Mime: Mimicking Centralized Stochastic Algorithms in Federated Learning

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Abstract

Federated learning is a challenging optimization problem due to the heterogeneity of the data across different clients. Such heterogeneity has been observed to induce *client drift* and significantly degrade the performance of algorithms designed for this setting. In contrast, centralized learning with centrally collected data does not experience such drift, and has seen great empirical and theoretical progress with innovations such as momentum, adaptivity, etc. In this work, we propose a general framework MIME which mitigates client-drift and adapts arbitrary centralized optimization algorithms (e.g. SGD, Adam, etc.) to federated learning. MIME uses a combination of control-variates and server-level statistics (e.g. momentum) at every client-update step to ensure that each local update mimics that of the centralized method run on iid data. Our thorough theoretical and empirical analyses strongly establish MIME's superiority over other baselines.

Introduction

Federated learning has become an important paradigm in large-scale machine learning where the training data remains distributed over a large number of clients, which may be mobile phones or network sensors [20, 19, 23, 24, 13]. A centralized model (referred to as server model) is then trained without ever transmitting client data over the network, providing basic levels of data privacy and security. In *cross-device* federated learning, the number of such clients may be extremely large e.g. there are more than 3.5 billion active android phones [9]. Thus, we potentially may never make even a single pass over the entire clients' data, giving rise to unique challenges [13]. In this work, we formalize and investigate stochastic optimization algorithms for cross-device federated learning.

The de facto standard algorithm for this setting is FEDAVG [23] which performs multiple local SGD updates on the available clients before communicating to the server. While this approach can reduce the total amount of communication required, performing multiple steps on the same client can lead to 'over-fitting' to its atypical local data, a phenomenon known as *client drift* [15]. Further, algorithmic innovations such as momentum [30, 4], adaptivity [17, 41, 42], and clipping [38, 39, 43] are critical for success in deep learning applications and need to be incorporated into the client updates, replacing the SGD update of FEDAVG. Perhaps due to such deficiencies, there exists a large gap in performance between the centralized setting (where data is centrally collected on the server) and the federated setting [44, 22, 10, 11, 15, 27].

To overcome such deficiencies, we propose a new framework MIME which mitigates client drift and adapts arbitrary centralized optimization algorithms (e.g. SGD with momentum, Adam, etc.) to the federated setting. In each local client update, MIME uses global statistics (e.g. momentum) and an SVRG style correction [12] in order to mimic the updates of the centralized algorithm run on iid data. These global statistics are computed only at the server level and kept fixed throughout the local steps, ensuring that they do not get biased by the atypical local data of a single client.

Contributions. We summarize our main results below.

- We formalize the cross-device federated learning problem, and propose a new framework MIME which can adapt arbitrary centralized algorithms to this setting.
- We prove that incorporating momentum based variance reduction [4, 33] into each local client update reduces client drift and obtains the optimal statistical rates.
- Further, we show that using additional local steps improves the optimization term. This is the first result to demonstrate the usefulness of local steps for general functions.
- Finally, we also propose a simpler variant MIMELITE with similar empirical performance to MIME. We experimentally verify that both MIME and MIMELITE are faster than FEDAVG.

Related work. A lot of recent effort in federated learning focuses on analyzing FEDAVG. For identical clients FEDAVG coincides with parallel SGD analyzed by [45] who proved asymptotic convergence. In [28], and more recently in [29, 26, 16, 37], sharper and refined analyses of the same method (sometimes called local SGD) for identical functions. Their analysis was extended to heterogeneous clients in [36, 40, 15, 16, 18]. Matching upper and lower bounds, proving FEDAVG can be slower than even SGD for heterogeneous data due to *client-drift*, were given recently in [15].

An alternative algorithm called SCAFFOLD was proposed by [15] for heterogeneous data which uses control-variates (similar to SVRG) to correct for drift. However, their algorithm requires stateful clients and is hence only applicable when there is a small number of clients (e.g. hospitals), but not to the cross-device setting studied here. In their setting, [15] also prove the usefulness of local updates, but only for quadratic functions. We generalize their techniques to all functions. In a different approach, [11] and [35] observed that using *server momentum* significantly improves over vanilla FEDAVG. This idea was generalized by [27] who replace the server update with an arbitrary optimizer (e.g. Adam). However, these methods only modify the server update while using SGD for the client updates. MIME, on the other hand, ensures that every *local client update* resembles the optimizer e.g. MIME would apply momentum in every client update and not just at the server level.

2 Problem setup

This section formalizes the problem of cross-device federated learning. We first examine some key challenges of this setting (cf. [13]) to ensure our formalism captures the difficulty:

- 1. Very large overhead during communication between the server and the clients implying that the key metric of concern is the number of communication rounds.
- 2. Each client will likely participate at most once due to the extremely large number of clients. Further, each individual client may have very little data of its own.
- 3. There may be large heterogeneity (non-iid-ness) in the data present on the different clients.

We then, in the fewest number of client-server communication rounds, want to minimize

$$f(\boldsymbol{x}) = \mathbb{E}_{i \sim \mathcal{D}} \left[f_i(\boldsymbol{x}) := \frac{1}{n_i} \sum_{\nu=1}^{n_i} f_i(\boldsymbol{x}; \zeta_{i,\nu}) \right]. \tag{1}$$

Here, f_i represents the loss function on client i and $\{\zeta_{i,1},\ldots,\zeta_{i,n_i}\}$ the local data. Since the number of clients is extremely large whereas size of local data is small, we represent the former as an expectation and the latter as a finite sum. In each round, the algorithm samples a subset of clients (of size S) and computes some updates to the server model. There is some inherent tension between the second and the third challenge we outlined above: if there exists a client with arbitrarily different data whom we may never encounter during training, then there is no hope to actually minimize f. Thus for (1) to be tractable, we need to assume some bounded dissimilarity between different f_i .

(A1) G^2 -BGD or bounded gradient dissimilarity: there exists $G \ge 0$ such that

$$\mathbb{E}_{i \sim \mathcal{D}}[\|\nabla f_i(\boldsymbol{x}) - \nabla f(\boldsymbol{x})\|^2] \le G^2, \ \forall \boldsymbol{x}.$$

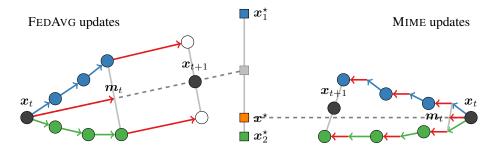


Figure 1: Client-drift in FEDAVG (left) and MIME (right) is illustrated for 2 clients with 3 local steps and momentum parameter $\beta=0.5$. The local SGD updates of FEDAVG (shown using arrows for client 1 and client2) move towards the average of client optima $\frac{x_1^*+x_2^*}{2}$ which can be quite different from the true global optimum x^* . Server momentum only speeds up the convergence to the wrong point in this case. In contrast, MIME uses unbiased momentum and applies it locally at every update. This keeps the updates of MIME closer to the true optimum x^* .

(A2) δ -BHD or bounded Hessian dissimilarity: Almost surely, f is δ -weakly convex i.e. $\nabla^2 f_i(x) \succeq -\delta I$ and the loss function of any client i satisfies

$$\|\nabla^2 f_i(\boldsymbol{x};\zeta) - \nabla^2 f(\boldsymbol{x})\| \le \delta, \ \forall \boldsymbol{x}.$$

In addition, we some standard assumptions that f is bounded from below by f^{\star} and that f(x) is L-smooth. Note that is all $f_i(\cdot;\zeta)$ is L-smooth, (A2) is always satisfied with $\delta \leq 2L$. However, in realistic examples we expect $\delta \ll L$.

3 Using momentum to reduce client drift

In this section we examine the tension between reducing communication by running multiple client updates each round, and degradation in performance due to client drift [15]. To simplify discussion, we assume a single client is sampled each round and that clients use full-batch gradients.

Server-only approach. A simple way to avoid the issue of client drift is to take no local steps. We sample a client $i \sim \mathcal{D}$ and run SGDm with momentum parameter β and step size η :

$$\mathbf{x}_{t} = \mathbf{x}_{t-1} - \eta \left((1 - \beta) \nabla f_{i}(\mathbf{x}_{t-1}) + \beta \mathbf{m}_{t-1} \right),$$

$$\mathbf{m}_{t} = (1 - \beta) \nabla f_{i}(\mathbf{x}_{t-1}) + \beta \mathbf{m}_{t-1}.$$
 (2)

Here, the gradient $\nabla f_i(\boldsymbol{x}_t)$ is *unbiased* i.e. $\mathbb{E}[\nabla f_i(\boldsymbol{x}_t)] = \nabla f(\boldsymbol{x}_t)$ and hence we are guaranteed convergence. However, this strategy can be communication-intensive and we are likely to spend all our time waiting for communication with very little time spent on computing the gradients.

FedAvg approach. To reduce the overall communication rounds required, we need to make more progress in each round of communication. Starting from $y_0 = x_{t-1}$, FEDAVG [23] runs multiple SGD steps on the sampled client $i \sim \mathcal{D}$

$$\mathbf{y}_{k} = \mathbf{y}_{k-1} - \eta \nabla f_{i}(\mathbf{y}_{k-1}) \text{ for } k \in [K],$$
(3)

and then a pseudo-gradient $\tilde{g}_t = -(y_K - x_t)$ replaces $\nabla f_i(x_{t-1})$ in the SGDm algorithm (2). This is referred to as server-momentum since it is computed and applied only at the server level [11]. However, such updates give rise to *client-drift* resulting in performance worse than the naive server-only strategy (2). This is because by using multiple local updates, (3) starts over-fitting to the local client data, optimizing $f_i(x)$ instead of the actual global objective f(x). The net effect is that FEDAVG moves towards an incorrect point (see Fig 1, left). If K is sufficiently large, approximately

$$egin{aligned} oldsymbol{y}_K &\leadsto oldsymbol{x}_i^\star, \ ext{where } oldsymbol{x}_i^\star := rg \min_{oldsymbol{x}} f_i(oldsymbol{x}) \ &\Rightarrow \mathbb{E}_{i \sim \mathcal{D}}[ilde{oldsymbol{g}}_t] &\leadsto (oldsymbol{x}_t - \mathbb{E}_{i \sim \mathcal{D}}[oldsymbol{x}_i^\star]) \,. \end{aligned}$$

Further, the server momentum is based on \tilde{g}_t and hence is also biased. Thus, it cannot correct for the client drift. We next see how a different way of using momentum could mitigate client drift.

Algorithm 1 Mime and MimeLite

```
input: initial x and s, learning rate \eta and base algorithm \mathcal{B} = (\mathcal{U}, \mathcal{V})
for each round t = 1, \dots, T do
    sample subset S of clients
   communicate (x, s) to all clients i \in S
   communicate c \leftarrow \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \nabla f_i(x) (only for Mime)
   on client i \in \mathcal{S} in parallel do
       initialize local model y_i \leftarrow x
       for k = 1, \dots, K do
           sample mini-batch ζ from local data
            y_i \leftarrow y_i - \eta \mathcal{U}(\nabla f_i(y_i; \zeta) - \nabla f_i(x; \zeta) + c, s) (Mime)
            y_i \leftarrow y_i - \eta \mathcal{U}(\nabla f_i(y_i; \zeta), s) (MimeLite)
       end for
       compute full local-batch gradient \nabla f_i(\boldsymbol{x})
       communicate (y_i, \nabla f_i(x))
   end on client
   m{x} \leftarrow rac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} m{y}_i, and m{s} \leftarrow \mathcal{V}\Big(rac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} 
abla f_i(m{x}), \ m{s}\Big)
```

Mime approach. FEDAVG experiences client drift because both the momentum and the client updates are biased. To fix the former, we compute momentum using only global statistics as in (2):

$$\boldsymbol{m}_t = (1 - \beta) \nabla f_i(\boldsymbol{x}_{t-1}) + \beta \boldsymbol{m}_{t-1}. \tag{4}$$

To reduce the bias in the local updates, we will apply this unbiased momentum every step:

$$y_k = y_{k-1} - \eta((1-\beta)\nabla f_i(y_{k-1}) + \beta m_{t-1}) \text{ for } k \in [K], \text{ where } .$$
 (5)

Note that the momentum term is kept fixed during the local updates i.e. there is no local momentum used, only global momentum is applied locally. Since m_{t-1} is a moving average of unbiased gradients computed over multiple clients, it intuitively is a good approximation of the general direction of the updates. By taking a convex combination of the local gradient with m_{t-1} , the update (5) is potentially also less biased. In this way MIME combines the communication benefits of taking multiple local steps and prevents client-drift (see Fig 1, right). Sec. 5 makes this intuition precise.

