}{n}$, where n is a nonzero integer by the Inverse Function Theorem (Theorem 14.1) and the conclusion of the previous steps.

Step 4. One can show that the claim holds when $r = \frac{m}{n}$, where m is an integer and n is a non-zero integer, by the Chain Rule (Theorem 13.5) and the conclusion of the previous steps.

Examples

The following examples and exercises further illustrate differentiation.

EXERCISE 14.3. For each $n \in \{0, 1, 2\}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

These functions are all differentiable at all points x in $\mathbb{R} \setminus \{0\}$, and hence continuous there. Decide whether these functions are continuous or differentiable at 0.

ANSWER. We discuss these functions separately.

(1) We claim that f_0 is not continuous at 0, since $\lim_{x\to 0} f_0(x)$ does not exist. In particular, f_0 is not differentiable at 0 (thanks to Theorem 12.6). To prove the claim, define sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ by

$$a_n = \frac{1}{n\pi}, \qquad b_n = \frac{1}{(2n + \frac{1}{2})\pi}.$$

Then $a_n, b_n \neq 0$ for each n, and $a_n, b_n \to 0$ as $n \to \infty$. But

$$f_0(a_n) = 0, \qquad f_0(b_n) = 1$$

for each n, so $\lim_{n\to\infty} f_0(a_n) = 0 \neq 1 = \lim_{n\to\infty} f_0(b_n)$. Hence, $\lim_{x\to 0} f_0(x)$ does not exist, as claimed.

(2) We claim that f_1 is continuous at 0, but is not differentiable at 0. Indeed,

$$-|x| \leq f_1(x) \leq |x| \quad \text{for all } x \in \mathbb{R}.$$

We know $\lim_{x\to 0} |x| = 0$ because the modulus function is continuous at 0, and it follows from the Sandwich Theorem that $\lim_{x\to 0} f_1(x) = 0$; further, $f_1(0) = 0$, and so f_1 is continuous at 0. But

$$\frac{f_1(0+h) - f_1(0)}{h} = \frac{h\sin(1/h) - 0}{h} = \sin(1/h),$$

and the limit $\lim_{h\to 0} \sin(1/h)$ does not exist, so f_1 is not differentiable at 0.

(3) We claim that f_2 is differentiable at 0, and in light of Theorem 12.6, f_2 is also continuous at 0. To show the differentiability, observe that

$$\frac{f_2(0+h) - f_2(0)}{h} = \frac{f_2(h)}{h} = \frac{h^2\sin(1/h)}{h} = h\sin(1/h),$$

and we showed that this tends to 0 as h tends to 0 when we looked at f_1. It follows that f_2 is differentiable at 0 and $f_2'(0) = 0$. $\triangle$

Example 14.4. Recall that the function $\sin : [-\pi/2, \pi/2] \to [-1, 1]$ is differentiable at every point in $(-\pi/2, \pi/2)$. By Theorem 9.5, the inverse function $\sin^{-1}$ is a well-defined continuous function on $[-1, 1]$. Write $f(x) = \sin(x)$ and $g(y) = \sin^{-1}(y)$, and take $x \in [-\pi/2, \pi/2]$ and $y = f(x)$. Then $y \in (-1, 1)$ if and only if $x \in (-\pi/2, \pi/2)$. For such y, by Theorem 14.1, g is differentiable at y, and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1 - \sin^2(x)}} = \frac{1}{\sqrt{1 - y^2}}.$$

(we take the positive square root because cos is positive when $x \in (-\pi/2, \pi/2)$.) So

$$\frac{d}{dy}(\sin^{-1} y) = \frac{1}{\sqrt{1 - y^2}}.$$

LECTURE 15

Rolle's Theorem

In this lecture, we first consider the relation between the sign of the derivative and the monotonicity of a function. Then we reach Rolle's Theorem, a fundamental result concerning differentiable functions.

Monotonicity at a point

The reader may well be familiar with the intuitive idea that if the derivative of a function f is positive at a point x_0, then the function is increasing near x_0. We now formalize this.

DEFINITION 15.1. A function f is *strictly increasing at the point* x_0 if there exists δ in $\mathbb{R}^+$ such that

$$f(x_0+h) > f(x_0) \quad \text{when } 0 < h < \delta \qquad \text{and} \qquad f(x_0+h) < f(x_0) \quad \text{when } -\delta < h < 0.$$

Further, f is *strictly decreasing at the point* x_0 if there exists δ in $\mathbb{R}^+$ such that

$$f(x_0+h) < f(x_0) \quad \text{when } 0 < h < \delta \qquad \text{and} \qquad f(x_0+h) > f(x_0) \quad \text{when } -\delta < h < 0.$$

REMARK 15.2. It follows that f is *strictly increasing* at x_0 if and only if there exists δ in $\mathbb{R}^+$ such that

$$\frac{f(x_0+h)-f(x_0)}{h} > 0 \quad \text{whenever } 0 < |h| < \delta.$$

Similarly, f is *strictly decreasing* at x_0 if and only if there exists δ in $\mathbb{R}^+$ such that

$$\frac{f(x_0+h)-f(x_0)}{h} < 0 \quad \text{whenever } 0 < |h| < \delta.$$

THEOREM 15.3. *Suppose that f is differentiable at x_0. If $f'(x_0) > 0$, then f is strictly increasing at x_0, and if $f'(x_0) < 0$, then f is strictly decreasing at x_0.*

PROOF. In Lemma 13.4, we showed that if a function $f : (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$, then there exists a function $F : (a,b) \to \mathbb{R}$ that satisfies

$$f(x) = f(x_0) + (x - x_0)\, F(x) \quad \text{for all } x \in (a,b)$$

and that is continuous at x_0; the value $F(x_0)$ is the derivative of f at x_0. By substituting $h = x - x_0$, we see that

$$\frac{f(x_0+h)-f(x_0)}{h} = F(x_0+h)$$

for all nonzero h such that $x_0 + h \in (a,b)$.

