Numer. Math. (2001) 87: 701–735
Digital Object Identifier (DOI) 10.1007/s002110000202

Numerische
Mathematik

L^1-Stability and error estimates for approximate Hamilton-Jacobi solutions

Chi-Tien Lin[1], **Eitan Tadmor**[2]

[1] Department of Applied Mathematics, Providence University, Shalu 43301, Taiwan,
e-mail: ctlin@pu.edu.tw

[2] Department of Mathematics, UCLA, Los Angeles, CA 90095, USA,
e-mail: tadmor@math.ucla.edu

Received April 20, 1998 / Revised version received November 8, 1999 /
Published online August 24, 2000 –

Summary. We study the L^1-stability and error estimates of general approximate solutions for the Cauchy problem associated with multidimensional Hamilton-Jacobi (H-J) equations. For strictly convex Hamiltonians, we obtain *a priori* error estimates in terms of the truncation errors and the initial perturbation errors. We then demonstrate this general theory for two types of approximations: approximate solutions constructed by the vanishing viscosity method, and by Godunov-type finite difference methods. If we let ϵ denote the 'small scale' of such approximations (– the viscosity amplitude ϵ, the spatial grad-size Δx, etc.), then our L^1-error estimates are of $\mathcal{O}(\epsilon)$, and are sharper than the classical L^∞-results of order one half, $\mathcal{O}(\sqrt{\epsilon})$. The main building blocks of our theory are the notions of the semi-concave stability condition and L^1-measure of the truncation error. The whole theory could be viewed as a multidimensional extension of the Lip'-stability theory for one-dimensional nonlinear conservation laws developed by Tadmor et. al. [34,24,25]. In addition, we construct new Godunov-type schemes for H-J equations which consist of an exact evolution operator and a *global* projection operator. Here, we restrict our attention to linear projection operators (first-order schemes). We note, however, that our convergence theory applies equally well to *nonlinear* projections used in the context of modern high-resolution conservation laws. We prove semi-concave stability and obtain L^1-bounds on their associated truncation errors; L^1-convergence of order one then follows. Second-order (central) Godunov-type schemes are

---

Correspondence to: E. Tadmor

also constructed. Numerical experiments are performed; errors and orders are calculated to confirm our L^1-theory.

Mathematics Subject Classification (1991): 65M12

1 Introduction and preliminaries

We study the L^1-stability and derive L^1-error estimates for general approximate solutions of the Cauchy problems for the multidimensional Hamilton-Jacobi (H-J) equations with *Hamiltonian* H:

$$\begin{cases} \partial_t\varphi + H(\nabla_x\varphi) = 0, \\ \varphi(x,0) = \varphi_0(x). \end{cases} \tag{1.1}$$

These equations arise from such areas as the calculus of variations, control theory, and differential games. Typically, solutions for H-J equations experience a loss of regularity which is similar to the loss of regularity for the primitive of solutions to nonlinear conservation laws:

$$\begin{cases} \partial_t u + \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(u) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

Indeed, in the one-dimensional case, H-J equations (1.1) and conservation laws (1.2) can be made formally equivalent to each other by setting $H = f_1$, $\varphi = \int^x u$ and differentiating the H-J equations (1.1) w.r.t. the spatial variable x.

Solutions of H-J equations are continuous; yet, in the generic case, even with smooth initial conditions, these H-J solutions form discontinuous derivatives in a finite time. Solutions with such kind of discontinuities are not unique. Therefore, analogous to conservation laws, it is necessary to introduce the concept of the entropy-like condition to facilitate the selection of a unique solution, which leads to the so-called *viscosity solution*. For convex Hamiltonians, the viscosity solution – characterized by a *semi-concave stability* condition, was first introduced by Kruzkov [14]. Indeed, such a viscosity solution coincides with the limit solution obtained by the vanishing viscosity method. For general Hamiltonians, the definition of the viscosity solution and the question of well-posedness (in L^∞) were formulated and systematically studied by Crandall, Evans, Lions, Souganidis, ... [21, 6, 4, 32]. There is an enormous amount of activity which is based on these studies, and for references to the literature of viscosity solutions for H-J equations we refer the reader to [21, 1]. In particular, we refer to the pioneering work [6] and the user's guide [5].

We note that the general notion of a viscosity solution coincides with Kruzkov's earlier definition in the convex case. Indeed, for one-dimensional convex Hamiltonians and $u_0 = \varphi_0'$, φ is the viscosity solution of H-J equations (1.1) if and only if $u = \frac{\partial}{\partial x}\varphi$ is the entropy solution of conservation laws (1.2). In the multidimensional case, however, this kind of one-to-one correspondence no longer exists. Instead, $\nabla_x \varphi$ satisfies a weakly hyperbolic *system* of conservation laws [14,13]. In view of these arguments, we can think of viscosity solutions of the H-J equations (1.1) as primitives of entropy solutions for the conservation laws (1.2). Based on this idea, concepts used for conservation laws can be passed to H-J equations (e.g., [28,22, 11,13]). Before turning to a description of our main convergence and L^1-stability results for H-J equations, we therefore provide a brief overview of the corresponding convergence results for conservation laws.

1.1 Convergence results for conservation laws: A brief overview

We consider the conservation laws (1.2) with compactly supported BV initial data $u_0(x)$. For one-dimensional convex flux, Oleinik [27] introduced the so-called *One-Sided Lipschitz Condition* (OSLC) as the entropy condition. This condition singles out the unique entropy solution, which is characterized by the method of vanishing viscosity, i.e., the limit of $\{u^\epsilon\}$ –the vanishing viscosity approximations associated with viscosity amplitude ϵ. In general, we let $\{u^\epsilon\}$ denote an *arbitrary* family of approximate solutions depending on a 'small scale' ϵ, e.g., the vanishing viscosity approximations with viscosity amplitude ϵ, finite difference and finite element solutions based on grid-cells of size $\epsilon = \Delta x$, spectral methods depending on $N = \frac{1}{\epsilon}$ modes, etc. Typically, such approximate methods result in a convergence rate of order one half, when measured in the L^1-norm, i.e., $\|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^1} = \mathcal{O}(\sqrt{\epsilon t})$ [17–19,30]. Still, numerical evidence indicates that these approximate solutions converge with order one in the sense that ϵ (and not $\sqrt{\epsilon}$) is the small scale in these approximate methods. We ask whether this first-order convergence rate can be quantified within an appropriate measure of the error, possibly different than the L^1-norm.

For one-dimensional convex conservation laws, Tadmor and his coworkers proved an $\mathcal{O}(\epsilon)$ convergence rate for general ϵ-approximate solutions satisfying the OSLC [34,24,31,35,25]. The novelty of this approach is the use of *a priori* estimates measured in terms of the Lip'-size of the truncation and initial errors, i.e.,

$$\begin{aligned}&\|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{Lip'}\\ &\quad = C(T)\{\|u^\epsilon_t + f(u^\epsilon)_x\|_{Lip'(x,t)} + \|u_0(\cdot) - u^\epsilon_0(\cdot)\|_{Lip'}\}, \quad 0 \le t \le T.\end{aligned}$$

This result yields the classical Kuznetsov's [17–19] L^1-convergence rate of order $\mathcal{O}(\sqrt{\epsilon})$. We will extend this approach to the multidimensional H-J equations (1.1).

As a well known example in this context we mention the class of monotone finite difference approximations, which are at most first-order accurate [9]. Recently, the optimal L^1-convergence rate for conservation laws was shown to be one half [38,29], which indicates that L^1 might not be the appropriate topology for measuring the convergence rate for conservation laws. Indeed, if there are only finitely many discontinuities, (– and we can only compute such solutions!), then the L^1-convergence rate is one [38,36]. This first-order rate is consistent with the first-order convergence results when measured in the Lip'-norm. Moreover, Kurganov [16] demonstrated second-order Lip'-convergence rate for second-order schemes which otherwise fail to achieve second order rate when measured in the L^1-norm.

We now turn to overview the corresponding convergence results for H-J equations.

1.2 Convergence results for H-J equations

We consider the Cauchy problem for the multidimensional H-J equation (1.1) with initial condition $\varphi_0(x)$. The question of well-posedness associated with the viscosity solutions of these H-J equations were systematically studied by Kruzkov, Crandall, Evans, Lions, Souganidis, ... [14,6,4,21,33]. Moreover, Crandall and Lions [7] proved an L^∞-convergence rate of order one half for approximate H-J solutions, based on vanishing viscosity regularization and on monotone finite difference methods (their result can be extended to general Hamiltonians $H(x,t,\nabla_x\varphi)$ [32].) Can we expect a first-order convergence result for H-J equations, similar to the Lip'-theory for convex one-dimensional conservation laws?

To answer this question, we appeal to the relationship between H-J equations and convex conservation laws. We recall that the main ingredients of the Lip'-convergence results for approximate solutions to such conservation laws, $\{u^\epsilon\}$, are a one-sided Lipschitz stability condition – requiring $\frac{\partial}{\partial x}u^\epsilon \leq Const$, and a consistency condition – measuring the Lip'-size of the truncation error. If we regard viscosity solutions to H-J equations as primitives of entropy solutions to conservation laws, then the OSLC turns into the semi-concave stability condition, $D_x^2\varphi^\epsilon \leq Const$, and the counterpart of the Lip'-space is the L^1-measure of the error. That is, we seek an L^1-error estimate under the assumption of a semi-concave stability condition on the approximate H-J solutions.

Indeed, we shall show the convergence of general ϵ-approximate solutions for H-J problems with strictly convex Hamiltonians and subject to compactly supported (or periodic) C^3 initial data. And, moreover, when measured in the L^1-norm, we have an *a priori* error estimate in terms of the truncation errors and initial perturbation errors, which reads

$$\begin{aligned}&\|\varphi(\cdot,t)-\varphi^\epsilon(\cdot,t)\|_{L^1}\\&\quad=C(T)\left(\|\partial_t\varphi^\epsilon+H(\nabla_x\varphi^\epsilon)\|_{L^1(x,t)}+\|\varphi_0-\varphi_0^\epsilon\|_{L^1}\right),\quad 0\le t\le T.\end{aligned}$$

In the first half of this paper, we demonstrate our L^1-theory for vanishing viscosity regularization, in Sect. 2.2, and to general Godunov-type finite difference approximations, in Sect. 2.3. In each case, L^1-error estimates are shown to be of order one (as opposed to the classical L^∞-estimates of order one half.) In the second half of the paper we describe the construction of new Godunov-type schemes, and their numerical simulations confirm the L^1-theory.

1.3 Construction of new Godunov-type schemes

We recall that a Godunov-type scheme consists of successive application of a discrete projection operator – possibly even a nonlinear projection, followed by the exact evolution operator. A key feature in Godunov-type schemes is that the projection operator should be defined *globally*, overall the computational domain. In the context of conservation laws, for example, the cell-averaging operator is the canonical Godunov choice for such globally defined projections. Unfortunately, we are aware of no *globally* defined projection which is utilized in the context of H-J equations; instead, Godunov-type schemes are currently designed with *local* projections with overlapping supports, e.g., [28], which do not fit into Godunov framework.

In the second half of this paper, we turn to design new Godunov-type schemes which employ convex combinations of global projections. We term these schemes as *generalized* Godunov-type schemes, and we note that, thanks to convexity, our L^1-theory applies. At this stage, we restrict ourselves to the first-order Lax-Friedrichs (LxF) type projections which are represented by a convex combination of pointwise interpolation projections. These schemes are monotone and hence are of at most first-order accurate. Indeed, we shall prove that these schemes satisfy the semi-concave stability condition and derive L^1-bounds on their associated truncation errors. Hence, their L^1-convergence rate is of order one. In addition to these new first-order Godunov-type schemes, we also construct second-order Godunov-type schemes based on central stencils along the lines of [23,12,10]. The error analysis together with the numerical simulations show that the L^1-norm is indeed an appropriate measure for the convergence rate of approximate solutions to convex H-J equations.

This paper is organized as follows. In Sect. 2, we state and prove our theory on L^1-stability and error estimates. We start with the general theory in Sect. 2.1. We then demonstrate our theory for the method of vanishing viscosity in Sect. 2.2 and the Godunov-type finite difference methods in Sect. 2.3. In both cases, L^1-convergence rates are shown to be of order one. We also extend our results to more general Hamiltonians, $H(x, t, \nabla_x \varphi)$ in Sect. 2.4. In Sect. 3, we construct new first-order Godunov-type schemes which consist of an exact evolution operator and a *global* projection operator. We also prove convergence results for these schemes. A second-order central Godunov-type scheme is constructed in Sect. 3.3. In Sect. 4, numerical simulations are performed and L^1-error estimates are calculated which support our theoretical results.

Notations. A brief summary of our notations is in order. $H = H(p)$ is the Hamiltonian, depending on the gradient of φ, $p := \nabla_x \varphi$. The gradient and Hessian of $H(\cdot)$ are denoted respectively by $D_p H \equiv H'(p)$ and $D_p^2 H \equiv H_{pp}$; similarly, $D_x^2 \varphi$ stands for the Hessian of $D_x^2 \varphi := \{\frac{\partial^2 \phi}{\partial x_i \partial x_j}\}$. $W^s(L^p)$ denote the usual L^p-Sobolev space of order s. We are particularly interested in the case $p = 1$ where L^1 is substituted, as usual, by the space of measures $\mathcal{M}$; in particular, $W^s(\mathcal{M})$ coincide with the Lipschitz space of order s, $Lip(s, L^1)$.

2 Convergence and error estimates of approximate H-J solutions

We consider multidimensional H-J equations with strictly convex Hamiltonians and $C_0^3(\mathbb{R}^d)$ compactly supported (or periodic) initial conditions, that is,

$$\left\{ \begin{array}{l} \partial_t \varphi + H(\nabla_x \varphi) = 0, \; 0 < \alpha \le D_p^2 H(p) \le \beta < \infty, \\ \varphi_(x, 0) = \varphi_0(x), \qquad supp\, \varphi_0 \subset \Omega_0. \end{array} \right. \tag{2.1}$$

We develop our main result on L^1-stability and error estimate in Sect. 2.1 (Theorem 2.1). We then apply this result to the vanishing viscosity method in Sect. 2.2 (Theorem 2.2) and to finite difference approximations in Sect. 2.3 (Theorem 2.3). In Sect. 2.4, our results are generalized to more general Hamiltonians (Theorems 2.4 and 2.5). In each case, the L^1-convergence rate measured in terms of the small scale of the problem, ϵ (– the viscosity amplitude, the grid-size, ...), and it is shown to be of order one, i.e., of order $\mathcal{O}(\epsilon)$. The whole theory is a natural multidimensional extension of the analogous Lip'-results for one-dimensional convex conservation laws developed by Tadmor, Nessyahu and Tassa [34,24,25].

2.1 Main results

The unique viscosity solution of the H-J equation (2.1) can be identified by the one-sided concavity condition, $D_x^2 \varphi \le Const$, which is analogous to the OSLC for the one-dimensional conservation laws (1.2). (A detailed discussion could be found in the Appendix. Consult also [14,21].) This leads to our

Definition 2.1 *(Semi-concave stability.) Let $\{\varphi^\epsilon\}$ be a family of approximate solutions for the H-J equation (2.1). It is called* semi-concave stable *if there exists a $k(t) \in L^1[0,T]$, $T < \infty$, such that*

$$D_x^2 \varphi^\epsilon(x,t) \le k(t), \qquad 0 \le t \le T. \tag{2.1}$$

Remarks.

2.1. *Semi concavity of the initial data*. We assume that the initial condition, $\varphi_0(x)$, is (at least) semi-concave, $D_x^2\varphi_0^\epsilon(x) \le m$, so that the corresponding exact viscosity solution satisfies the semi-concave stability condition (2.1) with $k(t) = \frac{1}{\alpha t + 1/m} \in L^1[0,T]$ for all $T < \infty$. We note in passing that for more general initial conditions with less than such C^3 semi-concave regularity, one encounters only the weaker semi-concave condition $D_x^2\varphi^\epsilon(x,t) \le \frac{Const}{t}$ which could be treated along the lines of [26].

2.2. *On the notion of solutions*. Let us point out that the semi-concavity of smooth approximate solutions, φ^ϵ, implies their $W^2(L^1)$ bounds, uniformly w.r.t. ϵ. To this end, we consider the second directional derivative $W_\xi(x,t) := \langle \xi, D_x^2\varphi^\epsilon(x,t)\xi\rangle$. Integrate the identity

$$|W_\xi(x,t)| \equiv 2W_\xi^+(x,t) - W_\xi(x,t)$$

over a bounded support, say $\Omega(t)$. The semi-concavity of $\varphi^\epsilon(\cdot,t)$, (2.1), implies $\int_\Omega W_\xi^+(x,t)dx \le |\Omega| k(t)$. By Green's theorem, $\int_\Omega W_\xi (x,t)dx$ depends solely on the *bounded* boundary data

$$\int_{\Omega(t)} \langle \xi, D_x^2\varphi^\epsilon(x,t)\xi\rangle dx = \int_{\partial\Omega(t)} \frac{\partial \varphi^\epsilon}{\partial \xi}\langle n, \xi\rangle dS. \tag{2.2}$$

Indeed, the last boundary term vanishes in the periodic case and the $W^2(L^1)$ bound follows. In the case of compactly supported initial data, the unique semi-concave solution of (2.1) is *realized* as the vanishing limit of the smooth viscosity regularization, $\varphi = \lim_{\epsilon\downarrow 0}\varphi^\epsilon(x,t)$, consult (2.13) below. Since the exact viscosity solution remains compactly supported, $supp\ \varphi(\cdot,t) \subset \Omega(t)$, it follows that the essential mass of its viscosity approximation, $\varphi^\epsilon(\cdot,t)$, concentrates on a bounded ball $B_r(t) \supseteq \Omega^\epsilon(t)$, in the sense that

$$\int_{\partial B_r(t)} \left|\frac{\partial \varphi^\epsilon(\cdot,t)}{\partial \xi}\langle n,\xi\rangle\right| dS \le Const_T, \quad t \le T.$$

The $W^2(L^1(\Omega(t)))$ bound of $\varphi^\epsilon(\cdot,t)$ follows by integration of (2.2) over $B_r(t)$.

Finally, letting $\epsilon \downarrow 0$ it follows that the second derivatives of the semi-concave solution of (2.1) are *bounded measures*. Using the refined semi-discrete concavity stated in lemma 5.2 this could be quantified in terms of the class of functions $W^2(\mathcal{M})$,

$$W^2(\mathcal{M}) := \{\phi \,|\, \sup_x |\phi(x+h\xi) - 2\phi(x) + \phi(x-h\xi)| \\ \leq Const.h^2, \quad \forall |\xi| = 1\},$$

corresponding to the BV class of solutions for nonlinear conservation laws. We seek solutions in this class of functions,

$$\|\varphi(\cdot,t)\|_{W^2(\mathcal{M})} \\ := \sup_{h,\xi} \frac{1}{h^2}\|\varphi(x+h\xi,t) - \varphi(x,t) + \varphi(x-h\xi,t)\|_{L^1(x)},$$

noting that $\|\varphi(\cdot,t)\|_{W^2(\mathcal{M})} \leq Const.$ It follows that these solutions are uniformly differentiable inside their support.

2.3. *Temporal regularity*. We proceed formally by time differentiation of (2.1),

$$\partial_{tt}\varphi + \langle H', D_x^2\varphi H'\rangle = 0,$$

and the $W^2(\mathcal{M})$ spatial regularity of φ implies the corresponding second order temporal regularity

$$sup_h \frac{1}{h^2}\|\varphi(x,t+h) - 2\varphi(x,t) + \varphi(x,t-h)\|_{L^1} \\ \leq Const. \tag{2.3}$$

Equipped with the notion of semi-concave stability, we are ready for our main theorem.

Theorem 2.1 *Let $\{\varphi_1^\epsilon\}$ and $\{\varphi_2^\epsilon\}$ be two semi-concave stable families of approximate solutions of H-J equation (2.1). Let*

$$F_j^\epsilon := \partial_t\varphi_j^\epsilon + H(\nabla_x\varphi_j^\epsilon), \qquad j = 1,2, \tag{2.4}$$

denote their truncation errors. Then, for a finite time T, there are constants $C_0 = C_0(T)$ and $C_1 = C_1(T)$ such that the following a priori estimate holds

$$\|\varphi_1^\epsilon(\cdot,t) - \varphi_2^\epsilon(\cdot,t)\|_{L^1} \\ \leq C_0\|\varphi_1^\epsilon(\cdot,0) - \varphi_2^\epsilon(\cdot,0)\|_{L^1} \\ +C_1\|F_1^\epsilon(\cdot,\cdot) - F_2^\epsilon(\cdot,\cdot)\|_{L^1(x,t)}, \qquad 0 \leq t \leq T. \tag{2.5}$$

Before we turn to the proof of Theorem 2.1, we state three immediate corollaries.

Corollary 2.1 *(L^1-stability.) Let $\varphi_1^\epsilon := \varphi_1$ and $\varphi_2^\epsilon := \varphi_2$ be the semi-concave stable (exact) viscosity solutions of the H-J equations (2.1) subject to different $C_0^3(I\!R^n)$ compactly supported (or periodic) initial conditions. Then*

$$\|\varphi_1(\cdot,t) - \varphi_2(\cdot,t)\|_{L^1} \leq C_0\|\varphi_1(\cdot,0) - \varphi_2(\cdot,0)\|_{L^1}, \qquad 0 \leq t \leq T.$$

We point out that Corollary 2.1 provides an L^1-stability result for the viscosity solution of convex H-J equations, which is to be compared to the classical L^∞-stability result for general Hamiltonians.

Corollary 2.2 *(L^1-error estimate.) Let φ be the viscosity solution and $\{\varphi^\epsilon\}$ be a family of semi-concave stable approximate solutions of the H-J equation (2.1), subject to initial condition $\varphi(x,0)$. Then we have*

$$\|\varphi(\cdot,t) - \varphi^\epsilon(\cdot,t)\|_{L^1} \leq C_1\|\partial_t\varphi^\epsilon + H(\nabla_x\varphi^\epsilon)\|_{L^1(x,t)}, \qquad 0 \leq t \leq T.$$

Of course, a similar statement to Corollary 2.2 holds in the L^∞-setup; the L^∞-norm, however, is too strong measure for the vanishing size of the truncation error.

Corollary 2.3 *(Perturbed Hamiltonians.) Let φ^ϵ be the viscosity solution of H-J equation (2.1) with Hamiltonian $H^\epsilon := H + \epsilon H_1$. Then, we have the following a priori error estimate*

$$\|\varphi(\cdot,t) - \varphi^\epsilon(\cdot,t)\|_{L^1} \leq \epsilon C_1\|H_1(\nabla_x\varphi^\epsilon)\|_{L^1(x,t)}, \qquad 0 \leq t \leq T.$$

We note in passing that using Corollary 2.3, we can relax the C^2-regularity assumption on the Hamiltonians in Theorem 2.1.

We now turn to the proof of Theorem 2.1.

Proof. Let $e(x,t) := \varphi_1^\epsilon(x,t) - \varphi_2^\epsilon(x,t)$ denote the error and set F as the difference between the corresponding truncation errors, $F := F_1^\epsilon - F_2^\epsilon$. From the definition of the truncation error (2.4), we have

$$\partial_t e + G \cdot \nabla_x e = F. \tag{2.6}$$

This is a transport equation, subject to the initial condition $e(x,0) := \varphi_1^\epsilon(x,0) - \varphi_2^\epsilon(x,0)$, where the velocity field, G, is given by the average

$$G := \int_0^1 \nabla_p H(\eta\nabla_x\varphi_1^\epsilon + (1-\eta)\nabla_x\varphi_2^\epsilon)d\eta. \tag{2.7}$$

To study the stability of this transport equation (2.6), we consider its dual equation,

$$\begin{cases} \partial_t\psi + \nabla_x \cdot (G\psi) = 0, \\ \psi(x,T) = \psi_T(x). \end{cases} \tag{2.8}$$

Here the backward initial condition, $\psi_T(x)$, is smooth and compactly supported in $\Omega(T)$. Taking the $L^2(x)$-inner product of e and ψ, we find from (2.6), (2.8), and Green's identity that

$$\frac{d}{dt}\langle e,\psi\rangle + \int_{\partial\Omega(t)} (e\psi)\langle G,n\rangle ds = \langle F,\psi\rangle.$$

The boundary term of the last equality vanishes due to compactly supported conditions, $\psi_T(x)$, and we are left with

$$\langle e(\cdot,T),\psi_T(\cdot)\rangle = \langle e(\cdot,0),\psi(\cdot,0)\rangle + \int_0^{\mathrm{T}} \langle F,\psi\rangle dt. \tag{2.9}$$

The first term of the RHS of (2.9) is bounded above by

$$|\langle e(\cdot,0),\psi(\cdot,0)\rangle| \le \|e(\cdot,0)\|_{L^1}\|\psi(\cdot,0)\|_{L^\infty},$$

and the second term does not exceed

$$\left|\int_0^{\mathrm{T}} \langle F(\cdot,t),\psi(\cdot,t)\rangle dt\right| \le \|F(\cdot,\cdot)\|_{L^1(x,t)} \cdot \sup_t \|\psi(\cdot,t)\|_{L^\infty}.$$

Hence, (2.9) yields

$$\|e(\cdot,T)\|_{L^1} := \sup \frac{|\langle e,\psi_T\rangle|}{\|\psi_T\|_{L^\infty}} \le C_0(T)\|e(\cdot,0)\|_{L^1} + C_1(T)\|F(\cdot,\cdot)\|_{L^1(x,t)},$$

where $C_0 \le C_1$ are given by

$$C_0(T) = \sup_{\psi_T} \frac{\|\psi(\cdot,0)\|_{L^\infty}}{\|\psi_T\|_{L^\infty}}, \qquad C_1(T) = \sup_{\psi_T} \frac{\sup_t \|\psi(\cdot,t)\|_{L^\infty}}{\|\psi_T\|_{L^\infty}}.$$

It remains to estimate $\|\psi(\cdot,t)\|_{L^\infty}$ in terms of its prescribed value $\psi_T(x)$ for all $0 \le t \le T$. To study the L^∞-stability of the last problem (2.8), we multiply it by $sgn(\psi)$ and evaluate at the maximum point of $|\ \psi\ |$, say x_0, to find

$$\frac{d}{dt}\|\psi(\cdot,t)\|_{L^\infty} + (\nabla_x \cdot G)(x_0,t)\|\psi(\cdot,t)\|_{L^\infty} = 0.$$

Gronwall's inequality (or direct L^p iterations with $p \uparrow \infty$) implies:

$$\|\psi(\cdot,t)\|_{L^\infty} \le \|\psi_T\|_{L^\infty} \exp\left(\int_t^{\mathrm{T}} \sup_x \nabla_x \cdot G(\cdot,s) ds\right), \qquad 0 \le t \le T.$$

By definition of the average velocity G in (2.7),

$$\nabla_x G = \int_0^1 \left(\eta \sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial^2 \varphi_1^\epsilon}{\partial x_i \partial x_j} + (1-\eta)\sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial^2 \varphi_2^\epsilon}{\partial x_i \partial x_j}\right) d\eta.$$

For the first sum of the RHS with the φ_1^ϵ term, we have,

$$\begin{aligned}
&\sum_{i,j} \frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial^2 \varphi_1^\epsilon}{\partial x_i \partial x_j} \\
&= tr\left(D_p^2 H \cdot D_x^2 \varphi_1^\epsilon\right) \\
&= tr\left((D_p^2 H)^{\frac{1}{2}} \cdot D_x^2 \varphi_1^\epsilon \cdot (D_p^2 H)^{\frac{1}{2}}\right) \\
&\le d \times \left(\text{maximum eigenvalue of } (D_p^2 H)^{\frac{1}{2}} \cdot D_x^2 \varphi_1^\epsilon \cdot (D_p^2 H)^{\frac{1}{2}}\right) \\
&\le d \times k(t) \times \beta. \qquad (2.10)
\end{aligned}$$

Similarly for the φ_2^ϵ term of the RHS. Thus,

$$\sup_x \nabla_x \cdot G(\cdot, t) \le d \times k(t) \times \beta,$$

which results in the desired upper bound of $\|\psi(\cdot, t)\|_{L^\infty}$,

$$\|\psi(\cdot, t)\|_{L^\infty} \le \|\psi_T(\cdot)\|_{L^\infty} \exp(d\beta K(T)) < \infty,$$

where

$$K(T) := \int_0^{\mathrm{T}} k(s) ds.$$

Hence the proof is completed with

$$C_0(T) = C_1(T) = \exp(d\beta K(T)). \quad \blacksquare \qquad (2.11)$$

We end this subsection with three remarks on possible extensions.

Remarks.

2.1. *weak convexity*. We note that the strict convexity assumption guarantees the existence of $(D_p^2 H)^{-\frac{1}{2}}$ in (2.10). However, since the final estimates (2.11) involving $C_0(T)$ and $C_1(T)$ which are independent of the convexity bound α, our main result (2.5) holds for weakly convex Hamiltonians as well.

2.2. *Time dependence*. Our result can easily carry over to the following H-J equation:

$$\partial_t \varphi + H(t, \nabla_x \varphi) = 0, \qquad (2.12)$$

where $H(t, p)$ is strictly convex in p.

2.3. *Boundary conditions*. Our L^1-theory holds for C^3 initial-boundary problems for bounded domains. In this case, additional compatibility conditions are needed to guarantee the existence of viscosity solutions with no boundary layers which violate the semi-concave stability condition. (Consult [21].)

2.2 Applications to viscosity regularization

In this subsection, we apply our results to approximate solutions constructed by viscosity regularization, that is, solutions of the vanishing viscosity method:

$$(2.13)\quad \begin{cases} \partial_t \varphi^\epsilon + H(\nabla_x \varphi^\epsilon) = \epsilon \Delta \varphi^\epsilon, \; 0 < \alpha \le D_p^2 H(p) \le \beta < \infty, \\ \varphi^\epsilon(x,0) = \varphi_0(x), \qquad\qquad supp\, \varphi_0 \subset \Omega_0. \end{cases}$$

Here, the initial data $\varphi_0(x)$ is $C_0^3(I\!R^d)$ compactly supported (or periodic). In this case, the approximate viscosity solutions and the exact viscosity solution are known to be semi-concave stable and our theory yields an L^1-convergence result of order $\mathcal{O}(\epsilon)$. To be more precise,

Theorem 2.2 *Let $\{\varphi^\epsilon\}$ be the family of approximate viscosity solutions of (2.13). Then, φ^ϵ converge to the exact viscosity solution and, for any fixed T, the following error estimate holds.*

$$(2.14)\qquad \|\varphi^\epsilon(\cdot,t) - \varphi(\cdot,t)\|_{L^1} \le C_1(T)\epsilon, \qquad\qquad 0 \le t \le T.$$

The proof of Theorem 2.2 is immediate: By Corollary 2.2, we have to show that the L^1-size of the truncation error, $\|\epsilon\Delta\varphi^\epsilon\|_{L^1(x,t)}$, is of order $\mathcal{O}(\epsilon)$. Indeed, the semi-concavity of φ^ϵ implies by lemma 5.1, the $W^2(\mathcal{M})$ regularity of φ^ϵ along the lines of our remark 2.1 above. More precisely, we have

$$(2.15)\quad \int_{\Omega(t)} |\Delta\varphi^\epsilon(\cdot,t)|dx \le Const_T, \quad Const_T = 2d|\Omega(t)|k(t),$$

which completes the proof of Theorem 2.2. ∎

Remarks.

2.1. *Initial perturbations*. One can also consider perturbed initial data $\varphi^\epsilon(x,0) = \varphi_0^\epsilon(x)$. Our results show that if the initial L^1-error is of order $\mathcal{O}(\epsilon)$, i.e. $\|\varphi_0^\epsilon - \varphi_0\|_{L^1} = \mathcal{O}(\epsilon)$, then the final estimate remains $\mathcal{O}(\epsilon)$ at later time.

2.2. *L^1 vs. L^∞ error estimates*. Our L^1-error estimate of order $\mathcal{O}(\epsilon)$ should be compared with the classical L^∞-result of order $\mathcal{O}(\sqrt{\epsilon})$ by Crandall and Lions [7]. The latter follows from our L^1-result by one-sided interpolation theory [36].

2.3. *Extension for time dependent Hamiltonians*. Our L^1-theory can carry over to the viscous H-J equations associated (2.12) with time-dependent Hamiltonians, $H(t, \nabla_x\varphi)$. Note that their viscosity solutions satisfy the semi-concave stability condition (2.1) [15].

2.3 Applications to finite difference approximations

In this subsection, we apply our L^1-convergence theory to Godunov-type finite difference approximations of the H-J equations (2.1). In this context, we consider the so-called *Godunov-type schemes*, which can be viewed as a finite volume method. The original Godunov scheme [8] is the forerunner for all subsequent Godunov-type methods which are based on a transport-projection algorithm. The transport part is evolved in terms of the exact solution operator. In its original context of conservation laws, one realizes the projection part by the approximate cell averages [8]. In the present context of H-J equations, however, the projection part is realized by point values rather than cell averages. This is consistent within the view that the H-J solutions correspond to the primitive of solutions of conservation laws. Applications to Godunov-type schemes for H-J equations can be found in [28, 11, 22] and the references therein.

For simplicity, we treat two-dimensional regular rectangular grids, which consist of cells of size $\Delta x \times \Delta y$ and satisfy the following regularity condition,

$$0 < c \leq \frac{\Delta x}{\Delta y} \leq C < \infty,$$

thus excluding "flat" cells.

A Godunov-type scheme then takes the following recursive form:

$$\varphi^{\Delta x}(\cdot, t) = \begin{cases} E(t - t^{n-1})\varphi^{\Delta x}(\cdot, t^{n-1}) & t^{n-1} < t < t^n \\ P_{\Delta x}\varphi^{\Delta x}(\cdot, t^n - 0) & t = t^n \end{cases} \qquad n = 1, 2, \ldots \tag{2.16}$$

subject to initial data

$$\varphi^{\Delta x}(\cdot, 0) = P_{\Delta x}\varphi_0(x).$$

(For simplicity, we assume a uniform time step of size Δt so that the solution proceeds over fixed time interval, $t^n = n\Delta t$.) Here, $E(\cdot)$ is the exact solution operator associate with the H-J equations (2.1), and P is any discrete projection (which is tagged by the discrete spatial grid-size in the x-direction, Δx.)

A typical example for such a projection operator, $P_{\Delta x}$, is the *linear* interpolant based on the point values given at the grid-points. We should emphasize, however, that our theory applies equally well to *nonlinear* projections, such as the nonlinear projections used in the context of modern high-resolution schemes for conservation laws. Moreover, our theory applies to so-called *generalized* Godunov-type schemes which are based on convex combinations of several projection operators of the type mentioned above. Such so-called generalized Godunov-type schemes are constructed in Sect. 3.

We now turn to our study on Godunov-type approximations. In order to implement our theory, we need to verify the following properties:

#1 to measure the L^1-size of the truncation error, and
#2 the semi-concave stability of Godunov-type approximate solutions.

We start with the first property. The following lemma measures the L^1-size of the truncation error in Godunov-type schemes.

Lemma 2.1 *Let $\{\varphi^{\Delta x}\}$ be a family of solutions of the Godunov-type scheme (2.16). Then the truncation error associated with $\varphi^{\Delta x}$ satisfies*

$$
\begin{aligned}
&\int_0^{\mathrm{T}} \|\partial_t \varphi^{\Delta x} + H(\nabla_x \varphi^{\Delta x})\|_{L^1(x)} dt \\
&\qquad \le \frac{T}{\Delta t} \max_{0<t^n\le T} \|(I-P_{\Delta x})\varphi^{\Delta x}(\cdot, t^n - 0)\|_{L^1(x)}.
\end{aligned} \tag{2.17}
$$

Proof. Since $\varphi^{\Delta x}$ is the solution of (1.1) in $\Omega \times [t^{n-1}, t^n)$, for every $\psi \in \mathcal{C}_0^1(\Omega \times [0,T])$,

$$
\int_\Omega \int_{t^{n-1}}^{t^n - 0} [-\varphi^{\Delta x}\partial_t \psi + H(\nabla_x \varphi^{\Delta x})\psi] dt dx = 0.
$$

Therefore, integration by parts shows that for any compactly supported ψ,

$$
\begin{aligned}
&\int_0^{\mathrm{T}} \int_\Omega [\partial_t \varphi^{\Delta x} + H(\nabla_x \varphi^{\Delta x})]\psi dx dt \\
&= \int_\Omega \sum_{n=1}^N \int_{t^{n-1}}^{t^n} [\partial_t \varphi^{\Delta x} + H(\nabla_x \varphi^{\Delta x})]\psi dt dx \\
&= \int_\Omega \sum_{n=1}^N \left[\varphi^{\Delta x}(x, t^n - 0)\psi(x,t^n) - \varphi^{\Delta x}(x, t^{n-1})\psi(x, t^{n-1})\right] dx \\
&= \int_\Omega \sum_{n=1}^N \left[\varphi^{\Delta x}(x, t^n - 0) - \varphi^{\Delta x}(x,t^n)\right] \psi(x,t^n) dx \\
&= \int_\Omega \sum_{n=1}^N \left[(I-P_{\Delta x})\varphi^{\Delta x}(x, t^n - 0)\right] \psi(x,t^n) dx \\
&\le N \max_n \|(I-P_{\Delta x})\varphi^{\Delta x}(x, t^n - 0)\|_{L^1} \|\psi\|_{L^\infty(x,t)}.
\end{aligned} \tag{2.18}
$$

The proof is completed by noting that at $T = N\,\Delta t$,

$$\|\partial_t\varphi^{\Delta x} + H(\nabla_x\varphi^{\Delta x})\|_{L^1(x,[0,T])}$$
$$= \sup_{\psi} \frac{\int_0^T \int_\Omega [\partial_t\varphi^{\Delta x} + H(\nabla_x\varphi^{\Delta x})]\psi dxdt}{\|\psi\|_{L^\infty(x,t)}},$$

does not exceed the upper-bound in (2.17). ■

Lemma 2.1 tells us how to measure the L^1-truncation error solely by the properties of the projection $P_{\Delta x}$ as an approximate identity operator, but otherwise, it is independent of the intricate behavior of the evolution operator E. Therefore, a CFL restriction, $\Delta t \cdot \max(\frac{1}{\Delta x}|H_u(\nabla\varphi)|, \frac{1}{\Delta y}|H_v(\nabla\varphi)|) \leq Const$, is not required at this stage. Note that Lemma 2.1 above applies to linear as well as nonlinear projections.