4 Mime framework

In this section we describe how to adapt arbitrary centralized algorithms (and not just SGDm) to the federated learning problem (1) while ensuring there is no client-drift. Algorithm 1 describes two variants MIME and MIMELITE, which consists of three components i) a base algorithm we are trying to mimic, ii) how we compute the global statistics, and iii) the local client updates.

Base algorithm. We assume the centralized base algorithm we are imitating can be decomposed into two steps: an *update step* \mathcal{U} which updates the parameters \boldsymbol{x} , and a *statistics step* $\mathcal{V}(\cdot)$ which keeps track of global statistics \boldsymbol{s} . Each step of the base algorithm $\mathcal{B} = (\mathcal{U}, \mathcal{V})$ uses a gradient \boldsymbol{g} to

$$egin{aligned} oldsymbol{x} \leftarrow oldsymbol{x} - \eta \, \mathcal{U}(oldsymbol{g}, oldsymbol{s}) \,, \ oldsymbol{s} \leftarrow \mathcal{V}(oldsymbol{g}, oldsymbol{s}) \,. \end{aligned}$$
 (BASEALG)

V may track multiple statistics which we represent collectively as s. While SGDm (2) is clearly of this form, Appendix A shows this for other algorithms like Adam, etc.

Compute statistics globally, apply locally. When updating the statistics of the base algorithm, we use only the gradient computed at the server parameters. Further, they remain fixed throughout the local updates of the clients. This ensures that these statistics remain unbiased and representative of the global function $f(\cdot)$. At the end of the round, the server performs

$$s \leftarrow \mathcal{V}\left(\frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \nabla f_i(\boldsymbol{x}), \ s\right), \quad \text{ where } \nabla f_i(\boldsymbol{x}) = \frac{1}{n_i} \sum_{\nu=1}^{n_i} \nabla f_i(\boldsymbol{x}; \zeta_{i,\nu}).$$
 (STATS)

Note that we use full-batch gradients computed at the server parameters x, not client parameters y_i .

Table 1: Number of communication rounds required to reach $\|\nabla f(x)\|^2 \le \epsilon$ for L-smooth functions (log factors are ignored) with S clients sampled each round. G^2 bounds the gradient dissimilarity (A1), and δ bounds the Hessian dissimilarity (A2). MIME matches the optimal statistical rates (first term in the rates) of the server-only methods while improving the optimization (second) term (typically $\delta \ll L$). FEDAVG is slower than the server-only methods due to additional drift terms.

Algorithm	Non-convex	μ -Strongly convex	Assumptions
SERVER-ONLY			
SGD [8]	$\frac{G^2}{S\epsilon^2} + \frac{L}{\epsilon}$	$\frac{G^2}{\mu S \epsilon} + \frac{L}{\mu}$	G^2 -BGD
MVR [4]	$rac{G^2}{S\epsilon^2} + rac{L}{\epsilon} \ \left(rac{G}{\sqrt{S}\epsilon} ight)^{rac{3}{2}} + rac{L}{\epsilon}$	_	G^2 -BGD
FEDAVG ¹			
FedSGD [15]	$\frac{G^2}{S\epsilon^2} + \left \frac{G}{\epsilon^{3/2}} \right + \frac{L}{\epsilon}$	$\frac{G^2}{\mu S \epsilon} + \frac{G}{\mu \sqrt{\epsilon}} + \frac{L}{\mu}$	G^2 -BGD
MIME ²			
MimeSGD	$\frac{G^2}{S\epsilon^2} + \frac{\delta}{\epsilon}$	$rac{G^2}{\mu S\epsilon} + rac{\delta}{\mu}$	G^2 -BGD, δ -BHD
MimeMVR	$rac{G^2}{S\epsilon^2} + rac{\delta}{\epsilon} \ \left(rac{G}{\sqrt{S}\epsilon} ight)^{rac{3}{2}} + rac{\delta}{\epsilon}$	_	G^2 -BGD, δ -BHD
Lower bound [1]	$\Omega(\frac{G}{\sqrt{S}\epsilon})^{\frac{3}{2}}$	$\Omega(\frac{G^2}{S\epsilon})$	G^2 -BGD

 $[\]frac{1}{1} \text{ Requires } K \geq \frac{\sigma^2}{G^2} \text{ number of local updates with within-client variance of } \sigma^2.$ $\frac{1}{2} \text{ Requires } K \geq L/\delta \text{ number of local updates.}$

Local client updates. Each client $i \in \mathcal{S}$ performs K updates using \mathcal{U} of the base algorithm and a minibatch gradient. There are two variants possible corresponding to MIME and MIMELITE differentiated using colored boxes. Starting from $y_i \leftarrow x$, repeat the following K times

$$egin{aligned} oldsymbol{y}_i &\leftarrow oldsymbol{y}_i - \eta \mathcal{U}(oldsymbol{g}, oldsymbol{s}) \,, \,\, ext{where} \ oldsymbol{g} &\leftarrow \nabla f_i(oldsymbol{y}_i; \zeta) - \nabla f_i(oldsymbol{x}; \zeta) + rac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \nabla f_i(oldsymbol{x}) \,\,\, ext{or} \,\, rac{
abla f_i(oldsymbol{y}_i; \zeta)}{|\mathcal{S}|} \,. \end{aligned}$$

While MIMELITE simply uses the local minibatch gradient, MIME uses an SVRG style correction [12]. This is done to mitigate the effect of using minibatch gradients instead of full gradients over the local client data. However, in deep learning applications we found that MIMELITE closely matches the performance of MIME, though our theory needs the SVRG correction.

Finally, in practical federated learning applications, there are two modifications made: we weigh all averages across the clients by the number of datapoints n_i , and we perform K epochs instead of K steps [23]. The former modifies the objective (1) with f_i being weighted by n_i , and the latter has been empirically observed to perform better, but lacks strong justification.

Theoretical analysis of Mime 5

Table 1 summarizes the rates of server-only methods, FEDAVG, and MIME (new results highlighted in blue). Our theory focuses on MIME with base algorithm of SGD with momentum based variance reduction¹ (MimeMVR) since it obtains optimal rates. It is also possible to derive a more general reduction-type result proving that MIME converges whenever the base algorithm it is mimicking converges, which we leave for future work.

Theorem I. For L-smooth f with G^2 gradient dissimilarity (A1), δ Hessian dissimilarity (A2) and $F:=(f(\boldsymbol{x}^0)-f^\star)$, the output of MimeMVR after T rounds \boldsymbol{x}^{out} satisfies $\mathbb{E}\|\nabla f(\boldsymbol{x}^{out})\|^2 \leq \epsilon$ for

• PL-Strongly convex without momentum: for
$$\eta = \tilde{\mathcal{O}}\Big(\min\Big(\frac{1}{\delta K + \mu K + L}\,, \frac{1}{\mu T}\Big)\Big)$$
, $\beta = 0$, and

¹The momentum based variance reduction (MVR), introduced by [4], is a modification of the standard SGDm algorithm to make it amenable to analysis. All our theory uses MVR, while our experiments use SGDm.

$$T = \tilde{\mathcal{O}}\left(\frac{LG^2}{\mu S\epsilon} + \frac{L + \delta K}{\mu K} F \log\left(\frac{1}{\epsilon}\right)\right),\,$$

• Non-convex without momentum: for $\eta = \mathcal{O}\left(\min\left(\frac{1}{\delta K + L}, \left(\frac{SF}{G^2TK^2}\right)^{1/2}\right)\right)$, $\beta = 0$, and $T = \mathcal{O}\left(\frac{LG^2F}{S\epsilon^2} + \frac{(L + \delta K)F}{\epsilon K}\right)$,

$$T = \mathcal{O}\left(\frac{LG^2F}{S\epsilon^2} + \frac{(L + \delta K)F}{\epsilon K}\right),\,$$

• Non-convex with momentum: for $\eta = \mathcal{O}\left(\min\left(\frac{1}{\delta K + L}, \left(\frac{SF}{G^2TK^3}\right)^{1/3}\right)\right)$, $\beta = 1 - \mathcal{O}(\eta^2 \delta^2 K^2)$,

$$T = \mathcal{O}\left(\left(\frac{(1+\delta)G^2F}{S\epsilon^2}\right)^{3/4} + \frac{(L+\delta K)F}{\epsilon K}\right).$$

Additional proofs and theorems have been relegated to Appendices E-F. On comparing the rates in Theorem I with those of FEDAVG in Table 1, we see that MIME does not have the additional drift terms. In the rest of this section, we give proof sketches of the main components of Theorem I: i) how momentum reduces the effect of client drift, ii) how local steps can take advantage of Hessian similarity, and iii) why the SVRG correction improves constants.

Improving the statistical term via momentum. Note that the statistical (first) term in Theorem I without momentum $(\beta=0)$ for the convex case is $\frac{LG^2}{\mu S\epsilon}$. This is (up to constants) optimal and cannot be improved. For the non-convex case however using $\beta=0$ gives the usual rate of $\frac{LG^2}{S\epsilon^2}$. However, this can be improved to $\left(\frac{(1+\delta)G^2F}{S\epsilon^2}\right)^{3/4}$ using momentum. This matches a similar improvement in the centralized setting [4, 33] and is in fact optimal [1]. Let us examine why momentum improves the statistical term. Assume that we sample a single client i_t in round t and that we use full-batch gradients. Also let the local client update at step k round t be of the form

$$\mathbf{y} \leftarrow \mathbf{y} - \eta \mathbf{d}_k$$
. (6)

The ideal choice of update is of course $d_k^\star = \nabla f(\boldsymbol{y})$ but however this is unattainable. Instead, MIME with momentum $\beta = 1 - a$ uses $\boldsymbol{d}_k^{\text{SGDm}} = \tilde{\boldsymbol{m}}_k \leftarrow a \nabla f_i(\boldsymbol{y}) + (1-a) \boldsymbol{m}_{t-1}$ where \boldsymbol{m}_{t-1} is the momentum computed at the server. The variance of this update can then be bounded as

$$\mathbb{E}\|\tilde{\boldsymbol{m}}_{k} - \nabla f(\boldsymbol{y})\|^{2} \lesssim a^{2} \mathbb{E}\|\nabla f_{i_{t}}(\boldsymbol{y}) - \nabla f(\boldsymbol{y})\|^{2} + (1-a) \mathbb{E}\|\boldsymbol{m}_{t-1} - \nabla f(\boldsymbol{y})\|^{2}$$

$$\approx a^{2}G^{2} + (1-a) \mathbb{E}\|\boldsymbol{m}_{t-1} - \nabla f(\boldsymbol{x}_{t-2})\|^{2} \approx aG^{2}.$$

The last step follows by unrolling the recursion on the variance of m. We also assumed that η is small enough that $y \approx x_{t-2}$. This way, momentum can reduce the variance of the update from G^2 to (aG^2) by using past gradients computed on different clients. To formalize the above sketch requires slightly modifying the momentum algorithm similar to [4], and is carried out in Appendix F.

