Advanced Macroeconomics I: Growth and Innovation

Lecture 2:
Neoclassical Growth (Ramsey)

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Course Outline Advanced Macroeconomics I

1. Neoclassical Growth Model: Solow-Swan
2. Neoclassical Growth Model: Ramsey-Cass-Koopmans
   - Government Financing and Ricardian Equivalence
3. Overlapping Generations Models
   - Pension systems
4. Extensions of the Neoclassical Growth Model
   - AK-Models
   - Growth through Externalities
   - Public Sector Investments
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Neoclassical Growth Model (Ramsey, Cass & Koopmans)

The Solow model is dynamic, but it lacks an explicit microeconomic foundation. Individuals should adjust their savings behaviour to market conditions, notably the interest rate. If, for example, the savings rate s is above the "Golden Rule" rate, consumption could be increased in a costless and sustained way by a reduction of s.

We study the model of Ramsey, Cass & Koopmans for two reasons:

1. *Substance:* How much "should" one save and how do the dynamics change if individuals act optimally? Relevant per se and for policy questions.
2. *Method:* How can an intertemporal optimization problem in continuous time be solved?

The Method: Dynamic programming

Intertemporal –or dynamic– optimization problems form the backbone of modern macroeconomics. Optimizing individuals choose a sequence of decisions to maximize an intertemporal utility function.

Maximization is carried out under *constraints* that determine future possibilities conditional on today's decisions.

Each intertemporal optimization problem has two components:

1. An *intertemporal objective/utility function*

$$\int_0^\infty e^{-\rho t} u(c(t)) dt$$

2. A *transition equation* that links the (potentially stochastic) future state variables to the current state variables and the decisions of the individuals
Example: Tomorrow's accumulated capital depends on the current capital stock as well as saving/consumption decisions of individuals.

$$\dot{k}(t) = f(k(t)) - c(t)$$

The Method: Dynamic programming (2)

In an intertemporal decision problem, all potential future decisions are taken simultaneously with current consumption and savings decisions, possibly conditional on the state of the economy.

Dynamic optimization problems can be classified as follows:

1. *Time interval:* discrete or continuous
2. *Uncertainty:* deterministic or stochastic
3. *Time horizon:* finite or infinite

Utility function

How does an individual value intertemporal utility?
We assume the following specification

$$
\begin{aligned}
U &= \sum_{t=0}^{T} \beta^t u(c_t) \\
U &= \int_0^{\infty} e^{-\rho t} u(c(t)) dt
\end{aligned}
$$

$U =$ discounted total utility, $u(.) =$ instantaneous utility, twice continuously differentiable with $u' > 0$ and $u'' < 0$. T can also be ∞.
Often the following assumptions are made:

Time separability:

- Total lifetime utility of a stream of consumption is the *discounted sum/integral* of time/instantaneous utility. This excludes habit formation (for which future utility depends on current utility).

Utility function (ctd.)

time consistency= preferences of today stay the same in the future if no new information are added (e.g. job loss, etc)

Time discounting: For $\beta < 1$ or $\rho > 0$ individuals are *impatient*.
The rate of time preference ρ is defined, implicitly, as $\beta = \frac{1}{1+\rho}$. Geometric discounting ensures *time consistency* (why?). In equilibrium, the rate of time preference will have a close link to the interest rate of an economy.

Homotheticity: The marginal rate of substitution between consumption in period t and $t+s$ is

In a model where competitive consumers optimize homothetic utility functions subject to a budget constraint, the ratios of goods

$$\mathrm{MRS}_{(c_{t+s},c_t)} = \frac{\partial u(c)}{\partial c_{t+s}} \Big/ \frac{\partial u(c)}{\partial c_t}$$

Instantaneous utility u is homothetic if for all $\lambda > 0$: $\mathrm{MRS}_{(c_{t+s},c_t)} = \mathrm{MRS}_{(\lambda c_{t+s},\lambda c_t)}$.
Homotheticity means that the savings rate is independent of wealth (discuss this!).

Constraints

Constraints (side conditions) usually have the following form:

- *Production:* for ex. $y_t = f(k_t)$ or $y(t) = f(k(t))$
- *Resources:* for ex. $c_t + i_t = y_t$ (output can be consumed or invested)
- *Investment:* for ex. $k_{t+1} = (1-\delta)k_t + i_t$ or $\dot{k}(t) = i(t) - \delta k(t)$
  The latter is also the *transition equation* of the problem.

