1Mech — Mechanics

Mechanics exercises 2 (weeks 3 and 4)

This sheet's assessed question is number 4.

1. The force on a particle of mass m and charge e, moving with velocity $\dot{\mathbf{r}}$ under the influence of a constant magnetic field $\mathbf{B}$ is $e\dot{\mathbf{r}} \times \mathbf{B}$. This is the only force acting on the particle, which starts at $\mathbf{0}$ with velocity $\mathbf{V}$ initially.

   (a) Show that

   $$m\ddot{\mathbf{r}} \quad = \quad e\dot{\mathbf{r}} \times \mathbf{B},$$

   and give appropriate initial conditions.

   (b) Hence show that

   $$m\dot{\mathbf{r}} \quad = \quad e\mathbf{r} \times \mathbf{B} + m\mathbf{V}.$$

   (c) If $\mathbf{B} = (0, 0, B)$ in Cartesians, write down an expression for $\mathbf{r} \times \mathbf{B}$.

   (d) Hence find the position of the particle $x(t)$, $y(t)$, $z(t)$ where $\mathbf{r} = (x, y, z)$, for initial velocity $\mathbf{V} = (V_1, 0, V_2)$.

   (e) What shape is the particle path?

   [Hint: Don't panic - this question looks a lot harder than it is initially, you should be able to get started OK! Recall the definition of the vector cross product and split into components. Combine your x and y equations to form a single second order ODE you can then solve.]

2. (a) Starting from expressions for the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ in the r and θ directions, where r and θ are polar coordinates, derive the radial and transverse components of acceleration.

   (b) If a particle of mass m moves under a central force of the form $\mathbf{F} = F(r)\mathbf{e}_r$, prove that $r^2\dot{\theta} = h$ is constant, and find the governing equation for the particle path in terms of $u = 1/r$ and θ.

3. Find the value of the constant $h = r^2\dot{\theta}$, and suitable initial conditions for $u(\theta) = 1/r$, $du/d\theta$ for the following particles under the action of a central force.

   (a) The particle is initially at $r = a$, moving with radial velocity v and transverse (also known as angular) velocity $a\omega$.

   (b) The particle is initially at $r = b$, moving away from the origin with speed V in a direction which makes an angle $\pi/4$ with the outward pointing radial vector.

   (c) The particle is initially at $r = c$, moving with speed w in a direction making an angle $\pi/3$ with the **inward** pointing radial vector.

4. **Assessed, marked out of 20. To earn full marks, your answer must be well presented with clear explanations of key steps.**

   A particle of mass m is moving in the **i**-**j** plane. Let the polar coodinates (r, θ) and basis vectors $\mathbf{e}_r, \mathbf{e}_\theta$ be defined in the usual way.

(a) Show that
$$\mathbf{i} = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta, \quad \mathbf{j} = \sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta.$$

(b) The particle is subject to a force $F(r,\theta)\mathbf{e}_r$, exerted along a massless tether with one end fixed at the origin; and the force of gravity, $m\mathbf{g} = -mg\mathbf{j}$. Show that the equations of motion for the particle are:
$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r,\theta) - mg\sin\theta, \tag{1}$$
$$\frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = -mg\cos\theta. \tag{2}$$

You may assume the following expression for the particle's acceleration:
$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta.$$

(c) Assume that the force is central, meaning $F(r,\theta) = F(r)$ with no θ-dependence. Assume also that the particle follows a circular path, meaning $r = a > 0$ is constant and $\dot{\theta} > 0$ for all t. Differentiate (1) with respect to t, to obtain
$$\ddot{\theta} = f(\theta), \tag{3}$$
where f is a function you should determine.

(d) Show that equations (2) and (3) contradict each other.

(e) The following is an interpretation of the result in part (d). Whenever we derive a model that is inconsistent, somewhere along the line we have made an invalid assumption. Here, only two assumptions could possibly be invalid: that the force is central, or that the particle follows a circular path. In conclusion, if F depends only on r then the particle cannot follow a circular path, and if the particle must follow a circular path then F must have a θ-dependence.

Let us insist that the particle follows a circular path of radius $a > 0$ and allow F to depend on θ. If $F(r,\theta) = mrG(\theta)$ for some function G, differentiate (1) with respect to t to obtain
$$\ddot{\theta} = -\frac{G'(\theta)}{2} + f(\theta), \tag{4}$$
where f is the same as in part (c). Hence, find a function G that makes the model consistent, i.e., making (2) equivalent to (4).