2RCA Problem Sheet 3

Questions marked (SUM) are assessed. Solutions to these questions should be submitted by the deadline of 17:00 Thursday 7th March.

1) Let $S \subseteq \mathbb{C}$ be a set.
   a) Show that the set of interior points of S is open.
   b) Show that the set of boundary points of S is closed.
2)
   a) Prove (or read the proof in the lecture notes) that a set $S \subseteq \mathbb{C}$ is closed if and only if it contains all its boundary points.
   b) Prove that a set $S \subseteq \mathbb{C}$ is open if and only if it contains none of its boundary points.
3) Sketch the set $S = \{z \in \mathbb{C}: |z - 2 + 5i| < 7,\ \mathrm{Im}(z) > 0\}$ in the complex plane. Is S open, connected, simply connected, a domain or bounded? What are the interior and boundary points of S?
4) (SUM) Sketch the set
   $$\mathrm{S} = \{z \in \mathbb{C}: (|z - 1 + i| < 3 \ or\ |z - 1 + i| \ \geq\ 10),\ \mathrm{Re(z)} > 0, \mathrm{Im}(z) > 0\}$$
   in the complex plane. Is S open, closed, connected, simply connected, a domain or bounded? What are the interior and boundary points of S?
5) For the following $S \subseteq \mathbb{C}$ and complex functions $f: S \to \mathbb{C}$ write down the real and imaginary parts of f. In other words, find functions $u, v: \{(x, y) \in \mathbb{R}^2: x + iy \in S\} \to \mathbb{R}$ such that $f(x + iy) = u(x, y) + iv(x, y)$ for all $z = x + iy \in S$.
   a) $S = \mathbb{C}, f(z) = z^2 + \overline{z}.$
   b) $S = \mathbb{C} \setminus \{0\}, f(z) = \frac{z}{\overline{z}+i}$
   c) $S = \mathbb{C}, f(z) = e^z.$
6) Give an example of a set $S \subseteq \mathbb{C}$ so that every complex number is a boundary point of S.
7) (SUM) Using the definition of limit, show that
   a) $\lim\limits_{z \to -i} \frac{1}{(z+i)^5} = \infty.$
   b) $\lim\limits_{z \to 3i} \frac{z^2 - 2iz + 3}{z - 3i} = 4i$
8) Determine the following limits:
   a) $\lim\limits_{z \to i} \frac{z^3 + iz^2 - z + 3i}{z - i}$
   b) $\lim\limits_{z \to \infty} \frac{z^2}{iz^3 + 3z - 1}$
   c) $\lim\limits_{z \to 2i} \frac{1}{z^2 + 4}$

9) Let $p \in \mathbb{C}$ and consider the function $f: \mathbb{C} \setminus \{p\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z-p}.$$

Using the definition of the derivative of f show that

$$f'(z) = -\frac{1}{(z-p)^2}.$$

10) (SUM) For the following complex functions f check whether f satisfies the Cauchy-Riemann equations. Determine where each function f is differentiable.

a) $f(x+iy) = xy^3$

b) $f(z) = (\overline{z} + i)^2$

c) $f(x+iy) = e^y \sin x + ie^y \cos x$

11) Let $S \subseteq \mathbb{C}$ be a domain and suppose that both $f: S \to \mathbb{C}$ and $\overline{f}: S \to \mathbb{C}$ are holomorphic. Prove that f is constant.

12) (SUM) For $a, b, c, d \in \mathbb{R}$ let $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$. Find a description of the set of all 4-tuples (a, b, c, d) for which u is harmonic.

13) Prove that the function $u: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by $u(x,y) = \log(x^2 + y^2)$ is harmonic. Find a harmonic conjugate of u.

14) For $T_1 = \{z \in \mathbb{C} : Re(z) = 0\}$ and $T_2 = \{z \in \mathbb{C} : Im(z) = 0\}$ investigate the limits $\lim\limits_{\substack{z \to 0 \\ z \in T_1}} \frac{z^2}{\overline{z}z}$ and $\lim\limits_{\substack{z \to 0 \\ z \in T_2}} \frac{z^2}{\overline{z}z}$. Does the limit $\lim\limits_{z \to 0} \frac{z^2}{\overline{z}z}$ exist?

15) Decide whether the limit $\lim\limits_{z \to 0} \frac{z}{|z|}$ exists.