Chapter 0 – Ordinary differential equations: introductory material

(key concepts, most of which you should have already covered)

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1 Important definitions

$$F(x, y, y', y'', ..., y^{(n)}) = 0, \tag{1}$$

where $y' = \dfrac{dy}{dx}$, $y'' = \dfrac{d^2y}{dx^2}$ and $y^{(n)} = \dfrac{d^ny}{dx^n}$, is an ordinary differential equation (ODE) of **order** n (the highest derivative of the dependent variable present in the equation). Recall that here y and x are the dependent and independent variables respectively.

If a function $\phi(x)$ can be substituted for y in (1) and it satisfies the equation for all x on an interval I, then $\phi(x)$ is a **solution** to (1) on I.

Example: Show that $\phi(x) = e^{4x}$ is a solution to

$$\frac{dy}{dx} = 4y, \tag{2}$$

on the interval $I = (-\infty, \infty)$.

Answer:

An ODE is **linear** if the dependent variable and all of its derivatives appear in a linear fashion[1].

[1] Recall that if a function $f(x)$ is linear then $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$ where α, β are real constants.

The most general 2nd order linear ODE is given by

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = d(x), \tag{3}$$

(we will come across ODEs in this form many times during this half of the module). For example,

$$\sin(x)\frac{d^2y}{dx^2} + \left(1 - \frac{1}{x}\right)\frac{dy}{dx} + x^4y = x^{\frac{2}{3}}\left(1 - \frac{7\cos(x)}{x}\right)$$

is linear, while

$$\frac{d^2y}{dx^2} + y^2 = 0$$

is nonlinear because of the y^2 term.

An ODE is **homogeneous** if the dependent variable appears in all the terms, and **inhomogeneous** otherwise. For example

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \tag{4}$$

is homogeneous, but (3) is not if $d(x) \neq 0$.

An ODE is **autonomous** if the independent variable does not appear explicitly in the equation. For example

$$\frac{d^4y}{dx^4} + \frac{dy}{dx} = y^4$$

is autonomous, while

$$\frac{d^4y}{dx^4} + \frac{dy}{dx} = x^4$$

is not autonomous.

In order to find unique solutions to ODEs, we must impose extra conditions on their solutions. If we specify the dependent variable and/or some of its derivatives at a particular point these are termed **initial conditions**. If we specify these conditions at different points they are **boundary conditions**. In most cases, we require n conditions to obtain a unique solution to an nth order ODE (if this solution exists). Without any conditions, we can seek the general solution which usually contains n arbitrary constants – more on this in Chapter 1.

2 Recap of some methods for solving 1st order ODEs

2.1 Separable equations

If an ODE takes the form

$$\frac{dy}{dx} = f(y)g(x),$$

it is called **separable** and can be solved by rearranging and integrating:

$$\begin{aligned}\frac{1}{f(y)}\frac{dy}{dx} &= g(x),\\ \int \frac{1}{f(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx,\\ \int \frac{1}{f(y)}\,dy &= \int g(x)\,dx + c,\end{aligned}$$

where c is a constant of integration. This method can be very useful for nonlinear equations (if, of course, they are separable!).

Example: Solve

$$\frac{dy}{dx} = \frac{x}{y}.$$

Answer:

Example: The following equation, called the logistic equation, is sometimes used to describe population growth in biology:

$$\frac{dy}{dt} = y(1-y),$$

where $y = y(t)$ represents the number or density of a population at time t. Find its solution.

Answer:

2.2 1st order linear ODEs and the integrating factor method

If we wish to solve the equation

$$b(x)\frac{dy}{dx} + c(x)y = d(x), \quad b(x) \neq 0$$

we can use the following procedure:

(a) write the equation in the form

$$\frac{dy}{dx} + p(x)y = q(x) \tag{5}$$

(we can do this because $b(x) \neq 0$);

(b) calculate the integrating factor $\mu(x) = e^{\int p(x)\,dx}$,

$$\begin{aligned}(\text{so that } \mu'(x) &= p(x)e^{\int p(x)\,dx},\\ &= p(x)\mu(x)\ );\end{aligned}$$

(c) multiply (5) through by the integrating factor $\mu(x)$:

$$\begin{aligned}\mu(x)\frac{dy}{dx} + \mu(x)p(x)y &= \mu(x)q(x),\\ \mu(x)\frac{dy}{dx} + \mu'(x)y &= \mu(x)q(x),\\ \implies \frac{d}{dx}(\mu(x)y) &= \mu(x)q(x); \qquad (6)\end{aligned}$$

(d) integrate (6) and solve for y by dividing through by $\mu(x)$:

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)q(x)\,dx; \tag{7}$$

(e) apply initial condition if given.

Example: Solve

$$\frac{1}{x}\frac{dy}{dx} - \frac{2}{x^2}y = x\cos(x),$$

subject to

$$y(\pi/2) = 3, \quad x \neq 0.$$

Answer:

(a)

(b)

(c)

(d)

(e)

3 General method for solving 2nd order homogeneous linear ODEs with constant coefficients

$$ay''(x) + by'(x) + cy(x) = 0, \quad a \neq 0, b, c \in \mathbb{R} \tag{8}$$

has the general solution

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \quad \text{if} \quad r_1 \neq r_2,$$
$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x} \quad \text{if} \quad r_1 = r_2,$$

where r_1, r_2 are the roots of the **characteristic equation** $ar^2 + br + c = 0$. We arrive at this by looking for a solution in the form $y = e^{rx}$.

4 General method for solving 2nd order inhomogeneous linear ODEs with constant coefficients

$$ay'' + by' + cy = d(x) \tag{9}$$

(a) determine the general solution to the corresponding homogeneous equation (8) as described above with $y_h = c_1 u_1(x) + c_2 u_2(x)$,

(b) find a particular solution of the given inhomogeneous equation (9), y_p, using the **method of undetermined coefficients**,

(c) the general solution of (9) is then $y = y_p + c_1 u_1 + c_2 u_2$.

We can readily find a particular solution to (9) if $d(x)$ takes one of the following forms:

- $d(x)$ is a polynomial;
- $d(x) = e^{rx}$;
- $d(x) = \sin(\rho x)$ or $\cos(\rho x)$;

- $d(x)$ is the product of any of the above;
- $d(x)$ is a linear combination of any of the above.

The form of the particular solution for each of these possibilities (that must be substituted into (9) to derive the required unknowns) can be obtained from Table 1.

Table 1: The required forms of the particular solution of $ay'' + by' + cy = d(x)$ for various $d(x)$. k is the smallest of $0, 1, 2$ that ensures no term in y_p is a solution of the corresponding homogeneous equation (you will probably find it easier to understand this by doing examples yourself).

$d(x)$	$y_p(x)$
$P_n(x) = d_0 + d_1x + ... + d_nx^n$	$x^k(D_0 + D_1x + ... + D_nx^n)$
$P_n(x)e^{rx}$	$x^k(D_0 + D_1x + ... + D_nx^n)e^{rx}$
$P_n(x)e^{rx}\sin\rho x$ or $P_n(x)e^{rx}\cos\rho x$	$x^k[(D_0 + D_1x + ... + D_nx^n)e^{rx}\cos\rho x+$ $(E_0 + E_1x + ... + E_nx^n)e^{rx}\sin\rho x]$

Example: Solve $y'' + 9y = x^2e^{3x} + 6$.

Answer:

(a)

(b)

(c)

5 The superposition principle

5.1 Linear homogeneous ODEs

The above methods for solving 2nd order linear ODEs with constant coefficients rely on the superposition principle. This states that if $y = u_1(x)$ and $y = u_2(x)$ are two solutions to a linear homogeneous ODE (for example, (4)) then

$$y = \alpha_1 u_1(x) + \alpha_2 u_2(x)$$

is also a solution to the same ODE for any constants α_1, α_2.

Proof for 2nd order homogeneous ODEs

If $y = u_1(x)$ and $y = u_2(x)$ are both solutions to (4) then

$$a(x)\frac{d^2u_1}{dx^2} + b(x)\frac{du_1}{dx} + c(x)u_1 = 0, \tag{10}$$

$$a(x)\frac{d^2u_2}{dx^2} + b(x)\frac{du_2}{dx} + c(x)u_2 = 0. \tag{11}$$

Adding $\alpha_1 \times (10)$ to $\alpha_2 \times (11)$ gives

$$a(x)\frac{d^2}{dx^2}(\alpha_1 u_1 + \alpha_2 u_2) + b(x)\frac{d}{dx}(\alpha_1 u_1 + \alpha_2 u_2) + c(x)(\alpha_1 u_1 + \alpha_2 u_2) = 0.$$

As $(\alpha_1 u_1 + \alpha_2 u_2)$ satisfies the above,

$$y = \alpha_1 u_1(x) + \alpha_2 u_2(x)$$

must also be a solution of (4).

5.2 Linear inhomogeneous ODEs

If $y = y_p(x)$ is a particular solution to an inhomogeneous linear ODE (for example, (3)) and $y = u_1(x)$ and $y = u_2(x)$ are solutions to the homogeneous version of the same ODE then

$$y = y_p(x) + \alpha_1 u_1(x) + \alpha_2 u_2(x)$$

is a solution to the inhomogeneous linear ODE for any constants α_1, α_2.

Proof for 2nd order inhomogeneous ODEs

This is left as an exercise.