Credit Spread Decomposition Notes

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Comparison: Real-World vs. Risk-Neutral Expectations

Real-World Expectation $\mathbb{E}^{\mathbb{P}}[\cdot]$	**Risk-Neutral Expectation** $\mathbb{E}^{\mathbb{Q}}[\cdot]$
Based on the actual probability distribution observed in the real world.	Based on an artificial "risk-neutral" probability distribution used for pricing.
Answers: "What do we expect to happen under real-world dynamics?"	Answers: "What is the fair price of a future payoff in a no-arbitrage market?"
Used for forecasting, risk management, and historical simulation.	Used to price financial instruments such as bonds and derivatives.
Accounts for real-world risk preferences and expected returns.	Assumes investors are indifferent to risk; all assets grow at the risk-free rate in expectation.
Example: $\mathbb{E}^{\mathbb{P}}[S_T]$ forecasts the expected future asset price.	Example: Price = $\mathbb{E}^{\mathbb{Q}}$[discounted payoff]

Abstract

The empirical evidence showing that a corporate bond's expected loss is only a small portion of a bond's credit spread is called the credit spread puzzle. This paper, using a reduced-form credit risk model, characterizes a risky bond's credit spread. This characterization provides a more general measure of a risky bond's credit risk and it shows that, in an arbitrage-free market, a bond's credit risk is only a fraction of the credit spread and not linearly related to the one-year, risk-neutral expected loss, resolving the credit spread puzzle.

The credit spread puzzle refers to the empirical observation that corporate bonds offer yields significantly higher than risk-free Treasury bonds, even when the expected loss from default is very small. To illustrate, consider a zero-coupon corporate bond that pays \$100 at maturity in one year. Suppose there is a 1% probability of default, and in the event of default, the bond pays nothing (i.e., zero recovery). Under the risk-neutral measure $\mathbb{Q}$, the expected payoff is:

$$\mathbb{E}^{\mathbb{Q}}[\text{Payoff}] = 0.99 \cdot 100 + 0.01 \cdot 0 = 99$$

If investors were only compensated for expected default losses, this bond should trade at approximately \$99. In contrast, a risk-free Treasury bond maturing in one year would pay \$100 with certainty and—assuming zero interest rates—would also trade at \$100. But due to the credit spread puzzle phenomenon, the observed spread between corporate and Treasury bond prices in reality is often significantly larger than the

$1 expected loss implied by default probabilities. To address the credit spread puzzle, the paper adopts a reduced-form credit risk model in which defaults occur randomly with intensity $\lambda(t)$. This default intensity is not directly observable, but it can be estimated from market instruments like bond prices and CDS spreads. These prices embed information about perceived credit risk, allowing the model to extract implied default probabilities.

Using this framework, the authors define a new, more general measure of a risky bond's credit risk. Crucially, they show that even in an arbitrage-free market, a bond's credit risk accounts for only a fraction of its total credit spread. Moreover, the credit spread is not a linear function of the bond's one-year expected loss, which contradicts many prior modeling assumptions. This more general nonlinear relationship helps resolve the credit spread puzzle by explaining why credit spreads remain large even when expected losses are small.

Introduction

The determinants of risky bond credit spreads are an often studied empirical topic in finance, with still no consensus on the importance of the bond's expected loss in this decomposition. The low explanatory power of a bond's expected loss in explaining the credit spread is referred to as the "credit spread puzzle" (Feldhutter and Schaefer, 2018; Bai *et al.*, 2020). The approaches to analyzing the credit spread are one of the two: (i) based on a structural model for credit risk, to run a linear regression which decomposes the credit spread into various firm and market explanatory variables (e.g., Bai *et al.*, 2020; Campbell and Taksler, 2003; Collin-Dufresne *et al.*, 2001; Davies, 2008; Feldhutter and Schaefer, 2018; Huang and Huang, 2012); or (ii) based on a reduced-form credit risk model, to run a linear regression that characterizes the credit spread using the estimated default probabilities and other firm and market explanatory variables (e.g., Elton *et al.*, 2001; Giesecke *et al.*, 2011).

This section explains two major empirical approaches used by researchers to analyze the *credit spread puzzle*—the well-documented observation that corporate bond credit spreads are significantly larger than what would be implied by expected default losses alone. The primary goal of these studies is to understand what drives the size of the credit spread, defined as the additional yield that corporate bonds offer over risk-free Treasuries.

The **first approach** is based on a *structural credit risk model*, rooted in corporate finance theory, such as the Merton model. In these models, defaults arise when the market value of a firm's assets falls below a threshold defined by its liabilities. Researchers use firm-level data (e.g., leverage, volatility, asset value) to estimate default probabilities from this structural framework. They then run a *linear regression* of the form:

$$\text{Credit Spread}_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + \varepsilon_i$$

Here, X_{ij} represents explanatory variables derived from firm and market characteristics—such as leverage ratio, asset volatility, interest rate levels, or market-wide risk indicators like the VIX. The term ε_i represents the regression residual—the portion of the credit spread not explained by the included variables—capturing noise, omitted factors, or model imperfections. The regression attempts to quantify how much of the observed credit spread can be attributed to these factors. Examples of this approach include Bai *et al.* (2020), Campbell and Taksler (2003), and Collin-Dufresne *et al.* (2001).

The **second approach** uses a *reduced-form credit risk model*, in which default is modeled as a random event driven by a stochastic *intensity process* $\lambda(t)$, estimated directly from market prices—such as bond yields or credit default swap (CDS) spreads—without requiring information about the firm's balance sheet. From the estimated intensity, researchers derive default probabilities, then run a regression with the same goal: to explain the credit spread using the estimated probability of default and other control variables. The same linear regression framework applies:

$$\text{Credit Spread}_i = \beta_0 + \beta_1 \widehat{\text{DefaultProb}}_i + \beta_2 Z_{i1} + \cdots + \beta_k Z_{ik} + \varepsilon_i$$

where $\widehat{\text{DefaultProb}}_i$ is the market-implied probability of default and Z_{ij} includes additional firm-level and macroeconomic factors. Notable studies using this method include Elton *et al.* (2001), Giesecke *et al.* (2011), and Feldhutter and Schaefer (2018).

In both approaches, the core idea is to *decompose the credit spread* into weighted contributions from various explanatory variables using regression. This allows researchers to identify how much of the credit spread is attributable to actual credit risk (i.e., expected losses due to default) versus other sources—such as liquidity risk, risk premia, and macroeconomic uncertainty. The difference lies in how default risk is modeled: either structurally through firm fundamentals, or statistically through market-implied intensities.

Unfortunately, structural model-based default probabilities have been shown in other contexts to be misspecified (e.g., Campbell *et al.*, 2008, 2011; Jarrow, 2011), calling into question the conclusions drawn from the structural approach to the credit spread decomposition. Also, the reduced-form credit risk approach is misspecified because it assumes a simple linear relation between the credit spread, default probabilities, and other market and firm explanatory variables. We show below that this relation is complex and nonlinear.

A model is said to be misspecified when its assumptions, structure, or included variables do not accurately reflect the true underlying process that generates the data. This can lead to biased estimates, incorrect inferences, or poor predictive performance.

The purpose of this paper is to provide a new decomposition of a bond's credit spread that resolves the credit spread puzzle and which can be used to understand the existing empirical evidence. This characterization, and the new credit risk measure derived herein, can be the basis for new empirical research on the determinants of a risky bond's credit spread.

To obtain this new decomposition, we use the risky bond valuation model contained in Hilscher *et al.* (2023). We show that the bond's credit spread is a nonlinear and complex function of the firm's default probability, recovery rate, risk premium, default risk premium, liquidity premium, and promised cash flows. This decomposition also provides a new credit risk measure for a risky bond, replacing the bond's one-year, risk-neutral expected loss. This new measure is needed because it is shown that the bond's credit spread is not a linear function of its one-year, risk-neutral expected loss.

This decomposition provides a theoretical explanation for the credit spread puzzle because: (i) the credit spread is not a linear function of the bond's one-year, risk-neutral expected loss; and (ii) the credit risk, although linear in this new credit risk measure, is still only a fraction of the credit spread. The remaining fraction corresponds to liquidity risk, which is a nonlinear function of the bond's default risk premium, liquidity premium, recovery rate, and the promised coupon and principal payments. It is important to emphasize that although this paper is theoretical, the risky debt pricing model from which the credit spread decomposition is obtained has been empirically validated with respect to the corporate bond market prices (Hilscher *et al.*, 2023). Consequently, this empirical validation implies that the credit spread decomposition provided herein is consistent with the observed and market determined credit spreads.

The *one-year risk-neutral expected loss* is the expected value of losses due to default occurring within the next year, as priced by the market. It is computed under the *risk-neutral probability measure* $\mathbb{Q}$, which reflects how investors price credit risk, including compensation for uncertainty and risk aversion.
Let the following variables be defined:

- F = face value (promised payoff) of the bond at maturity.
- R = recovery rate (fraction of F paid in the event of default), where $0 \leq R \leq 1$.
- τ = random default time.
- $T = 1$ = one-year time horizon.
- $\mathbf{1}_{\{\tau \leq 1\}}$ = indicator function equal to 1 if default occurs within one year, 0 otherwise.
- $\mathbb{E}^{\mathbb{Q}}[\cdot]$ = expectation under the risk-neutral probability measure $\mathbb{Q}$.

Then the **one-year risk-neutral expected loss**, denoted $\mathrm{RNEL}_{1\mathrm{yr}}$, is:

$$\mathrm{RNEL}_{1\mathrm{yr}} = \mathbb{E}^{\mathbb{Q}}\left[F \cdot (1 - R) \cdot \mathbf{1}_{\{\tau \leq 1\}}\right]$$

Assume the following inputs:

- $F = 100$ (the bond promises to pay \$100 at maturity),
- $R = 0$ (zero recovery in case of default),
- $\mathbb{Q}(\tau \leq 1) = 0.04$ (the market-implied probability of default within one year is 4%).