Ingredients of the Ramsey Model

The model encompasses two interacting types of agents: Firms and households.

- Firms:
  - are owned by individuals/households
  - neoclassical production function, with constant returns to scale in labor and capital, labor-augmenting technical progress $Y = F(K, AL)$
  - to simplify: no depreciation ($F(.)$ is measured in net terms)
  - firms maximize profits $\pi = F(K, AL) - wAL - rK$, price of output good normalized to one
  - competitive environment, price taking behavior
  - to get stationary variables in steady state, we express all variables again per unit of efficiency labor.

Ingredients of the Ramsey Model (ctd.)

- Firms maximize profits:
  As a consequence, the factor prices rate of return to capital $r(t)$ and the wage rate $w(t)$ correspond to the marginal products of capital and labor, respectively:

$$
\begin{aligned}
r(t) &= \frac{\partial F(K, AL)}{\partial K} = \frac{1}{\partial K}\left[ALf\left(\frac{K}{AL}\right)\right] = ALf'(k)\frac{1}{AL} = f'(k) \\
w(t) &= \frac{\partial F(K, AL)}{\partial AL} = ... = f(k) - kf'(k)
\end{aligned}
$$

- Per worker the wage is $A(t)w(t)$

Individuals/Households

Remark I:

In contrast to Romer, we use the dynamic programming approach of Hamilton.

Remark II:

We express the optimization problem of individuals in intensive form. If $C(t)$ is consumption per worker, $c(t) \equiv C(t)/A(t)$ is consumption per efficiency unit. The corresponding derivations can be found in Romer.

Individuals/Households (ctd.)

- Households grow at rate n: no growth in HH members, but in numbers of HH only
  $\Rightarrow$ for expositional purposes, we use $n = 0$ (easy exercise to study this extension)
- Household earn labor and capital income.
  They offer their labor force and savings (as capital) to firms for wage $w(t)$ and interest rate $r(t)$, respectively.
- In each period, the income is split between consumption and savings
- HH maximize their lifetime utility by choosing their consumption/savings pattern

Individuals/Households (ctd.)

- HH maximize their lifetime utility with:

$$U = \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} dt, \tag{1}$$

-1 is written, if theta = 1, apply rule d'Hopital, then
ln C is the limited case

where

- $u(C(t)) = \frac{C(t)^{1-\theta}-1}{1-\theta}$ is instantaneous CRRA utility.
- When $\theta = 1$, $u(C(t)) = \ln(C(t))$ (use Bernoulli-d'Hôpital rule).
-

CRRA is only utility
function which gives
interest constant =
constant savings rate

$$\text{CRRA} \equiv \frac{-C(t)u''(C(t))}{u'(C(t))} = \theta$$

- CRRA = constant relative risk aversion, constant and equal to θ. (Discuss: what is risk here)
  The coefficient $1/\theta$ measures the intertemporal elasticity of substitution (CIES = constant elasticity of intertemporal substitution).

theta measures how much we are willing to substitute
consumption in intertemporal timeframes
if theta = 1, consumer is an intermediate persopn
(Cobb-Douglas function)

Side-Note: Knife-Edge Assumptions in Growth Theory

We use several "knife-edge" assumptions to generate balanced growth

- Purely labor-augmenting technical progress, $F(K, A, L) = F(K, AL)$
- Technology grows at constant rate g
- CRRA utility which implies that the savings rate is constant when the interest rate $f'(k)$ is constant

Optimization:

- *Profit maximization* of firms: $w(t)$ and $r(t)$!
- Households choose their consumption path $C(t) = A(t)c(t)$ to maximize their intertemporal utility U under the *budget constraints*

$$\dot{a}(t) = w(t) + r(t)a(t) - ga(t) - c(t) \tag{2}$$

  - consumption $c(t)$ is a *control variable*
  - the level of savings $a(t)$ is a *state variable* in this case, the aggregated k is given
- In equilibrium, the following conditions must be satisfied
  - $a(t) = k(t)$
  - $r(t) = f'(k(t))$

Ramsey Model: The Control Problem with a Recipe

- For detailed derivations see the handout
- *Hamiltonian* of a households' optimization problem in the Ramsey model with CRRA utility (in efficiency units):

$$\mathcal{H} = \frac{A(t)^{1-\theta}c(t)^{1-\theta} - 1}{1-\theta} + \lambda(t)\{\underbrace{w(t) + r(t)a(t) - ga(t) - c(t)}_{=\dot{a}(t)}\}$$

utility funcion in efficiency units

flow budget constrain = change of assets over time

The first order conditions (FOC) can be written as (we do not write again the $\dot{a}(t)$ - equation):

