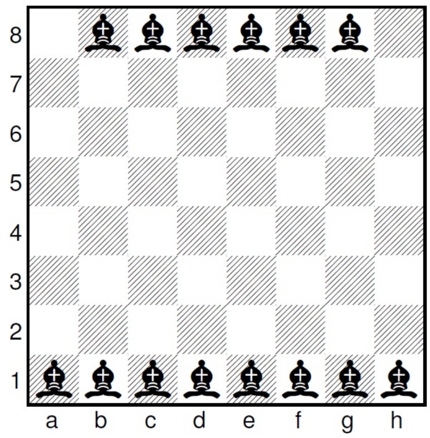
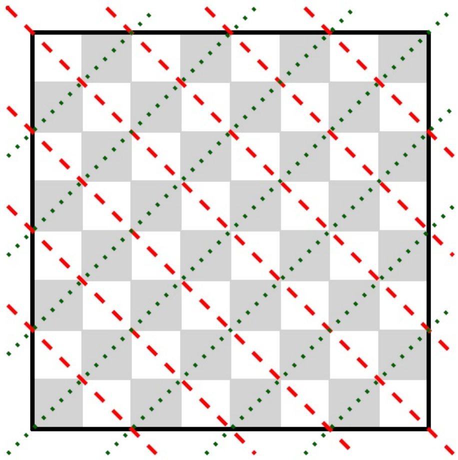
Problem Sheet 1 — Model Solutions and Feedback

Question 1. (SUM) What is the greatest number of bishops which can be placed on a chessboard so that no bishop is attacking another? (In other words, what is the greatest number of squares you can choose on an 8×8 grid without having two squares on the same diagonal.) In your answer you should show how this number can be achieved and also prove that a greater number is not possible.

Solution. 14. The left-hand diagram below shows one example of how this can be accomplished.





To see that you cannot do better, observe that the squares of the board may be partitioned into 14 diagonals as shown in the right hand diagram (with seven light-squared diagonals from top left to bottom right, and seven dark-squared diagonals from bottom left to top right). By the Pigeonhole Principle (with the 15 bishops as the pigeons and the 14 diagonals as the pigeonholes), if 15 bishops are placed on the board then some two must lie on the same diagonal. □

Feedback. A good answer should clearly do three things:

(i) State prominently that the correct answer is 14.

(ii) Give an example to show how you can place 14 bishops without any attacking another. This is a necessary component of the solution, since if you only show that you can't do 15 then you leave open the possibility that the answer is 13 or fewer. The best way to do this is though a clear diagram, as this is much easier to understand than trying to explain in words.

(iii) Prove that however you place 15 bishops on the board you must have a pair which attack each other (this shows that the answer can't be more than 14). Assuming you are following the solution above, you should:

- state clearly that you are using the pigeonhole principle for this;
- make clear what your pigeons and pigeonholes are (however, it's not necessary to explicitly use the words 'pigeon' and 'pigeonhole' as long as these are clear). A diagram really helps to illustrate how the board can be divided into 14 diagonals for your pigeonholes;
- state the conclusion of the principle (that two bishops must be on the same diagonal) and that this gives the desired conclusion (that these two bishops attack each other).

I don't know of any other concise way to solve this problem, but there are other ways you can split up the board, or arrange 14 bishops, which also work.

Question 2. (SUM) Using a similar argument to the proof of Theorem 1.2, or otherwise, prove the following useful variant of the Pigeonhole Principle:

> Let $k, \ell_1, \dots, \ell_k \in \mathbb{N}$. If at least $\ell_1 + \dots + \ell_k + 1$ pigeons are placed in k pigeonholes numbered $1, \dots, k$, then there exists some $i \leq k$ for which there are at least $\ell_i + 1$ pigeons in pigeonhole i.

Solution. Suppose for a contradiction that there exist natural numbers k and $\ell_1, \dots, \ell_k$ for which the statement does not hold. This means that we can arrange $\ell_1 + \dots + \ell_k + 1$ pigeons into pigeonholes $1, \dots, k$ such that for each i there are at most ℓ_i pigeons in pigeonhole i. Fix such an arrangement, and let x_i be the number of pigeons in the pigeonhole i, so $\sum_{i=1}^k x_i = \ell_1 + \dots + \ell_k + 1$ (since there are this many pigeons in total), and $x_i \leq \ell_i$ for each $1 \leq i \leq k$ (since there are at most ℓ_i pigeons in pigeonhole i). Summing the latter inequalities for each $1 \leq i \leq k$ then gives

$$\ell_1 + \dots + \ell_k + 1 = \sum_{i=1}^{k} x_i \leq \ell_1 + \dots + \ell_k,$$

so cancelling yields $1 \leq 0$, a contradiction. $\square$

Feedback. To proceed with a proof by contradiction, as in the model solution, I think conceptually the key steps are as follows.