First suppose $f'(x_0) > 0$, in which case $F(x_0) > 0$. Since F is continuous at x_0, by Lemma 6.5 there exists $\delta \in \mathbb{R}^+$ such that $F(x_0 + h) > 0$ whenever $|h| < \delta$. This means

$$\frac{f(x_0 + h) - f(x_0)}{h} > 0$$

whenever $0 < |h| < \delta$ and by Remark 15.2, this is equivalent to f being strictly increasing at x_0.

A similar argument shows that if $f'(x_0) < 0$ then f is strictly decreasing at x_0. $\square$

A function f may be strictly increasing or strictly decreasing at a point x_0 where $f'(x_0) = 0$ or $f'(x_0)$ may not even be defined. For instance, if $f(x) = x^3$, then $f'(0) = 0$ but f is strictly increasing and if $f(x) = x^{1/3}$, then $f'(0)$ is not defined but f is strictly increasing.

Differentiability on an interval

So far we have only discussed the notion of differentiability of *a function at a point*. Now we define the differentiability of a function on an open interval.

Definition 15.4. Suppose that $a, b \in \mathbb{R}$ and $a < b$. A function $f : (a, b) \to \mathbb{R}$ is *differentiable* if it is differentiable at all points in (a, b).

As with continuous functions, we can state several general theorems about functions that are differentiable in intervals.

Corollary 15.5. *Suppose that the functions $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are differentiable. Then*

(1) *$f + g$ is differentiable;*
(2) *$f - g$ is differentiable;*
(3) *$f\,g$ is differentiable; and if in addition, $g(x) \neq 0$ for all $x \in (a, b)$, then*
(4) *f/g is differentiable.*

Corollary 15.6. *Suppose that $f : (a, b) \to (c, d)$ and $g : (c, d) \to \mathbb{R}$ are differentiable functions. Then the composed function $g \circ f : (a, b) \to \mathbb{R}$ is differentiable.*

Rolle's Theorem

One of the most important theorems in the theory of functions of a real variable is Rolle's Theorem. It underpins a lot of what we know about functions.

Theorem 15.7 (Rolle's Theorem). *Suppose that the function f is continuous on $[a, b]$ and differentiable on (a, b), and that $f(a) = f(b)$. Then there is a point $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By the Boundedness Theorem (Theorem 8.3), since f is continuous on $[a, b]$, the supremum $\sup(f([a, b]))$ and the infimum $\inf(f([a, b]))$ exist, and they are both attained. We write them as M and m.

Since $m \leq f(a) = f(b) \leq M$ by definition, then it must be true that $m < f(a) = f(b)$, or that $f(a) = f(b) < M$, or that $m = f(a) = f(b) = M$.

Consider first the possibility that $M > f(a)$. In this case, by the Boundedness Theorem, there is a point $c \in [a, b]$ such that $f(c) = M$; further, $c \in (a, b)$, since $M > f(a) = f(b)$. Now $f'(c)$ exists since f is differentiable on (a, b). We claim that $f'(c) = 0$. If it were true that $f'(c) > 0$ or $f'(c) < 0$, then by Theorem 15.3, f would be strictly increasing or strictly decreasing at c. In either case there would be δ in $\mathbb{R}^+$ and points x in $(c - \delta, c + \delta)$, such that

$$f(x) > f(c),$$

which is impossible. Thus it cannot be true that $f'(c) > 0$ or $f'(c) < 0$, that is, $f'(c) = 0$.

Consider next the possibility that $m < f(a)$. In this case, a similar argument shows that there is a point $c \in (a, b)$ such that $f(c) = m$. If it were true that $f'(c) > 0$ or $f'(c) < 0$, then again by Theorem 15.3, f would be strictly increasing or strictly decreasing at c, and again there would be δ in $\mathbb{R}^+$ and points x in $(c - \delta, c + \delta)$ such that

$$f(x) < f(c),$$

which is impossible. Thus $f'(c) = 0$ in this case too.

The last possibility is that $m = M$. In this case, the function f is constant, and the derivative of a constant function is 0, so $f'(c) = 0$ for all $c \in (a, b)$.

Thus in each case, there is a point c with the required property. □

Corollary 15.8. *If $f : (a, b) \to \mathbb{R}$ is differentiable, then between any two different zeros of f there is a zero of f'.*

Proof. If x_1 and x_2 are two distinct zeros of f, then $f(x_1) = f(x_2)$, so, by Rolle's Theorem, there exists c in the interval (x_1, x_2) (or in (x_2, x_1) if $x_2 < x_1$) such that $f'(c) = 0$, that is, c is a root of f' lying between x_1 and x_2. □

Corollary 15.9. *If $f : (a, b) \to \mathbb{R}$ is differentiable and has n distinct zeros, then f' has* at least *$n - 1$ distinct zeros.*

Equivalently, if f' has m distinct zeros, then f has at most *$m+1$ distinct zeros.*

LECTURE 16

The Mean Value Theorem

In this lecture we state and prove the Mean Value Theorem. This is the most important and far-reaching consequence of Rolle's Theorem, which is actually a *generalization* of Rolle's Theorem. We also consider connections between differentiability and monotonicity on an interval, and compare this with monotonicity at a point.

