DIFFERENTIAL EQUATIONS

2DE & 2DE3

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Introduction

You will have already seen that that there is an organised and well-defined theory available for the solution of linear differential equations with constant coefficients. There is no such organized theory for the analytic solution of nonlinear ordinary differential equations. Methods which are useful in certain cases are not applicable in other situations. Some methods may not provide a solution at all, but will indicate certain properties of the solution.

In this part of 2DE/2DE3 Differential Equations, we will be studying dynamical systems, which we will define below, and, in particular, ways to explore the properties of the solutions without being able to derive an analytic expression for them. For this, we will focus on the so called Phase Plane Analysis technique.

Dynamical systems arise in the study of many different types of physical systems, for example, fluid and solid mechanics, chemistry, biology, ecology and economics. In this introduction, we will explore some of this context to illustrate why understanding the solutions of such systems is of a particular relevance to the world today.

0.1 Preliminary definitions

Definition 1. Differentiation

Given the function $y = y(x)$, where y is the dependent variable and x is the independent variable, we define

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad \left(\text{or } y_x = \frac{\mathrm{d}y}{\mathrm{d}x}\right),$$

to be the derivative of y with respect to x.

We also define higher order derivatives in the same way, for example

$$y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \quad y''' = \frac{\mathrm{d}^3y}{\mathrm{d}x^3}, \quad \cdots, \quad y^{(n)} = \frac{\mathrm{d}^ny}{\mathrm{d}x^n}$$

or

$$y_{xx} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \quad y_{xxx} = \frac{\mathrm{d}^3y}{\mathrm{d}x^3}, \quad \cdots$$

When differentiating with respect to time, we use the notation

$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t}, \quad \ddot{y} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \cdots$$

Note that the dependant and independent variables can be quite different things, e.g. when modelling the mortality rate of COVID-19 as a function of income, the dependent variable may be referred to as M, and the independent variable as I. The derivative of M with respect to I is then written as

$$\frac{\mathrm{d}M}{\mathrm{d}I}.$$

When modelling real life phenomena, one often obtains equation(s) which include derivatives, called differential equations. We are particularly interested in ordinary differential equations. We will differentiate between linear and linear differential equations and differential equations that are autonomous and non-autonomous according to the definitions below.

Definition 2. Ordinary differential equations

An equation of the form

$$F\left(x, y, y', \cdots, y^{(n)}\right) = 0, \tag{1}$$

is called an ***ordinary differential equation*** *(ODE) of order* n.

Definition 3. Linear

The differential equation (1) is ***linear*** *if the dependent variable and its derivatives appear to the power* 1 *(with no products of the dependent variable and its derivatives allowed, e.g. no* yy' *terms), and* ***nonlinear*** *otherwise.*

For example, the Van der Pol equation, describing the current in a triode oscillator,

$$\ddot{x} + 3\left(x^2 - 1\right)\dot{x} + x = 0, \tag{2}$$

is a second order nonlinear ordinary differential equation, nonlinear because of the x^2 term.

Definition 4. Autonomous

An ordinary differential equation is ***autonomous*** *if the independent variable does not appear explicitly in the equation.*

For example

$$y_{xxx} + \left(y_x\right)^2 = y, \tag{3}$$

is autonomous, while

$$y_x = x, \tag{4}$$

is not. Further, it is instructive to note that equation(3) is a third order nonlinear autonomous ordinary equation, while (4) is a first order linear nonautonomous ordinary differential equation.

We'll be studying the properties of solutions to such differential equations, so let us be clear what we mean by that:

Definition 5. Solution

A function $y = y(x)$ which, when substituted into

$$F\left(x, y, y', \cdots, y^{(n)}\right) = 0,$$

turns this equation into an identity is called a ***solution*** *of this equation.*

For example

$$y(x) = \frac{1}{4}x^2,$$

is a particular solution of the differential equation (3).

As indicated, in this section of the module, we are in particular studying dynamical systems. They are defined as follows:

Definition 6. Dynamical System

An n-dimensional, autonomous, dynamical system is a system of first order ordinary differential equations [for the dependent real variables $x_1,\ x_2,\ \cdots,\ x_n$ as functions of the independent real variable t] of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \cdots, x_n), \\ \dot{x}_2 &= f_2(x_1, \cdots, x_n), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \cdots, x_n), \end{aligned} \tag{5}$$

for $t \in I \subseteq \mathbb{R}$, $(x_1, \cdots, x_n)^T \in \mathbb{R}^n$, and where $f_i : \mathbb{R}^n \to \mathbb{R}\ \ (i = 1, \cdots, n)$ are continuous (cts) functions with cts first partial derivatives, and $x_i : I \to \mathbb{R}\ \ (i = 1, \cdots, n)$ are differentiable functions on the closed interval I.

On writing the dynamical system (5) in vector form we obtain

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad (\mathbf{x}, t) \in \mathbb{R}^n \times I, \tag{6}$$

where

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_n(\mathbf{x}))^T.$$

We define the solution of a dynamical system as

Definition 7. Solution to Dynamical System

A solution of (5) on I is a set of differentiable functions

$$x_1^*(t),\ x_2^*(t), \cdots, x_n^*(t) : I \to \mathbb{R},$$

which satisfy (5) identically throughout I.

We can write an n^{th}-order ($n > 1$) differential equation as as system of n first order ordinary differential equations. Consider the n^{th}-order equation

$$x^{(n)} = f\left(x, x', \cdots, x^{(n-1)}\right). \tag{7}$$

Equation (7) can be written as the system of n first order ODEs, namely

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\vdots \\ x_{n-1}' &= x_n, \\ x_n' &= f\left(x_1, \cdots, x_n\right). \end{aligned}$$

Example: Van der Pol equation

Take the Van der Pol equation (2) which is of the form $\ddot{x} = f\left(x, \dot{x}\right)$. We can replace the first order derivative $\dot{x}$ by y to obtain the system of two first order differential equations

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -x - 3\left(x^2 - 1\right) y. \end{aligned}$$

Note:

We note that any nonautonomous system

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, t\right),$$

with $\mathbf{x} \in \mathbb{R}^n$ can be written as an autonomous system of the form (5) with $\mathbf{x} \in \mathbb{R}^{n+1}$ by letting $x_{n+1} = t$ and $\dot{x}_{n+1} = 1$.

For example, consider the nonautonomous first order differential equation

$$\dot{x} = x + t.$$

On letting $x_1 = x$ and $x_2 = t$, we obtain the autonomous system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2, \\ \dot{x}_2 &= 1. \end{aligned}$$

In this module we will be studying first order dynamical systems (one equation), also referred to as dynamical systems on a line, as well as two dimensional dynamical systems (two equations) which are referred to as dynamical systems on the plane.

0.2 Examples of real-life inspired dynamical systems

In this section, we will illustrate the importance of dynamical systems in modelling real-life problems. The intention is to provide context and motivation, and there is no expectation that you would memorise these examples. All examples are taken from "Differential Equations" by James E. Brannan and William E. Boyce.

0.2.1 Examples of one-dimensional dynamical systems

► Example 1 ◄ Mass falling under gravity with drag

The motion of a small mass falling under gravity with drag is described by the one-dimensional dynamical system

$$m\dot{v} = mg - \gamma v, \tag{8}$$

where m is the mass of the object, v is the velocity of the falling object at time t, g is the acceleration due to gravity and γ is the drag coefficient.

If the object is larger, and hence falling more rapidly, the dynamical system

$$m\dot{v} = mg - \gamma v^2, \tag{9}$$

is a better approximation.

► Example 2 ◄ Population of field mice with owls present

The population of field mice in an environment where owls are present can be modelled by the one-dimensional dynamical system

$$\dot{p} = rp - k, \tag{10}$$

where p is the population of the field mice at time t, r is the growth rate of the field mice population and k is the predation rate by the owls, which here is assumed to be constant. In the absence of any predators, the population is assumed to be growing proportionally to the current population, with constant growth rate r.

► Example 3 ◄ The internal temperature in a building

The internal temperature in a building can be modelled by the one-dimensional dynamical system

$$\dot{u} = -k\,(u - T(t))\,, \tag{11}$$

where u is the internal temperature at time t, k is a positive constant and $T(t)$ is the external temperature.

► Example 4 ◄ The amount of a drug present in the bloodstream

The amount of a drug, administered intravenously, in the bloodstream can be modelled by the one-dimensional dynamical system

$$\dot{q} = \alpha - \beta q, \tag{12}$$

where q is the amount of the drug in the bloodstream, α is the amount of the drug being administered intravenously and β is the constant rate at which the drug is absorbed by body tissues.

► Example 5 ◄ Radioactive decay

A radioactive material typically disintegrates at a rate proprotional to the amount currently present, so is modelled by the one-dimensional dynamical system

$$\dot{Q} = -rQ, \tag{13}$$

where Q is the amount of the radioactive material present at time t and r is the constant decay rate.

▶ Example 6 ◀ Electric circuit

The charge q on the capacitor in an electric circuit containing a capacitor, resistor and a battery can be modelled by the one-dimensional dynamical system

$$R\dot{q} = V - \frac{q}{C}, \tag{14}$$

where R is the resistance, V is the constant voltage supplied by the battery and C is the capacitance.

▶ Example 7 ◀ Mortgage

The balance on a mortgage at time t, $S(t)$, can modelled by the one-dimensional dynamical system

$$\dot{S} = rS - 12k, \tag{15}$$

where r is the annual interest rate and k is the monthly payment rate.

▶ Example 8 ◀ Logistic population growth

Population growth is often modelled by exponential growth, but when the population becomes too large, the growth rate will be impacted because of a shortage of a variety of necessary resources. A typical model for such population growth uses the dynamical system

$$\dot{y} = r\left(1 - \frac{y}{K}\right)y, \tag{16}$$

where y is the population of a given species at time t, r is the intrinsic growth rate and K is the saturation level, i.e. the upper bound for a growing population. It is also referred to as the environmental carrying capacity for the given species. Equation (16) is often referred to as the logistic equation or the Verhulst equation.

▶ Example 9 ◀ **Logistic population growth with a threshold**

Some species need a sufficiently large population (above a threshold T) to enable growth. The population of such species can be modelled by the dynamical system

$$\dot{y} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y, \tag{17}$$

where y is the population of a given species at time t, $r > 0$ is the intrinsic growth rate, K is the saturation level, i.e. the upper bound for a growing population, and T is the threshold below which the population will decline. Note that $0 < T < K$.

▶ Example 10 ◀ **Tumor growth**

An alternative population growth model, also used for bacterial growth or cancer cell growth uses the dynamical system

$$\dot{y} = -r \ln\left(\frac{K}{y}\right)y, \tag{18}$$

where y is the tumor size at time t, $r > 0$ specifies growth in proportion to cell population size and $K > 0$ is the carrying capacity. Equation (18) is also known as the Gompertz equation.

▶ Example 11 ◀ **Spread of diseases**

Dynamical systems are also used in the study of epidemology, i.e. the spread of contagious diseases. A simple appraoch uses the dynamical system

$$\dot{y} = -\alpha y\left(1 - y\right), \tag{19}$$

where y is the proportion of infectious individuals at time t and α is a positive proportionality constant.

▶ Example 12 ◀ Bernoulli's model for smallpox

David Bernoulli modelled smallpox back in 1760 to check the effectiveness of the new inoculation programme. The model describes well any disease that yields lifelong immunity to any survivors. It uses the dynamical system

$$\dot{y} = -\alpha y \left(1 - \beta y\right), \tag{20}$$

where y is ratio of the number of people who have not had smallpox at time t over the number of people who have survived smallpox for t years, α is the rate at which smallpox is contracted and β is the rate at which people who contract smallpox die from the disease.

▶ Example 13 ◀ Chemical Reactions

Consider the chemical reaction

$$P + Q \to X, \tag{21}$$

whereby the compounds P and Q are combined to form compound X. This can be modelled by the dynamical system

$$\dot{x} = \alpha \left(p - x\right)\left(q - x\right), \tag{22}$$

where x is the concentration of compound X at time t, α is the rat at which the reaction occurs, p is the initial concentration of compound P and q is the initial concentration of compound Q.

▶ Example 14 ◀ Fish stock

Consider a species of fish, whose population is described by the Verhulst equation (16), but where the fish is harvested. The dynamical system describing such situation is given by

$$\dot{y} = r\left(1 - \frac{y}{K}\right) y - H(y,t), \tag{23}$$

where $H(y,t)$ is the harvesting rate. Where the harvesting is in proportion to the size of the fish population, one can say that $H(y,t) = Ey$, where E is the total effort made to harvest the species of fish. The model can than be described as a "constant effort harvesting" model. If, on the other hand, one harvest a set quota, irrespective of the fish's population size, $H(y,t) = h$, with h the constant harvesting rate.

▶ Example 15 ◀ Dispensing of fluids using a drip dispenser

In the design of drip dispensers, which should allow water to drip out at a constant rate, Torricelli's principle leads to the dynamical system

$$A(h)\dot{h} = -\alpha a\sqrt{2gh}, \tag{24}$$

where h is the height of the liquid surface above the dispenser outlet at time t, $A(h)$ is the cross-sectional area of the dispenser at height h, a is the area of the outlet and α is a measured contraction coefficient that accounts for the observed fact that the cross section of the smooth outlet flow stream is smaller than a. g is the usual gravitational constant.

▶ Example 16 ◀ Ground water contamination

Near industrial sites, we often find toxic compounds in the environment, and in some cases they can filter to beneath the natural water table and contaminate the ground water. The dynamical system,

$$\dot{m} = -\alpha m^{\beta}, \tag{25}$$

is used to model this, where m is the total mass of the toxic compound in the source region, α is a constant ratio of various parameters of the initial state and $\beta > 0$ is an empirically derived constant describing the relationship between the concentration of dissolved toxic compound leaving the source region and the total mass of the toxic compound in the source region.

0.2.2 Examples of two-dimensional dynamical systems

▶ Example 17 ◀ Predator-prey model

In the study of the population of a population $x(t)$ of an animal species which are prey to a predator species $y(t)$, both populations can be modelled by the Lotka-Volterra equations, which form a two-dimensional dynamical system:

$$\begin{aligned} \dot{x} &= ax - \alpha xy, \\ \dot{y} &= -cy + \gamma xy. \end{aligned} \tag{26}$$

The constants a, α, c and γ come from empirical observations and are species-specific.

▶ Example 18 ◀ **Non-linear pendulum**

The angle an oscillating pendulum makes with the vertical direction, θ, satisfies the second order equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin(\theta) = 0, \tag{27}$$

with g the gravitational constant and L the length of the pendulum. This can be written as a two-dimensional dynamical system by setting $\theta(t) = x(t)$, and introduce a second variable $y(t)$ such that

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{g}{L}\sin(x). \end{aligned} \tag{28}$$

This is an example of a non-linear two-dimensional dynamical system.

When damping is added, e.g. by friction in the rotating joint, a damping terms needs to be added, as in

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{g}{L}\sin(x) - \frac{c}{mL}y, \end{aligned} \tag{29}$$

where m is the mass of the pendulum and c is a damping coefficient.

▶ Example 19 ◀ **Spread of diseases**

Some diseases, like COVID-19, are largely spread by carriers, people who can transmit the disease but exhibit no symptoms themselves. The dynamical system modelling this situation is given by

$$\begin{aligned} \dot{x} &= -\alpha x y, \\ \dot{y} &= -\beta y, \end{aligned} \tag{30}$$

where x is the proportion of those still susceptible to catch the virus, and y the proportion of carriers in the population. α is a constant of proportionality indicating the rate at which the disease spreads whilst β is the rate at which carriers are removed from the population.

▶ Example 20 ◀ **Bernoulli's model for smallpox revisited**

Bernoulli's model for the impact of smallpox can be modelled also by the two-dimensional dynamic system

$$\begin{aligned} \dot{x} &= -\left(\beta + \mu(t)\right) x, \\ \dot{n} &= -\nu\beta x - \mu(t) n, \end{aligned} \tag{31}$$

, where x is the number of people who have not had smallpox at time t, n is the number of people who have survived smallpox for t years, β is the rate at which smallpox is contracted and ν is the rate at which people who contract smallpox die from the disease. $\mu(t)$ is the death rate from all causes other than smallpox. One can recombine the two equations in the two-dimensional dynamical system to form the one-dimensional system (20).

▶ Example 21 ◀ **Interconnected tanks**

Hee, we consider the flow os salty water through two interconnected tanks. The total volume in each tanks remains the same, as the total flows in and out of each tank remain the same. The amount of salt in tank 1 is given by $Q_1(t)$ and the amount of salt in tank 2 is given by $Q_2(t)$. Their behaviour over time can be modelled by the two-dimensional dynamical system

$$\begin{aligned} \dot{Q}_1 &= -aQ_1 + bQ_2 + c, \\ \dot{Q}_2 &= dQ_1 - eQ_2 + f, \end{aligned} \tag{32}$$

where a is the rate at which salt is drained from tank 1, b is the rate at which salt is flowing from tank 2 to tank 1, c is the rate at which salt is added to tank 1, d is the rate at which salt is flowing from tank 1 to tank 2, e is the ratew at which salt is drained from tank 2 and f is the rate at which salt is added to tank 2.

▶ Example 22 ◀ **Parallel LRC circuit**

A parralel LRC circuit, consisting of a capacitor, resistor and inductor can be modelled through the two-dimensional dynamical system

$$\begin{aligned} C\dot{v} &= -i - \frac{v}{R}, \\ \dot{i} &= v, \end{aligned} \tag{33}$$

where $v(t)$ is the voltage in the capacitor, $i(t)$ is the current through the inductor, C is the capacitance and R is the resitance.

▶ Example 23 ◀ **Satellite attitude control system**

In a simple model, the attitude control system of a satellite can be modelled by a two-dimensional dynamical system,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\alpha x - \beta y, \end{aligned} \tag{34}$$

where $x(t)$ is proper attitude angle. The thrustor torque is given by $-\alpha x - \beta y$ where α and β are so-called gain constants.

► Example 24 ◄ Two-compartment model in physiology

When administering a drug, some tissues rapidly diffuse with the drug whilst others react more slowly. Thisa is often modelled by creating two "compartments" for the different type of tissues, one for blood and other fluids that interact quickly with the drug, and another for the slower reacting tissue. If we call x the concentration of the drug in the first compartment (blood) and y the concentration of the drug in the second compartment (tissue), than we can use the two-dimensional dynamical system,

$$\begin{aligned} \dot{x} &= -(k_{01}+k_{21})\,x + k_{12}y + d(t), \\ \dot{y} &= k_{21}x - k_{12}y. \end{aligned} \tag{35}$$

Here k_{12} is the fraction per unit time of drug transfered from compartment 1 to compartment 2, k_{12} is the fraction per unit time of drug transfered from the second compartment to the first and k_{01} is the fraction per unit time of drug eliminated from the system. $d(t)$ is the rate at which the drug is administered into the bloodstream.

► Example 25 ◄ Population of competing species

When two species live in the same confined environment, competing for the same resources as food or water, a possible model, assuming both species individually can be modelled using the logistic dynamical system (16), the joint populations can be modelled by a two-dimensional dynamical system,

$$\begin{aligned} \dot{x} &= r\left(1 - \frac{x}{K} - \alpha y\right)x, \\ \dot{y} &= s\left(1 - \frac{y}{K} - \beta x\right)y, \end{aligned} \tag{36}$$

where α and β are a measure of the degree to which species x interferes with species y and vice versa.

▶ Example 26 ◀ Brusselator

Certain chemical reactions can exhibit oscillatory behaviour. One model, called the Brusselator, can be described by the two-dimensional dynamical system,

$$\begin{aligned} \dot{x} &= \alpha - (1+\beta)\,x + x^2 y, \\ \dot{y} &= \beta x - x^2 y, \end{aligned} \tag{37}$$

where α and β are some constants.

▶ Example 27 ◀ Fitzhugh-Nagumo model in neuroscience

The Fitzhugh-Nagumo equations model the transmission of neural impulses along a axon. It is described by the two-dimensional dynamical system,

$$\begin{aligned} \dot{v} &= v - \frac{v^3}{3} - w + e, \\ \tau \dot{w} &= v + a - bw, \end{aligned} \tag{38}$$

where v and w are the two variables, e is the external stimulus. τ, a and b are given constants.

Chapter 1

Dynamical systems on the line

1.1 Introduction

In this chapter, we'll discuss linear system of one dimension, also referred to as dynamical systems on a line. These are characterised by the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y). \tag{1.1}$$

We'll restrict ourselves to autonomous sysems as non-autonomous one-dimensional systems can be written as autonomous systems in two dimensions, which we'll discuss later.

We should remember that the general solution of any first order ODE (before imposing any initial or boundary conditions) is defined up to an arbitrary constant of integration. In fact, if we have a system of n first order ODEs, then we obtain n arbitrary constants, and thus require n **initial** or **boundary** conditions in order to obtain a unique solution to the system. Imposing a particular condition (or set of conditions) on our problem(s) thus impacts what our solution will look like. The general solution of the system can be thought of as representing **all** possible solutions to the problem.

Let us consider the following example to help us understand the approach taken in this chapter and course.

1.1.1 Example

Consider the following dynamical system on the line:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y. \tag{1.2}$$

which can easily be solved as

$$\begin{aligned} \frac{\mathrm{d}y}{y} &= -\mathrm{d}t \\ \Leftrightarrow ln(y) &= -t + c \\ \Leftrightarrow y(t) &= e^{-t}e^{c} \\ \Leftrightarrow y(t) &= Ce^{-t}, \end{aligned}$$

where $C \in \mathbb{R}$ is a undetermined parameter.

As indicated above, we obtain an infinte set of solutions. If we have an initial condition, we can fix the unknown C to get the unique solution that satisfies both equation (1.2) and the stated initial condition. So for $y(0) = 2$, we obtain that $C = 2$ and the unique solution is given by

$$y(t) = 2e^{-t}.$$

.

A plot of the solutions is given for $C = -2, -\frac{1}{2}, 0\frac{1}{2}$, and, 2 in Figure (1.1).

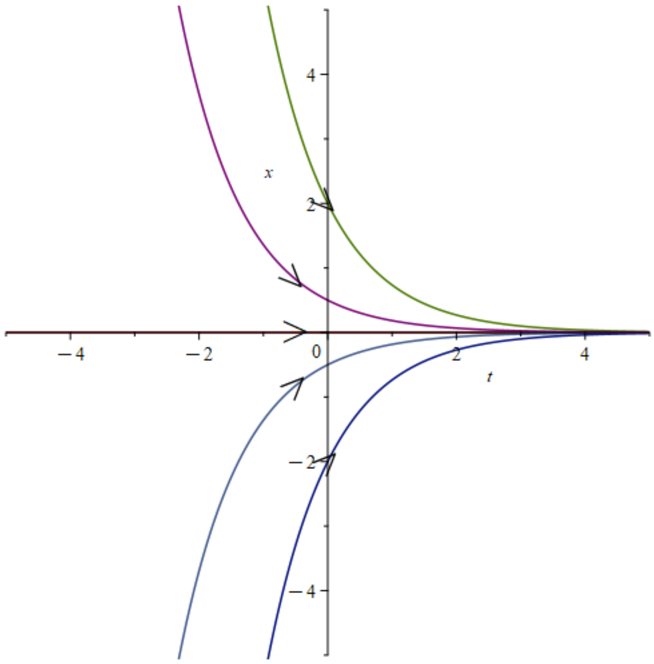


Figure 1.1: Plot of some solutions of the linear dynamical system (1.2).

If we study these solutions, we can see from the Figure (1.1) that there is one solution ($y = 0$) which is constant over time, hence $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$, and, that all other solutions tend to 0 when time goes to infinity:

$$\lim_{t \to \infty} y(t) = 0.$$

Now often we cannot easily find the analytic solution to the dynamical system. Equation (1.2) merely indicates the slope of the solution at a particular value of y. Note that the slope of the solution curve is the same for a given value of y, hence for all different solutions passing through this value of y. In Figure (1.2), we have added some of those slopes unto the solutions.

In the case that we do not know the analytic solutions, we're only left by the slopes. It is more helpful to plot these at more values of y and repeat them for different times t (even though they do not depend on t explicitly.)

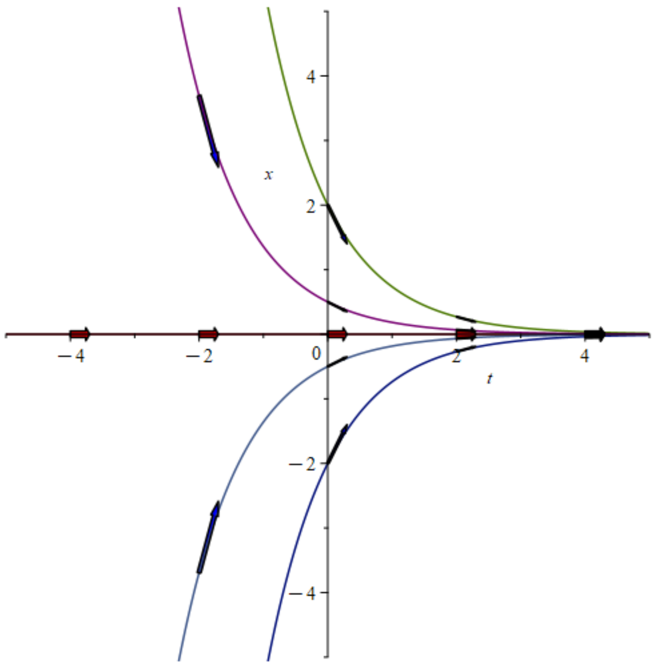


Figure 1.2: Plot of some solutions with a selection of slopes of the linear dynamical system (1.2).

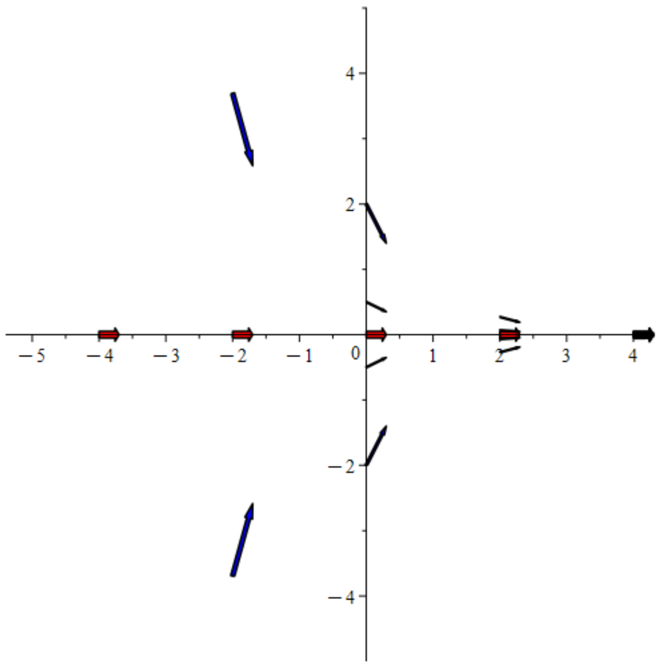


Figure 1.3: Plot of some slopes as given in the linear dynamical system (1.2).

This is what we will study in this chapter. We'll need to understand how the slopes vary with y in general, and consider the special cases where the slope is zero, and therefore

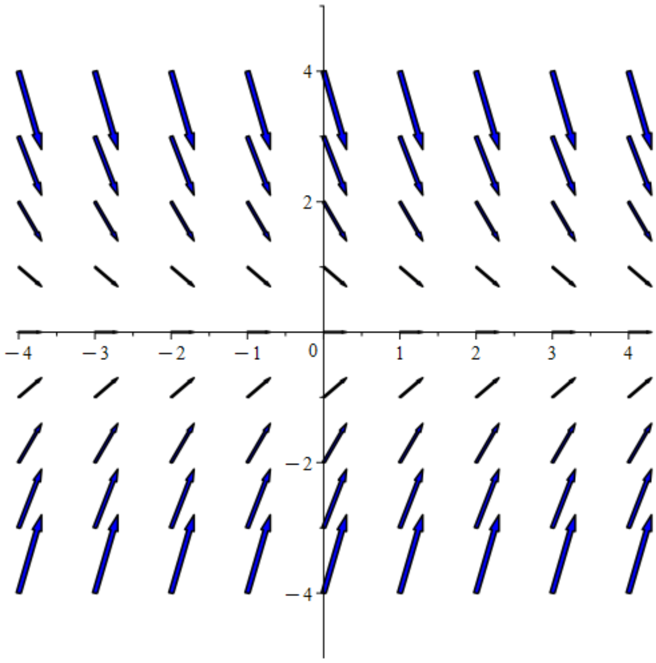


Figure 1.4: Plot of many slopes as given in the linear dynamical system (1.2).

the solution is a constant.

Definition 8. Direction field
A direction field is a set of directed slopes as given by a linear dynamical system.

So, Figure (1.4) is an example of a direction field.

We can use a direction field to sketch a solution, by reminding ourselves that in a first approximation

$$y\left(t_\circ + \delta t\right) = y\left(t_\circ\right) + \delta t \dot{y},$$

where,

$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t}.$$

.

So we can start at a point and draw the solution in small steps, following the slope at each point we're at.

1.2 Equilibria and Phase lines

Definition 9. Equilibrium Point *The equilibrium points of the first order ordinary differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y) \tag{1.3}$$

are those values for y *for which*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0. \tag{1.4}$$

In other words, those solutions which do not change (are constant) for all t. We also call these points **fixed**, **critical**, or **singular** points of the system.

Phase lines provide an easy way to understand how the solutions to first order autonomous differential equations behave. We can construct the phase line for the general first order ODE (1.1) by following these steps:

1. Find the **equilibrium points** of the system.
2. Either side of each of the equilibrium points, work out the sign of $\frac{\mathrm{d}y}{\mathrm{d}t}$.
3. Draw a line representing the y-axis and label the equilibrium points. Then, since $\frac{\mathrm{d}y}{\mathrm{d}t}$ can only change sign at an equilibrium point (if at all), we can draw an arrow indicating the sign of $\frac{\mathrm{d}y}{\mathrm{d}t}$ between the equilibrium points.

▶ Example 28 ◀ Phase Line

Consider the problem,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^3 - y = -y\,(1-y)\,(y+1)\,. \tag{1.5}$$

This has three equilibrium points, located at $y = 0$, 1 and -1. We can then find the sign of $\dot{y}$ between the equilibrium points:

	$-y$	$1-y$	$y+1$	$\dot{y}$
$y > 1$	$-$	$-$	$+$	$+$
$0 < y < 1$	$-$	$+$	$+$	$-$
$-1 < y < 0$	$+$	$+$	$+$	$+$
$y < -1$	$+$	$+$	$-$	$-$

Looking at our example phase line (Figure 1.5), we can clearly see how solutions to (1.5) with different initial conditions ($y\,(0) = y_0$) evolve.

For example, when $y_0 > 1$ the solution will grow to infinity with increasing t, and if $y_0 < -1$ then the solution will decay to minus infinity. However, if we pick y_0 such that $-1 < y_0 < 0$ (or $0 < y_0 < 1$) then the solution will decay (or grow) to zero. If we choose our initial condition to be one of the equilibrium points, then the solution will remain constant for all t.

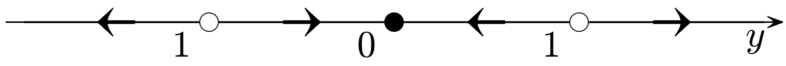


Figure 1.5: Plot of the phase line for (1.5)

▶ Example 29 ◀ Phaseline for Nonlinear ODE
Consider the problem,

$$\dot{x}\,(t) = \sin x, \quad x\,(0) = x_0. \tag{1.6}$$

This is one example of a **non-linear** first order ODE which can be solved exactly. The solution is given by

$$x = 2\arctan(Ce^t), \tag{1.7}$$

where C is an arbitrary constant. This solution is exact, but how easily can you obtain information from it? It is quicker to get information from a phase line. The equilibrium solutions of (1.6) are given by $x = n\pi$, $n \in \mathbb{Z}$, and we can plot the right-hand-side of equation (1.6) along with its horizontal phase line in Figure 1.6.

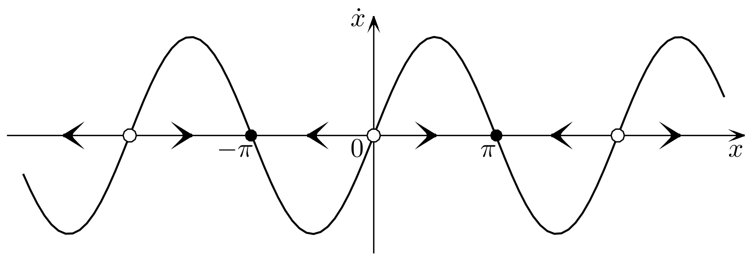


Figure 1.6: Plot of the phase line for (1.6)

We can see that the behaviour of the solutions to (1.6) is much clearer when presented in graphical form (Figure 1.6) rather than exact form (1.7).

Imagine that the system (1.6) describes a physical process, a fluid flow for example, what does Figure 1.6 tell us about the behaviour of the process?

