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A Credit Spread Decomposition: A Resolution of the Credit Spread Puzzle

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The empirical evidence showing that a corporate bond's expected loss is only a small portion of a bond's credit spread is called the credit spread puzzle. This paper, using a reduced-form credit risk model, characterizes a risky bond's credit spread. This characterization provides a more general measure of a risky bond's credit risk and it shows that, in an arbitrage-free market, a bond's credit risk is only a fraction of the credit spread and not linearly related to the one-year, risk-neutral expected loss, resolving the credit spread puzzle.

Keywords: Credit spread; credit risk; default probabilities; recovery rates; bond prices.

JEL Classifications: G12, E43.

1. Introduction

The determinants of risky bond credit spreads are an often studied empirical topic in finance, with still no consensus on the importance of the bond's expected loss in this decomposition. The low explanatory power of a bond's expected loss in explaining the credit spread is referred to as the "credit spread puzzle" (Feldhutter and Schaefer, 2018; Bai *et al.*, 2020). The approaches to analyzing the credit spread are one of the two: (i) based on a structural model for credit risk, to run a linear regression which decomposes the credit spread into various firm and market explanatory variables (e.g., Bai *et al.*, 2020; Campbell and Taksler, 2003; Collin-Dufresne *et al.*, 2001; Davies, 2008; Feldhutter and Schaefer, 2018; Huang and Huang, 2012); or (ii) based on a reduced-form credit risk model, to run a linear regression that characterizes the credit spread using the estimated default probabilities and other firm and market explanatory variables (e.g., Elton *et al.*, 2001; Giesecke *et al.*, 2011).

Unfortunately, structural model-based default probabilities have been shown in other contexts to be misspecified (e.g., Campbell *et al.*, 2008, 2011; Jarrow, 2011), calling into question the conclusions drawn from the structural approach to the credit spread decomposition. Also, the reduced-form credit risk approach is misspecified because it assumes a simple linear relation between the credit spread, default probabilities, and other market and firm explanatory variables. We show below that this relation is complex and nonlinear.

The purpose of this paper is to provide a new decomposition of a bond's credit spread that resolves the credit spread puzzle and which can be used to understand the existing empirical evidence. This characterization, and the new credit risk measure derived herein, can be the basis for new empirical research on the determinants of a risky bond's credit spread.

To obtain this new decomposition, we use the risky bond valuation model contained in Hilscher *et al.* (2023). We show that the bond's credit spread is a nonlinear and complex function of the firm's default probability, recovery rate, risk premium, default risk premium, liquidity premium, and promised cash flows. This decomposition also provides a new credit risk measure for a risky bond, replacing the bond's one-year, risk-neutral expected loss. This new measure is needed because it is shown that the bond's credit spread is not a linear function of its one-year, risk-neutral expected loss.

This decomposition provides a theoretical explanation for the credit spread puzzle because: (i) the credit spread is not a linear function of the bond's one-year, risk-neutral expected loss; and (ii) the credit risk, although linear in this new credit risk measure, is still only a fraction of the credit spread. The remaining fraction corresponds to liquidity risk, which is a nonlinear function of the bond's default risk premium, liquidity premium, recovery rate, and the promised coupon and principal payments. It is important to emphasize that although this paper is theoretical, the risky debt pricing model from which the credit spread decomposition is obtained has been empirically validated with respect to the corporate bond market prices (Hilscher *et al.*, 2023). Consequently, this empirical validation implies that the credit spread decomposition provided herein is consistent with the observed and market determined credit spreads.

Section 2 gives the model used to price risky debt, Sec. 3 characterizes the credit spread, and Sec. 4 concludes.

2. The Model

We assume a continuous-time, continuous trading market with a finite time horizon $[0, T]$ that is frictionless, competitive, and satisfies no arbitrage and no dominance.[1] Traded are a term structure of default-free bonds and a risky coupon bond. We are interested in characterizing the credit spread of this risky coupon bond. The frictionless market assumption is relaxed below to include illiquid corporate bond markets.