Improving the optimization term via local steps. The optimization (second) term in Theorem I for the convex case is $\frac{\delta K + L}{\mu K}$ and for the non-convex case (with or without momentum) is $\frac{\delta K + L}{\epsilon K}$. In contrast, the optimization term of the server-only methods is L/μ and L/ϵ respectively. Since in most cases $\delta \ll L$, the former can be significantly smaller than the latter. This rate also suggests that the best choice of number of local updates is L/δ i.e. we should perform more client updates when they have more similar Hessians. This generalizes results of [15] from quadratics to all functions.

This improvement is due to a careful analysis of the bias in the gradients computed during the local update steps. Note that for client parameters y_{k-1} , the gradient $\mathbb{E}_{i_t}[\nabla f_{i_t}(y_{k-1})] \neq \nabla f(y_{k-1})$ since y_{k-1} was also computed using the same loss function f_{i_t} . In fact, only the first gradient computed at x_{t-1} is unbiased. Dropping the subscripts k and t, we can bound this bias as:

$$\begin{split} \mathbb{E}_i[\nabla f_i(\boldsymbol{y})] - \nabla f(\boldsymbol{y}) &= \underbrace{\nabla f_i(\boldsymbol{y}) - \nabla f_i(\boldsymbol{x})}_{\approx \nabla^2 f_i(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})} + \underbrace{\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}_i)}_{\approx \nabla^2 f(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{y}_i)} + \underbrace{\mathbb{E}_i[\nabla f_i(\boldsymbol{x})] - \nabla f(\boldsymbol{x})}_{=0 \text{ since unbiased}} \\ &\approx (\nabla^2 f_i(\boldsymbol{x}) - \nabla^2 f(\boldsymbol{x}))(\boldsymbol{y}_i - \boldsymbol{x}) \approx \delta(\boldsymbol{y}_i - \boldsymbol{x}) \,. \end{split}$$

Thus, the Hessian dissimilarity (A2) control the bias, and hence the usefulness of local updates. This holds true for both MimeSGD and MimeMVR with proofs in Appendices E, F.

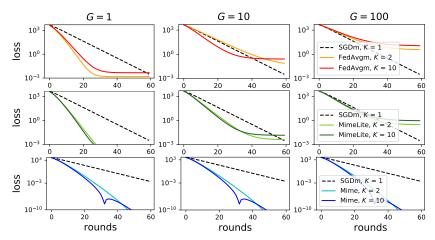


Figure 2: **SGDm** (dashed black), **FedSGDm** (top), **MimeLiteSGDm** (middle), and **MimeSGDm** (bottom) on simulated data, all with momentum ($\beta=0.5$). FedAvg gets slower as the gradient-dissimilarity (G) increases (to the right). MimeLite shows a similar pattern, but is consistently better than FedAvg. Mime is significantly faster than both, and its performance is identical as we vary heterogeneity (G). In all cases, using K=2 steps gives similar performance to K=10.

Mini-batches via SVRG correction. In our previous discussion about momentum and local steps, we assumed that the clients compute full batch gradients and that only one client is sampled per round. However, in practice a large number (S) of clients are sampled and further the clients use mini-batch gradients. The SVRG correction reduces this within-client variance since

$$Var\left(\nabla f_i(\boldsymbol{y}_i;\zeta) - \nabla f_i(\boldsymbol{x};\zeta) + \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \nabla f_i(\boldsymbol{x})\right) \lesssim L^2 \|\boldsymbol{y}_i - \boldsymbol{x}\|^2 + \frac{G^2}{S} \approx \frac{G^2}{S}.$$

Here, we used the smoothness of $f_i(\cdot; \zeta)$ and assumed that $y_i \approx x$ since we don't move too far within a single round. Thus, the SVRG correction allows us to use minibatch gradients in the local updates while still ensuring that the variance is of the order G^2/S .

6 Experimental analysis

We run experiments on simulated and real (EMNIST62 and CIFAR100) datasets to confirm our theory. Our main findings are i) MIME outperforms FEDAVG across all settings, ii) its SVRG correction is useful for convex problems, and iii) momentum improves performance for non convex problems.

We consider four algorithms: SERVER-ONLY, FEDAVG, MIME, and MIMELITE. Each of these adapt base optimizers SGD, SGDm, and Adam. The SERVER-ONLY method computes a full batch gradient on each of the sampled clients and uses their aggregate directly in the base optimizer (akin to (2)). For FEDAVG, we follow [27] who run multiple epochs of SGD on each client sampled, and then aggregate the net client updates. This aggregated update is used as a pseudo-gradient in the base optimizer (called server optimizer). The learning rate for the server optimizer is fixed to 1 as in [35, 32]. This is done to ensure all algorithms have the same number of hyper-parameters. Finally, MIME and MIMELITE follow Algorithm 1 and also run a fixed number of epochs on the client. All aggregation is weighted by the number of samples on the client as is standard [23, 27].

6.1 Simulated convex experiments

Our simulated experiments use two clients each with a simple scalar quadratic loss, same as the lower-bound example in [15]. We use full-batch gradients with both clients participating every round. The simulated data has Hessian dissimilarity $\delta=1$ (A2) and smoothness L=2. We vary the gradient dissimilarity (A1) as $G\in[1,10,100]$. All the algorithms use momentum with $\beta=0.5$ and their learning rates were tuned up to a tolerance of 5E-3 to ensure lowest loss after 60 rounds.

The results are collected in Fig. 2. When G is small, we see that FEDAVG can outperform the SERVER-ONLY (SGDm) baseline, though its loss quickly plateaus. On increasing G, FEDAVG becomes even slower. MIMELITE differs from FEDAVG only in how the momentum is used. In all settings, it slightly outperforms FEDAVG though even it sees a substantial slow down as we increase

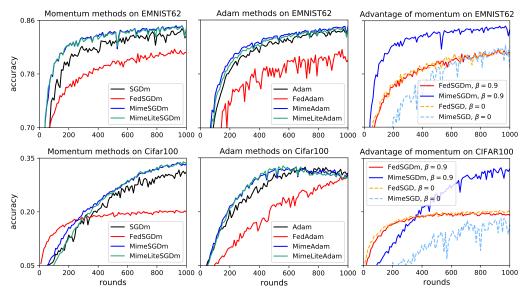


Figure 3: **Server-only**, **FedAvg**, **Mime**, and **MimeLite** with SGDm (left) and Adam (middle) run on EMNIST62 (top) and CIFAR100 (bottom). Mime and MimeLite have very similar performance and are consistently the best. FedAvg is often even worse than the server-only baselines. Also, Mime makes better use of momentum than FedAvg, with a large increase in performance (right).

G. This reflects our theory which predicts that for convex cases, momentum does not give significant gains. MIME, on the other hand, is substantially faster than all other methods and is even unaffected by changing G. This is because the SVRG correction is extremely useful in this simple setting to reduce the variance and completely eliminate client drift. Finally, note that in all cases, there is no significant difference between K=2 and 10, exactly as predicted by our theory since $L/\delta=2$.

6.2 Real world datasets

We run extensive experiments on the federated EMNIST62 [2] with a 2 layer 300u-100 MLP, and federated CIFAR100 [27] with Resnet20. For SGD, we search for β over [0, 0.9, 0.99] and for Adam, we fix $\beta_1 = 0.9, \beta_2 = 0.99, \epsilon$ =1E-3. The learning rate for all methods was individually tuned. All methods take 10 local epochs on the clients, with 20 clients sampled per round. We refer to Appendix B for additional experiments and details. The results are collected in Fig 3.

Mime \approx MimeLite > Server-only > FedAvg. In all cases, Mime and MimeLite have the best performance. FedAvg is most of the time (except with Adam on Cifar100) slower than even the naive server-only methods which make no local updates. This perfectly mirrors our theory that Mime > server-only > FedAvg. The performance of MimeLite can be attributed to the observation that the SVRG correction may not be necessary in deep learning [5, 34].

With momentum > without momentum. Fig. 3 (right) examines the impact of momentum on FedAvg and Mime. Momentum slightly improves the performance of FedAvg, whereas it has a significant impact on the performance of Mime. This is also in line with our theory and confirms that Mime's strategy of applying it locally at every client update makes better use of momentum.

Fixed statistics > **updated statistics.** Finally, we check how the performance of Mime changes if instead of keeping the momentum fixed throughout a round, we let it change. The momentum is reset at the end of the round ignoring the changes the clients make to it. Appendix B shows that this consistently *worsens* the performance, confirming that it is better to keep the statistics fixed.

Together, the above observations validate all aspects of Mime (and MimeLite) design: compute statistics at the server level, and apply them unchanged at every client update.

7 Conclusion

Our work initiated a formal study of the cross-device federated learning problem. We argued that the natural heterogeneity among the clients gives rise to client drift and significantly hampers the performance of approaches such as FEDAVG. We then showed how momentum can be an excellent

tool to overcome this client drift if used correctly. Based on this observation, we introduced a new framework MIME which not only overcomes client drift, but also adapts arbitrary centralized algorithms such as Adam to the federated setting without any additional hyper-parameters. We demonstrated the superiority of MIME via strong convergence guarantees and empirical evaluations.

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Appendix

A Additional algorithmic details

TI 1 1 2 D	1 1 1.1 1	. 1 . (2.1)	1
Table 2: Decomposing	base algorithms into	a parameter update (14) a	nd statistics tracking (\mathcal{V}) .

Algorithm	Tracked statistics s	Update step \mathcal{U}	Tracking step V
SGD	_	$oldsymbol{x} - \eta oldsymbol{g}$	_
SGDm	m	$oldsymbol{x} - \eta((1-eta)oldsymbol{g} + etaoldsymbol{m})$	$\boldsymbol{m} = (1 - \beta)\boldsymbol{g} + \beta \boldsymbol{m}$
RMSProp	$oldsymbol{v}$	$oldsymbol{x} - rac{\eta}{\epsilon + \sqrt{oldsymbol{v}}} oldsymbol{g}$	$\boldsymbol{v} = (1 - \beta)\boldsymbol{g}^2 + \beta \boldsymbol{v}$
Adam	$m{m}, m{v}$	$oldsymbol{x} - rac{\eta}{\epsilon + \sqrt{oldsymbol{v}}}((1 - eta_1)oldsymbol{g} + eta_1oldsymbol{m})$	$egin{aligned} oldsymbol{m} &= (1-eta_1) oldsymbol{g} + eta_1 oldsymbol{m} \ oldsymbol{v} &= (1-eta_2) oldsymbol{g}^2 + eta_2 oldsymbol{v} \end{aligned}$

B Additional experimental details

B.1 Effect of changing statistics

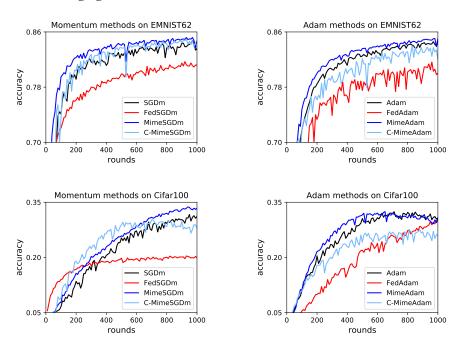


Figure 4: **Server-only**, **FedAvg**, **Mime**, and **C-Mime** with SGDm (left) and Adam (right) run on EMNIST62 (top) and CIFAR100 (bottom). C-Mime changes the statistics (momentum for SGDm, and first two moments for Adam) using the local client updates. These changes are discarded at the end of the round and the statistics are reset using only the server level gradients as in Mime. Clearly, C-Mime is always worse than Mime. This shows that adapting the statistics during the local client updates makes them too biased, and it best to keep them fixed during each round like Mime does.

