Chapter 1 – Some differential equation theory

(including existence and uniqueness, the general solution and the Wronskian)

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Before attempting to solve an ODE, it is important to establish whether or not the ODE actually has a solution. If it does, is this solution unique or might there be multiple possible solutions? Is the solution valid across the whole domain or is it restricted to a particular interval? Sometimes these questions are easily answered from the ODE itself but we shall see through this chapter that the existence of solutions to differential equations is not always immediately evident.

1 Existence and uniqueness

1.1 Existence and uniqueness theorem for solutions to 1st order linear ODEs

Theorem 1 *If the functions $p(x)$ and $q(x)$ are continuous on an open interval $I : \alpha < x < \beta$ containing the point $x = x_0$, then there exists a unique function $y = u(x)$ that satisfies the differential equation*

$$\frac{dy}{dx} + p(x)y(x) = q(x) \tag{1}$$

for each x in I, and that also satisfies the initial condition

$$y(x_0) = y_0 \tag{2}$$

where y_0 is an arbitrary prescribed initial value.

Idea behind the proof: we saw in the Recap chapter that if a solution to (1) exists, it is given by

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)\,dx + c\right) \tag{3}$$

where

$$\mu(x) = e^{\int p(x)\,dx}. \tag{4}$$

It is necessary to establish that (3) exists and that it is unique.

If $p(x)$ is continuous on I then $\mu(x)$ is defined on I and is a nonzero differentiable function. Since $\mu(x)$ and $q(x)$ are continuous on I, the function $\mu(x)q(x)$ is integrable and (3) exists and is differentiable on I. Therefore y as given by (3) exists and is differentiable. We can therefore substitute it into (1) to verify that it is indeed a solution on I, i.e. a solution exists.

Finally, the initial condition (2) determines the constant c uniquely so there is only one solution of the initial value problem given by (1) and (2), i.e. the solution is unique.

1.2 Existence and uniqueness theorem for solutions to 1st order nonlinear ODEs

Theorem 2 *The initial value problem*

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{5}$$

has a unique solution in some interval $x_0 - h < x < x_0 + h$ *contained in* $\alpha < x < \beta$ *if* f *and* $\dfrac{\partial f}{\partial y}$ *are continuous in some rectangle* $\alpha < x < \beta$, $\gamma < y < \delta$ *containing the point* (x_0, y_0).

Note that while this theorem can tell us that a unique solution may exist on a somewhat arbitrary interval, we often need to tackle the ODE directly to gain more information about this interval.

Example 1: Solve the initial value problem

$$\frac{dy}{dx} - y^{\frac{1}{3}} = 0, \quad y(0) = 0, \quad x \geq 0.$$

Answer:

Example 2: Solve the initial value problem

$$\frac{dy}{dx} - y^2 = 0, \quad y(0) = 1$$

and determine the interval on which the solution exists.

Answer:

See Figure 1 for the solutions when $y(0) = \pm 2$.

The above is a good example of problems associated with nonlinear ODEs: here the singularities of the solution depend in an essential way on the initial conditions as well as on the ODE.

Idea behind the proof of Existence and Uniqueness for 1st order nonlinear ODEs.

- Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad \text{with} \quad y(0) = 0 \tag{6}$$

  (if another initial point is given we can make a change of variables that will make the initial point the origin).

- Assume there is a function $y = u(x)$ that satisfies the initial value problem (6),

- then $f(x, u(x))$ is a continuous function of x only, and we can integrate $\dfrac{dy}{dx} = f(x, y)$ from the initial point $x = 0$ to an arbitrary value of x using $u(0) = 0$, i.e.