2.1. *Default-free bonds*

This subsection analyzes default-free bonds.

Let $p(0, t)$ denote the time-0 price of a zero-coupon bond paying a sure dollar at time $t \in [0, T]$.

Let r_t be the default-free spot rate of interest at time $t \in [0, T]$.

Let $B_T(0)$ denote the time-0 price of a default-free coupon bond with the coupon rate $c \in [0, 1]$, a maturity of T, and a principal equal to \$1. Coupons are paid at times $t = 1, \ldots, T$.

By the first and third fundamental theorems of asset pricing, there exist risk-neutral probabilities[2] $\mathbb{Q}$ such that

$$p(0, t) = E\left(e^{-\int_0^t r_s ds}\right)$$

for $t \in [0, T]$, where $E(\cdot)$ denotes expectation under the risk-neutral probabilities $\mathbb{Q}$.

For the default-free coupon bond, this implies the well-known expression

$$\begin{aligned} B_T(0) &= E\left(\sum_{t=1}^{T} ce^{-\int_0^t r_s ds} + 1e^{-\int_0^T r_s ds}\right) \\ &= \sum_{i=1}^{T} cp(0, t) + p(0, T). \end{aligned}$$

[1] No dominance is needed to guarantee the existence of risk-neutral martingale probabilities, see Chap. 2 of Jarrow (2022).

[2] For simplicity of the presentation, we omit all the measurability and integrability conditions in the statements of the subsequent notation and results. See Chaps. 6 and 7 of Jarrow (2022) for these conditions.

The zero-coupon bond's yield R_t at time 0 is defined by the following expression:

$$p(0, t) = e^{-R_t \cdot t}$$

for $t \in [0, T]$.

The coupon bond's time-0 yield $y_T(c)$, which depends on the bond's maturity T, is defined by the equation

$$B_T(0) = \sum_{t=1}^{T} c e^{-y_T(c)t} + 1 e^{-y_T(c)T}.$$

We include the coupon rate in the definition of the default-free coupon bond's yield $y_T(c)$ because this expression will appear later in the credit spread decomposition for a default-free bond with a different coupon rate, but with the yield as given above.

2.2. *The risky coupon bond*

This subsection studies the risky coupon bond issued by a credit entity. For simplicity, we call this credit entity a firm.

Let $D_T^{\alpha}(0)$ denote the time-0 price of a risky coupon bond with the coupon rate $C \in [0, 1]$, a maturity of T, a principal equal to \$1, and a liquidity discount α. The coupons are paid at times $t = 1, \ldots, T$. We don't explicitly include the coupon rate C in the bond's value to simplify the notation. It is fixed for the remainder of the paper.

Because corporate bond markets are illiquid relative to Treasuries, risky bond prices typically reflect a liquidity discount (Jarrow and Turnbull, 1997; Duffie and Singleton, 1999; Cherian *et al.*, 2004). Consequently, we assume that a risky bond's arbitrage-free price reflects a liquidity discount of α_t. For simplicity, we assume that the liquidity discount $\alpha_t = \alpha \geq 0$ is a constant. We apply the illiquidity discount to all of the bond's cash flows. Including such a liquidity discount modifies the pricing formula to incorporate the impact of various market frictions such as transaction costs, trading constraints, and the effects of differential taxes on coupon income versus capital gains.

Let τ denote the first default time of the bond's promised payments. We assume that $\tau > 0$.

We assume that the point process $N_t \in \{0, 1\}$ which jumps from 0 to 1 at the default time τ is a Cox process with stochastic intensity $\tilde{\lambda}_t(X_t)$ for $t \in [0, T]$ under the statistical probabilities where X_t represents a vector of state variables. The state variables represent firm-specific characteristics (e.g., debt/equity ratio) and relevant macro-variables (e.g., the default-free spot rate). Here, $\tilde{\lambda}_t(X_t)$ represents the conditional probability of default over $[t, t + dt]$ given no default prior to t.