Then the one-year risk-neutral expected loss is:

$$\mathrm{RNEL}_{1\mathrm{yr}} = 100 \cdot (1 - 0) \cdot 0.04 = 4$$

So, the expected loss under the risk-neutral measure is \$4. This means that if investors were only being compensated for default risk (and no other risks), the bond would trade at approximately:

$$\text{Price} = F - \mathrm{RNEL}_{1\mathrm{yr}} = 100 - 4 = 96$$

This paper resolves the credit spread puzzle by showing that the credit spread on a risky bond is not linearly related to its one-year, risk-neutral expected loss. Previous empirical models often approximated the credit spread using a scalar multiple of the expected loss under the risk-neutral measure. Formally, they assumed:

$$\text{Credit Spread} \approx \alpha \cdot \mathrm{RNEL}_{1\mathrm{yr}}$$

for some constant α. However, the paper demonstrates that this approximation is fundamentally flawed.

The authors define a more general measure of **credit risk** as the portion of a bond's price that is attributable to the possibility of default.

In this framework, credit risk enters the model *linearly*. That is, holding all else fixed, if the model-implied default intensity doubles, then the credit risk component of the bond price also doubles. This linearity refers to the way default-related inputs scale within the pricing equation of the model.

However, the paper emphasizes two key findings:

- The total credit spread is **not** a linear function of the one-year risk-neutral expected loss. The expected loss alone is a poor proxy for what actually drives spreads in the market.
- Even after defining credit risk precisely through their model, this credit risk component explains only a *fraction* of the observed credit spread.

The remaining portion of the credit spread is attributed to **liquidity risk**. This includes factors such as the bond's default risk premium, liquidity premium, recovery assumptions, and promised payments. Crucially, this liquidity risk enters the pricing relationship *nonlinearly* — meaning its effect on spreads is more complex and cannot be captured by a simple proportional change in input variables.

The Model

We assume a continuous-time, continuous trading market with a finite time horizon $[0, T]$ that is frictionless, competitive, and satisfies no arbitrage and no dominance.[1] Traded are a term structure of default-free bonds and a risky coupon bond. We are interested in characterizing the credit spread of this risky coupon bond. The frictionless market assumption is relaxed below to include illiquid corporate bond markets.

We consider a simplified, idealized financial market model in which trading occurs continuously over a finite time interval $[0, T]$. This market is assumed to be frictionless, meaning there are no transaction costs, bid-ask spreads, or taxes, and it is competitive—no individual trader can influence market prices. Additionally, the market satisfies the conditions of no arbitrage (i.e., there are no opportunities to make a riskless profit) and no dominance (i.e., no asset always outperforms all others in every state of the world). Within this market, two types of bonds are traded: a term structure of default-free (risk-free) bonds, and a risky coupon bond that pays periodic interest but carries credit risk due to the possibility of default. The primary objective is to characterize the credit spread of the risky bond, which is the yield premium investors require over the risk-free bond to compensate for the default risk. While the model begins with a frictionless assumption, the framework is later extended to incorporate illiquid corporate bond markets where such ideal conditions do not apply.

Default-free bonds

This subsection analyzes default-free bonds.

Let $p(0, t)$ denote the time-0 price of a zero-coupon bond paying a sure dollar at time $t \in [0, T]$.

Let r_t be the default-free spot rate of interest at time $t \in [0, T]$.

Let $B_T(0)$ denote the time-0 price of a default-free coupon bond with the coupon rate $c \in [0, 1]$, a maturity of T, and a principal equal to \$1. Coupons are paid at times $t = 1, \ldots, T$.

By the first and third fundamental theorems of asset pricing, there exist risk-neutral probabilities[2] $\mathbb{Q}$ such that

$$p(0, t) = E\left(e^{-\int_0^t r_s ds}\right)$$

for $t \in [0, T]$, where $E(\cdot)$ denotes expectation under the risk-neutral probabilities $\mathbb{Q}$.

For the default-free coupon bond, this implies the well-known expression

$$\begin{aligned} B_T(0) &= E\left(\sum_{t=1}^{T} ce^{-\int_0^t r_s ds} + 1e^{-\int_0^T r_s ds}\right) \\ &= \sum_{i=1}^{T} cp(0, t) + p(0, T). \end{aligned}$$

Feature	Zero-Coupon Bond	Default-Free Coupon Bond
Coupons?	No coupons	Pays regular coupons
Payment Timing	Pays only once at maturity	Pays coupons periodically + final principal at maturity
Final Payment	Pays \$1 at time t	Pays \$$c$ each period + \$1 at time T
Notation (Price)	$p(0,t)$	$B_T(0)$
Risk	No credit risk (assumed default-free)	Also default-free (guaranteed payment)

Table 1: Comparison between Zero-Coupon and Default-Free Coupon Bonds

Understanding Zero-Coupon Bond Pricing

We begin by recalling the formula for compound interest, which describes how an investment grows over time. If you invest \$1 today at an annual interest rate r, compounded n times per year, then after t years the future value is given by:

$$\text{Future value} = \left(1 + \frac{r}{n}\right)^{nt}$$

As the compounding frequency increases, we take the limit as $n \to \infty$, arriving at the continuous compounding formula:

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$$

Thus, under continuous compounding, the future value of \$1 invested at rate r for t years is:

$$\text{Future value} = e^{rt}$$

This expression uses a *positive* exponent because it represents money growing forward in time.

Now consider the reverse situation. Suppose you expect to receive \$1 at a future time t, and want to know what that is worth today, assuming the same interest rate r. This is the concept of *present value*, and involves reversing the compounding process. We solve the equation:

$$1 = \text{Present value} \cdot e^{rt} \quad \Rightarrow \quad \text{Present value} = e^{-rt}$$

Therefore, the present value of \$1 to be received at time t, under continuous discounting at rate r, is:

$$p(0,t) = e^{-rt}$$

This is where the *negative exponent* comes from: it reflects the fact that future money is worth less today, due to the time value of money. We are effectively undoing exponential growth to compute a value in today's dollars.

This logic directly applies to the pricing of a zero-coupon bond. A zero-coupon bond pays exactly \$1 at a future time t, with no interim payments. If its continuously compounded yield is R_t, then the price of the bond today (at time 0) is simply:

$$p(0,t) = e^{-R_t \cdot t}$$

The subscript t in R_t indicates that the yield is associated with a bond that matures at time t.

Risk-Neutral Valuation of Zero-Coupon Bonds

We now move beyond constant interest rates and explore the more realistic case where interest rates change randomly over time. Our goal is to understand the formula:

$$p(0,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r_s\,ds}\right]$$

This equation tells us how to compute the present value $p(0,t)$ of a zero-coupon bond that pays \$1 at future time t, when interest rates are stochastic and evolve over time.

Previously, when interest rates were constant, we used the formula:

$$p(0,t) = e^{-R_t \cdot t}$$

This assumes a deterministic rate R_t known in advance. But in reality, interest rates fluctuate and are not known ahead of time. To account for this uncertainty, we introduce a time-dependent process r_s, which represents the short-term interest rate at time $s \in [0,t]$.

The quantity r_s denotes the instantaneous interest rate at time s. This means:

- r_0 is the interest rate today.
- $r_1, r_2, \ldots, r_t$ are interest rates in the future.

These future interest rates are unknown and modeled as a stochastic process.

When interest rates are constant, we discount future cash flows with e^{-rt}. If rates change continuously, we must account for every infinitesimal change along the path. We do this using an integral:

$$\int_0^t r_s\,ds$$

This integral adds up all the small interest rate contributions between time 0 and t. The corresponding discount factor is then:

$$e^{-\int_0^t r_s\,ds}$$

This gives the present value of \$1 received at time t, assuming we knew the full path of r_s from 0 to t.

Since we don't know the future values of r_s, we cannot compute this directly. Instead, we take an expected value over all the possible paths that interest rates could follow. But we don't use a regular average—we use a *risk-neutral expectation*, denoted by $\mathbb{E}^{\mathbb{Q}}[\cdot]$.

The risk-neutral measure $\mathbb{Q}$ is a mathematical construct used in finance that ensures no-arbitrage pricing. Under this measure, asset prices are equal to the expected present value of their future cash flows. So the price of the zero-coupon bond is:

$$p(0,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r_s\,ds}\right]$$

This formula averages the discount factors across all possible future interest rate paths, under the risk-neutral probabilities.

- $p(0,t)$: price today of a zero-coupon bond maturing at time t
- r_s: short-term interest rate at time $s \in [0,t]$
- $\int_0^t r_s\,ds$: accumulated interest from time 0 to t
- $e^{-\int_0^t r_s\,ds}$: discount factor accounting for time-varying interest rates

- $\mathbb{E}^{\mathbb{Q}}[\cdot]$: expectation under the risk-neutral probability measure $\mathbb{Q}$

If interest rates are constant, so that $r_s = r$ for all s, then:

$$\int_0^t r_s\,ds = rt \quad \text{and} \quad p(0,t) = e^{-rt}$$

Thus, the risk-neutral formula reduces to the familiar exponential discounting case. This confirms that our general formula is consistent with the classical setting when rates are known and fixed.

Understanding the Price of a Default-Free Coupon Bond

We now examine how to compute the time-0 price of a default-free coupon bond. The full expression is:

$$B_T(0) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{t=1}^{T} c \cdot e^{-\int_0^t r_s\,ds} + 1 \cdot e^{-\int_0^T r_s\,ds}\right)$$

This simplifies to:

$$B_T(0) = \sum_{t=1}^{T} c \cdot p(0,t) + p(0,T)$$

The expression $B_T(0)$ represents the price at time 0 of a default-free coupon bond that matures at time T.

