1)

a. The centre is $2i$. Ratio test: $\left|\frac{a_n}{a_{n+1}}\right| = \frac{3^{n+1}n}{3^n(n+1)} = \frac{3n}{n+1} \to 3$ as $n \to \infty$. So $R = 3$.

b. The centre is $-5 - i$. Setting $w = (z + 5 + i)^3$ the given power series may be rewritten as $\sum_{n=0}^{\infty} \frac{1}{n} w^n$. Using the ratio test we may compute the radius of this power series as $\lim_{n\to\infty} \frac{n+1}{n} = 1$. So $\sum_{n=0}^{\infty} \frac{1}{n} w^n = \sum_{n=0}^{\infty} \frac{1}{n} (z + 5 + i)^{3n}$ converges if $|w| = |z + 5 + i|^3 < 1$ and does not converge if $|w| = |z + 5 + i|^3 > 1$. Hence, $\sum_{n=0}^{\infty} \frac{1}{n} (z + 5 + i)^{3n}$ converges if $|z + 5 + i| < 1$ and does not converge if $|z + 5 + i| > 1$. Therefore the radius of convergence is $R = 1$.

c. The centre is $-i$. We have $|a_n|^{\frac{1}{n}} = \frac{1}{2}$ if n is odd and $|a_n|^{\frac{1}{n}} = \frac{1}{4}$ if n is even. Hence $limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2}$, so $R = \frac{1}{\frac{1}{2}} = 2$.

2)

a. Start by rewriting the equation as $e^{-y}e^{ix} = \sqrt{2}\, e^{-\frac{i\pi}{2}}$. Taking the absolute value of both sides of this equation we get $e^{-y} = \sqrt{2}$ and so $y = -\frac{1}{2}\log 2$. This leaves us with the task of solving the equation $e^{ix} = e^{-\frac{i\pi}{2}}$, which has solutions $x = -\frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$. So the solution set is given by $\{\left(-\frac{\pi}{2} + 2k\pi\right) + i\left(-\frac{1}{2}\log 2\right) : k \in \mathbb{Z}\}$.

b. Rewrite the equation as $\frac{e^z - e^{-z}}{2} = i$ and multiply through by e^z to show that e^z is a root of the complex polynomial $p(u) = u^2 - 2iu - 1 = (u - i)^2$. We conclude that $e^z = i$. Writing $z = x + iy$, this equation becomes $e^x e^{iy} = e^{\frac{i\pi}{2}}$, which by the method used in part a has solutions $x = 0$ and $y = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$. So the solution set is given by $\{i\left(\frac{\pi}{2} + 2k\pi\right) : k \in \mathbb{Z}\}$.

c. Rewrite as $\frac{e^{iz} + e^{-iz}}{2} = 5$ and multiply through by e^{iz} to show that e^{iz} is a root of the complex polynomial $p(u) = u^2 - 10u + 1 = (u - 5)^2 - 24 = \left(u - 5 - 2\sqrt{6}\,\right)\left(u - 5 + 2\sqrt{6}\right)$. We conclude that $e^{iz} = 5 \pm 2\sqrt{6}$. Writing $z = x + iy$, this equation becomes $e^{ix}e^{-y} = \left(5 \pm 2\sqrt{6}\right)e^{i0}$, which by the method of part a has solutions $y = -\log(5 \pm 2\sqrt{6})$ and $x = 2k\pi$ for $k \in \mathbb{Z}$. Hence, the solution set is given by $\{2k\pi - i\log(5 \pm 2\sqrt{6}) : k \in \mathbb{Z}\}$.

3) A suitable parameterisation is given by $\gamma(t) = t(2 - i), t \in [0,1]$. Now, using the formula $\int_\gamma f(z)dz = \int_0^1 f\left(\gamma(t)\right)\gamma'(t)dt$, we get

$$\int_\gamma z^2 + 5\overline{z}\,dz = \int_0^1 \left(\left(t(2-i)\right)^2 + 5\left(t(2+i)\right)\right)(2-i)dt = \int_0^1 (2-i)^3t^2 + 125t\,dt$$
$$= \frac{(2-i)^3}{3}t^3 + \frac{25}{2}t^2|_0^1 = \frac{(2-i)^3}{3} + \frac{25}{2}.$$

4) Start by writing $\sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = -\frac{i}{2}(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x))$ and then separate the RHS into its real and imaginary parts to get $\sin(x+iy) = \frac{e^y+e^{-y}}{2}\sin x + i\frac{e^y-e^{-y}}{2}\cos x = \cosh y \sin x + i\sinh y\cos x$. The derivation of the identity for $\cos(x+iy)$ is similar: start by writing
$$\cos(x+iy) = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{1}{2}\left(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right)$$
$$= \frac{e^y+e^{-y}}{2}\cos x + i\frac{-e^y+e^{-y}}{2}\sin x = \cosh y\cos x - i\sinh y\sin x.$$