Let $\delta_t \in [0, 1]$ be the stochastic recovery rate if default happens at time $t \in [0, T]$ on the bond's promised cash flows. To be consistent with practice, we assume that a recovery rate is only received on the promised principal and not any of the promised future coupon payments. If default happens between coupon payment dates, i.e., $t - 1 < \tau \leq t$, then the recovery rate is paid at time t.

Under the assumption of no arbitrage and no dominance, there exist risk-neutral probabilities $\mathbb{Q}$ such that

$$D_T^{\alpha}(0) = \sum_{t=1}^{T} CE\left[1_{\{\tau>t\}}e^{-\int_0^t r_u du}\right]e^{-\alpha t} + E\left[1_{\{\tau>T\}}e^{-\int_t^T r_u du}\right]e^{-\alpha T}$$
$$+ \sum_{t=1}^{T} E\left[1_{\{t-1<\tau\leq t\}}\delta_t e^{-\int_0^t r_u du}\right]e^{-\alpha t}, \tag{1}$$

where $E(\cdot)$ denotes expectation under the risk-neutral probabilities $\mathbb{Q}$ [the proof is in Hilscher *et al.* (2023)] and the risk-neutral default intensity $\lambda_t(X_t)$ satisfies $\lambda_t(X_t) = \psi_t\tilde{\lambda}_t(X_t)$ with ψ_t being the default risk premium process (Jarrow, 2022, Chap. 7).

For later use, we note that the risk-neutral probabilities of default satisfy the following relations:

$$\mathbb{Q}(\tau = t) = E\left(\lambda_t(X_t)e^{-\int_0^t \lambda_s(X_s)ds}\right)$$

and

$$\mathbb{Q}(\tau > t) = E\left(e^{-\int_0^t \lambda_s(X_s)ds}\right)$$

for $t \in (0, T]$. It is easily seen that the one-year risk-neutral default probability is not approximately equal to $\lambda_0(X_0)$, i.e.,

$$\mathbb{Q}(\tau \leq 1) = 1 - E\left(e^{-\int_0^1 \lambda_t(X_t)dt}\right) \neq 1 - e^{-\lambda_0(X_0)} \approx \lambda_0(X_0),$$

because the time interval of a year is too long, given the randomness in the state variables, for $E\left(e^{-\int_0^1 \lambda_t(X_t)dt}\right) \approx e^{-\lambda_0(X_0)\cdot 1}$.

To simplify expression (1), we add the following assumption.

(Conditional independence) The default-free spot rate r_t, the default time τ, and the recovery rate process δ_t are conditionally independent under the risk-neutral probabilities given the information at time $t \in [0, T]$.

This is a weak assumption on the default-free spot rate, the default time, and the recovery rate because it imposes very little structure on their evolutions under the statistical probabilities, as distinct from the risk-neutral probabilities. First, conditioned on the information at time t, a large default intensity could imply a lower recovery rate under both $\mathbb{P}$ and $\mathbb{Q}$. Second, under the statistical probabilities, these processes need not be conditionally independent over $(t, T]$ due to default and recovery rate risk premiums. This implies that the default probability and recovery rate (conditioned on the information at time t) under $\mathbb{P}$, could be negatively correlated over $(t, T]$, which has been observed in the historical time series data.

This assumption implies that the expectations of the products in expression (1) can be separated as follows:

$$\begin{aligned} D_T^\alpha(0) = &\sum_{t=1}^{T} e^{-\alpha t} C\mathbb{Q}(\tau > t)p(0, t) + e^{-\alpha T}\mathbb{Q}(\tau > T)p(0, T) \\ &+ d_0 \int_0^T \mathbb{Q}(\tau = s)e^{-\alpha s}p(0, s)\,ds, \end{aligned} \tag{2}$$

where

$$d_0 = E[\delta_\tau]$$

is the risk-neutral expected recovery rate at default.