- The **flow** is to the right when $\dot{x} > 0$, and to the left when $\dot{x} < 0$

- The greater the value of $|\dot{x}|$, the greater the **velocity** of the flow
- When $\dot{x} = 0$ the velocity of the flow is 0, these are called **equilibrium points** (or stationary points, etc)

We see that there are two types of **fixed points** in our problem:

- Solid circles - Where the 'fluid' flows towards the equilibrium points. These are **stable** equilibrium points (often called attractors or sinks).
- Open circles - Where the 'fluid' flows away from the equilibrium points. These are **unstable** equilibrium points (often called repellers or sources).

We can now use Figure 1.6 to answer questions about what will happen to the solution to the problem (1.6) for different initial conditions. For example,

- $-\pi < x_0 < 0$: If we choose any x_0 in this range, then we see that the solution will decay, with $x(t) \to -\pi^+$ as $t \to \infty$.
- $x_0 = 0$: If $x_0 = 0$, then $x(t) = 0$ for all t.
- $0 < x_0 < \pi$: If we choose any x_0 in this range, then we see that the solution will grow, with $x(t) \to \pi^-$ as $t \to \infty$.

This allows us to build up a picture of the behaviour of solutions to the system (1.6) for any given initial condition x_0.

We can extend these ideas to any one-dimensional system $\dot{x} = f(x)$. We just need to plot $f(x)$ against x, and then use this to sketch the vector field on the real line.

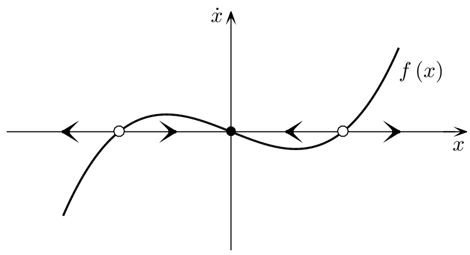


Figure 1.7: Equilibrium points and stability for $\dot{x} = f(x)$

We can again imagine $f(x)$ as describing the flow of some imaginary fluid along the real line. We call this fluid the **phase fluid**, and the real line the **phase space**. If $f(x) > 0$ then the flow is to the right, and if $f(x) < 0$ then the flow is to the left. To find a solution to $\dot{x} = f(x)$ we place an imaginary particle (**phase point**) at an initial condition x_0 and see how it is carried along by the flow.

What we have learnt so far is a graphical method to determine the stability of fixed points. However it will be useful to have a quantitative method to establish stability. To do this we use **linear stability analysis**.

1.3 Stability of equilibrium points

First, we'll give a more formal definition of a stable equilibrium point.

Definition 10. Asymptotic stability. *An equilibrium point x_e is said to be asymptotically stable if there exists a $\nu > 0$ such that for all $x(0)$ with*

$$|x(0) - x_e| < \nu,$$

then,

$$|x(t) - x_e| \to 0 \quad as \quad t \to \infty.$$

Consider the system

$$\dot{x} = f\left(x\right), \tag{1.8}$$

which has an equilibrium point $x = x^*$, and let $\eta\left(t\right)$ be some small perturbation away from the point x^*, so that

$$x\left(t\right) = x^* + \eta\left(t\right). \tag{1.9}$$

To investigate whether this perturbation grows or decays (whether the solution moves away from, or collapse onto, the equilibrium point) we need to consider the rate of change of η with respect to time.

$$\dot{\eta} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x - x^*\right) = \dot{x}, \tag{1.10}$$

since x^* is constant. From this we can deduce that

$$\dot{\eta} = f\left(x\right) = f\left(x^* + \eta\left(t\right)\right). \tag{1.11}$$

Taylor expanding $f\left(x^* + \eta\right)$ about our equilibrium point $x = x^*$ we find that

$$\dot{\eta} = f\left(x^*\right) + \eta f'\left(x^*\right) + \cdots \tag{1.12}$$

Now, provided that $f'\left(x^*\right) \neq 0$, then we can ignore the smaller terms in the Taylor expansion and write

$$\dot{\eta} \approx \eta f'\left(x^*\right). \tag{1.13}$$

Equation (1.13) is linear in η, and is called the **linearisation about** the point x^*. Since η is our small perturbation from our equilibrium point, and $\dot{\eta}$ denotes how the perturbation changes with time. This is a very useful result. Indeed, we can solve (1.13) explicitly

$$\eta(t) = \eta_0 \exp(f'(x^*)t).$$

This tells us that, if $f'(x^*) > 0$:

$$\eta(t) \to +\infty, \text{ if } \eta_0 > 0, \text{ and } \eta(t) \to -\infty, \text{ if } \eta_0 < 0,$$

hence the perturbation will grow in modulus over time, namely x^* is an **unstable** equilibrium point. If instead $f'(x^*) < 0$:

$$\eta(t) \to 0,$$

the perturbation will decrease in modulus over time, namely x^* is a **stable** equilibrium point.

Thus in order to classify the stability of equilibrium points on the line, we need only consider the gradient f' at x^*.

If $f'(x^*) = 0$ then linear stability analysis fails and we need to consider other arguments to establish stability. Consider the phase lines in Figure 1.8. We can clearly see that the equilibrium points are (a) unstable and (b) stable. The final plot, $f(x) = x^2$, is neither stable nor unstable. We call this type of equilibrium point **half-stable**.

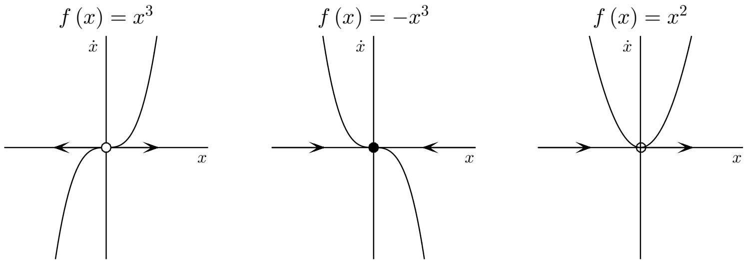


Figure 1.8: Examples where linear stability analysis fails

► Example 30 ◄ Linear stability analysis

Using linear stability analysis, determine the stability of the equilibrium points for the nonlinear ODE (1.6),

$$\dot{x}(t) = \sin x, \quad x(0) = x_0. \tag{1.14}$$

We have previously shown that the equilibrium points are given by $x^* = n\pi$, $n \in \mathbb{Z}$, and have found the stability of these equilibrium points using graphical arguments. Now let us consider the linear stability approach. We have that,

$$f'(x) = \cos x, \tag{1.15}$$

and so, at the equilibrium points $x = x^*$,

$$f'(x^*) = f'(n\pi) = \cos(n\pi), \tag{1.16}$$

$$= \begin{cases} 1 & \text{for } n \text{ even,} \\ -1 & \text{for } n \text{ odd.} \end{cases} \tag{1.17}$$

Hence, those equilibrium points $x^* = n\pi$ with n odd are stable, and those with n even are unstable. As we can see, this agrees with the graphical argument used to construct Figure 1.6.

1.4 Direction fields

One of the disadvantages of phase lines is that they only tell us about direction of flow, they give no information about the rate of change of solutions. For example, we have shown that a solution to $\dot{x}\,(t) = \sin x$ with initial condition $x(0) = x_0 = 1$ will approach $x = \pi$ as $t \to \infty$, but we don't know how the path it takes to get there. To find this information, we need to know the direction of the flow at many points in time.

A given phase point will move with time according to some function $x\,(t)$ (defined by $\dot{x}\,(t) = f\,(x\,(t))$. The path traced out by the movement of the phase point is called a **trajectory**, and represents the solution to the differential equation $\dot{x} = f\,(x)$ with initial condition $x(0) = x_0$. Direction fields provide a way of representing the direction of trajectories at any point in time and space, without needing to solve the differential equation analytically. Take the first order ODE

$$\dot{x} = f\,(x\,(t))\,, \tag{1.18}$$

We can construct the direction field for the (1.18) by picking a point $x\,(t)$ and plotting, at that point, a line segment with gradient $f\,(x\,(t))$. If we repeat this for many points $x\,(t)$, then we build up a picture of the direction field corresponding to all solutions to (1.18).

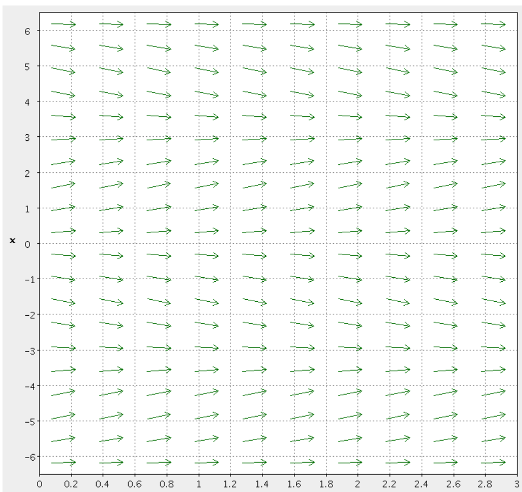


Figure 1.9: The direction field for (1.6)

▶ Example 31 ◀ Let's continue with the problem,

$$\dot{x}\,(t) = \sin x, \quad x\,(0) = x_0. \tag{1.19}$$

We can draw the direction field by considering the value of $\dot{x}$ at each point.

Then, after choosing an initial condition x_0, we can plot the trajectory of $x(t)$ by following the direction of the flow. This allows us to plot the solution of (1.6), as shown in Figure 1.10.

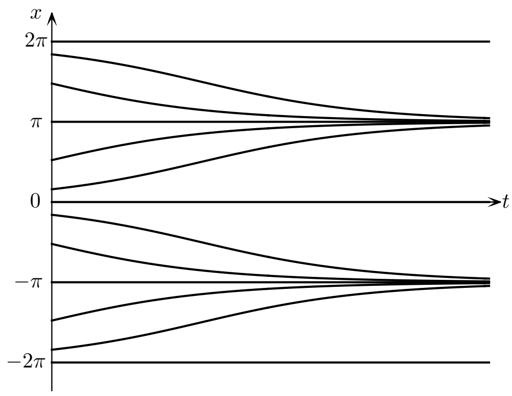


Figure 1.10: Plot of the solution curves for (1.6)

We can see from our sketch that the solution changes significantly depending on what we pick as our initial solution. We can also see that the solution curves never intersect, no matter what we choose as the initial condition, if they did we would no longer have uniqueness of our solution!

A picture showing all possible qualitatively different trajectories of the solution (such as Figure 1.10 is known as a **phase portrait**.

▶ Example 32 ◀ Consider the problem,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y, \quad y \in \mathbb{R},\ t \geq 0, \tag{1.20}$$

which we discussed at the beginning of the chapter.

It is clear that we only have one equilibrium point at $y = 0$. We can also work out the value of $\dot{y}$ at several points, for example at $y = 0$, $y = \pm 1$, $y = \pm 2$ and so on. This allows us to easily plot the direction field for (1.20), which in turn allows us to plot solution curves for a selection of initial conditions.

It is now clear from Figure 1.11 that the only equilibrium point of this system ($y = 0$) is stable.

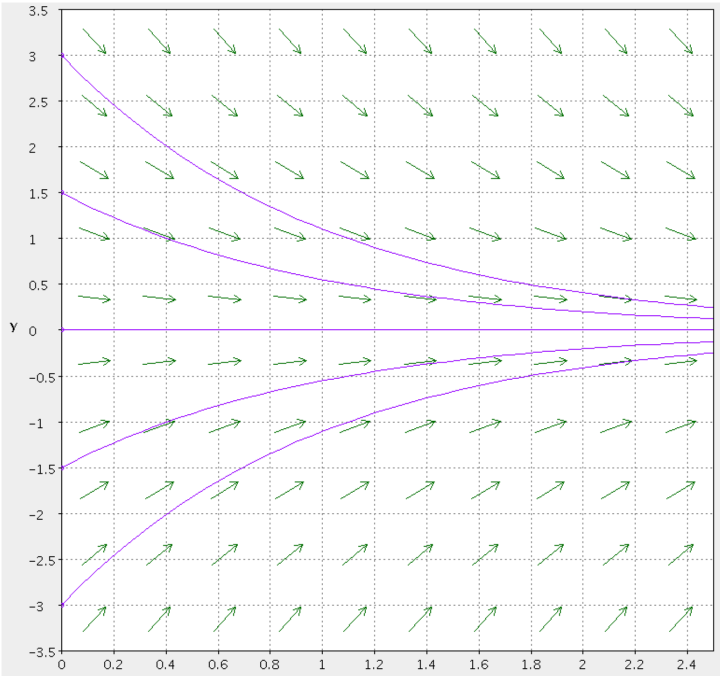


Figure 1.11: Plot of the direction field for system (1.20)

▶ Example 33 ◀ Find all equilibrium points and classify their stability for the system

$$\dot{x} = -x^2 + 3x - 2, \quad x(0) = 0. \tag{1.21}$$

We can write

$$f(x) = -x^2 + 3x - 2 = (2 - x)(x - 1). \tag{1.22}$$

The equilibrium points are given by $f(x) = 0$, i.e. when $x = 1$ and $x = 2$. We can now sketch $f(x)$ and use that to classify the stability of the equilibrium points. It is clear from Figure 1.12 that $x = 1$ and $x = 2$ are unstable and stable equilibrium points respectively, but what do the solution curves look like? We plot them in Figure 1.13.

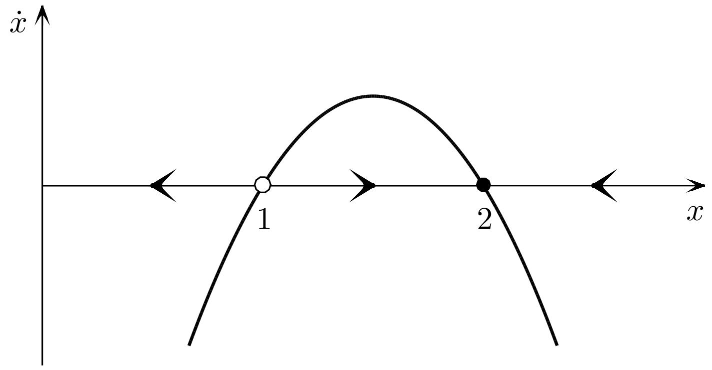


Figure 1.12: Phase line for the system (1.21)

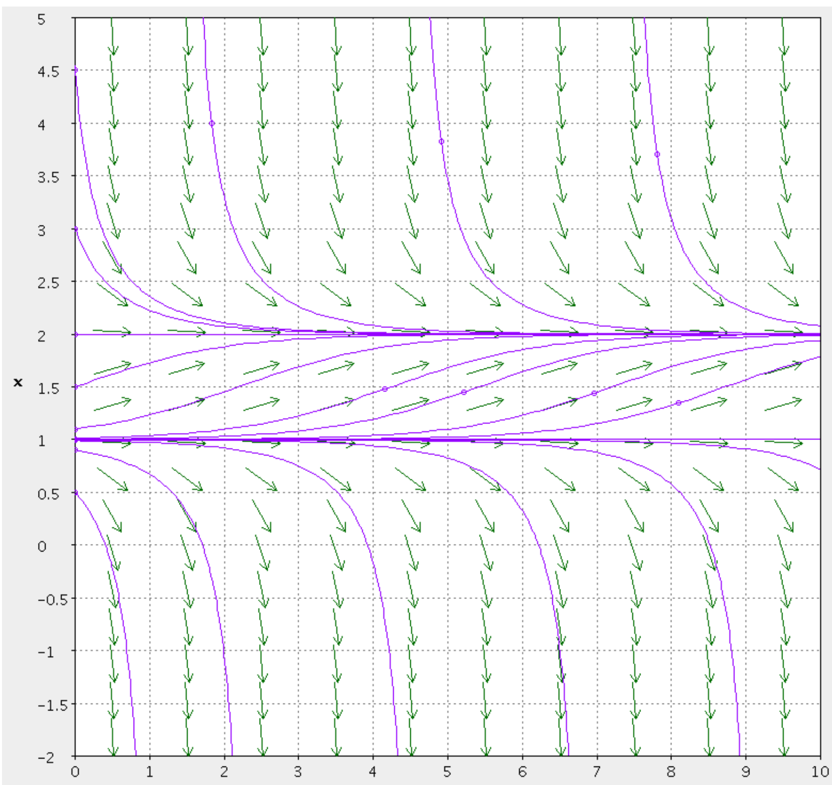


Figure 1.13: Solution curves and direction fields for the system (1.21)

In the following section, we'll be looking at those one-dimensional dynamical systems which have a parameter in the function $f(x)$ on the right hand side, and where the characteristics hence can change with a change in this parameter.

1.5 Bifurcations

As we've shown so far, the behaviour of solutions to dynamical systems on the line is very limited - solutions can only tend to $\pm\infty$ or to an equilibrium point. So what makes them interesting? The answer is their *dependence on parameters*. As we vary parameters in our problem, the qualitative structure of our solutions can change. Such changes in structure are called **bifurcations**, and the points at which they occur are known as **bifurcation points**.

Bifurcations are important when modelling physical systems, they represent the transition between states as some *control parameter* is varied. Consider, for example, the application of force to a beam, if the force applied is too large, the beam will buckle. The bifurcation here is the change in shape of the beam, the control parameter in this problem is the force applied, and the bifurcation point is the force applied in the instant that the beam begins to buckle.

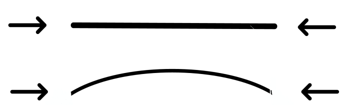


Figure 1.14: Buckling of a beam when force is applied from each side

In this section we will discuss three types of bifurcations, beginning with **Saddle-Node** bifurcations.

1.5.1 Saddle-node bifurcation

Saddle-node bifurcations are the basic mechanism by which equilibrium points are created or destroyed. Consider the first order system

$$\dot{x} = x^2 + c, \tag{1.23}$$

where c, constant, is a control parameter. We see that when c changes sign from negative to positive the solution transitions from having two equilibrium points to having none, as shown in Figure 1.15.

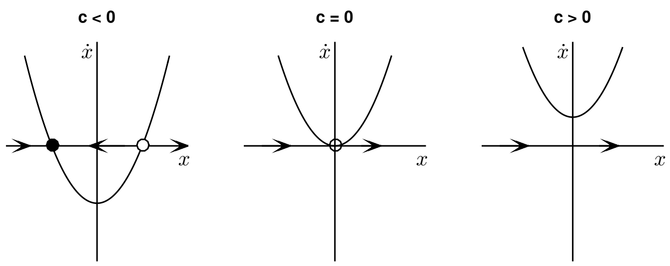


Figure 1.15: The phase line for (1.23)

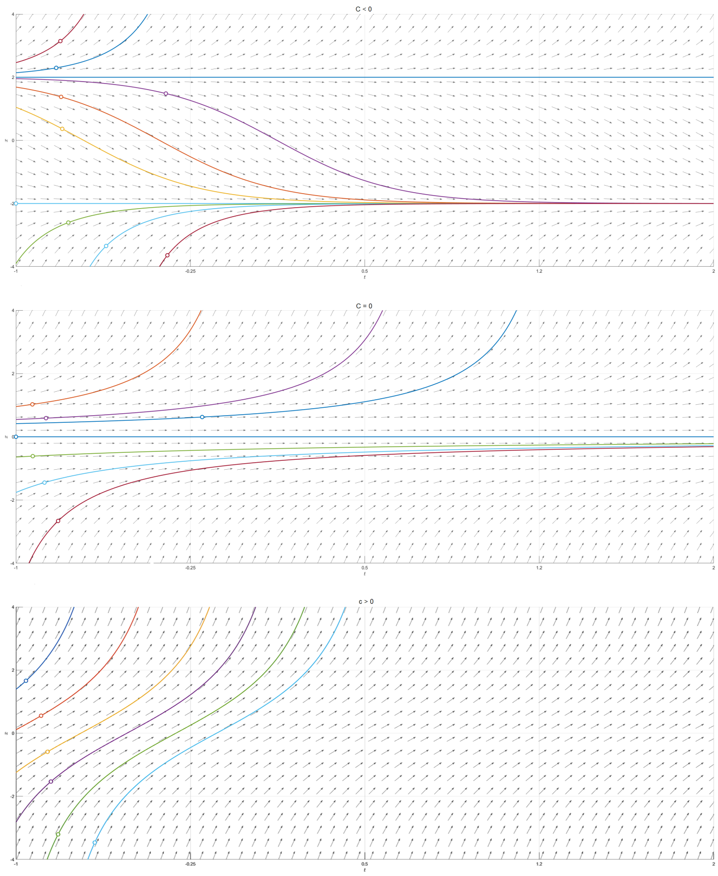


Figure 1.16: Solution & direction field for all case $c < 0$, $c = 0$ and $c > 0$.

The value $c = 0$ at which (1.23) transitions from having two equilibrium points to having none is the bifurcation point of this system.

We can show all the bifurcations of a system by plotting all (c, x) pairs which correspond to the equilibrium points of the system. If we consider the system (1.23), then this is equivalent to plotting all points $c = -x^2$. Such a plot is called the **bifurcation diagram** of the system. Figure (1.17) is the bifurcation diagram for system (1.23).h

Definition 11. Saddle-node Bifurcation *A saddle-node bifurcation is a local bifurcation in which two equilibria collide and annihilate each other.*

$\boxed{\blacktriangleright \textbf{ Example 34 } \blacktriangleleft}$ Consider the dynamical system

$$\dot{x} = 1 + cx + x^2 \tag{1.24}$$

Establish whether there are any bifurcations in the system and where they are located.

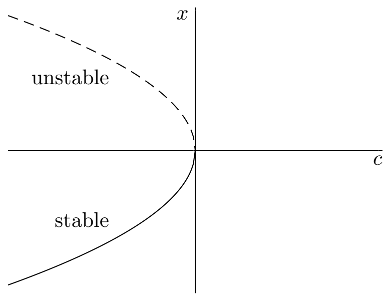


Figure 1.17: Bifurcation diagram for the system (1.23)

The equilibrium points of (1.24) are given by

$$x^* = \frac{-c \pm \sqrt{c^2 - 4}}{2}. \tag{1.25}$$

We have the following cases:

(i) If $c^2 - 4 < 0$ ($-2 < c < 2$), then (1.24) has no equilibrium points

(ii) If $c^2 - 4 = 0$ ($c = \pm 2$), then (1.24) has one equilibrium point

(iii) If $c^2 - 4 > 0$ ($c > 2$, or $c < -2$), then (1.24) has two equilibrium points

We can plot the different cases, and use Figure 1.18 to determine stability. Looking at Figure 1.18 it becomes clear that the point $c = 2$ is a bifurcation point of the system (1.24). Task: Show on a graph that $c = -2$ is also a bifurcation point of (1.24).

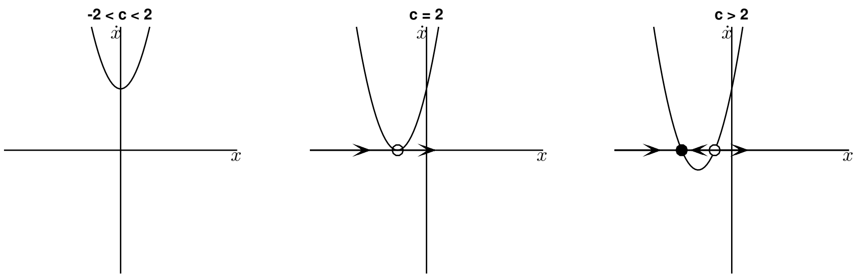


Figure 1.18: The phase line for (1.24)

Notice that

$$\lim_{c \to \pm\infty} \frac{-c + \sqrt{c^2 - 4}}{2} = 0.$$

So if we plot the values of the equilibrium points versus the value of c, we obtain the bifurcation diagram in Figure 1.19, where the green lines indicate a curve of stable equilibrium points, the dashed red lines a curve of unstable equilibrium points and the blue circles the two halfstable equilibria that correspond to the bifurcation points $c = \pm 2$.

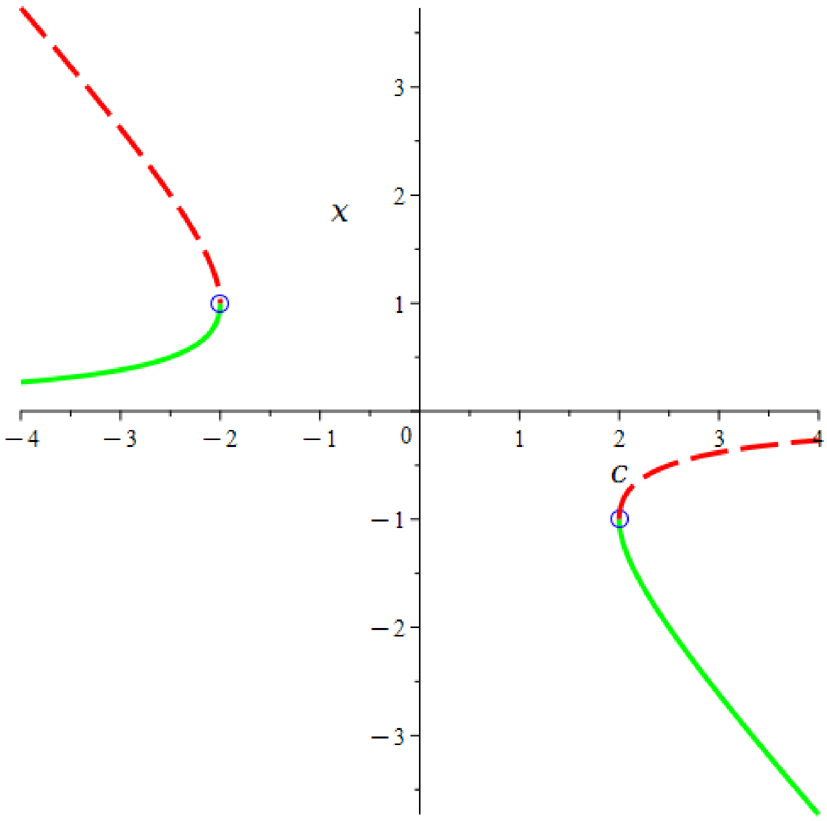


Figure 1.19: The bifurcation diagram for the system (1.24)

1.5.2 Transcritical bifurcation

Some models require that we always have at least one equilibrium point which can never be destroyed. As an example, any attempt to model population growth must always have an equilibrium point when the population is zero! However these 'fixed' equilibrium points can undergo a change of stability, the basic mechanism for such a change is a transcritical bifurcation.

The normal form for a transcritical bifurcation is

$$\dot{x} = cx - x^2, \quad c \text{ constant.} \tag{1.26}$$

We can see that the system (1.26) has an equilibrium point at $x^* = 0$, for all c. From Figure 1.20 we see that this point is stable when $c < 0$, and unstable when $c > 0$. There is a second equilibrium point at $x^* = c$, which is unstable when $c < 0$ and stable when $c > 0$.

Again, the qualitative behaviour of the solutions is quite different for each of these cases, as seen in Figure 1.21.

Because of this behaviour we say that an exchange of stabilities has occurred between the two equilibrium points. The point at which the exchange of stabilities occurs ($c = 0$) is the transcritical bifurcation point.

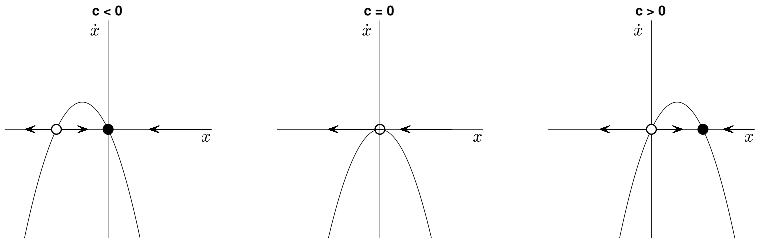


Figure 1.20: Transcritical bifurcation

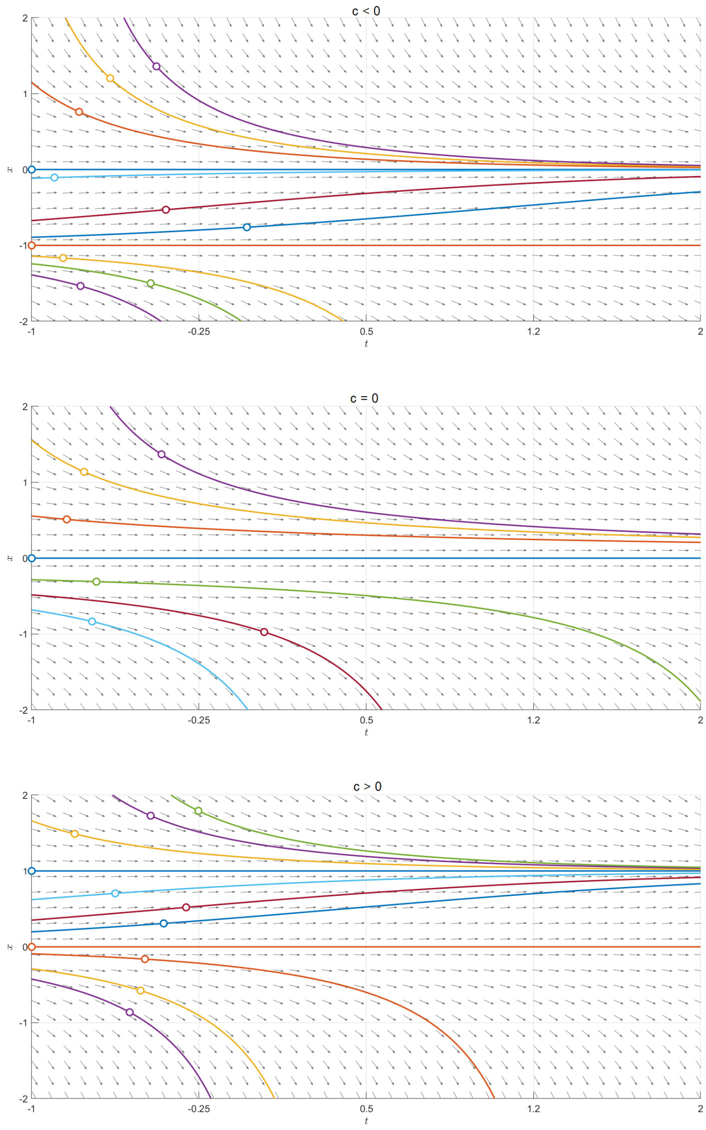


Figure 1.21: Solution & direction field for all case $c < 0$, $c = 0$ and $c > 0$.

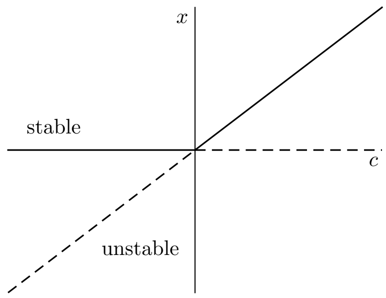


Figure 1.22: Bifurcation diagram for the system (1.26)

The bifurcation diagram for the dynamical system (1.26) is depicted in Figure 1.22.

Definition 12. Transcritical Bifurcation *A transcritical bifurcation is a local bifurcation in which an exchange of stabilities occurs between two equilibrium points.*

1.5.3 Pitchfork bifurcation

The final type of bifurcation we will deal with is the pitchfork bifurcation, which usually occurs in problems where there is some symmetry. For example, many problems have some spatial symmetry between left and right, and in these cases equilibrium points are often created and destroyed in symmetrical pairs. If we consider again the buckling beam in Figure 1.14, when there is equal force applied to each side (below the threshold) we have an equilibrium point corresponding to zero deflection of the beam. However, when enough force is applied for the beam to buckle, then it can either deflect upwards or downwards, resulting in two stable equilibrium points (corresponding to upward or downward deflection of the beam) and a (now unstable) equilibrium point of zero deflection. We will consider two types of pitchfork bifurcation, beginning with the more simple case - supercritical pitchfork bifurcation.

Supercritical pitchfork bifurcation

The normal form for a supercritical pitchfork bifurcation is

$$\dot{x} = cx - x^3. \tag{1.27}$$

We can see that the supercritical pitchfork bifurcation is invariant under the transformation $x \to -x$ (if we replace x by $-x$ in (1.27) we get the same equation) - this is the symmetry we have just discussed.

Figure 1.23 shows the onset of bifurcation. When the control parameter $c \leq 0$, we have a single stable equilibrium point at $x^* = 0$. When $c > 0$ the equilibrium point at $x^* = 0$ becomes unstable with two new stable points appearing at $x^* = \pm\sqrt{c}$.

This leads to quite different qualitative solutions for the carious cases in c, as can be seen in Figure 1.27.

The naming of this type of bifurcation becomes clear when we look at the bifurcation diagram, Figure 1.25.