- First, you need to understand what the presented statement is actually saying. I find a good approach for this is to try to write it with some specific values of the variables – for example, you could take $k = 3$, $\ell_1 = 1, \ell_2 = 2$ and $\ell_3 = 4$, and write out what the statement is saying in this instance. This should not form part of your final answer, but hopefully will help you to understand fully what you are trying to prove.

- Second, work out precisely what the negation of the statement we are trying to prove is. We are trying to show that for every way that $\ell_1 + \dots + \ell_k + 1$ pigeons could be placed in the k pigeonholes, there exists some $i \leq k$ for which there are at least $\ell_i + 1$ pigeons in pigeonhole i. So the negation of this is that it is possible to place $\ell_1 + \dots + \ell_k + 1$ pigeons in k pigeonholes so that there is no $i \leq k$ for which there are at least $\ell_i + 1$ pigeons in pigeonhole i, that is, that for every $i \leq k$ we have at most ℓ_i pigeons in pigeonhole i. You should state this clearly in your answer, but more importantly you should make sure you are comfortable translating between a statement like this and its negation.

- Finally, you argue for a contradiction between the expression for the total number of pigeons and our supposed bound that there are at most ℓ_i pigeons in hole i. To explain this clearly it is helpful to introduce notation for the number of pigeons in hole i, e.g. with a variable x_i as in the model solution.

Do note that there are a number of variations on this approach you could use, like the following: if there is some $i \leq k - 1$ for which there are $\ell_i + 1$ pigeons in hole i, then we are done. So we may assume there are at most ℓ_i pigeons in hole i for each $i \leq k-1$, and therefore at least $\ell_1 + \dots + \ell_k + 1) - (\ell_1 + \dots + \ell_{k-1}) = \ell_k + 1$ pigeons in hole k, as required. However, it's important that with either approach we are allowing for possibilities. Don't say e.g. "we can put ℓ_1 pigeons in hole 1, and then ℓ_2 pigeons in hole 2, and so forth", since these is no requirement that you actually do this (you don't get to choose where the pigeons are!)

Question 3. Prove that some member of the following sequence is divisible by 2023.

$$7, 77, 777, 7777, 77777, 777777, 7777777, 77777777, \dots$$

Hint: consider the differences $x_i - x_j$ for elements x_i and x_j in the sequence above. It may help to consider the example from the first lecture in which the Pigeonhole Principle was applied to show that in any sequence of r integers there is a consecutive subsequence whose sum is divisible by r.

Solution. Let x_r denote the rth member of the sequence, so $x_r = \overbrace{77\ldots7}^{r}$. Since there are 2023 distinct remainders when dividing by 2023, by the Pigeonhole Principle at least two of the integers $x_1, x_2, \ldots, x_{2024}$ must have the same remainder when divided by 2023. Choose x_i and x_j with this property and so that $i < j$, and observe that

$$x_j - x_i = \overbrace{77\ldots7}^{j} - \overbrace{77\ldots7}^{i} = \overbrace{77\ldots7}^{j-i}\overbrace{00\ldots0}^{i} = 10^i x_{j-i}.$$

Since 2023 divides $x_j - x_i$ but $\gcd(10^i, 2023) = 1$, it follows that 2023 divides x_{j-i}, so x_{j-i} is a member of the sequence which is divisible by 2023. $\square$

Feedback. A good answer should:

(i) Apply the pigeonhole principle to say that two elements of the sequence have the same remainder on division by 2023.

(ii) Choose two such integers, and point out that their difference is then divisible by 2023.

(iii) Say that this difference is another member of the sequence multiplied by a power of 10.

(iv) Point out that since $\gcd(10, 2023) = 1$ (this bit is crucial – the solution wouldn't work with 2022 in place of 2023 – though it can be phrased differently), this implies that the aforementioned member of the sequence is divisible by 2023, completing the proof.

Note in particular the introduction of the notation x_r to avoid having to keep talking about 'members of the sequence'.

The challenge in this question is that at first sight it is not immediately clear if considering the term $x_j - x_i$ is the right thing to do, since $x_j - x_i$ is not one of the numbers $x_1, x_2, \ldots$. However, knowing that $x_j - x_i$ is divisible by 2023 gives us information about x_{j-i}, namely it is divisible by 2023 also, and the reasoning for this is quite similar to the example mentioned in the hint.

Question 4. A set X has five elements, each of which is a natural number, and the sum of the elements of X is at most 30.

(a) How many non-empty proper subsets of X are there?

(b) Show that the sum of the members of any non-empty proper subset $A \subsetneq X$ must be between 1 and 29.