$$u(x) = \int_0^x f(s, u(s))\, ds. \tag{7}$$

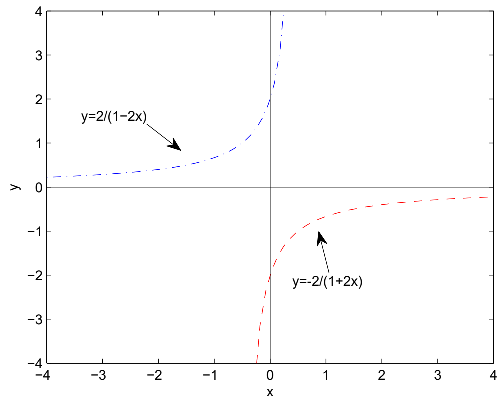

Figure 1: The solution to the initial value problem of Example 2 with $y(0) = 2$ (dot-dash blue line) and $y(0) = -2$ (dashed red line). For $y(0) = 2$, the solution exists on $-\infty < x < \dfrac{1}{2}$ and blows up to infinity as $x \to \dfrac{1}{2}$ from the left. For $y(0) = -2$, the solution exits on $-\dfrac{1}{2} < x < \infty$ and blows up to negative infinity as $x \to -\dfrac{1}{2}$ from the right.

- This is an integral equation that provides another relation that can be satisfied by any solution of (6) and vice versa, i.e. the initial value problem (6) and the integral equation (7) are equivalent in that any solution of one is also a solution of the other. We can then proceed to work with (7) rather than (6) to prove the Existence and Uniqueness theorem for 1st order nonlinear equations by showing that there is a unique solution to (7) on an interval $-h < x < h$.
- Use the method of successive approximations (or Picard's iteration method):
  - choose an initial function $u_0(x) = 0$ for example. Then $u_0(x)$ satisfies the initial condition of (6) but not necessarily (and most likely not) the differential equation of (6)
  - obtain a next approximation u_1 by substituting $u_0(s)$ for $u(s)$ in (7):
  $$u_1(x) = \int_0^x f(s, u_0(s))\, ds$$
  - let
  $$u_2(x) = \int_0^x f(s, u_1(s))\, ds$$
  etc...
  - $$u_{n+1}(x) = \int_0^x f(s, u_n(s))\, ds$$
  - each $u_i(x)$ will satisfy the initial condition but not necessarily the differential equation.
- If at any stage $u_{k+1}(x) = u_k(x)$ then $u_k(x)$ is a solution of (7) and therefore also of (6). Otherwise the entire infinite sequence $u_0, u_1, ..., u_n, ...$ must be considered.

It then needs to be established that

- all members of the sequence u_i exist on some interval,
- the sequence converges to some limit

$$u(x) = \lim_{n\to\infty} u_n(x),$$

- $u(x)$ is continuous and satisfies (7) and therefore also satisfies (6),
- $u(x)$ is the unique solution of (7).

Full details of this proof can be found in advanced textbooks on differential equations. We omit the full details here to concentrate on methods for obtaining solutions to differential equations when they do exist.

1.3 General existence and uniqueness theorem for nth order linear ODEs

Theorem 3 *The nth order linear initial value problem*

$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + ... + b_0(x)y = d(x) \tag{8}$$

$$y(a) = y_0, \quad y'(a) = y_1, \quad ... \quad , \quad y^{(n-2)}(a) = y_{n-2}, \quad y^{(n-1)}(a) = y_{n-1}, \quad a \in I, \tag{9}$$

(where I is some open interval), has exactly one solution if the functions $b_0(x), ..., b_{n-1}(x)$ and $d(x)$ are continuous on I. The solution exists throughout the interval I.

It can help to understand this theorem by thinking of the solution in terms of a Taylor series:

$$\begin{aligned} y(x) &= y(a) + (x-a)y'(a) + \frac{(x-a)^2}{2!}y''(a) + ... \\ &= \sum_{j=0}^{\infty} \frac{y^{(j)}(a)}{j!}(x-a)^j. \end{aligned}$$

If we substitute (9) into our ODE equation in (8) (i.e. let $x = a$ in (8)), we can use this to obtain $y^{(n)}(a)$. Then differentiate (8) and substitute in $y(a), ..., y^{(n)}(a)$ and use this to obtain $y^{(n+1)}(a)$. If we keep doing this we can obtain everything we need to compute the complete Taylor series for $y(x)$ as given above.

We shall see later in the course that obtaining series approximations to solutions is often a valuable approach to solving differential equations.

No single theorem such as the above guarantees a unique solution to boundary value problems.

