Question # 1:[8+4=12 marks]

Part I[4+4=8 marks]

Given $f(x) = xe^x$

a. Find nth degree Maclaurin's polynomial $P_n(x)$ of $f(x)$.

b. Find minimum degree of Maclaurin's polynomial to calculate $2e^2$ with an error of less than 10^{-6}.

$\left(Hint: \frac{n+3}{(n+1)!} < 1\right)$

11.5

a. Solution:

$f(x) = xe^x$

Maclaurin's polynomial $P_n(x)$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0)$$

$f(0) = 0 \times e^0 = 0$

$f'(x) = e^x + xe^x \Rightarrow f'(0) = e^0 + 0 \times e^0 = 1$

$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x \Rightarrow f''(0) = (2 \times e^0) + (0 \times e^0) = 2$

$f'''(x) = 2e^x + e^x + xe^x = 3e^x + xe^x \Rightarrow f'''(0) = 3$

$f^{(n)}(x) = ne^x + xe^x \Rightarrow f^{(4)}(0) = n$

4 + 3.5 =

Thus, Maclaurin's polynomial $P_n(x)$ of $f(x) = xe^x$ is

$$xe^x = 0 + x(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(3) + \cdots + \frac{x^n}{n!}(n)$$

$$xe^x = x + \frac{x^2}{1!} + \frac{x^3}{2!} + \cdots + \frac{x^n}{(n-1)!}$$

b. Solution:

For $2e^2$, we need to put the following value in $f(x) = xe^x$

$x = 2$, $a = 0$

Error $< 10^{-6}$

'n' is to be found

$R_n(x) \leq \frac{x^n}{n!} f^{(n)}(c)$ n+1 n+1

$\frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c)$

$R_n(x) \leq \frac{2^n}{n!}(ne^c + ce^c)$

$0 < c < 2$

We have $R_n(x) \le \frac{2^n}{n!}(ne^c + ce^c)$

c has the maximum value of 2

$$R_n(x) \le \frac{2^n}{n!}(n e^2 + 2e^2)$$

$$R_n(x) \le \frac{2^n}{n!} e^2 (n+2)$$

$$R_n(x) \le \frac{2^n e^2 (n+2)}{n!} < 10^{-6}$$

$$\frac{2^n (n+2)}{n!} < \frac{10^{-6}}{e^2}$$

Part II[4 marks]

Find a bound for the number of iterations of Bisection Method necessary to achieve an approximation with accuracy 5×10^{-4} to the solution of $x^3 + 2x^2 - x + 10 = 0$ lying in the interval [1,2].

Solution:

$f(x) = x^3 + 2x^2 - x + 10$

Accuracy $= 5 \times 10^{-4}$

Interval $= [1, 2]$

Thus, $a = 1$

$b = 2$

$f(1) = 12$

$f(2) = 24$

4

/1

Here, the function does not change sign over the interval. Thus, the choice of interval is wrong and the error bound ~~formula cannot~~ for Bisection Method cannot be applied in this interval.

An appropriate interval would have been $[-4, -3]$ where

$f(-4) = (-4)^3 + 2(-4)^2 + 4 + 10 = -18$

$f(-3) = (-3)^3 + 2(-3)^2 + 3 + 10 = 4$

Question # 2:[4+5=9 marks]

9

Given $f(x) = (x-2)^2 - lnx$

(a) Locate two interval(s) in which two root(s) of $f(x) = 0$ lie. Show by graphical method.