In general, since $\alpha > 0$, the bond's value is decreased as the liquidity premium α increases, i.e., $D_T^\alpha < D_T^0$.

This expression can be further simplified using the mean-value theorem for integrals. This implies that there exists an $\hat{s} \in [0, T]$ such that

$$\int_0^T \mathbb{Q}(\tau = s)e^{-\alpha s}p(0, s)\,ds = p(0, \hat{s})e^{-\alpha \hat{s}} \int_0^T \mathbb{Q}(\tau = s)\,ds,$$

where $\int_0^T \mathbb{Q}(\tau = s)\,ds = \mathbb{Q}(\tau \leq T)$. Hence, we have the alternative expression

$$\begin{aligned} D_T^\alpha(0) = &\sum_{t=1}^{T} e^{-\alpha t} C\mathbb{Q}(\tau > t)p(0, t) + e^{-\alpha T}\mathbb{Q}(\tau > T)p(0, T) \\ &+ d_0 p(0, \hat{s})e^{-\alpha \hat{s}}\mathbb{Q}(\tau \leq T). \end{aligned} \tag{3}$$

The risky bond's yield Y_T, which depends on its maturity T, is defined by

$$D_T^\alpha(0) = \sum_{t=1}^{T} Ce^{-Y_T t} + 1e^{-Y_T T}.$$

3. The Credit Spread

This section characterizes the risky coupon bond's credit spread. The risky bond's credit spread is defined by

$$\kappa_T := Y_T - y_T(c).$$

The credit spread is defined relative to a default-free coupon bond with the same maturity as the risky coupon bond. However, as the notation makes explicit, the coupon rates will typically be different between the default-free (c) and risky bonds (C).

This definition implies the following expression is satisfied:

$$D_T^\alpha(0) = \sum_{t=1}^{T} Ce^{-(y_T(c)+\kappa_T)t} + 1e^{-(y_T(c)+\kappa_T)T}. \tag{4}$$

Remark 1. (Different credit spreads) The above definition matches the maturity of the risky and default-free coupon bonds. In practice and in some of the empirical literature, the credit spread κ_T^* is defined relative to a default-free coupon bond of close but unequal maturity, e.g.,

$$D_T^\alpha(0) = \sum_{t=1}^{T} Ce^{-(y_{10}(c)+\kappa_T^*)t} + 1e^{-(y_{10}(c)+\kappa_T^*)T},$$

where the 10-year default-free coupon bond yield $y_{10}(c)$ is used, for example, for all $T \geq 7$ maturity risky coupon bonds. The following analysis extends to any such alternative definition.

Sometimes in the literature, the credit spread is defined relative to risky and default-free zero-coupon bonds (e.g., Bai *et al.*, 2020; Elton *et al.*, 2001; Feldhutter and Schaefer, 2018). This introduces additional error in the empirical methodology because zero-coupon bond credit spreads are unobservable and must be estimated from coupon bond prices. This completes the remark.

3.1. *A simple case*

To illustrate the intuition underlying the characterization of a bond's credit spread in the general model, we first characterize this spread in a simple model. This simple model is the basis for many of the common misunderstandings about the decomposition of the credit spread. For the simple model, we assume that the risk-neutral default intensity and recovery rate are constants, i.e., $\lambda_t = \lambda > 0$ and $\delta_t = \delta \in [0, 1)$.

In addition, we assume that the default-free term structure of interest rates is deterministic and flat so that the default-free zero-coupon and default-free coupon bond yields are equal, i.e.,

$$r_0 = R_t = y_T(c)$$

for all $t \in [0, T]$, which implies that the default-free zero-coupon bond's price is

$$p(0, t) = e^{-r_0 t}$$

for all $t \in [0, T]$.

Under these simplifying assumptions,

$$D_T^{\alpha}(0) = \sum_{t=1}^{T} Ce^{-\lambda t - \alpha t} p(0, t) + 1e^{-\lambda T - \alpha T} p(0, T) + \delta p(0, \hat{s}) e^{-\alpha \hat{s}} (1 - e^{-\lambda T}),$$

and using the definition of the credit spread,

$$\begin{aligned}&\sum_{t=1}^{T} Ce^{-\kappa_T t} p(0, t) + 1e^{-\kappa_T T} p(0, T)\\ &\quad = \sum_{t=1}^{T} Ce^{-\lambda t - \alpha t} p(0, t) + 1e^{-\lambda T - \alpha T} p(0, T) + \delta p(0, \hat{s}) e^{-\alpha \hat{s}} (1 - e^{-\lambda T}).\end{aligned}$$

This expression gives κ_T as an implicit function of the other parameters. To obtain an analytic solution, using the approximation $e^{-z} \approx 1 - z$, we can rewrite this expression as

$$\begin{aligned}&\sum_{t=1}^{T} C(1 - \kappa_T t) p(0, t) + (1 - \kappa_T T) p(0, T)\\ &\quad \approx \sum_{t=1}^{T} C(1 - (\lambda + \alpha) t) p(0, t) + (1 - (\lambda + \alpha) T) p(0, T) + \delta e^{-\alpha \hat{s}} p(0, \hat{s}) \lambda T.\end{aligned}$$

The solution, for small values of $(\kappa_T, \alpha, \lambda)$, is

$$\kappa_T \approx \lambda(1 - \delta) + \alpha + \lambda\delta \left(1 - \frac{e^{-\alpha \hat{s}} p(0, \hat{s}) T}{\sum_{t=1}^{T} Ctp(0, t) + Tp(0, T)}\right). \tag{5}$$

As seen, the credit spread can be decomposed into three terms. The first $\lambda(1 - \delta)$ is the one-year, risk-neutral expected loss. Here, because the default intensity is nonrandom, $\int_0^1 \lambda du = \lambda$. The second is the liquidity premium, and the third is an adjustment for the timing of the various coupon and principal cash flows. Note that a risky zero-coupon bond with $C = 0$ will still have a third term, adjusting for the fact that a recovery on the risky bond's principal is paid at the default time, which could occur earlier than time T.

This decomposition provides *an explanation of the credit spread puzzle.* Indeed, using the one-year, risk-neutral expected loss as the credit risk measure, we see here that it is only a fraction of the credit spread, even in this highly simplified case. Typically, the credit spread is much larger than the one-year, risk-neutral expected loss, i.e.,

$$\kappa_T > \lambda(1-\delta),$$

because $\alpha > 0$ and

$$\frac{e^{-\alpha\hat{s}}p(0,\hat{s})\,T}{\sum_{t=1}^{T} Ctp(0,t) + Tp(0,T)} < 1.$$

Finally, recalling that the risk-neutral default probability satisfies $\lambda = \psi\tilde{\lambda}$, where ψ is the default risk premium, this decomposition also demonstrates that the credit spread is a nonlinear function of the bond's default risk premium.

3.2. *The general case*

Given the insights from the simple case, this subsection provides the credit spread decomposition for the general model. The logic replicates that of the simple case, and to simplify the presentation, the derivation of the decomposition is contained in the Appendix.

We note that in the general case, the recovery rate, default-free spot rate, and default probabilities are stochastic processes. Recall that the bond price is given by expression (2). The credit spread decomposition is given in the following theorem.

Theorem 1. (Credit spread decomposition) *For small* $\kappa_T, \alpha > 0$,

$$\kappa_T \approx \frac{1}{\theta_B}\left(\frac{\hat{B}_T(0) - D_T^0(0)}{\hat{B}_T(0)}\right) + \alpha\frac{\theta_D}{\theta_B}\,\frac{D_T^0(0)}{\hat{B}_T(0)}, \tag{6}$$

where

$$D_T^0(0) = \sum_{t=1}^{T} C\mathbb{Q}(\tau > t)p(0,t) + \mathbb{Q}(\tau > T)p(0,T) + d_0\int_0^T \mathbb{Q}(\tau = s)p(0,s)\,ds,$$