2 The general solution to nth order linear ODEs

2.1 Homogeneous nth order linear ODEs

Take the nth order homogeneous linear ODE:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_0(x)y = 0 \tag{10}$$

with the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad ... \quad , \quad y^{(n-2)}(x_0) = y_{n-2}, \quad y^{(n-1)}(x_0) = y_{n-1} \tag{11}$$

where $y_0, ..., y_{n-1}$ are constants (this is an initial value problem).

If $a_{n-1}(x), ..., a_0(x)$ are continuous in some interval around $x = x_0$ then the Existence and Uniqueness Theorem for nth order linear ODEs tells us there is a unique solution satisfying both (10) and (11).

Suppose we know n solutions $u_1(x), ..., u_n(x)$ to (10) (without considering (11) yet). Since (10) is homogeneous, the superposition principle tells us that a linear combination of these functions will also be a solution to (10), i.e.

$$y = \alpha_1 u_1(x) + ... + \alpha_n u_n(x)$$

is a solution of (10) for any constants $\alpha_1, ..., \alpha_n$. As we are not considering the initial conditions in (11), we have infinitely many solutions to (10) based on n arbitrary constants.

However, if we are looking for the unique solution that will also satisfy the initial conditions (11), then we require

$$\begin{aligned}
\alpha_1 u_1(x_0) + ... + \alpha_n u_n(x_0) &= y_0, \\
\alpha_1 u_1'(x_0) + ... + \alpha_n u_n'(x_0) &= y_1, \\
&\vdots \\
\alpha_1 u_1^{(n-1)}(x_0) + ... + \alpha_n u_n^{(n-1)}(x_0) &= y_{n-1}.
\end{aligned}$$

This can be written in matrix form as

$$\begin{pmatrix} u_1(x_0) & u_2(x_0) & ... & ... & u_n(x_0) \\ u_1'(x_0) & u_2'(x_0) & ... & ... & u_n'(x_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-1)}(x_0) & u_2^{(n-1)}(x_0) & ... & ... & u_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}. \tag{12}$$

The system of equations given by (12) has a unique solution if and only if the determinant of the square matrix on the left hand side is non-zero (think back to the linear algebra you studied in Year 1).

If, on the other hand, the determinant of the matrix is zero then it is possible to express at least one of the solutions $u_1(x), ..., u_n(x)$ as a linear combination of the others, in which case the solutions are **linearly dependent**.

If the determinant is nonzero, the solutions $u_1(x), ..., u_n(x)$ are **linearly independent** and they form what is called a **fundamental set of solutions**.

In general, we call the **Wronskian**

$$W = \begin{vmatrix} u_1(x) & u_2(x) & \dots & \dots & u_n(x) \\ u_1'(x) & u_2'(x) & \dots & \dots & u_n'(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & \dots & \dots & u_n^{(n-1)}(x) \end{vmatrix}$$

When the solutions $u_1(x), ..., u_n(x)$ form a fundamental set (i.e. $W \neq 0$ and the solutions are linearly independent), **the general solution** to (10) can be written as

$$y = \alpha_1 u_1(x) + ... + \alpha_n u_n(x)$$

where the $\alpha_1, ..., \alpha_n$ are arbitrary constants. Then for any set of initial conditions (11), $\alpha_1, ..., \alpha_n$ can be determined uniquely (using (12)) to satisfy these conditions.

2.2 Inhomogeneous nth order linear ODEs

The general solution to the nth order inhomogeneous ODE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_0(x)y = d(x) \tag{13}$$

takes the form

$$y = y_p(x) + \alpha_1 u_1(x) + ... + \alpha_n u_n(x)$$

where $y_p(x)$ is a particular solution of (13) and $u_1(x), ..., u_n(x)$ are a linearly independent (fundamental) set of solutions to the homogeneous version of (13) (i.e. (10)). Note that this is precisely what you have seen before in the theory of 2nd order linear ODEs with constant coefficients.

Example: Show that $u_1(x) = x^{\frac{1}{2}}$ and $u_2(x) = x^{-1}$ form a fundamental set of solutions of

$$2x^2y'' + 3xy' - y = 0, \quad x > 0.$$

Answer: