Credit Spread Decomposition Notes

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Abstract

The empirical evidence showing that a corporate bond's expected loss is only a small portion of a bond's credit spread is called the credit spread puzzle. This paper, using a reduced-form credit risk model, characterizes a risky bond's credit spread. This characterization provides a more general measure of a risky bond's credit risk and it shows that, in an arbitrage-free market, a bond's credit risk is only a fraction of the credit spread and not linearly related to the one-year, risk-neutral expected loss, resolving the credit spread puzzle.

The credit spread puzzle refers to the empirical observation that corporate bonds offer yields significantly higher than risk-free Treasury bonds, even when the expected loss from default is very small. To illustrate, consider a zero-coupon corporate bond that pays \$100 at maturity in one year. Suppose there is a 1% probability of default, and in the event of default, the bond pays nothing (i.e., zero recovery). Under the real-world measure $\mathbb{P}$, the expected payoff is:

$$\mathbb{E}^{\mathbb{P}}[\text{Payoff}] = 0.99 \cdot 100 + 0.01 \cdot 0 = 99$$

If investors were only compensated for expected default losses, this bond should trade at approximately \$99. In contrast, a risk-free Treasury bond maturing in one year would pay \$100 with certainty and—assuming zero interest rates—would also trade at \$100.

To address the credit spread puzzle, the paper adopts a reduced-form credit risk model in which defaults occur randomly with intensity $\lambda(t)$. This default intensity is not directly observable, but it can be estimated from market instruments like bond prices and CDS spreads. These prices embed information about perceived credit risk, allowing the model to extract implied default probabilities.

Using this framework, the authors define a new, more general measure of a risky bond's credit risk. Crucially, they show that even in an arbitrage-free market, a bond's credit risk accounts for only a fraction of its total credit spread. Moreover, the credit spread is not a linear function of the bond's one-year expected loss, which contradicts many prior modeling assumptions. This more general nonlinear relationship helps resolve the credit spread puzzle by explaining why credit spreads remain large even when expected losses are small.

Introduction

The determinants of risky bond credit spreads are an often studied empirical topic in finance, with still no consensus on the importance of the bond's expected loss in this decomposition. The low explanatory power of a bond's expected loss in explaining the credit spread is referred to as the "credit spread puzzle" (Feldhutter and Schaefer, 2018; Bai *et al.*, 2020). The approaches to analyzing the credit spread are one of the two: (i) based on a structural model for credit risk, to run a linear regression which decomposes the credit spread into various firm and market explanatory variables (e.g., Bai *et al.*, 2020; Campbell and Taksler, 2003; Collin-Dufresne *et al.*,

2001; Davies, 2008; Feldhutter and Schaefer, 2018; Huang and Huang, 2012); or (ii) based on a reduced-form credit risk model, to run a linear regression that characterizes the credit spread using the estimated default probabilities and other firm and market explanatory variables (e.g., Elton *et al.*, 2001; Giesecke *et al.*, 2011).

This section explains two major empirical approaches used by researchers to analyze the *credit spread puzzle*—the well-documented observation that corporate bond credit spreads are significantly larger than what would be implied by expected default losses alone. The primary goal of these studies is to understand what drives the size of the credit spread, defined as the additional yield that corporate bonds offer over risk-free Treasuries.

The **first approach** is based on a *structural credit risk model*, rooted in corporate finance theory, such as the Merton model. In these models, defaults arise when the market value of a firm's assets falls below a threshold defined by its liabilities. Researchers use firm-level data (e.g., leverage, volatility, asset value) to estimate default probabilities from this structural framework. They then run a *linear regression* of the form:

$$\text{Credit Spread}_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + \varepsilon_i$$

Here, X_{ij} represents explanatory variables derived from firm and market characteristics—such as leverage ratio, asset volatility, interest rate levels, or market-wide risk indicators like the VIX. The term ε_i represents the regression residual—the portion of the credit spread not explained by the included variables—capturing noise, omitted factors, or model imperfections. The regression attempts to quantify how much of the observed credit spread can be attributed to these factors. Examples of this approach include Bai *et al.* (2020), Campbell and Taksler (2003), and Collin-Dufresne *et al.* (2001).