5) Use parameterisations of the form $\gamma_1(t) = (1-t)1 + ti$, $t\in[0,1]$ and $\gamma_2(t) = (1-t)i + t(-1), t\in[0,1]$ for the two line segments and $\gamma_3(t) = e^{i(\pi+t)}$, $t\in[0,\pi]$ for the circular arc. Then Γ is parameterised by the join $\gamma = \gamma_1+\gamma_2+\gamma_3$ of γ_1,γ_2 and γ_3. Using successively the formula for the join of two paths, we may compute the formula for $\gamma = (\gamma_1+\gamma_2)+\gamma_3$ as $\gamma(t) = (1-t)1 + ti$, if $t\in[0,1], \gamma(t) = (2-t)i + (t-1)(-1)$, if $t\in[1,2]$, and $\gamma(t) = e^{i(\pi+t-2)}$ if $t\in[2,2+\pi]$. Since each of γ_1,γ_2 and γ_3 are smooth paths, their join γ is a piecewise smooth contour. Hence, Γ is a piecewise smooth contour. Γ is closed because we have $\gamma(0) = \gamma(2+\pi)$ and Γ is simple because whenever $\gamma(s) = \gamma(t)$ occurs for $s,t\in[0,2+\pi]$, we either have $s=t$ or $\{s,t\} = \{0,2+\pi\}$. To check this formally observe that each of γ_1,γ_2 and γ_3 is injective and the only points which lie in the images of more than one γ_i are their endpoints $1 = \gamma_1(0) = \gamma_3(\pi)$, $i = \gamma_1(1) = \gamma_2(0)$ and $-1 = \gamma_2(1) = \gamma_3(0)$. It follows that $\gamma(s) = \gamma(t)$ may occur only when $s=t$ or $\gamma(s) = \gamma(t) \in \{1, i, -1\}$. We observe that $t=1$ is the only solution of $\gamma(t) = i$, $t=2$ is the only solution of $\gamma(t) = -1$ and $\gamma(s) = \gamma(t) = 1$ whenever $s,t\in\{0,2+\pi\}$.

6)
  a. We parameterise Γ by the contour $\gamma: [0,1]\to\mathbb{C},\ \gamma(t) = i+t$. So, by definition, the contour integral is given by
$$\int_\Gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt$$
$$= \int_0^1 ((i+t)^2 + 2(-i+t))1dt$$
$$= \int_0^1 t^2 + 2t - 1dt + i\int_0^1 2(t-1)dt = \frac{1}{3} - i.$$
  b. We parameterise Γ by the contour $\gamma: [0,\pi]\to\mathbb{C}, \gamma(t) = i + e^{it}$. So, by definition, the contour integral is given by

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_0^\pi f(\gamma(t))\gamma'(t)dt \\ &= \int_0^\pi \overline{(i+e^{it})}^3 ie^{it}dt \\ &= \int_0^\pi \left(-i+e^{-it}\right)^3 ie^{it}dt \\ &= \int_0^\pi -e^{it} - 3i + 3e^{-it} + ie^{-2it}dt \\ &= -\frac{1}{i}e^{it} - 3it - \frac{3}{i}e^{-it} - \frac{1}{2}e^{-2it}|_0^\pi = (-8-3\pi)i.\end{aligned}$$

7) For the holomorphic (and therefore continuous) function $f\colon \mathbb{C} \setminus \{-i, i\} \to \mathbb{C}$ defined by $f(z) = \frac{e^z}{z^2+1}$ we derive an upper bound on $|f(z)|$ for $z \in \Gamma$. First observe that, for $z = x + iy \in \Gamma$ we have $|x|, |y| \leq |z| = 5$ and so

$$|e^z| = |e^x||e^{iy}| = |e^x| = e^x \leq e^{|x|} \leq e^5.$$

Moreover, we have

$$|z^2+1| \geq |z^2| - 1 = |z|^2 - 1 = 25 - 1 = 24.$$

We conclude that

$$|f(z)| = \frac{|e^z|}{|z^2+1|} \leq \frac{e^5}{24} =: M$$

for all $z \in \Gamma$. Letting L denote the length of Γ we now observe that $L = \frac{5\pi}{3}$. Since the conditions of the ML-Lemma are satisfied, we may apply the ML-Lemma to conclude that

$$|\int_\Gamma f(z)dz| \leq ML = \frac{e^5}{24}\frac{5\pi}{3} = \frac{5\pi e^5}{72}.$$

8)

a. We may compute the roots of the complex polynomial in the denominator as $i \pm 1$. Therefore, by the algebra of differentiability and the fact that the denominator is never zero when $|z| < |i \pm 1| = \sqrt{2}$, the expression inside the integral defines a holomorphic function $f(z)$ on the set $\Omega = B(0, \sqrt{2})$. Observe that $\Gamma \subseteq \Omega$. Therefore, the conditions of the Cauchy-Goursat Theorem are satisfied and we get $\int_\Gamma f(z)dz = 0$.

