LECTURE 16

Eigenvectors

We consider again the eigenvalue equation

$$f(\mathbf{v}) = \lambda \mathbf{v},$$

where $f \in \mathcal{L}(V)$, $\lambda \in \mathbb{F}$. Let us recall also its matrix formulation in the two equivalent forms considered previously:

$$A\mathbf{x} = \lambda \mathbf{x} \iff (\lambda I - A)\mathbf{x} = \mathbf{0}.$$

16.1 Eigenspaces

The first observation is that $\mathbf{v}$ belongs to a certain subspace of V.

Proposition 16.1 Let $(\lambda, \mathbf{v})$ be an eigenpair of f. Then $\mathbf{v} \in E_\lambda := \ker(f - \lambda id_{\mathrm{V}})$. Moreover, $E_\lambda \leq V$.

Proof. Since $\lambda \mathbf{v} = (\lambda id_{\mathrm{V}})\mathbf{v}$, we find that

$$f(\mathbf{v}) - (\lambda id_{\mathrm{V}})(\mathbf{v}) = \mathbf{0} \iff (f - \lambda id_{\mathrm{V}})(\mathbf{v}) = \mathbf{0} \iff \mathbf{v} \in \ker(f - \lambda id_{\mathrm{V}}).$$

Finally, since$f, id_{\mathrm{V}} \in \mathcal{L}(V)$, the map $f - \lambda id_{\mathrm{V}} \in V$, by closure in $\mathcal{L}(V)$. The result then follows, as the kernel of an endomorphism on V is a subspace of V. ■

The subspace property in the previous result suggests the next definition.

Definition 16.1 — Eigenspace. The subspace E_λ is the eigenspace of f associated with eigenvalue λ.

Note that for any λ, E_λ is non-trivial, since it contains at least one non-zero vector: an eigenvector associated with λ. This means that dim $E_\lambda \geq 1$. Let us derive further properties of eigenspaces.

Proposition 16.2 Let $(\lambda, \mathbf{v}), (\lambda', \mathbf{v}')$ denote two distinct eigenpairs. Then $E_\lambda \cap E_{\lambda'} = \{\mathbf{0}_{\mathrm{V}}\}$.

Proof. Assume, for a contradiction, that there exists a nonzero $\mathbf{u}$ such that $\mathbf{u} \in E_\lambda \cap E_{\lambda'}$. Then

$$f(\mathbf{u}) = \lambda \mathbf{u} = \lambda' \mathbf{u} \implies (\lambda - \lambda')\mathbf{u} = \mathbf{0}_{\mathrm{V}} \implies \lambda = \lambda',$$

which is the contradiction we sought. ■

We immediately obtain the following corollary.

Corollary 16.3 Eigenvectors corresponding to different eigenvalues (i.e., from different eigenspaces) are linearly independent.

Another corollary is included below.

Corollary 16.4 Let dim $E_\lambda = 1$ for all $\lambda \in \operatorname{sp} f$. Then V is a direct sum of eigenspaces:

$$V = \bigoplus_{\lambda \in \operatorname{sp} f} E_\lambda.$$

In particular, the eigenvectors form a basis for V.

Can we actually have dim $E_\lambda > 1$? The answer is provided by the following example:

Example 16.1 Let $V(\mathbb{F})$ be an n-dimensional vector space and let $f = id_V$. Then its matrix representation is I_n, which has a single eigenvalue $\lambda = 1$, with algebraic multiplicity n. Moreover, each canonical vector $\mathbf{e}_i \in \mathbb{R}^n$ is an eigenvector for λ, so that $E_\lambda = \mathbb{R}^n$ and dim $E_\lambda = n$.

It is clear from this example that the algebraic multiplicity of λ is related to the dimension of E_λ. Let us look at this more closely.

16.2 Geometric multiplicity

Definition 16.2 The **geometric multiplicity** of λ is denoted by $\gamma(\lambda)$ and is defined to be the dimension of its associated eigenspace: $\gamma(\lambda) :=$ dim E_λ.

Let $B_\lambda = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ be a basis for E_λ. Then each element $\mathbf{v}_j$ of B_λ is an eigenvector of f. Therefore, the geometric multiplicity of λ can be viewed as the number of linearly independent eigenvectors associated with λ.

The following result provides some initial insight into the existence of eigenspaces of dimension greater than one.

Proposition 16.5 Let $f \in \mathcal{L}(V(\mathbb{F}))$, where $V(\mathbb{F})$ is an n-dimensional vector space. Then every eigenvalue λ of f has geometric multiplicity no greater than the algebraic multiplicity: $\gamma(\lambda) \leq \alpha(\lambda)$.

Proof. Let $1 \leq r = \gamma(\lambda)$ and $B_\lambda = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ be a basis for E_λ, i.e., $f(\mathbf{v}_k) = \lambda \mathbf{v}_k$ for $k = 1, 2, \dots, r$. Let us choose now a basis B for V containing B_λ. Then the matrix representation of f takes the form

$$A = \begin{bmatrix} \lambda I_r & B \\ O & C \end{bmatrix}.$$

Hence, using the properties of determinants,

$$p_A(t) = \det(tI - A) = \det(tI_r - \lambda I_r) \cdot \det(tI_{n-r} - C) = (t - \lambda)^r p_C(t),$$

where $p_C \in \mathcal{P}_{n-r}(\mathbb{F})$ is the characteristic polynomial of C. Therefore, $\alpha(\lambda) \geq r$, since there are at least r factors $t - \lambda$ of $p_A(t)$. Thus, $\gamma(\lambda) = |B_\lambda| = r \leq \alpha(\lambda)$. ∎

An important consequence of the above result is that the sum of geometric multiplicities is no greater than n:

$$\sum_{k=1}^{r} \gamma(\lambda_k) \leq \sum_{k=1}^{r} \alpha(\lambda_k) = n.$$

Since the geometric multiplicity $\gamma(\lambda)$ can be viewed as the number of linearly independent eigenvectors associated with λ, we deduce that the total number of linearly independent eigenvectors of a linear map can be less than n.

Can we have indeed $\gamma(\lambda) \leq \alpha(\lambda)$ for some λ? The answer is yes; here is an example we encountered when we discussed invariant subspaces and which we recall below.

Example 16.2 Let $V = \mathbb{R}^2$ and let

$$f(\mathbf{v}) = A\mathbf{v}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then the eigenvalues are $\lambda_1 = \lambda_2 = \lambda = 1$, but we find that $\gamma(\lambda) = 1 < \alpha(\lambda) = 2$, since

$$(\lambda I - A)\mathbf{v} = \mathbf{0} \iff \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \mathbf{v} = \mathbf{0} \iff \begin{cases} v_1 \in \mathbb{R} \\ v_2 = 0 \end{cases} \iff \mathbf{v} \in \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} =: E_\lambda.$$

Note. This is an example of so-called **Jordan block** (of size 2, with eigenvalue $\lambda = 1$).

There are plenty examples where the geometric multiplicity can take any value between 1 and n.

Exercise 16.1 The following matrices have a single eigenvalue $\lambda = 1$. In each case, find the geometric multiplicity $\gamma(\lambda)$.

$$\text{i. } A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{ii. } B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$