$$d_0 = E[\delta_\tau],$$

$$\hat{B}_T(0) := \sum_{t=1}^{T} Ce^{-y_T(c)t} + 1e^{-y_T(c)T},$$

$$\theta_B := \sum_{t=1}^{T} \frac{Ce^{-y_T(c)t}}{\hat{B}_T(0)}\, t + \frac{e^{-y_T(c)T}}{\hat{B}_T(0)}\, T,$$

$$\theta_D := \sum_{t=1}^{T} \frac{C\mathbb{Q}(\tau > t)p(0,t)}{D_T^0(0)}\, t + \frac{\mathbb{Q}(\tau > T)p(0,T)}{D_T^0(0)}\, T$$
$$+ \int_0^T \frac{\mathbb{Q}(\tau = s)\, d_0 p(0,s)}{D_T^0(0)}\, s ds.$$

In this decomposition, $\hat{B}_T(0)$ corresponds to the value of a default-free coupon bond paying the coupon rate C but using the yield $y_T(c)$ based on an equivalent-maturity default-free coupon bond paying the coupon rate c. θ_B corresponds to the duration of $\hat{B}_T(0)$. Note that in general, $\hat{B}_T(0) \neq B_T(0)$ unless $c = C$. Finally, θ_D is a measure of the risky coupon bond's duration explicitly considering default in the timing of the cash flows.

The general decomposition of the credit spread more closely corresponds to the intuition that the credit spread should be decomposable into just two components: credit risk and liquidity risk. We now explain why the first term $\frac{1}{\theta_B}\left(\frac{\hat{B}_T(0) - D_T^0(0)}{\hat{B}_T(0)}\right)$ is a measure of credit risk and the second term $\alpha \frac{\theta_D}{\theta_B} \frac{D_T^0(0)}{\hat{B}_T(0)}$ is a measure of liquidity risk. Note that a more general measure of credit risk is necessary because the one-year, risk-neutral expected loss $[\lambda_0(X_0)(1 - \delta_0)]$ is not easily visible in the credit spread when the default intensity and recovery rates are stochastic processes.

First, the quantity $\left(\frac{\hat{B}_T(0) - D_T^0(0)}{\hat{B}_T(0)}\right)$ is the percentage difference between the default-free and risky coupon bond values both with the same coupon rates and maturities. Alternatively stated, this represents the holding period return over $[0, T]$ based on the price differential between the two bonds, assuming that the risky bond does not default — so that it pays the same coupons and principal as the default-free bond. Normalizing by the default-free bond's duration, $\frac{1}{\theta_B}$, over the time period $[0, T]$ makes this holding period return per year. The second term $\alpha \frac{\theta_D}{\theta_B} \frac{D_T^0(0)}{\hat{B}_T(0)}$ is easily seen to be a measure of liquidity risk because it is the liquidity premium scaled by the different prices and durations of the two coupon bonds.

With respect to the credit spread puzzle, this decomposition shows that credit risk is still only a fraction of the credit spread. The remainder of the spread is due to the liquidity premium. Because the credit risk component is only a fraction of the credit spread, and the decomposition is not linear in the one-year, risk-neutral expected loss [approximately $\lambda_0(X_0)(1-\delta_0)$], this decomposition *provides an explanation for the credit spread puzzle.*

With respect to the existing empirical literature, two additional observations are worth emphasizing. First, the credit spread κ_T is a function of the default probabilities $\mathbb{Q}(\tau > t)$ for $t \in [(0, T)]$, which depend on the risk-neutral default intensity $\lambda_t(X_t)$, via the risky bond's price $D_T^0(0)$ and the risk-neutral expected recovery rate d_0. It is not linear in the (approximate) one-year risk-neutral intensity $\lambda_0(X_0)$. This implies that running a linear regression with the credit spread as the dependent variable and with the independent variables reflecting market conditions, the credit entity's balance sheet, the recovery rate d_0, plus the average one-year risk-neutral intensity $\lambda_0(X_0)$ will be misspecified with low explanatory power (e.g., Elton *et al.*, 2001; Giesecke *et al.*, 2011).