The Mean Value Theorem

THEOREM 16.1 (Mean Value Theorem). *Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

PROOF. Define $\phi : [a,b] \to \mathbb{R}$ by

$$\phi(x) = f(x) - \Big(\frac{f(b)-f(a)}{b-a}\Big)x.$$

Then ϕ is continuous on $[a,b]$ and differentiable on (a,b), by the algebra of continuous and differentiable functions (Corollaries 5.6 and 15.5). Furthermore,

$$\begin{aligned}\phi(a)-\phi(b) &= f(a) - \Big(\frac{f(b)-f(a)}{b-a}\Big)a - \Bigg(f(b) - \Big(\frac{f(b)-f(a)}{b-a}\Big)b\Bigg)\\ &= f(a)-f(b) - \Big(\frac{f(b)-f(a)}{b-a}\Big)(a-b)\\ &= 0\end{aligned}$$

and hence $\phi(a)=\phi(b)$. By Rolle's Theorem, there exists $c \in (a,b)$ such that $\phi'(c)=0$. Hence

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0,$$

which is equivalent to the desired conclusion. $\square$

Monotonicity on an interval

Corollary 16.2. *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*

(1) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*
(2) *If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.*
(3) *If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.*

Proof. For (1), take an arbitrary point $x \in (a, b]$, and consider f as a function on $[a, x]$; it is automatic that f is continuous on $[a, x]$ and differentiable on (a, x).

By the Mean Value Theorem applied to f on $[a, x]$, there exists $c \in (a, x)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(c) = 0.$$

Therefore $f(x) = f(a)$. Since x was chosen arbitrarily, f is constant on $[a, b]$.

For (2), (3), take $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$, and consider f as a function on $[x_1, x_2]$; it is automatic that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value Theorem applied to f on $[x_1, x_2]$, there exists $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

In case (2), $f'(x) > 0$ for all $x \in (a, b)$ by hypothesis, so in particular $f'(c) > 0$, and hence $f(x_2) > f(x_1)$; that is, f is strictly increasing. In case (3), we argue similarly to conclude that f is strictly decreasing. □

Remark 16.3. The converse of Parts (2) and (3) of Corollary 16.2 are not true. For instance, suppose that $f(x) = x^3$; then f is strictly increasing, but $f'(0) = 0$. However, it can be shown that if f is increasing and differentiable, then $f'(x) \geq 0$ for all x.

To see the difference between being strictly increasing at a point and being strictly increasing on an interval. Define

$$f(x) = \begin{cases} \dfrac{x}{2} + x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $f'(0) = 1/2$ (we computed the derivative of the hard part of this function in Exercise 14.3), and, when $x \neq 0$,

$$f'(x) = \frac{1}{2} + 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right)(-1/x^2) = \frac{1}{2} + 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Define $a_n = 1/(2\pi n)$, for any positive integer n. Then $f(a_n) = 1/2 - 1 = -1/2$. This means that f is strictly increasing at the point 0, but there are small intervals arbitrarily close to 0 in which f is strictly decreasing.

The Mean Value Theorem guarantees the *existence* of a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c),$$

but it does not tell us *anything* about how we might find such a point c. Often the actual point c is not important, as the following example illustrates.

EXERCISE 16.4. Show that, if $-\pi/4 \leq a < b \leq \pi/4$, then

$$|\tan b - \tan a| < 2\,|b - a|\,.$$

ANSWER. The function $x \mapsto \tan x$ is continuous and differentiable on $(-\pi/2, \pi/2)$, and hence continuous on $[a, b]$ and differentiable on (a, b), because $[a, b] \subset (-\pi/2, \pi/2)$ (and $(a, b) \subset (-\pi/2, \pi/2)$). Hence by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$\frac{\tan b - \tan a}{b - a} = \sec^2 c,$$

and this implies that

$$|\tan b - \tan a| = (\sec^2 c)\,|b - a|\,.$$

Since $\cos^2 x > 1/2$ for all $x \in (-\pi/4, \pi/4)$,

$$|\tan b - \tan a| < 2\,|b - a|\,,$$

as claimed. △

The Mean Value Theorem is very useful for proving inequalities of this type.

The Generalized Mean Value Theorem

We now come to a useful generalization of the Mean Value Theorem, known as the Generalized Mean Value Theorem, or Cauchy's Mean Value Theorem.

THEOREM 16.5 (Generalized Mean Value Theorem). *Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}\,.$$

Proof. We define a function $\phi : [a,b] \to \mathbb{R}$ by

$$\phi(x) = f(x) - \left(\frac{f(b) - f(a)}{g(b) - g(a)} \right) g(x) \quad \forall x \in [a,b].$$

First, observe that $g(b) \neq g(a)$, and thus ϕ is well-defined (indeed, if $g(b) = g(a)$, using Rolle's theorem we have that there exists $c \in (a,b)$ such that $g'(c) = 0$, but this contradicts the fact that we are assuming that $g'(x) \neq 0$ for all $x \in (a,b)$. Thus $g(b) \neq g(a)$).

Notice also that ϕ is continuous on $[a,b]$ and differentiable on (a,b), by the algebra of continuous and differentiable functions (Corollaries 5.6 and 15.5). Furthermore, $\phi(a) = \phi(b)$. Hence by Rolle's Theorem, there exists $c \in (a,b)$ such that $\phi'(c) = 0$; that is,

$$f'(c) - \left(\frac{f(b) - f(a)}{g(b) - g(a)} \right) g'(c) = 0,$$

which is equivalent to the desired conclusion, since $g'(c) \neq 0$. $\square$

The following argument is *erroneous*. By the Mean Value Theorem there exists a point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

and

$$\frac{g(b) - g(a)}{b - a} = g'(c);$$

dividing the first identity by the second gives the desired conclusion.

This argument is erroneous because, in general, the point c that works for f is different to the point c that works for g.