The **second approach** uses a *reduced-form credit risk model*, in which default is modeled as a random event driven by a stochastic *intensity process* $\lambda(t)$, estimated directly from market prices—such as bond yields or credit default swap (CDS) spreads—without requiring information about the firm's balance sheet. From the estimated intensity, researchers derive default probabilities, then run a regression with the same goal: to explain the credit spread using the estimated probability of default and other control variables. The same linear regression framework applies:

$$\text{Credit Spread}_i = \beta_0 + \beta_1 \widehat{\text{DefaultProb}}_i + \beta_2 Z_{i1} + \cdots + \beta_k Z_{ik} + \varepsilon_i$$

where $\widehat{\text{DefaultProb}}_i$ is the market-implied probability of default and Z_{ij} includes additional firm-level and macroeconomic factors. Notable studies using this method include Elton *et al.* (2001), Giesecke *et al.* (2011), and Feldhutter and Schaefer (2018).

In both approaches, the core idea is to *decompose the credit spread* into weighted contributions from various explanatory variables using regression. This allows researchers to identify how much of the credit spread is attributable to actual credit risk (i.e., expected losses due to default) versus other sources—such as liquidity risk, risk premia, and macroeconomic uncertainty. The difference lies in how default risk is modeled: either structurally through firm fundamentals, or statistically through market-implied intensities.

> Unfortunately, structural model-based default probabilities have been shown in other contexts to be misspecified (e.g., Campbell *et al.*, 2008, 2011; Jarrow, 2011), calling into question the conclusions drawn from the structural approach to the credit spread decomposition. Also, the reduced-form credit risk approach is misspecified because it assumes a simple linear relation between the credit spread, default probabilities, and other market and firm explanatory variables. We show below that this relation is complex and nonlinear.

A model is said to be misspecified when its assumptions, structure, or included variables do not accurately reflect the true underlying process that generates the data. This can lead to biased estimates, incorrect inferences, or poor predictive performance.

The purpose of this paper is to provide a new decomposition of a bond's credit spread that resolves the credit spread puzzle and which can be used to understand the existing empirical evidence. This characterization, and the new credit risk measure derived herein, can be the basis for new empirical research on the determinants of a risky bond's credit spread.

To obtain this new decomposition, we use the risky bond valuation model contained in Hilscher *et al.* (2023). We show that the bond's credit spread is a nonlinear and complex function of the firm's default probability, recovery rate, risk premium, default risk premium, liquidity premium, and promised cash flows. This decomposition also provides a new credit risk measure for a risky bond, replacing the bond's one-year, risk-neutral expected loss. This new measure is needed because it is shown that the bond's credit spread is not a linear function of its one-year, risk-neutral expected loss.

This decomposition provides a theoretical explanation for the credit spread puzzle because: (i) the credit spread is not a linear function of the bond's one-year, risk-neutral expected loss; and (ii) the credit risk, although linear in this new credit risk measure, is still only a fraction of the credit spread. The remaining fraction corresponds to liquidity risk, which is a nonlinear function of the bond's default risk premium, liquidity premium, recovery rate, and the promised coupon and principal payments. It is important to emphasize that although this paper is theoretical, the risky debt pricing model from which the credit spread decomposition is obtained has been empirically validated with respect to the corporate bond market prices (Hilscher *et al.*, 2023). Consequently, this empirical validation implies that the credit spread decomposition provided herein is consistent with the observed and market determined credit spreads.

The *one-year risk-neutral expected loss* is the expected value of losses due to default occurring within the next year, as priced by the market. It is computed under the *risk-neutral probability measure* $\mathbb{Q}$, which reflects how investors price credit risk, including compensation for uncertainty and risk aversion.
Let the following variables be defined:

- F = face value (promised payoff) of the bond at maturity.
- R = recovery rate (fraction of F paid in the event of default), where $0 \leq R \leq 1$.
- τ = random default time.
- $T = 1$ = one-year time horizon.
- $\mathbf{1}_{\{\tau \leq 1\}}$ = indicator function equal to 1 if default occurs within one year, 0 otherwise.
- $\mathbb{E}^{\mathbb{Q}}[\cdot]$ = expectation under the risk-neutral probability measure $\mathbb{Q}$.