Second, with respect to both the bond's risk premium and the bond's default risk premium, it is embedded in the credit spread in a complex and nonlinear fashion via the risk-neutral default intensity process $\lambda_t(X_t) = \psi_t \tilde{\lambda}_t(X_t)$, where ψ_t is the default risk premium process. We note that the bond's risk premium obtained via taking expectations with respect to the risk-neutral probabilities $\mathbb{Q}$ is a result of its interest rate risk, the state variables X_t generating the randomness in the default intensity, and the stochastic recovery rate process δ_t [readers can refer to Jarrow *et al.* (2005) for a discussion of these different risk premiums]. This implies that these two risk premiums are important in understanding the credit spread, as has been documented in the empirical literature (e.g., Bai *et al.*, 2020; Elton *et al.*, 2001; Huang and Huang, 2012). However, adding proxies for these risk premiums as independent variables in a linear regression for credit spreads will result in a misspecified model (e.g., Elton *et al.*, 2001; Giesecke *et al.*, 2011).

4. Conclusion

This paper used an empirically validated reduced-form model to characterize a corporate bond's credit spread as a function of risk-neutral default probabilities, recovery rates, liquidity premium, and coupon/principal promised cash flows. This characterization yielded a new measure of credit risk and it provided a decomposition that resolved the credit spread puzzle, showing that the new measure of credit risk is only a fraction of a bond's credit spread and not linear in the one-year, risk-neutral expected loss.

Appendix

Proof of Theorem For the arbitrage-free prices, using the approximation $e^{-z} \approx 1 - z$ gives

$$
\begin{aligned}
D_T^{\alpha}(0) &\approx \sum_{t=1}^{T} (1-\alpha t) C\mathbb{Q}(\tau > t)p(0,t) + (1-\alpha T)\mathbb{Q}(\tau > T)p(0,T) \\
&\quad + d_0 \int_0^T \mathbb{Q}(\tau = s)(1-\alpha s)p(0,s)\,ds \\
&= D_T^0(0) - \alpha\left(\sum_{t=1}^{T} tC\mathbb{Q}(\tau > t)p(0,t) + T\mathbb{Q}(\tau > T)p(0,T)\right. \\
&\quad \left. + d_0 \int_0^T \mathbb{Q}(\tau = s)sp(0,s)\,ds\right).
\end{aligned}
$$

Define the duration of the risky coupon bond, *with* $\alpha = 0$, as

$$
\begin{aligned}
\theta_D &:= \sum_{t=1}^{T} \frac{C\mathbb{Q}(\tau > t)p(0,t)}{D_T^0(0)}\, t + \frac{\mathbb{Q}(\tau > T)p(0,T)}{D_T^0(0)}\, T \\
&\quad + \int_0^T \frac{\mathbb{Q}(\tau = s)\,d_0 p(0,s)}{D_T^0(0)}\, s ds.
\end{aligned}
$$

Then, the substitution yields

$$
D_T^{\alpha}(0) \approx D_T^0(0)(1 - \alpha\theta_D).
$$

Here, in this approximation, we must have $(1 - \alpha\theta_D) \geq 0$ because $D_T^{\alpha}(0) \geq 0$.