(c) Show that there are two non-empty proper subsets of X whose sums are the same.

(d) Show that there are two *disjoint* non-empty proper subsets of X whose sums are the same.

(e) Adapt your arguments to show that the same conclusion holds if X instead has 60 elements, each of which is an natural number with at most 16 digits.

Remark: the final part verifies the claim I made at the very start of the first lecture, since if we colour one of the subsets red and the other blue, then the sum of the red numbers and the sum of the blue numbers are equal.

Solution. For any non-empty subset $A \subseteq X$, let $S(A)$ denote the sum of the members of A, and note that $S(A)$ must be a positive integer.[1]

(a) 30. Indeed, X has 5 elements, so has $2^5 = 32$ subsets. One of these, $\emptyset$, is empty, and another, X, is not a proper subset of X, leaving 30 non-empty proper subsets of X.

[1] If there is an object or phrase which you will refer to often in your argument, it will normally make your proof clearer and more concise to give it a name. In this case, this introductory sentence allows me to write $S(A)$ instead of 'the sum of the members of A'; since this is something which appears frequently the argument which follows is shortened significantly and is easier to follow.

(b) Let A be a proper non-empty subset of X. Since A is non-empty it contains at least one element, so $S(A) \geq 1$. Moreover, since A is a proper subset of X, there must be some $x \in X$ which is not a member of A. So $S(A) \leq S(X) - x \leq 30 - 1 = 29$, as required.

(c) Since there are 30 proper non-empty subsets $A \subsetneq X$ by (a), but there are only 29 possibilities for $S(A)$ for such subsets, the Pigeonhole Principle implies that there must be at least two (distinct) proper non-empty subsets of X which have the same value of $S(A)$, that is, the same sum.

(d) By (c) we may choose proper non-empty subsets $A, B \subseteq X$ so that $A \neq B$ and $S(A) = S(B)$. Let $C = A \setminus (A \cap B)$ and $D = B \setminus (A \cap B)$. Since $A \cap B$ is a subset of both A and B, we have $S(C) = S(A) - S(A \cap B) = S(B) - S(A \cap B) = S(D)$, that is, C and D have equal sum.

It remains to show that C and D are disjoint proper non-empty subsets of X. For this, first observe that $C \subseteq A \subsetneq X$ and $D \subseteq B \subsetneq X$ so C and D are proper subsets of X. Now suppose that C and D are not disjoint; then we may choose $x \in C \cap D$. Since $C \subseteq A$ and $D \subseteq B$ this implies that $x \in A \cap B$, so $x \notin C$, contradicting the assumption that $x \in C \cap D$. So C and D must indeed be disjoint. Finally, suppose that $C = \emptyset$. Then $A \subseteq A \cap B$ (by definition of C), so $A \subseteq B$. But since $A \neq B$ this implies that $A \subsetneq B$, so $S(A) < S(B)$, a contradiction. If $D = \emptyset$ we obtain a contradiction similarly, so we conclude that C and D are non-empty, completing the proof.

(e) If X has 60 elements, then it has $2^{60} - 2 > 10^{18}$ proper non-empty subsets. On the other hand, if each element of X is a natural number with at most 16 digits, then the sum of the elements of any subset of X is at most $60 \times 10^{16} < 10^{18}$. So just as in (c) we may conclude that X has two proper non-empty subsets with the same sum, and then (d) follows exactly as before (the numbers in the question has no part in the deduction of (d) from (c)). □

Feedback. Hopefully your answers to (a) and (b) caused you to think of the Pigeonhole Principle for (c); ideally you will become sufficiently familiar with it as a tool that you would try this approach for (c) even if you hadn't just done (a) and (b). For (d) you then use your answer for (c) – this is typical for multipart questions (that later parts use the answers to earlier parts) and is something you should look out for to suggest the way to proceed with a question.

Note the use of the notation $S(A)$ to avoid having to keep talking about "the sum of the elements of". Introducing this kind of notation can make solutions a lot shorted and easier to follow – try to do this whenever something (a function, a statement, an operation etc) is used frequently in your work, but make sure to state the definition clearly. Also observe the use of set notation in (d) rather than talking about "removing common elements", and also how we defined new sets C and D rather than modifying the sets A and B from (c) whilst keeping the names – in general this tends to be much clearer, and if you change a mathematical object you should try to use a new name for the new value instead of continuing to use the old name.

For (a) it is also perfectly fine just to write down all 30 non-empty proper subsets and count them.

For (d), you are asked to find sets which are disjoint, non-empty, proper subsets of X, so your answer should justify that the sets you find have these properties, as in the model solution. For example, how do you know that removing the common elements hasn't removed all elements of one or both sets? How do you know the new sets are not equal to one another?