Then the **one-year risk-neutral expected loss**, denoted $\text{RNEL}_{1\text{yr}}$, is:

$$\text{RNEL}_{1\text{yr}} = \mathbb{E}^{\mathbb{Q}}\left[F \cdot (1 - R) \cdot \mathbf{1}_{\{\tau \leq 1\}}\right]$$

Assume the following inputs:

- $F = 100$ (the bond promises to pay \$100 at maturity),
- $R = 0$ (zero recovery in case of default),
- $\mathbb{Q}(\tau \leq 1) = 0.04$ (the market-implied probability of default within one year is 4%).

Then the one-year risk-neutral expected loss is:

$$\text{RNEL}_{1\text{yr}} = 100 \cdot (1 - 0) \cdot 0.04 = 4$$

So, the expected loss under the risk-neutral measure is \$4. This means that if investors were only being compensated for default risk (and no other risks), the bond would trade at approximately:

$$\text{Price} = F - \text{RNEL}_{1\text{yr}} = 100 - 4 = 96$$

This paper resolves the credit spread puzzle by showing that the credit spread on a risky bond is not linearly related to its one-year, risk-neutral expected loss. Previous empirical models often approximated the credit spread using a scalar multiple of the expected loss under the risk-neutral measure. Formally, they assumed:

$$\text{Credit Spread} \approx \alpha \cdot \text{RNEL}_{1\text{yr}}$$

for some constant α. However, the paper demonstrates that this approximation is fundamentally flawed.

The authors define a more general measure of **credit risk** as the portion of a bond's price that is attributable to the possibility of default.

In this framework, credit risk enters the model *linearly*. That is, holding all else fixed, if the model-implied default intensity doubles, then the credit risk component of the bond price also doubles. This linearity refers to the way default-related inputs scale within the pricing equation of the model.

However, the paper emphasizes two key findings:

- The total credit spread is **not** a linear function of the one-year risk-neutral expected loss. The expected loss alone is a poor proxy for what actually drives spreads in the market.
- Even after defining credit risk precisely through their model, this credit risk component explains only a *fraction* of the observed credit spread.

The remaining portion of the credit spread is attributed to **liquidity risk**. This includes factors such as the bond's default risk premium, liquidity premium, recovery assumptions, and promised payments. Crucially, this liquidity risk enters the pricing relationship *nonlinearly* — meaning its effect on spreads is more complex and cannot be captured by a simple proportional change in input variables.

The Model

We assume a continuous-time, continuous trading market with a finite time horizon $[0, T]$ that is frictionless, competitive, and satisfies no arbitrage and no dominance.[1] Traded are a term structure of default-free bonds and a risky coupon bond. We are interested in characterizing the credit spread of this risky coupon bond. The frictionless market assumption is relaxed below to include illiquid corporate bond markets.

We consider a simplified, idealized financial market model in which trading occurs continuously over a finite time interval $[0, T]$. This market is assumed to be frictionless, meaning there are no transaction costs, bid-ask spreads, or taxes, and it is competitive—no individual trader can influence market prices. Additionally, the market satisfies the conditions of no arbitrage (i.e., there are no opportunities to make a riskless profit) and no dominance (i.e., no asset always outperforms all others in every state of the world). Within this market, two types of bonds are traded: a term structure of default-free (risk-free) bonds, and a risky coupon bond that pays periodic interest but carries credit risk due to the possibility of default. The primary objective is to characterize the credit spread of the risky bond, which is the yield premium investors require over the risk-free bond to compensate for the default risk. While the model begins with a frictionless assumption, the framework is later extended to incorporate illiquid corporate bond markets where such ideal conditions do not apply.

Default-free bonds

This subsection analyzes default-free bonds.

Let $p(0, t)$ denote the time-0 price of a zero-coupon bond paying a sure dollar at time $t \in [0, T]$.