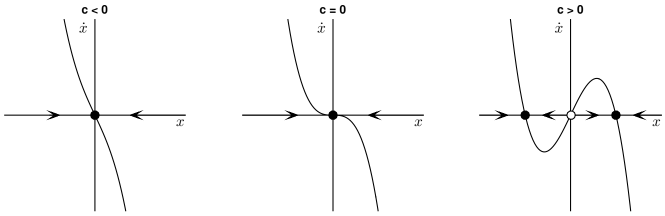


Figure 1.23: Supercritical pitchfork bifurcation

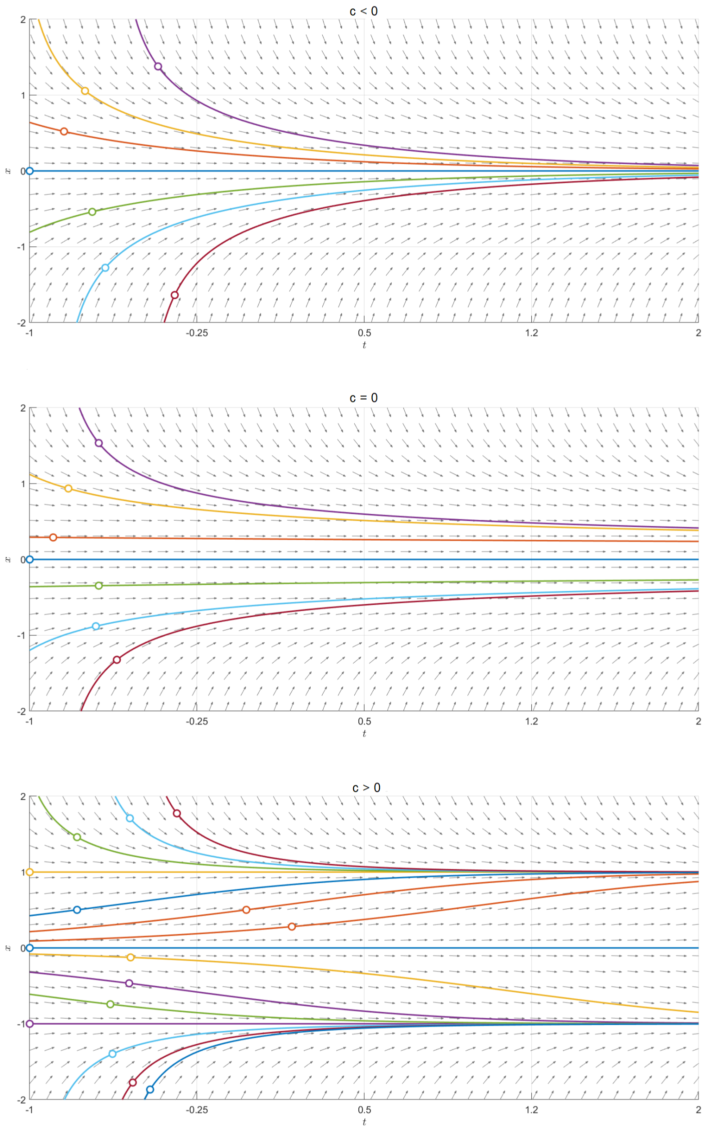


Figure 1.24: Solution & direction field for all case $c < 0$, $c = 0$ and $c > 0$.

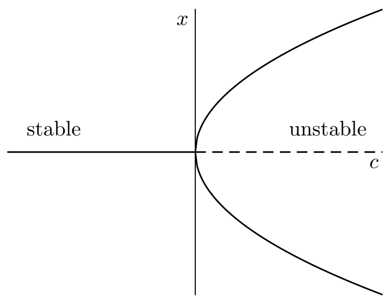


Figure 1.25: Bifurcation diagram for the system (1.27)

Definition 13. Supercritical Pitchfork Bifurcation *A supercritical pitchfork bifurcation is a local bifurcation in which there is a transition from a single stable equilibrium point into three equilibria points where the new symmetric equilibria are stable and the original equilibrium point becomes unstable.*

Subcritical pitchfork bifurcation

The normal form for a subcritical pitchfork bifurcation is

$$\dot{x} = cx + x^3. \tag{1.28}$$

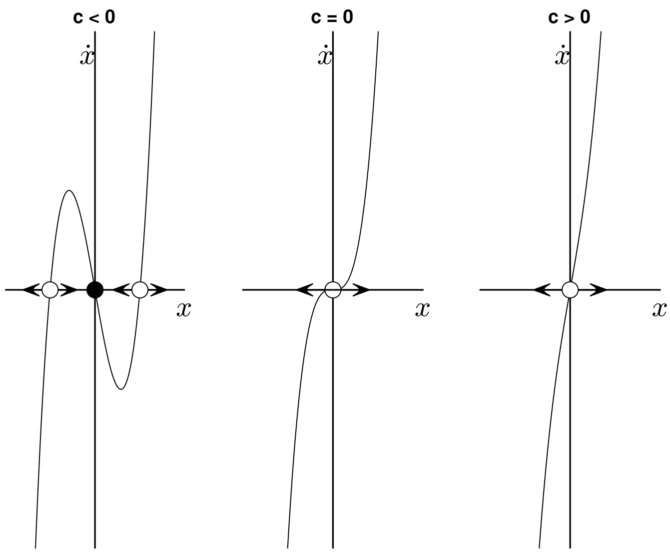


Figure 1.26: Subcritical pitchfork bifurcation

Here, we see that the pitchfork is inverted compared to the supercritical case. The non-zero equilibrium points $x^* = \pm\sqrt{-c}$ now only exist before the bifurcation ($c = 0$)

and are unstable. Most importantly, the equilibrium point $x^* = 0$ is stable for $c < 0$, and becomes unstable for $c \geq 0$. However, now the non-zero branches contribute to the instability.

This leads to quite different qualitative solutions for the carious cases in c, as can be seen in Figure 1.27.

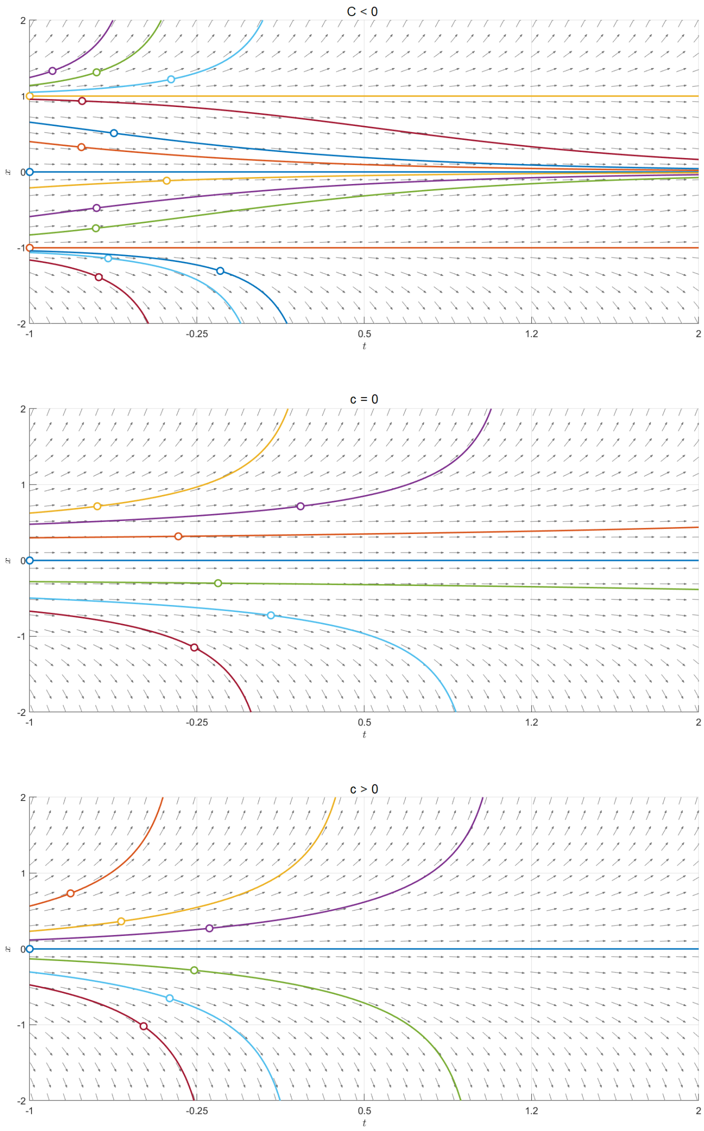


Figure 1.27: Solution & direction field for all case $c < 0$, $c = 0$ and $c > 0$.

The bifurcation diagram is given in Figure 1.28.

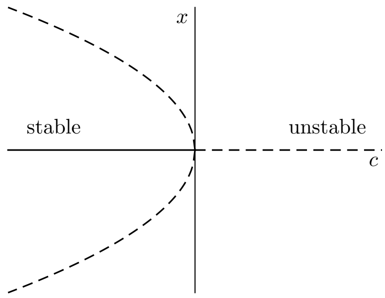


Figure 1.28: Bifurcation diagram for the system (1.28)

Definition 14. Subcritical Pitchfork Bifurcation *A subcritical pitchfork bifurcation is a local bifurcation in which there is a transition from a single unstable equilibrium point into three equilibria points where the new symmetric equilibria are unstable and the original equilibrium point becomes stable.*

▶ Example 35 ◀ Consider the system

$$\dot{x} = x^3 - 3x + c, \quad c \text{ constant}, \tag{1.29}$$

which can be factored as

$$\dot{x} = x\left(x^2 - 3\right) + c.$$

How many equilibrium points does the system (1.29) have for different values of the control parameter c?

The easiest way to see this is to plot the phase line for (1.29) for a selection of values of c, as shown in Figure 1.29.

Establish the stability of the equilibrium points and, by drawing the bifurcation diagram, state what type of bifurcation is occurring.

We can see from Table 1.1 that there is a bifurcation happening at both $c = -2$ and $c = 2$. Let's first consider the case $c = -2$.

c	# of equilibrium points
$c < -2$	1
$-2 < c < 2$	3
$c > 2$	1

Table 1.1: Number of equilibrium points for given values of c in (1.29)

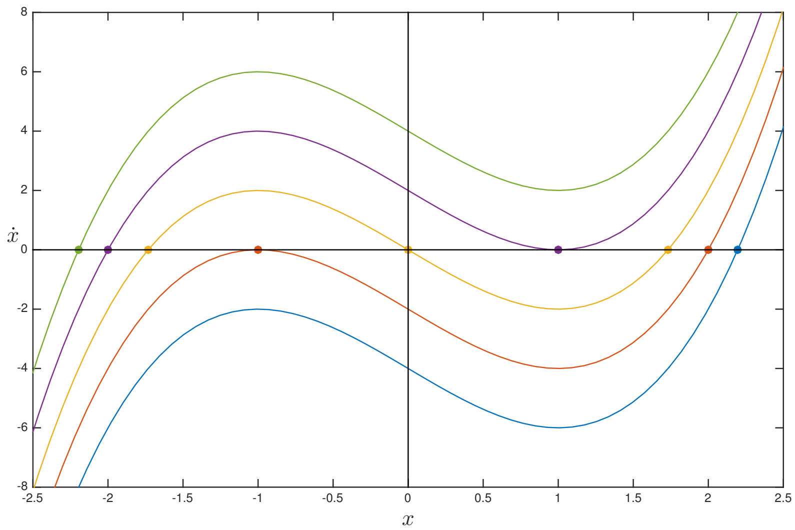


Figure 1.29: Phase lines for the system (1.29), with a selection of values of c

When $c = -2$, the equilibrium points of the system x^* are given by

$$x^3 - 3x - 2 = 0,$$
$$\Rightarrow (x-2)(x+1)^2 = 0.$$

We can use linear stability analysis to investigate the nature of the equilibrium points $x^* = -1$ and $x^* = 2$. We recall that, via linear stability analysis, we need to only consider the gradient of $\dot{x}$ at the equilibrium points to classify the nature of the equilibrium points. We have that,

$$f(x) = x^3 - 3x - 2,$$
$$\Rightarrow f'(x) = 3x^2 - 3,$$
$$\Rightarrow f'(-1) = 0, \quad f'(2) = 9.$$

Thus, the righthand equilibrium point $x^* = 2$ is unstable. We need to be more careful with the point $x^2 = -1$ since $f'(-1) = 0$. We note that, given the shape of f', the flow is always to the left, and so $x = x^*$ is a half-stable equilibrium point.

When $c < -2$, the system (1.29) has a single equilibrium point which can be shown to be unstable. While for $-2 < c < 2$ the system (1.29) has three equilibrium points, the righthand point is unstable, the middle point is stable and the left hand point is unstable. Since the righthand point is an unstable equilibrium point for all values of c, we consider the remaining points and conclude that there is a bifurcation point at $c = -2$, and that this is an example of a saddle-node bifurcation.

We can repeat this analysis for the remaining cases after which it becomes clear that we have another bifurcation point when $c = 2$, which is also an example of a saddle-node bifurcation.

The bifurcation diagram is given in Figure 1.30

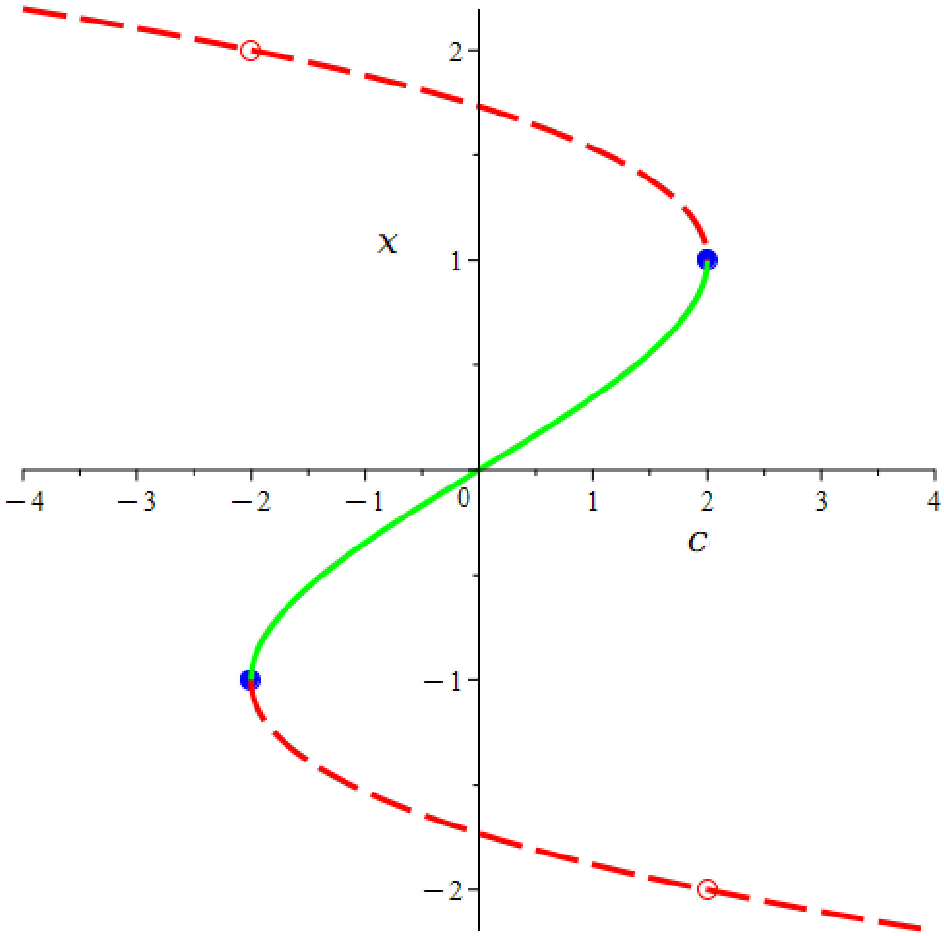


Figure 1.30: Bifurcation Diagram for the dynamical system (1.29)

Chapter 2

Dynamical systems on the plane: Introduction

So far we have only considered problems in one dimension - dynamical systems on the line. However it is much more common in applied mathematics to come across higher order systems of ODEs. In this chapter we will discuss 2-dimensional, autonomous, dynamical systems. Otherwise known as dynamical systems on the plane.

We can write a general 2-dimensional system of ODEs in the form,

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{2.1}$$

with $t \in I$, $(x, y) \in D$, and where P, $Q : \mathbb{R}^2 \to \mathbb{R}$. The functions P and Q are continuous and have continuous first partial derivatives. In most cases we will assume all partial derivatives of P and Q exist and are continuous. In general (2.1) cannot be solved to give $(x(t), y(t))$ explicitly as functions of t. Instead we need to consider methods of obtaining **qualitative** information about the solutions of (2.1) without having to obtain an explicit integral of the equations. In Chapter 1 we constructed phase-lines and introduced the concept of a **phase plane**. In this chapter we will consider general techniques to draw phase planes.

2.1 The phase plane

Just as any solution to the 1-dimensional system $\dot{x} = f(x)$ may be represented by a **phase line**, any solution of a 2-dimensional system (2.1) may be represented on a plane with Cartesian axes (x, y). Then, starting from some fixed value, as t increases $(x(t), y(t))$ traces out a **smooth**, **directed curve** in the plane, called a **trajectory** (or **phase path**). The collection of all possible trajectories of (2.1) is called the **phase portrait** of the system.

2.1.1 Trajectories

On a trajectory we may regard y as a function of x, i.e.

$$y(t) = y(x(t)),$$

therefore, by the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}, \tag{2.2}$$

on a path, and using (2.1) we obtain a nonautonomous ordinary differential equation for the trajectories,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Q(x,y)}{P(x,y)}. \tag{2.3}$$

Trajectories of (2.1) are solutions of (2.3). We see by the existence-uniqueness theorem that there can only be one solution to (2.3) at each point (x, y), except where $\frac{Q(x,y)}{P(x,y)}$ is undetermined, i.e. where $P(x_e, y_e) = 0$. When at Such points $Q(x_e, y_e) = 0$ as well, they are called equilibrium solutions of the system (2.1).

2.1.2 Example 1

Consider the second order differential equation

$$\ddot{x} + x = 0, \tag{2.4}$$

which is a simple model for an harmonic oscillator.

We write $\dot{x} = y$, and substitute into (2.4) to obtain the dynamical system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x, \end{aligned} \tag{2.5}$$

where $P = y$ and $Q = -x$. If we write

$$\frac{dy}{dx} = -\frac{x}{y},$$

we can integrate this to obtain that

$$x^2(t) + y^2(t) = C^2.$$

As the constant is arbitrary, I've chosen to write it as a square for ease later. Hence we do know the solutions curves of this system in the (x, y)-plane. One can also readily see that

$$\begin{aligned} x(t) &= C\,\cos(t) \\ y(t) &= C\,\sin(t), \end{aligned}$$

are an explicit solution to the system.

The solutions can be plotted in the (x, y)-plane as in Figure 2.1, also referred to as a **phase portrait**:

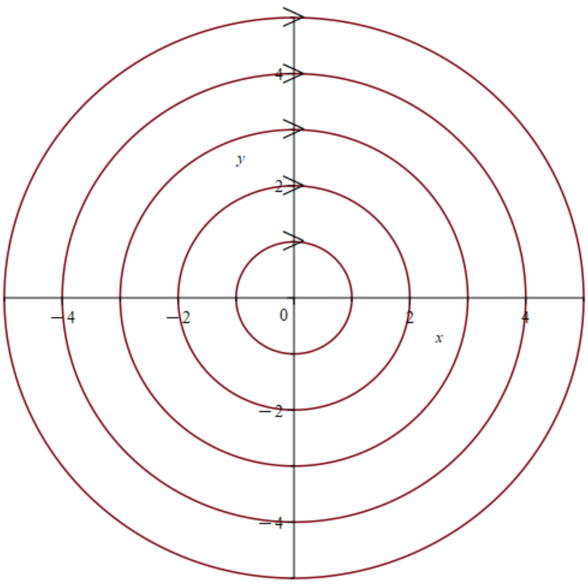


Figure 2.1: Solution curves for system (2.5).

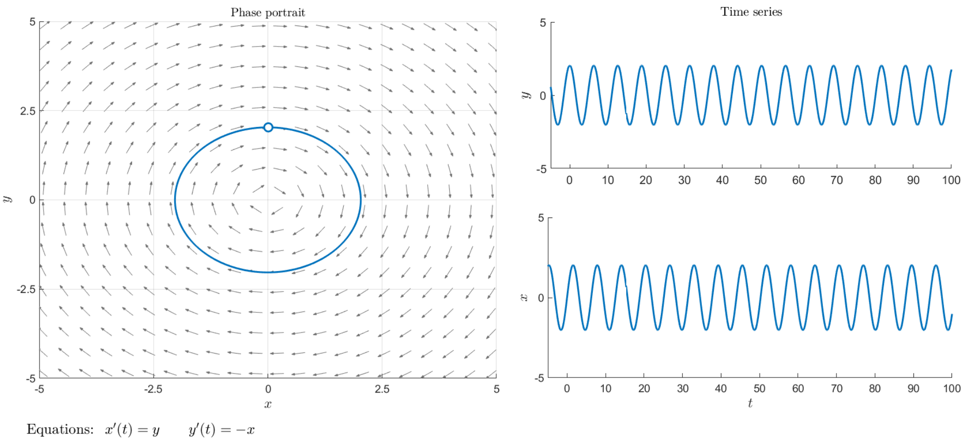
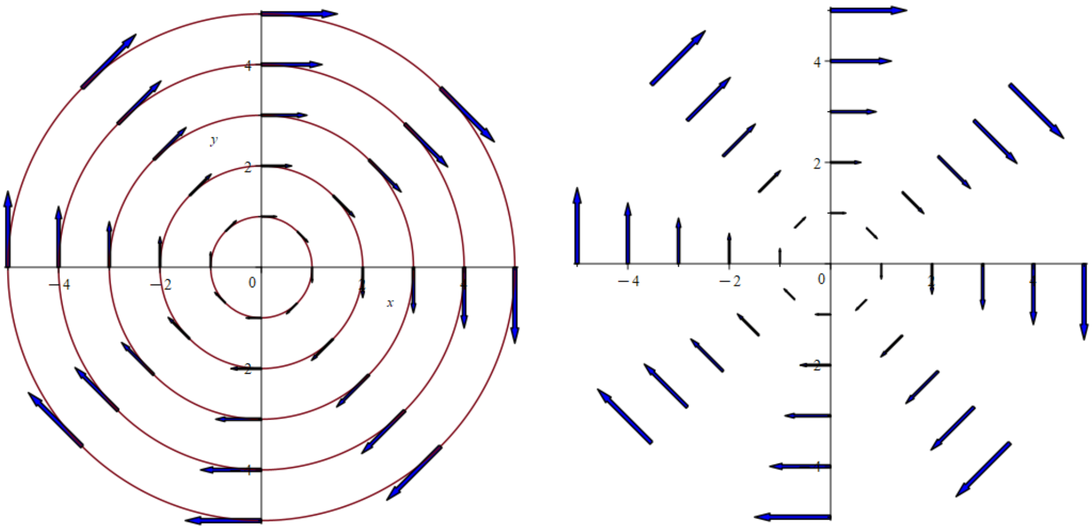


Figure 2.2: Solution curve, $x(t)$ and $y(t)$ for system (2.5).

From these, we can deduce the individual evolution of $x(t)$ and $y(t)$ with time t, as illustrated in Figure 2.2

We can see that in this case, the solution is oscillatory.

We note that in the phaseplot, the solution curves are circles with centre $(0, 0)$. But what can we do if we did not have the solution. The system (2.5) only gives the slopes at the solution curves at a given point. This is illustrated in Figure 2.3



Solutions and direction vectors. Only direction vectors.

Figure 2.3: Solution curves with selected directions for system (2.5).

The picture would become more clearly if the directions were plotted on a finer grid, like in Figure 2.4

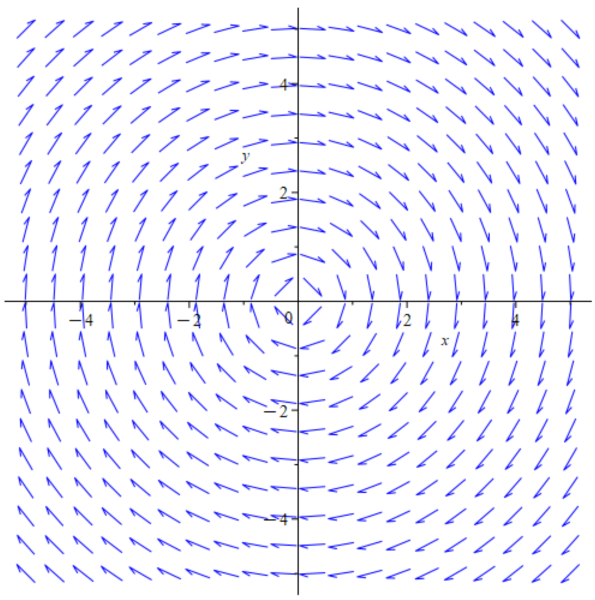


Figure 2.4: Direction vectors for for system (2.5).

This clearly suggest closed curves, but how can we be sure? How can we capture the essential elements of this direction field plot without calculating so many directions? What is happening at $(0, 0)$?

2.1.3 Example 2

Consider the second order differential equation

$$\ddot{x} - x = 0. \tag{2.6}$$

We write $\dot{x} = y$, and substitute into (2.6) to obtain the dynamical system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x, \end{aligned} \tag{2.7}$$

where $P = y$ and $Q = x$. If we write

$$\frac{dy}{dx} = \frac{x}{y},$$

we can integrate this to obtain that

$$x^2(t) - y^2(t) = C^2.$$

As the constant is arbitrary, I've chosen to write it as a square for ease later. Hence we do know that solutions curves of this system in the (x, y)-plane are hyperbola with centre at $(0, 0)$. These solutions can be plotted in the (x, y)-plane as in Figure 2.5, also referred to as a **phase portrait**:

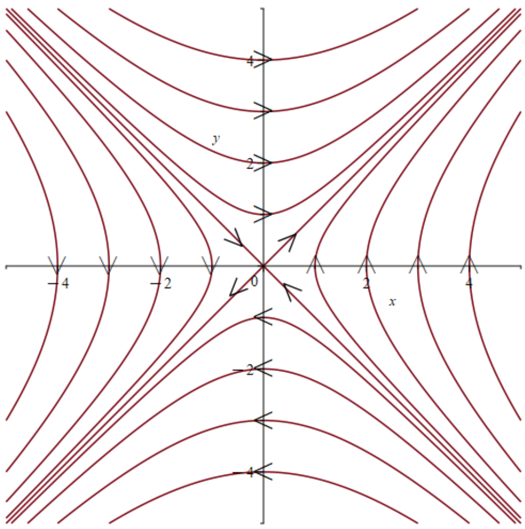


Figure 2.5: Solution curves for system (2.7).

From these, we can deduce the individual evolution of $x(t)$ and $y(t)$ with time t, as illustrated in Figure 2.6 and Figure 2.7

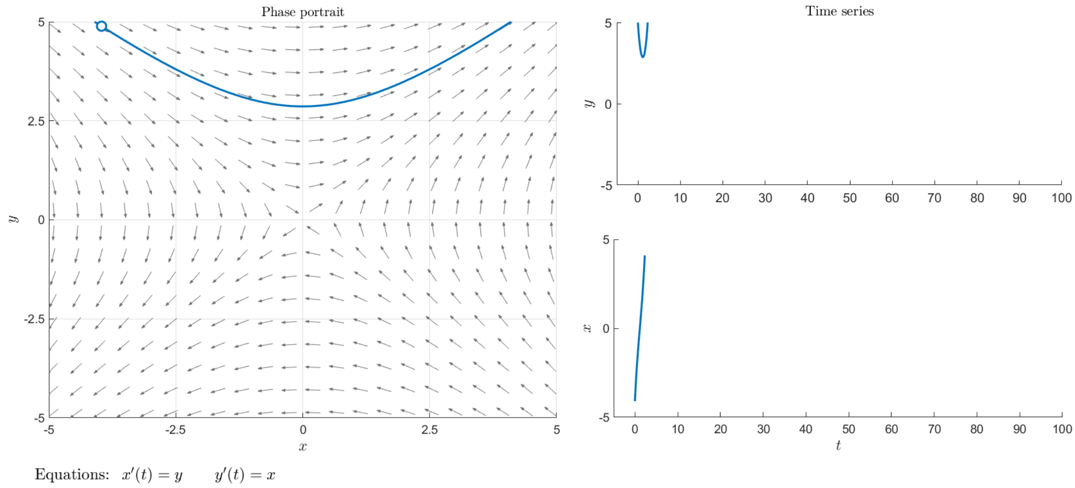


Figure 2.6: A solution curve, $x(t)$ and $y(t)$ for system (2.7).

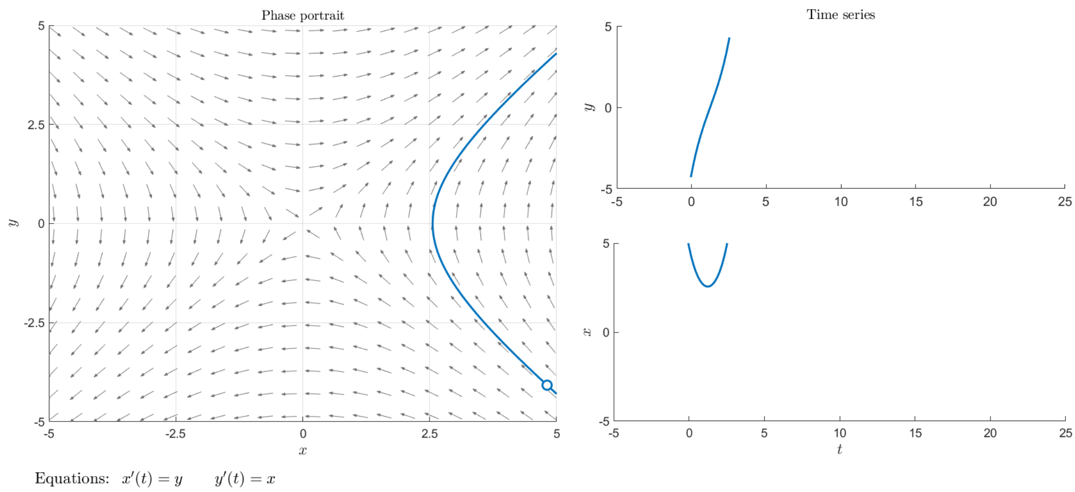
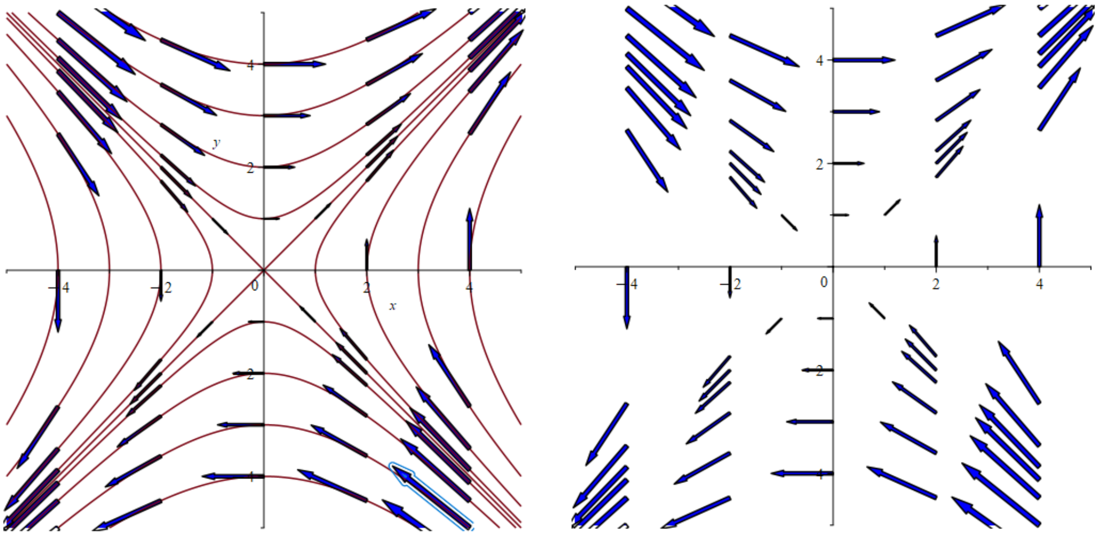


Figure 2.7: A solution curve, $x(t)$ and $y(t)$ for system (2.7).

But what can we do if we did not have the solution. The system (2.7) only gives the slopes at the solution curves at a given point. This is illustrated in Figure 2.8

The picture would become more clearly if the directions were plotted on a finer grid, like in Figure 2.9

This clearly suggests certain trajectories or solution curves. How can we capture the essential elements of this direction field plot without calculating so many directions? What is happening at $(0, 0)$? Are there any solution curves along straight line segments?



Solutions and direction vectors. Only direction vectors.

Figure 2.8: Solution curves with selected directions for system (2.7).