B.2 Description of datasets

We use Tensorflow federated datasets [31] to generate the datasets. Our federated learning simulation code is written in Jax [7]. Our Resnet20 model is based off of [6] (Resnet v1), and following [10, 27] we replace batch norm with group norm with 2 groups. Black and white was reversed in EMNIST62 (i.e. subtracted from 1) to make them similar to MNIST. CIFAR100 used the usual pre-processing (normalization and centering), and data augmentation (random crop and horizontal flipping) following [21].

Table 3: Details about the datasets used and experiment setting.

	EMNIST62	CIFAR100
Clients	3,400	500
Examples	671,585	50,000
Sampled clients	20	20
Batch size	20	20
Number of epochs	10	10
Model	2 layer MLP (300-100)	Resnet20

B.3 Hyperparameter search

The learning rate is searched over a grid of

$$\eta \in [1 \times 10^{1}, 1, 1 \times 10^{-1}, 1 \times 10^{-2}, 1 \times 10^{-3}, 1 \times 10^{-4}, 1 \times 10^{-5}].$$

For SGDm, we search for the momentum parameter over

$$\beta \in [0, 0.9, 0.99]$$
.

For Adam, we fix $\beta_1 = 0.9$, $\beta_2 = 0.99$, and $\varepsilon = 1 \times 10^{-3}$ similar to [27]. None of the algorithms use weight decay, clipping etc.

B.4 Comparison with previous results

Since there are so few baselines for the cross-device setting, it is unclear what test accuracy should be targeted. Our results qualitatively match those of [27]. They compared FedSGD, FedSGDm, and FedAdam and found that i) these methods were comparable for EMNIST62, and ii) FedAdam was the best on CIFAR100. Our experiments also show a similar result. However, we have additional baselines of server-only methods (SGDm, Adam) which are strong competitors, outperforming the afore mentioned methods. This shows that including server-only methods is an important baseline to consider as well. A caveat in comparing our results with those of [27] is that we fix the server learning rate for FEDAVG to be 1. This was done so that all methods have equal number of hyperparameters. This is also the default behavior in TensorFlow Federated [32] and is also recommended in [35]. We believe a large-scale experimental evaluation of all methods is important, though it is outside the scope of the current work.

C Technicalities

We examine some additional definitions and introduce some technical lemmas.

C.1 Additional definitions

We make precise a few definitions and explain some of their implications.

(A3) f is L-smooth and satisfies:

$$\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\| \le L\|\boldsymbol{x} - \boldsymbol{y}\|, \text{ for any } \boldsymbol{x}, \boldsymbol{y}.$$
 (7)

The assumption (A3) also implies the following quadratic upper bound on f

$$f(\mathbf{y}) \le f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{L}{2} \|\mathbf{y} - \mathbf{x}\|^2.$$
 (8)

(A4) f is μ -PL strongly convex [14] for $\mu \ge 0$ if it satisfies:

$$\|\nabla f(\boldsymbol{x})\|^2 \geq 2\mu(f(\boldsymbol{x}) - f^*).$$

Note that PL-strong convexity is much weaker than the standard notion of strong-convexity [14].

If f_i (and not just f) is twice-differentiable and satisfies (A3), then we have $\|\nabla^2 f_i(x) - \nabla^2 f(x)\| \le 2L$ for any x.

C.2 Some technical lemmas

Now we cover some technical lemmas which are useful for computations later on. First, we state a relaxed triangle inequality true for the squared ℓ_2 norm.

Lemma 1 (relaxed triangle inequality). Let $\{v_1, \ldots, v_{\tau}\}$ be τ vectors in \mathbb{R}^d . Then the following are true:

1.
$$\|\mathbf{v}_i + \mathbf{v}_j\|^2 \le (1+c)\|\mathbf{v}_i\|^2 + (1+\frac{1}{c})\|\mathbf{v}_j\|^2$$
 for any $c > 0$, and

2.
$$\|\sum_{i=1}^{\tau} \mathbf{v}_i\|^2 \le \tau \sum_{i=1}^{\tau} \|\mathbf{v}_i\|^2$$
.

Proof. The proof of the first statement for any c > 0 follows from the identity:

$$\|\mathbf{v}_i + \mathbf{v}_j\|^2 = (1+c)\|\mathbf{v}_i\|^2 + (1+\frac{1}{c})\|\mathbf{v}_j\|^2 - \|\sqrt{c}\mathbf{v}_i + \frac{1}{\sqrt{c}}\mathbf{v}_j\|^2$$
.

For the second inequality, we use the convexity of $oldsymbol{x} o \|oldsymbol{x}\|^2$ and Jensen's inequality

$$\left\| \frac{1}{ au} \sum_{i=1}^{ au} oldsymbol{v}_i
ight\|^2 \leq rac{1}{ au} \sum_{i=1}^{ au} \left\| oldsymbol{v}_i
ight\|^2.$$

Next we state an elementary lemma about expectations of norms of random vectors.

Lemma 2 (separating mean and variance). Let $\{\Xi_1, \ldots, \Xi_\tau\}$ be τ random variables in \mathbb{R}^d which are not necessarily independent. First suppose that their mean is $\mathbb{E}[\Xi_i] = \xi_i$ and variance is bounded as $\mathbb{E}[\|\Xi_i - \xi_i\|^2] \leq \sigma^2$. Then, the following holds

$$\mathbb{E}[\|\sum_{i=1}^{\tau} \Xi_i\|^2] \le \|\sum_{i=1}^{\tau} \xi_i\|^2 + \tau^2 \sigma^2.$$

Now instead suppose that their conditional mean is $\mathbb{E}[\Xi_i|\Xi_{i-1},\ldots\Xi_1]=\xi_i$ i.e. the variables $\{\Xi_i-\xi_i\}$ form a martingale difference sequence, and the variance is bounded by $\mathbb{E}[\|\Xi_i-\xi_i\|^2] \leq \sigma^2$ as before. Then we can show the tighter bound

$$\mathbb{E}[\|\sum_{i=1}^{\tau} \Xi_i\|^2] \le 2\|\sum_{i=1}^{\tau} \xi_i\|^2 + 2\tau\sigma^2.$$

Proof. For any random variable X, $\mathbb{E}[X^2] = (\mathbb{E}[X - \mathbb{E}[X]])^2 + (\mathbb{E}[X])^2$ implying

$$\mathbb{E}[\|\sum_{i=1}^{\tau} \Xi_i\|^2] = \|\sum_{i=1}^{\tau} \xi_i\|^2 + \mathbb{E}[\|\sum_{i=1}^{\tau} \Xi_i - \xi_i\|^2].$$

Expanding the above expression using relaxed triangle inequality (Lemma 1) proves the first claim:

$$\mathbb{E}[\|\sum_{i=1}^{\tau} \Xi_i - \xi_i\|^2] \le \tau \sum_{i=1}^{\tau} \mathbb{E}[\|\Xi_i - \xi_i\|^2] \le \tau^2 \sigma^2.$$

For the second statement, ξ_i is not deterministic and depends on Ξ_{i-1}, \ldots, Ξ_1 . Hence we have to resort to the cruder relaxed triangle inequality to claim

$$\mathbb{E}[\|\sum_{i=1}^{\tau} \Xi_i\|^2] \le 2\|\sum_{i=1}^{\tau} \xi_i\|^2 + 2\mathbb{E}[\|\sum_{i=1}^{\tau} \Xi_i - \xi_i\|^2]$$

and then use the tighter expansion of the second term:

$$\mathbb{E}[\|\sum_{i=1}^{\tau} \Xi_i - \xi_i\|^2] = \sum_{i,j} \mathbb{E}[(\Xi_i - \xi_i)^{\top} (\Xi_j - \xi_j)] = \sum_i \mathbb{E}[\|\Xi_i - \xi_i\|^2] \le \tau \sigma^2.$$

The cross terms in the above expression have zero mean since $\{\Xi_i - \xi_i\}$ form a martingale difference sequence.

D Properties of functions with δ bounded Hessian dissimilarity

We now study two lemmas which hold for any functions which satisfy (A2). The first is closely related to the notion of smoothness (A3).

Lemma 3 (similarity). The following holds for any two functions $f_i(\cdot; \zeta)$ and $f(\cdot)$ satisfying (A2), and any x, y:

$$\|\nabla f_i(\boldsymbol{y};\zeta) - \nabla f_i(\boldsymbol{x};\zeta) + \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^2 \le \delta^2 \|\boldsymbol{y} - \boldsymbol{x}\|^2$$
.

Proof. Consider the function $\Psi(z) := f_i(z;\zeta) - f(z)$. By the assumption (A2), we know that $\|\nabla^2\Psi(z)\| \le \delta$ for all z i.e. Ψ is δ -smooth. By standard arguments based on taking limits [25], this implies that

$$\|\nabla \Psi(\boldsymbol{y}) - \nabla \Psi(\boldsymbol{x})\| < \delta \|\boldsymbol{y} - \boldsymbol{x}\|.$$

Plugging back the definition of Ψ into the above inequality proves the lemma.

Next, we see how weakly-convex functions satisfy a weaker notion of "averaging does not hurt".

Lemma 4 (averaging). Suppose f is δ -weakly convex. Then, for any $\gamma \geq \delta$, and a sequence of parameters $\{y_i\}_{i \in S}$ and x:

$$\frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} f(\boldsymbol{y}_i) + \frac{\gamma}{2} \|\boldsymbol{x} - \boldsymbol{y}_i\|^2 \geq f(\bar{\boldsymbol{y}}) + \frac{\gamma}{2} \|\boldsymbol{x} - \bar{\boldsymbol{y}}\|^2, \text{ where } \bar{\boldsymbol{y}} := \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \boldsymbol{y}_i.$$

Proof. Since f is δ -weakly convex, $\Phi(z) := f(z) + \frac{\gamma}{2} \|z - x\|^2$ is convex. This proves the first claim since $\frac{1}{|S|} \sum_{i \in S} \Phi(y_i) \leq \Phi(\bar{y})$.

E Analysis of MimeSGD (without momentum)

Let us rewrite the MimeSGD update using notation convenient for analysis. In each round t, we sample clients \mathcal{S}^t such that $|\mathcal{S}^t| = S$. The server communicates the server parameters \boldsymbol{x}^{t-1} as well as the average gradient across the sampled clients \boldsymbol{c}^{t-1} defined as

$$\boldsymbol{c}^{t-1} = \frac{1}{S} \sum_{i \in S^t} \nabla f_i(\boldsymbol{x}^{t-1}). \tag{9}$$

Note that computing c^{t-1} itself requires two rounds of communication. It is indeed possible to actually use $c^{t-1} = \frac{1}{S} \sum_{i \in \mathcal{S}^t} \nabla f_i(\boldsymbol{x}^{t-2})$ instead as we will see in the next section. For now, we ignore this since it only changes the communication complexity by a constant.

Then each client $i \in \mathcal{S}^t$ makes a copy $\boldsymbol{y}_{i,0}^t = \boldsymbol{x}^{t-1}$ and perform K local client updates. In each local client update $k \in [K]$, the client samples a dataset $\zeta_{i,k}^t$ and

$$\boldsymbol{y}_{i,k}^{t} = \boldsymbol{y}_{i,k-1}^{t} - \eta(\nabla f_i(\boldsymbol{y}_{i,k-1}^{t}; \zeta_{i,k}^{t}) - \nabla f_i(\boldsymbol{x}^{t-1}; \zeta_{i,k}^{t}) + \boldsymbol{c}^{t-1}).$$
(10)

After K such local updates, the server then aggregates the new client parameters as

$$\boldsymbol{x}^t = \frac{1}{S} \sum_{i \in \mathcal{S}^t} \boldsymbol{y}_{i,K}^t \,. \tag{11}$$

Variance of update. Consider the local update at step k on client i, dropping superscript t

$$\mathbf{y}_{i,k} = \mathbf{y}_{i,k-1} - \eta \mathbf{d}_{i,k}$$
, where $\mathbf{d}_{i,k} := \nabla f_i(\mathbf{y}_{i,k-1}; \zeta_{i,k}) - \nabla f_i(\mathbf{x}; \zeta_{i,k}) + \mathbf{c}$.

Lemma 5. Given that assumptions (A1) and (A2) are satisfied, each client update satisfies

$$\mathbb{E}\|\boldsymbol{d}_{i,k}\|^{2} \leq \frac{3G^{2}}{S} + 3\delta^{2}\|\boldsymbol{y}_{i,k-1} - \boldsymbol{x}\|^{2} + 3\|\nabla f(\boldsymbol{y}_{i,k-1})\|^{2}.$$

Proof. Starting from the definition of $d_{i,k}$ and the relaxed triangle inequality,

$$\|\boldsymbol{d}_{i,k}\|^{2} = \|\nabla f_{i}(\boldsymbol{y}_{i,k-1};\zeta_{i,k}) - \nabla f_{i}(\boldsymbol{x};\zeta_{i,k}) + \boldsymbol{c}\|^{2}$$

$$= \|\nabla f_{i}(\boldsymbol{y}_{i,k-1};\zeta_{i,k}) - \nabla f_{i}(\boldsymbol{x};\zeta_{i,k}) + \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}_{i,k-1}) + (\boldsymbol{c} - \nabla f(\boldsymbol{x})) + \nabla f(\boldsymbol{y}_{i,k-1})\|^{2}$$

$$\leq 3\|\nabla f_{i}(\boldsymbol{y}_{i,k-1};\zeta_{i,k}) - \nabla f_{i}(\boldsymbol{x};\zeta_{i,k}) + \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}_{i,k-1})\|^{2} + 3\|\boldsymbol{c} - \nabla f(\boldsymbol{x})\|^{2} + 3\|\nabla f(\boldsymbol{y}_{i,k-1})\|^{2}$$

$$\leq 3\delta^{2}\|\boldsymbol{y}_{i,k-1} - \boldsymbol{x}\|^{2} + 3\|\boldsymbol{c} - \nabla f(\boldsymbol{x})\|^{2} + 3\|\nabla f(\boldsymbol{y}_{i,k-1})\|^{2}.$$

We used Lemma 3 to bound the first term. Taking expectations on both sides to bound the second term via (A1) yields the lemma.

Distance moved in each round. We show that the distance moved by a client in each round during the K updates can be controlled. To further reduce the burden of notation, we will drop he subscript i, k and refer $y_{i,k-1}$ simply as y and $y_{i,k}$ as y^+ .

Lemma 6. For update following (10) for $\eta \leq \frac{1}{4K\delta}$ satisfying (A1) and (A2), we have at any step k,

$$\mathbb{E}\|\boldsymbol{y}^{+} - \boldsymbol{x}\|^{2} \leq \left(1 + \frac{2}{K}\right)\|\boldsymbol{y} - \boldsymbol{x}\|^{2} + 6K\eta^{2}\frac{G^{2}}{S} + 6K\eta^{2}\|\nabla f(\boldsymbol{y})\|^{2}.$$

Proof. Starting from the update (10) and the relaxed triangle inequality Lemma 1 with $c = K \ge 1$,

$$\mathbb{E}\|\boldsymbol{y}^{+} - \boldsymbol{x}\|^{2} = \mathbb{E}\|\boldsymbol{y} - \eta \boldsymbol{d} - \boldsymbol{x}\|^{2} \\
\leq (1 + \frac{1}{c}) \,\mathbb{E}\|\boldsymbol{y} - \boldsymbol{x}\|^{2} + (1 + c)\eta^{2} \,\mathbb{E}\|\boldsymbol{d}\|^{2} \\
\leq (1 + \frac{1}{K}) \,\mathbb{E}\|\boldsymbol{y} - \boldsymbol{x}\|^{2} + 3(1 + K)\eta^{2} \frac{G^{2}}{S} + 3(1 + K)\eta^{2}\delta^{2}\|\boldsymbol{y} - \boldsymbol{x}\|^{2} + 3(1 + K)\eta^{2}\|\nabla f(\boldsymbol{y})\|^{2} \\
\leq (1 + \frac{1}{K} + 6K\eta^{2}\delta^{2}) \,\mathbb{E}\|\boldsymbol{y} - \boldsymbol{x}\|^{2} + 6K\eta^{2} \frac{G^{2}}{S} + 6K\eta^{2}\|\nabla f(\boldsymbol{y})\|^{2}.$$

The second to last step used the variance bound in Lemma 5. The proof now follows from the restriction on step-size since $16K^2\eta^2\delta^2 \le 1$.

Progress in one client update. We now have the tools required to keep track of the progress made in one round.

Lemma 7. For any constant $\mu \geq 0$ and each step of MimeSGD with step size $\eta \leq \min\left(\frac{1}{18L}, \frac{1}{756\delta K}, \frac{1}{42\mu K}\right)$, and given that (A1)–(A3) hold, we have

$$\frac{\eta}{4} \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 \le A_{i,k-1}^t - A_{i,k}^t + \frac{(255KL\eta^2)G^2}{2S},$$

where we define

$$\begin{split} A_{i,k}^t &:= \mathbb{E}[f(\boldsymbol{y}_{i,k}^t)] + \delta \Big(1 + \tfrac{3}{K}\Big)^{K-k} \, \mathbb{E}\|\boldsymbol{y}_{i,k}^t - \boldsymbol{x}^{t-1}\|^2 \,, \text{ and} \\ A_{i,k-1}^t &:= \mathbb{E}[f(\boldsymbol{y}_{i,k-1}^t)] + \delta (1 - \mu \eta) \Big(1 + \tfrac{3}{K}\Big)^{K-k+1} \, \mathbb{E}\|\boldsymbol{y}_{i,k-1}^t - \boldsymbol{x}^{t-1}\|^2 \,. \end{split}$$

Proof. The assumption that f is L-smooth implies a quadratic upper bound (8). Using this in our case, we have

$$\mathbb{E}[f(\boldsymbol{y}^{+})] - \mathbb{E}[f(\boldsymbol{y})] \leq -\eta \, \mathbb{E}[\langle \nabla f(\boldsymbol{y}), \boldsymbol{d} \rangle] + \frac{L\eta^{2}}{2} \, \mathbb{E} \|\boldsymbol{d}\|^{2}$$

$$= \underbrace{-\eta \, \mathbb{E}[\langle \nabla f(\boldsymbol{y}), \nabla f_{i}(\boldsymbol{y}; \zeta) - \nabla f_{i}(\boldsymbol{x}; \zeta) + \boldsymbol{c} \rangle]}_{\mathcal{T}_{1}} + \underbrace{\frac{L\eta^{2}}{2} \, \mathbb{E} \|\boldsymbol{d}\|^{2}}_{\mathcal{T}_{2}}.$$

Let us examine the terms \mathcal{T}_1 and \mathcal{T}_2 separately. By our variance bound Lemma 5, we have that

$$\mathcal{T}_2 \leq \frac{3L\eta^2 G^2}{2S} + \frac{3L\eta^2 \delta^2}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \frac{3L\eta^2}{2} \|\nabla f(\boldsymbol{y})\|^2.$$

To simplify \mathcal{T}_1 , the biggest obstacle is that $\mathbb{E}[\nabla f_i(\boldsymbol{y};\zeta)] \neq \nabla f(\boldsymbol{y})$ since \boldsymbol{y} itself depends on the sampling of the client i. Only the server gradient is unbiased and $\mathbb{E}[\boldsymbol{c}] = \nabla f(\boldsymbol{x})$. Instead we will use the similarity of the functions as in Lemma 3:

$$\mathcal{T}_{1} = -\eta \mathbb{E}[\langle \nabla f(\boldsymbol{y}), \nabla f_{i}(\boldsymbol{y}; \zeta) - \nabla f_{i}(\boldsymbol{x}; \zeta) + \nabla f(\boldsymbol{x}) \rangle]$$

$$\leq -\frac{\eta}{2} \mathbb{E} \|\nabla f(\boldsymbol{y})\|^{2} + \frac{\eta}{2} \|\nabla f_{i}(\boldsymbol{y}; \zeta) - \nabla f_{i}(\boldsymbol{x}; \zeta) + \nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|^{2}$$

$$\leq -\frac{\eta}{2} \mathbb{E} \|\nabla f(\boldsymbol{y})\|^{2} + \frac{\eta \delta^{2}}{2} \mathbb{E} \|\boldsymbol{y} - \boldsymbol{x}\|^{2}.$$

The first inequality above used that for any a,b, the following holds $-2ab=(a-b)^2-a^2-b^2\leq (a-b)^2-a^2$. The second used the similarity Lemma 3. Combining the terms \mathcal{T}_1 and \mathcal{T}_2 together, we have

$$\mathbb{E}[f(\boldsymbol{y}^+)] - \mathbb{E}[f(\boldsymbol{y})] \le \frac{(3L\eta^2 - \eta)}{2} \mathbb{E}\|\nabla f(\boldsymbol{y})\|^2 + \frac{(\eta\delta^2 + 3L\eta^2\delta^2)}{2} \mathbb{E}\|\boldsymbol{y} - \boldsymbol{x}\|^2 + \frac{3L\eta^2G^2}{2S}.$$

To bound the distance between \boldsymbol{y} and \boldsymbol{x} , we use Lemma 6 multiplied on both sides by $\delta \left(1+\frac{3}{K}\right)^{K-k}$. Note that $\delta \leq \delta \left(1+\frac{3}{K}\right)^{K-k} \leq 21\delta$. This gives us for any constant $\mu \geq 0$

$$\delta\left(1+\frac{3}{K}\right)^{K-k} \mathbb{E}\|\boldsymbol{y}^{+}-\boldsymbol{x}\|^{2} \leq \delta\left(1+\frac{3}{K}\right)^{K-k} \left(1+\frac{2}{K}\right)\|\boldsymbol{y}-\boldsymbol{x}\|^{2} + 6K\delta\left(1+\frac{3}{K}\right)^{K-k} \eta^{2} \frac{G^{2}}{S} + 6K\delta\left(1+\frac{3}{K}\right)^{K-k} \eta^{2} \|\nabla f(\boldsymbol{y})\|^{2}$$

$$\leq \delta(1-\mu\eta)\left(1+\frac{3}{K}\right)^{K-(k-1)} \|\boldsymbol{y}-\boldsymbol{x}\|^{2} + 6K\delta\left(1+\frac{3}{K}\right)^{K-k} \eta^{2} \frac{G^{2}}{S} + 6K\delta\left(1+\frac{3}{K}\right)^{K-k} \eta^{2} \|\nabla f(\boldsymbol{y})\|^{2} + \left(1+\frac{3}{K}\right)^{K-k} (\mu\eta\delta - \frac{\delta}{K})\|\boldsymbol{y}-\boldsymbol{x}\|^{2}$$

$$\leq \delta\left(1+\frac{3}{K}\right)^{K-(k-1)} \|\boldsymbol{y}-\boldsymbol{x}\|^{2} + 126K\delta\eta^{2} \frac{G^{2}}{S} + 126K\delta\eta^{2} \|\nabla f(\boldsymbol{y})\|^{2} + (21\mu\eta\delta - \frac{\delta}{K})\|\boldsymbol{y}-\boldsymbol{x}\|^{2}.$$

Adding the two bounds, we get the following recursion

$$\underbrace{\mathbb{E}[f(\boldsymbol{y}^{+})] + \delta\left(1 + \frac{3}{K}\right)^{K-k} \mathbb{E}\|\boldsymbol{y}^{+} - \boldsymbol{x}\|^{2}}_{=:A_{i,k}} \leq \underbrace{\mathbb{E}[f(\boldsymbol{y})] + \delta\left(1 + \frac{3}{K}\right)^{K-(k-1)} (1 - \mu\eta)\|\boldsymbol{y} - \boldsymbol{x}\|^{2}}_{=:A_{i,k-1}} + \frac{(252K\delta\eta^{2} + 3L\eta^{2} - \eta)}{2} \mathbb{E}\|\nabla f(\boldsymbol{y})\|^{2} + \left(\frac{(\eta\delta^{2} + 3L\eta^{2}\delta^{2} + 42\mu\eta\delta)}{2} - \frac{\delta}{K}\right) \mathbb{E}\|\boldsymbol{y} - \boldsymbol{x}\|^{2} + \frac{(3L\eta^{2} + 252K\delta\eta^{2})G^{2}}{2S}$$

Now, note that our constraint on the step-size $\eta \leq \min(\frac{1}{18L}, \frac{1}{756\delta K})$ implies that $252K\delta\eta^2 + 3L\eta^2 \leq \frac{\eta}{2}$ and $K(\eta\delta^2 + 3L\eta^2\delta^2 + 42\mu\eta\delta) \leq 2\delta$. Plugging this into the above bound and recalling that $\delta \leq L$ finishes the proof.

Convergence for PL strongly-convex functions. We will unroll the one step progress Lemma 7 to compute a linear rate.

Theorem II. Suppose that (A1)–(A4) are satisfied for $\mu > 0$. Then the updates of MimeSGD with step-size $\eta = \min(\eta_{\max}, \tilde{\mathcal{O}}\left(\frac{1}{\mu TK}\right))$ for $\eta_{\max} = \min\left(\frac{1}{18L}, \frac{1}{756\delta K}, \frac{1}{42\mu K}\right)$ satisfy

$$\mathbb{E}\|\nabla f(\boldsymbol{x}^{out})\|^2 \leq \tilde{\mathcal{O}}\bigg(\frac{LG^2}{\mu TS} + \frac{F}{\eta_{\text{max}}} \exp\bigg(-\frac{\mu}{18L + 756\delta K + 42\mu K}TK\bigg)\bigg)$$

where we define $F:=f(\boldsymbol{x}^0)-f^\star$, $\bar{\boldsymbol{y}}_k^t$ is chosen to be $\boldsymbol{y}_{i,k}^t$ for $i\in\mathcal{S}^t$ uniformly at random, and the output \boldsymbol{x}^{out} to be $\bar{\boldsymbol{y}}_k^t$ with probability proportional to $(1-\frac{\eta\mu}{4})^{KT-kt}$.

Proof. Note that by PL strong convexity (A4), we have

$$\frac{\eta}{4} \|\nabla f(\boldsymbol{y})\|^2 \leq \frac{\eta}{8} \|\nabla f(\boldsymbol{y})\|^2 + \frac{\eta \mu}{4} (f(\boldsymbol{y}) - f^{\star}).$$

Using this, we can tighten the one step progress Lemma 7 as

$$\frac{\eta}{8} \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2} \leq \underbrace{\left(1 - \frac{\mu\eta}{4}\right) \mathbb{E}[f(\boldsymbol{y}_{i,k-1}^{t}) - f^{\star}] + \delta\left(1 - \frac{\mu\eta}{4}\right) \left(1 + \frac{3}{K}\right)^{K-k+1} \mathbb{E} \|\boldsymbol{y}_{i,k-1}^{t} - \boldsymbol{x}^{t-1}\|^{2}}_{=:\left(1 - \frac{\mu\eta}{4}\right) \Phi_{i,k-1}^{t}} \\ - \underbrace{\mathbb{E}[f(\boldsymbol{y}_{i,k}^{t}) - f^{\star}] + \delta\left(1 + \frac{3}{K}\right)^{K-k} \mathbb{E} \|\boldsymbol{y}_{i,k}^{t} - \boldsymbol{x}^{t-1}\|^{2}}_{=:\Phi^{t}} + \frac{(255KL\eta^{2})G^{2}}{2S}},$$

Now take a weighted sum over the steps k using weights $(1 - \frac{\eta \mu}{4})^{K-k}$

$$\frac{\eta}{8} \sum_{k \in [K]} (1 - \frac{\eta \mu}{4})^{K - k} \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 \le \Phi_{i,0}^t - \left(1 - \frac{\mu \eta}{4}\right)^K \Phi_{i,K}^t + \sum_{k \in [K]} (1 - \frac{\eta \mu}{4})^{K - k} \frac{(255KL\eta^2)G^2}{2S}.$$

By the initialization $y_{i,0}^t = x^{t-1}$ and hence $\Phi_{i,0}^t = \mathbb{E} f(x^{t-1}) - f^*$ and further by the averaging Lemma 4, we have

$$\frac{1}{S} \sum_{i \in S} \Phi_{i,K}^t \ge \mathbb{E} f(\boldsymbol{x}^t) - f^{\star}.$$

Hence, on averaging over the clients we get the one round progress lemma

$$\begin{split} \frac{\eta}{8S} \sum_{k \in [K]} \sum_{i \in S^t} (1 - \frac{\eta \mu}{4})^{K-k} \, \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 &\leq \mathbb{E} \, f(\boldsymbol{x}^{t-1}) - f^\star - \left(1 - \frac{\mu \eta}{4}\right)^K (\mathbb{E} \, f(\boldsymbol{x}^t) - f^\star) \\ &+ \sum_{k \in [K]} (1 - \frac{\eta \mu}{4})^{K-k} \frac{(255KL\eta^2)G^2}{2S} \, . \end{split}$$

Now further taking a weighted average over the rounds $t \in [T]$ with weights proportional to $(1 - \frac{\eta \mu}{4})^{tK}$ gives

$$\begin{split} \frac{\eta}{8S} \sum_{t \in [T]} \sum_{k \in [K]} \sum_{i \in \mathcal{S}^t} (1 - \tfrac{\eta \mu}{4})^{KT - kt} \, \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 & \leq \mathbb{E} \, f(\boldsymbol{x}^0) - f^\star \\ & + \sum_{t \in [T]} \sum_{k \in [K]} (1 - \tfrac{\eta \mu}{4})^{KT - kt} \frac{(255KL\eta^2)G^2}{2S} \, . \end{split}$$

Finally, choosing the right step size, similar to Lemma 23 of [15] yields the desired rate. \Box

Convergence for general functions. We will unroll the one step progress Lemma 7 to compute a sublinear rate.

Theorem III. Suppose that (A1)–(A3) are satisfied. Then the updates of MimeSGD with step-size $\eta = \min\left(\eta_{\max}, \frac{\sqrt{FS}}{\sqrt{255K^2LG^2T}}\right)$ for $\eta_{\max} = \min\left(\frac{1}{18L}, \frac{1}{756\delta K}\right)$ satisfy

$$\frac{1}{KTS} \sum_{k \in [K]} \sum_{t \in [T]} \sum_{i \in \mathcal{S}^t} \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 \le \mathcal{O} \left(\frac{G\sqrt{LF}}{\sqrt{TS}} + \frac{(L + \delta K)F}{TK} \right).$$

where we define $F := f(\mathbf{x}^0) - f^*$.

Proof. By summing over the equations from Lemma 7 for all local steps in one round we obtain

$$\frac{\eta}{2} \sum_{k=1}^K \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 \leq A_{i,0}^t - A_{i,K}^t + \frac{255K^2L\eta^2G^2}{2S} \,.$$

By the initialization $\boldsymbol{y}_{i,0}^t = \boldsymbol{x}^{t-1}$, hence $A_{i,0}^t = A_{j,0}^t = \mathbb{E}[f(\boldsymbol{x}^{t-1})]$ for all $i,j \in \mathcal{S}^t$. Furthermore, by Lemma 4

$$\frac{1}{|\mathcal{S}^t|} \sum_{i \in \mathcal{S}_t} A_{i,K}^t \ge \mathbb{E}[f(\boldsymbol{x}^t)] + \delta \|\boldsymbol{x}^{t-1} - \boldsymbol{x}^t\|^2 \ge \mathbb{E}[f(\boldsymbol{x}^t)] = A_{i,0}^{t+1}$$

This means that we can keep unrolling over all rounds, obtaining

$$\frac{\eta}{2S} \sum_{t=1}^T \sum_{k=1}^K \sum_{i \in S^t} \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 \leq A_{i,0}^1 - A_{i,K}^T + \frac{255TK^2L\eta^2G^2}{2S} \,.$$

By noting $A_{i,0}^1 - A_{i,K}^T = (f(\boldsymbol{x}^0) - f^\star) - (\mathbb{E}[f(\boldsymbol{x}^T) - f^\star) \leq F$ and the choice of the stepsize the theorem follows.

F Analysis of MimeMVR (with momentum based variance reduction)

In this section we see how to use momentum based variance reduction [4] to reduce the variance of the updates and improve convergence. It should be noted that MVR does not exactly fit the MIME framework (BASEALG) since it requires computing gradients at two points on the same batch. However, it is straightforward to extend the idea of MIME to MVR as we will now do. We use MVR as a theoretical justification for why the usual momentum works well in practice. An interesting future direction would be to adapt the algorithm and analysis of [3], which does fit the framework of MIME.

MimeMVR algorithm. Now, we formally describe the MimeMVR algorithm. In each round t, we sample clients \mathcal{S}^t such that $|\mathcal{S}^t| = S$. The server communicates the server parameters \boldsymbol{x}^{t-1} , the momentum \boldsymbol{m}^{t-1} and the average gradient across the sampled clients \boldsymbol{c}^{t-1} defined as

$$\mathbf{c}^{t-1} = \frac{1}{S} \sum_{i \in S^t} \nabla f_i(\mathbf{x}^{t-2}).$$
 (12)

Note that both c^{t-1} and m^{t-1} use gradients and parameters from previous rounds (different from the previous section).

Then each client $i \in \mathcal{S}^t$ makes a copy $\boldsymbol{y}_{i,0}^t = \boldsymbol{x}^{t-1}$ and perform K local client updates. In each local client update $k \in [K]$, the client samples a dataset $\zeta_{i,k}^t$ and

$$\mathbf{y}_{i,k}^{t} = \mathbf{y}_{i,k-1}^{t} - \eta \mathbf{d}_{i,k}^{t}, \text{ where}
\mathbf{d}_{i,k}^{t} = a(\nabla f_{i}(\mathbf{y}_{i,k-1}^{t}; \zeta_{i,k}^{t}) - \nabla f_{i}(\mathbf{x}^{t-1}; \zeta_{i,k}^{t}) + \mathbf{c}^{t} - 1) + (1 - a)\mathbf{m}^{t-1}
+ (1 - a)(\nabla f_{i}(\mathbf{y}_{i,k-1}^{t}; \zeta_{i,k}^{t}) - \nabla f_{i}(\mathbf{x}^{t-1}; \zeta_{i,k}^{t})).$$
(13)

After K such local updates, the server then aggregates the new client parameters as

$$\boldsymbol{x}^t = \frac{1}{S} \sum_{j \in \mathcal{S}^t} \boldsymbol{y}_{j,K}^t \,. \tag{14}$$

The momentum term is updated at the end of the round for $a \ge 0$ as

$$\boldsymbol{m}^{t} = \underbrace{a(\frac{1}{S}\sum_{j \in \mathcal{S}^{t}} \nabla f_{j}(\boldsymbol{x}^{t-1})) + (1-a)\boldsymbol{m}^{t-1}}_{\text{SGDm}} + \underbrace{(1-a)(\frac{1}{S}\sum_{j \in \mathcal{S}^{t}} \nabla f_{j}(\boldsymbol{x}^{t-1}) - \nabla f_{j}(\boldsymbol{x}^{t-2}))}_{\text{correction}}.$$
(15)

As we can see, the momentum update of MVR can be broken down into the usual SGDm update, and a correction. Intuitively, this correction term is very small since f_i is smooth and $\boldsymbol{x}^{t-1} \approx \boldsymbol{x}^{t-2}$. Another way of looking at the update (15) is to note that if all functions are identical i.e. $f_j = f_k$ for any j, k, then (15) just becomes the usual gradient descent. Thus MimeMVR tries to maintain an exponential moving average of only the variance terms, reducing its bias. We refer to [4] for more detailed explanation of MVR.