LECTURE 17

L'Hôpital's Rule and Taylor's Theorem

In this lecture, we meet L'Hôpital's Rule and Taylor's Theorem. Both are essentially consequences of Rolle's Theorem (we will deduce L'Hôpital's Rule from the Generalised Mean Value Theorem, which itself was proved via Rolle's Theorem in the previous lecture).

In order to state Taylor's Theorem, we must introduce derivatives at endpoints of intervals and higher derivatives.

L'Hôpital's Rule

Our main application of the Generalized Mean Value Theorem is L'Hôpital's Rule.

COROLLARY 17.1 (0/0 form of L'Hôpital's Rule). *Suppose that f and g are differentiable on (a, b). Suppose further that $x_0 \in (a, b)$ and that $g'(x) \neq 0$ for all $x \in (a, b) \setminus \{x_0\}$. If $f(x_0) = g(x_0) = 0$ and*

$$\frac{f'(x)}{g'(x)} \to A \quad \textit{as } x \to x_0,$$

then

$$\frac{f(x)}{g(x)} \to A \quad \textit{as } x \to x_0.$$

PROOF. We wish to show that

$$\frac{f(x)}{g(x)} \to A \quad \text{as } x \to x_0.$$

To this end, we take arbitrary $\epsilon \in \mathbb{R}^+$. Since

$$\frac{f'(x)}{g'(x)} \to A \quad \text{as } x \to x_0,$$

there exists $\delta \in \mathbb{R}^+$ such that

$$\left|\frac{f'(x)}{g'(x)} - A\right| < \epsilon \quad \text{whenever } 0 < |x - x_0| < \delta. \tag{17.12}$$

Now suppose that $a < x < b$ and $x_0 < x < x_0 + \delta$. The functions f and g are continuous on $[x_0, x]$ and differentiable on (x_0, x), by hypothesis, and so by the Generalized Mean Value

Theorem, there exists $c \in (x_0, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)}.$$

Now $x_0 < x < x_0 + \delta$ and $x_0 < c < x$, so $x_0 < c < x_0 + \delta$, and hence by (17.12),

$$\left|\frac{f(x)}{g(x)} - A\right| = \left|\frac{f'(c)}{g'(c)} - A\right| < \epsilon.$$

Therefore

$$\frac{f(x)}{g(x)} \to A \quad \text{as } x \to x_0+.$$

A similar argument, where we consider x in $(x_0 - \delta, x_0)$, gives

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to x_0-,$$

and the desired conclusion follows. □

Exercise 17.2. Find $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$.

Answer. Let $f(x) = 1 - \cos x$ and $g(x) = x^2$. Since the cosine function and polynomials are differentiable on $\mathbb{R}$, it follows that f and g are differentiable on $\mathbb{R}$ by the algebra of differentiable functions, with $f'(x) = \sin x$ and $g'(x) = 2x$. Clearly we have $f(0) = g(0) = 0$ and $g'(x) \neq 0$ whenever $x \neq 0$. Also,

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}$$

using the algebra of limits and the standard limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Hence, by L'Hôpital's Rule (Corollary 17.1), we get

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{1}{2}.$$

△

More definitions related to differentiability

For reasons that will soon become clear, we would like to have a definition of differentiability for a function defined on *any* interval, not just an *open* interval.

Definition 17.3. A function $f : [a, b] \to \mathbb{R}$ is *differentiable* if it is differentiable on the open interval (a, b), *and* both of the limits

$$\lim_{h\to 0+} \frac{f(a + h) - f(a)}{h} \quad \text{and} \quad \lim_{h\to 0-} \frac{f(b + h) - f(b)}{h}$$

exist. We denote these one sided limits by $f'(a)$ and $f'(b)$ respectively.

Similarly a function $f : [a, b) \to \mathbb{R}$ is differentiable if it is differentiable on (a, b) and the right hand limit

$$\lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$

exists, and a function $f : (a, b] \to \mathbb{R}$ is differentiable if it is differentiable on (a, b) and the left hand limit

$$\lim_{h \to 0-} \frac{f(b+h) - f(b)}{h}$$

exists.

The notion of *higher derivatives* will also be important for us.

DEFINITION 17.4. Suppose that f is differentiable with derivative f' on an interval, and that f' is itself differentiable. Then we denote the derivative of f' by f'', and call it the *second derivative* of f. Continuing in this way (differentiability permitting), we obtain functions

$$f, f', f'', f''', \dots, f^{(n)}$$

each of which is the derivative of the one before. We call $f^{(n)}$ the *nth derivative of f*. If $f^{(n)}$ exists for *all* positive integers n, then we say that f is *infinitely differentiable.*

THEOREM 17.5 (Taylor's Theorem). *Suppose that $f, f', f'', \dots, f^{(n)}$ all exist and are continuous on $[a, b]$, and that $f^{(n+1)}$ exists on (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Equivalently, if $f, f', \dots, f^{(n)}$ exist and are continuous on $[a, a+t]$ and $f^{(n+1)}$ exists on $(a, a+t)$, then there exists $\theta \in (0, 1)$ such that

$$f(a+t) = f(a) + f'(a)t + \frac{f''(a)}{2}t^2 + \cdots + \frac{f^{(n)}(a)}{n!}t^n + \frac{f^{(n+1)}(a+\theta t)}{(n+1)!}t^{n+1}.$$

PROOF. Define the function $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(x) = f(x) + (b-x)\, f'(x) + \cdots + \frac{(b-x)^n}{n!} f^{(n)}(x) + K\left(\frac{b-x}{b-a}\right)^{n+1},$$

where

$$K = f(b) - f(a) - f'(a)(b-a) - \cdots - \frac{f^{(n)}(a)}{n!}(b-a)^n.$$

Since $f, f', \dots, f^{(n)}$ exist and are continuous on $[a, b]$, it follows (from Theorem 5.4) that ϕ is continuous on $[a, b]$. Since $f^{(n+1)}$ exists on (a, b), it follows (from Theorem 13.1) that ϕ is differentiable on (a, b). Moreover, $\phi(a) = \phi(b)$, and hence by Rolle's Theorem, there exists $c \in (a, b)$ such that $\phi'(c) = 0$.

Hence

$$0 = \phi'(c) = \frac{(b-c)^n}{n!}\, f^{(n+1)}(c) - (n+1)K\frac{(b-c)^n}{(b-a)^{n+1}},$$

which can be rewritten as

$$K = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

as required. $\square$

LECTURE 18

More on Taylor's Theorem and L'Hôpital's Rule

In this lecture, we continue to our development of Taylor's Theorem and show how Taylor's Theorem can be helpful in justifying a certain hypothesis in L'Hôpital's Rule.

More on Taylor's Theorem

REMARK 18.1. (1) If f is a polynomial of degree at most n, then $f^{(n+1)}$ is identically zero. Therefore, in this case, Taylor's Theorem says

$$f(a+t) = f(a) + f'(a)\,t + \frac{f''(a)}{2!}\,t^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,t^n.$$

(2) When $n = 0$, Taylor's Theorem is exactly the Mean Value Theorem. It is instructive to compare the proofs of these two theorems, and the Generalized Mean Value Theorem.

(3) Implicitly $b > a$ (and thus $t > 0$). However, the theorem still holds when $b < a$ (and thus $t < 0$) provided that we replace the interval $[a, b]$ by $[b, a]$ (and thus $[a, a+t]$ by $[a+t, a]$), and replace similarly the open intervals that appear.

(4) The value c (and thus θ) depends on f, a, b and n (or f, a, t and n). If any of these are to be varied, we should make this dependence clear by writing, for example, c_n, $c_n(t)$, ... (or θ_n, $\theta_n(t)$, ...).

In conclusion, Taylor's Theorem says that we may approximate certain functions $t \mapsto f(a+t)$ by the polynomial

$$f(a) + f'(a)\,t + \frac{f''(a)}{2}\,t^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,t^n,$$

with an error (or remainder) term

$$R_n(t) = \frac{f^{(n+1)}(a+\theta t)}{(n+1)!}t^{n+1}.$$

Here θ depends on f, n, a and t.

REMARK 18.2. We sometimes wish to do calculations of the form

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\cdots=\lim_{x\to x_0}\frac{f^{(n)}(x)}{g^{(n)}(x)}=\frac{f^{(n)}(x_0)}{g^{(n)}(x_0)},$$

when at each stage except the last we have a limit of the form $0/0$ and use L'Hôpital's Rule, and where $g^{(n)}(x_0)\neq 0$, so at the last stage we can use the Algebra of Limits. To be able to apply L'Hôpital's Rule the kth time, we need to know that $g^{(k)}(x)\neq 0$ when x is close to, but different from x_0. This condition follows automatically from the following lemma.

LEMMA 18.3. *Suppose that g, g', g'', ..., $g^{(n)}$ exist and are continuous on (a,b), and that for some point $x_0\in(a,b)$,*

$$g^{(k)}(x_0)=0 \text{ whenever } 0\le k\le n-1 \qquad \text{and} \qquad g^{(n)}(x_0)\neq 0.$$

Then there exists $\delta\in\mathbb{R}^+$ such that

$$g^{(k)}(x)\neq 0 \quad \text{whenever } 0\le k\le n \text{ and } 0<|x-x_0|<\delta.$$

The proof of this lemma is omitted; it can be shown to be a consequence of Taylor's Theorem. The next example shows off the usefulness of this lemma.

EXERCISE 18.4. Prove

$$\lim_{x\to 0}\frac{x^2-2(1-\cos(x))}{x^2(1-\cos(x))}=\frac{1}{6}.$$

ANSWER. Let $f(x)=x^2-2(1-\cos(x))$ and $g(x)=x^2(1-\cos(x))$.

Since the sine, cosine and polynomial functions are differentiable, and sums and products of differentiable functions are differentiable, it follows that f and g are infinitely differentiable. Moreover,

$$f'(x)=2x-2\sin x,\ f''(x)=2-2\cos x,\ f'''(x)=2\sin x,\ f''''(x)=2\cos x$$

and

$$g'(x)=x^2\sin x+2x(1-\cos x),\ g''(x)=x^2\cos x+4x\sin x+2(1-\cos x)$$
$$g'''(x)=-x^2\sin x+6x\cos x+6\sin x,\ g''''(x)=-x^2\cos x-8x\sin x+12\cos x.$$

So $f^{(k)}(0)=0$ and $g^{(k)}(0)=0$ for each $k=0,1,2,3$, and $f^{(4)}(0)=2$, and $g^{(4)}(0)=12$. So, using Lemma 18.3, there exists (a,b) such that $g^{(j)}(x)\neq 0$ for all $x\in(a,b)\setminus\{0\}$ for $j=1,2,3,4$. Hence, by the $0/0$ form of L'Hôpital's Rule (repeatedly) and the Algebra of Limits, it follows that

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\lim_{x\to 0}\frac{f'''(x)}{g'''(x)}=\lim_{x\to 0}\frac{f''''(x)}{g''''(x)}=\frac{1}{6}.$$

△

Note that, without appealing to Lemma 18.3, it would be difficult to give a *direct* proof that there exists an interval (a, b) containing zero such that $g^{(j)}(x) \neq 0$ for all $x \in (a, b) \setminus \{0\}$ and each $j = 1, 2, 3, 4$.

LECTURE 19

Maxima, Minima and Stationary Points

In this lecture, we use the derivative to find maxima and minima of functions. This involves finding stationary points, that is, points where the derivative is zero. Not all stationary points correspond to maxima or minima. To understand how a function behaves near a stationary point, we use Taylor's Theorem.

Maxima and minima

Definition 19.1. Suppose that the real function f is defined on the open interval (a, b) and that $x_0 \in (a, b)$. We say that f has a *local maximum* at x_0 if there exists $\delta \in \mathbb{R}^+$ such that

$$f(x) \leq f(x_0) \quad \text{whenever } x \in (x_0 - \delta, x_0 + \delta),$$

and that f has a *global maximum* at x_0 if

$$f(x) \leq f(x_0) \quad \text{whenever } x \in (a, b).$$

Similarly, we say that f has a *local minimum* at x_0 if there exists $\delta \in \mathbb{R}^+$ such that

$$f(x) \geq f(x_0) \quad \text{whenever } x \in (x_0 - \delta, x_0 + \delta),$$

and that f has a *global minimum* at x_0 if

$$f(x) \geq f(x_0) \quad \text{whenever } x \in (a, b).$$

Stationary points

Theorem 19.2. *Suppose that the real function f, defined on the open interval (a, b), has a local maximum or minimum at $x_0 \in (a, b)$. If f is differentiable at x_0, then*

$$f'(x_0) = 0.$$

Proof. If $f'(x_0) \neq 0$, then by Theorem 15.3, f is strictly increasing or strictly decreasing at x_0, and so f cannot have a local maximum or a local minimum at x_0. □

Remark 19.3. We have already used this argument in proving Rolle's Theorem.

Definition 19.4. Suppose that f is defined on an open interval (a, b). A point $x_0 \in (a, b)$ is a *stationary point* of f if f is differentiable at x_0 and $f'(x_0) = 0$.

Not all stationary points are local maxima or minima; for instance, the function $x \mapsto x^3$ has a stationary point at 0, which is neither a local maximum or minimum. So we cannot classify stationary points in terms of local maxima and minima alone. Our next result describes the possibilities for stationary points for many differentiable functions.

Theorem 19.5 (Classification of stationary points). *Suppose that f, f', f'', $\ldots$, $f^{(n+1)}$ exist and are continuous on (a, b), and that for some point $x_0 \in (a, b)$,*

$$f^{(k)}(x_0) = 0 \quad \text{when } 1 \leq k \leq n, \text{ while} \quad f^{(n+1)}(x_0) \neq 0,$$

for some positive integer n.

(1) *In the case that $n + 1$ is even,*
   If $f^{(n+1)}(x_0) > 0$, then f has a local minimum at x_0
   If $f^{(n+1)}(x_0) < 0$, then f has a local maximum at x_0.
(2) *In the case that $n + 1$ is odd,*
   If $f^{(n+1)}(x_0) > 0$, then f is strictly increasing at x_0
   If $f^{(n+1)}(x_0) < 0$, then f is strictly decreasing at x_0.

Proof. Since $f^{(n+1)}(x_0) \neq 0$, it suffices to consider the two cases $f^{(n+1)}(x_0) > 0$ and $f^{(n+1)}(x_0) < 0$. We shall prove only $f^{(n+1)}(x_0) > 0$, leaving the latter as an exercise in consolidating ideas.

Since $f^{(n+1)}$ is continuous at x_0, by Lemma 6.5, there exists $\delta \in \mathbb{R}^+$ such that

$$f^{(n+1)}(x) > 0 \;\text{ whenever }\; |x - x_0| < \delta. \tag{19.13}$$

Now, by Taylor's Theorem, for each x with $|x - x_0| < \delta$ there exists c lying strictly between x and x_0, such that

$$f(x) = f(x_0) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Here we have used the fact that $f^{(k)}(x_0) = 0$ for all $1 \leq k \leq n$. Since $|c - x_0| < \delta$, by (19.13) we have that $f^{(n+1)}(c) > 0$.

Now, if $n + 1$ is *even*, then the error term satisfies

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \geq 0$$

whenever $|x - x_0| < \delta$, which is equivalent to the statement that

$$f(x) \geq f(x_0) \;\text{ whenever }\; |x - x_0| < \delta;$$

i.e. f has a local minimum at x_0.

If $n+1$ is *odd*, then

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^n > 0$$

whenever $0 < |x-x_0| < \delta$ (since n is even). By Remark 15.2, this is equivalent to the statement that f is strictly increasing at x_0, as required. $\square$

REMARK 19.6. Using an analogue of Lemma 13.4 for higher derivatives, we can weaken the hypotheses of Theorem 19.5. In particular, suppose that f, f', f'', $\ldots$, $f^{(n)}$ exist on (a,b), and that for some point $x_0 \in (a,b)$,

$$f^{(k)}(x_0) = 0 \quad \text{when } 1 \leq k \leq n, \text{ while} \quad f^{(n+1)}(x_0) \neq 0, \tag{19.14}$$

for some positive integer n. Then (1) and (2) of Theorem 19.5 hold.

LECTURE 20

Taylor Series and Power Series

In this lecture, we spend more time on Taylor polynomials, and see that taking more terms in the polynomials may lead to better approximations to the original functions. This leads to the topic of Taylor series, and then naturally onto the topic of power series.

Taylor series

Suppose that f and *all* its derivatives exist on (a, b), and that $x_0 \in (a, b)$. By Taylor's Theorem, for all t such that $x_0 + t \in (a, b)$ and all $n \in \mathbb{N}$,

$$f(x_0 + t) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\, t^k + R_n(t),$$

where

$$R_n(t) = \frac{f^{(n+1)}(x_0 + \theta t)}{(n+1)!} t^{n+1}$$

for some $\theta \in (0, 1)$. *If* $R_n(t) \to 0$ as $n \to \infty$ for all t such that $x_0 + t \in (a, b)$, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}\, t^k$$

converges, to $f(x_0 + t)$. This series is called the *Taylor Series of f centred at x_0*. Sometimes we set $x = x_0 + t$, and write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

In this case, the restriction $x_0 + t \in (a, b)$ becomes $x \in (a, b)$.

Remark 20.1. Taylor Series centred at 0 (that is, series for which the centre x_0 is 0) are usually known as *Maclaurin Series*, for historical reasons.

Exercise 20.2. Find the Maclaurin Series of the function $x \mapsto \sin x$.

Answer. Write $f(x) = \sin(x)$. We first observe that f has derivatives of all orders on $\mathbb{R}$, and

$$\begin{aligned} f(x) &= \sin(x) & f^{(4)}(x) &= \sin(x) \\ f'(x) &= \cos(x) & f^{(5)}(x) &= \cos(x) \\ f''(x) &= -\sin(x) & f^{(6)}(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) & & \dots \end{aligned}$$

and so on, repeating periodically. We set $x = 0$. This gives

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k, \qquad k = 0, 1, 2 \dots \\ (-1)^k & \text{if } n = 2k+1, \qquad k = 0, 1, 2 \dots \end{cases}$$

Now, taking the centre x_0 to be 0 and fixing t,

$$|R_n(t)| = \left| \frac{t^{n+1}}{(n+1)!} \right| \left| f^{(n+1)}(\theta t) \right| \leq \frac{|t|^{n+1}}{(n+1)!},$$

since $|\sin(\theta t)| \leq 1$ and $|\cos(\theta t)| \leq 1$. Therefore, since

$$\frac{|t|^{n+1}}{(n+1)!} \to 0 \quad \text{as } n \to \infty$$

(this is an exercise in its own right!), it follows from the Sandwich Theorem that $R_n(t) \to 0$ as $n \to \infty$, and so

$$\sin(t) = f(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

is the Maclaurin series of the sine function, which is valid *for all* $t \in \mathbb{R}$. △

A similar argument yields the Maclaurin series for the cosine function (see Problem Sheet 5).

Remark 20.3. There are some functions, which have derivatives of all orders, but for which $R_n(t) \not\to 0$ as $n \to \infty$, and these functions cannot be expanded in Taylor Series. A classic example is the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

This function has derivatives of all orders, and moreover

$$f^{(k)}(0) = 0 \quad \text{for all } k \in \mathbb{N},$$

and so in particular

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 \quad \text{for all } x \in \mathbb{R},$$

and this is rather different to $f(x)$.

A rapid recap on power series

In discussing Taylor Series above, we dealt with series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where $(a_n)_{n\geq 0}$ is a sequence of real numbers. We shall refer to such series as *power series*. We consider the power series $\sum_{n=0}^{\infty} a_n x^n$ as a function

$$S \to \mathbb{R}; \qquad x \mapsto \sum_{n=0}^{\infty} a_n x^n,$$

where S is the *set of convergence* given by

$$S := \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

For such a power series, the *radius of convergence* is given by

$$R = \begin{cases} \sup\{|x| : x \in S\} & \text{if } S \text{ is bounded,} \\ \infty & \text{if } S \text{ is unbounded,} \end{cases}$$

and a fundamental result (from 1RAC) is that

$$(-R, R) \subseteq S \subseteq [-R, R]$$

and the power series converges absolutely in $(-R, R)$.

LECTURE 21

Metric Spaces. Distances. Convergence and Continuity

We will see how some of the classical notions for real numbers and functions of a real variable can be made sense of in a much wider context - For example, we might want to consider continuous functions on a more general set X (the elements of X could now be functions themselves!). What do we mean by continuity in this context?

Let us reflect on the definition of continuity of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ at a point. Recall the following definition:

Definition. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say that f is continuous at a point $a \in \mathbb{R}$ if: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \qquad \text{whenever} \qquad |x - a| < \delta.$$

Roughly speaking, a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a point $a \in \mathbb{R}$ if we can make the distance between $f(x)$ and $f(a)$ as small as we please by choosing x so that its distance from a is sufficiently small, in other words, to say a function f is continuous at $a \in \mathbb{R}$ is to say that we can make $f(x)$ and $f(a)$ arbitrarily "close" by making x and a "close".

What is it that makes the notion of continuity really work? The notion of "closeness", that is the notion of "distance".

Notice that the above definition makes no reference to the fact that f is defined on $\mathbb{R}$ with values in $\mathbb{R}$ except for the definition of "distance" in $\mathbb{R}$. Thus, if we can find an appropriate notion of "distance" on $\mathbb{R}^n$, we may give the definition of what it means for a function f to be a continuous function on $\mathbb{R}^n$, or any other spaces rather than $\mathbb{R}$.

Distances

The precise definition of a distance or metric is the following:

Definition 21.1. Let $X \neq \emptyset$. A function $d : X \times X \longrightarrow \mathbb{R}$ satisfying the following properties

(M1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
(M2) $d(x, y) = d(y, x)$
(M3) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality)

for all x, y and z in X, is called a distance or metric on X. The pair (X, d) is called a metric space.

We continue giving some examples of different distances in different sets $X \neq \emptyset$

Examples.