This bond pays:

- A coupon payment of amount c at times $t = 1, 2, \ldots, T$,
- A final principal repayment of \$1 at time T.

The sum

$$\sum_{t=1}^{T} c \cdot e^{-\int_0^t r_s\,ds} + 1 \cdot e^{-\int_0^T r_s\,ds}$$

represents the present value of all future payments. Each coupon payment c made at time t is discounted using the stochastic discount factor

$$e^{-\int_0^t r_s\,ds}.$$

The final principal payment of \$1 at maturity T is also discounted in the same way, using

$$e^{-\int_0^T r_s\,ds}.$$

The term $e^{-\int_0^t r_s\,ds}$ gives the present value of receiving \$1 at time t. However, the bondholder is not receiving \$1, but rather \$$c$. To account for this, we scale the discount factor by c, giving:

$$\text{Present value of } \$c = c \cdot e^{-\int_0^t r_s\,ds}$$

Example: If \$1 in the future is worth \$0.95 today, then the value of receiving \$5 is:

$$5 \cdot 0.95 = 4.75$$

The same principle applies in this context.

Because interest rates r_s are uncertain and change over time, we do not know the exact value of $\int_0^t r_s\,ds$. Therefore, we take the expectation of each discounted cash flow using the risk-neutral probability measure $\mathbb{Q}$:

$$\mathbb{E}^{\mathbb{Q}}[\cdots]$$

This gives us the fair, arbitrage-free value of the bond under stochastic interest rates.

From the earlier result for the price of a zero-coupon bond, we know:

$$p(0,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t r_s\,ds}\right]$$

So we can substitute this into the earlier formula to get:

$$B_T(0) = \sum_{t=1}^{T} c \cdot p(0,t) + p(0,T)$$

Each coupon is priced using the price of a zero-coupon bond that matures at that time, and the final principal is priced using a zero-coupon bond maturing at time T.

A coupon bond can be viewed as a collection of many zero-coupon bonds. Each future cash flow is treated like a separate zero-coupon bond, discounted independently to the present. The total bond price is the sum of all these present values.

Term	**Meaning**
$c \cdot e^{-\int_0^t r_s ds}$	Present value of a coupon payment at time t
$\sum_{t=1}^{T} \cdot$	Adds up all coupon present values
$e^{-\int_0^T r_s ds}$	Present value of the final principal
$\mathbb{E}^{\mathbb{Q}}[\cdot]$	Average over all future interest rate paths
$p(0,t)$	Price of a zero-coupon bond maturing at time t

The zero-coupon bond's yield R_t at time 0 is defined by the following expression:

$$p(0,t) = e^{-R_t \cdot t}$$

for $t \in [0, T]$.

The coupon bond's time-0 yield $y_T(c)$, which depends on the bond's maturity T, is defined by the equation

$$B_T(0) = \sum_{t=1}^{T} ce^{-y_T(c)t} + 1e^{-y_T(c)T}.$$

We include the coupon rate in the definition of the default-free coupon bond's yield $y_T(c)$ because this expression will appear later in the credit spread decomposition for a default-free bond with a different coupon rate, but with the yield as given above.

We now turn our attention to the yield of a coupon bond. This yield is:

- Different from the zero-coupon case,
- Not directly observable,

- And must be solved for implicitly.

Let $y_T(c)$ denote the yield-to-maturity of a coupon bond with:

- Maturity T,
- Coupon rate c.

This yield $y_T(c)$ is defined to be the fixed rate such that, when used to discount all of the bond's future cash flows, the present value equals the bond's current market price.

The yield satisfies the following equation:

$$B_T(0) = \sum_{t=1}^{T} c \cdot e^{-y_T(c)\cdot t} + 1 \cdot e^{-y_T(c)\cdot T}$$

Let us break this expression down. The bond pays a coupon of amount c at each time $t = 1, 2, \ldots, T$, and a final principal of \$1 at maturity T. Each of these cash flows is discounted using the same fixed yield $y_T(c)$, resulting in terms of the form $e^{-y_T(c)\cdot t}$. The exponent is linear in time (not an integral) because the yield is assumed to be constant. The equation defines $y_T(c)$ as the specific value that equates the total present value of the bond's cash flows to its current market price $B_T(0)$.

This equation defines $y_T(c)$ *implicitly*; the yield is not directly computed from a closed-form formula. Instead, it must be solved for numerically in practice.

You can think of $y_T(c)$ as the constant rate that, if used to discount the bond's actual payments, would result in its current price.

We write $y_T(c)$ to emphasize that the yield depends on:

- The number of periods T,
- The size of the coupon c.

Changing the coupon alters the cash flows, and therefore changes the yield that solves the equation. This dependence is crucial when comparing bonds with different coupon rates.

A zero-coupon bond has an explicit yield formula, given by

$$p(0,t) = e^{-R_t t} \quad \Rightarrow \quad R_t = -\frac{1}{t} \ln p(0,t),$$

which allows us to solve directly for the yield R_t from the bond price.

In contrast, a coupon bond defines its yield implicitly through the equation

$$B_T(0) = \sum_{t=1}^{T} c \cdot e^{-y_T(c)t} + e^{-y_T(c)T}.$$

This yield, denoted $y_T(c)$, is known as the *yield-to-maturity* — a hypothetical constant rate that, when applied to all of the bond's future payments, exactly reproduces the bond's market price.

The risky coupon bond

This subsection studies the risky coupon bond issued by a credit entity. For simplicity, we call this credit entity a firm.

Let $D_T^{\alpha}(0)$ denote the time-0 price of a risky coupon bond with the coupon rate $C \in [0,1]$, a maturity of T, a principal equal to \$1, and a liquidity discount α. The coupons are paid at times $t = 1, \dots, T$. We don't explicitly include the coupon rate C in the bond's value to simplify the notation. It is fixed for the remainder of the paper.

Because corporate bond markets are illiquid relative to Treasuries, risky bond prices typically reflect a liquidity discount (Jarrow and Turnbull, 1997; Duffie and Singleton, 1999; Cherian *et al.*, 2004). Consequently, we assume that a risky bond's arbitrage-free price reflects a liquidity discount of α_t. For simplicity, we assume that the liquidity discount $\alpha_t = \alpha \geq 0$ is a constant. We apply the illiquidity discount to all of the bond's cash flows. Including such a liquidity discount modifies the pricing formula to incorporate the impact of various market frictions such as transaction costs, trading constraints, and the effects of differential taxes on coupon income versus capital gains.

Let τ denote the first default time of the bond's promised payments. We assume that $\tau > 0$.

We assume that the point process $N_t \in \{0, 1\}$ which jumps from 0 to 1 at the default time τ is a Cox process with stochastic intensity $\tilde{\lambda}_t(X_t)$ for $t \in [0, T]$ under the statistical probabilities where X_t represents a vector of state

variables. The state variables represent firm-specific characteristics (e.g., debt/equity ratio) and relevant macro-variables (e.g., the default-free spot rate). Here, $\tilde{\lambda}_t(X_t)$ represents the conditional probability of default over $[t, t + dt]$ given no default prior to t.

Let $\delta_t \in [0,1]$ be the stochastic recovery rate if default happens at time $t \in [0, T]$ on the bond's promised cash flows. To be consistent with practice, we assume that a recovery rate is only received on the promised principal and not any of the promised future coupon payments. If default happens between coupon payment dates, i.e., $t - 1 < \tau \leq t$, then the recovery rate is paid at time t.

We now describe the framework used to model the default of a risky bond. Let τ denote the (random) time of default, which represents the first time the firm fails to meet its promised payments. We assume that $\tau > 0$, meaning that default does not occur immediately at time zero.

To track whether default has occurred by time t, we introduce a point process $N_t \in \{0,1\}$, defined as

$$N_t = \begin{cases} 0 & \text{if } t < \tau \\ 1 & \text{if } t \geq \tau \end{cases}$$

This process jumps from 0 to 1 exactly once, at the time of default τ.

The default time τ is modeled as a Cox process, which is also referred to as a *doubly stochastic Poisson process*. Unlike a standard Poisson process, which has a constant arrival rate λ, a Cox process allows the arrival rate to be stochastic and time-varying.

Specifically, the Cox process is characterized by a *stochastic intensity* function $\tilde{\lambda}_t(X_t)$, where X_t is a vector of underlying state variables (such as firm-specific and market-wide economic factors). The intensity $\tilde{\lambda}_t(X_t)$ determines the instantaneous probability per unit time of default occurring at time t, given survival up to that time. Unlike a Poisson process, which has the memoryless property and treats future arrivals as independent of the past, a Cox process requires conditioning on survival. To understand why, consider an analogy from life insurance: if a person is alive at age 70, the probability they die in the next year is different from someone who is alive at 40 — even if both face the same disease. The fact that someone has survived to a later age carries information about their underlying health. The same logic applies in credit risk modeling. The fact that a firm has not defaulted by time t is informative. As a result, the conditional probability of default at time t must be updated to reflect this survival, and the arrival of default is no longer independent of the past. Whether the chance of default increases or decreases with time depends on the intensity process $\tilde{\lambda}_t(X_t)$: if the firm's financials deteriorate over time (i.e., X_t worsens), then $\tilde{\lambda}_t$ may increase, leading to a higher likelihood of default as time passes. But if the firm performs well and market conditions improve, $\tilde{\lambda}_t$ may decrease, indicating lower default risk the longer the firm survives. This loss of independence — and the presence of a time-varying, state-dependent intensity — is what distinguishes the Cox process from a standard Poisson process.

In this setup, the statistical probability of default within the time interval $[0,T]$ is governed by the stochastic evolution of $\tilde{\lambda}_t(X_t)$ for $t \in [0,T]$, under the risk-neutral measure. This allows the model to reflect realistic behavior, where the likelihood of default evolves dynamically based on observable market and firm conditions.

For clarity, the key elements of the framework are:

- τ: the random default time, with $\tau > 0$,
- N_t: the default indicator process defined by the jump from 0 to 1 at τ,
- $\tilde{\lambda}_t(X_t)$: the stochastic intensity function driving the Cox process,
- X_t: a vector of state variables affecting the default intensity.

This structure enables the pricing of risky bonds to incorporate uncertainty in default timing and allows the default risk to respond to evolving economic information in a flexible and tractable way.

Under the assumption of no arbitrage and no dominance, there exist risk-neutral probabilities $\mathbb{Q}$ such that

$$\begin{aligned} D_T^{\alpha}(0) = & \sum_{t=1}^{T} CE\left[1_{\{\tau>t\}} e^{-\int_0^t r_u du}\right] e^{-\alpha t} + E\left[1_{\{\tau>T\}} e^{-\int_t^T r_u du}\right] e^{-\alpha T} \\ & + \sum_{t=1}^{T} E\left[1_{\{t-1<\tau\leq t\}} \delta_t e^{-\int_0^t r_u du}\right] e^{-\alpha t}, \end{aligned} \tag{1}$$

where $E(\cdot)$ denotes expectation under the risk-neutral probabilities $\mathbb{Q}$ [the proof is in Hilscher *et al.* (2023)] and the risk-neutral default intensity $\lambda_t(X_t)$ satisfies $\lambda_t(X_t) = \psi_t \tilde{\lambda}_t(X_t)$ with ψ_t being the default risk premium process (Jarrow, 2022, Chap. 7).

We aim to compute the present value at time 0 of a risky coupon bond. This bond pays coupons at fixed times $t = 1, 2, \ldots, T$, may default at some random time τ, and includes a liquidity discount rate α which reduces the present value of all future payments. If the bond survives until maturity T, it pays a principal of 1. If it defaults earlier, a recovery payment δ_t is made at the time of default.

The present value of the bond is denoted $D_T^\alpha(0)$. It consists of three main components: expected coupon payments conditional on survival, the expected principal repayment at maturity conditional on no default, and the expected recovery payment in case of default.

The first component is the present value of coupon payments received only if the firm has not defaulted before time t. It is given by:

$$\sum_{t=1}^{T} C \cdot \mathbb{E}\left[\mathbf{1}_{\{\tau>t\}} e^{-\int_0^t r_u\,du}\right] e^{-\alpha t}$$

Here, C is the coupon payment, $\nVdash_{\{\tau>t\}}$ is an indicator function that equals 1 if the firm survives past time t, and $e^{-\int_0^t r_u\,du}$ discounts the payment from time t to time 0 using the short-rate process r_u. The outer expectation accounts for uncertainty in both the interest rates and the time of default. The factor $e^{-\alpha t}$ introduces an additional discount due to illiquidity, reducing the value of each payment beyond the time value of money. This reflects the idea that even if a payment is made, it is less valuable if the bond is hard to sell, trade, or hedge. Just as $e^{-\int_0^t r_u du}$ accounts for the erosion of value through time, $e^{-\alpha t}$ models the erosion of value through trading friction and market illiquidity.

The second component is the present value of the final principal repayment, provided the firm does not default by time T. It is given by:

$$\mathbb{E}\left[\mathbf{1}_{\{\tau>T\}} e^{-\int_0^T r_u\,du}\right] e^{-\alpha T}$$

This expression is structurally the same as the coupon terms, but applies to the lump-sum principal payment at the end. Again, it is discounted both by the interest rate path and the liquidity penalty.

The third component accounts for the recovery payment δ_t made in case the firm defaults during a specific time interval $(t-1, t]$. This component is:

$$\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}_{\{t-1<\tau\leq t\}} \delta_t \cdot e^{-\int_0^\tau r_u\,du}\right] e^{-\alpha t}$$

In this term, the indicator function equals 1 only if default occurs during the interval from $t-1$ to t. The recovery payment is discounted from the actual (random) time of default τ, and once again receives an illiquidity penalty based on the upper bound t. Note that the liquidity discount $e^{-\alpha t}$ applies even to recovery payments, reflecting that such payouts may also be hard to monetize quickly. These recovery payments are treated symmetrically in valuation terms, even if their magnitude and timing are uncertain.

Combining the three components, the full formula becomes:

$$D_T^\alpha(0) = \sum_{t=1}^{T} C \cdot \mathbb{E}\left[\mathbf{1}_{\{\tau>t\}} e^{-\int_0^t r_u\,du}\right] e^{-\alpha t} + \mathbb{E}\left[\mathbf{1}_{\{\tau>T\}} e^{-\int_0^T r_u\,du}\right] e^{-\alpha T} + \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}_{\{t-1<\tau\leq t\}} \delta_t \cdot e^{-\int_0^\tau r_u\,du}\right] e^{-\alpha t}$$

The exponential illiquidity factor $e^{-\alpha t}$ acts analogously to the time discounting factor $e^{-\int_0^t r_u\,du}$, but reflects non-time risks — specifically, friction in trading or selling the bond. Together, the two exponentials form a compound discount that lowers the present value of each future payment more than time alone would. This aligns with our earlier understanding that illiquidity imposes an implicit cost on the investor, requiring a higher yield (or equivalently, lowering the present price).

Finally, the formula may also include a specification of the risk-neutral default intensity:

$$\lambda_t(X_t) = \psi_t \cdot \tilde{\lambda}_t(X_t)$$

Here, $\tilde{\lambda}_t(X_t)$ is the real-world (physical measure) default intensity, which may depend on firm characteristics X_t, while ψ_t captures the market price of credit risk. The product λ_t is the risk-neutral intensity used in expectation calculations under the measure $\mathbb{Q}$. This distinction is essential when converting from real-world probabilities to pricing-relevant expectations.

For later use, we note that the risk-neutral probabilities of default satisfy the following relations:

$$\mathbb{Q}(\tau = t) = E\left(\lambda_t(X_t)e^{-\int_0^t \lambda_s(X_s)\,ds}\right)$$

and

$$\mathbb{Q}(\tau > t) = E\left(e^{-\int_0^t \lambda_s(X_s)\,ds}\right)$$

We consider a random time of default τ and seek to understand the probability of default occurring at a specific time t. The central quantity of interest is the cumulative default probability, defined as:

$$F(t) := \mathbb{Q}(\tau \leq t)$$

which gives the probability that default has occurred by time t. The probability density function of τ is then the derivative of this cumulative function:

$$f(t) := \frac{d}{dt}F(t) = \mathbb{Q}(\tau = t)$$

This result follows directly from the **Fundamental Theorem of Calculus (FTC)**, which consists of two parts.

Part I states that if a function $F(x)$ is defined as the integral of a continuous function $f(x)$ from a constant lower bound a to a variable upper bound x, then:

$$F(x) = \int_a^x f(x)\,dx \quad \Rightarrow \quad \frac{d}{dx}F(x) = f(x)$$

Part II states that if $f(x)$ is continuous on the interval $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$ (that is, $F'(x) = f(x)$), then the definite integral of $f(x)$ from a to b is given by:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This theorem provides a bridge between differentiation and integration: it tells us that the derivative of the area under a curve (i.e., the integral) up to t is simply the height of the curve at t.

In our context, if we assume that τ has a well-defined density function $f(t)$, then:

$$F(t) = \mathbb{Q}(\tau \leq t) = \int_0^t f(s)\,ds$$

and so:

$$\frac{d}{dt}\mathbb{Q}(\tau \leq t) = f(t)$$

However, we are not given $f(t)$ directly. Instead, we often work with the *survival function*:

$$\mathbb{Q}(\tau > t)$$

which is related by:

$$\mathbb{Q}(\tau \leq t) = 1 - \mathbb{Q}(\tau > t)$$

and so:

$$\frac{d}{dt}\mathbb{Q}(\tau \leq t) = -\frac{d}{dt}\mathbb{Q}(\tau > t)$$

In intensity-based credit models, where the default intensity is stochastic and given by $\lambda_t(X_t)$ for some underlying process X_t, the survival function is:

$$\mathbb{Q}(\tau > t) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s(X_s)\, ds\right)\right]$$

This formula may appear abstract at first, so we take time to unpack and derive its meaning.

Imagine that time is divided into very small intervals, say $[s, s + ds]$. At each moment in time, there is a small chance the firm defaults, and that probability is approximately given by $\lambda_s(X_s)\, ds$, where $\lambda_s(X_s)$ is called the *intensity process* or *hazard rate*. This is a random process adapted to some underlying state variable X_s, such as a firm's financial health or macroeconomic condition.

Surviving until time t means not defaulting at any point in the interval $[0, t]$. In the simple case where $\lambda_s(X_s) = \lambda$ is constant, the probability of survival in each short interval is about $1 - \lambda\, ds$, and the total survival probability is the product of all these infinitesimal survival chances:

$$\prod_{s=0}^{t}(1 - \lambda\, ds)$$

Note: We multiply by ds because a single point in continuous time carries no measure and therefore contributes no probability mass on its own. The term $(1 - \lambda\, ds)$ represents the infinitesimal probability of surviving over a small time interval of length ds, and can be interpreted as the area of a narrow rectangle beneath the hazard rate curve. By taking the product of these infinitesimal survival probabilities over the interval $[0, t]$, we approximate the overall survival probability as the continuous-time limit of this product, capturing the cumulative effect of survival over time.

Taking the limit as the interval size $ds \to 0$, this product becomes an exponential:

$$\lim_{n\to\infty} \prod_{i=1}^{n} \left(1 - \lambda \cdot \frac{t}{n}\right) = e^{-\lambda t}$$

Proof

We aim to evaluate the limit:

$$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n$$

Define the function:

$$f(n) := \left(1 - \frac{x}{n}\right)^n$$

To simplify, take natural logarithms:

$$\ln f(n) = n \cdot \ln\left(1 - \frac{x}{n}\right)$$

Using the Taylor expansion for $\ln(1 - y)$ around $y = 0$:

$$\ln(1 - y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \cdots$$

Substitute $y = \frac{x}{n}$:

$$\ln f(n) = n \cdot \left(-\frac{x}{n} - \frac{x^2}{2n^2} - \frac{x^3}{3n^3} - \cdots \right) = -x - \frac{x^2}{2n} - \frac{x^3}{3n^2} - \cdots$$

Taking the limit as $n \to \infty$, all higher-order terms vanish:

$$\lim_{n\to\infty} \ln f(n) = -x \Rightarrow \lim_{n\to\infty} f(n) = e^{-x}$$

Therefore:

$$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}$$

This result implies:

$$\lim_{n\to\infty} \prod_{i=1}^{n} \left(1 - \lambda \cdot \frac{t}{n}\right) = \left(1 - \frac{\lambda t}{n}\right)^n \xrightarrow{n\to\infty} e^{-\lambda t}$$

$\square$

For constant hazard rate λ, the survival function is:

$$\mathbb{Q}(\tau > t) = e^{-\lambda t}$$

This is the well-known exponential survival distribution. When the hazard rate varies with time, but is still deterministic, the formula generalizes to:

$$\mathbb{Q}(\tau > t) = \exp\left(-\int_0^t \lambda(s)\, ds\right)$$

This accounts for varying levels of risk at different times. But in a stochastic model, where $\lambda_s = \lambda_s(X_s)$ depends on an underlying random process X_s, the survival probability becomes a random variable. To obtain an actual number for the probability of survival, we take the expected value over all paths of X_s:

$$\mathbb{Q}(\tau > t) = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s(X_s)\, ds\right)\right]$$

This is the survival probability up to time t. It is the expected value of the exponential of the negative cumulative hazard. We define:

$$F(t) := \mathbb{Q}(\tau \leq t) = 1 - \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s(X_s)\, ds\right)\right]$$

To compute the density, we differentiate $F(t)$:

$$\frac{d}{dt}F(t) = \frac{d}{dt}\left(1 - \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s(X_s)\, ds\right)\right]\right)$$

We now apply the chain rule. Let:

$$Z(t) := \exp\left(-\int_0^t \lambda_s(X_s)\, ds\right)$$

Then:

$$\frac{d}{dt}Z(t) = \frac{d}{dt}\left(e^{-u(t)}\right)$$

where $u(t) = \int_0^t \lambda_s(X_s)\, ds$.

Using the chain rule:

$$\frac{d}{dt}Z(t) = -\frac{d}{dt}u(t) \cdot e^{-u(t)} = -\lambda_t(X_t) \cdot \exp\left(-\int_0^t \lambda_s(X_s)\, ds\right)$$

Therefore:

$$\frac{d}{dt}\mathbb{Q}(\tau > t) = \frac{d}{dt}\mathbb{E}[Z(t)] = \mathbb{E}\left[\frac{d}{dt}Z(t)\right] = -\mathbb{E}\left[\lambda_t(X_t)\cdot\exp\left(-\int_0^t \lambda_s(X_s)\,ds\right)\right]$$

Finally, we obtain:

$$\frac{d}{dt}\mathbb{Q}(\tau \leq t) = \mathbb{E}\left[\lambda_t(X_t)\cdot\exp\left(-\int_0^t \lambda_s(X_s)\,ds\right)\right]$$

This is the probability density of default occurring at time t: the firm must survive up to time t, and then default at that exact instant with intensity $\lambda_t(X_t)$.

for $t \in (0, T]$. It is easily seen that the one-year risk-neutral default probability is not approximately equal to $\lambda_0(X_0)$, i.e.,

$$\mathbb{Q}(\tau \leq 1) = 1 - E\left(e^{-\int_0^1 \lambda_t(X_t)\,dt}\right) \neq 1 - e^{-\lambda_0(X_0)} \approx \lambda_0(X_0),$$

because the time interval of a year is too long, given the randomness in the state variables, for $E\left(e^{-\int_0^1 \lambda_t(X_t)\,dt}\right) \approx e^{-\lambda_0(X_0)\cdot 1}$.

To simplify expression (1), we add the following assumption.

To understand why the approximation $1 - e^{-\lambda_0(X_0)} \approx \lambda_0(X_0)$ is commonly used, we begin by recalling the Taylor expansion of the exponential function e^{-x} around $x = 0$:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

Subtracting both sides from 1 gives:

$$1 - e^{-x} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots$$

When x is small, the higher-order terms involving x^2, x^3, and beyond become negligible, so we may write:

$$1 - e^{-x} \approx x$$

Applying this with $x = \lambda_0(X_0)$ yields:

$$1 - e^{-\lambda_0(X_0)} \approx \lambda_0(X_0)$$

This approximation is the reason one might be tempted to write the one-year cumulative risk-neutral default probability as:

$$\mathbb{Q}(\tau \leq 1) = 1 - \mathbb{E}\left(e^{-\int_0^1 \lambda_t(X_t)dt}\right) \approx 1 - e^{-\lambda_0(X_0)} \approx \lambda_0(X_0)$$

The temptation arises because if $\lambda_t(X_t)$ does not vary much over $t \in [0, 1]$, then the integral $\int_0^1 \lambda_t(X_t)dt$ is approximately equal to $\lambda_0(X_0)\cdot 1$, and therefore:

$$\exp\left(-\int_0^1 \lambda_t(X_t)dt\right) \approx \exp\left(-\lambda_0(X_0)\right)$$

Taking expectations then gives:

$$\mathbb{E}\left(e^{-\int_0^1 \lambda_t(X_t)dt}\right) \approx e^{-\lambda_0(X_0)} \quad \Rightarrow \quad \mathbb{Q}(\tau \leq 1) \approx 1 - e^{-\lambda_0(X_0)} \approx \lambda_0(X_0)$$

However, this approximation is only valid under the assumption that the hazard rate remains nearly constant over the year. In reality, $\lambda_t(X_t)$ is a stochastic process and varies over time. Because of this randomness, the integral $\int_0^1 \lambda_t(X_t)dt$ becomes a nontrivial random variable, and the exponential of its negative is also random. Therefore, the approximation:

$$\mathbb{Q}(\tau \leq 1) = 1 - \mathbb{E}\left(e^{-\int_0^1 \lambda_t(X_t)dt}\right) \not\approx 1 - e^{-\lambda_0(X_0)} \approx \lambda_0(X_0)$$

breaks down. The failure occurs because the interval of one year is too long for the approximation to hold; too much variation in the hazard process accumulates.

> (Conditional independence) The default-free spot rate r_t, the default time τ, and the recovery rate process δ_t are conditionally independent under the risk-neutral probabilities given the information at time $t \in [0, T]$.
>
> This is a weak assumption on the default-free spot rate, the default time, and the recovery rate because it imposes very little structure on their evolutions under the statistical probabilities, as distinct from the risk-neutral probabilities. First, conditioned on the information at time t, a large default intensity could imply a lower recovery rate under both $\mathbb{P}$ and $\mathbb{Q}$. Second, under the statistical probabilities, these processes need not be conditionally independent over $(t, T]$ due to default and recovery rate risk premiums. This implies that the default probability and recovery rate (conditioned on the information at time t) under $\mathbb{P}$, could be negatively correlated over $(t, T]$, which has been observed in the historical time series data.
>
> This assumption implies that the expectations of the products in expression (1) can be separated as follows:
>
> $$\begin{aligned} D_T^{\alpha}(0) = & \sum_{t=1}^{T} e^{-\alpha t} C \mathbb{Q}(\tau > t) p(0, t) + e^{-\alpha T} \mathbb{Q}(\tau > T) p(0, T) \\ & + d_0 \int_0^T \mathbb{Q}(\tau = s) e^{-\alpha s} p(0, s) ds, \end{aligned} \tag{2}$$
>
> where
>
> $$d_0 = E[\delta_\tau]$$
>
> is the risk-neutral expected recovery rate at default.

The conditional independence assumption states that the default-free short rate process r_t, the default time τ, and the recovery rate process δ_t are conditionally independent under the risk-neutral measure $\mathbb{Q}$, given the information available at time $t \in [0, T]$. In practical terms, this means that once we condition on all observable information at time t (such as market states, firm-specific variables, etc.), the short rate, the timing of default, and the recovery rate evolve independently. Knowing the value of the short rate at time t gives no additional information about when default will occur. Similarly, knowing the recovery rate tells us nothing about the short rate or the probability of default. This assumption only holds under $\mathbb{Q}$ and is used to simplify pricing mathematics. However, this assumption is considered weak because it ignores real-world interactions between these quantities. Under the statistical measure $\mathbb{P}$, default intensity and recovery rates are not independent. For example, a firm with high default risk often simultaneously has a low expected recovery rate — a negative correlation that has been repeatedly observed in historical time series. Moreover, investors typically demand separate risk premia for default and recovery, which further couples these processes. As a result, while the independence assumption is computationally convenient under $\mathbb{Q}$, it fails to capture essential economic relationships under $\mathbb{P}$.

We begin with the full risky bond valuation formula under the risk-neutral measure $\mathbb{Q}$:

$$
\begin{aligned}
D_T^\alpha(0) = & \sum_{t=1}^{T} C \cdot \mathbb{E}^\mathbb{Q}\left[\mathbf{1}_{\{\tau>t\}} \cdot e^{-\int_0^t r_u\,du}\right] e^{-\alpha t} + \mathbb{E}^\mathbb{Q}\left[\mathbf{1}_{\{\tau>T\}} \cdot e^{-\int_0^T r_u\,du}\right] e^{-\alpha T} \\
& + \sum_{t=1}^{T} \mathbb{E}^\mathbb{Q}\left[\mathbf{1}_{\{t-1<\tau\leq t\}} \cdot \delta_t \cdot e^{-\int_0^t r_u\,du}\right] e^{-\alpha t}
\end{aligned} \tag{1}
$$

We now simplify each term by applying conditional independence assumptions and standard properties of expectation.

Term 1: Expected discounted coupon conditional on survival at time t

We analyze:

$$\mathbb{E}^\mathbb{Q}\left[\mathbf{1}_{\{\tau>t\}} \cdot e^{-\int_0^t r_u\,du}\right]$$

This is an expectation of a product of two random variables:

- $\mathbf{1}_{\{\tau>t\}}$ depends on the default time τ,
- $e^{-\int_0^t r_u\,du}$ depends on the path of the short rate r_u up to time t.

Unless we assume that τ is conditionally independent of the interest rate process r_u under $\mathbb{Q}$, we cannot factor this expectation. Under the independence assumption, however, we apply:

$$\mathbb{E}^\mathbb{Q}[A \cdot B] = \mathbb{E}^\mathbb{Q}[A] \cdot \mathbb{E}^\mathbb{Q}[B]$$

So we get:

$$\mathbb{E}^\mathbb{Q}\left[\mathbf{1}_{\{\tau>t\}} \cdot e^{-\int_0^t r_u\,du}\right] = \mathbb{Q}(\tau > t) \cdot p(0,t)$$

Substituting this into Equation (1), the coupon term becomes:

$$\sum_{t=1}^{T} e^{-\alpha t} C \cdot \mathbb{Q}(\tau > t) \cdot p(0,t)$$

Term 2: Discounted principal paid at T if the bond survives

We examine:

$$\mathbb{E}^\mathbb{Q}\left[\mathbf{1}_{\{\tau>T\}} \cdot e^{-\int_0^T r_u\,du}\right]$$

This is structurally identical to the first term, with $t = T$. Applying the same reasoning under the conditional independence assumption:

$$\mathbb{E}^\mathbb{Q}\left[\mathbf{1}_{\{\tau>T\}} \cdot e^{-\int_0^T r_u\,du}\right] = \mathbb{Q}(\tau > T) \cdot p(0,T)$$

The corresponding simplified term becomes:

$$e^{-\alpha T} \cdot \mathbb{Q}(\tau > T) \cdot p(0,T)$$

Term 3: Discounted expected recovery if default occurs during year t

We now examine the third term from Equation (1), which represents the recovery payment if default occurs during the interval $(t-1, t]$:

$$\sum_{t=1}^{T} \mathbb{E}^\mathbb{Q}\left[\mathbf{1}_{\{t-1<\tau\leq t\}} \cdot \delta_t \cdot e^{-\int_0^t r_u\,du}\right] e^{-\alpha t}$$

This term consists of the product of three components: The indicator $\mathbf{1}_{\{t-1<\tau\leq t\}}$ flags whether the default event, represented by the random variable τ, occurs in the year $(t-1,t]$. The recovery amount δ_t is a random variable representing the recovery value associated with year t, and it is applied whenever default is observed in that interval. The exponential $e^{-\int_0^t r_u\,du}$ discounts the recovery back to time 0, assuming the payment is made at the end of year t.

To simplify this expression, we assume that δ_t, the indicator $\mathbf{1}_{\{t-1<\tau\leq t\}}$, and the interest rate path $\{r_u\}_{u\leq t}$ are conditionally independent under the risk-neutral measure $\mathbb{Q}$. This allows us to factor the expectation:

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{t-1<\tau\leq t\}}\cdot\delta_t\cdot e^{-\int_0^t r_u\,du}\right]=\mathbb{Q}(t-1<\tau\leq t)\cdot\mathbb{E}^{\mathbb{Q}}[\delta_t]\cdot p(0,t)$$

Letting $d_0:=\mathbb{E}^{\mathbb{Q}}[\delta_t]$, the total contribution from recovery becomes:

$$\sum_{t=1}^{T} d_0\cdot\mathbb{Q}(t-1<\tau\leq t)\cdot p(0,t)\cdot e^{-\alpha t}$$

In the continuous-time limit, we replace the discrete sum with an integral over $s\in[0,T]$, assume the recovery process becomes continuous and time-homogeneous, and denote the constant expected recovery by $d_0:=\mathbb{E}^{\mathbb{Q}}[\delta_s]$. The term becomes:

$$d_0\cdot\int_0^T\mathbb{Q}(\tau=s)\cdot p(0,s)\cdot e^{-\alpha s}\,ds$$

This matches the third term in Equation (2), and represents the expected present value of recovery under default, using conditional independence to factor and simplify the expression. **Putting all three terms together, we obtain the simplified expression:**

$$\begin{aligned}D_T^{\alpha}(0)=&\sum_{t=1}^{T}e^{-\alpha t}C\cdot\mathbb{Q}(\tau>t)\cdot p(0,t)+e^{-\alpha T}\cdot\mathbb{Q}(\tau>T)\cdot p(0,T)\\&+d_0\cdot\int_0^T\mathbb{Q}(\tau=s)e^{-\alpha s}p(0,s)\,ds\end{aligned}\tag{2}$$

Each simplification critically relies on conditional independence. Without those assumptions, one cannot break the expectations apart, and a full joint model of (τ,δ_t,r_t) would be necessary — an intractable task in practice.

In general, since $\alpha>0$, the bond's value is decreased as the liquidity premium α increases, i.e., $D_T^{\alpha}<D_T^{0}$.

This expression can be further simplified using the mean-value theorem for integrals. This implies that there exists an $\hat{s}\in[0,T]$ such that

$$\int_0^T\mathbb{Q}(\tau=s)e^{-\alpha s}p(0,s)\,ds=p(0,\hat{s})e^{-\alpha\hat{s}}\int_0^T\mathbb{Q}(\tau=s)\,ds,$$

where $\int_0^T\mathbb{Q}(\tau=s)\,ds=\mathbb{Q}(\tau\leq T)$. Hence, we have the alternative expression

$$\begin{aligned}D_T^{\alpha}(0)=&\sum_{t=1}^{T}e^{-\alpha t}C\mathbb{Q}(\tau>t)p(0,t)+e^{-\alpha T}\mathbb{Q}(\tau>T)p(0,T)\\&+d_0p(0,\hat{s})e^{-\alpha\hat{s}}\mathbb{Q}(\tau\leq T).\end{aligned}\tag{3}$$

The risky bond's yield Y_T, which depends on its maturity T, is defined by

$$D_T^{\alpha}(0)=\sum_{t=1}^{T}Ce^{-Y_T t}+1e^{-Y_T T}.$$

The Generalized Mean Value Theorem for Integrals

$$\int_a^b f(s)w(s)\,ds = f(\hat{s})\int_a^b w(s)\,ds \quad \text{for some } \hat{s} \in [a,b].$$

This is a weighted version of the Mean Value Theorem (MVT) for integrals. To fully understand its meaning and justification, we rebuild the result from first principles.

The standard MVT for integrals states that if $f(s)$ is continuous on $[a,b]$, then:

$$\int_a^b f(s)\,ds = f(\hat{s})(b-a) \quad \text{for some } \hat{s} \in [a,b].$$

This result asserts that the area under the curve $f(s)$ is equal to the value of the function at some intermediate point $\hat{s}$, multiplied by the width of the interval. Even if the function is not constant, the total area behaves as if the function had been constant at the average value $f(\hat{s})$.

We now generalize the result to include a weighting function. Let $w(s) \geq 0$ be an integrable function, and let $f(s)$ be continuous on $[a,b]$. Then:

$$\int_a^b f(s)w(s)\,ds = f(\hat{s})\int_a^b w(s)\,ds \quad \text{for some } \hat{s} \in [a,b].$$

This is the weighted version of the MVT. It states that there exists a point $\hat{s}$ in the interval $[a,b]$ where the function $f(s)$ achieves its average value under the weighting $w(s)$. The total weighted area under the curve is thus equal to the value of the function at that representative point multiplied by the total weight.

Proof of the Generalized Mean Value Theorem for Integrals

Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and let $w : [a,b] \to \mathbb{R}$ be an integrable, non-negative function such that $w(s) \geq 0$ for all $s \in [a,b]$, and $w \not\equiv 0$. Then there exists a point $\hat{s} \in [a,b]$ such that:

$$\int_a^b f(s)w(s)\,ds = f(\hat{s})\int_a^b w(s)\,ds.$$

The condition $w \not\equiv 0$ means that $w(s)$ is *not identically zero* on the interval $[a,b]$; that is, there exists some measurable subset of $[a,b]$ where $w(s) > 0$. A *measurable subset* means a region for which the size (or length) is well-defined in the sense of Lebesgue measure. This ensures that the total weight is strictly positive:

$$\int_a^b w(s)\,ds > 0.$$

Importantly, this does not require $w(s) > 0$ everywhere. It only requires that there exists some region of nonzero weight — i.e., a part of the interval $[a,b]$ where $w(s) > 0$ — and that this region has positive measure (i.e., it's not just a single point, which would have zero measure). This condition guarantees that the denominator in the weighted average is nonzero and the theorem is well-defined.

$$\int_a^b w(s)\,ds > 0.$$

Since f is continuous on the closed interval $[a,b]$, and $[a,b]$ is a compact set in $\mathbb{R}$, the Extreme Value Theorem guarantees that f attains its minimum and maximum values on this interval. (A set is *compact* if it is both closed and bounded — that is, it includes its endpoints a and b, and lies entirely between two finite numbers.)

Let:

$$A := \min_{s\in[a,b]} f(s), \qquad B := \max_{s\in[a,b]} f(s).$$

Then for all $s \in [a,b]$, we have:

$$A \le f(s) \le B.$$

Multiplying by $w(s) \ge 0$, we obtain:

$$Aw(s) \le f(s)w(s) \le Bw(s).$$

Integrating both sides over $[a,b]$:

$$\int_a^b Aw(s)\,ds \le \int_a^b f(s)w(s)\,ds \le \int_a^b Bw(s)\,ds.$$

This simplifies to:

$$A \cdot \int_a^b w(s)\,ds \le \int_a^b f(s)w(s)\,ds \le B \cdot \int_a^b w(s)\,ds.$$

Define:

$$I := \int_a^b w(s)\,ds > 0, \qquad F := \int_a^b f(s)w(s)\,ds.$$

Then:

$$A \le \frac{F}{I} \le B.$$

Since f is continuous on $[a,b]$, the Intermediate Value Theorem guarantees that f takes on every value between A and B. In particular, there exists $\hat{s} \in [a,b]$ such that:

$$f(\hat{s}) = \frac{F}{I}.$$

Rewriting:

$$\int_a^b f(s)w(s)\,ds = f(\hat{s}) \int_a^b w(s)\,ds.$$

$\square$

Application of the Generalized Mean Value Theorem for Integrals

In the context of bond valuation, consider the integral:

$$\int_0^T \mathbb{Q}(\tau = s) e^{-\alpha s} p(0,s)\,ds.$$

This arises in the pricing of a risky bond, where $\mathbb{Q}(\tau = s)$ is the risk-neutral default density, $e^{-\alpha s}$ is the illiquidity discount factor, and $p(0,s)$ is the time-zero price of a zero-coupon bond maturing at time s.

Define:

$$w(s) := \mathbb{Q}(\tau = s), \quad f(s) := e^{-\alpha s} p(0,s).$$

Applying the weighted MVT, we obtain:

$$\int_0^T \mathbb{Q}(\tau = s) e^{-\alpha s} p(0,s)\,ds = p(0,\hat{s}) e^{-\alpha \hat{s}} \int_0^T \mathbb{Q}(\tau = s)\,ds.$$

But the integral on the right-hand side is simply:

$$\int_0^T \mathbb{Q}(\tau = s)\,ds = \mathbb{Q}(\tau \le T),$$

which represents the total risk-neutral probability that default occurs on or before time T. Therefore, the final simplification is:

$$\int_0^T \mathbb{Q}(\tau = s) e^{-\alpha s} p(0,s)\,ds = p(0,\hat{s}) e^{-\alpha \hat{s}} \cdot \mathbb{Q}(\tau \le T).$$

Interpreting the Yield-to-Maturity Equation

The expression

$$D^{\alpha}_T(0) = \sum_{t=1}^{T} Ce^{-Y_T t} + 1 \cdot e^{-Y_T T}$$

defines the risky bond's yield-to-maturity Y_T. This formula assumes that all cash flows are paid with certainty, i.e., there is no default. The bond is priced as though it were risk-free, and the yield Y_T serves as the discount rate that equates the present value of those deterministic cash flows to the observed bond price. In contrast, the full model-based expression for $D^{\alpha}_T(0)$ incorporates survival probabilities, recovery in the event of default, and a liquidity premium α. It includes terms such as $\mathbb{Q}(\tau > t)$, $p(0,t)$, and expected recovery. This model produces a bond price that reflects all sources of risk and discounting. The connection between these two expressions is not a derivation, but a definition. Once the model-based bond price $D^{\alpha}_T(0)$ is computed, we *define* the yield Y_T as the unique number that solves the simplified, default-free pricing equation above. This process is often referred to as "reverse engineering the yield," because we are asking: what discount rate Y_T would explain the model-implied bond price if the bond were default-free? The purpose of this definition is to summarize the effects of credit risk, recovery, and liquidity into a single interpretable statistic. The yield Y_T tells us what constant rate of return the bond offers under the illusion of certainty.

The Credit Spread

This section characterizes the risky coupon bond's credit spread. The risky bond's credit spread is defined by

$$\kappa_T := Y_T - y_T(c).$$

The credit spread is defined relative to a default-free coupon bond with the same maturity as the risky coupon bond. However, as the notation makes explicit, the coupon rates will typically be different between the default-free (c) and risky bonds (C).

This definition implies the following expression is satisfied:

$$D^{\alpha}_T(0) = \sum_{t=1}^{T} Ce^{-(y_T(c)+\kappa_T)t} + 1e^{-(y_T(c)+\kappa_T)T}. \tag{4}$$

We define the *credit spread* κ_T of a risky bond as

$$\kappa_T := Y_T - y_T(c),$$

where Y_T denotes the yield on the risky bond, and $y_T(c)$ denotes the yield on a default-free bond with identical maturity T and coupon rate c. The quantity κ_T reflects the additional compensation investors require for bearing credit risk. The risky bond pays a coupon C, while the default-free comparison bond pays coupon c. In general, C and c may differ — for example, riskier bonds may offer higher coupons to attract investors — but they need not be different. The spread κ_T is well-defined regardless of whether $C = c$ or not.

Given this definition, the pricing identity becomes

$$D^{\alpha}_T(0) = \sum_{t=1}^{T} Ce^{-(y_T(c)+\kappa_T)t} + e^{-(y_T(c)+\kappa_T)T}.$$

This is not a modeling assumption, but a pricing equivalence: the left-hand side $D^{\alpha}_T(0)$ reflects the model-based price of the risky bond, incorporating credit and liquidity risk. The right-hand side expresses the same value using a default-free bond pricing formula, but discounted at the adjusted yield $y_T(c) + \kappa_T$.

Remark 1. (Different credit spreads) The above definition matches the maturity of the risky and default-free coupon bonds. In practice and in some of the empirical literature, the credit spread κ_T^* is defined relative to a default-free coupon bond of close but unequal maturity, e.g.,

$$D_T^{\alpha}(0) = \sum_{t=1}^{T} Ce^{-(y_{10}(c)+\kappa_T^*)t} + 1e^{-(y_{10}(c)+\kappa_T^*)T},$$

where the 10-year default-free coupon bond yield $y_{10}(c)$ is used, for example, for all $T \geq 7$ maturity risky coupon bonds. The following analysis extends to any such alternative definition.

Sometimes in the literature, the credit spread is defined relative to risky and default-free zero-coupon bonds (e.g., Bai *et al.*, 2020; Elton *et al.*, 2001; Feldhutter and Schaefer, 2018). This introduces additional error in the empirical methodology because zero-coupon bond credit spreads are unobservable and must be estimated from coupon bond prices. This completes the remark.

Remark 1 discusses alternative ways to define the credit spread κ_T. In particular, it focuses on definitions that use a fixed-maturity benchmark rather than a maturity-matched default-free bond. The displayed equation expresses the price of a risky coupon bond, $D_T^{\alpha}(0)$, using a discount rate of the form $y_{10}(c) + \kappa_T^*$. Here, $y_{10}(c)$ is the yield on a 10-year default-free coupon bond with coupon rate c, and κ_T^* is the credit spread defined relative to this fixed reference point. This differs from the usual approach, which compares bonds of the same maturity. Instead, the 10-year yield is applied to all risky bonds with maturity $T \geq 7$. The spread κ_T^* adjusts this common yield to match the observed price of each risky bond. This method is often used in empirical studies because full yield curves are not always available. Fixing the reference yield simplifies comparisons across different bonds. It is important to note that $y_{10}(c)$ is not directly observed. It is inferred from market prices by solving the standard pricing equation for a 10-year Treasury bond, namely

$$P_{10}(c) = \sum_{t=1}^{10} c \cdot e^{-y_{10}(c)t} + e^{-y_{10}(c)\cdot 10}.$$

.

The quantity $y_T(c)$ is not a closed-form inversion of the bond pricing equation. In practice, solving for $y_T(c)$ typically requires a numerical method such as Newton–Raphson or bisection, a financial calculator or yield solver, or a built-in function in code libraries (e.g., `np.irr()` or `bond_yield_to_maturity()`). As a result, both the yield $y_{10}(c)$ and the resulting spread κ_T^* are estimated quantities. The final paragraph points out another approach used in the literature: defining credit spreads using zero-coupon bonds instead of coupon bonds. However, this introduces additional error. Zero-coupon bond prices and spreads are not directly observable and must be estimated from coupon bond prices using bootstrapping techniques.

A simple case

To illustrate the intuition underlying the characterization of a bond's credit spread in the general model, we first characterize this spread in a simple model. This simple model is the basis for many of the common misunderstandings about the decomposition of the credit spread. For the simple model, we assume that the risk-neutral default intensity and recovery rate are constants, i.e., $\lambda_t = \lambda > 0$ and $\delta_t = \delta \in [0, 1)$.

In addition, we assume that the default-free term structure of interest rates is deterministic and flat so that the default-free zero-coupon and default-free coupon bond yields are equal, i.e.,

$$r_0 = R_t = y_T(c)$$

for all $t \in [0, T]$, which implies that the default-free zero-coupon bond's price is

$$p(0, t) = e^{-r_0 t}$$

for all $t \in [0, T]$.

It may seem puzzling that a zero-coupon bond and a coupon-paying bond can have the same yield, given that their cash flow structures are so different. One pays nothing until maturity; the other makes periodic payments along the way. How can their effective returns be equal? The answer is that the bond with earlier payments also has a higher price. The bond that defers all cash flows is cheaper. When both are priced using the same discount rate—under a flat, default-free term structure—the differences in timing are exactly offset by differences in price. As a result, the investor earns the same yield to maturity on both instruments, despite their structural differences.