Momentum variance bound. We compute the variance of the server momentum m^{t-1} . Define the variance term $V^t = m^t - \nabla f(x^{t-1})$. Then its expected norm can be bounded as follows.

Lemma 8. For the momentum update (15), given (A1) and (A2), the following holds for any $a \in [0,1]$ and $V^t := m^t - \nabla f(x^{t-1})$

$$\mathbb{E}{\left\|\boldsymbol{V}^{t}\right\|^{2}} \leq \left(1-a\right)\mathbb{E}{\left\|\boldsymbol{V}^{t-1}\right\|^{2}} + 2\delta^{2}\,\mathbb{E}{\left\|\boldsymbol{x}^{t-1}-\boldsymbol{x}^{t-2}\right\|^{2}} + \frac{2a^{2}G^{2}}{S}\,.$$

Proof. Starting from the momentum update (15),

$$\begin{split} V^t &= (1-a)V^{t-1} \\ &+ (1-a) \left(\frac{1}{S} \sum_{j \in \mathcal{S}^t} (\nabla f_j(\boldsymbol{x}^{t-1}) - \nabla f_j(\boldsymbol{x}^{t-2})) - \nabla f(\boldsymbol{x}^{t-1}) + \nabla f(\boldsymbol{x}^{t-2}) \right) \\ &+ a \left(\frac{1}{S} \sum_{j \in \mathcal{S}^t} (\nabla f_j(\boldsymbol{x}^{t-1}) - \nabla f(\boldsymbol{x}^{t-1}) \right). \end{split}$$

Now, the term V^{t-1} does not have any information from round t and hence is statistically independent of the rest of the terms. Further, the rest of the terms have mean 0. Hence, we can separate out the zero mean noise terms from the V^{t-1} following Lemma 2 and then the relaxed triangle inequality Lemma 1 to claim

$$\mathbb{E}\|V^{t}\|^{2} = (1-a)^{2} \mathbb{E}\|V^{t-1}\|^{2}
+ 2(1-a)^{2} \left\| \frac{1}{S} \sum_{j \in S^{t}} (\nabla f_{j}(\boldsymbol{x}^{t-1}) - \nabla f_{j}(\boldsymbol{x}^{t-2})) - \nabla f(\boldsymbol{x}^{t-1}) + \nabla f(\boldsymbol{x}^{t-2}) \right\|^{2}
+ 2a^{2} \left\| \frac{1}{S} \sum_{j \in S^{t}} (\nabla f_{j}(\boldsymbol{x}^{t-1}) - \nabla f(\boldsymbol{x}^{t-1}) \right\|^{2}
\leq (1-a)^{2} \mathbb{E}\|V^{t-1}\|^{2} + 2(1-a)^{2} \delta^{2} \|\boldsymbol{x}^{t-1} - \boldsymbol{x}^{t-2}\|^{2} + \frac{2a^{2}G^{2}}{S}.$$

The inequality used the Hessian similarity Lemma 3 to bound the second term and the heterogeneity bound (A1) to bound the last term. Finally, note that $(1-a)^2 \le (1-a) \le 1$ for $a \in [0,1]$.

Update variance bound. Now we examine the variance of our update in each local step $d_{i,k}^t$.

Lemma 9. For the client update (13), given (A1) and (A2), the following holds for any $a \in [0,1]$

$$\mathbb{E}\|\boldsymbol{d}_{i,k}^{t} - \nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2} \leq 3\mathbb{E}\|V^{t-1}\|^{2} + 3\delta^{2}\mathbb{E}\|\boldsymbol{y}_{i,k-1}^{t} - \boldsymbol{x}^{t-2}\|^{2} + \frac{3a^{2}G^{2}}{S}.$$

Proof. Starting from the client update (13), we can rewrite it as

$$\begin{aligned} \boldsymbol{d}_{i,k}^t - \nabla f(\boldsymbol{y}_{i,k-1}^t) &= (1-a)V^{t-1} \\ &\quad + \left(\nabla f_i(\boldsymbol{y}_{i,k-1}^t; \zeta_{i,k}^t) - \nabla f_i(\boldsymbol{x}^{t-2}; \zeta_{i,k}^t)\right) - \nabla f(\boldsymbol{y}_{i,k-1}^t + \nabla f(\boldsymbol{x}^{t-2})\right) \\ &\quad + a\left(\frac{1}{S}\sum_{j \in \mathcal{S}^t} \nabla f_j(\boldsymbol{x}^{t-2}) - \nabla f(\boldsymbol{x}^{t-2})\right). \end{aligned}$$

We can use the relaxed triangle inequality Lemma 1 to claim

$$\begin{split} \mathbb{E}\|\boldsymbol{d}_{i,k}^{t} - \nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2} &= 3(1-a)^{2} \, \mathbb{E}\|\boldsymbol{V}^{t-1}\|^{2} \\ &+ 3(1-a)^{2} \left\| \nabla f_{i}(\boldsymbol{y}_{i,k-1}^{t};\boldsymbol{\zeta}_{i,k}^{t}) - \nabla f_{i}(\boldsymbol{x}^{t-2};\boldsymbol{\zeta}_{i,k}^{t})) - \nabla f(\boldsymbol{y}_{i,k-1}^{t} + \nabla f(\boldsymbol{x}^{t-2}) \right\|^{2} \\ &+ 3a^{2} \left\| \frac{1}{S} \sum_{j \in S^{t}} (\nabla f_{j}(\boldsymbol{x}^{t-2}) - \nabla f(\boldsymbol{x}^{t-2}) \right\|^{2} \\ &\leq 3 \, \mathbb{E}\|\boldsymbol{V}^{t-1}\|^{2} + 3\delta^{2} \|\boldsymbol{y}_{i,k-1}^{t} - \boldsymbol{x}^{t-2}\|^{2} + \frac{3a^{2}G^{2}}{S} \, . \end{split}$$

The last inequality used the Hessian similarity Lemma 3 to bound the second term and the heterogeneity bound (A1) to bound the last term. Also, $(1-a)^2 \le 1$ since $a \in [0,1]$.

Distance moved in each step. We show that the distance moved by a client in each step during the client update can be controlled.

Lemma 10. For MimeMVR updates (13) with $\eta \leq \frac{1}{6K\delta}$ and given (A1) and (A2), the following holds

$$\Delta_{i,k}^{t} \leq \left(1 + \frac{1}{K}\right) \Delta_{i,k-1}^{t} + 18\eta^{2} K a^{2} \frac{G^{2}}{S} + 18\eta^{2} K \left\| V^{t-1} \right\|^{2} + 6\eta^{2} K \left\| \nabla f(\boldsymbol{y}_{i,k-1}^{t}) \right\|^{2},$$

where we define
$$\Delta_{i,k}^t := \max\Bigl(\mathbb{E}\|oldsymbol{y}_{i,k}^t - oldsymbol{x}^{t-2}\|^2, \mathbb{E}\|oldsymbol{y}_{i,k}^t - oldsymbol{x}^{t-1}\|^2, \mathbb{E}\|oldsymbol{x}^{t-1} - oldsymbol{x}^{t-2}\|^2\Bigr).$$

Proof. Starting from the MimeMVR update (13) and the relaxed triangle inequality with c = 2K,

$$\begin{split} \mathbb{E}\|\boldsymbol{y}_{i,k}^{t} - \boldsymbol{x}^{t-2}\|^{2} &= \mathbb{E}\|\boldsymbol{y}_{i,k-1}^{t} - \eta \boldsymbol{d}_{i,k}^{t} - \boldsymbol{x}^{t-2}\|^{2} \\ &\leq \left(1 + \frac{1}{2K}\right) \mathbb{E}\|\boldsymbol{y}_{i,k-1}^{t} - \boldsymbol{x}^{t-2}\|^{2} + (2K+1)\eta^{2} \, \mathbb{E}\|\boldsymbol{d}_{i,k}^{t}\|^{2} \\ &\leq \left(1 + \frac{1}{2K}\right) \mathbb{E}\|\boldsymbol{y}_{i,k-1}^{t} - \boldsymbol{x}^{t-2}\|^{2} + 6K\eta^{2} \, \mathbb{E}\|\boldsymbol{d}_{i,k}^{t} - \nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2} \\ &\quad + 6K\eta^{2} \, \mathbb{E}\|\nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2} \\ &\leq \left(1 + \frac{1}{2K} + 18K\eta^{2}\delta^{2}\right) \mathbb{E}\|\boldsymbol{y}_{i,k-1}^{t} - \boldsymbol{x}^{t-2}\|^{2} \\ &\quad + 18K\eta^{2} \, \mathbb{E}\|V^{t-1}\|^{2} + \frac{18K\eta^{2}a^{2}G^{2}}{S} + 6K\eta^{2} \, \mathbb{E}\|\nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2} \, . \end{split}$$

The last inequality used the update variance bound Lemma 9. We can simplify the expression further since $\eta \leq \frac{1}{6K\delta}$ implies $18K\eta^2\delta^2 \leq \frac{1}{2K}$. Similar computations for $\mathbb{E}\|\boldsymbol{y}_{i,k}^t - \boldsymbol{x}^{t-1}\|^2$ yield the lemma.

Progress in one step. Now we have all the tools required to compute the progress made in each round.

Lemma 11. For any step of MimeMVR with step size $\eta \leq \min(\frac{1}{L}, \frac{1}{40\delta K})$ and momentum parameter $a = 1536\eta^2\delta^2K^2$. Then, given that (A1)–(A3) hold, we have

$$\begin{split} \mathbb{E}[f(\boldsymbol{y}_{i,k}^{t})] + \frac{3\eta}{a} \, \mathbb{E}\|\boldsymbol{V}^{t}\|^{2} + \frac{8\eta\delta^{2}K}{a} \bigg(1 + \frac{2}{K}\bigg)^{K-k} \Delta_{i,k}^{t} \\ & \leq \mathbb{E}[f(\boldsymbol{y}_{i,k-1}^{t})] + \frac{3\eta}{a} \, \mathbb{E}\|\boldsymbol{V}^{t-1}\|^{2} + \frac{8\eta\delta^{2}K}{a} \bigg(1 + \frac{2}{K}\bigg)^{K-(k-1)} \Delta_{i,k-1}^{t} \\ & - \frac{\eta}{4} \, \mathbb{E}\|\nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2} + \frac{11136\eta^{3}\delta^{2}K^{2}G^{2}}{S} \,. \end{split}$$

Proof. The assumption that f is L-smooth implies a quadratic upper bound (8).

$$f(\boldsymbol{y}_{i,k}^{t}) - f(\boldsymbol{y}_{i,k-1}^{t}) \leq -\eta \langle \nabla f(\boldsymbol{y}_{i,k-1}^{t}), \boldsymbol{d}_{i,k}^{t} \rangle + \frac{L\eta^{2}}{2} \|\boldsymbol{d}_{i,k}^{t}\|^{2}$$

$$= -\frac{\eta}{2} \|\nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2} + \frac{L\eta^{2} - \eta}{2} \|\boldsymbol{d}_{i,k}^{t}\|^{2} + \frac{\eta}{2} \|\boldsymbol{d}_{i,k}^{t} - \nabla f(\boldsymbol{y}_{i,k-1}^{t})\|^{2}.$$