Let r_t be the default-free spot rate of interest at time $t \in [0, T]$.

Let $B_T(0)$ denote the time-0 price of a default-free coupon bond with the coupon rate $c \in [0, 1]$, a maturity of T, and a principal equal to \$1. Coupons are paid at times $t = 1, \ldots, T$.

By the first and third fundamental theorems of asset pricing, there exist risk-neutral probabilities[2] $\mathbb{Q}$ such that

$$p(0, t) = E\left(e^{-\int_0^t r_s ds}\right)$$

for $t \in [0, T]$, where $E(\cdot)$ denotes expectation under the risk-neutral probabilities $\mathbb{Q}$.

For the default-free coupon bond, this implies the well-known expression

$$B_T(0) = E\left(\sum_{t=1}^{T} c e^{-\int_0^t r_s ds} + 1 e^{-\int_0^T r_s ds}\right)$$

$$= \sum_{i=1}^{T} cp(0, t) + p(0, T).$$

Feature	Zero-Coupon Bond	Default-Free Coupon Bond
Coupons?	No coupons	Pays regular coupons
Payment Timing	Pays only once at maturity	Pays coupons periodically + final principal at maturity
Final Payment	Pays \$1 at time t	Pays \$$c$ each period + \$1 at time T
Notation (Price)	$p(0, t)$	$B_T(0)$
Risk	No credit risk (assumed default-free)	Also default-free (guaranteed payment)

Table 1: Comparison between Zero-Coupon and Default-Free Coupon Bonds

Understanding Zero-Coupon Bond Pricing

We begin by recalling the formula for compound interest, which describes how an investment grows over time. If you invest \$1 today at an annual interest rate r, compounded n times per year, then after t years the future value is given by:

$$\text{Future value} = \left(1 + \frac{r}{n}\right)^{nt}$$

As the compounding frequency increases, we take the limit as $n \to \infty$, arriving at the continuous compounding formula:

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$$

Thus, under continuous compounding, the future value of \$1 invested at rate r for t years is:

$$\text{Future value} = e^{rt}$$

This expression uses a *positive* exponent because it represents money growing forward in time.

Now consider the reverse situation. Suppose you expect to receive \$1 at a future time t, and want to know what that is worth today, assuming the same interest rate r. This is the concept of *present value*, and involves reversing the compounding process. We solve the equation:

$$1 = \text{Present value} \cdot e^{rt} \quad \Rightarrow \quad \text{Present value} = e^{-rt}$$

Therefore, the present value of \$1 to be received at time t, under continuous discounting at rate r, is:

$$p(0,t) = e^{-rt}$$

This is where the *negative exponent* comes from: it reflects the fact that future money is worth less today, due to the time value of money. We are effectively undoing exponential growth to compute a value in today's dollars.

This logic directly applies to the pricing of a zero-coupon bond. A zero-coupon bond pays exactly \$1 at a future time t, with no interim payments. If its continuously compounded yield is R_t, then the price of the bond today (at time 0) is simply:

$$p(0,t) = e^{-R_t \cdot t}$$

The subscript t in R_t indicates that the yield is associated with a bond that matures at time t.

Risk-Neutral Valuation of Zero-Coupon Bonds

We now move beyond constant interest rates and explore the more realistic case where interest rates change randomly over time. Our goal is to understand the formula:

$$p(0,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r_s\, ds}\right]$$

This equation tells us how to compute the present value $p(0,t)$ of a zero-coupon bond that pays \$1 at future time t, when interest rates are stochastic and evolve over time.

Previously, when interest rates were constant, we used the formula:

$$p(0,t) = e^{-R_t \cdot t}$$

This assumes a deterministic rate R_t known in advance. But in reality, interest rates fluctuate and are not known ahead of time. To account for this uncertainty, we introduce a time-dependent process r_s, which represents the short-term interest rate at time $s \in [0,t]$.

The quantity r_s denotes the instantaneous interest rate at time s. This means:

- r_0 is the interest rate today.

- $r_1, r_2, \ldots, r_t$ are interest rates in the future.

These future interest rates are unknown and modeled as a stochastic process.

