

## 5 CONSTANT JERK EQUATIONS FOR A TRAJECTORY GENERATOR

### *Purpose:*

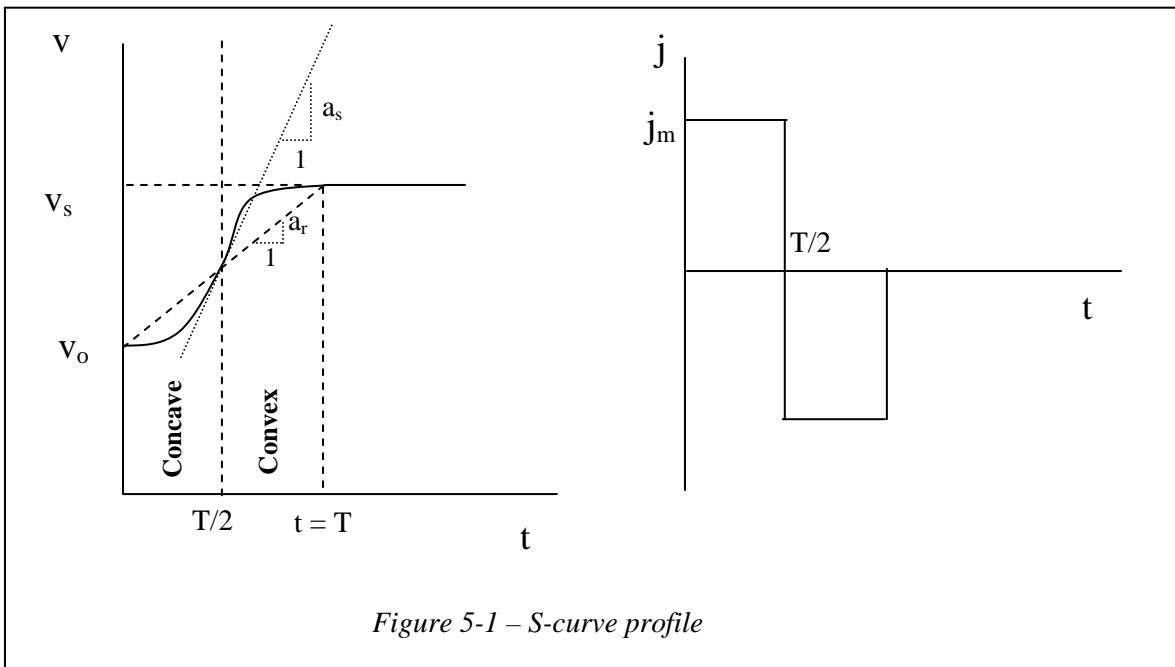
The purpose of this material is to impart an understanding of polynomial transitions for a TG (trajectory generator). The ideal constant jerk S-curve (jerk is the derivative of acceleration and a measure of impact) can be represented by a second-order polynomial in velocity (3<sup>rd</sup> in position). Its shape is governed by the motion conditions at the start and end of the transition.

An S-curve with an intermediate constant acceleration (linear portion) is often used to reduce the time to make large speed changes. The jerk can be used to determine how much of the rise or fall period can be made under constant acceleration.

### 5.1 Ideal S-Curve

Referencing Figure 5-1, we assume a perfect S-curve with no linear transition period, and also examine the rise portion from  $v_o$  to  $v_s$  (the set speed) where  $v_o = 0$  if we are starting from rest. The following observations can be made:

1. The area under the **concave** and **convex** portions of the S-curve is the same as the area under the constant acceleration ramp ( $a_r$ ). Thus, the constant ramp profile can be used to assist in making trajectory decisions for the S-curve profile, such as predicting the time  $T$  to move to the set speed.



2. The maximum S-curve accel (midpoint of accel period and specified as  $a_s$ ) exceeds the constant ramp accel value which is constant and specified as  $a_r$ . Thus, the S-curve accel must be checked against the robot maximum and adjusted accordingly.

The form assumed for the S-curve velocity profile is

$$v(t) = c_0 + c_1 t + c_2 t^2 \quad (5.1)$$

giving the acceleration and **constant jerk** equations:

$$a(t) = c_1 + 2 c_2 t \quad (5.2)$$

$$j(t) = 2 c_2 \quad (5.3)$$

The rise motion can be divided into 2 periods - a **concave** period followed by a **convex** period.