Next, for the bond's credit spread relation

$$
D_T^{\alpha}(0) = \sum_{t=1}^{T} Ce^{-(\kappa_T + y_T(c))t} + e^{-(\kappa_T + y_T(c))T}
$$

and again using the approximation $e^{-z} \approx 1 - z$, we have

$$
\begin{aligned}
D_T^{\alpha}(0) &\approx \sum_{t=1}^{T} C(1-\kappa_T t)e^{-y_T(c)t} + (1-\kappa_T T)e^{-y_T(c)T} \\
&= \hat{B}_T(0) - \kappa_T\left(\sum_{i=1}^{T} Cte^{-y_T(c)t} + Te^{-y_T(c)T}\right),
\end{aligned}
$$

where

$$
\hat{B}_T(0) := \sum_{t=1}^{T} Ce^{-y_T(c)t} + 1e^{-y_T(c)T}.
$$

Define the default-free bond's duration

$$\theta_B := \sum_{t=1}^{T} \frac{Ce^{-y_T(c)t}}{\hat{B}_T(0)} t + \frac{e^{-y_T(c)T}}{\hat{B}_T(0)} T.$$

Substitution gives

$$D_T^{\alpha}(0) \approx \hat{B}_T(0)(1 - \kappa_T \theta_B).$$

Combining the two expressions gives

$$D_T^{0}(0)(1 - \alpha\theta_D) \approx \hat{B}_T(0)(1 - \kappa_T \theta_B).$$

Algebra gives the final result. □

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References

Bai, J., R. Goldstein, and F. Yang, 2020, Is the Credit Spread Puzzle a Myth?, *Journal of Financial Economics* 137, 297–319.

Campbell, J. Y., J. Hilscher, and J. Szilagyi, 2008, In Search of Distress Risk, *The Journal of Finance* 63(6), 2899–2939.

Campbell, J. Y., J. Hilscher, and J. Szilagyi, 2011, Predicting Financial Distress and the Performance of Distressed Stocks, *Journal of Investment Management* 9(2), 14–34.

Campbell, J. Y., and G. B. Taksler, 2003, Equity Volatility and Corporate Bond Yields, *The Journal of Finance* 58, 2321–2350.

Cherian, J., E. Jacquier, and R. Jarrow, 2004, A Model of the Convenience Yields in On-the-Run Treasuries, *Review of Derivatives Research* 7, 79–97.

Collin-Dufresne, P., R. Goldstein, and J. Martin, 2001, The Determinants of Credit Spread Changes, *The Journal of Finance* 56(5), 2177–2207.

Davies, A., 2008, Credit Spread Determinants: An 85 Year Perspective, *Journal of Financial Markets* 11, 180–197.

Duffie, D., and K. Singleton, 1999, Modeling Term Structures of Defaultable Bonds, *The Review of Financial Studies* 12(4), 687–720.

Elton, E., M. Gruber, D. Agrawal, and C. Mann, 2001, Explaining the Rate Spread on Corporate Bonds, *The Journal of Finance* 56(1), 247–277.

Feldhutter, P. and S. Schaefer, 2018, The Myth of the Credit Spread Puzzle, *The Review of Financial Studies* 31(8), 2897–2942.

Giesecke, K., F. Longstaff, S. Schaefer, and I. Strebulaev, 2011, Corporate Bond Default Risk: A 150-Year Perspective, *Journal of Financial Economics* 102, 233–250.

Hilscher, J., R. Jarrow, and D. van Deventer, 2023, The Valuation of Corporate Coupon Bonds, *Journal of Financial and Quantitative Analysis*, forthcoming.

Huang, J., and M. Huang, 2012, How Much of the Corporate-Treasury Yield Spread is due to Credit Risk?, *The Review of Asset Pricing Studies* 2(2), 153–202.

Jarrow, R., 2011, Credit Market Equilibrium Theory and Evidence: Revisiting the Structural Versus Reduced Form Credit Risk Model Debate, *Finance Research Letters* 8, 2–7.

Jarrow, R., 2022, *Continuous-Time Asset Pricing Theory: A Martingale-Based Approach*, second edition, Springer, Cham.

Jarrow, R., D. Lando and F. Yu, 2005, Default Risk and Diversification: Theory and Empirical Applications, *Mathematical Finance* 15(1), 1–26.

Jarrow, R., and S. Turnbull, 1997, An Integrated Approach to the Hedging and Pricing of Eurodollar Derivatives, *Journal of Risk and Insurance* 64(2), 271–299.