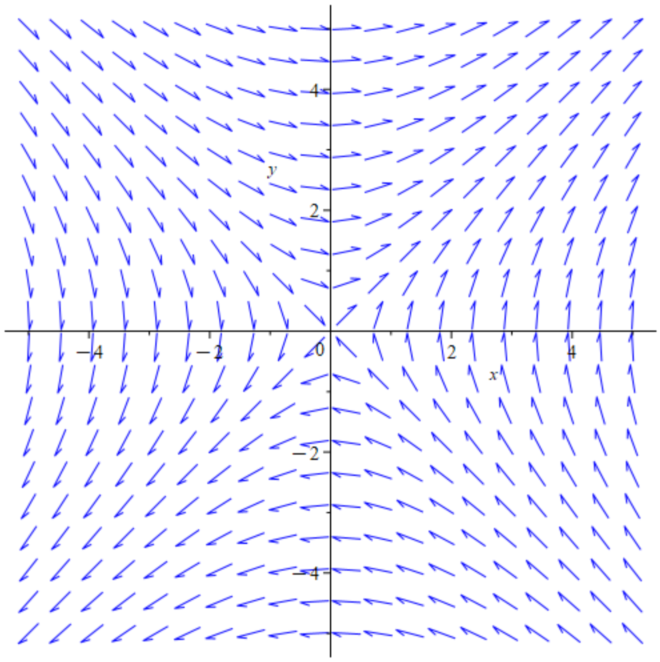


Figure 2.9: Direction vectors for for system (2.7).

In the first instance, we'll be looking at certain properties of features in the phase portrait as well as techniques to plot the direction field by hand. In the next chapter, we'll be looking at the stability properties of the different types of equilibria for a linear two-dimensional dynamical system,, before expanding this to non-linear two-dynamical systems in the following chapter.

2.2 Properties of a phase portrait

2.2.1 Equilibrium solutions

For 1-dimensional systems we defined equilibrium points to be the points where $\dot{x} = 0$. We have a similar definition for 2-dimensional systems. Constant solutions of the system (2.1), given by $x(t) \equiv x_e$, and $y(t) \equiv y_e$, are known as **equilibrium solutions**.

Definition 15. Equilibrium Solution *A point* $(x, y) = (x_e, y_e)$ *is an equilibrium solution of*

$$\dot{x} = P(x, y),$$
$$\dot{y} = Q(x, y),$$

if both

$$\begin{aligned} \dot{x}_e(t) &= P(x_e, y_e) = 0, \\ \dot{y}_e(t) &= Q(x_e, y_e) = 0. \end{aligned} \tag{2.8}$$

The phase path in the phase plane for an equilibrium solution is degenerate, being just a point (x_e, y_e) called an **equilibrium point** (or critical point) of the system.

A point in the phase plane which is **not** an equilibrium point of (2.1) is called an **ordinary point**.

Equilibrium solutions come in many forms and we will soon be able to classify them. Just as in the one-dimensional case, we will be interested in the stability properties of these equilibrium points.

2.2.2 Stability of equilibrium points

In Chapter 1 we introduced the concept of stability. We said that the equilibrium points towards which the solutions converged were stable, and those which solutions evolved away from were unstable. For dynamical systems on the plane we say that an equilibrium point is **stable** if all trajectories starting close to the equilibrium point remain close to it for all $t > 0$.

In the plane this concept is formally phrased by introducing the *norm* $||\mathbf{x}||$ of a vector $\mathbf{x} = (x, y)$ as

$$||\mathbf{x}|| = \sqrt{x^2 + y^2}$$

:

Definition 16. Asymptotic stability. *An equilibrium point* $\mathbf{x}_e$ *is said to be asymptotically stable if there exists a* $\nu > 0$ *such that for all* $\mathbf{x}(0)$ *with*

$$\|\mathbf{x}(0) - \mathbf{x}_e\| < \nu,$$

then,

$$\|\mathbf{x}(t) - \mathbf{x}_e\| \to 0 \quad as \quad t \to \infty.$$

This notion is often too strong for the plane. A better definition we use is the following:

Definition 17. Lyapunov stability. *An equilibrium point* $\mathbf{x}_e$ *is said to be Lyapunov stable if given any* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $\mathbf{x}(0)$ *with*

$$\| \mathbf{x}(0) - \mathbf{x}_e \| < \delta,$$

then,

$$\| \mathbf{x}(t) - \mathbf{x}_e \| < \epsilon \quad \textit{for all} \quad t > 0.$$

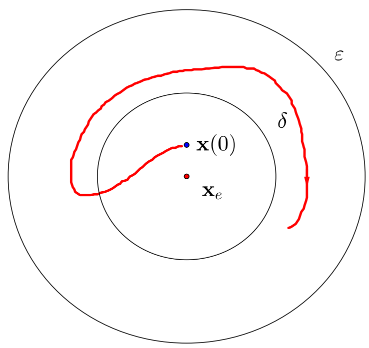


Figure 2.10: Liapunov stability. $\mathbf{x}(t)$ remains close to $\mathbf{x}_e$ for all t.

An equilibrium point that is neither Lyapunov stable or asymptotically stable is **unstable**.

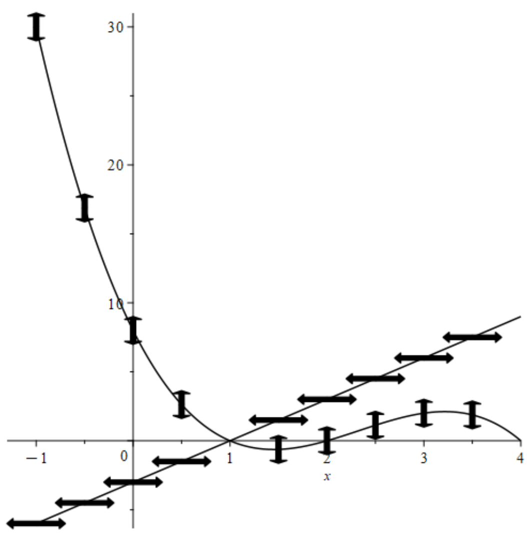


Figure 2.11: Horizontal and vertical isoclines

2.2.3 Isoclines

The curves in the phase plane along which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is constant are called **isoclines**. Specifically, a curve in the phase plane upon which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $(Q(x, y) = 0)$ is called a **horizontal isocline**, and a curve in the phase plane upon which $\frac{\mathrm{d}y}{\mathrm{d}x} = \infty$ $(P(x, y) = 0)$ is called a **vertical isocline**. The vector field $(\dot{x}, \dot{y})$ is directed horizontally on the horizontal isocline (zero gradient) and vertically on the vertical isocline (infinite gradient). This is illustrated in Figure 2.11, where the actual direction of the flow (left/right or up/down) is to be determined from the equations of the dynamical system.

Note that the vertical and horizontal isoclines intersect at equilibrium points (since at an equilibrium point, both $P(x_e, y_e)$ and $Q(x_e, y_e) = 0$).

► Example 36 ◄ Consider the system

$$\begin{aligned} \dot{x} &= 1, \\ \dot{y} &= 1 - y^2. \end{aligned} \tag{2.9}$$

Determine its isoclines.

In this case,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - y^2 = (1 + y)(1 - y).$$

Isoclines are given by curves on which $\frac{\mathrm{d}y}{\mathrm{d}x} = c$, constant, so $c = \frac{1-y^2}{1} = 1 - y^2$, and hence $y = \pm(1 - c)^{1/2}$. In particular, the horizontal and vertical isoclines are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - y^2 = \begin{cases} 0, & y = \pm 1, \\ \infty, & \text{no real values of } y. \end{cases}$$

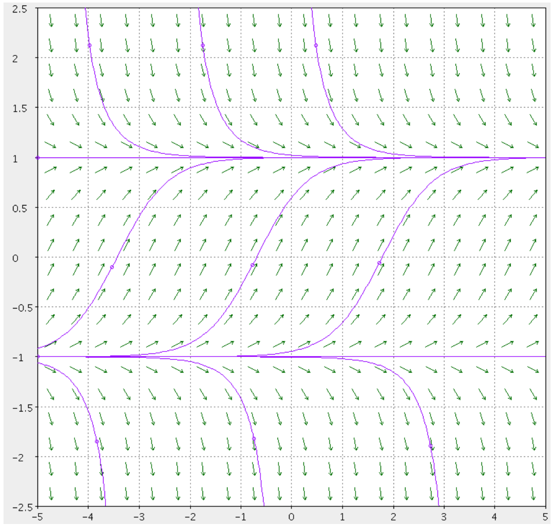


Figure 2.12: The isoclines of (2.9) are horizontal lines

On the horizontal isoclines, the gradient to the trajectories is horizontal and since $\dot{x} = 1 > 0$, the direction of travel is always towards the right. As in this case the gradient is constant along horizontal lines, we can plot a selection of these, and from this plot the trajectories which correspond to solutions of (2.9) Notice that there are no equilibrium solutions for this system.

Note that when a horizontal isocline is itself a horizontal line ($y = c$) then any trajectory that starts from a point on that horizontal isocline will remain on that isocline. Similarly, if a vertical asymptote is a vertical line ($x = c$), then any trajectory starting from a point on the vertical isocline will remain on that vertical isocline.

Theorem 1. *When a horizontal isocline of a two-dimensional dynamical system (2.1)is itself an horizontal line, than there will be solution curves or trajectories along the line segments that make up the horizontal isocline. Similarly, When a vertical isocline of a two-dimensional dynamical system is itself an vertical line, than there will be solution curves or trajectories along the line segments that make up the vertical isocline.*

Note that as solution curves or trajectories cannot cross another trajectory(uniqueness!), such isoclines will separate areas in the phase portrait that will have qualitatively different trajectories. Such a trajectory we will refer to in this course as a separatrix:

Definition 18. Separatrix *A trajectory or solution curve in the phase portrait of a dynamical system (2.1) which separates regions where the trajectories are qualitatively different, is called a* **separatrix**.

Once we have identified the horizontal and vertical isoclines, we know where $\dot{x}$ and $\dot{y}$ change sign, we can identify the sign of $\dot{x}$ and $\dot{y}$ in each region defined by the horizontal and vertical isoclines, and derive into which quadrant of the plane the flow is moving. For example, for the dynamicl system (2.9), we see this illustrated in Figure 2.13:

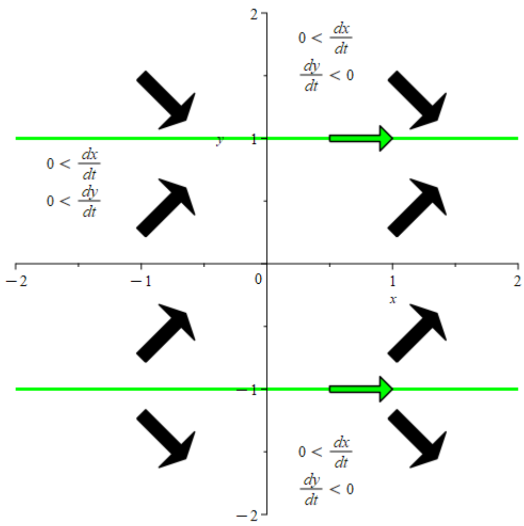


Figure 2.13: The indicative direction of flow in each region defined by the horizontal and vertical isoclines of (2.9).

We can also identify any straigth lines that contain solution curves or trajectories. Such straight lines will be isoclines as the gradient of the flow along them will be constant, and hence, in addition to satisfying the condition of an isocline, i,.e.,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Q(x,y)}{P(x,y)} = m, \tag{2.10}$$

we will need to verify that $y = mx + q$ itself satisfies the dynamical system. One way to do this is to ensure that when we substitute $y = mx + q$ into condition (2.10), the resulting relationship is satisfied for all possible values of the dependent variable x. This normally leads to a number of equations in m and q, all of which needs to be satisfied, otherwise no such straigth line solutions will exist. This analysis wuld also find horizontal isoclines that are horizontal lines ($m = 0$), but one will need to check sparately for any vertical isoclines that are vertical lines.

▶ Example 37 ◀ Consider the second order differential equation

$$\ddot{x} + x = 0. \tag{2.11}$$

Find its horizontal and vertical isoclines.

We studied this sytem before, so we know the solution curves are circles with centre at $((0,0))$. The two-dimensional dynamical system equivalent to differential equation (2.11) is given by

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x,\end{aligned} \tag{2.12}$$

where $P = y$ and $Q = -x$. We consider the horizontal and vertical isoclines:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} = \begin{cases} 0, & x = 0, \\ \infty, & y = 0. \end{cases}$$

So $x = 0$ ia an horizontal isocline and $y = 0$ is a vertical isocline. To know the direction of the flow along those iscline, we study the sign of the right-hand-side of the non-zero derivative. For the horizontal isocline, $x = 0$ (and thus $\dot{y} = 0$),

$$\dot{x} = y \quad \begin{cases} > 0, & y > 0, \\ < 0, & y < 0. \end{cases}$$

Hence x increases or decreases with increasing t when y is positive or negative respectively.

When $y = 0$ (and thus $\dot{x} = 0$),

$$\dot{y} = -x \quad \begin{cases} > 0, & x < 0, \\ < 0, & x > 0. \end{cases}$$

Hence y increases or decreases with increasing t when x is negative or positive respectively.

This allows us to check for the sign of $\dot{x}$ and $\dot{y}$ in each region defined by the horizontal and vertical isocline, and, hence the quadrant to which the direction of the flow points. The result is illustrated in Figure 2.14

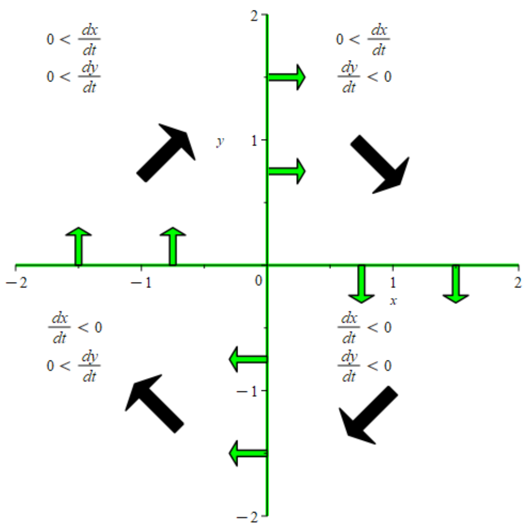


Figure 2.14: The indicative direction of flow in each region defined by the horizontal and vertical isoclines of (2.12).

If we check for straight line solutions, we obtain:

$$\begin{aligned} -\frac{x}{y} &= m \\ \Leftrightarrow -\frac{x}{mx+q} &= m \\ \Leftrightarrow -x &= m^2x + mq \\ \Leftrightarrow \left(m^2+1\right)x + mq &= 0, \end{aligned}$$

which can only be satisfied for all values of x when $m^2+1=0$ and $mq=0$, which is not possible, hence no such straigth line solutions do exist.

This allows us to sketch the phase portrait of system (2.12), as in Figure (2.15).

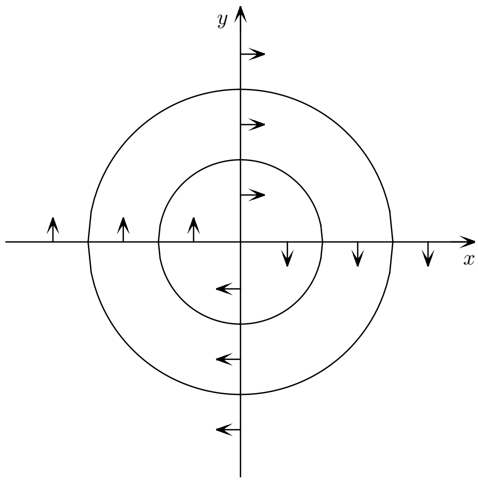


Figure 2.15: Phase portrait of system (2.12)

▶ Example 38 ◀

$$\begin{aligned} \dot{x} &= (x-1)\,y, \\ \dot{y} &= (x+2)\,(y-3)\,, \end{aligned} \tag{2.13}$$

Find all horizontal and vertical isoclines and check for any straight line trajectories.

For a horizontal isocline,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+2)\,(y-3)}{(x-1)\,y} = 0, \tag{2.14}$$

which happens for $x=-2$ and $y=3$.

For $x = -2$,

$$\dot{x} = -3y \quad \begin{cases} < 0, & y > 0, \\ > 0, & y < 0. \end{cases}$$

Hence x increases or decreases with increasing t when y is negative or positive respectively.

For $y = 3$

$$\dot{x} = 3\,(x-1) \quad \begin{cases} < 0, & x < 1, \\ > 0, & x > 1. \end{cases}$$

Hence x increases or decreases with increasing t when x is larger than 1 or smaller than 1 respectively. Notice that this horizontal isocline is a horizontal line, so trajectories will align with the linesegments on $y = 3$, and these will be separatrices.

For a vertical isocline,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+2)\,(y-3)}{(x-1)\,y} = \infty, \tag{2.15}$$

which happens for $x = 1$ and $y = 0$.

For $x = 1$,

$$\dot{y} = 3\,(y-3) \quad \begin{cases} < 0, & y < 3, \\ > 0, & y > 3. \end{cases}$$

Hence y increases or decreases with increasing t when y is larger than 3 or smaller than 3 respectively. Notice that this vertical isocline is a vertical line, so trajectories will align with the linesegments on $y = 3$, and these will be separatrices.

For $y = 0$

$$\dot{y} = -3\,(x+2) \quad \begin{cases} < 0, & x > -2, \\ > 0, & x < -2. \end{cases}$$

Hence y increases or decreases with increasing t when x is smaller than -2 or larger than -2 respectively.

Now we can check for the sign of $\dot{x}$ and $\dot{y}$ in each region defined by the horizontal and vertical isocline, and, hence the quadrant to which the direction of the flow points. The result is illustrated in Figure 2.16

To look for straight line trajectories, we need to consider

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+2)\,(mx+q-3)}{(x-1)\,(mx+q)} = m, \tag{2.16}$$

This can be rewritten as

$$m\,(m-1)\,x^2 - \big(m^2 + (2-q)\,m + (q-3)\big)\,x - (mq + 2\,(q-3)) = 0,$$

which must hold for all values of x so that

$$\begin{aligned} m\,(m-1) &= 0, \\ m^2 + (2-q)\,m + (q-3) &= 0, \\ mq + 2\,(q-3) &= 0. \end{aligned} \tag{2.17}$$

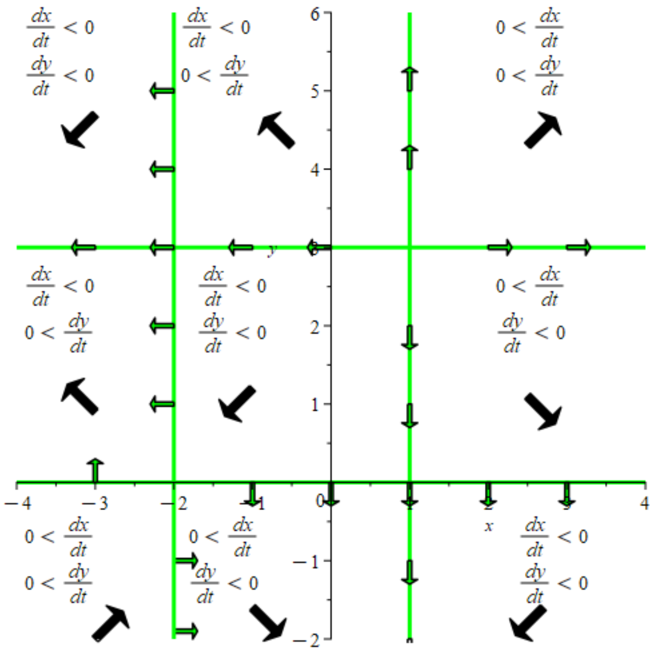


Figure 2.16: The indicative direction of flow in each region defined by the horizontal and vertical isoclines of (2.13).

The solution with $m = 0, q = 3$ returns the horizontal isocline $y = 3$ which we already noted above contains straight line trajectories.

For $m = 1$, the second equation in (2.17) is satisfied whatever the value of q, and the third equation yields $q = 2$. So the line $y = x + 2$ consists of line segments which are trajectories. As we try to sketch a trajectory starting on this line, the direction in which it evolves will align with the direction of the line $y = x + 2$ and hence it will remain on this line.

To determine in what way the flow is moving along this line, we analyse the equations (2.13):

$$\dot{x} = (x-1)(x+2) \quad \begin{cases} < 0, & -2 < x < 1, \\ > 0, & x < -2 or x > 1, \end{cases}$$

and

$$\dot{y} = (x-1)(x+2) \quad \begin{cases} < 0, & -2 < x < 1, \\ > 0, & x < -2 or x > 1. \end{cases}$$

Hence, the flow is upwards and to the right (increasing x and y with t) when $x < -2$ or $x > 1$. It is downwards and towards the left (decreasing x and y with t) when $-2 < x < 1$.

Note that $x = 1$, $y = 3$ and $y = x + 2$ act as separatrices.

The horizontal and vertical isoclines as well as $y = x + 2$ are depicted in Figure 2.17

Note that the three separatrices divide the plane in six different regions, as labelled on Figure 2.17. There may be other separatrices, particularly associated with the equilibrium

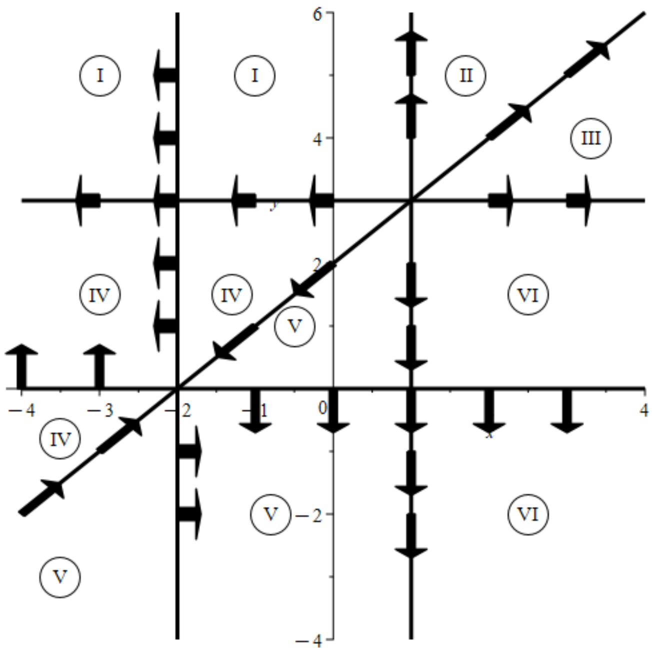


Figure 2.17: Phase portrait of system (2.13)

points (here found at $(-2,0)$ and $(1,3)$.

2.2.4 Geometrical properties of the phase plane

a. Through each ordinary point (x_o, y_o) on the phase plane there is one, and only one, trajectory. This trajectory can be extended and may remain bounded, or have $\|\mathbf{x}(t)\| \to \infty$ in finite or infinite time, t. These represent the solution curves of the system (2.1)

b. Trajectories do not intersect in the phase plane at ordinary points, which follows from the uniqueness of solutions.

c. Trajectories may only approach equilibrium points as $t \to \infty$ or $t \to -\infty$, which follows again from the uniqueness of solutions.

d. Suppose there is a trajectory $\mathcal{C}$ joining the points A with coordinates (x_A, y_A) and B with coordinates (x_B, y_B). The time taken to traverse $\mathcal{C}$ between A and B is finite when $\mathcal{C}$ is composed of ordinary points.

e. Suppose a trajectory $\mathcal{C}$ approaches an equilibrium point (x_e, y_e). The time taken to reach the equilibrium point from any other point on the path is unbounded.

f. Closed trajectories in the phase plane which are composed entirely of ordinary points represent **periodic** solutions of (2.1) (See later in the course).

2.2.5 How to sketch a phase portrait

Ignoring equilibrium point properties for the moment, we now have plenty of information needed to sketch the phase portrait of a linear system. To do this we must do the following steps:

1. Locate the equilibrium points;
2. Locate the horizontal and vertical isoclines; Determine the direction of the flow on them;
3. Check direction of flow in each region defined by the horizontal and vertical isoclines;
4. Check for any straight line solutions, and if any, determine the direction of flow on them;
5. Identify all separatrices;
6. If the information is scarce in certain regions, you can establish the direction of the flow along given curves. If not yet featured as an isocline or straightlin solutions, the co-ordinate axes are ideal, the lines $y = x$ or $y = -x$ often also provide useful information;
7. If need be, the direction of the flow at selected points can also be added.

► Example 39 ◄ Consider the dynamical system

$$\begin{aligned} \dot{x} &= 1 - xy, \\ \dot{y} &= xy - y. \end{aligned} \tag{2.18}$$

Determine the horizontal and vertical isoclines, the direction of flow in each region defined by these isoclines. Check also for straight line solutions and check the direction of the flow along $y = x$ and $y = -x$. Determine the location of any equilibrium points.

Chapter 3

Linear dynamical systems on the plane

We will start this chapter with the study of linear two-dynamical systems of the form

$$\begin{aligned}\dot{x} &= ax + by, \\ \dot{y} &= cx + dy,\end{aligned} \tag{3.1}$$

for some constants a, b, c, d. We can write a linear system in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{3.2}$$

where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.3}$$

We note that there is an equilibrium point at $(0, 0)$. For

$$\det(\mathbf{A}) = ad - bc \neq 0,$$

this will be the only equilibrium point. The horizontal isocline can be found at $cx+dy = 0$ and the vertical isocline at $ax+by = 0$, both of these are straight lines through the origin $(0, 0)$.

When looking for straight line solutions, the straight line solutions not through the origin, i.e. $q \neq 0$, only exist when $\det(\mathbf{A}) = 0$. We will discuss these exceptional cases later. When $\det(\mathbf{A}) \neq 0$, the procedure outlined in the previous chapter, when looking for solutions of the form $y = mx$, will lead to a quadratic equation in the slope m, which may or may not have real solutions.

To understand the equilibrium point properties we first need to revise some theory.

3.1 Solutions

The general solution for a linear two-dynamical system of equations is given by

$$\mathbf{x}(t) = \alpha \mathbf{v} e^{\lambda_1 t} + \beta \mathbf{w} e^{\lambda_2 t}, \tag{3.4}$$

where α, β are constants, and $\mathbf{v}$, $\mathbf{w}$ are the eigenvectors corresponding to the eigenvalues $\lambda_{1,2}$, as long as $\lambda_1 \neq \lambda_2$. We will discuss the special case when $\lambda_1 = \lambda_2$ at the end of this section. Let us first revise the theory on eigenvalues and eigenvectors.

3.1.1 Eigenvalues and Eigenvectors

Thus we need to determine the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Given

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we want to find the eigenvectors $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} \neq \mathbf{0}$ and the eigenvalues λ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}, \tag{3.5}$$

where x and y are both not zero. On rearranging (3.5) we obtain

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{v} = 0, \tag{3.6}$$

where $\mathbf{I}$ is the identity matrix. We note that if $(\mathbf{A} - \lambda \mathbf{I})$ has an inverse then

$$\mathbf{v} = (\mathbf{A} - \lambda \mathbf{I})^{-1}\,\mathbf{0},$$

which only has one solution, the trivial solution, $\mathbf{v} = \mathbf{0}$ which is not allowed. Therefore, we require λ to be such that $(\mathbf{A} - \lambda \mathbf{I})$ has no inverse. Thus, we require

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \tag{3.7}$$

Equation (3.7) is known as the **characteristic equation** for the matrix $\mathbf{A}$.

We obtain the eigenvalues of $\mathbf{A}$ by evaluating

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0.$$

The characteristic equation associated with (3.2) is given by the quadratic polynomial

$$\lambda^2 - \lambda\ \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = 0, \tag{3.8}$$

the roots of which are given by

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) \pm \frac{1}{2}\sqrt{\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})}, \tag{3.9}$$

where $\mathrm{tr}(\mathbf{A})$ is the trace of the matrix $\mathbf{A}$ ($\mathrm{tr}(\mathbf{A}) = a + d$). If we define $\tau = \mathrm{tr}(\mathbf{A})$ and $\delta = \det(\mathbf{A})$, we can rewrite the expression for the eigenvalues (3.9) as

$$\lambda_{1,2} = \frac{1}{2}\tau \pm \frac{1}{2}\sqrt{\tau^2 - 4\delta}, \tag{3.10}$$

We have three possibilities for the types of eigenvalues of the matrix $\mathbf{A}$:

1. If $\operatorname{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) > 0$, i.e. $\tau^2 - 4\delta > 0$, then the characteristic equation (3.9) has two distinct real roots $\lambda_{1,2}$.
2. If $\operatorname{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) = 0$, i.e. $\tau^2 = 4\delta$, then the characteristic equation (3.9) has the repeated real root $\lambda_{1,2} = \lambda = \frac{1}{2}\operatorname{tr}(\mathbf{A})$.
3. If $\operatorname{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) < 0$, i.e. $\tau^2 - 4\delta < 0$ then the characteristic equation (3.9) has two complex conjugate roots $\lambda_{1,2} = \mu \pm i\gamma$, for some constants μ and γ.

► Example 40 ◄ Given $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$, determine the eigenvalues and corresponding eigenvectors.

We begin by finding the eigenvalues of $\mathbf{A}$, and evaluate

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Leftrightarrow \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} &= 0, \\ \Leftrightarrow \lambda^2 - 2\lambda - 3 &= 0, \\ \Leftrightarrow \lambda_1 = -1, \quad \lambda_2 &= 3. \end{aligned}$$

We now calculate the corresponding eigenvectors. Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ be the eigenvector corresponding to $\lambda_1 \, (= -1)$. Now (3.6) becomes

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.11}$$

Writing (3.11) in terms of simultaneous equations we obtain

$$\begin{aligned} 2x + y &= 0, \\ 4x + 2y &= 0, \end{aligned}$$

giving $x = -\frac{1}{2}y$. Therefore,

$$\mathbf{v} = y \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}.$$

Any multiple of an eigenvector is also an eigenvector, so for simplicity we take

$$\mathbf{v} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}.$$

A similar calculation can be carried out for $\mathbf{w}$, the eigenvector corresponding to λ_2, to obtain

$$\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

3.1.2 Solutions

For the case that both eigenvalues are different ($\lambda_1 \neq \lambda_2$) we can use the general solution (3.4) to study the trajectories. If we write the eigenvectors $\mathbf{v}$ and $\mathbf{w}$ as

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{3.12}$$

we note that

$$\frac{y}{x} = \frac{\alpha v_2 e^{\lambda_1 t} + \beta w_2 e^{\lambda_2 t}}{\alpha v_1 e^{\lambda_1 t} + \beta w_1 e^{\lambda_2 t}} \tag{3.13}$$

We can use (3.13) to sketch the solution trajectories of the linear system (3.2).

For the case $\lambda_1 = \lambda_2$, there is only one eigenvalue λ_1 with one corresponding eigenvector $\mathbf{v}$. One can then find a **generalised** eigenvector of rank 2 by solving

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{w} = \mathbf{v}, \tag{3.14}$$

so that

$$(\mathbf{A} - \lambda \mathbf{I})^2\,\mathbf{w} = 0, \tag{3.15}$$

In this case, the solutions to the linear system (3.2) are given by

$$\mathbf{x} = \alpha \mathbf{v} e^{\lambda_1 t} + \beta \left(\mathbf{v} t e^{\lambda_1 t} + \mathbf{w} e^{\lambda_1 t} \right). \tag{3.16}$$

Often though, in this case, the matrix $\mathbf{A}$ is diagonal, and there is an easier way to study the solutions by treating the two-dimensional dynamical system as two uncoupled differential equations.

The following sections will look at the classification of the equilibria, in function of the nature of the eigenvalues. We will begin with the more regular cases, where $\det(\mathbf{A}) \neq 0$, to ensure that we always have a unique equilibrium at the origin $(x, y) = (0, 0)$. This also excludes the possibility of a zero eigenvalue. The nature of this equilibrium point is determined by the eigenvalues of the matrix $\mathbf{A}$.

3.2 Classification of a linear system with $\det(\mathbf{A}) \neq 0$.

3.2.1 $\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) > 0$

We have two real and distinct eigenvalues, with $\lambda_1 > \lambda_2$ (we are free to choose this, if $\lambda_2 > \lambda_1$ then just set $\lambda_2 = \lambda_1$, $\mathbf{w} = \mathbf{v}$, and vice versa). Equation (3.13) then implies that, if both α, $\beta \neq 0$,

$$\frac{y}{x} \to \begin{cases} v_2/v_1 & \text{as } t \to \infty \\ w_2/w_1 & \text{as } t \to -\infty, \end{cases} \tag{3.17}$$

or

$$\frac{y}{x} = \begin{cases} v_2/v_1 & \text{if } \beta = 0, \\ w_2/w_1 & \text{if } \alpha = 0. \end{cases} \tag{3.18}$$

Thus, for one of α or β zero, we have two straight line solutions. These are in the direction of the corresponding eigenvector.

We then have the following three subcases:

- $\mathbf{0 > \lambda_1 > \lambda_2}$ This will be the case if $\tau < 0$ and $\sqrt{\tau^2 - 4\delta} < |\tau|$, which is the case if and only if $\delta > 0$.
- $\mathbf{\lambda_1 > \lambda_2 > 0}$ This will be the case if $\tau > 0$ and $\sqrt{\tau^2 - 4\delta} < |\tau|$, which is the case if and only if $\delta > 0$.
- $\mathbf{\lambda_1 > 0 > \lambda_2}$ This will be the case if $\sqrt{\tau^2 - 4\delta} > |\tau|$, which is the case if and only if $\delta < 0$.