$f(x) = (x-2)^2 - \ln x$

$\downarrow$ $f_1(x)$ $\downarrow$ $f_2(x)$

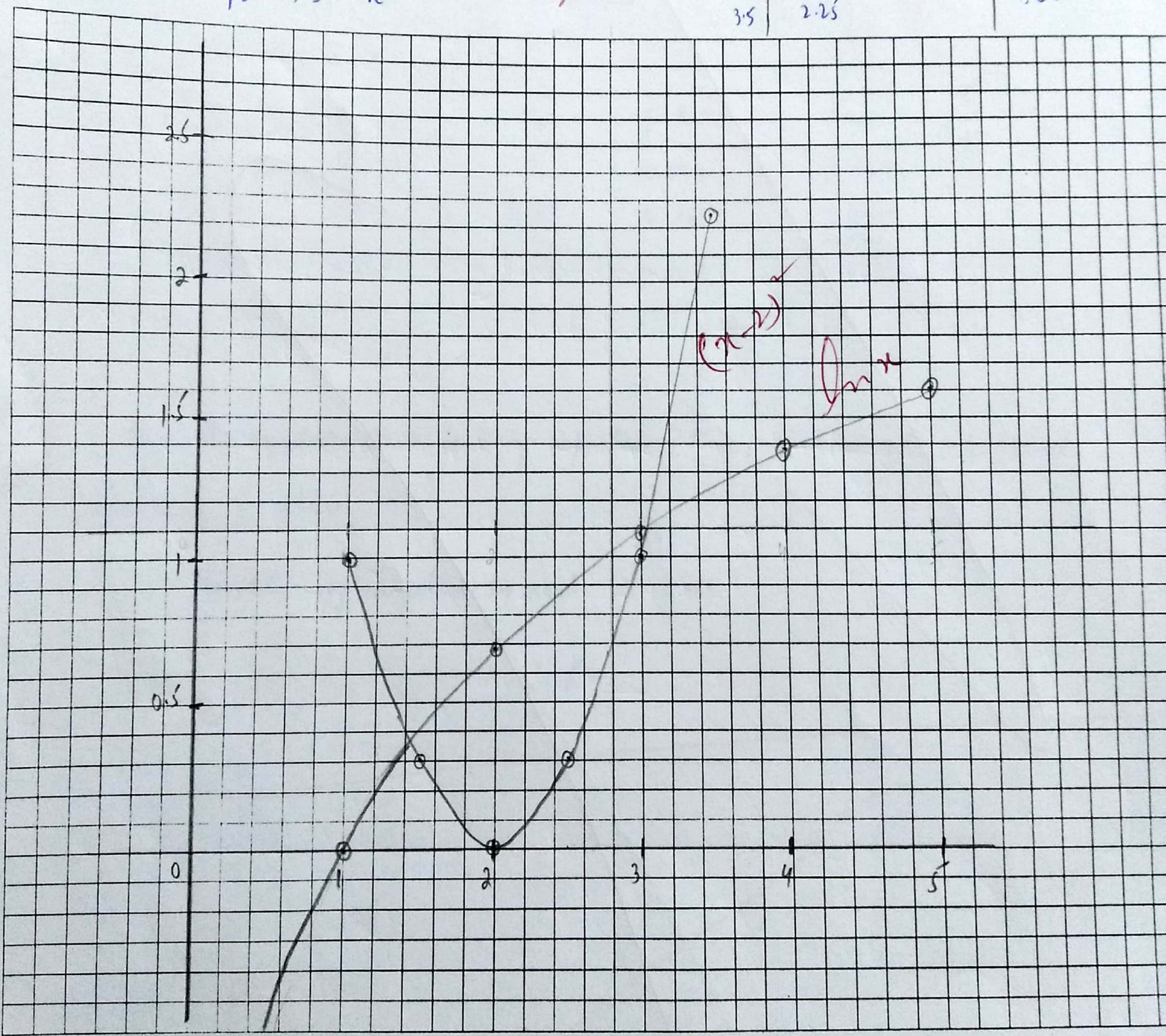
Thus, $f_1(x) = (x-2)^2$

$f_2(x) = \ln x$

4

x	$(x-2)^2$
1	1
1.5	0.25
2	0
2.5	0.25
3	1
3.5	2.25

x	$\ln x$
1	0
2	0.69
3	1.09
4	1.38
5	1.60



Thus, the two intervals are [1, 2] and [3, 4]

Scanned by CamScanner

(b) Use Secant method to find approximation to the root of $(x-2)^2 - \ln x = 0$ accurate to within 10^{-3}. Use initial guess $x_0 = 2$ and $x_1 = 3$.

Solution:

$f(x) = (x-2)^2 - \ln x$

Accuracy $= 10^{-3} = 0.001$

$x_0 = 2$, $x_1 = 3$

$$x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$$

I_n	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$	x_{n+2}
1	2	-0.693	3	-0.098	3.164
2	3	-0.098	3.164	0.203	3.053
3	3.164	0.203	3.053	-0.007	3.056

5

Since the approximation is being repeated, thus, we have got our desired accuracy (within 10^{-3})

Thus, the approximation to root is 3.056

$x = 3.056$

Question # 3:[9 marks]

Consider the following system of nonlinear equations.

$$3x - \cos(xy) - \frac{1}{2} = 0$$

$$e^{-xy} + 20x - 1 = 0$$

9

Use Newton Raphson method to calculate first approximation X_1 of the solution of given system with initial guess $X_0 = (x_0, y_0) = (1, -1)$. Perform calculations upto 3 decimal places.

Solution:

$$f_1(x,y) = 3x - \cos(xy) - \frac{1}{2}$$

$$f_2(x,y) = e^{-xy} + 20x - 1$$

$$X_0 = (x_0, y_0) = (1,-1) \quad \text{or} \quad X_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We have $\quad X_{n+1} = X_n - J^{-1}(X_n)\, F(X_n)$

$$F(X_n) = \begin{bmatrix} 3x - \cos(xy) - \frac{1}{2} \\ e^{-xy} + 20x - 1 \end{bmatrix}$$

$$J(X_n) = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x}\right|_{X_n} & \left.\frac{\partial f_1}{\partial y}\right|_{X_n} \\ \left.\frac{\partial f_2}{\partial x}\right|_{X_n} & \left.\frac{\partial f_2}{\partial y}\right|_{X_n} \end{bmatrix}$$

For X_1,

$$X_1 = X_0 - J^{-1}(X_0)\, F(X_0) \quad -①$$

So,

$$F(X_0) = \begin{bmatrix} 1.959 \\ 21.718 \end{bmatrix}$$

$$J(X_n) = \begin{bmatrix} 3 + y\sin(xy)|_{X_n} & x\sin(xy)|_{X_n} \\ -ye^{-xy} + 20|_{X_n} & -xe^{-xy}|_{X_n} \end{bmatrix}$$

$$J(X_0) = \begin{bmatrix} 3.841 & -0.841 \\ 22.718 & -2.718 \end{bmatrix} \quad , \quad |J(X_0)| = 8.666 \neq 0$$

$$J^{-1}(X_0) = \frac{1}{8.666} \begin{bmatrix} -2.718 & 0.841 \\ -22.718 & 3.841 \end{bmatrix}$$

$$= \begin{bmatrix} -0.313 & 0.097 \\ -2.621 & 0.443 \end{bmatrix}$$

Thus, putting values in ①

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -0.313 & 0.097 \\ -2.621 & 0.443 \end{bmatrix} \begin{bmatrix} 1.959 \\ 21.718 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1.493 \\ 4.486 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -0.493 \\ -5.486 \end{bmatrix}$$

Scanned by CamScanner