Examples sheet 4 – Linear Algebra

The exercises below correspond to material from Lectures 13–17. Selected exercises will be covered in the Examples class scheduled in week 9. Solutions will be available on Canvas.

ISOMORPHISMS.

1. Let V be an n-dimensional vector space with basis B_V.

(a) Find the matrix representation of the coordinate map φ_V with respect to B_V and the canonical (or standard) basis for $\mathbb{R}^n$.

(b) Let $B'_n = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ denote another basis for $\mathbb{R}^n$. Find the matrix representation of φ_V with respect to B_V and B'_n.

Are these matrices invertible? Is φ_V an isomorphism?

2. (a) Show that the set $\mathcal{M}_{2,2} := \mathcal{M}_{2,2}(\mathbb{R}) := \mathbb{R}^{2\times 2}$ of 2-by-2 matrices with real entries is a vector space over $\mathbb{R}$ when equipped with the standard operations of matrix addition and scalar-matrix multiplication.

(b) Find a basis for $\mathcal{M}_{2,2}$ and state its dimension.

(c) Show further that $\mathcal{M}_{2,2}$ is isomorphic to $\mathbb{R}^4$ by identifying a one-to-one correspondence between the basis from part **(a)** and the canonical basis of $\mathbb{R}^4$.

3. Let $\mathcal{M}_{2,2}^{\mathrm{sym}}$ denote the set of symmetric 2-by-2 matrices, i.e., matrices with equal off-diagonal entries.

(a) Show that $\mathcal{M}_{2,2}^{\mathrm{sym}} \leq \mathcal{M}_{2,2}$.

(b) Find a basis for $\mathcal{M}_{2,2}^{\mathrm{sym}}$ and state its dimension.

(c) Show further that $\mathcal{M}_{2,2}^{\mathrm{sym}}$ is isomorphic to $\mathbb{R}^3$ by identifying a one-to-one correspondence between the basis from part **(a)** and the canonical basis of $\mathbb{R}^3$.

4. Let $f \in \mathcal{L}(V, W)$. Identify the isomorphisms in the following list. [You may want to consider the dimensions of V and W.]

(a) $V = \mathcal{P}_n(\mathbb{R}), W = \mathcal{P}_{n+1}(\mathbb{R}), f(p) = I(p) := \int p(x)\mathrm{d}x$.

(b) $V = \mathcal{P}_n(\mathbb{R}), W = \mathcal{P}^0_{n+1}(\mathbb{R}) := \{p \in \mathcal{P}_{n+1}(\mathbb{R}), p(0) = 0\},\ f(p) = I(p)$.

(c) $V = \mathcal{P}_n(\mathbb{R}), W = \mathcal{P}_{n+2}(\mathbb{R}), f(p) = I(I(p))$.

(d) $V = \mathcal{P}_n(\mathbb{R}), W = \mathcal{P}^0_{n+2}(\mathbb{R}) := \{p \in \mathcal{P}_{n+1}(\mathbb{R}), p(0) = 0\},\ f(p) = I(I(p))$.

(e) $V = \mathcal{P}_n(\mathbb{R}), W = \mathcal{P}^{00}_{n+2}(\mathbb{R}) := \{p \in \mathcal{P}_{n+1}(\mathbb{R}), p(0) = p'(0) = 0\},\ f(p) = I(I(p))$.

5. Show that $\mathbb{R}^{m\times n}$ is a vector space over $\mathbb{R}$. What is its dimension?

6. Let $f\colon V \to W$ be an isomorphism. Let $S \subseteq V$. Prove the following statements.

(a) S is a spanning set for V if and only if $f(S)$ is a spanning set for W.

(b) S is linearly independent if and only if $f(S)$ is.

(c) S is a basis for V if and only if $f(S)$ is a basis for W.

(d) U is a subspace of V if and only if $f(U)$ is a subspace of W with same dimension.

7. Let $V(\mathbb{F}), W(\mathbb{F})$ have bases B_V, B_W and dimensions n, m, respectively. Define the map $\mathrm{m}\colon \mathcal{L}(V,W) \to \mathbb{F}^{m\times n}$ that associates with a linear map f its matrix representation relative to B_V, B_W:

$$\mathrm{m}(f) = A_{VW}.$$

Use the isomorphism criterion to check that m is an isomorphism and therefore that $\mathcal{L}(V,W) \cong \mathbb{F}^{m\times n}$.