(a) $\mathbf{0 > \lambda_1 > \lambda_2}$

In this case, equation (3.4) gives that

$$|x|\ \&\ |y| \to \begin{cases} 0 & \text{as } t \to \infty, \\ \infty & \text{as } t \to -\infty. \end{cases}$$

The equilibrium point is a **stable node** (see Figure 3.1).

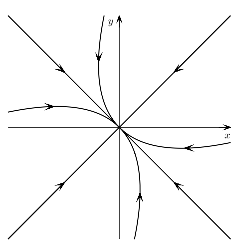


Figure 3.1: Stable node

▶ Example 41 ◀ Given the two-dimensional system

$$\begin{aligned} \dot{x} &= -5x + y, \\ \dot{y} &= x - 5y, \end{aligned} \tag{3.19}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

(b) $\boldsymbol{\lambda_1 > \lambda_2 > 0}$

In this case, equation (3.4) gives that

$$|x| \,\&\, |y| \to \begin{cases} \infty & \text{as } t \to \infty, \\ 0 & \text{as } t \to -\infty. \end{cases}$$

The equilibrium point is an **unstable node** (see Figure 3.2).

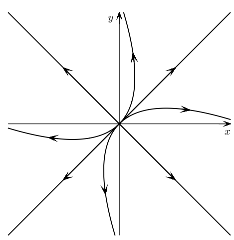


Figure 3.2: Unstable node

(c) $\boldsymbol{\lambda_1 > 0 > \lambda_2}$

In this case, equation (3.4) gives that, provided α, $\beta \neq 0$,

$$|x| \,\&\, |y| \to \begin{cases} \infty & \text{as } t \to \infty, \\ \infty & \text{as } t \to -\infty, \end{cases}$$

the only trajectory which goes into the origin is when $\alpha = 0$, and the only trajectory which goes out from the origin is when $\beta = 0$. The equilibrium point is a **saddle point** (see Figure 3.3).

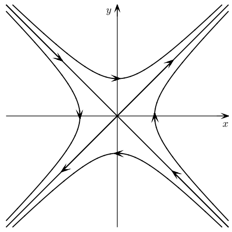


Figure 3.3: Saddle point

▶ Example 42 ◀ Given the two-dimensional system

$$\begin{aligned} \dot{x} &= -x + 5y, \\ \dot{y} &= 5x - y, \end{aligned} \tag{3.20}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

3.2.2 $\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) = 0$

The case $\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) = \mathbf{0}$ is equivalent to $(a-d)^2 = 4bc$. We will consider two sub-cases:

(a) **The matrix A is diagonal** ($b = c = 0,\ a = d \neq 0$):
In this case, the system can be written as

$$\dot{x} = ax, \quad \dot{y} = ay, \tag{3.21}$$

which has the general solution

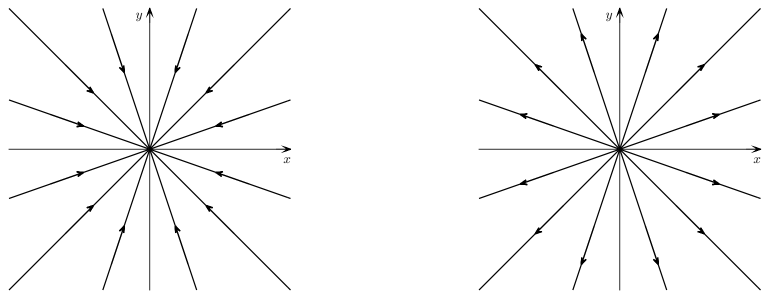
$$x = \alpha e^{at}, \quad y = \beta e^{at}, \tag{3.22}$$

for some constants α and β. We then have that

$$\frac{y}{x} = \frac{\beta}{\alpha}, \tag{3.23}$$

so all trajectories are straight lines. The equilibrium point is a stable star if $a < 0$ (Figure 3.4a), and an unstable star if $a > 0$ (Figure 3.4b).

(b) **Otherwise**:



(a) Stable star (b) Unstable star

Figure 3.4: Stable and unstable stars

The solution is given by (3.16):

$$\mathbf{x} = \alpha \mathbf{v} e^{\lambda_1 t} + \beta \left(\mathbf{v} t e^{\lambda_1 t} + \mathbf{w} e^{\lambda_1 t} \right).$$

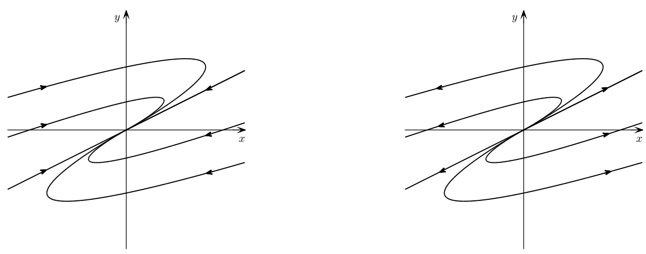
If $\beta = 0$, then

$$\frac{y}{x} = \frac{v_2}{v_1}, \tag{3.24}$$

and the linear system has a straight line solution, which is the only one. If $\beta \neq 0$, then

$$\frac{y}{x} \to \frac{v_2}{v_1} \quad \text{as} \quad t \to \infty, \tag{3.25}$$

The equilibrium point is a degenerate node, stable if $\lambda_1 < 0$ (Figure 3.5a), and unstable if $\lambda_1 > 0$ (Figure 3.5b).



(a) Stable degenerate node (b) Unstable degenerate node

Figure 3.5: Stable and unstable degenerate nodes

▶ Example 43 ◀ Given the two-dimensional system

$$\begin{aligned} \dot{x} &= 2x + y, \\ \dot{y} &= 2y, \end{aligned} \tag{3.26}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

▶ Example 44 ◀ Given the two-dimensional system

$$\begin{aligned} \dot{x} &= -2x + y, \\ \dot{y} &= -2y, \end{aligned} \tag{3.27}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

3.2.3 $\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) < 0$

The eigenvalues λ_1, λ_2, are complex conjugate pairs, as are the corresponding eigenvectors, $\mathbf{w} = \mathbf{v}^*$. We write the solutions to the linear system (3.2) as

$$\mathbf{x}(t) = \alpha \mathbf{v} e^{\lambda_1 t} + \beta \mathbf{w} e^{\lambda_2 t},$$

or, with $\lambda_1 = \lambda_R + i\lambda_I$,

$$\mathbf{x}(t) = \alpha \mathbf{v} e^{\lambda_R t}(\cos(\lambda_I t) + i\sin(\lambda_I t)) + \beta \mathbf{w} e^{\lambda_R t}(\cos(\lambda_I t) - i\sin(\lambda_I t)), \tag{3.28}$$

which clearly shows the rotating nature of these solutions. If we look at the magnitude of the typical term,

$$\left|\alpha \mathbf{v} e^{\lambda_R t}(\cos(\lambda_I t) + i\sin(\lambda_I t))\right| = |\alpha|\,|\mathbf{v}|\left|e^{\lambda_R t}\right|\left|(\cos(\lambda_I t) + i\sin(\lambda_I t))\right|, \tag{3.29}$$

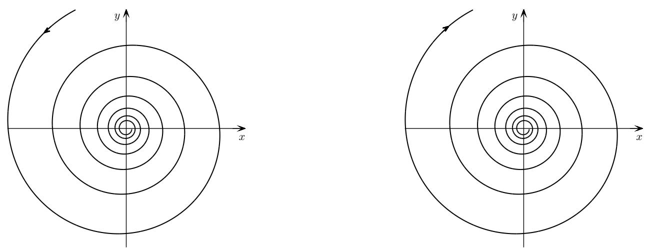
the first term is a constant, the magnitude of the complex eigenvector is also some constant and the final term is a point on the unit circle. So the behaviour of the magnitude of our solution with time depends entirely on the factor

$$\left|e^{\lambda_R t}\right|,$$

or more precisely on the sign of λ_R, the real part of the eigenvalue.

(a) $\lambda_R \neq 0$

If $\lambda_R < 0$, the magnitude of the solution goes to 0 for $t \to \infty$ and the equilibrium point is a stable focus or spiral (Figure 3.6a). Similarly, if $\lambda_R > 0$, the magnitude of the solution goes to ∞ when $t \to \infty$ and the equilibrium point is an unstable focus or spiral (Figure 3.6b).



(a) Stable focus (b) Unstable focus

Figure 3.6: Stable and unstable foci

▶ Example 45 ◀ Given the two-dimensional system

$$\begin{aligned} \dot{x} &= -x - 3y, \\ \dot{y} &= x - 2y, \end{aligned} \tag{3.30}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

▶ Example 46 ◀ Given the two-dimensional system

$$\begin{aligned} \dot{x} &= x - 3y, \\ \dot{y} &= x + 2y, \end{aligned} \tag{3.31}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

(b) $\lambda_R = 0$

Finally, if $\mathrm{Re}\,(\lambda) = 0$, the magnitude of the solution does not change with time and the only change is oscillatory. The trajectories are therefore closed curves, either circles or ellipses. The equilibrium point is a centre, and said to be *neutrally stable* (Figure 3.7).

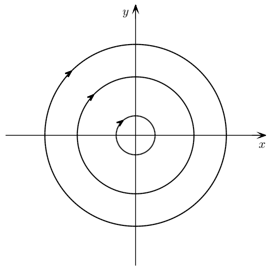


Figure 3.7: Centre

▶ Example 47 ◀ Given the two-dimensional system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x, \end{aligned} \tag{3.32}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

3.3 Classification of a linear system with $\det(\mathbf{A}) = 0$.

3.3.1 Equilibria and solutions

So far we have only considered 2D systems where the matrix is non-singular, i.e. the determinant is non-zero. Here we look at the special cases where the determinant equals zero:

$$\det(\mathbf{A}) = ad - bc = 0, \tag{3.33}$$

or

$$ad = bc \Leftrightarrow \frac{a}{c} = \frac{b}{d} = k,$$

for some $k \in \mathbb{R}$ if $c \neq 0$ and $d \neq 0$.

This means that the right hand side of the equation for $\dot{y}$ is a multiple of the right hand side of the equation for $\dot{x}$:

$$ax + by = k(cx + dy).$$

Hence, there is not a single equilibrium point, but all points on the line

$$ax + by = 0$$

are equilibrium points.

For completeness, we need to look at the cases where either $c = 0$ or $d = 0$ or both. If $c = 0$ than from (3.33) we see that either $a = 0$ or $d = 0$. If $d = 0$, the equality (3.33) tells us that either $b = 0$ or $c = 0$. This leads to three special cases to consider:

1. $c = a = 0$. The system reduces to

$$\begin{aligned}\dot{x} &= by,\\ \dot{y} &= dy,\end{aligned}$$

   and every point on the line $y = 0$ is an equilibrium point.

2. $c = d = 0$. The system reduces to

$$\begin{aligned}\dot{x} &= ax + by,\\ \dot{y} &= 0,\end{aligned}$$

   and every point on the line $ax + by = 0$ is an equilibrium point.

3. $d = b = 0$. The system reduces to

$$\begin{aligned}\dot{x} &= ax,\\ \dot{y} &= cx,\end{aligned}$$

   and every point on the line $x = 0$ is an equilibrium point.

We exclude the case where $\mathbf{A} = \mathbf{0}$ as in this case, every point is an equilibrium point and the we have a completely static situation (nothing changes), which we can't really label a dynamical system.

For $\det \mathbf{A} = 0$, the characteristic equation for the matrix (3.9) becomes

$$\lambda^2 - \lambda \operatorname{tr}(\mathbf{A}) = 0 \Leftrightarrow \lambda (\lambda - \operatorname{tr}(\mathbf{A})) = 0. \tag{3.34}$$

Hence the eigenvalues are given by $\lambda_1 = 0$ and $\lambda_2 = \operatorname{tr}(\mathbf{A})$.

The eigenvector corresponding to the eigenvalue $\lambda_1 = 0$ is given by

$$\mathbf{v} = \begin{pmatrix} 1 \\ -\frac{a}{b} \end{pmatrix}, \quad (b \neq 0) \quad \text{or} \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (b = 0)\,. \tag{3.35}$$

This corresponds with the straight line of equilibrium points.

The eigenvector corresponding to the eigenvalue $\lambda_2 = \operatorname{tr}(\mathbf{A})$ is given by

$$\mathbf{w} = \begin{pmatrix} 1 \\ \frac{c}{a} \end{pmatrix}, \quad (a \neq 0) \quad \text{or} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (a = 0)\,. \tag{3.36}$$

The general solution (3.4) of the two-dimensional dynamical system, i.e.

$$\mathbf{x}(t) = \alpha \mathbf{v} e^{\lambda_1 t} + \beta \mathbf{w} e^{\lambda_2 t}, \tag{3.37}$$

leads to

$$\frac{y}{x} = \frac{\alpha v_2 e^{\lambda_1 t} + \beta w_2 e^{\lambda_2 t}}{\alpha v_1 e^{\lambda_1 t} + \beta w_1 e^{\lambda_2 t}} \tag{3.38}$$

which for $\lambda_1 = 0$ reduces to

$$\frac{y}{x} = \frac{\alpha v_2 + \beta w_2 e^{\lambda_2 t}}{\alpha v_1 + \beta w_1 e^{\lambda_2 t}} \tag{3.39}$$

For $\beta = 0$, we obtain the line of equilibrium points. For $\alpha = 0$, we obtain a straight-line solution

$$y = \frac{w_2}{w_1} x,$$

aligned with the second eigenvector $\mathbf{w}$.

3.3.2 $\operatorname{tr}(\mathbf{A}) \neq 0$

If $\lambda_2 < 0$, we can see from equation (3.39) that

$$\frac{y}{x} \to \begin{cases} v_2/v_1 & \text{as } t \to \infty \\ w_2/w_1 & \text{as } t \to -\infty, \end{cases} \tag{3.40}$$

or, when $\lambda_2 > 0$,

$$\frac{y}{x} \to \begin{cases} w_2/w_1 & \text{as } t \to \infty \\ v_2/v_1 & \text{as } t \to -\infty, \end{cases} \tag{3.41}$$

In both cases ($\lambda_2 \neq 0$), we can write

$$e^{\lambda_2 t} = \frac{1}{\beta w_1}\left(x - \alpha v_1\right),$$

so that

$$y = \alpha v_2 + \frac{w_2}{w_1}\left(x - \alpha v_1\right) = \alpha\left(v_2 - \frac{w_2}{w_1}v_2\right) + \frac{w_2}{w_1}x.$$

Hence, the trajectories are straight lines, with slopes aligned to the eigenvector of the non-zero eigenvalue, where the solution either tends to the line of equilibrium points ($\lambda_2 < 0$) (in which case the equilibrium points are asymptotically stable) or away from the line of equilibrium points ($\lambda_2 > 0$) (in which case the equilibrium points are unstable).

We classify the equilibria in this case as a **stable** ($\lambda_2 < 0$) or **unstable** ($\lambda_2 > 0$) **comb**.

▶ Example 48 ◀ Consider the dynamical system

$$\begin{aligned} \dot{x} &= 2x, \\ \dot{y} &= x. \end{aligned} \tag{3.42}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

The dynamical system (3.42) has equilibrium points on the line $x = 0$. In this case

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.43}$$

which has eigenvalues $\lambda_1 = 0$, and $\lambda_2 = 2$, with associated eigenvectors

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{3.44}$$

Following the earlier analysis, the trajectories are given by

$$y = \gamma + \frac{1}{2}x,$$

where γ is an undetermined constant, so the trajectories are straight lines parallel to the direction given by the eigenvector $\mathbf{w}$. If you try to find the horizontal or vertical isoclines you'll recover the line of equilibrium points. We know all parallel lines have the same

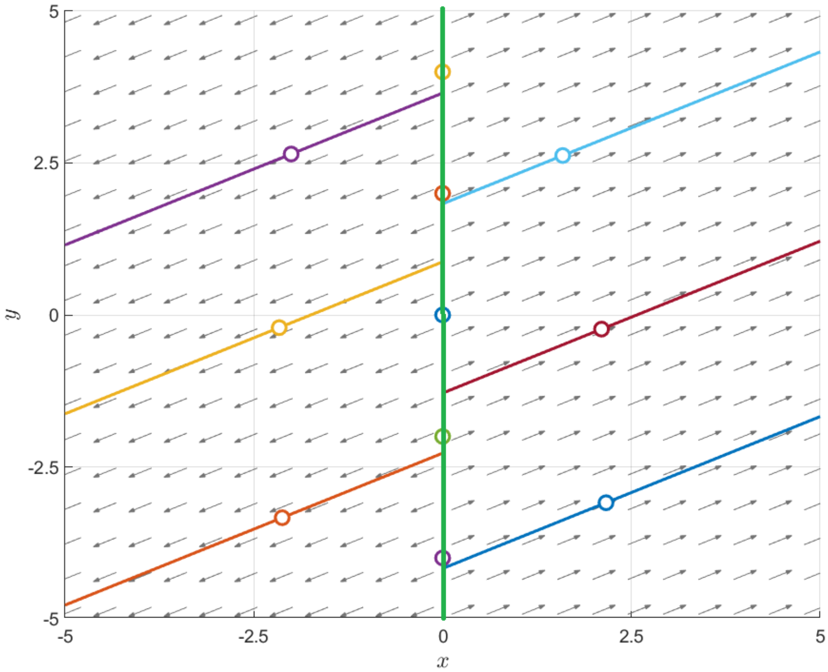


Figure 3.8: Phase portrait of the dynamical system (3.42). The bolder green line is the line of equilibrium points.

slope but not whether the points move up and to the right, or down and to the left. To determine the direction of change, let us consider the line $y = 0$. On this line,

$$\dot{x} = 2x \quad \begin{cases} > 0, & x > 0, \\ < 0, & x < 0, \end{cases}$$

and

$$\dot{y} = x \quad \begin{cases} > 0, & x > 0, \\ < 0, & x < 0. \end{cases}$$

So solutions move down and to the left for $x < 0$ and hence everywhere to the left of the line of equilibrium points, and to the right and up for $x > 0$ and hence everywhere to the right of the line of equilibrium points.

The phaseportrait is given in Figure 3.8.

As the solutions move away from the equilibrium points towards ∞, this is an unstable comb.

▶ Example 49 ◀ Consider the dynamical system

$$\begin{aligned}\dot{x} &= 3x + 2y,\\ \dot{y} &= -6x - 4y.\end{aligned} \tag{3.45}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

The dynamical system (3.45) has equilibrium points on the line $y = -\frac{3x}{2}$. In this case

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix}, \tag{3.46}$$

which has eigenvalues $\lambda_1 = 0$, and $\lambda_2 = -1$, with associated eigenvectors

$$\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{3.47}$$

Following the earlier analysis, the trajectories are given by

$$y = \gamma - 2x,$$

where γ is an undetermined constant, so the trajectories are straight lines parallel to the direction given by the eigenvector $\mathbf{w}$. If you try to find the horizontal or vertical isoclines you'll again recover the line of equilibrium points. To determine the direction of change, let us consider the line $y = 0$. On this line,

$$\dot{x} = 3x \quad \begin{cases} > 0, & x > 0, \\ < 0, & x < 0, \end{cases}$$

and

$$\dot{y} = -6x \quad \begin{cases} < 0, & x > 0, \\ > 0, & x < 0. \end{cases}$$

So solutions move down and to the right for $x > 0, y = 0$ and hence everywhere to the right of the line of equilibrium points, and to the left and up for $x < 0, y = 0$ and hence everywhere to the left of the line of equilibrium points.

The phaseportrait is given in Figure 3.9.

As the solutions move towards the equilibrium points, this is a stable comb.

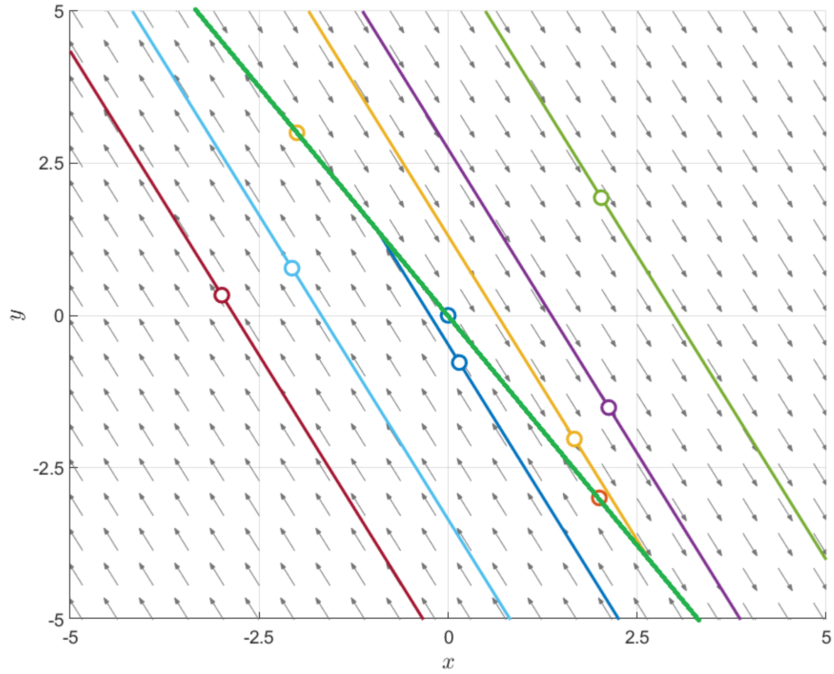


Figure 3.9: Phase portrait of the dynamical system (3.45). The bolder green line is the line of equilibrium points.

3.3.3 $\operatorname{tr}(\mathbf{A}) = 0$

The only remaining case is when $\lambda_2 = \operatorname{tr}(\mathbf{A}) = 0$. Since $\mathbf{A} \neq \mathbf{O}$, this cannot be the case if the matrix $\mathbf{A}$ is diagonal. Hence, the zero eigenvalue has multiplicity 2 but only has one eigenvector, $\mathbf{v}$, aligned with the line of equilibrium points. We can find a generalised eigenvector $\mathbf{w}$ to complement $\mathbf{v}$.

In this case, the solutions to the linear system (3.1) are given by

$$\mathbf{x} = \alpha \mathbf{v} e^{\lambda_1 t} + \beta \left(\mathbf{v} t e^{\lambda_1 t} + \mathbf{w} e^{\lambda_1 t} \right),$$

or,

$$\mathbf{x} = \alpha \mathbf{v} + \beta (\mathbf{v} t + \mathbf{w}) = (\alpha \mathbf{v} + \beta \mathbf{w}) + \beta \mathbf{v} t.$$

Than

$$\frac{y}{x} = \frac{\alpha v_2 + \beta w_2 + \beta v_2 t}{\alpha v_1 + \beta w_1 + \beta v_1 t}, \tag{3.48}$$

If $\beta = 0$, then

$$\frac{y}{x} = \frac{v_2}{v_1}, \tag{3.49}$$

which represents the line with equilibrium points.

If $\beta \neq 0$, than time dependence is linear (not exponential) and the trajectories tend to infinity when $t \to \pm\infty$. Also

$$t = \frac{1}{\beta v_1} (x - \alpha v_1 - \beta w_1),$$

so that

$$y = (\alpha v_2 + \beta w_2) + \frac{v_2}{v_1}(x - \alpha v_1 - \beta w_1),$$

or

$$y = \beta\left(w_2 - \frac{v_2}{v_1}w_1\right) + \frac{v_2}{v_1}x.$$

So the trajectories are straight lines parallel to the line with equilibrium points. This phase portrait is classified as a **shear**.

Note that when $v_1 = 0$, the trajectories are defined by vertical lines, $x = \beta w_1$.

► Example 50 ◄ Consider the dynamical system

$$\begin{aligned} \dot{x} &= 2x - 4y, \\ \dot{y} &= x - 2y. \end{aligned} \tag{3.50}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

The dynamical system (3.50) has equilibrium points on the line $y = \frac{x}{2}$. In this case

$$\mathbf{A} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}, \tag{3.51}$$

which has eigenvalues $\lambda_1 = 0$, and $\lambda_2 = 0$, with associated eigenvector

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We need to solve

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{w} = \mathbf{v},$$

or, as $\lambda = 0$,

$$\mathbf{A}\mathbf{w} = \mathbf{v},$$

which gives the generalised eigenvector

$$\mathbf{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Following the earlier analysis, the trajectories are given by

$$y = \gamma + \frac{1}{2}x,$$

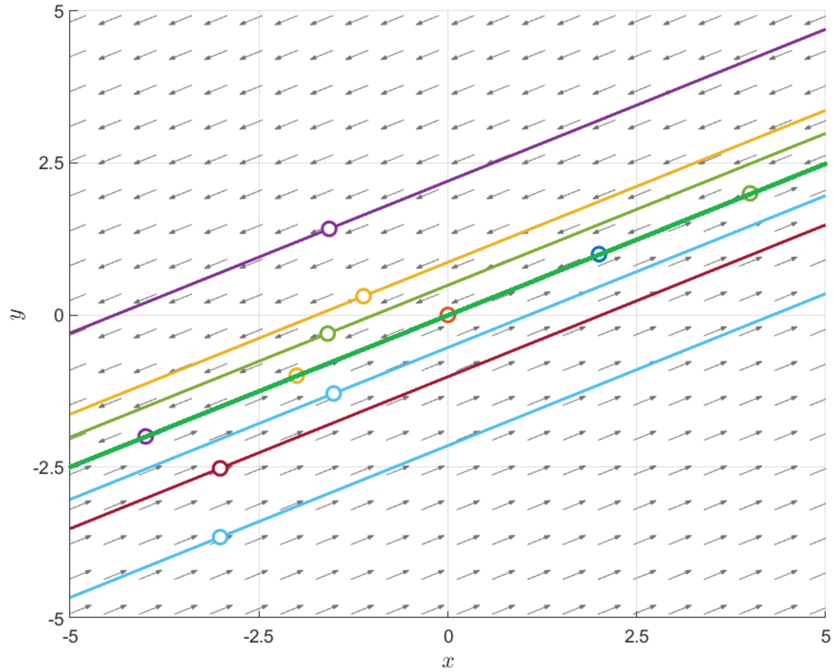


Figure 3.10: Phase portrait of the dynamical system (3.50). The bolder green line is the line of equilibrium points.

where γ is an undetermined constant, so the trajectories are straight lines parallel to the line with equilibrium points ($y = \frac{x}{2}$). If you try to find the horizontal or vertical isoclines you'll recover the line of equilibrium points. We know all parallel lines have the same slope but not whether the points move up and to the right, or down and to the left. To determine the direction of change, let us consider the line $x = 0$. On this line,

$$\dot{x} = -4y \qquad \begin{cases} < 0, & y > 0, \\ > 0, & y < 0, \end{cases}$$

and

$$\dot{y} = -2y \qquad \begin{cases} < 0, & y > 0, \\ > 0, & y < 0. \end{cases}$$

So solutions move down and to the left for $x = 0, y > 0$ and hence everywhere above the line of equilibrium points, and to the right and up for $x = 0, y < 0$ and hence everywhere below the line of equilibrium points.

The phaseportrait is given in Figure 3.10.

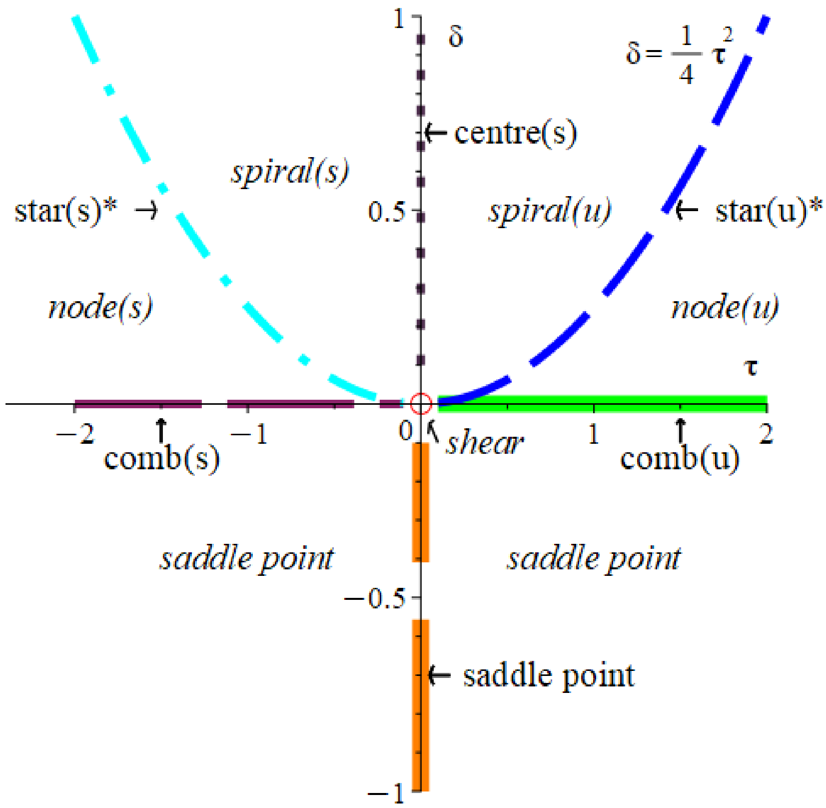


Figure 3.11: Overview of nature of equilibrium points. (s) denotes a stable equilibrium and (u) denotes an unstable equilibrium. The two curve segments labelled "star*" could also be degenerate nodes.

3.4 Summary: classification of equilibrium points

All the results can now be grouped together in a single diagram, given in Figure 3.11. The parabola represents all cases where $\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) = 0$. Above the parabola, $\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) < 0$ and we have complex conjugate eigenvalues. Underneath the parabola, we have real eigenvalues.

To summarize:

1. $\delta = \det(\mathbf{A}) > 0$. Here, there are several possiblities:

   (a) $\delta = \det(\mathbf{A}) > \frac{1}{4}\mathrm{tr}^2(\mathbf{A}) = \frac{1}{4}\tau^2$.
   The eigenvalues of $\mathbf{A}$ are complex conjugate, $\lambda_\pm = \mu \pm i\gamma$. There are three cases to consider:

      i. $\tau > 0$.
      In this case, $\mu = \frac{1}{2}\tau > 0$ and the equilibrium points is a **unstable focus**, also called a **unstable spiral**.

ii. $\tau = 0$.
   In this case, $\mu = \frac{1}{2}\tau = 0$ and we have two opposite purely imaginary eigenvalues, so the equilibrium points is a **centre**.
iii. $\tau < 0$.
   In this case, $\mu = \frac{1}{2}\tau < 0$ and the equilibrium points is a **stable focus**, also called a **stable spiral**.

(b) $\delta = \det(\mathbf{A}) = \frac{1}{4}\mathrm{tr}^2(\mathbf{A}) = \frac{1}{4}\tau^2$.
   The two eigenvalues are real and equal. There are two cases to consider:
   i. If the matrix $\mathbf{A}$ is diagonal then the equilibrium point is a **star**. This node is **stable** if $\tau < 0$, **unstable** if $\tau > 0$.
   ii. If the matrix $\mathbf{A}$ is **not** diagonal then the equilibrium point is a **degenerate node**. This node is **stable** if $\tau < 0$, **unstable** if $\tau > 0$. There is only one independent eigenvector in this case.

(c) $\delta = \det(\mathbf{A}) < \frac{1}{4}\mathrm{tr}^2(\mathbf{A}) = \frac{1}{4}\tau$. The eigenvalues of $\mathbf{A}$ are real, different, and of the same sign. In this case the equilibrium point is a **stable node** if $\tau < 0$ and an **unstable node** if $\tau > 0$.

2. $\delta = \det(\mathbf{A}) = 0$.

   These are the degenerate cases where at least one of the eigenvalues is zero, and the second is given by $\lambda_2 = \mathrm{tr}(\mathbf{A}) = \tau$. There is a line of equilibrium points. We consider two cases:

   (a) $\tau \neq 0$. The equilibrium points form an **unstable comb** when $\tau > 0$ and a **stable comb** when $\tau < 0$.

   (b) $\tau = 0$. Both eigenvalues are 0 and the line of equilibrium points is referred to as a **shear**.

3. $\delta = \det(\mathbf{A}) < 0$.

   The eigenvalues of $\mathbf{A}$ are real and of opposite sign. In this case the equilibrium point is a **saddle point**. Even if $\tau = \mathrm{tr}(\mathbf{A}) = 0$, the equilibrium point is a saddle points, as this simply means the two real eigenvalues of different sign are each other's opposite so their sum is 0.