## 5.2 Concave Period

The concave conditions are

$$v(0) = v_o$$

$$a(0) = 0$$

$$a(T/2) = a_s$$

$$j(0) = j_m$$

where  $j_m$  is the jerk set for the profile (near the maximum allowed for the robot), and  $a_s$  is the maximum acceleration encountered at the S-curve inflection point. Applying these conditions to solve for the unknown constants in (5.1) and (5.2), we get:

$$c_0 = v_o \quad (5.4a)$$

$$c_1 = 0 \quad (5.4b)$$

$$c_2 = j_m / 2 = a_s / T \quad (5.4c)$$

Equation (4c) gives an important relation:

$$j_m = 2 a_s / T \quad (5.5)$$

Given that the jerk is limited to the value  $j_m$ ,  $a_s$  and  $T$  must be chosen such that

$$2 a_s / T \leq j_m \quad (5.6)$$

Applying (5.4) we get the equations for  $s$  (position),  $v$ , and  $a$  along the **concave** portion of the S-curve

$$s(t) = v_o t + j_m t^3 / 6 \quad (5.7)$$

$$v(t) = v_o + j_m t^2 / 2 \quad (5.8)$$

$$a(t) = j_m t \quad (5.9)$$

**Note:** It is assumed that  $s$  is 0 at the beginning of the S-move. Thus,  $s$  represents a position delta.

These equations can be applied for  $0 \leq t \leq T/2$ . Note also that at  $T = T/2$ ,  $v = (v_s + v_o)/2$ .

**Observations:**

1. If we let  $\Delta v = v_s - v_o$  and define  $a_r = \Delta v/T$  to be the acceleration of a constant acceleration ramp from  $v_o$  to  $v_s$ , then we note that  $a_s$  is twice  $a_r$ . It is also true that  $T = 2 \Delta v / a_s$ .
2. The trapezoidal profile can be used to predict the time and distance required to transition the accel and decel periods of the ideal S-curve. This exercise is commonly called **motion or path planning**.

**Other cases of importance:**

1. Given  $v < v_h$ , where  $v_h = (v_s + v_o)/2$  is the velocity at the inflection or halfway point, find  $t$  and  $s$ . This is done by solving (5.8):

$$t = \sqrt{2 \Delta v / j_m}$$

where  $\Delta v = v(t) - v_o$ .  $s(t)$  is now found from (5.7).

2. At  $T/2$ , we note that

$$s(T/2) = s_h = [v_o + a_s^2 / (6 j_m)] a_s / j_m$$

where  $s_h$  is the distance to the inflection point from the current condition. Note that this distance is in terms of  $v_o$ ,  $a_s$ , and  $j_m$  and not  $T$ .

3. Given any  $s < s_h$ , determine  $t$ . This involves a cubic solution for  $t$ .

### 5.3 Convex Period

This period applies for  $T/2 \leq t \leq T$ . Letting time be zero measured from the beginning of the convex period ( $0 \leq t \leq T/2$ ), the pertinent motion conditions are:

$$v(0) = v_h = (v_s + v_o)/2$$

$$a(0) = a_s$$

$$a(T/2) = 0$$

$$j(0) = -j_m$$

where  $-j_m$  is the jerk set for the profile, and  $a_s$  is the maximum acceleration encountered at the S-curve inflection point.

**Note: The jerk changes sign depending on whether we are in a concave or convex period.**

$$\text{For concave period} \quad j = j_m \qquad \text{For convex period} \quad j = -j_m$$

Applying the motion conditions to solve for the unknown constants in (5.1) and (5.2), we get:

$$c_0 = v_h \quad (5.10a)$$

$$c_1 = a_s \quad (5.10b)$$

$$c_2 = -j_m/2 = -a_s/T \quad (5.10c)$$

Applying (5.10) we get the equations for s (position), v, and a along the **convex** portion of the S-curve

$$s(t) = v_h t + a_s t^2/2 - j_m t^3/6 \quad (5.11)$$

$$v(t) = v_h + a_s t - j_m t^2/2 \quad (5.12)$$

$$a(t) = a_s - j_m t \quad (5.13)$$

*Note: It is assumed that s is 0 at the beginning of the S-move. Thus, s represents a position delta.*

**Other cases of importance:**

1. Given  $v_h \leq v < v_s$ , find t and s. This is done by solving (5.12) for t and then substituting t into (5.11) to get s. We find t from:

$$t = [a_s - \sqrt{a_s^2 - 2 j_s \Delta v_h}] / j_m \quad (5.14)$$

where  $\Delta v_h = (v - v_h)$ , and then go to (5.11) to get s(t).

2. Over the time period T/2 the distance change measured from the halfway point to the ending point of the S-profile is

$$s(T/2) = [v_h + a_s^2/(3 j_m)] a_s / j_m \quad (5.15)$$

Adding in the distance at the halfway point gives the total distance traversed in the S-curve, including both concave and convex sections. This simplifies to the total distance

$$S = (v_s^2 - v_o^2)/a_s \quad (5.16)$$

3. Given s within the convex profile, determine t. This involves a cubic solution for t and will be considered in the motion planning document.

## 5.4 Max Jerk Considerations

An ideal S-curve cannot transition smoothly between any speed change using the same jerk value. Given a jerk  $j_m$ , a starting speed  $v_o$ , and the ending speed  $v_s$ , we can determine  $v_1$  and  $v_2$ , where these are the velocities that end the concave transition and begin the convex transition at max jerk.

$$v_1 = v_o + a_s^2/(2j_m) \quad (5.17)$$

$$v_2 = v_s - a_s^2/(2j_m) \quad (5.18)$$

By setting  $v_1 = v_2$ , we determine the max jerk for a given  $a_s$  and  $\Delta v = v_s - v_o$ :

$$j_m = a_s^2 / \Delta v \quad (5.19)$$

If  $v_1 > v_2$  (overlap), we can determine an intermediate transition point using speed and acceleration continuity.

#### 5.4.1 Speed Transition

If  $v_1 > v_2$ , then this case, referred to as a *speed transition*, can be solved by noting that the velocity and acceleration for the previous concave curve and the new convex curve must be equal at  $T_t$  where the velocity is  $v_t$ . In this case, the velocity change is sufficiently small that we cannot reach the maximum acceleration  $a_s$  by applying maximum jerk transitions. Nevertheless, there exists a point where the concave profile will be tangent to the convex profile. This point will lie between  $v_o$  and  $v_s$ . At this point the acceleration and speed of both profiles are the same, although there will be a sign change in the jerk.

The pertinent equations are:

$$v_o + a_o T_t + j_m T_t^2 / 2 = v_s - j_m (T - T_t)^2 / 2 \quad (5.20)$$

$$a_o + j_m T_t = j_m (T - T_t) \quad (5.21)$$

Solving these we get:

$$T = [-a_o + \sqrt{2 a_o^2 + 4 \Delta v j_m}] / j_m \quad (5.22)$$

$$T_t = (j_m T - a_o) / (2 j_m) \quad (5.23)$$

where  $\Delta v = (v_s - v_o)$ .

## 5.5 S-Curve with Linear Period

A linear transition period intermediate to the S-curve transitions can be used to reduce the time to transition to the set speed. The max jerk formulation requires that linear periods be used when  $\Delta v$  is large. In the max jerk formulation, the length of the linear transition is determined by the velocity changes spent in the max jerk concave and convex periods. Once these changes in velocity have been subtracted from the total  $\Delta v$ , the remaining velocity change is left for a linear transition - see Figure 5-2.

The desired maximum S accel ( $a_s$ ) is known, as is the maximum jerk ( $j_m$ ).

#### 5.5.1 Phase 1 - Concave (S to linear: $0 \leq t \leq t_1$ )

The motion conditions are:

$$v(0) = v_o$$

$$a(0) = 0$$

$$a(t_1) = a_s$$

$$j(0) = j_m$$

Applying these equations, we get the following solutions for the polynomial coefficients:

$$c_0 = v_0 \quad (5.24a)$$

$$c_1 = 0 \quad (5.24b)$$

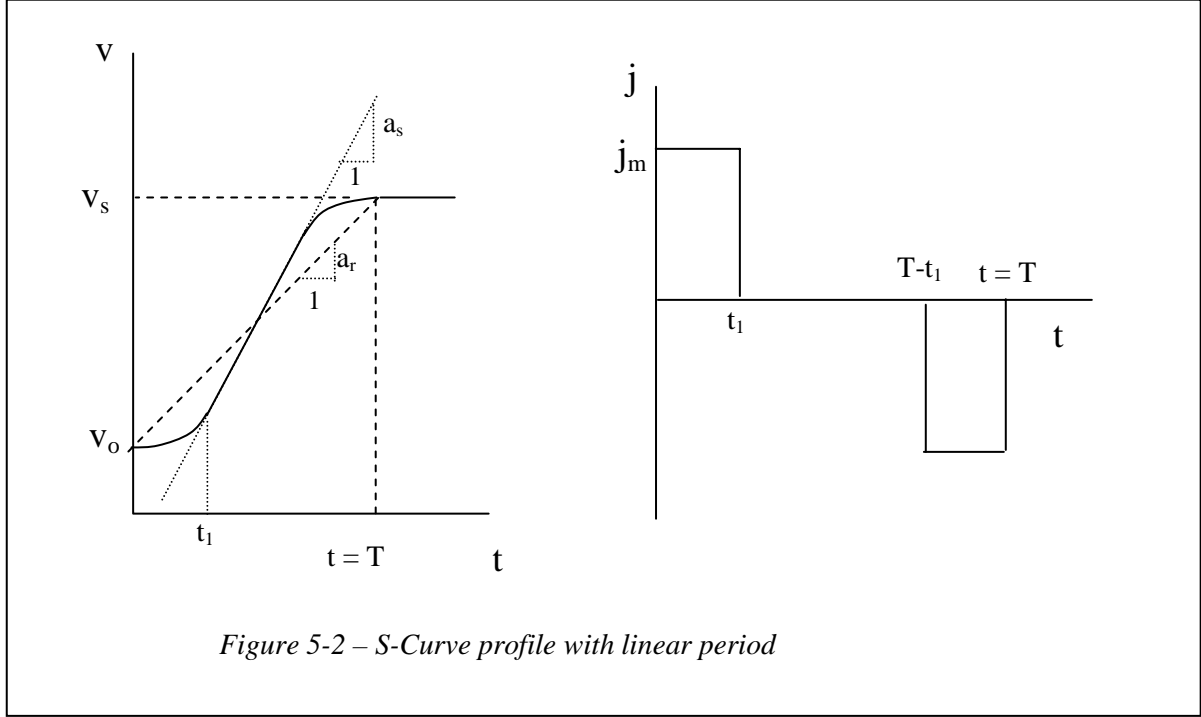


Figure 5-2 – S-Curve profile with linear period

$$c_2 = j_m / 2 = a_s / (2t_1) \quad (5.24c)$$

Equations (5.24) apply for the period from  $0 \leq t \leq t_1$  where  $t_1 = a_s / j_m$ . These will give

$$s(t) = v_0 t + j_m t^3 / 6 \quad (5.25)$$

$$v(t) = v_0 + j_m t^2 / 2 \quad (5.26)$$

$$a(t) = j_m t \quad (5.27)$$

At the end of the concave period:

$$s(t_1) = s_1 = [v_0 + a_s^2 / (6 j_m)] a_s / j_m \quad (5.28)$$

$$v(t_1) = v_1 = v_0 + a_s^2 / (2 j_m) \quad (5.29)$$

$$a(t_1) = a_s = j_m t_1 \quad (5.30)$$

### 5.5.2 Phase 2 - Linear: $0 \leq t \leq T - 2t_1$

The linear portion is governed by

$$s(t) = v_1 t + a_s t^2 / 2 \quad (5.30)$$

$$v(t) = v_1 + a_s t \quad (5.31)$$

where  $0 \leq t \leq (T - 2 t_1)$  and  $v(0) = v_1$  is given by (5.29). The position change at the end of the linear period are given by

$$s(T - 2 t_1) = s_2 = v_1 (T - 2 t_1) + a_s (T - 2 t_1)^2 / 2 \quad (5.32)$$

where  $s$  is interpreted as the position change measured from the beginning of the linear portion. The speed at the end of the linear period can be found from an equation similar to (5.29):

$$v_2 = v_s - a_s^2 / (2 j_m) \quad (5.33)$$

### 5.5.3 Phase 3 - Final Convex S blend

The equations are determined from:

$$v(0) = v_2$$

$$a(0) = a_s$$

$$v(t_1) = v_s$$

$$a(t_1) = 0$$

$$j(0) = - j_m$$

$t_1$  is used as the terminal time because the time to transition the convex portion is the same as the time to transition the concave portion. Solving for the coefficients:

$$c_0 = v_2 \quad (5.34a)$$

$$c_1 = a_s \quad (5.34b)$$

$$c_2 = - j_m / 2 = - a_s / (2 t_1) \quad (5.34c)$$

These give the following equations for the convex profile:

$$s(t) = v_2 t + a_s t^2 / 2 - j_m t^3 / 6 \quad (5.35)$$

$$v(t) = v_2 + a_s t - j_m t^2 / 2 \quad (5.36)$$

$$a(t) = a_s - j_m t \quad (5.37)$$

where  $0 \leq t \leq t_1$ . The distance change over the convex curve is

$$s(t_1) = [v_2 + a_s^2 / (3 j_m)] a_s / j_m \quad (5.38)$$

## 5.6 Deceleration Period

The equations for the deceleration period are similar to those for the acceleration period, except the coefficient sign has to be changed in several places. This exercise is left for the reader.

## 5.7 Motion Planning

Consider a move that starts from rest, moves to a set speed, and then decelerates to rest. To determine whether constant acceleration and constant deceleration (linear) periods are required depends on the maximum speed setting and the max jerk and acceleration/deceleration settings. How much time and distance (if any) is allocated to the constant speed transition at the maximum speed setting is determined by subtracting the distances required for the acceleration and deceleration periods from the total length of the move.

***Note:** Although not discussed here, there is a special case where the move distance is so small that the desired speed cannot be reached. This case requires additional equations be derived for the attainable speed.*

## 5.8 Summary

Constant jerk S-curves are used to transition robot moves. The S-curve is used to eliminate discontinuities in acceleration that are detrimental to robot performance.

The most general type of move involves convex and concave S-curve periods, a linear acceleration (or decel) period where jerk is zero, and a constant speed period that transitions most of the move distance.