ENDOMORPHISMS.

8. Let $f \in \mathcal{L}(V)$. Identify the automorphisms in the following list. [You may want to consider their matrix representations.]

(a) $V = \mathbb{E}^2$, $f(\mathbf{A})$ is the vector obtained by rotating $\mathbf{A}$ through an angle θ.

(b) $V = \mathbb{E}^2$, $f(\mathbf{A})$ is the vector obtained by reflecting (mirroring) $\mathbf{A}$ in the line $x = y$.

(c) $V = \mathbb{E}^2$, $f(\mathbf{A})$ is the vector obtained by projecting $\mathbf{A}$ on the line $y = 0$.

(d) $V = \mathcal{P}_2(\mathbb{R}), f(p) = p + p'$.

(e) $V = \mathcal{P}_2(\mathbb{R}), f(p) = xp'$.

(f) $V = \mathcal{P}_2(\mathbb{R}), f(p) = x^2p''$.

9. True or false: $\mathrm{Aut}(V) \leq \mathcal{L}(V)$.

10. Show that matrix similarity is an equivalence relation on the space of square matrices.

11. Show that $f \in \mathcal{L}(V)$ is an automorphism if and only if $\ker f = \{\mathbf{0}_{\mathrm{V}}\}$.

INVARIANCE

12. Show that $U \leq V$ is invariant under $f \in \mathcal{L}(V)$ for the following choices of U:

(a) $U = \{\mathbf{0}\}$;

(b) $U = V$;

(c) $U = \ker f$;

(d) $U = \mathrm{im}\, f$.

13. Let $f \in \mathcal{L}(V)$ and assume $V = U \oplus W$, with U, W f-invariant subspaces of V.

(a) Show that f has a block-diagonal matrix representation:

$$A = \begin{bmatrix} A_U & O \\ O & A_W \end{bmatrix},$$

where A_U, A_W are square matrices of dimensions summing up to $\dim V$.

(b) Discuss the matrix representation when $V = \bigoplus_k U_k$, where U_k are f-invariant subspaces of V.

14. Let U be f-invariant, where f is invertible. Show that U is f^{-1}-invariant.

EIGENVALUES AND EIGENVECTORS.

15. The following matrices have a single eigenvalue $\lambda = 1$. In each case, find the geometric multiplicity $\gamma(\lambda)$.

$$\text{i.}\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{ii. } B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

16. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 & 4 \\ -3 & 3 & -1 \\ 1 & -4 & 0 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Write down the diagonal canonical form of A over $\mathbb{C}$.

(c) Write down the block-diagonal canonical form of A over $\mathbb{R}$.

17. Let $A \in \mathbb{R}^{n\times n}$ be non-singular. Show that the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A. What can you say about the eigenvectors of A^{-1}?

18. Let $A, B \in \mathbb{R}^{n\times n}$ be non-singular. True or false:

(a) $\lambda(AB) = \lambda(BA)$;

(b) $\lambda(AB) = \lambda(A)\lambda(B)$;

(c) $\lambda(A+B) = \lambda(A) + \lambda(B)$.

19. Let $A \in \mathbb{R}^{n\times n}$ and let $\mathrm{sp}(A) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A\}$ denote the spectrum of A.

(a) Show that if A is lower triangular, then $\mathrm{sp}(A) = \{a_{ii} : i = 1, \ldots, n\}$.

(b) Show that if

$$A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} A_1 & \\ B & A_2 \end{bmatrix},$$

then

$$\mathrm{sp}(A) = \mathrm{sp}(A_1) \cup \mathrm{sp}(A_2).$$

20. Let $A \in \mathbb{R}^{n\times n}$ be diagonalisable: $A = VDV^{-1}$, where $V \in \mathbb{R}^{n\times n}$ is the matrix of eigenvectors and D is a diagonal matrix.

(a) Show that $A^m = VD^mV^{-1}$.

(b) Given $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$, let $p(A)$ denote the matrix

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.$$

Show that $p(A) = Vp(D)V^{-1}$.

(c) Evaluate $p_A(A)$, where $p_A(t)$ is the characteristic polynomial of A.