3.5 Linear dynamical systems not of the standard form

What do we do if the linear system we wish to study is not of the form (3.1)? Consider the system

$$\begin{aligned} \dot{x} &= Ax + By + \epsilon, \\ \dot{y} &= Cx + Dy + \delta, \end{aligned} \tag{3.52}$$

where ϵ and δ are constants. The system (3.52) admits the unique equilibrium point

$$(x_e, y_e) = \left(\frac{(B\delta - D\epsilon)}{(AD - BC)}, \frac{(C\epsilon - A\delta)}{(AD - BC)} \right).$$

In order to follow the methods of the previous sections we need to transform (3.52) to standard form. We accomplish this by the transformation

$$X = x - x_e, \quad Y = y - y_e. \tag{3.53}$$

On writing (3.52) in terms of X and Y we obtain

$$\begin{aligned} \dot{X} &= AX + BY, \\ \dot{Y} &= CX + DY, \end{aligned} \tag{3.54}$$

which is now in the form (3.1). However, it is important to note that once the phase portrait of (3.54) has been determined we will need to need to revert back to the original coordinates x and y and then sketch the phase portrait of the original system.

3.6 Further examples

► Example 51 ◄ Consider the dynamical system

$$\begin{aligned} \dot{x} &= -2x - 2y, \\ \dot{y} &= -x - 3y. \end{aligned} \tag{3.55}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

The dynamical system (3.55) has an equilibrium point at $(0, 0)$. In this case

$$\mathbf{A} = \begin{pmatrix} -2 & -2 \\ -1 & -3 \end{pmatrix}, \tag{3.56}$$

which has eigenvalues $\lambda_1 = -1$, and $\lambda_2 = -4$, with associated eigenvectors

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.57}$$

Thus the equilibrium point $(0,0)$ is a stable node (since the eigenvalues are real and negative). To plot the phase plane, we note that the straight line paths are given by $\mathbf{v}$ and $\mathbf{w}$, and that, since the equilibrium point is stable, all trajectories approach $(0,0)$ as $t \to \infty$.

The horizontal isocline is given by $y = -\frac{1}{3}x$. The direction field along the horizontal isocline is given by

$$\dot{x} = -2x - 2\left(-\frac{1}{3}x\right) = -\frac{4}{3}x \quad \begin{cases} > 0, & x < 0, \\ < 0, & x > 0. \end{cases}$$

The vertical isocline is given by $y = -x$. The direction field along the vertical isocline is given by

$$\dot{y} = -x - 3(-x) = 2x \quad \begin{cases} > 0, & x > 0, \\ < 0, & x < 0. \end{cases}$$

The resulting phase portrait is given in Figure 3.12.

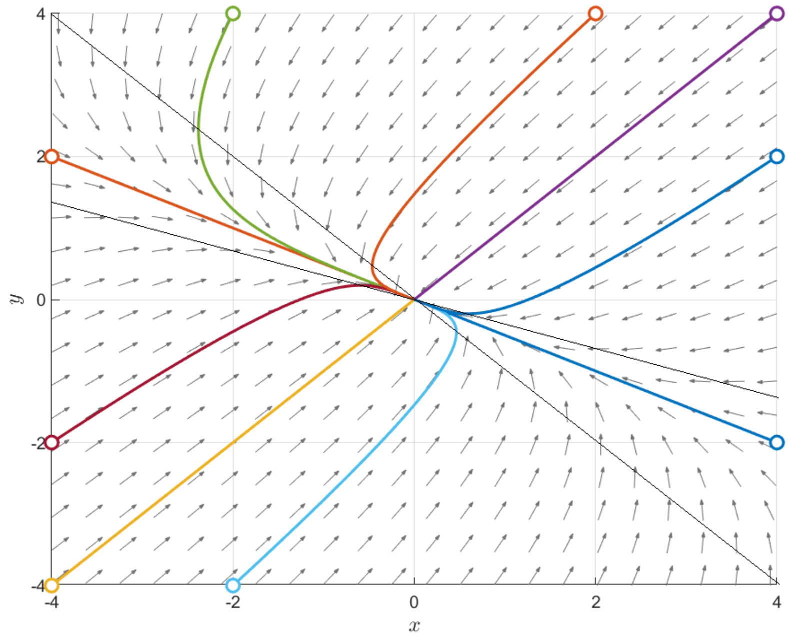


Figure 3.12: Phase portrait of system (3.55). We note that the equilibrium point at $(0,0)$ is a stable node. The eigenvectors $\mathbf{v}$ and $\mathbf{w}$ are given in orange \ blue and yellow \ purple with additional phase paths being given in a variety of colours. We observe that all phase paths (other than the phase paths given by $\mathbf{w}$) approach the equilibrium point along $\mathbf{v}$ as $t \to \infty$.

▶ Example 52 ◀ Consider the dynamical system

$$\begin{aligned}\dot{x} &= 2x + 4y - 8,\\ \dot{y} &= 3x + 3y - 9.\end{aligned} \tag{3.58}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

The dynamical system (3.58) has an equilibrium point at $(2, 1)$. So we transform the dynamical sysem by substituting

$$\begin{aligned}X &= x - 2,\\ Y &= y - 1.\end{aligned} \tag{3.59}$$

In the new variables, the system becomes

$$\begin{aligned}\dot{X} &= 2X + 4Y,\\ \dot{Y} &= 3X + 3Y.\end{aligned} \tag{3.60}$$

Hence

$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}, \tag{3.61}$$

which has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -1$, with the associated eigenvectors

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \tag{3.62}$$

In this case the equilibrium point $(X, Y) = (0, 0)$ is a saddle point, and so is the original equilibrium point $(x, y) = (2, 1)$.

The horizontal isocline is given by $y = -x+3$. The direction field along the horizontal isocline is given by

$$\dot{x} = 2x + 4\left(-x + 3\right) - 8 = -2\left(x - 2\right) \quad \begin{cases} > 0, & x < 2, \\ < 0, & x > 2. \end{cases}$$

The vertical isocline is given by $y = -\frac{x}{2} + 2$. The direction field along the vertical isocline is given by

$$\dot{y} = 3x + 3\left(-\frac{x}{2} + 2\right) - 9 = \frac{3}{2}\left(x - 2\right) \quad \begin{cases} > 0, & x > 2, \\ < 0, & x < 2. \end{cases}$$

The phase portrait of (3.58) is given in Figure 3.13.

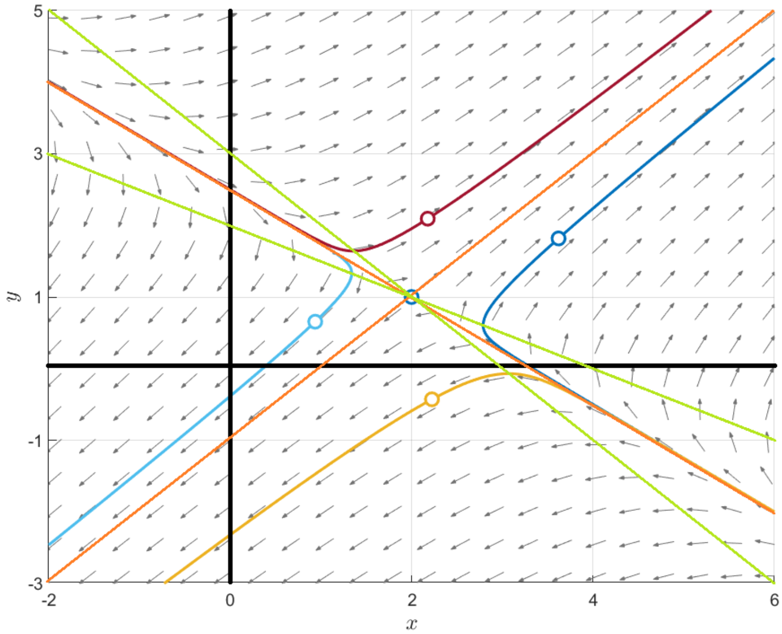


Figure 3.13: Phase portrait of system (3.63).

We note that the equilibrium point at $(2, 1)$ is a saddle point. We observe that all phase paths approach the equilibrium point along $\mathbf{w}$ (the eigenvectors are given in orange) as $t \to \infty$. The horizontal and vertical isoclines are added in light green.

All solutions $(x(t), y(t))$ starting above straight line solutions associated with $(\lambda_2, \mathbf{w})$ will tend to (∞, ∞) as $t \to \infty$. All solutions $(x(t), y(t))$ starting below straight line solutions associated with $(\lambda_2, \mathbf{w})$ will tend to $(-\infty, -\infty)$ as $t \to \infty$.

► Example 53 ◄ Consider the dynamical system

$$\begin{aligned} \dot{x} &= x - y + 3, \\ \dot{y} &= 4x - 3y + 10. \end{aligned} \tag{3.63}$$

Determine, through the eigenvalues and eigenvectors, the type of equilibrium, and, sketch the phase portrait, using the techniques given in the previous chapter.

The dynamical system (3.63) has an equilibrium point at $(-1, 2)$. So we transform the dynamical sysem by substituting

$$\begin{aligned} X &= x + 1, \\ Y &= y - 2. \end{aligned} \tag{3.64}$$

In the new variables, the system becomes

$$
\begin{aligned}
\dot{X} &= X - Y, \\
\dot{Y} &= 4X - 3Y.
\end{aligned}
\tag{3.65}
$$

Hence

$$
\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}, \tag{3.66}
$$

which has a single eigenvalue $\lambda_{1,2} = -1$, with the associated eigenvector

$$
\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{3.67}
$$

In this case the equilibrium point $(X, Y) = (0, 0)$ is a stable degenerate node, and so is the original equilibrium point $(x, y) = (-1, 2)$.

The horizontal isocline is given by $y = \frac{4}{3}x + \frac{10}{3}$. The direction field along the horizontal isocline is given by

$$
\dot{x} = x - \left(\frac{4}{3}x + \frac{10}{3}\right) + 3 = -\frac{1}{3}(x+1) \quad \begin{cases} > 0, & x < -1, \\ < 0, & x > -1. \end{cases}
$$

The vertical isocline is given by $y = x + 3$. The direction field along the vertical isocline is given by

$$
\dot{y} = 4x - 3(x+3) + 10 = x + 1 \quad \begin{cases} > 0, & x > -1, \\ < 0, & x < -1. \end{cases}
$$

The phase portrait of (3.63) is given in Figure 3.14.

We note that the equilibrium point at $(-1, 2)$ is a stable degenerate node. We observe that all phase paths approach the equilibrium point as $t \to \infty$ along $\mathbf{v}$ (the eigenvector given in orange).

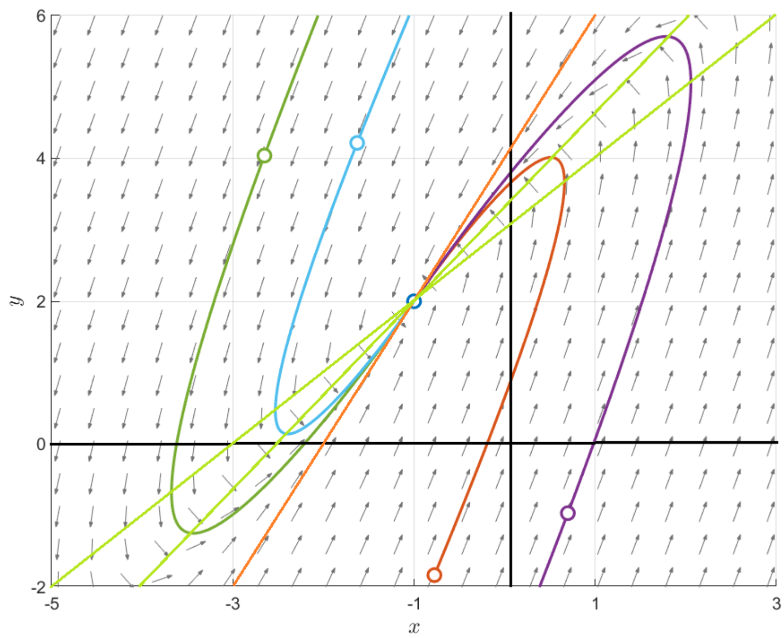


Figure 3.14: Phase portrait of system (3.63).

Chapter 4

Nonlinear dynamical systems

Having investigated the behaviour of linear dynamical systems, we will now consider 2-dimensional, autonomous dynamical systems of the form,

$$\begin{aligned} \dot{x} &= P\left(x, y\right), \\ \dot{y} &= Q\left(x, y\right), \end{aligned} \tag{4.1}$$

with $t \in I \subseteq \mathbb{R}$, $(x, y) \in D \subseteq \mathbb{R}^2$, and where P, $Q : \mathbb{R}^2 \to \mathbb{R}$. The functions P and Q are continuous with continuous first partial derivatives. In most cases we will assume that all partial derivatives of P and Q exist and are continuous. We begin by considering the **local behaviour** of the system (4.1) close to it's equilibrium points.

4.1 Local Analysis

In Chapter 1 we used linear stability analysis to investigate the behaviour of solutions close to equilibrium points. We will use similar methods here to investigate the behaviour of the system (4.1) close to it's equilibrium points $(x, y) = (x_e, y_e)$.

The first order Taylor series expansions of P and Q, about an equilibrium point, are given by

$$\begin{aligned} P\left(x, y\right) &= P\left(x_e, y_e\right) + \left(x - x_e\right) \left.\frac{\partial P}{\partial x}\right|_{(x_e, y_e)} + \left(y - y_e\right) \left.\frac{\partial P}{\partial y}\right|_{(x_e, y_e)} + \cdots \\ Q\left(x, y\right) &= Q\left(x_e, y_e\right) + \left(x - x_e\right) \left.\frac{\partial Q}{\partial x}\right|_{(x_e, y_e)} + \left(y - y_e\right) \left.\frac{\partial Q}{\partial y}\right|_{(x_e, y_e)} + \cdots \end{aligned} \tag{4.2}$$

These approximations are valid provided we are **close** to the equilibrium point. We recall that the definition of an equilibrium point is that $P\left(x_e, y_e\right) = 0$ and $Q\left(x_e, y_e\right) = 0$, so that we can write (4.1) as

$$\begin{aligned} \dot{x} &= a\left(x - x_e\right) + b\left(y - y_e\right) + \text{smaller terms}, \\ \dot{y} &= c\left(x - x_e\right) + d\left(y - y_e\right) + \text{smaller terms}, \end{aligned} \tag{4.3}$$

where

$$a = \frac{\partial P}{\partial x}_{(x_e, y_e)}, \quad b = \frac{\partial P}{\partial y}_{(x_e, y_e)}, \quad c = \frac{\partial Q}{\partial x}_{(x_e, y_e)}, \quad d = \frac{\partial Q}{\partial y}_{(x_e, y_e)}. \tag{4.4}$$

If we ignore the smaller terms in (4.3) we obtain the **linearisation** of (4.1) as

$$\begin{aligned} \dot{x} &= a\,(x - x_e) + b\,(y - y_e)\,, \\ \dot{y} &= c\,(x - x_e) + d\,(y - y_e)\,. \end{aligned} \tag{4.5}$$

If we write

$$X = x - x_e, \quad Y = y - y_e, \tag{4.6}$$

then we obtain the associated linear system

$$\begin{aligned} \dot{X} &= aX + bY, \\ \dot{Y} &= cX + dY, \end{aligned} \tag{4.7}$$

which has a unique equilibrium point at $(0, 0)$. We can write the linear system (4.7) in the form

$$\dot{\mathbf{X}} = \mathbf{J}\mathbf{X}, \tag{4.8}$$

where $\mathbf{J}$ is the Jacobian matrix, evaluated at $(x, y) = (x_e, y_e)$, given by

$$\mathbf{J} = \left.\begin{pmatrix} \dfrac{\partial P}{\partial x} & \dfrac{\partial P}{\partial y} \\[2ex] \dfrac{\partial Q}{\partial x} & \dfrac{\partial Q}{\partial y} \end{pmatrix}\right|_{(x_e, y_e)} . \tag{4.9}$$

It is reasonable to expect that the behaviour of the nonlinear system close to the equilibrium point will be approximated by the behaviour of the associated linear system. But some linear results depend on the eigenvalues being a particular value and that might mean the small terms ignored in the linear analysis may come into play.

It can be shown that the following property holds:

Theorem 2. ***Linearisation**: The behaviour of the phase paths in the neighbourhood of an equilibrium point of the nonlinear dynamical system (4.1) is the same as that of the associated linearised system* ***except*** *when the linearised system has a zero eigenvalue or purely imaginary eigenvalues.*

Note when the eigenvalues are purely imaginary the equilibrium point of the nonlinear system may be a centre or a spiral.

There is one particular case, however, when we can be more precise about what happens when the eigenvalues are purely imaginary values and hence indicate a centre.

Theorem 3. *If the nonlinear equation*

$$\ddot{x} + f\,(x) = 0, \tag{4.10}$$

has an equilibrium point in the phase plane where the linearised equations indicate a centre, then the nonlinear equation also has a centre at this equilibrium point.

Proof. Write (4.10) as a 2-dimensional dynamical system by writing,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f(x). \end{aligned} \tag{4.11}$$

Suppose there exists an equilibrium point at $(x_e, 0)$ (that is, $f(x_e) = 0$) and that the linearised system indicates that this point is a centre. With Jacobian

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -f'(x_e) & 0 \end{pmatrix},$$

and eigenvalues given by $\lambda^2 = -f'(x_e)$, this means that $f'(x_e) > 0$. Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{f(x)}{y}. \tag{4.12}$$

On integrating we obtain

$$y^2 = -F(x) + c, \tag{4.13}$$

where c is a constant, and $F(x) = 2\int f(x)\,\mathrm{d}x$. The points of intersection with $x = x_e$ are given by

$$y^2 = -F(x_e) + c. \tag{4.14}$$

There can only be two points of intersection here (quadratic equation), so we can exclude the possibility of the equilibrium point being a spiral. □

4.2 Examples

▶ Example 54 ◀ Consider the following nonlinear dynamical system,

$$\begin{aligned} \dot{x} &= -xy + 1, \\ \dot{y} &= xy - y. \end{aligned} \tag{4.15}$$

Determine the nature of all equilibrium points and, following the guidance given in Chapter 2, sketch the phase portrait.

First, we must establish all equilibrium points of (4.15). The equilibrium points are solutions of

$$\begin{aligned} P(x, y) &= -xy + 1 = 0, \\ Q(x, y) &= xy - y = 0. \end{aligned} \tag{4.16}$$

The system (4.16) has a unique solution, given by $(x_e, y_e) = (1, 1)$.

Second, we calculate the Jacobian, evaluated at the equilibrium point $(x_e, y_e) = (1, 1)$, as

$$\mathbf{J} = \begin{pmatrix} -y & -x \\ y & x-1 \end{pmatrix}\Bigg|_{(1,1)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.17}$$

so that for

$$\begin{aligned} X &= x - x_e \, (= x - 1) \, , \\ Y &= y - y_e \, (= y - 1) \, , \end{aligned} \tag{4.18}$$

we obtain the linear system

$$\begin{aligned} \dot{X} &= -X - Y, \\ \dot{Y} &= X. \end{aligned} \tag{4.19}$$

The eigenvalues are $\lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, so the equilibrium point $(X, Y) = (0, 0)$ of the linear system is an stable spiral.

Third, we consider other features of the phase plane, such as isoclines. Considering the horizontal and vertical isoclines we have that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\,(x-1)}{1-xy} = \begin{cases} 0, & y = 0 \text{ or } x = 1 \,(\text{with } y \neq 1)\, , \\ \infty, & y = \frac{1}{x}. \end{cases} \tag{4.20}$$

What is the direction of flow on the isoclines? Let's consider the horizontal isocline. When $x = 1$ (with $y \neq 1$) we have

$$\dot{x} = 1 - y \begin{cases} > 0, & y < 1, \\ < 0, & y > 1. \end{cases} \tag{4.21}$$

Therefore, when $y > 1$, $\dot{x} < 0$ and x decreases as t increases, while when $y < 1$, $\dot{x} > 0$ and x increases as t increases.

When $y = 0$ we have

$$\dot{x} = 1 \Big\{ > 0, \quad \text{for all } y. \tag{4.22}$$

Therefore, when $y = 0$, $\dot{x} > 0$ and x increases as t increases.

For the vertical isocline, $y = \frac{1}{x}$, we have

$$\dot{y} = \frac{x-1}{x} \begin{cases} > 0, & x < 0, \\ < 0, & 0 < x < 1, \\ < 0, & x > 1. \end{cases} \tag{4.23}$$

Therefore, when $x < 0$, $\dot{y} > 0$ and y increases as t increases, when $0 < x < 1$,$\dot{y} < 0$ and y decreases as t increases, and,when $x > 1$, $\dot{y} > 0$ and y increases as t increases.

Further, it is useful to note that the curve $y = 0$ contains straight line trajectories. It may also help to check the gradient on the lines $y = x$ and $y = -x$.

Finally, combining all this information, we can now sketch the phase portrait for the system (4.15) shown in Figure 4.1.

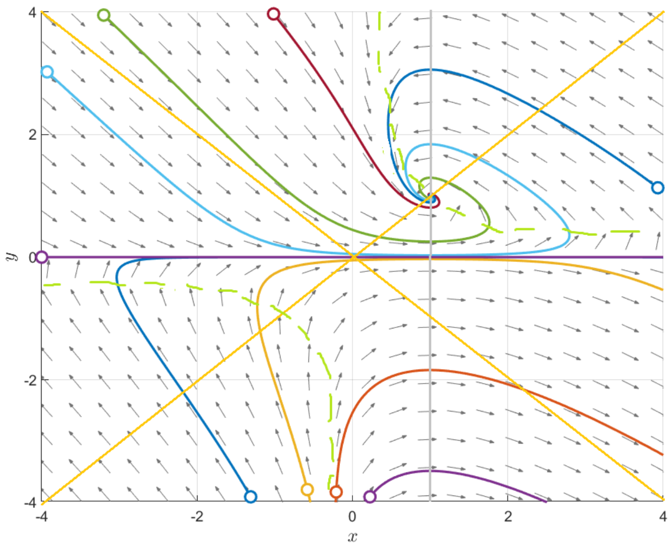


Figure 4.1: Phase portrait of the nonlinear dynamical system (4.15). Note equilibrium point $(1, 1)$ is a stable spiral, and $y = 0$ is a phase path.

If we consider a trajectory starting at an arbitrary initial condition $(x, y) = (x_0, y_0)$, we observe

(i) If $y_0 > 0$ then $(x(t), y(t)) \to (1, 1)$ as $t \to \infty$, via damped oscillations.

(ii) If $y_0 = 0$ then $(x(t), y(t)) \to (\infty, 0)$ as $t \to \infty$.

(iii) If $y_0 < 0$ then $(x(t), y(t)) \to (\infty, -\infty)$ as $t \to \infty$.

▶ Example 55 ◀ Consider the nonlinear dynamical system

$$\begin{aligned} \dot{x} &= -y - \left(x^2 + y^2\right) x, \\ \dot{y} &= x - \left(x^2 + y^2\right) y. \end{aligned} \tag{4.24}$$

Determine the nature of all equilibrium points and, following the guidance given in Chapter 2, sketch the phase portrait.

This system has a single equilibrium point at $(0, 0)$. To find the associated linear system we can either find the Jacobian matrix or, equivalently, just discard the nonlinear terms to find

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x, \end{aligned} \tag{4.25}$$

where $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, with eigenvalues $\lambda_{1,2} = \pm i$ indicating that $(0,0)$ is a centre. **However**, the linearisation theorem does not apply in this case, and $(0,0)$ is not necessarily a centre for the nonlinear system (despite being a centre for the linear system). Further investigation is required.

Writing the system (4.24) in polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{4.26}$$

with $r \geq 0$, and

$$\dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta, \quad \dot{\theta} = \frac{1}{r}\left(-\dot{x}\sin\theta + \dot{y}\cos\theta\right), \tag{4.27}$$

we obtain

$$\begin{aligned} \dot{r} &= -r^3, \\ \dot{\theta} &= 1. \end{aligned} \tag{4.28}$$

We see that, since $\dot{r} < 0$, $r(t)$ is decreasing with increasing t. Further, since $\dot{\theta} > 0$, $\theta(t)$ is increasing with increasing t (anticlockwise rotation). Therefore the equilibrium point at $(0,0)$ is a stable nonlinear spiral.

The horizontal and vertical isoclines are more difficult to plot, but useful information can be added by looking at the gradient along specifc lines like $x = 0$, $y = 0$, $y = x$ and $y = -x$.

When $x = 0$ we have

$$\begin{aligned} \dot{x} &= -y \begin{cases} > 0, & y < 0, \\ < 0, & y > 0. \end{cases} \\ \dot{y} &= -y^3 \begin{cases} > 0, & y < 0, \\ < 0, & y > 0. \end{cases} \end{aligned} \tag{4.29}$$

Therefore, when $x = 0$ and $y < 0$, $\dot{x}$ and $\dot{y}$ both increase as t increases, and when $x = 0$ and $y > 0$, $\dot{x}$ and $\dot{y}$ both decrease as t increases.

When $y = 0$ we have

$$\begin{aligned} \dot{x} &= -x^3 \begin{cases} > 0, & x < 0, \\ < 0, & x > 0. \end{cases} \\ \dot{y} &= x \begin{cases} < 0, & x < 0, \\ > 0, & x > 0. \end{cases} \end{aligned} \tag{4.30}$$

Therefore, when $y = 0$ and $x < 0$, $\dot{x}$ increases and $\dot{y}$ decreases as t increases, and when $y = 0$ and $x > 0$, $\dot{x}$ decreases and $\dot{y}$ increases as t increases.

When $y = x$ we have

$$\dot{x} = -x\left(1 + 2x^2\right)\begin{cases} > 0, & x < 0, \\ < 0, & x > 0. \end{cases}$$
$$\dot{y} = x\left(1 - 2x^2\right)\begin{cases} > 0, & x < -\frac{1}{\sqrt{2}}, \\ < 0, & -\frac{1}{\sqrt{2}} < x < 0, \\ > 0, & 0 < x < \frac{1}{\sqrt{2}}, \\ < 0, & x > \frac{1}{\sqrt{2}}. \end{cases} \tag{4.31}$$

Therefore, when $y = x$ and $x < -\frac{1}{\sqrt{2}}$, $\dot{x}$ and $\dot{y}$ both increase as t increases, when $y = x$ and $-\frac{1}{\sqrt{2}} < x < 0$, $\dot{x}$ increases and $\dot{y}$ decreases as t increases, when $y = x$ and $0 < x < \frac{1}{\sqrt{2}}$, $\dot{x}$ decreases and $\dot{y}$ increases as t increases, and, finally, when $y = x$ and $x > \frac{1}{\sqrt{2}}$, $\dot{x}$ and $\dot{y}$ both decrease as t increases.

When $y = -x$ we have

$$\dot{x} = x\left(1 - 2x^2\right)\begin{cases} > 0, & x < -\frac{1}{\sqrt{2}}, \\ < 0, & -\frac{1}{\sqrt{2}} < x < 0, \\ > 0, & 0 < x < \frac{1}{\sqrt{2}}, \\ < 0, & x > \frac{1}{\sqrt{2}}. \end{cases}$$
$$\dot{y} = x\left(1 + 2x^2\right)\begin{cases} < 0, & x < 0, \\ > 0, & x > 0. \end{cases} \tag{4.32}$$

Therefore, when $y = -x$ and $x < -\frac{1}{\sqrt{2}}$, $\dot{x}$ increases and $\dot{y}$ decreases as t increases, when $y = -x$ and $-\frac{1}{\sqrt{2}} < x < 0$, $\dot{x}$ and $\dot{y}$ both decrease as t increases, when $y = -x$ and $0 < x < \frac{1}{\sqrt{2}}$, $\dot{x}$ and $\dot{y}$ both increase as t increases, and, finally, when $y = -x$ and $x > \frac{1}{\sqrt{2}}$, $\dot{x}$ decreases and $\dot{y}$ increases as t increases.

The full phase portrait is given in Figure 4.2.

All trajectories tend to $(0, 0)$ as $t \to \infty$.

So far we have only considered the local behaviour of a nonlinear system about a single equilibrium point. What happens when a system has more than one equilibrium point? We need to consider what happens locally to each, but also what happens globally.

► Example 56 ◄ Consider the dynamical system

$$\begin{aligned} \dot{x} &= x - 2y, \\ \dot{y} &= 4x - x^3, \end{aligned} \tag{4.33}$$

Determine the nature of all equilibrium points and, following the guidance given in Chapter 2, sketch the phase portrait.

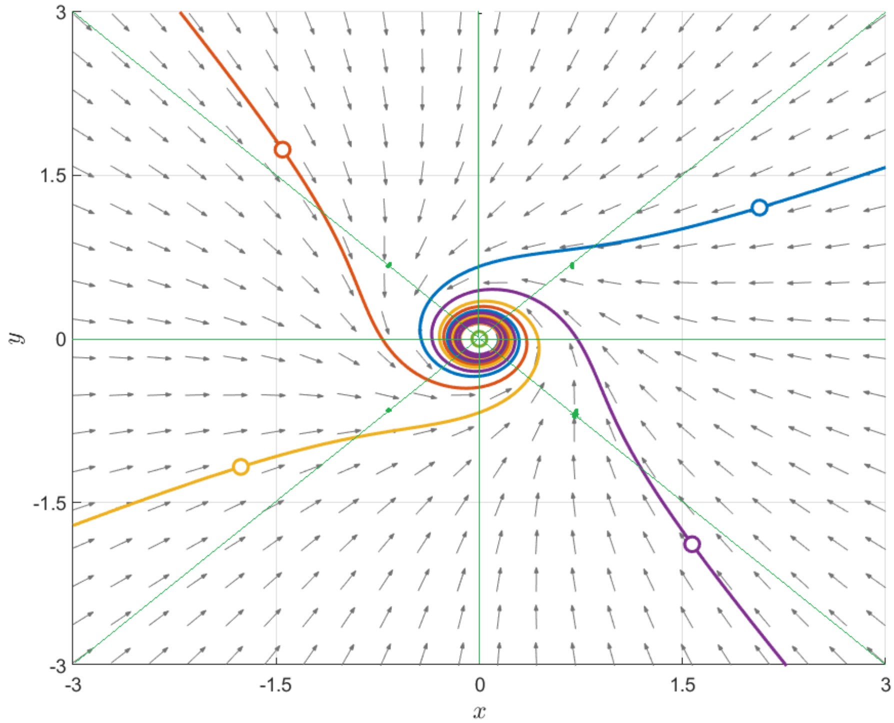


Figure 4.2: Phase portrait of system (4.24). In this case the equilibrium point is a nonlinear spiral

This system has three equilibrium points at $(0, 0)$, $(2, 1)$ and $(-2, -1)$. To begin, we must consider the *local* behaviour of the system (4.33) about each of the equilibrium points.

1. $(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$

The associated linear system is given by

$$\begin{aligned} \dot{x} &= x - 2y, \\ \dot{y} &= 4x, \end{aligned} \tag{4.34}$$

where $\mathbf{J} = \begin{pmatrix} 1 & -2 \\ 4 & 0 \end{pmatrix}$, with eigenvalues $\lambda_{1,2} = 1/2 \pm i\sqrt{31}/2$. Thus the equilibrium point of the linear system is an unstable spiral. Therefore, by the linearisation theorem, $(0, 0)$ is an unstable spiral for the nonlinear system (4.33).

2. $(\mathbf{x}, \mathbf{y}) = (\mathbf{2}, \mathbf{1})$

Following previous examples we write

$$x = 2 + X, \quad y = 1 + Y, \tag{4.35}$$

and, neglecting nonlinear terms, the associated linear system is given by

$$\begin{aligned} \dot{X} &= X - 2Y, \\ \dot{Y} &= -8X, \end{aligned} \tag{4.36}$$

where $\mathbf{J} = \begin{pmatrix} 1 & -2 \\ -8 & 0 \end{pmatrix}$, with eigenvalues $\lambda_{1,2} = 1/2 \pm \sqrt{65}/2$. Thus the equilibrium point $(X, Y) = (0, 0)$ of the linear system is a saddle point. Therefore, by the linearisation theorem, $(x, y) = (2, 1)$ is a saddle point nonlinear system (4.33).

We still need to determine the straight line paths in the (X, Y)-plane, we recall that to do this we need to either determine the associated eigenvectors $\mathbf{v}$, $\mathbf{w}$

$$J\mathbf{v} = \lambda_1 \mathbf{v}, \quad \text{and} \quad J\mathbf{w} = \lambda_2 \mathbf{w}. \tag{4.37}$$

Using the second row, we obtain

$$\mathbf{v} = \begin{pmatrix} 1 \\ k_1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ k_2 \end{pmatrix}, \tag{4.38}$$

where $k_i = -8/\lambda_i$.

2. $(\mathbf{x}, \mathbf{y}) = (-\mathbf{2}, -\mathbf{1})$

The associated linear system, along with all the analysis, is the same as the previous case. Thus the equilibrium point $(-2, -1)$ is a saddle point.

It remains to go back to the full nonlinear system and consider the horizontal and vertical isoclines, given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x - x^3}{x - 2y} = \begin{cases} 0, & x = 0, \text{ or } x = \pm 2, \\ \infty & y = x/2. \end{cases} \tag{4.39}$$

The direction of flow on the horizontal isocline $(x = -2)$ is given by

$$\dot{x} = -2\,(1 + y) \begin{cases} > 0, & y < -1, \\ < 0, & y > -1. \end{cases} \tag{4.40}$$

Therefore, when $y < -1$, $\dot{x} > 0$ and x increases as t increases, while when $y > -1$, $\dot{x} < 0$ and x decreases as t increases.