b. The function $f\colon \mathbb{C} \to \mathbb{C}, f(z) = \cosh 3z$ is holomorphic and defined on a simply connected set (the set $\mathbb{C}$). Moreover γ is a simple contour with image in this set $\mathbb{C}$. We also observe that the function $F\colon \mathbb{C} \to \mathbb{C}, F(z) =$

$\frac{1}{3}\sinh z$ is holomorphic and satisfies $F' = f$. Therefore, by Corollary 12.3 in the lecture notes, we have

$$\int_\Gamma f(z)dz = F\big(\gamma(\pi)\big) - F\big(\gamma(0)\big) = \frac{1}{3}\sinh(i-1) - \frac{1}{3}\sinh(i+1).$$

9) Since the two functions are holomorphic on $\mathbb{C}$ they are both equal to their Taylor series with centre 0 on $\mathbb{C}$, by Theorem 14.2 or Corollary 13.4. Now applying Corollary 13.4 with $z_0 = 0$ and Γ equal to the unit circle with centre 0 and the standard orientation, we may compute the coefficients in the Taylor series for both g and f as contour integrals along Γ. However, since f and g coincide on Γ the formulae for the Taylor coefficients of g coincides with those for f. Hence, Corollary 13.4 gives that g and f have the same coefficients in their Taylor series centred at 0. Hence, we have that f and g coincide on the whole of $\mathbb{C}$.

10) We observe that

$$\frac{1}{(z+2)(z+5)} = \frac{1}{3(z+2)} - \frac{1}{3(z+5)}.$$

Equation 1

a. Suppose $z \in B(1,1)$, so $|z-1| < 1$. We may write

$$\frac{1}{z+2} = \frac{1}{3+(z-1)} = \frac{1}{3}\frac{1}{1+\frac{z-1}{3}}$$
$$= \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^{n+1}}.$$

In the third equality we used that $|\frac{z-1}{3}| = \frac{|z-1|}{3} < 1$ in order to apply the formula $\frac{1}{1-p} = \sum_{n=0}^{\infty} p^n$ for a geometric series, valid for $p \in \mathbb{C}$ with $|p| < 1$. Similarly we get

$$\frac{1}{z+5} = \frac{1}{6+(z-1)} = \frac{1}{6}\frac{1}{1+\frac{z-1}{6}}$$
$$= \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{6^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{6^{n+1}}.$$

Therefore, plugging these expressions into Equation 1 and using the algebra of power series we get $\frac{1}{(z+2)(z+5)} = \sum_{n=0}^{\infty} c_n (z-1)^n$, where

$$c_n = \frac{(-1)^n}{3^{n+2}} - \frac{1}{3}\frac{(-1)^n}{6^{n+1}}.$$

b. We have

$$\frac{1}{z+2} = \frac{1}{z}\frac{1}{1+\left(\frac{2}{z}\right)}$$
$$= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^n}$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}} = \sum_{n=-\infty}^{-1}(-1)^{n+1}\frac{z^n}{2^{n+1}}.$$

In the third equality we used that $|\frac{2}{z}| = \frac{2}{|z|} < 1$ in order to apply the formula $\frac{1}{1-p} = \sum_{n=0}^{\infty} p^n$ for a geometric series, valid for $p \in \mathbb{C}$ with $|p| < 1$.

Similarly,

$$\frac{1}{z+5} = \frac{1}{z}\frac{1}{1+\frac{5}{z}}$$
$$= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{z^n}$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{5^n}{z^{n+1}} = \sum_{n=-\infty}^{-1}(-1)^{n+1}\frac{z^n}{5^{n+1}}.$$

Therefore, plugging these expressions into Equation 1 and using the algebra of Laurent series, we get

$$\frac{1}{(z+2)(z+5)} = \sum_{n=-\infty}^{-1} c_n z^n,$$

with

$$c_n = (-1)^{n+1}\frac{2^{-(n+1)} - 5^{-(n+1)}}{3}.$$

c. We may write

$$\frac{1}{z+5} = \frac{1}{3+(z+2)} = \frac{1}{3}\frac{1}{1+\frac{z+2}{3}}.$$

Since $|\frac{z+2}{3}| < 1$ the last expression may be written instead as

$$\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+2}{3}\right)^n.$$

Plugging this into Equation 1 we get

$$\frac{1}{(z+2)(z+5)} = \frac{1}{3(z+2)} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(z+2)^n}{3^{n+2}} = \sum_{n=-1}^{\infty} c_n(z-2)^n,$$

with $c_{-1} = \frac{1}{3}$ and $c_n = \frac{(-1)^{n+1}}{3^{n+2}}$ for $n \geq 0$.