The second equality used the fact that for any $a,b,-2ab=(a-b)^2-a^2-b^2$. The second term can be removed since $\eta \leq \frac{1}{L}$. Taking expectation on both sides and using the update variance bound Lemma 9,

$$\begin{split} \mathbb{E} \, f(\boldsymbol{y}_{i,k}^t) - \mathbb{E} \, f(\boldsymbol{y}_{i,k-1}^t) &\leq -\frac{\eta}{2} \, \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 + \frac{3\eta a^2 G^2}{2S} \\ &\quad + \frac{3\eta}{2} \, \mathbb{E} \|\boldsymbol{V}^{t-1}\|^2 + \frac{3\eta \delta^2}{2} \, \mathbb{E} \|\boldsymbol{y}_{i,k-1}^t - \boldsymbol{x}^{t-2}\|^2 \end{split}$$

Multiplying the momentum variance bound Lemma 8 by $\frac{3\eta}{a}$, we have

$$\frac{3\eta}{a} \left\| \|V^t \|^2 \le \frac{3\eta}{a} \left\| \|V^{t-1}\|^2 + \frac{6\eta\delta^2}{a} \left\| \|x^{t-1} - x^{t-2}\|^2 + \frac{6\eta a G^2}{S} - 3\eta \left\| \|V^{t-1}\|^2 \right\|.$$

We will also multiply the distance bound Lemma 10 by $\frac{8\eta\delta^2K}{a}\left(1+\frac{2}{K}\right)^{K-k}$. Note that for any $K\geq 1$ and $k\in[K]$, we have $1\leq\left(1+\frac{2}{K}\right)^{K-k}\leq 8$. Then we get

$$\begin{split} \frac{8\eta\delta^2K}{a} \left(1 + \frac{2}{K}\right)^{K-k} \Delta_{i,k}^t &\leq \frac{8\eta\delta^2K}{a} \left(1 + \frac{2}{K}\right)^{K-(k-1)} \Delta_{i,k-1}^t - \frac{8\eta\delta^2}{a} \Delta_{i,k-1}^t \\ &\quad + 1152\eta^3\delta^2K^2 a \frac{G^2}{S} + \frac{1152\eta^3\delta^2K^2}{a} \left\| \mathbb{E} \|V^{t-1}\|^2 + \frac{384\eta^3\delta^2K^2}{a} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 \,, \end{split}$$

where recall that we defined $\Delta_{i,k}^t := \max \left(\mathbb{E} \| \boldsymbol{y}_{i,k}^t - \boldsymbol{x}^{t-2} \|^2, \mathbb{E} \| \boldsymbol{y}_{i,k}^t - \boldsymbol{x}^{t-1} \|^2, \mathbb{E} \| \boldsymbol{x}^{t-1} - \boldsymbol{x}^{t-2} \|^2 \right)$. Combining the three inequalities together, we get

$$\begin{split} \mathbb{E} \, f(\boldsymbol{y}_{i,k}^t) + \frac{3\eta}{a} \, \mathbb{E} \|\boldsymbol{V}^t\|^2 + \frac{8\eta\delta^2 K}{a} \bigg(1 + \frac{2}{K} \bigg)^{K-k} \Delta_{i,k}^t \\ & \leq \mathbb{E} \, f(\boldsymbol{y}_{i,k-1}^t) + \frac{3\eta}{a} \, \mathbb{E} \|\boldsymbol{V}^{t-1}\|^2 + \frac{8\eta\delta^2 K}{a} \bigg(1 + \frac{2}{K} \bigg)^{K-(k-1)} \Delta_{i,k-1}^t \\ & + \bigg(\frac{384\eta^3 \delta^2 K^2}{a} - \frac{\eta}{2} \bigg) \, \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 \\ & + \bigg(1152\eta^2 \delta^2 K^2 + 6 + \frac{3a}{2} \bigg) \frac{a\eta G^2}{S} \\ & + \bigg(\frac{3\eta}{2} + \frac{1152\eta^3 \delta^2 K^2}{a} - 3\eta \bigg) \, \mathbb{E} \|\boldsymbol{V}^{t-1}\|^2 \\ & + \bigg(\frac{3}{2} + \frac{6}{a} - \frac{8}{a} \bigg) \eta \delta^2 \Delta_{i,k-1}^t \,. \end{split}$$

Note that $\frac{1152\eta^3\delta^2K^2}{a}=\frac{3\eta}{4}$ since we defined $a=1536\eta^2\delta^2K^2$. Further, $a\leq 1$ when defined this way since we assumed $\eta\leq\frac{1}{40\delta K}$. Similarly, the definition of a implies that $\frac{384\eta^3\delta^2K^2}{a}=\frac{\eta}{4}$. Thus, we can simplify the above expression as

$$\begin{split} \mathbb{E} \, f(\boldsymbol{y}_{i,k}^t) + \frac{3\eta}{a} \, \mathbb{E} \|\boldsymbol{V}^t\|^2 + \frac{8\eta\delta^2 K}{a} \bigg(1 + \frac{2}{K} \bigg)^{K-k} \Delta_{i,k}^t \\ & \leq \mathbb{E} \, f(\boldsymbol{y}_{i,k-1}^t) + \frac{3\eta}{a} \, \mathbb{E} \|\boldsymbol{V}^{t-1}\|^2 + \frac{8\eta\delta^2 K}{a} \bigg(1 + \frac{2}{K} \bigg)^{K-(k-1)} \Delta_{i,k-1}^t \\ & - \frac{\eta}{4} \, \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 + \frac{11136\eta^3 \delta^2 K^2 G^2}{S} \, . \end{split}$$

This proves the lemma.

Progress in one round. Let us sum over all the steps within a round to compute the progress made in a full round.

Lemma 12. For any round of MimeMVR with step size $\eta \leq \min(\frac{1}{L}, \frac{1}{40\delta K})$ and momentum parameter $a = 1536\eta^2\delta^2K^2$. Then, given that (A1)–(A3) hold, we have

$$\frac{\eta}{4KS} \sum_{k \in [K], j \in \mathcal{S}^t} \mathbb{E} \left\| \nabla f(\boldsymbol{y}_{i,k-1}^t) \right\|^2 \leq \Phi^{t-1} - \Phi^t + \frac{11136\eta^3 \delta^2 K^2 G^2}{S} \,,$$

where we define the sequence

$$\Phi^t := rac{1}{K} \mathbb{E}[f(oldsymbol{x}^t)] + rac{3\eta K}{a} \mathbb{E} \|V^t\|^2 + rac{8\eta \delta^2}{a} \mathbb{E} \|oldsymbol{x}^t - oldsymbol{x}^{t-1}\|^2$$

Proof. We start by summing Lemma 11 over the client updates

$$\begin{split} \frac{\eta}{4} \sum_{k \in [K]} \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 &\leq \mathbb{E}[f(\boldsymbol{y}_{i,0}^t)] + \frac{3\eta K}{a} \, \mathbb{E} \|V^{t-1}\|^2 + \frac{8\eta \delta^2 K}{a} \left(1 + \frac{2}{K}\right)^K \Delta_{i,0}^t \\ &- \mathbb{E}[f(\boldsymbol{y}_{i,K}^t)] - \frac{3\eta K}{a} \, \mathbb{E} \|V^t\|^2 + \frac{8\eta \delta^2 K}{a} \Delta_{i,K}^t \\ &+ \frac{11136\eta^3 \delta^2 K^3 G^2}{C} \,. \end{split}$$

Recall that we defined $\Delta_{i,k}^t := \max \left(\mathbb{E} \| \boldsymbol{y}_{i,k}^t - \boldsymbol{x}^{t-2} \|^2, \mathbb{E} \| \boldsymbol{y}_{i,k}^t - \boldsymbol{x}^{t-1} \|^2, \mathbb{E} \| \boldsymbol{x}^{t-1} - \boldsymbol{x}^{t-2} \|^2 \right)$. Because $\boldsymbol{y}_{i,0}^t = \boldsymbol{x}^{t-1}$, we can simplify

$$\mathbb{E}[f(\boldsymbol{y}_{i,0}^t)] + \frac{8\eta\delta^2K}{a}\Delta_{i,0}^t \leq \mathbb{E}[f(\boldsymbol{x}^{t-1})] + \frac{8\eta\delta^2K}{a}\mathbb{E}\|\boldsymbol{x}^{t-1} - \boldsymbol{x}^{t-2}\|^2$$

Then by the averaging Lemma 4, we have

$$\frac{1}{S} \sum_{j \in \mathcal{S}^t} \mathbb{E}[f(\boldsymbol{y}_{j,K}^t)] + \frac{8\eta \delta^2 K}{a} \Delta_{j,K}^t \ge \frac{1}{S} \sum_{j \in \mathcal{S}} \mathbb{E}[f(\boldsymbol{y}_{j,K}^t)] + \mathbb{E} \|\boldsymbol{x}^{t-1} - \boldsymbol{y}_{j,K}^t\|^2 \\
\ge \mathbb{E}[f(\boldsymbol{x}^t)] + \frac{8\eta \delta^2 K}{a} \mathbb{E} \|\boldsymbol{x}^{t-1} - \boldsymbol{x}^t\|^2.$$

So by averaging our recursion over the sampled clients, and diving our summation over the updates by K, we get

$$\begin{split} \frac{\eta}{4KS} \sum_{k \in [K], j \in S^t} \mathbb{E} \|\nabla f(\boldsymbol{y}_{i,k-1}^t)\|^2 &\leq \underbrace{\frac{1}{K} \, \mathbb{E}[f(\boldsymbol{x}^{t-1})] + \frac{3\eta}{a} \, \mathbb{E} \|V^{t-1}\|^2 + \frac{8\eta\delta^2}{a} \, \mathbb{E} \|\boldsymbol{x}^{t-1} - \boldsymbol{x}^{t-2}\|^2}_{=\Phi^{t-1}} \\ &- \underbrace{\frac{1}{K} \, \mathbb{E}[f(\boldsymbol{x}^t)] + \frac{3\eta K}{a} \, \mathbb{E} \|V^t\|^2 + \frac{8\eta\delta^2}{a} \, \mathbb{E} \|\boldsymbol{x}^t - \boldsymbol{x}^{t-1}\|^2}_{=\Phi^t} \\ &+ \underbrace{\frac{11136\eta^3 \delta^2 K^2 G^2}{S}}. \end{split}$$

Theorem IV (non-convex convergence of MimeMVR). Let us run MimeMVR with step size $\eta \leq \min\left(\frac{1}{15K}\left(\frac{FS}{T\delta^2G^2}\right)^{1/3}, \frac{1}{L}, \frac{1}{40\delta K}\right)$ and momentum parameter $a=1536\eta^2\delta^2K^2$. Then, given that (A1)–(A3) hold, we have

$$\frac{1}{KST} \sum_{t \in [T]} \sum_{k \in [K]} \sum_{i \in S^t} \mathbb{E} \left\| \nabla f(\boldsymbol{y}_{i,k-1}^t) \right\|^2 \leq \mathcal{O} \left(\left(\frac{(1+\delta)G^2F}{ST} \right)^{2/3} + \frac{(L+\delta K)F}{KT} \right),$$

where we define $F := f(\boldsymbol{x}^0) - f^*$.

Proof. Unroll the one round progress Lemma 12 and average over T rounds to get

$$\begin{split} \frac{1}{KST} \sum_{t \in [T]} \sum_{k \in [K]} \sum_{j \in \mathcal{S}^t} \mathbb{E} \|f(\boldsymbol{y}_{i,k-1}^t)\|^2 &\leq \frac{4(\Phi^0 - \Phi^T)}{\eta KT} + \frac{11136\eta^2 \delta^2 K^2 G^2}{S} \\ &\leq \frac{4(f(\boldsymbol{x}^0) - f^\star)}{\eta KT} + \frac{11136\eta^2 \delta^2 K^2 G^2}{S} \,. \end{split}$$

Our choice of step size now yields the desired rate.