The direction of flow on the horizontal isocline $(x = 0)$ is given by

$$\dot{x} = -2y \begin{cases} > 0, & y < 0, \\ < 0, & y > 0. \end{cases} \tag{4.41}$$

Therefore, when $y < 0$, $\dot{x} > 0$ and x increases as t increases, while when $y > 0$, $\dot{x} < 0$ and x decreases as t increases.

The direction of flow on the horizontal isocline $(x = 2)$ is given by

$$\dot{x} = 2\,(1 - y) \begin{cases} > 0, & y < 1, \\ < 0, & y > 1. \end{cases} \tag{4.42}$$

Therefore, when $y < 1$, $\dot{x} > 0$ and x increases as t increases, while when $y > 1$, $\dot{x} < 0$ and x decreases as t increases.

The direction of flow on the vertical isocline ($y = \frac{x}{2}$) is given by

$$\dot{y} = x\,(2-x)\,(2+x) \begin{cases} >0, & x<-2, \\ <0, & -2<x<0, \\ >0, & 0<x<2, \\ <0, & x>2. \end{cases} \tag{4.43}$$

Therefore, when $x < -2$, $\dot{y} > 0$ and y increases as t increases, when $-2 < x < 0$, $\dot{y} < 0$ and y decreases as t increases, when $0 < x < 2$, $\dot{y} > 0$ and y increases as t increases, and, when $x > 2$, $\dot{y} < 0$ and y decreases as t increases.

As $x = 0$ is an horizontal isocline, we only need to check the flow direction on $y = 0$ where,

$$\dot{x} = x \begin{cases} <0, & x<0, \\ >0, & x>0. \end{cases}$$

$$\dot{y} = x\,(2-x)\,(2+x) \begin{cases} >0, & x<-2, \\ <0, & -2<x<0, \\ >0, & 0<x<2, \\ <0, & x>2. \end{cases} \tag{4.44}$$

Hence, on $y = 0$, for $x < -2$, when$x < -2$, $\dot{x}$ decreases and $\dot{y}$ increases as t increases, when $-2 < x < 0$, $\dot{x}$ and $\dot{y}$ both decrease as t increases, when $0 < x < 2$, $\dot{x}$ and $\dot{y}$ both increase as t increases, and, finally, when $x > 2$, $\dot{x}$ increases and $\dot{y}$ decreases as t increases.

Combining all this information, we can now sketch the phase portrait of the nonlinear system (4.33), taking care to think about the global behaviour between each of the equilibrium points. See Figure 4.3.

We note that as we move away from the equilibrium points$(2, 1)$ and $(-2, -1)$ the straight line paths are affected by the nonlinear terms and become smooth curves called **manifolds**. Note also that the trajectories approaching or leaving a saddle point are called the stable and unstable manifolds respectively. These special trajectories play a significant role in establishing the **global** structure of the phase portrait.

We can see the crucial role they play if we study the qualitatively different trajectories. Those that go through a point above the manifolds entering and leaving the saddlepoint at $(2, 1)$ come from (∞, ∞), close to the manifold moving towards the saddle point and tend towards $(-\infty, \infty)$ along the manifold leaving the saddle point at $(2, 1)$. The trajectories to the right of the manifold coming into the saddlepoint at $(2, 1)$ and the manifold leaving the same saddlepoint and tending towards $(\infty, -\infty)$ all come from (∞, ∞), close to the manifold moving towards the saddle point and tend towards $(\infty, -\infty)$ along the manifold leaving the saddle point at $(2, 1)$. The trajectories through a point below the manifold leaving the other saddle point at $(-2, 1)$ and the manifold moving towards this same saddle point, all come from $(-\infty, -\infty)$, close to the manifold moving towards the saddle point at $(-2, 1)$ and tend towards $(\infty, -\infty)$ along the manifold leaving the saddle point at $(2, 1)$. Similar behaviour is seen for trajectories through a point to the left of the manifold

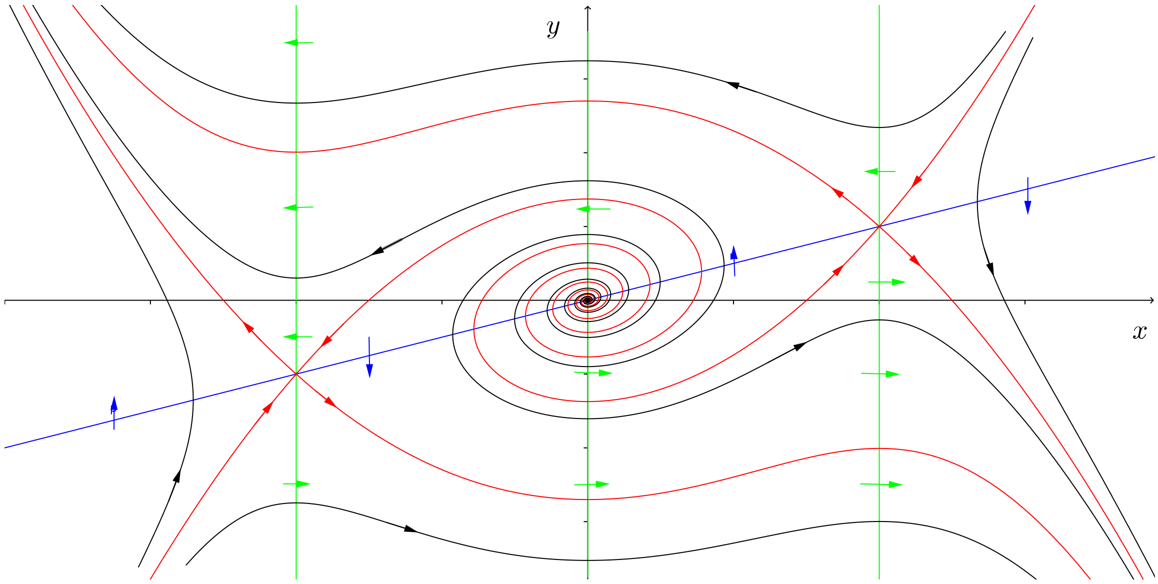


Figure 4.3: Phase portrait of system (4.33). Note the horizontal isoclines are coloured green, while the vertical isocline is coloured blue.

leaving the saddlepoint at $(-2, 1)$ and tending towards $(-\infty, \infty)$ and the manifold coming from $(-\infty, -\infty)$ and tending towards the same saddle point. These trajectories all come from $(-\infty, -\infty)$, close to the manifold moving towards the saddle point at $(-2, 1)$ and tend towards $(-\infty, \infty)$ along the manifold leaving the same saddle point.

Some of the remaining trajectories are defined by the manifold that leaves the unstable spiral at $(0, 0)$ and tends towards the saddle point at $(2, 1)$. The trajectories through points underneath this will spiral out from the equilibrium at $(0, 0)$ and tend towards $(\infty, -\infty)$ between the two manifolds tending in the same direction, one from each saddle point. Similarly, there is a manifold coming from the unstabe equilibrium at $(0, 0)$ and tending towards the saddlepoint at $(-2, 1)$, and trajectories through points above this will spiral out of the equilibrium at $(0, 0)$ and tend to $(-\infty, \infty)$ in between the two manifolds, one from each saddle point, tending in the same direction.

In the next chapter, we'll look at manifolds, like the last two mentioned, which connect equilibrium points.

▶ Example 57 ◀ Consider the nonlinear dynamical system

$$\begin{aligned}\dot{x} &= 3x + 2y + xy,\\ \dot{y} &= -6x - 4y,\end{aligned} \tag{4.45}$$

Determine the nature of all equilibrium points and, following the guidance given in Chapter 2, sketch the phase portrait.

This system has a single equilibrium point at $(0,0)$. The Jacobian is given by

$$\mathbf{J} = \begin{pmatrix} 3+y & 2+x \\ -6 & -4 \end{pmatrix}, \tag{4.46}$$

which at $(0,0)$ becomes

$$\mathbf{J} = \begin{pmatrix} 3 & 2 \\ -6 & -4 \end{pmatrix}, \tag{4.47}$$

which, as we've seen in the example of a linear system with a stable comb, has eigenvalues $\lambda_1 = 0$, and $\lambda_2 = -1$, with associated eigenvectors

$$\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{4.48}$$

In the linear case, this was a stable comb. But since one eigenvalue is zero, we cannot transfer this property to the non-linear case.

Let us consider the isoclines. The horizontal isocline is given by $y = -\frac{3}{2}x$. The direction field along the horizontal isocline is given by

$$\dot{x} = -3x^2 \quad \begin{cases} < 0, & x \neq 0, \\ = 0, & x = 0 \text{ (equilibrium point)}. \end{cases}$$

The vertical isocline is given by

$$y = \frac{-3x}{x+2},$$

which has a vertical asymptote at $x = -2$. For $x \to \pm\infty$, $y \to -3$, so there is an horizontal asymptote at $y = -3$. Also

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-6}{(x+2)^2},$$

so the vertical isocline is a strictly decreasing graph.

The direction field along the vertical isocline is given by

$$\dot{y} = -\frac{6x^2}{x+2} \begin{cases} < 0, & x > -2, \\ > 0, & x < -2. \end{cases}$$

Hence, movement along the vertical isocline to the left of $x = -2$ is upwards, and downwards to the right of $x = -2$.

For extra information, consider the flow along the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$. For $x = 0$,

$$\begin{aligned} \dot{x} &= 2y, \\ \dot{y} &= -4y, \end{aligned}$$

so the flow is upwards and to the left for $y < 0$ and downwards and to the right for $y > 0$.

For $y = 0$,

$$\begin{aligned} \dot{x} &= 3x, \\ \dot{y} &= -6x, \end{aligned}$$

so the flow is upwards and to the left for $x < 0$ and downwards and to the right for $x > 0$.

For $y = x$,

$$\begin{aligned} \dot{x} &= x\left(5 + x\right), \\ \dot{y} &= -10x, \end{aligned}$$

so the flow is upwards and to the right for $x < -5$, upwards and to the left for $-5 < x < 0$ and downwards and to the right for $x > 0$.

For $y = -x$,

$$\begin{aligned} \dot{x} &= x\left(1 - x\right), \\ \dot{y} &= -2x, \end{aligned}$$

with the flow upwards and to the left for $x < 0$, downwards and to the right for $0 < x < 1$ and downwards and to the left for $x > 1$.

With this information we can put together the phase portrait, which is depicted in Figure 4.4.

This type of configuration about the equilibrium is sometimes referred to as a Bogdanov-Takens equilibrium. All trajectories tend to $(-\infty, \infty)$ and, coming from close to the vertical asymptote, i.e. $(-2, \infty)$ as $t \to -\infty$, will swing around the equilibrium.

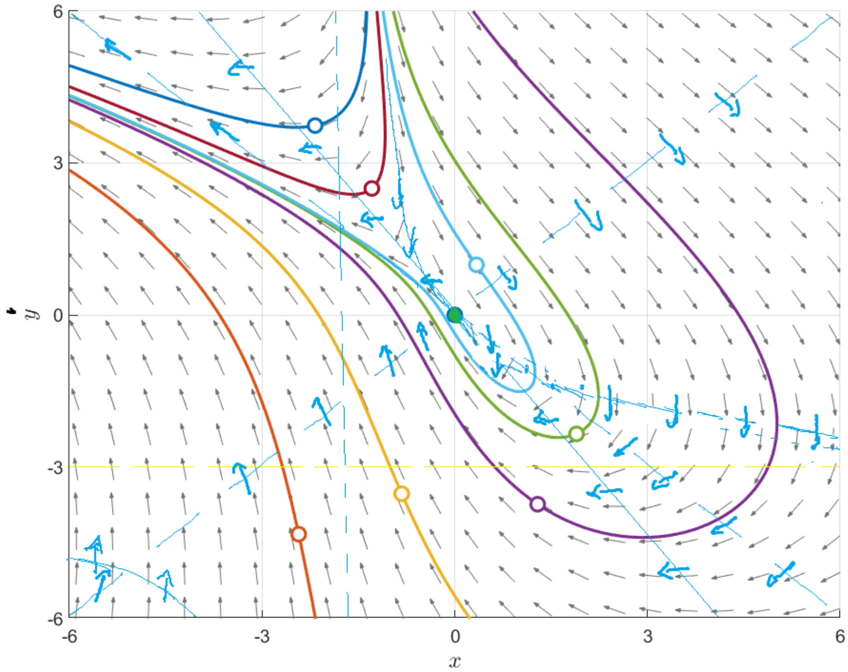


Figure 4.4: Phase portrait of the dynamical system (4.45).

Chapter 5

Periodic orbits, limit cycles, and the Poincaré-Bendixson theorem

5.1 Periodic orbits and limit cycle.

We mentioned in Chapter 3 that a closed trajectory in the phase plane, made entirely of ordinary points, represent **periodic orbits** of the dynamical system

$$\begin{aligned} \dot{x} &= P\left(x,y\right), \\ \dot{y} &= Q\left(x,y\right). \end{aligned} \tag{5.1}$$

Before investigating such trajectories, we first distinguish between two types of periodic orbit:

Definition 19. Non-isolated Periodic Orbit *A **non-isolated periodic orbit** of (5.1) is a periodic orbit such that in any sufficiently small neighbourhood of this periodic orbit, lie other periodic orbits.*

Definition 20. Limit Cycle *A **limit cycle** is a periodic orbit of (5.1) such that it is approached as $t \to \infty$ or as $t \to -\infty$. by all trajectories in a sufficiently small neighbourhood of itself.*

A limit cycle is either asymptotically stable or unstable.
A non-isolated periodic orbit is Lyapunov stable.

► Example 58 ◄ **Limit cycle:** Consider the dynamical system

$$\begin{aligned} \dot{x} &= x - y - \left(x^2+y^2\right)x, \\ \dot{y} &= x + y - \left(x^2+y^2\right)y. \end{aligned} \tag{5.2}$$

Determine if there are any periodic orbits or limit cycles in the phase portrait.

This system has an equilibrium point at $(0,0)$ which is an unstable focus (spiral). It is more illuminating to consider (5.2) when written in polar coordinates,

$$\begin{aligned}\dot{r} &= r\left(1-r^2\right),\\ \dot{\theta} &= 1.\end{aligned} \tag{5.3}$$

We note that $\dot{r} > 0$ when $0 < r < 1$, indicating that r increases with increasing t, while $\dot{r} < 0$ when $r > 1$ indicating that r decreases with increasing t. Therefore, we have a stable limit cycle at $r = 1$. Clearly $\dot{r} = 0$ when $r = 1$, and so we have a stable limit cycle when $r = 1$. We denote this limit cycle by $\mathcal{C}$. Further, $\dot{\theta} > 0$, indicating that θ increases with increasing t (anticlockwise rotation). The phase portrait of (5.2) is given in Figure 5.1. We see that for all initial conditions, except $(0,0)$, the solutions of (5.2) $(x(t), y(t)) \to \mathcal{C}$ as $t \to \infty$.

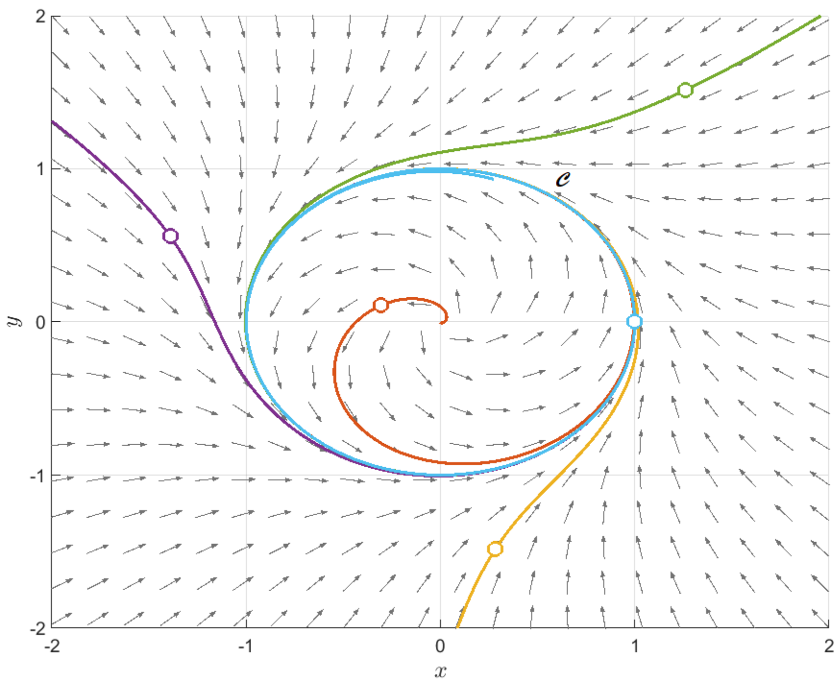


Figure 5.1: Phase portrait of system (5.2). The stable limit cycle, $\mathcal{C}$, is given in blue. Note that there is an unstable equilibrium point at $(0,0)$

► Example 59 ◄ **Non-isolated periodic orbits:** Consider the linear dynamical system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x.\end{aligned} \tag{5.4}$$

We can clearly see that the only equilibrium point is at $(0,0)$, and note that this is a linear centre. The trajectories of (5.4) are given by

$$x^2 + y^2 = r^2, \tag{5.5}$$

for some constant r. Therefore, the trajectories are concentric circles centred on $(0,0)$ of radius r, and are non-isolated periodic orbits.

We can use the following theorem to exclude the possibility of periodic orbits from particular regions in the phase plane. But let us first remind ourselves what it means for a region to be *simply connected.*

Definition 21. Simply connected *A closed region D is* ***simply connected*** *if*

1. *any two points within the region D can be connected by a continuous path that lies completely within the region,*
2. *any loop in the region D can be contracted to a point.*

So a region that is bordered by a single closed curve, e.g. an ellipse, is simply connected as long has it does not contain any holes.

Theorem 4. ***Bendixson's Negative Criterion****. Consider the dynamical system*

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y). \end{aligned} \tag{5.6}$$

Suppose that on a simply connected closed region D of the phase plane that the expression

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \tag{5.7}$$

is of one fixed sign, then D contains no periodic orbits of (5.1).

▶ Example 60 ◀ **Bendixson's Negative Criterion:** Is it possible for the following dynamical system to have periodic orbits?

$$\begin{aligned} \dot{x} &= y \equiv P(x,y), \\ \dot{y} &= -\left(1 + x^2 + x^4\right) y - x \equiv Q(x,y). \end{aligned} \tag{5.8}$$

Here

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 - \left(1 + x^2 + x^4\right) < 0, \tag{5.9}$$

on any simply connected region D. Thus, via Bendixson's negative criterion, the system cannot have any periodic orbits in the phase plane. System (5.8) has only one equilibrium

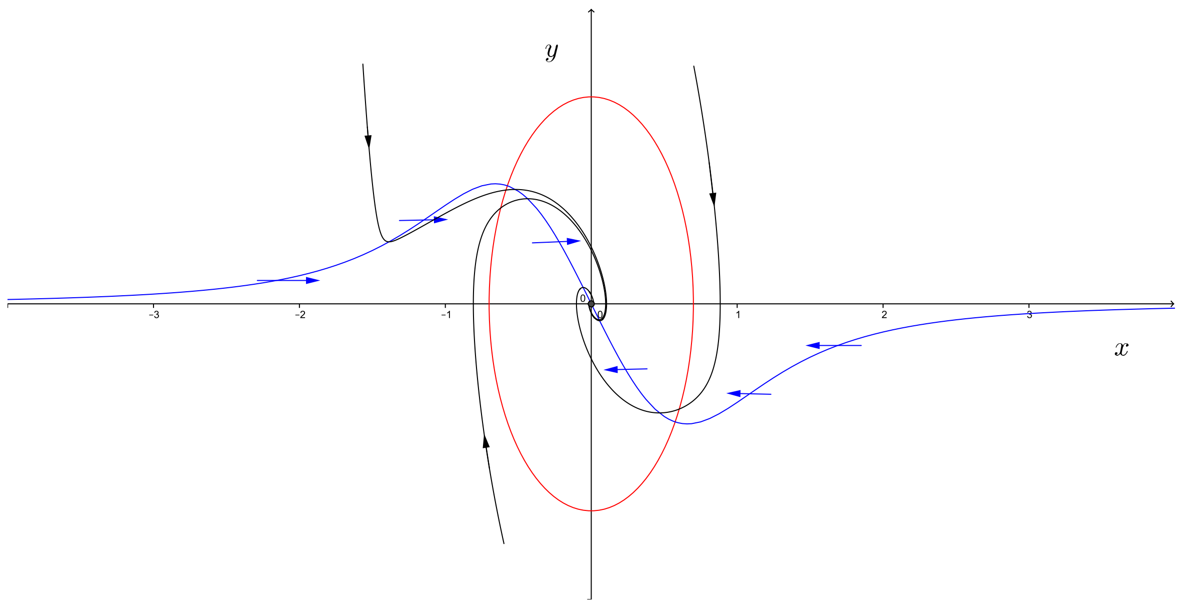


Figure 5.2: Phase portrait of system (5.8). The equilibrium point $(0, 0)$ is an stable spiral and the horizontal isocline is given in blue. We note that the family of disks $x^2 + y^2 \leq R,\ R > 0$ are invariant sets for the system, the circle $x^2 + y^2 = 4$ is given in red - note that because of the unequal scales on the axis, this appears as an ellipse.

point at $(0, 0)$ and linearisation about this point reveals that $(0, 0)$ is a stable spiral. Considering the isoclines,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(1 + x^2 + x^4)\, y + x}{y} = \begin{cases} 0, & y = -\frac{x}{1+x^2+x^4}, \\ \infty, & y = 0. \end{cases} \tag{5.10}$$

The global phase portrait of (5.8) is given in Figure 5.2.

We can now show that a system **doesn't** have any periodic orbits. What can we use to establish the existence of periodic orbits in a given system?

We need to discuss invariant sets first.

Definition 22. *Let E be a closed subset of the plane. E is said to be a* ***positive invariant*** *set of (5.1) if any trajectory starting in E at $t = t_0$, remains in E for all $t > t_0$. A negatively invariant set is defined similarly, with $t < t_0$.*

$\boxed{\blacktriangleright \textbf{ Example 61 } \blacktriangleleft}$ The equilibrium point (x_e, y_e) and the closed path C in Figure 5.3 are both positive and negative invariant sets. But these are rather special cases.

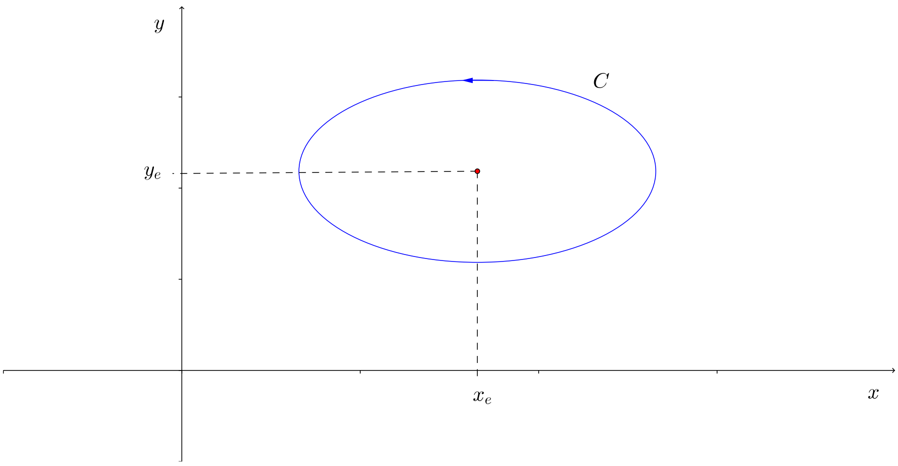


Figure 5.3: Invariant sets

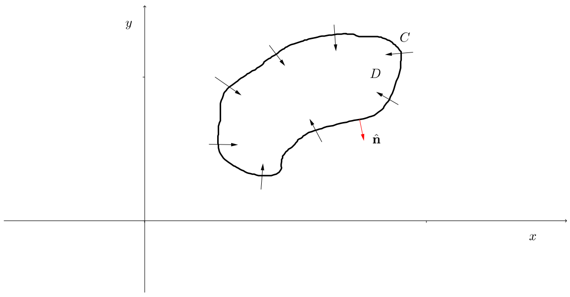


Figure 5.4: Positively invariant domain D

Theorem 5. *Let D be a connected domain in the plane with boundary C and with $\hat{\mathbf{n}}$ being the outward unit normal to D on C. Then $D \cup C$ is a positively invariant set for (5.1) provided $(\dot{x}, \dot{y}) \cdot \hat{\mathbf{n}} \leq 0$ for all $(x, y) \in C$.*

An illustration of a positively invariant region is given in Figure 5.4. Here $(P(x,y), Q(x,y)) \cdot \hat{\mathbf{n}} \leq 0$ for all $(x, y) \in C$ and each trajectory with $\mathbf{x}(0) \in D \cup C$ remains in D for all t. Further, any trajectory that enters D through the boundary C remains in D for all t.

▶ Example 62 ◀ Consider the dynamical system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1-x) - 2y. \end{aligned} \tag{5.11}$$

We note that system (5.11) has two equilibrium points at $(0, 0)$ and $(1, 0)$. We consider the region D with boundary $C = C_1 \cup C_2 \cup C_3$ (see Figure 5.5), where:

$$\begin{aligned} C_1 &= ((x, y) : x = 1,\ y \in [0, -1]), \\ C_2 &= ((x, y) : y = 0,\ x \in [0, 1]), \\ C_3 &= ((x, y) : y = -x,\ x \in [0, 1]). \end{aligned}$$

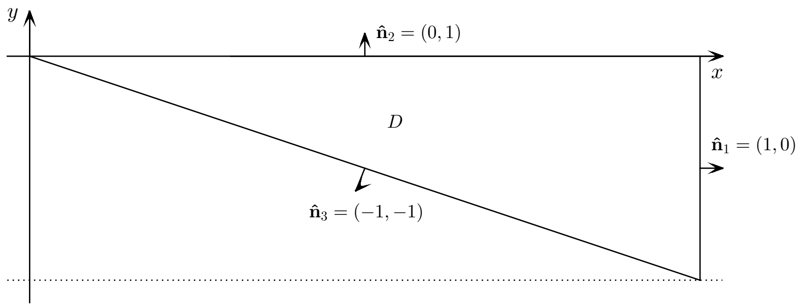


Figure 5.5: Positively invariant domain D

On C_2: $y = 0$, the governing equations are $\dot{x} = 0$, $\dot{y} = -x(1-x)$, and

$$(0, -x(1-x)) \cdot \hat{\mathbf{n}}_2 = -x(1-x) \leq 0 \quad \text{for} \quad x \in [0, 1].$$

On C_1: $x = 1$, the governing equations are $\dot{x} = y$, $\dot{y} = -2y$, and

$$(y, -2y) \cdot \hat{\mathbf{n}}_1 = y \leq 0 \quad \text{for} \quad y \in [0, -1].$$

On C_3: $y = -x$, the governing equations are $\dot{x} = y$, $\dot{y} = y^2 - y$, and

$$\left(y, y^2 - y\right) \cdot \hat{\mathbf{n}}_3 = -y^2 \leq 0 \quad \text{for} \quad y \in [0, -1].$$

Therefore, the region D is positively invariant.

Having information about the existence of bounded, invariant sets allows us to determine information about the long-time behaviour of the solutions of (5.1).

Theorem 6. ***Poincaré-Bendixson Theorem.*** *Let M be a closed, bounded, positively invariant region of the system (5.1), containing a finite number of equilibrium points. Let $\mathbf{p} \in M$ and denote by $\mathbf{x}(t; \mathbf{p})$ the trajectory which satisfies $\mathbf{x}(0) = \mathbf{p}$. Then one of the following possibilities holds as $t \to \infty$,*

(i) $\mathbf{x}(t; \mathbf{p})$ *approaches an equilibrium point in M.*

(ii) $\mathbf{x}(t; \mathbf{p})$ *approaches a periodic orbit in M.*

(iii) $\mathbf{x}(t; \mathbf{p})$ *is, itself, a periodic orbit in M or an equilibrium point in M*

We can use this theorem to determine the existence of periodic orbits for given dynamical systems as follows:

(i) Determine a closed bounded positively invariant region M for the system which contains **no** equilibrium points.

(ii) Let $\mathbf{p} \in M$ and $\mathbf{x}(t; \mathbf{p})$ be a trajectory in M. Consider the Poincaré-Bendixson Theorem. Case (i) cannot hold as there are no equilibrium points in M. Therefore, case (ii) or (iii) must occur, **both** of which imply the existence of a periodic orbit in M.

▶ Example 63 ◀ **Poincaré-Bendixson Theorem:** The Van der Pol equation describes the dynamics of a particular electric circuit as is given by

$$\ddot{x} + \left(3x^2 - 1\right)\dot{x} + x = 0. \tag{5.12}$$

Show that the Van der Pol equation has a periodic solution.

We first write the equation as a 2-dimensional dynamical system, by setting $y = \dot{x}$, to give

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\left(3x^2 - 1\right) y - x. \end{aligned} \tag{5.13}$$

System (5.13) has a unique equilibrium point at $(0, 0)$. On linearising about $(0, 0)$ we obtain the associated linear system as,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= y - x, \end{aligned} \tag{5.14}$$

where $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, with eigenvalues $\lambda_{1,2} = 1/2 \pm i\sqrt{3}/2$. Therefore the equilibrium point $(0, 0)$ of the associated linear system is an unstable spiral. The linearisation theorem then determines that system (5.13) has an unstable spiral at $(0, 0)$.

Now consider a trajectory $\mathcal{C}$ through $(\epsilon, 0)$, with $0 \leq \epsilon \ll 1$ so that the linearised theory holds. The trajectory $\mathcal{C}$ is shown in Figure 5.6. If we now draw a line between the point $(\epsilon, 0)$ and the point at which the trajectory next crosses the x-axes (with $x > 0$) we have formed a negatively invariant region for the system (5.13). Any trajectory starting within this region must exit the region by crossing this as $t \to \infty$.

We next consider the isoclines

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(3x^2 - 1)\, y + x}{y} = \begin{cases} 0, & y = \frac{x}{1 - 3x^2}, \\ \infty, & y = 0. \end{cases} \tag{5.15}$$

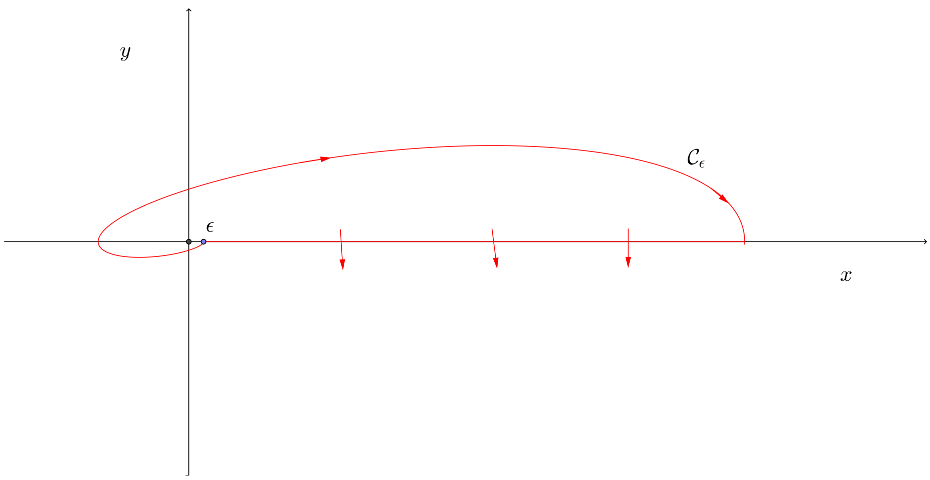


Figure 5.6: Phase portrait of system (5.13) close to the equilibrium point $(0, 0)$

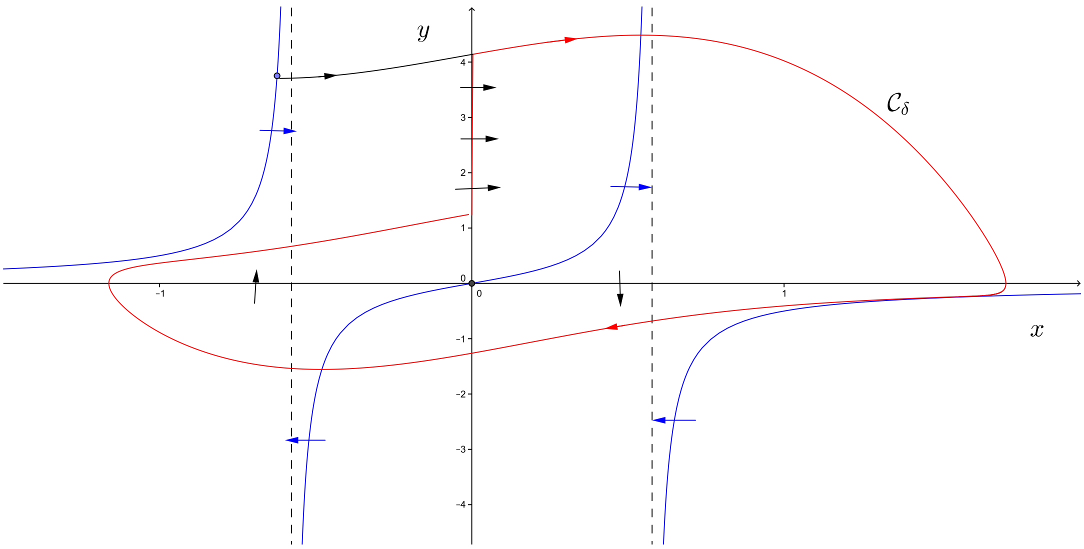


Figure 5.7: Phase portrait of system (5.13). The horizontal isocline is coloured blue and the path $\mathcal{C}_\delta$ is coloured red

Consider a trajectory starting on the horizontal isocline. Let $\mathcal{C}_\delta$ be the closed path given in red in Figure 5.7. Consider the closed bounded region D contained between $\mathcal{C}_\epsilon$ and $\mathcal{C}_\delta$. On $\mathcal{C}_\epsilon$, and $\mathcal{C}_\delta$, $(P, Q) \cdot \mathbf{n} \leq 0$, and D is positively invariant. However, D does not contain any equilibrium points, and via the Poincaré-Bendixson Theorem must contain at least one periodic orbit.