Under these simplifying assumptions,

$$D_T^\alpha(0) = \sum_{t=1}^{T} Ce^{-\lambda t - \alpha t} p(0, t) + 1e^{-\lambda T - \alpha T} p(0, T) + \delta p(0, \hat{s}) e^{-\alpha \hat{s}}(1 - e^{-\lambda T}),$$

We begin with the general expression for the illiquidity-adjusted price of a risky bond under the risk-neutral measure $\mathbb{Q}$ found on page 21:

$$D_T^\alpha(0) = \sum_{t=1}^{T} e^{-\alpha t} C\, \mathbb{Q}(\tau > t)\, p(0,t) + e^{-\alpha T}\, \mathbb{Q}(\tau > T)\, p(0,T) + d_0\, p(0,\hat{s})\, e^{-\alpha \hat{s}}\, \mathbb{Q}(\tau \le T).$$

Using the result found on page 16:

$$\mathbb{Q}(\tau > t) = e^{-\lambda t}.$$

Consequently, $\mathbb{Q}(\tau > T) = e^{-\lambda T}$ and $\mathbb{Q}(\tau \leq T) = 1 - e^{-\lambda T}$. We also assume that the risk-free interest rate is constant, $r_u = r_0$, for all $u \in [0, T]$, which implies that the risk-free discount factor is given by:

$$p(0,t) = e^{-\int_0^t r_u du} = e^{-r_0 t}.$$

Finally, we assume that the recovery rate is constant, so that $d_0 = \delta \in [0,1]$. We now apply these assumptions to each term of the general expression.

The coupon leg becomes:

$$\sum_{t=1}^{T} e^{-\alpha t} C \cdot \mathbb{Q}(\tau > t) \cdot p(0,t) = \sum_{t=1}^{T} C e^{-\lambda t - \alpha t} p(0,t).$$

The principal repayment term simplifies to:

$$e^{-\alpha T} \cdot \mathbb{Q}(\tau > T) \cdot p(0,T) = e^{-\lambda T - \alpha T} p(0,T).$$

The recovery term becomes:

$$d_0 \cdot p(0,\hat{s}) \cdot e^{-\alpha \hat{s}} \cdot \mathbb{Q}(\tau \leq T) = \delta p(0,\hat{s}) e^{-\alpha \hat{s}} (1 - e^{-\lambda T}).$$

Combining all terms, we obtain the final expression:

$$D_T^{\alpha}(0) = \sum_{t=1}^{T} C e^{-\lambda t - \alpha t} p(0,t) + e^{-\lambda T - \alpha T} p(0,T) + \delta p(0,\hat{s}) e^{-\alpha \hat{s}} (1 - e^{-\lambda T}).$$

and using the definition of the credit spread,

$$\sum_{t=1}^{T} C e^{-\kappa_T t} p(0,t) + 1 e^{-\kappa_T T} p(0,T)$$
$$= \sum_{t=1}^{T} C e^{-\lambda t - \alpha t} p(0,t) + 1 e^{-\lambda T - \alpha T} p(0,T) + \delta p(0,\hat{s}) e^{-\alpha \hat{s}} (1 - e^{-\lambda T}).$$

We define κ_T as the spread between the yield on a risky bond and the risk-free rate r_0.

$$\kappa_T = \lambda + \alpha$$

This spread reflects two components:

$$\text{default risk } (\lambda) \quad \text{and} \quad \text{illiquidity risk } (\alpha)$$

The total yield on the risky bond is:

$$y_T(c) + \kappa_T = r_0 + \lambda + \alpha$$

Thus, κ_T is the excess over the risk-free rate r_0 required to compensate investors for bearing default and illiquidity risk.

This expression gives κ_T as an implicit function of the other parameters. To obtain an analytic solution, using the approximation $e^{-z} \approx 1 - z$, we can rewrite this expression as

$$\sum_{t=1}^{T} C(1-\kappa_T t)p(0,t) + (1-\kappa_T T)p(0,T)$$
$$\approx \sum_{t=1}^{T} C(1-(\lambda+\alpha)t)p(0,t) + (1-(\lambda+\alpha)T)p(0,T) + \delta e^{-\alpha\hat{s}}p(0,\hat{s})\lambda T.$$

The solution, for small values of $(\kappa_T, \alpha, \lambda)$, is

$$\kappa_T \approx \lambda(1-\delta) + \alpha + \lambda\delta\left(1 - \frac{e^{-\alpha\hat{s}}p(0,\hat{s})T}{\sum_{t=1}^{T} Ctp(0,t) + Tp(0,T)}\right). \qquad (5)$$

The exponential function is approximated via a Maclaurin series, which is the Taylor expansion evaluated at the origin $z = 0$:

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots$$

If z is small, then the higher-order terms $z^2, z^3, \ldots$ are negligible, and we can use the first-order approximation:

$$e^{-z} \approx 1 - z$$

For example, letting $z = \lambda t + \alpha t$, we obtain:

$$e^{-(\lambda+\alpha)t} \approx 1 - (\lambda+\alpha)t$$

This is the approximation used in the derivation of the credit spread formula. It allows us to linearize the bond pricing expression and explicitly solve for κ_T, which otherwise would be trapped inside exponentials.

The approximation and analytic derivation for κ_T, the credit spread, begins with the observation that κ_T appears implicitly in the following pricing expression for the risky bond:

$$\sum_{t=1}^{T} Ce^{-\lambda t-\alpha t}p(0,t) + e^{-\lambda T-\alpha T}p(0,T) + \delta p(0,\hat{s})e^{-\alpha\hat{s}}(1-e^{-\lambda T})$$

To obtain a tractable analytic solution, we apply the first-order approximation $e^{-z} \approx 1 - z$ to each exponential term in the expression above.

The coupon terms become:

$$Ce^{-\lambda t-\alpha t}p(0,t) \approx C(1-(\lambda+\alpha)t)p(0,t)$$

The principal term becomes:

$$e^{-\lambda T-\alpha T}p(0,T) \approx (1-(\lambda+\alpha)T)p(0,T)$$

The recovery term simplifies as follows:

$$\delta p(0,\hat{s})e^{-\alpha\hat{s}}(1-e^{-\lambda T}) \approx \delta p(0,\hat{s})e^{-\alpha\hat{s}}\lambda T$$

Combining all three, the bond price is approximately:

$$\sum_{t=1}^{T} C(1-(\lambda+\alpha)t)p(0,t) + (1-(\lambda+\alpha)T)p(0,T) + \delta e^{-\alpha\hat{s}}p(0,\hat{s})\lambda T$$

We now derive an explicit formula for κ_T. Begin with the linearized version of the bond price:

$$\sum_{t=1}^{T} C(1-\kappa_T t)p(0,t) + (1-\kappa_T T)p(0,T)$$

Expanding this expression yields:

$$\sum_{t=1}^{T} Cp(0,t) - \kappa_T \sum_{t=1}^{T} Ctp(0,t) + p(0,T) - \kappa_T Tp(0,T)$$

Group terms:

$$\left[\sum_{t=1}^{T} Cp(0,t) + p(0,T)\right] - \kappa_T \left[\sum_{t=1}^{T} Ctp(0,t) + Tp(0,T)\right]$$

Define:

$$A = \sum_{t=1}^{T} Cp(0,t) + p(0,T), \quad B = \sum_{t=1}^{T} Ctp(0,t) + Tp(0,T)$$

The bond price becomes:

$$P \approx A - \kappa_T B$$

Solving for κ_T gives:

$$\kappa_T = \frac{A-P}{B}$$

Now substitute in the expression for P derived above, which includes default and recovery:

$$P = A - (\lambda+\alpha)B + \delta e^{-\alpha\hat{s}}p(0,\hat{s})\lambda T$$

Plugging into the equation for κ_T:

$$\kappa_T = \frac{A - \left[A - (\lambda+\alpha)B + \delta e^{-\alpha\hat{s}}p(0,\hat{s})\lambda T\right]}{B} = \lambda + \alpha - \frac{\delta e^{-\alpha\hat{s}}p(0,\hat{s})\lambda T}{B}$$

Thus by factoring:

$$\kappa_T \approx \lambda(1-\delta) + \alpha + \lambda\delta\left(1 - \frac{e^{-\alpha\hat{s}}p(0,\hat{s})T}{\sum_{t=1}^{T} Ctp(0,t) + Tp(0,T)}\right)$$

The denominator $\sum_{t=1}^{T} Ctp(0,t) + Tp(0,T)$ acts as a present-value-weighted average maturity of the bond's cash flows. It reflects the average timing, in discounted terms, of when the cash is expected to be received. This naturally emerges when applying the linearized exponential approximation, as each discount term is now weighted by time t.

As seen, the credit spread can be decomposed into three terms. The first $\lambda(1-\delta)$ is the one-year, risk-neutral expected loss. Here, because the default intensity is nonrandom, $\int_0^1 \lambda du = \lambda$. The second is the liquidity premium, and the third is an adjustment for the timing of the various coupon and principal cash flows. Note that a risky zero-coupon bond with $C=0$ will still have a third term, adjusting for the fact that a recovery on the risky bond's principal is paid at the default time, which could occur earlier than time T.

The third term in the credit spread expression,

$$\lambda\delta\left(1-\frac{e^{-\alpha\hat{s}}p(0,\hat{s})T}{\sum_{t=1}^{T}Ctp(0,t)+Tp(0,T)}\right),$$

adjusts for the timing of the recovery. It accounts for the fact that, in the event of default, the recovery on the bond's principal is received at the time of default rather than at maturity. Since this recovery may occur earlier than time T, its present value is higher. This timing advantage makes the bond more valuable, and the third term reduces the spread accordingly to reflect this.

If default occurs earlier, then $\hat{s}$ is smaller, which means the discount factor $p(0,\hat{s})$ is larger due to less time decay. As a result, the numerator in the third term of the credit spread increases, causing the entire fraction to increase. Since this fraction is subtracted from 1, the expression $1-$ (fraction) decreases. Therefore, the third term becomes smaller, reducing the overall value of κ_T.

Note: The denominator in the third term, $\sum_{t=1}^{T}Ctp(0,t)+Tp(0,T)$, remains unchanged even when default occurs early. This is because it reflects the present-value-weighted average timing of all scheduled coupon and principal payments, under the assumption that the bond pays in full to maturity. It serves as a fixed benchmark against which the earlier recovery is compared. The effect of early default is captured only in the numerator, which reflects the timing and present value of recovery.