When interest rates are constant, we discount future cash flows with e^{-rt}. If rates change continuously, we must account for every infinitesimal change along the path. We do this using an integral:

$$\int_0^t r_s \, ds$$

This integral adds up all the small interest rate contributions between time 0 and t. The corresponding discount factor is then:

$$e^{-\int_0^t r_s \, ds}$$

This gives the present value of \$1 received at time t, assuming we knew the full path of r_s from 0 to t.

Since we don't know the future values of r_s, we cannot compute this directly. Instead, we take an expected value over all the possible paths that interest rates could follow. But we don't use a regular average—we use a *risk-neutral expectation*, denoted by $\mathbb{E}^{\mathbb{Q}}[\cdot]$.

The risk-neutral measure $\mathbb{Q}$ is a mathematical construct used in finance that ensures no-arbitrage pricing. Under this measure, asset prices are equal to the expected present value of their future cash flows. So the price of the zero-coupon bond is:

$$p(0,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r_s \, ds}\right]$$

This formula averages the discount factors across all possible future interest rate paths, under the risk-neutral probabilities.

- $p(0,t)$: price today of a zero-coupon bond maturing at time t
- r_s: short-term interest rate at time $s \in [0,t]$
- $\int_0^t r_s \, ds$: accumulated interest from time 0 to t
- $e^{-\int_0^t r_s \, ds}$: discount factor accounting for time-varying interest rates
- $\mathbb{E}^{\mathbb{Q}}[\cdot]$: expectation under the risk-neutral probability measure $\mathbb{Q}$

If interest rates are constant, so that $r_s = r$ for all s, then:

$$\int_0^t r_s \, ds = rt \quad \text{and} \quad p(0,t) = e^{-rt}$$

Thus, the risk-neutral formula reduces to the familiar exponential discounting case. This confirms that our general formula is consistent with the classical setting when rates are known and fixed.

Understanding the Price of a Default-Free Coupon Bond

We now examine how to compute the time-0 price of a default-free coupon bond. The full expression is:

$$B_T(0) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{t=1}^{T} c \cdot e^{-\int_0^t r_s \, ds} + 1 \cdot e^{-\int_0^T r_s \, ds}\right)$$

This simplifies to:

$$B_T(0) = \sum_{t=1}^{T} c \cdot p(0,t) + p(0,T)$$

The expression $B_T(0)$ represents the price at time 0 of a default-free coupon bond that matures at time T.

This bond pays:

- A coupon payment of amount c at times $t = 1, 2, \ldots, T$,
- A final principal repayment of \$1 at time T.

The sum

$$\sum_{t=1}^{T} c \cdot e^{-\int_0^t r_s\, ds} + 1 \cdot e^{-\int_0^T r_s\, ds}$$

represents the present value of all future payments. Each coupon payment c made at time t is discounted using the stochastic discount factor

$$e^{-\int_0^t r_s\, ds}.$$

The final principal payment of \$1 at maturity T is also discounted in the same way, using

$$e^{-\int_0^T r_s\, ds}.$$

The term $e^{-\int_0^t r_s\, ds}$ gives the present value of receiving \$1 at time t. However, the bondholder is not receiving \$1, but rather \$$c$. To account for this, we scale the discount factor by c, giving:

$$\text{Present value of } \$c = c \cdot e^{-\int_0^t r_s\, ds}$$

Example: If \$1 in the future is worth \$0.95 today, then the value of receiving \$5 is:

$$5 \cdot 0.95 = 4.75$$

The same principle applies in this context.

Because interest rates r_s are uncertain and change over time, we do not know the exact value of $\int_0^t r_s\, ds$. Therefore, we take the expectation of each discounted cash flow using the risk-neutral probability measure $\mathbb{Q}$:

$$\mathbb{E}^{\mathbb{Q}}[\cdots]$$

This gives us the fair, arbitrage-free value of the bond under stochastic interest rates.

From the earlier result for the price of a zero-coupon bond, we know:

$$p(0,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r_s\, ds}\right]$$

So we can substitute this into the earlier formula to get:

$$B_T(0) = \sum_{t=1}^{T} c \cdot p(0,t) + p(0,T)$$

Each coupon is priced using the price of a zero-coupon bond that matures at that time, and the final principal is priced using a zero-coupon bond maturing at time T.