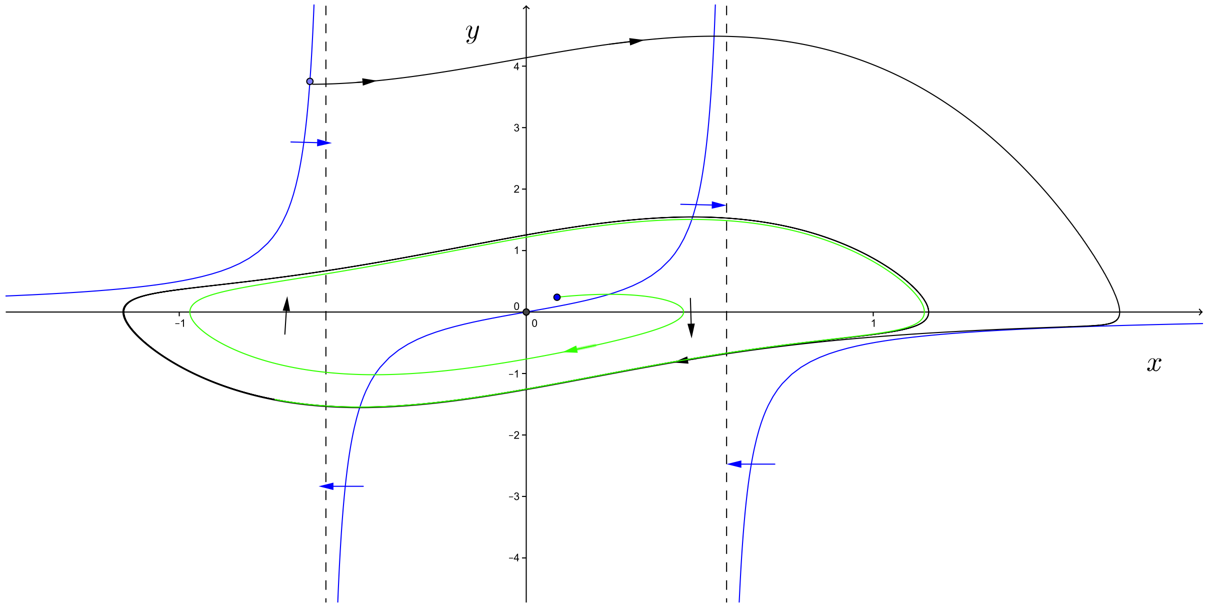


Figure 5.8: Phase portrait of system (5.13) displaying the existence of a stable periodic orbit. Clearly, the phase path coloured black approaches the periodic orbit from above as $t \to \infty$, while the phase path coloured green starting close to the equilibrium point approaches the periodic orbit from below as $t \to \infty$.

Figure 5.8 gives the phase portrait of system (5.13), which has been determined by numerical integration of the governing equations, indicating the presence of exactly one stable periodic orbit.

5.2 Homoclinic and heteroclinic orbits

Definition 23. Homoclinic Orbit *Let $\mathcal{C}_{\text{Hom}}$ be a trajectory of (5.1) and let $\mathbf{x}_e$ be an equilibrium point of the same system. When $\mathcal{C}_{\text{Hom}} \to \mathbf{x}_e$ both as $t \to \pm\infty$, then $\mathcal{C}_{\text{Hom}}$ is called a **Homoclinic orbit** of (5.1).*

The red trajectory in the phase portrait given in Figure 5.9 is an example of a homoclinic orbit.

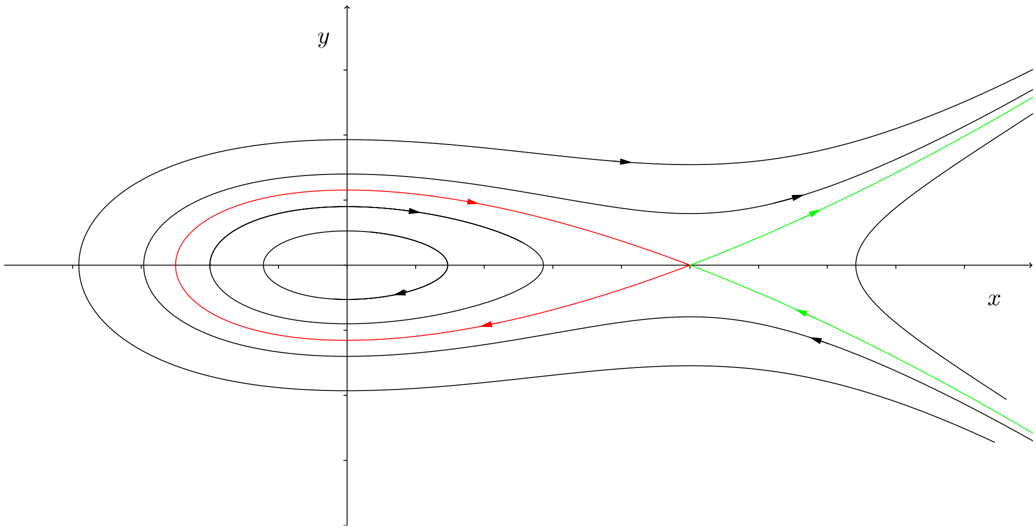


Figure 5.9: Phase portrait of system (5.17). The equilibrium point $(0,0)$ is a centre, while $(1,0)$ is a saddle point. Note the homoclinic orbit is given in red and corresponds to $C = \frac{1}{3}$.

Definition 24. Heteroclinic Orbit *Let $\mathcal{C}_{\text{Het}}$ be a trajectory of (5.1), and let $\mathbf{x}_{e_1}$ and $\mathbf{x}_{e_2}$ be two distinct equilibrium points of the same system. When $\mathcal{C}_{\text{Het}} \to \mathbf{x}_{e_1}$ as $t \to \infty$, and $\mathcal{C}_{\text{Het}} \to \mathbf{x}_{e_2}$ as $t \to -\infty$, then $\mathcal{C}_{\text{Het}}$ is called a **Heteroclinic orbit** of (5.1).*

We can see two heteroclinic orbits in Figure 4.3 which gives the phaseportrait of system (4.33). For $t \to -\infty$ they both tend to the unstable spiral at $(0,0)$, with one tending towards the saddle point at $(-2,-1)$ for $t \to \infty$ and the other to the second saddle point at $(2,1)$.

▶ Example 64 ◀ Consider the nonlinear pendulum equation

$$\ddot{x} + x - x^2 = 0. \tag{5.16}$$

Determine if there is a homoclinic orbit and if so, find the orbit.

We can write equation (5.16 as the 2-dimensional dynamical system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= x^2 - x,\end{aligned} \tag{5.17}$$

which has two equilibrium points at $(0, 0)$ (centre - see Theorem 2) and $(1, 0)$ (saddle point). The trajectories are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 - x}{y}, \tag{5.18}$$

which, on integrating, gives

$$y^2 = \frac{2}{3}x^3 - x^2 + C, \tag{5.19}$$

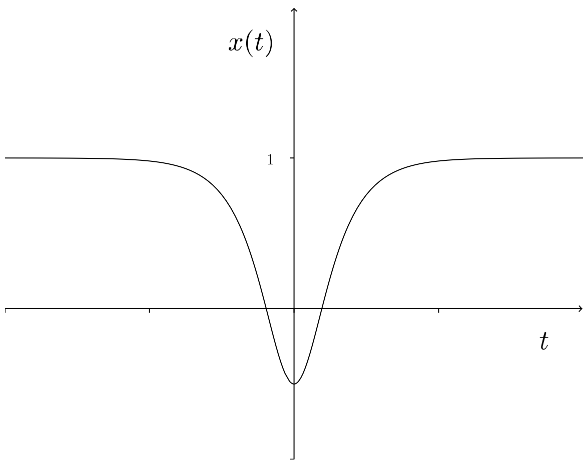


Figure 5.10: Plot of solution (5.21) against t with $\phi = 0$

with C constant. Equation (5.19) represents a one parameter family of solutions which are plotted in Figure 5.9 for a number of values of C. When $C = \frac{1}{3}$ we obtain the homoclinic orbit,

$$y^2 = \frac{2}{3}x^3 - x^2 + \frac{1}{3} = \frac{1}{3}(x-1)^2(2x+1), \tag{5.20}$$

depicted by the red trajectory in Figure 5.9. Clearly the trajectory $\mathcal{C}_{\mathrm{Hom}} \to (1, 0)$ as $t \to \pm\infty$. We recall that $y = \dot{x}$, and observe that we can integrate (5.20) to obtain

$$x(t) = 1 - \frac{3}{2}\mathrm{sech}^2\left(\frac{t}{2} + \phi\right), \tag{5.21}$$

where ϕ is a constant, which is plotted against t in Figure 5.10.

Finally, we note that for $0 < C < \frac{1}{3}$, the solution $x(t)$ of (5.16) is oscillatory in t, while for $C > \frac{1}{3}$ the solution $x(t) \to \infty$ as $t \to \pm\infty$.

Definition 25. Heteroclinic Cycle *An* ***heteroclinic cycle*** *is a closed sequence of* ***heteroclinic orbits****.*

▶ Example 65 ◀ Consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\sin(x), \end{aligned} \tag{5.22}$$

The phase portrait is given in Figure 5.11. The pattern clearly repeats itself, with regularly spaced equilibrium points. At $x = \pm\pi$ we have saddle points and we can draw an heteroclinic orbit from $x = -\pi$ to $x = \pi$ and back from $x = \pi$ to $x = -\pi$ via a different trajectory, forming a closed sequence of heteroclinic orbits, i.e. a heteroclinic cycle.

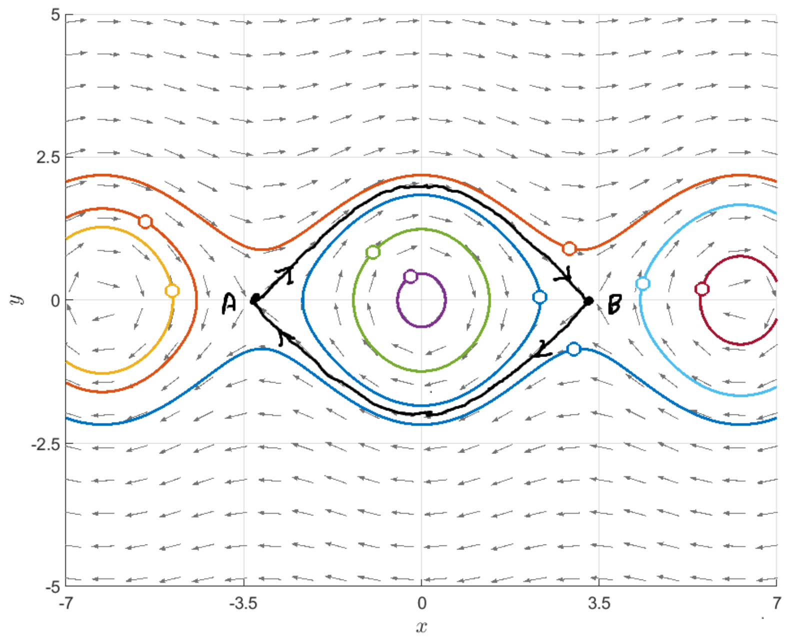


Figure 5.11: Phaseplot for the system (5.21). An heteroclinic cycle is indicated in black.

Finally, we are able to rule out the existence of homoclinic orbits or hetereoclinic cycles using the following Theorem.

Theorem 7. *Suppose that on a, simply connected region D of the phase plane, the expression*

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \tag{5.23}$$

is of one fixed sign, then D contains no homoclinic orbits or heteroclinic cycles of (5.1).

But all these often do not suffice to prro all details.

5.3 Example

Consider the nonlinear dynamical system

$$\begin{aligned}\dot{x} &= 2x - 4y + 2x^2,\\ \dot{y} &= x - 2y + y^2,\end{aligned} \tag{5.24}$$

which has an equilibrium point at $(0,0)$. If we substitute the expression obtained for x from the equation $x - 2y + y^2 = 0$ into the right hand side of the $\dot{x}$ equation and require that also to be zero, we get

$$2y^2\,(y-1)\,(y-3) = 0.$$

For $y = 0$ requiring that $\dot{y} = 0$ we get $x = 0$ which is the already know equilibrium. For $y = 1$ we similarly obtain that $x = 1$ and for $y = 3$ we find that $x = -3$. Hence there are three equilibrium points: $(0,0)\,,(1,1)$ and $(-3,3)$.

The Jacobian is given by

$$\mathbf{J} = \begin{pmatrix} 2+4x & -4 \\ 1 & 2\,(y-1) \end{pmatrix}. \tag{5.25}$$

At $(0,0)$ the Jacobian becomes

$$\mathbf{J} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}, \tag{5.26}$$

which, as we've seen in the example of a linear system with a shear, has eigenvalues $\lambda_1 = 0$, and $\lambda_2 = 0$, with associated eigenvector

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{5.27}$$

and generalised eigenvector

$$\mathbf{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \tag{5.28}$$

In the linear case, this was a shear.

At $(1,1)$ the Jacobian becomes

$$\mathbf{J} = \begin{pmatrix} 6 & -4 \\ 1 & 0 \end{pmatrix}, \tag{5.29}$$

with eigenvalues $\lambda_{1,2} = 3 \pm \sqrt{5}$, and respective eigenvectors

$$\mathbf{v} = \begin{pmatrix} 3+\sqrt{5} \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 3-\sqrt{5} \\ 1 \end{pmatrix}. \tag{5.30}$$

This is an unstable node as $3 - \sqrt{5} > 0$.

At $(-3, 3)$ the Jacobian becomes

$$\mathbf{J} = \begin{pmatrix} -10 & -4 \\ 1 & 4 \end{pmatrix}, \tag{5.31}$$

with eigenvalues $\lambda_{1,2} = -3 \pm 3\sqrt{5}$, and respective eigenvectors

$$\mathbf{v} = \begin{pmatrix} -7 + 3\sqrt{5} \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -7 - 3\sqrt{5} \\ 1 \end{pmatrix}. \tag{5.32}$$

This is a saddle point.

Let us consider the isoclines. The horizontal isocline is given by $x = y\,(2 - y)$, which can easily be plotted in the (y, x)-plane and then transformed to the (x, y)-plane The direction field along the horizontal isocline is given by

$$\dot{x} = 2y^2\,(y - 1)\,(y - 3) \quad \begin{cases} > 0, & y < 0 \quad \text{or} \quad 0 < y < 1 \quad \text{or} \quad y > 3, \\ < 0, & 1 < y < 3. \end{cases}$$

The vertical isocline is given by

$$y = \frac{x\,(x + 1)}{2},$$

The direction field along the vertical isocline is given by

$$\dot{y} = \frac{x^2\,(x - 1)\,(x + 3)}{4} e \quad \begin{cases} < 0, & -3 < x < 0 \quad \text{or} \quad 0 < x < 1, \\ > 0, & x < -3 \quad \text{or} \quad x > 1. \end{cases}$$

One can easily check that the horizontal and vertical isocline intersect only at the three equilibrium points.

For extra information, consider the flow along the lines $y = -x, x = 0$ and $y = 0$. For $y = -x$,

$$\begin{aligned} \dot{x} &= 2x\,(3 + x)\,, \\ \dot{y} &= x\,(3 + x)\,, \end{aligned}$$

so the flow is upwards and to the right for $x < -3$ or $x > 0$, and downwards and to the left for $-3 < x < 0$. Note that the slope along $y = -x$ is constant, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$.

For $x = 0$,

$$\begin{aligned} \dot{x} &= -4y, \\ \dot{y} &= y\,(y - 2)\,, \end{aligned}$$

with the flow upwards and to the right for $y < 0$, downwards and to the left for $0 < y < 2$ and upwards and to the left for $y > 2$.

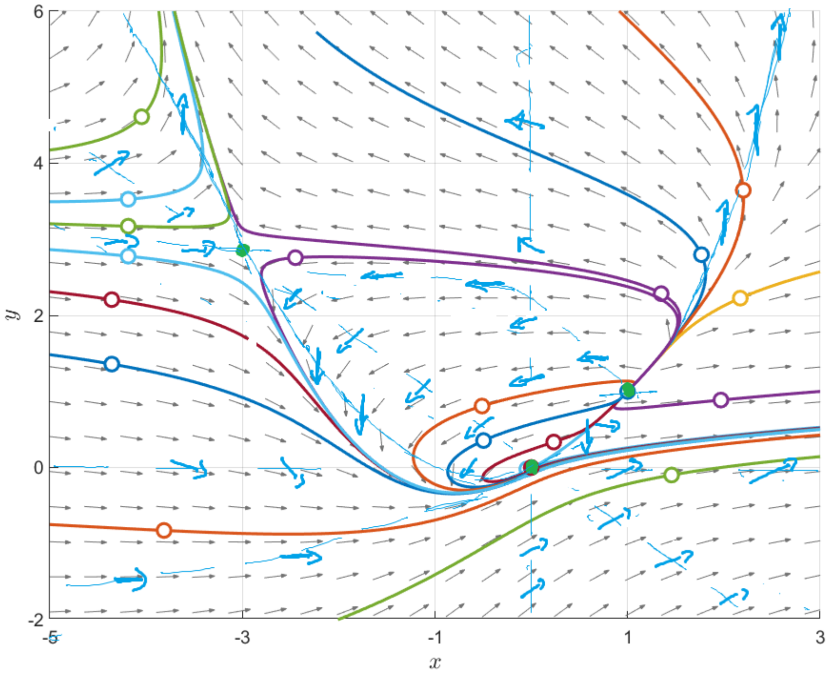


Figure 5.12: Phase portrait of the dynamical system (5.24).

For $y = 0$,

$$\dot{x} = 2x\,(x+1)\,,$$
$$\dot{y} = x,$$

with the flow downwards and to the right for $x < -1$, downwards and to the left for $-1 < x < 0$ and upwards and to the right for $x > 0$.

With this information we can put together the phase portrait, which is depicted in Figure 5.12. A closer look at the phaseportrait arounf $(0,0)$ is given in Figure 5.13 and an even closer look is provided in Figure 5.14.

This type of configuration about the equilibrium also looks like a Bogdanov-Takens equilibrium. There are no closed trajectories near any of the equilibrium points so no homoclinic orbits. There is an heteroclinic orbit from $(1,1)$ towards the saddle point at $(-3,3)$, and another one from the node at $(1,1)$ towards the equilibrium at $(0,0)$.

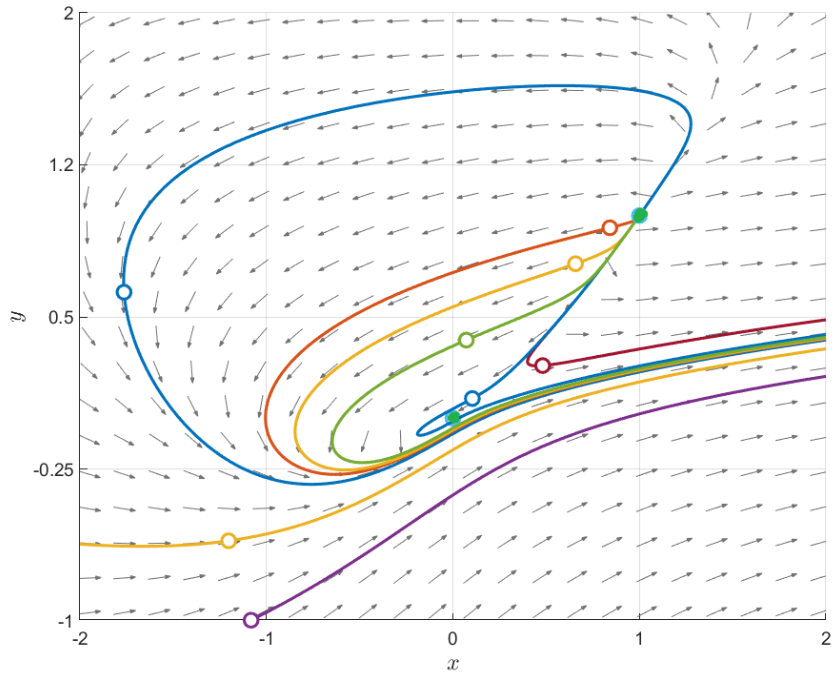


Figure 5.13: Phase portrait of the dynamical system (5.24) around the equilibrium at $(0, 0)$.

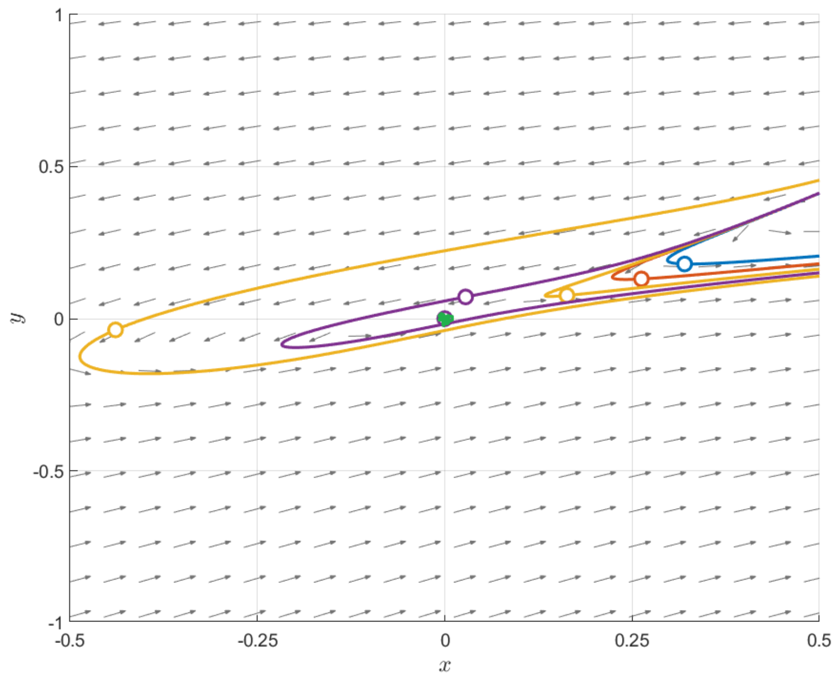


Figure 5.14: Phase portrait of the dynamical system (5.24) around the equilibrium at $(0, 0)$.

We can check that

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2y + 4x,$$

so is of the same sign on either side of the line $y = -2x$, so this will not help us in excluding any periodic orbits around $(0, 0)$.

The key manifolds (separatrices) are indicate in the figure below:

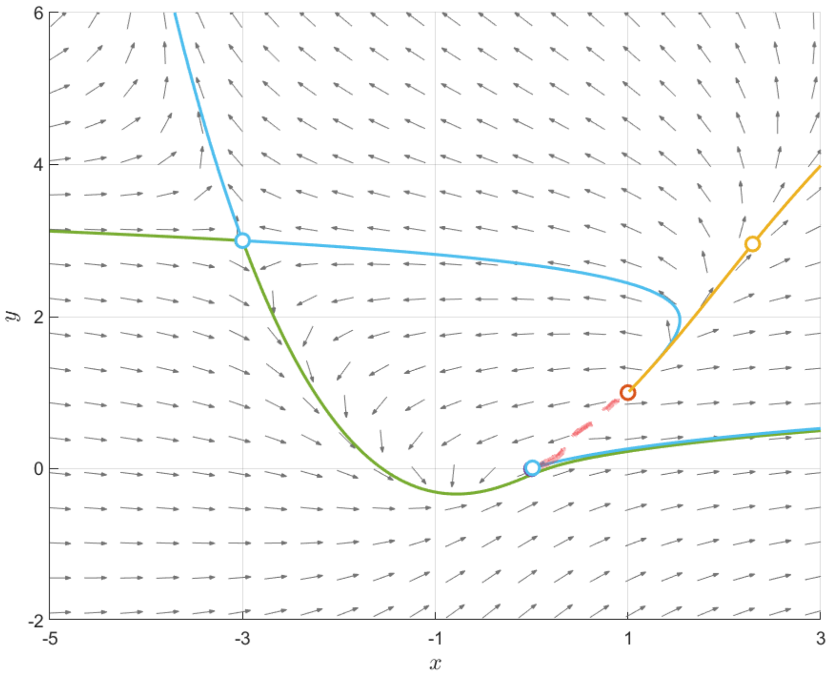


Figure 5.15: Key manifolds for the dynamical system (5.24).

Note that the manifold coming from the equilibrium $(-3, 3)$ (tending to $(-3, 3)$ for $t \to -\infty$), bypasses the equilibrium at $(0, 0)$ but runs eventually very close to a manifold coming from near $(0, 0)$.

Chapter 6

Hamiltonian systems

Hamiltonian systems form a large part of classical mechanics. They provide a more geometrical, but still equivalent, form of Newton's laws. Hamiltonian systems are often used to explain elements of celestial and plasma physics. In this chapter we will provide an overview of what constitutes a Hamiltonian system, and how we can extract information about the solutions of a Hamiltonian system.

Definition 26. Hamiltonian System *Suppose there is a function $H : \mathbb{R}^2 \to \mathbb{R}$ which is continuous with continuous first and second order partial derivatives, such that the dynamical system*

$$\begin{aligned} \dot{x} &= P\left(x, y\right), \\ \dot{y} &= Q\left(x, y\right), \end{aligned} \tag{6.1}$$

has the form

$$\begin{aligned} \dot{x} &= H_y\left(x, y\right), \\ \dot{y} &= -H_x\left(x, y\right), \end{aligned} \tag{6.2}$$

*then the dynamical system is said to be **Hamiltonian**, and $H\left(x, y\right)$ is the Hamiltonian function for the dynamical system.*

Theorem 8. *The dynamical system (6.1) is Hamiltonian*

$$\iff P_x + Q_y = 0 \quad \textit{for all} \quad (x, y) \in \mathbb{R}^2. \tag{6.3}$$

Proof. $\Rightarrow$: Suppose that (6.1) is Hamiltonian, then $P = H_y$ and $Q = -H_x$. Therefore, $P_x = H_{yx}$ and $Q_y = -H_{xy}$. hence,

$$P_x + Q_y = H_{xy} - H_{xy} = 0 \quad \text{for all} \quad (x, y) \in \mathbb{R}^2. \tag{6.4}$$

$\Leftarrow$: Suppose (6.1) is such that

$$P_x + Q_y = 0 \quad \text{for all} \quad (x, y) \in \mathbb{R}^2. \tag{6.5}$$

Hence $Q_y = -P_x$, and so,

$$Q(x,y) = -\int_0^y P_x(x,s)\,\mathrm{d}s + q(x), \tag{6.6}$$

for some differentiable function $q : \mathbb{R} \to \mathbb{R}$. Now set $H : \mathbb{R}^2 \to \mathbb{R}$, such that

$$H(x,y) = \int_0^y P(x,s)\,\mathrm{d}s - \int_0^x q(s)\,\mathrm{d}s. \tag{6.7}$$

Then,

$$H_y = P(x,y), \tag{6.8}$$

whilst

$$-H_x = -\int_0^y P_x(x,s)\,\mathrm{d}s + q(x) = Q(x,y). \tag{6.9}$$

Hence (6.1) is Hamiltonian, as required. □

Theorem 9. *The trajectories of a Hamiltonian system are given by $H(x,y) =$ constant.*

Proof. Let $(x(t), y(t))$ be a trajectory of (6.2). Let $H(t) = H(x(t), y(t))$ on the trajectory. Then,

$$\dot{H}(t) = H_x\dot{x} + H_y\dot{y} = H_xH_y - H_yH_x \equiv 0. \tag{6.10}$$

Implying that $H =$ constant on a trajectory. □

Example:

Show that the dynamical system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\sin x, \end{aligned} \tag{6.11}$$

is Hamiltonian, and determine the Hamiltonian function.

Here $P(x,y) = y$ and $Q(x,y) = -\sin x$. Therefore,

$$P_x + Q_y = 0 + 0 = 0, \tag{6.12}$$

implying that the system is Hamiltonian. Now,

$$\begin{aligned} H_y &= y, \\ H_x &= \sin x. \end{aligned} \tag{6.13}$$

On integrating H_y with respect to y we obtain

$$H(x,y) = \frac{1}{2}y^2 + g(x), \tag{6.14}$$

where $g(x)$ is a function which must be determined. On differentiating (6.14) w.r.t. x and comparing with (6.13), we find that

$$g'(x) = \sin x, \tag{6.15}$$

and therefore that

$$g(x) = -\cos x + C, \tag{6.16}$$

where C is an arbitrary constant. Thus,

$$H(x,y) = \frac{1}{2}y^2 - \cos x + C. \tag{6.17}$$

We can also say the following about the nature of the equilibrium points of a Hamiltonian system.

Theorem 10. *Let $\mathbf{x}_e$ be an equilibrium point of the Hamiltonian system (6.2). Suppose that the associated linearised system at $\mathbf{x} = \mathbf{x}_e$ has **no** zero eigenvalues, then the eigenvalues at $\mathbf{x}_e$ are either*

(i) real and of different sign, or

(ii) purely imaginary.

Moreover, in case (i) $\mathbf{x}_e$ is a saddle point, whilst in case (ii) $\mathbf{x}_e$ is a centre.

Proof. The Jacobian of a Hamiltonian system is given by

$$\mathbf{J} = \begin{pmatrix} H_{x,y} & H_{yy} \\ -H_{xx} & -H_{x,y} \end{pmatrix} \tag{6.18}$$

with characteristic equation

$$\lambda^2 = H_{x,y}^2 - H_{xx}H_{yy}. \tag{6.19}$$

which, for non-zero eigenvalues only allows for real and opposite or purely imaginary and complex conjugate solutions. □

Note that we can determine the nature of an equilibrium point (x_e, y_e) by evaluating the function $H_{x,y}^2 - H_{xx}H_{yy}$ at (x_e, y_e):

(i) If $\left(H_{x,y}^2 - H_{xx}H_{yy}\right)\big|_{(x_e,y_e)} > 0$, then (x_e, y_e) is a saddle point

(ii) If $\left(H_{x,y}^2 - H_{xx}H_{yy}\right)\big|_{(x_e,y_e)} < 0$, then (x_e, y_e) is a centre

Finally, we have that

Theorem 11. *Periodic orbits of the Hamiltonian system (6.2) are non-isolated.*

Definition 27. Conservative System *A **conservative** system is a system for which there exists a function $F : \mathbb{R}^2 \to \mathbb{R}$, which is continuous with continuous first and second derivatives, and such that the level curves of F ($F(x, y) =$ constant) define the trajectories of the system.*

On the trajectories of a conservative system suppose $y = y(x)$. Then

$$F(x, y(x)) = \text{constant}, \tag{6.20}$$

implying that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \tag{6.21}$$

Thus the trajectories of a conservative system are solutions of the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}. \tag{6.22}$$

But, these trajectories are the same as those for the system

$$\begin{aligned} \dot{x} &= F_y, \\ \dot{y} &= -F_x, \end{aligned} \tag{6.23}$$

which is Hamiltonian. Hence the phase portrait of a conservative system is always equivalent to that of an associated Hamiltonian system. Therefore, Theorems 10 and 11 also hold for conservative systems.

Example:

We now re-examine the nonlinear pendulum example (Chapter 5, equation (5.17). Clearly, since $P_x + Q_y = 0$, the system (5.17) is Hamiltonian, and it is straightforward to determine that the Hamiltonian function is given by

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + C, \tag{6.24}$$

with C constant. Further, Theorem 10 allows us to determine that the nature of the equilibrium point $(0, 0)$ is a centre.