We now turn to property #2 of semi-concave stability. Since the exact evolution operator E is semi-concave stable, the question of semi-concave stability is entirely due to the properties of the projection $P_{\Delta x}$. Typically, Godunov-type schemes are based on projections onto the space of piecewise polynomials. Such projections may fail to satisfy the semi-concave stability condition (2.1). Indeed, possible 'non-concave' jumps of the gradients of the projections may be introduced at the interfaces. This failure is due to the strict sense of our semi-concave stability which rules out 'non-concave' jumps at the level of our grid-size. Clearly, such jumps should be acceptable, and we therefore need to relax the semi-concave stability requirement (2.1), requiring, instead, the following discrete analogue along the lines of Lemma 5.1 in the Appendix.

Definition 2.2 *(Discrete semi-concave stability.) Let $\{\varphi^\epsilon\}$ be a family of approximate solutions for the H-J equation (2.1). It is called* discrete semi-concave stable *if there exists a $k(t) \in L^1(0,T)$, $T < \infty$, such that for all $h \geq h_0(\epsilon) > 0$ there holds*

$$D^2_{h,\xi}\varphi^\epsilon(x,t) := \frac{\varphi^\epsilon(x+h\xi,t) - 2\varphi^\epsilon(x,t) + \varphi^\epsilon(x-h\xi,t)}{h^2} \leq k(t),$$

(2.19) $\quad \forall|\xi| = 1.$

Remarks.

2.1. *A small spatial scale.* Of course, letting $h \downarrow 0$ in Definition 2.2, then (2.19) coincides with the standard Definition 2.1 of (continuous) semi-concave stability. The essence of Definition 2.2, however, is its restriction to a discrete semi-concavity at a small scale of order $h \geq h_0(\epsilon)$. Consider, for example, the Godunov-type approximate solution, $\varphi^{\Delta x}(\cdot,t)$, based on regular grid of rectangle size $\Delta x \sim \Delta y$. In this case, we require the discrete semi-concave stability (2.19) where

$h_0(\epsilon) \sim \Delta x$ denotes the small spatial scale, i.e., for all $|\xi| = 1$ and $h \geq Const \cdot \Delta x$,

$$D^2_{h,\xi}\varphi^{\Delta x}(x,t) := \frac{\varphi^{\Delta x}(x+h\xi,t) - 2\varphi^{\Delta x}(x,t) + \varphi^{\Delta x}(x-h\xi,t)}{h^2} \\ \leq k(t) \in L^1.$$

2.2. *The discrete semi-concave stability of Godunov-type schemes.* We demonstrate the advantage of the notion of *discrete* semi-concavity (Definition 2.2) over the continuous one (Definition 2.1) in the following example of Godunov-type scheme. If $E(t)$ is the *exact* viscosity solution operator, then according to Lemma 5.2,

$$\sup_x D^2_{h,\xi} E(t-t^{n-1})\varphi^{\Delta x}(\cdot,t^{n-1}) \\ \leq \sup_x D^2_{h,\xi}\varphi^{\Delta x}(\cdot,t^{n-1}) \qquad t > t^{n-1}. \tag{2.20}$$

(Recall that E is in fact semi-concave stable in *each* direction). Moreover, let $P_{\Delta x}$ denote the *linear* interpolant over a regular grid of rectangular size $\Delta x \sim \Delta y$; then for all $Const \cdot \min(\Delta x, \Delta y) \leq h \leq \min(\Delta x, \Delta y)$ there holds

$$\sup_x D^2_{h,\xi} P_{\Delta x} w(x) \leq \sup_x D^2_{h,\xi} w(x). \tag{2.21}$$

(To verify (2.21), note that since $D^2_{h,\xi} P_{\Delta x} w(\cdot)$ is piecewise linear, it attains its extrema at the grid points, $\sup_{\mathbf{x}_\nu} D^2_{h,\xi} P_{\Delta x} w(\mathbf{x}_\nu)$, and the latter is upper bounded by one of the interpolating directions along the x-axis, y-axis and diagonal axis, i.e., we even have the upper bound $\sup_{\mathbf{x}_\nu} D^2_{h,\xi} w(\mathbf{x}_\nu)$). We note that (2.21) fails for the corresponding continuous semi-concavity, because $P_{\Delta x}$ may introduce *convex* corners – no matter how small, and consequently

$$D^2_\xi P_{\Delta x} w(x) \not\leq D^2_\xi w(x).$$

Thus, the projection $P_{\Delta x}$, and hence the evolving Godunov-type solution $\varphi^{\Delta x}$, need not satisfy the (stricter) semi-concave stability Definition 2.1. In fact, the introduction of small non-convex jumps may lead to $D^2_x P_{\Delta x}\varphi_0(x) = \infty$. It is in this context that the discrete semi-concavity stability (stated in Lemma 5.2 in the Appendix) offers the right notion of stability at the level of our grid-size.

We conclude

Corollary 2.4 *(Discrete semi-concave stability of Godunov-type solutions.) A Godunov-type scheme (2.16) which employs a linear interpolant projection, $P_{\Delta x}$, based on a regular rectangular grid of size $\Delta x \sim \Delta y$ is discrete semi-concave stable, i.e., its solution satisfies*

$$\sup_x D^2_{h,\xi}\varphi^{\Delta x}(\cdot,t) \le \sup_x D^2_{h,\xi}\varphi^{\Delta x}(\cdot,0), \qquad \forall|\xi|=1.$$

How can the *discrete* semi-concavity be used together with our main result? Since the Godunov-type approximate solution, $\varphi^{\Delta x}$, need not satisfy the (continuous) semi-concave stability requirement (2.1), our main result in Sect. 3 do not apply directly. Instead, if $\varphi^{\Delta x}$ satisfies the discrete semi-concave stability, then we claim that one can find a near-by semi-concave stable approximate solution so that our main results apply. This is the context of

Theorem 2.3 *Let $\{\varphi^{\Delta x}\}$ be a family of solutions for the Godunov-type scheme (2.16), which satisfies the CFL condition*

$$\max(\lambda|H_u(\nabla\varphi)|, \mu|H_v(\nabla\varphi)|) \le \frac{1}{4},$$

where $\lambda := \frac{\Delta t}{\Delta x}$ and $\mu := \frac{\Delta t}{\Delta y}$ are the fixed mesh ratios. Assume

2.1. (Consistency.)

$$\begin{aligned} &\|(I-P_{\Delta x})\varphi^{\Delta x}(\cdot,t)\|_{L^1(x)} \\ &\qquad = \mathcal{O}(\Delta x)^2\|\varphi^{\Delta x}(\cdot,t)\|_{W^2(\mathcal{M}(x))}, \end{aligned} \tag{2.22}$$

and

2.2. (Near-by approximations.) There exist a family of 'near-by' semi-concave stable approximate solutions, $\{\psi^{\Delta x}(x,t)\}$. Here, 'near-by' means that the following holds

$$\begin{aligned} &\|D_t(\varphi^{\Delta x}(\cdot,t)-\psi^{\Delta x}(\cdot,t))\|_{L^1(x)} \\ &\quad +\|D_x(\varphi^{\Delta x}(\cdot,t)-\psi^{\Delta x}(\cdot,t))\|_{L^1(x)} \\ &\quad \le Const\cdot\Delta x\|\psi^{\Delta x}(\cdot,t)\|_{W^2(\mathcal{M}(x))}. \end{aligned} \tag{2.23}$$

Then $\varphi^{\Delta x}(\cdot,t)$ converges to the exact viscosity solution $\varphi(\cdot,t)$, and for any fixed T, the following estimate holds

$$\|\varphi^{\Delta x}(\cdot,t)-\varphi(\cdot,t)\|_{L^1(x)} \le C_T\Delta x, \qquad 0\le t\le T.$$

Note. The essence of this theorem, therefore, is that one can circumvent the lack of *continuous* semi-concave stability for $\varphi^{\Delta x}$, by constructing 'near-by' semi-concave stable approximations, $\{\psi^{\Delta x}\}$. Such near-by semi-concave stable approximation is guaranteed by the *discrete* semi-concave stability of

$\{\varphi^{\Delta x}\}$. The same procedure was used in the context of convex conservation laws in [25]. In that case, the situation was simplified, however, by constructing 'near-by' solutions by mollification, $\psi^{\Delta x} = B_{\Delta x} * \varphi^{\Delta x}$ (where $B_{\Delta x}$ is the standard dilated mollifier).

Proof. Since $\psi^{\Delta x}(x,t)$ is semi-concave stable our main result then applies, stating that

$$\|\psi^{\Delta x}(\cdot,t) - \varphi(\cdot,t)\|_{L^1(x)} \leq Const \cdot \left[\|\psi_t^{\Delta x} + H(\nabla_x \psi^{\Delta x})\|_{L^1(x,t)} + \|\psi_0^{\Delta x} - \varphi_0\|_{L^1(x)}\right]. \tag{2.24}$$

By assumption (2.23) $\|\varphi^{\Delta x} - \psi^{\Delta x}\|_{L^1(x)} = \mathcal{O}(\Delta x)$, and the result then follows if we can show that $\psi^{\Delta x}$ is $\mathcal{O}(\Delta x)$ away from the exact solution, $\|\psi^{\Delta x} - \varphi\|_{L^1} \leq Const \cdot \Delta x$.

First we note that $\|\psi_0^{\Delta x} - \varphi_0\|_{L^1} \leq \|\psi_0^{\Delta x} - \varphi_0^{\Delta x}\|_{L^1} + \|\varphi_0^{\Delta x} - \varphi_0\|_{L^1}$, so that the initial error on the right-hand side of (2.24) is of the desired order $\mathcal{O}(\Delta x)$.

Next, we treat the truncation error on the right-hand side of (2.24),

$$\begin{aligned}
&\|\psi_t^{\Delta x} + H(\nabla_x \psi^{\Delta x})\|_{L^1(x,t)} \\
&\leq \|\varphi_t^{\Delta x} + H(\nabla_x \varphi^{\Delta x})\|_{L^1(x,t)} + \|H(\nabla_x \varphi^{\Delta x}) - H(\nabla_x \psi^{\Delta x})\|_{L^1(x,t)} \\
&\quad + \|(\psi^{\Delta x} - \varphi^{\Delta x})_t\|_{L^1(x,t)} \\
&= I + II + III.
\end{aligned}$$

The first term, $I = \mathcal{O}(\Delta x)$, by Lemma 2.1 and the assumption of consistency (2.22). The second term, II, is also of order $\mathcal{O}(\Delta x)$, since $\psi^{\Delta x}$ is a 'near-by' approximation of $\varphi^{\Delta x}$ and (2.23) implies

$$II \leq \sup(H'(\cdot)) \cdot \|\nabla_x(\varphi^{\Delta x} - \psi^{\Delta x})\|_{L^1(x,t)} = \mathcal{O}(\Delta x).$$

And finally, the third term involves the error of a linear interpolation in time, and by (2.23) it does not exceed

$$III = \|(\psi^{\Delta x} - \varphi^{\Delta x})_t\|_{L^1(x,t)} \leq Const_t \cdot \Delta x, \quad t \leq T.$$

2.4 General Hamiltonians

In this subsection, we extend our results to more general Hamiltonians which depend on not only $\nabla_x \varphi$ but also spatial and temporal variables x and t, i.e., we consider L^1-stability and error estimates for the following H-J equations,

$$\begin{cases} \partial_t \varphi + H(x,t,\nabla_x \varphi) = 0, \\ \varphi_(x,0) = \varphi_0(x). \end{cases} \tag{2.25}$$

Here, $\varphi_0(x)$ is a $C_0^3(\mathbb{R}^n)$ compactly supported (or periodic) initial data, and the Hamiltonian $H(\cdot,\cdot,p)$ is assumed to be sufficiently smooth. Moreover, we assume that the Hessian of $H(x,t,p)$ w.r.t. x and p is strictly convex, i.e.,

$$0 < \alpha \le \begin{pmatrix} H_{xx} & H_{xp} \\ H_{px} & H_{pp} \end{pmatrix} \le \beta < \infty. \tag{2.26}$$

Analogous argument as in Sect. 2.1 and Sect. 2.2 leads to the following theorems whose proof is left to the reader.

Theorem 2.4 *Let $\{\varphi_1^\epsilon\}$ and $\{\varphi_2^\epsilon\}$ be two semi-concave stable families of approximate solutions of H-J equation (2.25). Let $F_j^\epsilon = \partial_t\varphi_j^\epsilon + H(x,t,\nabla_x\varphi_j^\epsilon)$, $j = 1,2$, denote the truncation errors. Then, for a finite time T, there are constants $c_0 = c_0(T)$ and $c_1 = c_1(T)$ such that the following a priori error estimate holds.*

$$\begin{aligned} \|\varphi_1^\epsilon(\cdot,t) - \varphi_2^\epsilon(\cdot,t)\|_{L^1} &\le c_0\|\varphi_1^\epsilon(\cdot,0) - \varphi_2^\epsilon(\cdot,0)\|_{L^1} \\ &\quad + c_1\|F_1^\epsilon(\cdot,\cdot) - F_2^\epsilon(\cdot,\cdot)\|_{L^1(x,t)}, \qquad 0 \le t \le T. \end{aligned} \tag{2.27}$$

As in Sect. 2.1, we have L^1-stability (Corollary 2.1), L^1-error estimate (Corollary 2.2) and L^1-results on perturbed Hamiltonians (Corollary 2.3) which are immediate consequence. We now summarize our results for viscosity regularization in the following.

Theorem 2.5 *Let $\{\varphi^\epsilon\}$ be a family of the viscosity regularization associated with the general H-J equations (2.25). Assume $\{\varphi^\epsilon(x,t)\}$ are smooth in x, for $0 \le t \le T$. Then $\{\varphi^\epsilon\}$ converges to the corresponding viscosity solution, φ, and the following error estimate holds.*

$$\|\varphi^\epsilon(\cdot,t) - \varphi(\cdot,t)\|_{L^1} = C_T\epsilon, \qquad 0 \le t \le T.$$

Remarks.

2.1. Theorem 2.4 holds for the initial-boundary H-J problems, if we assume additional compatibility conditions to ensure the existence of semi-concave viscosity solutions of (2.25) without boundary layers, consult [21].

2.2. Theorem 2.5 which asserts that the L^1-error estimate of order $\mathcal{O}(\epsilon)$ for convex Hamiltonians with variable coefficients, (2.25) and (2.26), should be compared with the corresponding classical L^∞-result of order $\mathcal{O}(\sqrt{\epsilon})$ for general Hamiltonians with variable coefficients by Souganidis [32].

2.3. Our L^1-theory holds for even more general Hamiltonians depending on x, t, u, and $p := \nabla_x u$. For example, we can consider Hamiltonians, $H(x,t,u,p)$, such that $\frac{\partial^2 H}{\partial u \partial p} \equiv 0$, $\frac{\partial H}{\partial u} \ge Const$, and strictly convex w.r.t. x, u, p. Note that the last two conditions rule out the possibility of discontinuous solutions and insure the regularity of the semi-concave viscosity solutions. [1,21].

3 New Godunov-type schemes

In Sect. 2.3, we proved that discrete semi-concave stable Godunov-type schemes converge to the exact viscosity solution, and we quantified their L^1-convergence rates. We recall that a Godunov-type scheme consists of successive application of a discrete projection operator – possibly even a nonlinear projection, followed by the exact evolution operator. A key feature in Godunov-type schemes is that the projection operator should be defined *globally*, overall the computational domain. In the context of conservation laws, for example, the cell-averaging operator is the canonical Godunov choice for such globally defined projections. Unfortunately, we are aware of no *globally* defined projection which is utilized in the context of H-J equations; instead, Godunov-type schemes are currently designed with *local* projections with overlapping supports, e.g., [28], which do not fit into Godunov framework.

In this section, we turn to design new Godunov-type schemes which employ convex combinations of global projections. We term these schemes as *generalized* Godunov-type schemes, and we note that, thanks to convexity, our L^1-theory applies. At this stage, we restrict ourselves to the first-order Lax-Friedrichs (LxF) type projections which are represented by a convex combination of pointwise interpolation projections. Generalized second-order schemes which employ nonlinear projections (along the lines of Nessyahu and Tadmor [23]) will be dealt at a later stage. We shall start constructing our schemes in Sect. 3.1 and prove their convergence in Sect. 3.2. In Sect. 3.3, we design a second-order scheme, which is a natural extension from our first-order Godunov-type scheme constructed in Sect. 3.1.

3.1 Construction of new (first-order) Godunov-type schemes

For simplicity, we demonstrate our construction in two-dimensional case. To approximate the H-J equation (1.1) by numerical schemes, we begin with discrete grid-function, φ^n_{jk}, which represents the point-value at $(x_j := j\Delta x, y_k := k\Delta y)$ and $t = t^n$. The computational grid consists of cells, $C_{j+\frac{1}{2},k+\frac{1}{2}} := \left\{(\xi,\eta) \,\middle|\, |\xi - x_{j+\frac{1}{2}}| \le \frac{\Delta x}{2},\ |\eta - y_{k+\frac{1}{2}}| \le \frac{\Delta y}{2}\right\}$ centered around $(x_{j+\frac{1}{2}}, y_{k+\frac{1}{2}})$. Each cell $C_{j+\frac{1}{2},k+\frac{1}{2}}$ can be divided into two triangles as shown in either Fig. 3.1 (A) or (B). We shall only construct our schemes based on the mesh divided into NW/SE triangles shown in Fig. 3.1 (A). Similar construction can apply to the NE/SW divided mesh in Fig. 3.1 (B). We use the standard notations $\Delta^+_x \varphi^n_{j,k} = \varphi^n_{j+1,k} - \varphi^n_{j,k}$, and $\Delta^+_y \varphi^n_{j,k} = \varphi^n_{j,k+1} - \varphi^n_{j,k}$.

We now turn to the construction of our first scheme.

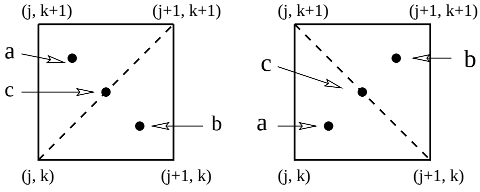

Fig. 3.1a,b. Staggered LxF schemes for the H-J equations. **a** NW/SE division; **b** NE/SW division

Algorithm 1. (Staggered LxF scheme.) At time t^n, on each cell, $C_{j+\frac{1}{2},k+\frac{1}{2}}$, choose points $\mathbf{a}$, $\mathbf{b}$ along the segment connecting (x_j, y_{k+1}) and (x_{j+1}, y_k) and are symmetric to each other w.r.t. the point $(x_{j+\frac{1}{2}}, y_{k+\frac{1}{2}})$ as shown in Fig. 3.1 (A). For example, we may choose point $\mathbf{a}$ to be $(x_{j+\frac{1}{4}}, y_{k+\frac{3}{4}})$ and pont $\mathbf{b}$ to be $(x_{j+\frac{3}{4}}, y_{k+\frac{1}{4}})$. We construct our first scheme by iteration of the following 3-step algorithm which advances from time t^n to t^{n+1}.

– Step $\mathbf{P}$ (Projection): We linearly interpolate over each triangle according to given point-values at its three vertices, e.g., $\varphi^n_{j,k}, \varphi^n_{j+1,k}, \varphi^n_{j+1,k+1}$. In particular, the point-value of this interpolant at the interior point, $\mathbf{a} = \left(x_{j+\frac{1}{4}}, y_{k+\frac{3}{4}}\right)$ is given by

$$\varphi^n_{j,k+1} + \frac{1}{4}\left(\varphi^n_{j,k} - 2\varphi^n_{j,k+1} + \varphi^n_{j+1,k+1}\right)$$

and

$$\varphi^n_{j+1,k} + \frac{1}{4}\left(\varphi^n_{j,k} - 2\varphi^n_{j+1,k} + \varphi^n_{j+1,k+1}\right),$$

at the interior point $\mathbf{b} = (x_{j+\frac{3}{4}}, y_{k+\frac{1}{4}})$.

– Step $\mathbf{E}$ (Evolution): Using the exact evolution operator, evolve the piecewise linear interpolant constructed in step $\mathbf{P}$. To follow the exact evolutions, we assume the CFL restriction, $\Delta t \cdot \max(\frac{1}{\Delta x}|H_u(\nabla\varphi)|, \frac{1}{\Delta y}|H_v(\nabla\varphi)|) \le \frac{1}{4}$. Then, the possible singularities at each boundary edge will not reach the interior points $\mathbf{a}$ and $\mathbf{b}$, and therefore, $H(\nabla\varphi)$ is constant at the points $\mathbf{a}$ and $\mathbf{b}$ for all $t^n \le t < t^{n+1}$. Next, we integrate the H-J equation (1.1) over this time period. The value of the point $\mathbf{a}$ at t^{n+1} is given by

$$\varphi^n_{j,k+1} + \frac{1}{4}\left(\varphi^n_{j,k} - 2\varphi^n_{j,k+1} + \varphi^n_{j+1,k+1}\right) \\ -\Delta t H\left(\frac{\Delta^+_x\varphi^n_{j,k+1}}{\Delta x}, \frac{\Delta^+_y\varphi^n_{j,k}}{\Delta y}\right).$$

and at $(\mathbf{b}, t^{n+1})$, by,

$$\begin{aligned}\varphi^n_{j+1,k} &+ \frac{1}{4}\left(\varphi^n_{j,k} - 2\varphi^n_{j+1,k} + \varphi^n_{j+1,k+1}\right)\\ &-\Delta t H\left(\frac{\Delta^+_x \varphi^n_{j,k}}{\Delta x}, \frac{\Delta^+_y \varphi^n_{j+1,k}}{\Delta y}\right)\end{aligned}$$

– Step $\mathbf{M}$ (Convex combination): On each cell, $C_{j+\frac{1}{2},k+\frac{1}{2}}$, the new point-value $\varphi^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}}$ is assigned to be the arithmetic mean of point-values evaluated in step $\mathbf{E}$ at $\mathbf{a}$ and $\mathbf{b}$.

The final scheme then reads

$$\begin{aligned}\varphi^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = &\frac{1}{4}\left(\varphi^n_{j,k} + \varphi^n_{j+1,k} + \varphi^n_{j,k+1} + \varphi^n_{j+1,k+1}\right)\\ &-\frac{\Delta t}{2}\left[H\left(\frac{\Delta^+_x \varphi^n_{j,k}}{\Delta x}, \frac{\Delta^+_y \varphi^n_{j+1,k}}{\Delta y}\right)\right.\\ &\left.+H\left(\frac{\Delta^+_x \varphi^n_{j,k+1}}{\Delta x}, \frac{\Delta^+_y \varphi^n_{j,k}}{\Delta y}\right)\right] .\end{aligned}$$

Remark. The final scheme for mesh (B) reads

$$\begin{aligned}\varphi^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = &\frac{1}{4}\left(\varphi^n_{j,k} + \varphi^n_{j+1,k} + \varphi^n_{j,k+1} + \varphi^n_{j+1,k+1}\right)\\ &-\frac{\Delta t}{2}\left[H\left(\frac{\Delta^+_x \varphi^n_{j,k}}{\Delta x}, \frac{\Delta^+_y \varphi^n_{j,k}}{\Delta y}\right)\right.\\ &\left.+H\left(\frac{\Delta^+_x \varphi^n_{j,k+1}}{\Delta x}, \frac{\Delta^+_y \varphi^n_{j+1,k}}{\Delta y}\right)\right] .\end{aligned}$$

Our schemes constructed above are staggered in the sense that the computational grid at time t^{n+1} shifts half grid-size in each direction. Staggered schemes may increase difficulty on the treatment of the numerical boundary conditions. However, staggered schemes can be changed into non-staggered ones by using the mechanism in [10] which results the following two algorithms.

Algorithm 2a. (Post-averaged non-staggered LxF scheme.)

– Step $\mathbf{PEM}$: On each cell, $C_{j+\frac{1}{2},k+\frac{1}{2}}$, we perform the three staggered LxF steps, i.e. operators $\mathbf{P}$, $\mathbf{E}$ and $\mathbf{M}$ in this order.
– Step $\mathbf{R}$ (Re-average.): We assign the value at each grid-point as the arithmetic mean of values of its related four centered points. For example, the value of point (x_j, y_k) as shown in Fig. 3.2 is the mean of point-values $\varphi^n_{j\pm\frac{1}{2},k\pm\frac{1}{2}}$.

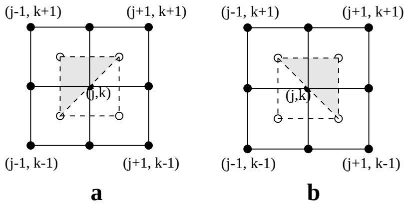

Fig. 3.2a,b. Non-staggered LxF schemes for the H-J equations

The main ingredient of converting a staggered version into a non-staggered one is that we shift backwards the computational grid at time t^{n+1}. Instead, we can shift the grid forwards at t^n, which leads to the following non-staggered version.

Algorithm 2b. (Pre-averaged non-staggered LxF scheme.)

– Step $\mathbf{R}$: We first assign values at centered points as the arithmetic mean of values of its related four grid-points. For example, the value of point $(x_{j-\frac{1}{2}}, y_{k+\frac{1}{2}})$ as shown in Fig. 3.2 is the mean of values $\varphi^n_{j-1,k}$, $\varphi^n_{j-1,k+1}$, $\varphi^n_{j,k}$, $\varphi^n_{j,k+1}$.
– Step $\mathbf{PEM}$: On each new cell, for example, the one has points $(x_{j\pm\frac{1}{2}}, y_{k\pm\frac{1}{2}})$ as its vertices, perform the staggered LxF scheme's procedure. (That is, we perform operators, $\mathbf{P}$, $\mathbf{E}$ and $\mathbf{M}$ in this order.)

Remarks.

3.1. We can identify these two non-staggered schemes, Algorithms 2a and 2b, by listing all steps **P, E, M**, and **R** from left to right into a row. We find:

$$\text{Algorithm 2a: } \mathbf{RPEM}\ldots\mathbf{RPEM} = (\mathbf{D_a})^{\mathbf{n}}$$
$$\text{Algorithm 2b: } \mathbf{PEMR}\ldots\mathbf{PEMR} = (\mathbf{D_b})^{\mathbf{n}}$$

where $\mathbf{D_b} := \mathbf{PEMR}$, and $\mathbf{D_a} := \mathbf{RPEM}$. Thus, we have

$$(\mathbf{D_a})^{\mathbf{n}}\mathbf{R} = \mathbf{R}(\mathbf{D_b})^{\mathbf{n}}.$$

3.2. Algorithms 2a and 2b have almost the same resolution which is less than that of Algorithm 1 due to the re-average operation, Step **R**.

Finally, we call our last scheme whose evolution operator is quite different from Algorithms 1, 2a, and 2b.

Algorithm 3. (The mini-Godunov scheme)

– Step **M**: We assign the values at $(x_{j+\frac{1}{2}}, y_k)$ to be $\frac{1}{2}(\varphi^n_{j,k} + \varphi^n_{j+1,k})$ and at $(x_j, y_{k+\frac{1}{2}})$ to be $\frac{1}{2}(\varphi^n_{j,k} + \varphi^n_{j,k+1})$.
– Step **P**: We linearly interpolate on each triangle based on the point-values given at its vertices, say $\varphi^n_{j,k}$, $\varphi^n_{j,k+\frac{1}{2}}$, $\varphi_{j+\frac{1}{2},k}$.

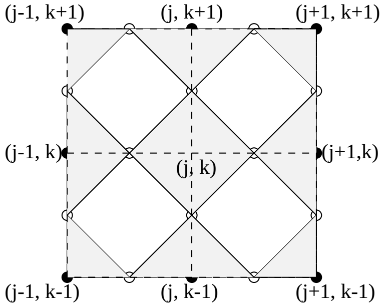

Fig. 3.3. The mini-Godunov scheme

– Step **E**: Evolve exactly based on the above linear interpolant in a time step. The exact formula is given in [28].

Remark. Although Algorithm 3 assume the same evolution operator as in the Godunov scheme designed by Osher and Shu [28], it can be interpolated globally because we take a smaller and non-overlapping computational grid. Consequently, the CFL number of Algorithm 3 is only half of the one for the Osher and Shu's scheme. Moreover, both schemes are nonstaggered.

3.2 Convergence results

In this subsection, we discuss the convergence of those schemes constructed in Sect. 3.1. We first note that Algorithms 1, 2a, 2b, and 3 are monotone, hence, they converge with rate $\mathcal{O}(\sqrt{\Delta x})$ when measured in the L^∞-norm [7]. To attain the L^1-convergence rate $\mathcal{O}(\Delta x)$, we need to prove that they satisfy the hypotheses of Theorem 2.3: the consistency condition, (2.22), and the existence of semi-concave stable near-by approximate solutions, (2.23).

Theorem 3.1 *The approximate solutions, $\{\varphi^{\Delta x}\}$, by the family of LxF schemes (Algorithms 1, 2a, and 2b) and the mini-Godunov scheme(Algorithm 3) constructed in Sect. 3.1 converge to the viscosity solution φ, and the following error estimate holds*

$$\|\varphi^{\Delta x}(\cdot,t)-\varphi(\cdot,t)\|_{L^1}=\mathcal{O}(\Delta x).$$

Proof. Here, for simplicity, we prove the convergence of the staggered LxF scheme, Algorithm 1. Similar arguments apply to the other three schemes. We proceed in several steps.

As a first step we note that the LxF solution $\varphi^{\Delta x}$ satisfies the discrete semi-concave stability, (2.19), since by Corollary 2.4, all Godunov-type schemes based on piecewise-linear reconstructions do.

As a second step we note, along the lines of Remark 2.1, that the discrete semi-concavity implies the piecewise-linear LxF solution, $\varphi^{\Delta x}(\cdot, t)$ is bounded in $W^2(\mathcal{M})$. As an alternative argument we note that the $W^2(\mathcal{M})$ regularity of $\varphi^{\Delta x}(\cdot, t)$ is due to the fact that the evolution operator **E**, and the piecewise linear interpolation **P** and **M** do not increase the $W^2(\mathcal{M})$ norm.

Granted the $W^2(\mathcal{M})$ regularity, we turn to the third step of checking the consistency, (2.22). If $P_{\Delta x} = \mathbf{MP}$ is the piecewise-linear projection, then we decompose

$$\|(P_{\Delta x} - \mathbf{I})\varphi^{\Delta x}\|_{L^1} \leq \|(\mathbf{MP} - \mathbf{P})\varphi^{\Delta x}\|_{L^1} + \|(\mathbf{P} - \mathbf{I})\varphi^{\Delta x}\|_{L^1}.$$

By Taylor expansion, the error of averaging in the first term on the RHS is controlled by $(\Delta x)^2\|\varphi^{\Delta x}\|_{W^2(\mathcal{M}(x))}$. By straightforward error estimate for (one-dimensional) linear interpolant, e.g., [3], the second term on the RHS is also bounded by $(\Delta x)^2\|\varphi^{\Delta x}\|_{W^2(\mathcal{M}(x))}$.

Finally, $\{\varphi^{\Delta x}\}$ satisfies the *discrete* semi-concavity but may fail to satisfy the stronger semi-concave stability condition (2.1). Here we appeal to Theorem 2.3, showing the existence of a semi-concave stable approximation, $\psi^{\Delta x}$, which is 'near-by' $\varphi^{\Delta x}$ in the sense that (2.23) holds. The existence of such a $\psi^{\Delta x}$ hinges on the *discrete* semi-concave stability of the LxF solutions, $\varphi^{\Delta x}$.

To construct the desired near-by approximation, we first restrict attention to one prefered direction, ξ, say the x-axis, $\xi = (1, 0)$. Using the gridvalues, $\{\varphi^{\Delta x}_{j,0}\}$, we construct a semi-concave stable piecewise *quadratic* interpolant $Q^{\Delta x}(x, 0)$ as follows. We first form the *average slopes*, $(\varphi^{\Delta x}_{j+1,k} - \varphi^{\Delta x}_{j,k})/\Delta x$; next we use these averages to construct piecewise linear approximations based on *max* limiter advocated in [2], and we conclude by *integrating* the piecewise-linear max reconstruction. Note that this is a nonlinear reconstruction. In this fashion, one arrives at a piecewise quadratic *interpolant* in each typical triangle based on its one-dimensional interpolation along the x-axis, y-axis and the diagonal axis. Denote this piecewise quadratic interpolant $\psi^{\Delta x}(x, y)$,

$$\begin{aligned}\psi^{\Delta x}(x,y) = \sum Q^{\Delta x,NW}(x_{j+\frac{1}{4}}, y_{k+\frac{3}{4}})\chi_{C^{NW}_{j+\frac{1}{2},k+\frac{1}{2}}} \\ +Q^{\Delta x,SE}(x_{j+\frac{3}{4}}, y_{k+\frac{1}{4}})\chi_{C^{SE}_{j+\frac{1}{2},k+\frac{1}{2}}}.\end{aligned}$$

Since the max limiter guarantees the one-sided Lipschitz condition for a piecewise-linear reconstruction, [2, Proposition 5], the quadratic primitive satisfies

$$D^2_\xi \psi^{\Delta x}(\cdot, t^n) \leq Const, \tag{3.1}$$

if ξ is one of the major directions, $\xi = (1, 0), (0, 1), (1, 1)$. By the property of max limiters, one verifies that (3.1) holds for *any* direction. Consult [20] for detailed discussion. We complete the proof by noting that the spatial error bound in (2.23) follows from the construction using max limiter. In between time levels we use linear interpolation,

$$\psi^{\Delta x}(x,t) = \sum_n \big[(t-t^n)\psi^{\Delta x}(x,t^{n+1})/\Delta t \\ +(t^{n+1}-t)\psi^{\Delta x}(x,t^n)/\Delta t\big]\,\chi_{[t^n,t^{n+1}]}$$

and the latter is indeed a 'near-by' approximation satisfying the temporal part of the error bound (2.23). ■

3.3 Second-order Godunov-type schemes

In this subsection, we design a second-order central scheme, based on the first-order LxF scheme, Algorithm 1 constructed in Sect. 3.1.

Algorithm 4.

$$\begin{aligned}\varphi^{n+1}_{j+\frac12,k+\frac12} = & \frac14(\varphi^n_{j,k}+\varphi^n_{j+1,k}+\varphi^n_{j,k+1}+\varphi^n_{j+1,k+1}) \\ & +\frac{1}{16}(\varphi'_{j,k}-\varphi'_{j+1,k}+\varphi'_{j,k+1}-\varphi'_{j+1,k+1}) \\ & +\frac{1}{16}(\grave{\varphi}_{j,k}-\grave{\varphi}_{j,k+1}+\grave{\varphi}_{j+1,k}-\grave{\varphi}_{j+1,k+1}) \\ & -\frac{\Delta t}{2}\left[H\left(\frac{\Delta^+_x\varphi^{n+\frac12}_{j,k}}{\Delta x},\frac{\Delta^+_y\varphi^{n+\frac12}_{j+1,k}}{\Delta y}\right)\right. \\ & \left.+H\left(\frac{\Delta^+_x\varphi^{n+\frac12}_{j,k+1}}{\Delta x},\frac{\Delta^+_y\varphi^{n+\frac12}_{j,k}}{\Delta y}\right)\right]\end{aligned}$$

where

$$\varphi^{n+\frac12}_{j,k} := \varphi_{j,k} - \frac{\Delta t}{2}H\Big(\frac{\varphi'_{j,k}}{\Delta x},\frac{\grave{\varphi}_{j,k}}{\Delta y}\Big),$$

and $\varphi'_{j,k}$ and $\grave{\varphi}_{j,k}$ are numerical differentials. For example, [23, 12, 10], we can choose

$$\varphi'_{j,k} = MM(\varphi_{j+1,k}-\varphi_{j,k},\varphi_{j,k}-\varphi_{j-1,k})$$

and

$$\grave{\varphi}_{j,k} = MM(\varphi_{j,k+1}-\varphi_{j,k},\varphi_{j,k}-\varphi_{j,k-1})$$

where MM denotes the Min-Mod non-linear limiter

$$MM\{x_1, x_2, ...\} = \begin{cases} \min_j\{x_j\} & \text{if } x_j > 0, \forall j \\ \max_j\{x_j\} & \text{if } x_j < 0, \forall j \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We finally note that the second-order Godunov-type scheme designed above is an analogous extension of the first-order LxF scheme, Algorithm 1, as the family of second-order central schemes were constructed by Tadmor and his coworkers based on the first-order LxF scheme for conservation laws [23, 12, 10]. For a detailed discussion on this scheme, consult [20].

4 Numerical experiments

In this section, we implement our first- and second-order generalized Godunov-type schemes constructed in Sect. 3. Our test problems is the 2D periodic Burgers-type equation, i.e.,

$$\begin{cases} \varphi_t + H(\varphi_x, \varphi_y) = 0, & -2 \leq x, y \leq 2 \\ \varphi(x, y, 0) = -\cos(\pi \frac{x+y}{2}), \end{cases} \tag{4.1}$$

with a strictly convex Hamiltonian $H(u, v) = \frac{(u+v+1)^2}{2}$. The singularity occurs at time $t = 1/\pi^2$. We recorded the error at $t_1 = 0.5/\pi^2$ (before singularity) and $t_2 = 1.5/\pi^2$ (after singularity).

The exact solution can be found through the solution of the Burgers' equation after changing variables, consult Shu and Osher [28] for details. Before singularity occurs, the exact solution can also be found by the characteristic method, [21]. Since the L^1-error of the piecewise linear interpolant is of order $\mathcal{O}(\Delta x)$, we only need to check the order of the error under the following discrete l^1 norm,

$$\|e(\cdot, \cdot, T)\|_{l^1} := \sum_{i,j} |e(x_i, y_j, T)| \Delta x \Delta y.$$

The staggered LxF schemes, Algorithm 1 (A) (for mesh (A)), the non-staggered LxF schemes, Algorithm 2a (A) and Algorithm 2b (A), and the mini-Godunov scheme, Algorithm 3, are referred to LxF, $LxFn$, $LxFc$, and $Godm$ respectively. Resolution results are shown in Fig. 4.1. Before the singularity, errors and orders measured in L^1 are listed in Table 1, and in Table 2 for L^∞-norm. After the singularity, errors and orders measured in L^1 are listed in Table 3, and in Table 4 for L^∞-norm.

The results quoted in Table 1 and Table 3 for the first-order Godunov-type schemes, Algorithms 1–3, show L^1-convergence rate of order $\mathcal{O}(\Delta x)$. These apply for both before and *after* the formation of singularity, and thus confirms

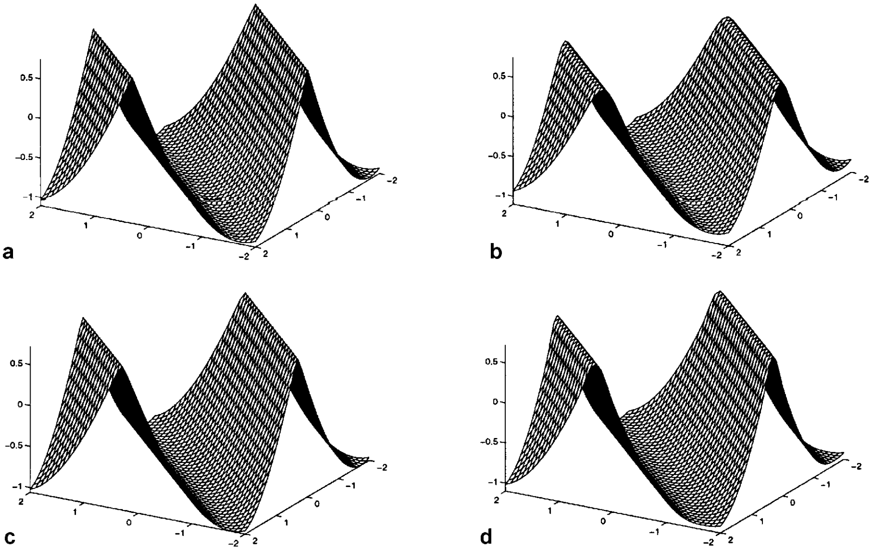

Fig. 4.1a–d. After the singularity: at $t = 1.5/\pi^2$, Mesh:40×40, **a** exact solution; **b** LxF; **c** Min-Godunov; **d** 2nd order central scheme

Table 1. L^1-Errors and orders before the singularity: at $t = 0.5/\pi^2$

	LxF		$LxFc$		$LxFn$		$Godm$	
N	error	order	error	order	error	order	error	order
20	1.2814	-	1.6957	-	1.6983	-	0.2996	-
40	0.5397	1.248	0.9941	0.770	0.9943	0.772	0.1534	0.965
80	0.2418	1.158	0.4779	1.057	0.4779	1.057	0.0778	0.980
160	0.1133	1.094	0.2411	0.987	0.2411	0.987	0.0390	0.995
320	0.0567	0.999	0.1191	1.018	0.1191	1.018	0.0196	0.996
640	0.0283	0.999	0.0601	0.985	0.0601	0.985	0.0098	0.999
1280	0.0141	1.012	0.0300	1.004	0.0300	1.004	0.0049	0.999

Table 2. L^∞-Errors and orders before the singularity: at $t = 0.5/\pi^2$

	LxF		$LxFc$		$LxFn$		$Godm$	
N	error	order	error	order	error	order	error	order
20	0.1651	-	0.2049	-	0.1944	-	0.0448	-
40	0.0754	1.131	0.1299	0.657	0.1267	0.618	0.0234	0.938
80	0.0361	1.064	0.0670	0.956	0.0659	0.944	0.0123	0.924
160	0.0172	1.066	0.0346	0.952	0.0343	0.941	0.0064	0.942
320	0.0088	0.971	0.0173	1.000	0.0172	0.994	0.0032	0.982
640	0.0044	0.987	0.0088	0.970	0.0088	0.967	0.0016	0.990
1280	0.0022	1.009	0.0044	1.000	0.0044	0.998	0.0008	0.995

Table 3. L^1-Errors and orders after the singularity: at $t = 1.5/\pi^2$

	LxF		$LxFc$		$LxFn$		$Godm$	
N	error	order	error	order	error	order	error	order
20	1.9204	-	3.4320	-	3.4856	-	0.6124	-
40	1.0867	0.821	1.9534	0.813	1.9703	0.823	0.3312	0.887
80	0.5103	1.091	1.0644	0.876	1.0433	0.917	0.1671	0.987
160	0.2507	1.025	0.5350	0.992	0.5357	0.962	0.0840	0.993
320	0.1263	0.990	0.2677	0.999	0.2678	1.000	0.0422	0.994
640	0.0629	1.004	0.1334	1.005	0.1334	1.005	0.0211	0.996
1280	0.0313	1.007	0.0667	0.999	0.0667	0.999	0.0106	0.997

Table 4. L^∞-Errors and orders after the singularity: at $t = 1.5/\pi^2$

	LxF		$LxFc$		$LxFn$		$Godm$	
N	error	order	error	order	error	order	error	order
20	0.1723	-	0.2957	-	0.3047	-	0.0734	-
40	0.0930	0.890	0.1644	0.847	0.1673	0.865	0.0378	0.958
80	0.0736	0.337	0.1621	0.021	0.1452	0.205	0.0192	0.978
160	0.0290	1.343	0.0796	1.027	0.0767	0.920	0.0097	0.986
320	0.0153	0.926	0.0362	1.138	0.0346	1.149	0.0049	0.994
640	0.0061	1.316	0.0140	1.372	0.0131	1.400	0.0024	0.997
1280	0.0029	1.072	0.0068	1.040	0.0064	1.037	0.0012	0.998

Table 5. Errors and orders for the 2nd-order scheme

	Time = $1/(2\pi^2)$				Time = $3/(2\pi^2)$			
	L^1		L^∞		L^1		L^∞	
N	error	order	error	order	error	order	error	order
20	0.35889	-	0.06713	-	0.54414	-	0.11403	-
40	0.09464	1.923	0.01852	1.858	0.18828	1.531	0.05742	0.990
80	0.02392	1.985	0.01166	0.668	0.04732	1.992	0.02200	1.384
160	0.00591	2.016	0.00468	1.317	0.01090	2.119	0.00990	1.290
320	0.00158	1.908	0.00196	1.255	0.00291	1.905	0.00512	0.812
640	0.00042	1.917	0.00081	1.276	0.00075	1.956	0.00323	0.666
1280	0.00011	1.953	0.00033	1.309	0.00019	1.962	0.00160	1.012

our main result regarding the optimality of the L^1-convergence rate. At the same time, Table 2 and Table 4 also record the same order of convergence rate when measured in the L^∞-norm. In particular, as quoted in Table 4, L^∞-convergence rate is of order $\mathcal{O}(\Delta x)$ after the formation of singularity. The discrepancy between the optimal L^∞-result of order $\mathcal{O}(\sqrt{\Delta x})$, [7,32] and the result of Table 4, is similar to the discrepancy in the L^1-error for convex conservation laws. Although the optimal result for conservation laws is of order $\mathcal{O}(\sqrt{\Delta x})$, e.g., [38,36,37], still, when computing with finitely many shock discontinuities, we find a convergence rate of order $\mathcal{O}(\Delta x)$, e.g., [38, 36]. In particular, since all we can compute are solutions with finitely many singularities, one can not distinguish between the convergence rates of first-order Godunov-type H-J solutions when measured either by L^1 or L^∞-norm – they both are of order $\mathcal{O}(\Delta x)$.

Table 5 quotes the results for the second-order Godunov-type scheme, Algorithm 4. Here the L^1-measure of the error achieves the expected convergence rate of order $\mathcal{O}(\Delta x)^2$, in contrast to a lower rate of order $\mathcal{O}(\Delta x)$ when measured in the L^∞-norm. Hence, we conclude that the L^1-norm is an appropriate measure for convergence rate of approximate solutions to convex H-J equations.

Acknowledgements. We thank Bo Li and Lihe Wang for their suggestions, and Ron DeVore and Stanley Osher for constructive discussions. Authors were partially supported by ONR Grant #N00014-91-J-1076, and NSF Grants #DMS97-06827 and #DMS96-23087.

5 Appendix: On the semi-concave stability of viscosity solutions

Although the semi-concave stability for vanishing viscosity regularizations – and hence, for the exact viscosity solution – is well-known [14, 21], we re-derive it here for completeness, and moreover, it will enable us to specify the exact error bound stated in (2.14).

Lemma 5.1 *(Semi-concavity) Let* $\{\varphi^\epsilon\}$ *be a family of solutions of the vanishing viscosity method (2.13):*

$$\begin{cases} \partial_t \varphi^\epsilon + H(\nabla_x \varphi^\epsilon) = \epsilon \Delta \varphi^\epsilon, & 0 < \alpha \le D_p^2 H(p) \le \beta < \infty. \\ \varphi^\epsilon(x, 0) = \varphi_0(x), \end{cases} \tag{5.1}$$

Then, $\{\varphi^\epsilon\}$ *is semi-concave, and the following upper-bound holds for any* $|\xi| = 1$*:*

$$W_\xi^+(t) \le k(t) = \frac{1}{W_\xi^+(0)^{-1} + \alpha t}; \qquad W_\xi^+(t) := \sup_x \langle \xi, D_x^2 \varphi^\epsilon(x,t) \xi \rangle^+. \tag{5.2}$$

In particular, we find

$$W^+(t) \le k(t) = \frac{1}{W^+(0)^{-1} + \alpha t}; \qquad W^+(t) := \sup_{|\xi|=1} W_\xi^+(t).$$

Proof. To this end, we start by differentiating the viscous H-J equation (5.1) twice w.r.t. x_i, x_j, and then take the inner product with a constant unit vector ξ to obtain:

$$\partial_t w + \langle \xi, D_x^2 \varphi^\epsilon \cdot D_p^2 H \cdot D_x^2 \varphi^\epsilon \xi \rangle + \langle \nabla_p H, \nabla_x w \rangle = \epsilon \Delta_x w. \tag{5.3}$$

Here $w := \frac{\partial^2}{\partial\xi^2}\varphi^\epsilon = \langle \xi, D_x^2\varphi^\epsilon\xi\rangle$ is the second directional derivative of φ^ϵ w.r.t. ξ. The strict convexity of H (which is bounded below by α in (2.1)) and the Cauchy-Schwartz inequality imply

$$\langle \xi, D_x^2\varphi^\epsilon \cdot D_p^2 H \cdot D_x^2\varphi^\epsilon\xi\rangle \geq \alpha\|D_x^2\varphi^\epsilon \cdot \xi\|^2 \geq \alpha\left\|\frac{\partial^2}{\partial\xi^2}\varphi^\epsilon\right\|^2 = \alpha\|w\|^2.$$

Returning to (5.3), we then find, using the usual cut-off arguments, that $w^+(x,t) := (\frac{\partial^2}{\partial\xi^2}\varphi^\epsilon)^+$ satisfies the inequality

$$\partial_t w^+ + \alpha(w^+)^2 + \langle \nabla_p H, \nabla_x w^+\rangle \leq \epsilon\Delta_x w^+.$$

Hence, $W_\xi^+(t) = \sup_x w^+(x,t)$ satisfies the following Ricatti equation,

$$\partial_t W_\xi^+ + \alpha(W_\xi^+)^2 \leq 0,$$

which in turn leads to the desired semi-concave stability estimate (5.2). ■

Remarks.

5.1. The semi-concave stability holds for vanishing viscosity regularizations, and hence the exact viscosity solutions, in each direction. Analogous results hold for OSLC stability for convex conservation laws.