1. In $\mathbb{R}$, the function $d(x, y) = |x - y|$, for any x and y in $\mathbb{R}$, is a distance on $\mathbb{R}$.
2. In $\mathbb{R}^2$, the function
$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$
for all (x_1, x_2), (y_1, y_2) in $\mathbb{R}^2$, is a distance on $\mathbb{R}^2$.
3. **Euclidean n-space**. Let $X = \mathbb{R}^n$ and d be the function defined by
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$
for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$. Then d is a distance on $\mathbb{R}^n$.
We call $\mathbb{R}^n$ with this distance the Euclidean n-space and the distance d is called the Euclidean distance on $\mathbb{R}^n$.
4. Let $\mathcal{C}([a, b])$ be the set of all continuous functions $f : [a, b] \longrightarrow \mathbb{R}$, that is
$$\mathcal{C}([a, b]) = \{f : [a, b] \longrightarrow \mathbb{R} \quad : \quad f \text{ is continuous}\}.$$
Let d be defined as follows
$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$
for any f and g in X. Then d is a metric on $\mathcal{C}([a, b])$ called the supremum metric (sup-metric for short) or the uniform metric.

Convergence of sequences

Recall the following definition of convergence of a sequence of real numbers.

Definition. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=1}^{\infty}$ converges to a point l in $\mathbb{R}$ if and only if
$$\forall \varepsilon > 0, \quad \text{there exists} \quad N \in \mathbb{N} \quad \text{such that} \quad |x_n - l| < \varepsilon, \quad \forall n \geq N.$$
We write $x_n \longrightarrow l$, as $n \longrightarrow \infty$ or $\lim_{n \to \infty} x_n = l$.

Informally: $(x_n)_{n=1}^{\infty}$ converges to x if the distance between x_n and x is as small as we want provided that we look at sufficiently large indexes n.

The above definition easily generalises to the following definition of convergence of a sequence to a point in a metric space.

DEFINITION 21.2. Let (X,d) be a metric space. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in X, and $l \in X$. We say that $\underline{(x_n)_{n=1}^{\infty} \text{ converges to } l}$ iff

$$\forall\, \varepsilon > 0, \qquad \text{there exists} \quad N \in \mathbb{N} \quad \text{such that} \quad d(x_n, l) < \varepsilon, \qquad \text{whenever} \quad n \geq N.$$

If $(x_n)_{n=1}^{\infty}$ converges to l in (X,d), we will write $x_n \longrightarrow l$, as $n \longrightarrow \infty$.

Examples.

1. Consider $\mathbb{R}$ with the Euclidean distance, that is

$$d(x,y) = |x-y|, \qquad \forall\, x, y \in \mathbb{R}.$$

Then, the above definition is nothing else that the "usual" definition of convergence of sequences of real numbers.

2. In $\mathbb{R}^2$ with the Euclidean distance, the sequence

$$\left\{ \left(\frac{1}{n}, \frac{1}{n^2} \right) \right\} \longrightarrow (0,0) \qquad \text{as} \qquad n \longrightarrow \infty.$$

Indeed: Fix any $\varepsilon > 0$. Notice that if $n \geq 1$ and $n \geq N$ for some $N \in \mathbb{N}$ to be chosen later, we have that

$$\begin{aligned} d\left(\left(\frac{1}{n}, \frac{1}{n^2} \right), (0,0) \right) &= \sqrt{\left(\frac{1}{n} - 0 \right)^2 + \left(\frac{1}{n^2} - 0 \right)^2} \\ &= \sqrt{\frac{1}{n^2} + \frac{1}{n^4}} \leq \frac{1}{n} + \frac{1}{n^2} \\ &\leq \frac{2}{n} \leq \frac{2}{N}. \end{aligned} \tag{21.15}$$

Thus, if one chooses $N \in \mathbb{N}$ such that $N > (2/\varepsilon)$, then from (21.15) we obtain that

$$d\left(\left(\frac{1}{n}, \frac{1}{n^2} \right), (0,0) \right) \leq \frac{2}{N} < \varepsilon, \quad \text{whenever} \quad n \geq N,$$

as required.

Continuity

The following definition generalizes the notion of continuity to the more general setting of metric spaces.

Definition 21.3. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \longrightarrow Y$ be a function. We say that $\underline{f \text{ is continuous at } a \in X}$ if $\forall\, \varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(x), f(a)) < \varepsilon, \qquad \text{whenever} \qquad d_X(x, a) < \delta.$$

Examples.

1. Consider $\mathbb{R}$ with the usual metric. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{1}{x}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0. \end{cases}$$

The function f is not continuous at 0.

2. Consider the space

$$\mathcal{C}([a, b]; \mathbb{R}) = \{f\ [a, b] \longrightarrow \mathbb{R}\ :\ f \text{ is continuous}\}$$

with the supremum metric, that is

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Define the map $T : X \longrightarrow \mathbb{R}$ as follows

$$T(f) = \int_a^b f(x)\, dx, \qquad \text{for all} \qquad f \in X.$$

Show that T is continuous.

Solution. We need to show that T is a continuous map at any given $f_0 \in X$. Let $f_0 \in X$ and $\varepsilon > 0$. Notice that

$$\begin{aligned} |T(f) - T(f_0)| &= \left| \int_a^b f(x)\, dx - \int_a^b f_0(x)\, dx \right| \\ &= \left| \int_a^b (f(x) - f_0(x))\, dx \right| \leq \int_a^b |f(x) - f_0(x)|\, dx \\ &\leq \int_a^b \sup_{x \in [a,b]} |f(x) - f_0(x)|\, dx = d(f, f_0) \left(\int_a^b dx \right) \\ &= d(f, f_0)(b - a) < \delta(b - a), \end{aligned} \tag{21.16}$$

whenever $d(f, f_0) < \delta$. Therefore, by taking (say) $\delta = \varepsilon/(b - a) > 0$, from (21.16) and this choice of δ, it follows that

$$|T(f) - T(f_0)| < \delta(b - a) = \varepsilon, \qquad \text{whenever} \qquad d(f, f_0) < \delta.$$

Thus, T is continuous at $f_0 \in X$, and since f_0 is an arbitrary function in X, we conclude that T is continuous on X.