A coupon bond can be viewed as a collection of many zero-coupon bonds. Each future cash flow is treated like a separate zero-coupon bond, discounted independently to the present. The total bond price is the sum of all these present values.

Term	Meaning
$c \cdot e^{-\int_0^t r_s ds}$	Present value of a coupon payment at time t
$\sum_{t=1}^{T} \cdot$	Adds up all coupon present values
$e^{-\int_0^T r_s ds}$	Present value of the final principal
$\mathbb{E}^{\mathbb{Q}}[\cdot]$	Average over all future interest rate paths
$p(0,t)$	Price of a zero-coupon bond maturing at time t

The zero-coupon bond's yield R_t at time 0 is defined by the following expression:

$$p(0,t) = e^{-R_t \cdot t}$$

for $t \in [0, T]$.

The coupon bond's time-0 yield $y_T(c)$, which depends on the bond's maturity T, is defined by the equation

$$B_T(0) = \sum_{t=1}^{T} ce^{-y_T(c)t} + 1e^{-y_T(c)T}.$$

We include the coupon rate in the definition of the default-free coupon bond's yield $y_T(c)$ because this expression will appear later in the credit spread decomposition for a default-free bond with a different coupon rate, but with the yield as given above.

Yield-to-Maturity of a Coupon Bond

We now turn our attention to the yield of a coupon bond. This yield is:

- Different from the zero-coupon case,
- Not directly observable,
- And must be solved for implicitly.

Let $y_T(c)$ denote the yield-to-maturity of a coupon bond with:

- Maturity T,
- Coupon rate c.

This yield $y_T(c)$ is defined to be the fixed rate such that, when used to discount all of the bond's future cash flows, the present value equals the bond's current market price.

The yield satisfies the following equation:

$$B_T(0) = \sum_{t=1}^{T} c \cdot e^{-y_T(c) \cdot t} + 1 \cdot e^{-y_T(c) \cdot T}$$

Let us break this expression down. The bond pays a coupon of amount c at each time $t = 1, 2, \ldots, T$, and a final principal of \$1 at maturity T. Each of these cash flows is discounted using the same fixed yield $y_T(c)$, resulting in terms of the form $e^{-y_T(c)\cdot t}$. The exponent is linear in time (not an integral) because the yield is assumed to be constant. The equation defines $y_T(c)$ as the specific value that equates the total present value of the bond's cash flows to its current market price $B_T(0)$.

This equation defines $y_T(c)$ *implicitly*; the yield is not directly computed from a closed-form formula. Instead, it must be solved for numerically in practice.

You can think of $y_T(c)$ as the constant rate that, if used to discount the bond's actual payments, would result in its current price.

We write $y_T(c)$ to emphasize that the yield depends on:

- The number of periods T,
- The size of the coupon c.

Changing the coupon alters the cash flows, and therefore changes the yield that solves the equation. This dependence is crucial when comparing bonds with different coupon rates.

A zero-coupon bond has an explicit yield formula, given by

$$p(0,t) = e^{-R_t t} \quad \Rightarrow \quad R_t = -\frac{1}{t}\ln p(0,t),$$

which allows us to solve directly for the yield R_t from the bond price.

In contrast, a coupon bond defines its yield implicitly through the equation

$$B_T(0) = \sum_{t=1}^{T} c \cdot e^{-y_T(c)t} + e^{-y_T(c)T}.$$

This yield, denoted $y_T(c)$, is known as the *yield-to-maturity* — a hypothetical constant rate that, when applied to all of the bond's future payments, exactly reproduces the bond's market price.

The risky coupon bond

This subsection studies the risky coupon bond issued by a credit entity. For simplicity, we call this credit entity a firm.

Let $D_T^{\alpha}(0)$ denote the time-0 price of a risky coupon bond with the coupon rate $C \in [0, 1]$, a maturity of T, a principal equal to \$1, and a liquidity discount α. The coupons are paid at times $t = 1, \ldots, T$. We don't explicitly include the coupon rate C in the bond's value to simplify the notation. It is fixed for the remainder of the paper.