5.2. Let m denote the semi-concave stable upper bound of the initial data, i.e. $D_x^2\varphi_0^\epsilon \leq m$. Then, semi-concave stability (2.1) follows with $k(t) = \frac{1}{m^{-1}+\alpha t}$, which in turn, applied to the constants, $C_0(T)$ and $C_1(T)$ in (2.11), yields Corollary 2.2 with

$$C_i(T) = (m^{-1} + \alpha T)^{\frac{d\beta}{\alpha}}, \qquad i = 0, 1.$$

Equipped with these bounds, we revisit our error estimates in Sect. 2.1. The truncation error for periodic boundary conditions is upper bounded in (2.15). Using this together with the L^1-error estimate in Corollary 2.2, and the explicit value of C_i's above, we arrive at the explicit L^1-error bound:

$$\|\varphi^\epsilon(\cdot,t) - \varphi(\cdot,t)\|_{L^1} \leq 2\epsilon \cdot |\Omega| \cdot (m^{-1} + \alpha T)^{\frac{d\beta}{\alpha}} \int_0^T k(t)dt, \quad 0 \leq t \leq T.$$

We note that, the last inequality with $\alpha = \beta$ yields

$$\|\varphi^\epsilon(\cdot,t) - \varphi(\cdot,t)\|_{L^1} = \mathcal{O}(\epsilon t^d \ln t).$$

5.3. The above arguments fail in case the upper semi-concave bound of the initial data $m=\infty$, for $k(t)=\frac{1}{\alpha t}\notin L^1[0,T]$. However, this kind of singularity will be "smoothed out" immediately as time evolves, (e.g., sharp corner of an initial 'cup' function can no longer exist when $t>0$.) This enables to extend our results to include such initial sharp corners ($m=\infty$) along the lines of [26].

Next we turn to show that the regularized (and hence the exact) solution of H-J equation (2.1) satisfy the following discrete version of semi-concave stability. This discrete version, which differs from Lemma 5.1, is essential for studying the semi-concave stability of finite difference approximations.

Lemma 5.2 *(Discrete Semi-concavity.) Let $\{\varphi^\epsilon(x)\}$ be the family of vanishing viscosity solutions of (5.1). Then, $\{\varphi^\epsilon(x)\}$ is discrete semi-concave stable in the sense that, for any given fixed $h>0$, the usual second-order discrete difference,*

$$D^2_{\xi,h}\varphi^\epsilon(x,t):=\frac{1}{h^2}\left\{\varphi^\epsilon(x+h\xi,t)-2\varphi^\epsilon(x,t)+\varphi^\epsilon(x-h\xi,t)\right\},$$

satisfies

$$W_{h,\xi}(t)\le W_{h,\xi}(0),\qquad W_{h,\xi}(t):=\sup_x D^2_{\xi,h}\varphi^\epsilon(x,t)\,. \tag{5.4}$$

Proof. To prove that $D^2_{\xi,h}\varphi^\epsilon(x,t)$ is upper bounded, we first take temporal derivative over $D^2_{\xi,h}\varphi(x,t)$. We obtain

$$\begin{aligned}
&\partial_t D^2_{\xi,h}\varphi^\epsilon(x,t)\\
&=\frac{1}{h^2}\left\{\partial_t\varphi^\epsilon(x+h\xi,t)-2\partial_t\varphi^\epsilon(x,t)+\partial_t\varphi^\epsilon(x-h\xi,t)\right\}\\
&=-\frac{1}{h^2}\left\{H(\nabla_x\varphi^\epsilon(x+h\xi,t))-2H(\nabla_x\varphi^\epsilon(x,t))\right.\\
&\quad\left.+H(\nabla_x\varphi^\epsilon(x-h\xi,t))\right\}+\epsilon\Delta_x D^2_{\xi,h}\varphi^\epsilon(x,t)\\
&=-\frac{1}{h^2}\left\{H_p(\nabla_x\varphi^\epsilon(x,t))\cdot\nabla_x\varphi^\epsilon(x,t,h,\xi)^+ + H_p(\nabla_x\varphi^\epsilon(x,t))\right.\\
&\quad\cdot\nabla_x\varphi^\epsilon(x,t,h,\xi)^- H_{pp}(\eta_1)(\nabla_x\varphi^\epsilon(x,t,h,\xi)^+)^2\\
&\quad\left.+H_{pp}(\eta_2)(\nabla_x\varphi^\epsilon(x,t,h,\xi)^-)^2\right\}+\epsilon\Delta_x D^2_{\xi,h}\varphi^\epsilon(x,t)
\end{aligned} \tag{5.5}$$

where

$$\nabla_x\varphi^\epsilon(x,t,h,\xi)^+:=\nabla_x\varphi^\epsilon(x+h\xi,t)-\nabla_x\varphi^\epsilon(x,t),$$

$$\nabla_x\varphi^\epsilon(x,t,h,\xi)^-:=\nabla_x\varphi^\epsilon(x,t)-\nabla_x\varphi^\epsilon(x-h\xi,t).$$

The last equation (5.5) is obtained by Taylor expansion of $H(\nabla_x\varphi^\epsilon(x \pm h\xi, t))$ w.r.t. $\nabla_x\varphi^\epsilon(x,t)$. Note that

$$\nabla_x\varphi^\epsilon(x + h\xi, t) - \nabla_x\varphi^\epsilon(x,t) = \nabla_x\varphi^\epsilon(x,t) - \nabla_x\varphi^\epsilon(x - h\xi, t)$$

and

$$\Delta_x D^2_{\xi,h}\varphi^\epsilon(x,t) \le 0,$$

when we evaluate at the maximum point of $D^2_{\xi,h}\varphi^\epsilon(x,t)$. By the strict convexity of H, the last equality (5.5) does not exceed

$$\begin{aligned}
&-\frac{1}{2h^2}\left\{H_{pp}(\eta_1)(\nabla_x\varphi^\epsilon(x,t,h,\xi)^+)^2 + H_{pp}(\eta_2)(\nabla_x\varphi^\epsilon(x,t,h,\xi)^-)^2\right\}\\
&\le -\frac{\alpha}{2h^2}\left\{(\nabla_x\varphi^\epsilon(x,t,h,\xi)^+)^2 + (\nabla_x\varphi^\epsilon(x,t,h,\xi)^-)^2\right\}\\
&\le 0\ .
\end{aligned}$$

Take sup on both sides which leads to

$$\frac{\partial}{\partial t}W_{h,\xi}(t) \le 0,$$

and hence completes the proof. ■

We close this Appendix with the following remarks.

Remarks.

5.1. Of course, by letting $\epsilon \downarrow 0$, the discrete semi concavity (5.4) implies a similar upper bound on the exact viscosity solution

$$W_\xi(t) := D^2_{x,\xi}\varphi(x,t) \le W_\xi(0).$$

(We ignore the decay factor $\frac{1}{W_\xi^{-1}(0)+\alpha t}$; in either case, we find L^1_{loc}-integrable bound.)

5.2. We note however, that the discrete semi-concave stability (5.4) is not equivalent with its continuous analogue (5.2). For example, in the 1D case Lemma 5.1 is a refinement of Oleinik's OSLC, yet the discrete version Theorem 5.2 states a new one-sided Lip-bound for convex conservation laws which was introduced in [16]. This discrete version is therefore, the appropriate framework to study the semi-concave stability of discrete Godunov-type schemes.

References

1. G. Barles. Solution de viscosite des equations de Hamilton-Jacobi. Springer, Berlin, 1994
2. Y. Brenier and S. Osher. The discrete one-side Lipschitz condition for convex scalar conservation laws. SIAM J. Numer. Anal. **25**, 8–23 (1988)
3. P.G. Ciarlet and P.-A. Raviart. General Lagrange and Hermite interpolation in R^n with applications to finite element methods. Arch. Rational Mech. Anal. **46**, 177–199 (1972)
4. M.G. Crandall, L. C. Evans, and P. L. Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. Trans. AMS **282**, 487–502 (1984)
5. M.G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. Bull. AMS **27**, 1–67 (1992)
6. M.G. Crandall and P. L. Lions. Viscosity solutions of Hamilton-Jacobi equations. Trans. AMS **277**, 1–42 (1983)
7. M.G. Crandall and P. L. Lions. Two approximations of solutions of Hamilton-Jacobi equations. Math. Comp. **43**, 1–19 (1984)
8. S.K. Godunov. A finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics. Mat. Sb. **47**, 271–290 (1959)
9. A. Harten, J.M. Hyman, and P.D. Lax. On finite-difference approximations and entropy conditions. Comm. Pure Appl. Math. **29**, 297–322 (1976)
10. G.S. Jiang, D. Levy, C.-T. Lin, S. Osher, and E. Tadmor. High-resolution non-oscillatory central schemes with non-staggered grids for hyperbolic conservation laws. SIAM J. Numer. Anal. **35**, 2147–2168 (1998)
11. G.S. Jiang and D. Peng. Weighted ENO schemes for Hamilton-Jacobi equations. SISSC. **21**, 2126–2143 (2000)
12. G.S. Jiang and E. Tadmor. Nonoscillatory central schemes for multidimensional hyperbolic conservation laws. SISSC. **19**, 1892–1917 (1998)
13. S. Jin and Z. Xin. Numerical passage from systems of conservation laws to Hamilton-Jacobi equations. SIAM J. Numer. Anal., pages 2385–2404 (1998)
14. S.N. Kruzkov. The Cauchy problem in the large for nonlinear equations and for certain quasilinear systems of the first order with several variables. Soviet Math. Dokl. **5**, 493–496 (1964)
15. S.N. Kruzkov. General solutions of nonlinear first order with several variables. II. Math. USSR Sbornik **1**, 93–116 (1967)
16. A. Kurganov. Conservation laws: Stability of numerical approximations and nonlinear regularization. PhD thesis, Tel-Aviv University, 1997
17. N.N. Kuznetsov. Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation. USSR Comp. Math. Math. Phys. **16**, 105–119 (1976)
18. N.N. Kuznetsov. On stable methods for solving nonlinear first-order partial differential equations in the class of discontinuous solutions. in Topics in Numerical Analysis III, J. J. H. Miller (ed.), Academic Press, London, pages 183–197 (1977)
19. N.N. Kuznetsov and S. A. Volosin. On monotone difference approximations for a first order quasilinear equation. Soviet. Math. Dokl. **17**, 1203–1206 (1976)
20. C.-T. Lin and E. Tadmor. High-resolution non-oscillatory central schemes for Hamilton-Jacobi equations. SISSC **21**, 2163–2186, (2000)
21. P.L. Lions. Generalized Solutions of Hamilton-Jacobi Equations. Pitman, London, 1982
22. P.L. Lions and P.E. Souganidis. Convergence of MUSCL and filtered schemes for scalar conservation laws and Hamilton-Jacobi equations. Numer. Math. **69**, 441–470 (1995)
23. H. Nessyahu and E. Tadmor. Non-oscillatory central differencing for hyperbolic conservation laws. JCP **87**, 408–463 (1990)

24. H. Nessyahu and E. Tadmor. The convergence rate of approximate solutions for nonlinear scalar conservation laws. SIAM Num. Anal. **29**, 1505–1519 (1992)
25. H. Nessyahu, E. Tadmor, and T. Tassa. The convergence rate of Godunov type schemes. SIAM Num. Anal. **31**, 1–16 (1994)
26. H. Nessyahu and T. Tassa. Convergence rate of approximate solutions to conservation laws with initial rarefactions. SIAM Num. Anal. **31**, 628–654 (1994)
27. O.A. Oleinik. Discontinuous solutions of nonlinear differential equations. Transl. AMS **26**, 95–172 (1963)
28. S. Osher and C.-W. Shu. High-order essential nonoscillatory schemes for Hamilton-Jacobi equations. SIAM J. Numer. Anal. **28**, 907–922 (1991)
29. F. Sabac. The optimal convergence rate of monotone finite difference methods for hyperbolic conservation laws. SIAM Num Anal. **34**, 2306–2318 (1997)
30. R. Sanders. On convergence of monotone finite difference schemes with variable spatial differencing. Math. Comp. **40**, 91–106 (1993)
31. S. Schochet and E. Tadmor. The regularized Chapman-Enskog expansion for scalar conservation laws. Arch. Rational Mech. Anal. **119**, 95–107 (1992)
32. P.E. Souganidis. Approximation schemes for viscosity solutions of Hamilton-Jacobi equations. J. Diff. Eqs. **59**, 1–43 (1985)
33. P.E. Souganidis. Existence of viscosity solutions of Hamilton-Jacobi equations. J. Diff. Eqs. **56**, 345–390 (1985)
34. E. Tadmor. Local error estimates for discontinuous solutions of nonlinear hyperbolic equations. SIAM Num. Anal. **28**, 891–906 (1991)
35. E. Tadmor. Total variation and error estimates for spectral viscosity approximations. Math. Comp. **60**, 245–256 (1993)
36. E. Tadmor and T. Tang. Pointwise error estimates for scalar conservation laws with piecewise smooth solutions. SIAM J. Numer. Anal. **36**, 1739–1756 (1999)
37. E. Tadmor and T. Tassa. On the piecewise regularity of entropy solutions to scalar conservation laws. Com. PDE **18**, 1631–1652 (1993)
38. T. Tang and Z.-H. Teng. The sharpness of Kuznetsov's O($\sqrt{\Delta x}$) L^1-error estimate for monotone difference schemes. Math. Comp. **64**, 581–589 (1995)