Because corporate bond markets are illiquid relative to Treasuries, risky bond prices typically reflect a liquidity discount (Jarrow and Turnbull, 1997; Duffie and Singleton, 1999; Cherian *et al.*, 2004). Consequently, we assume that a risky bond's arbitrage-free price reflects a liquidity discount of α_t. For simplicity, we assume that the liquidity discount $\alpha_t = \alpha \geq 0$ is a constant. We apply the illiquidity discount to all of the bond's cash flows. Including such a liquidity discount modifies the pricing formula to incorporate the impact of various market frictions such as transaction costs, trading constraints, and the effects of differential taxes on coupon income versus capital gains.

Let τ denote the first default time of the bond's promised payments. We assume that $\tau > 0$.

We assume that the point process $N_t \in \{0, 1\}$ which jumps from 0 to 1 at the default time τ is a Cox process with stochastic intensity $\tilde{\lambda}_t(X_t)$ for $t \in [0, T]$ under the statistical probabilities where X_t represents a vector of state

We now describe the framework used to model the default of a risky bond. Let τ denote the (random) time of default, which represents the first time the firm fails to meet its promised payments. We assume that $\tau > 0$, meaning that default does not occur immediately at time zero.

To track whether default has occurred by time t, we introduce a point process $N_t \in \{0, 1\}$, defined as

$$N_t = \begin{cases} 0 & \text{if } t < \tau \\ 1 & \text{if } t \geq \tau \end{cases}$$

This process jumps from 0 to 1 exactly once, at the time of default τ.

The default time τ is modeled as a Cox process, which is also referred to as a *doubly stochastic Poisson process*. Unlike a standard Poisson process, which has a constant arrival rate λ, a Cox process allows the arrival rate to be stochastic and time-varying.

Specifically, the Cox process is characterized by a *stochastic intensity* function $\tilde{\lambda}_t(X_t)$, where X_t is a vector of underlying state variables (such as firm-specific and market-wide economic factors). The intensity $\tilde{\lambda}_t(X_t)$

determines the instantaneous probability per unit time of default occurring at time t, given survival up to that time. Unlike a Poisson process, which has the memoryless property and treats future arrivals as independent of the past, a Cox process requires conditioning on survival. To understand why, consider an analogy from life insurance: if a person is alive at age 70, the probability they die in the next year is different from someone who is alive at 40 — even if both face the same disease. The fact that someone has survived to a later age carries information about their underlying health. The same logic applies in credit risk modeling. The fact that a firm has not defaulted by time t is informative. As a result, the conditional probability of default at time t must be updated to reflect this survival, and the arrival of default is no longer independent of the past. Whether the chance of default increases or decreases with time depends on the intensity process $\tilde{\lambda}_t(X_t)$: if the firm's financials deteriorate over time (i.e., X_t worsens), then $\tilde{\lambda}_t$ may increase, leading to a higher likelihood of default as time passes. But if the firm performs well and market conditions improve, $\tilde{\lambda}_t$ may decrease, indicating lower default risk the longer the firm survives. This loss of independence — and the presence of a time-varying, state-dependent intensity — is what distinguishes the Cox process from a standard Poisson process.

In this setup, the statistical probability of default within the time interval $[0, T]$ is governed by the stochastic evolution of $\tilde{\lambda}_t(X_t)$ for $t \in [0, T]$, under the risk-neutral measure. This allows the model to reflect realistic behavior, where the likelihood of default evolves dynamically based on observable market and firm conditions.

For clarity, the key elements of the framework are:

- τ: the random default time, with $\tau > 0$,
- N_t: the default indicator process defined by the jump from 0 to 1 at τ,
- $\tilde{\lambda}_t(X_t)$: the stochastic intensity function driving the Cox process,
- X_t: a vector of state variables affecting the default intensity.

This structure enables the pricing of risky bonds to incorporate uncertainty in default timing and allows the default risk to respond to evolving economic information in a flexible and tractable way.