- $$\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial\mathcal{H}}{\partial a(t)} = \rho\lambda(t) - \lambda(t)\left(r(t) - g\right) = \lambda(t)(\rho - r(t) + g)$$

  or rewritten as

  shadow price of the capital/assets
  $$\frac{\dot{\lambda}(t)}{\lambda(t)} = \rho - r(t) + g$$

Ramsey Model: The Control Problem with a Recipe (2)

- 2. *condition for optimal consumption:*

$$\frac{\partial \mathcal{H}}{\partial c(t)} = 0 = A(t)^{1-\theta} c(t)^{-\theta} - \lambda(t)$$

or, alternatively,

$$\lambda(t) = A(t)^{1-\theta} c(t)^{-\theta}$$

in growth rates*

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = (1-\theta)\underbrace{g}_{\text{g=A/A}} - \theta \frac{\dot{c}(t)}{c(t)}$$

- 3. *transversality condition:*

1/theta = intertemporal rate of substitution

$$lim_{t \to \infty} \lambda(t) exp(-\rho t) a(t) = 0$$

*recall that the growth rate of a variable $x(t)$ can be computed as the time derivative of the variable's logarithm, $\frac{\dot{x}(t)}{x(t)} = \frac{d}{dt} ln(x(t))$.

Ramsey Model: The Control Problem with a Recipe (3)

We can now plug the asset pricing equation

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \rho - r(t) + g$$

into the condition for optimal consumption (in growth rates)

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = (1-\theta)g - \theta\frac{\dot{c}(t)}{c(t)}$$

We obtain the famous **Euler equation**:

1/theta = intertemporal rate of substitution

in efficiency units

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho}{\theta} - g$$

In per capita levels

theta describes how much the difference between rho and r is exploided

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho}{\theta}$$

if r is higher, people save more and consume more in the future

if rho is higher, the more impatient people are and consume now and save less

Equilibrium Conditions and Discussion

We can now substitute the equilibrium conditions ($a(t) = k(t)$ and $r(t) = f'(k(t))$) into the Euler equation to obtain

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho}{\theta} - g$$

- The Euler equation shows the optimal consumption growth over time. The growth rate is larger for larger returns to capital (low capital stock).
- Optimal consumption growth does <u>not</u> depend on the wage profile.
  wage matters for consumption today, but not for decision between consumption today and tomorrow
- The optimality conditions have been derived in a decentralized economy setting. They are, thus, not *a priori* Pareto optimal.

Equilibrium Conditions and Discussion (2)

In this model, we can show that a benevolent social planner would choose the same allocation. This follows from the fact that the *First Welfare Theorem* holds in this setup.
You can easily verify that the outcome is the same by using the aggregate budget constraint

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

There are still two unsolved problems:

1. The initial value of consumption (we only know how consumption at different points in time relate to each other)
2. The dynamics of the economy

Dynamics of the Economy

We analyze the dynamics of the model by means of a **phase diagram**:

- In a $k(t) - c(t)$ diagram we show for each possible combination of $k(t)$ and $c(t)$ the gradient (the direction of movement) in the two variables, $\dot{k}(t)$ and $\dot{c}(t)$.

Phase diagram

The equations for $\dot{k}(t)$ und $\dot{c}(t)$ follow from the FOC:

$$\dot{c}(t) = c(t)\frac{f'(k(t)) - \rho - \theta g}{\theta}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

In a long run steady state we also know that $\dot{c}(t) = \dot{k}(t) = 0$.

Phase Diagram

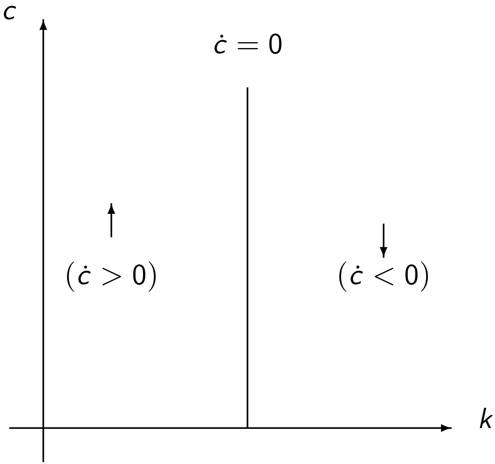
These two conditions

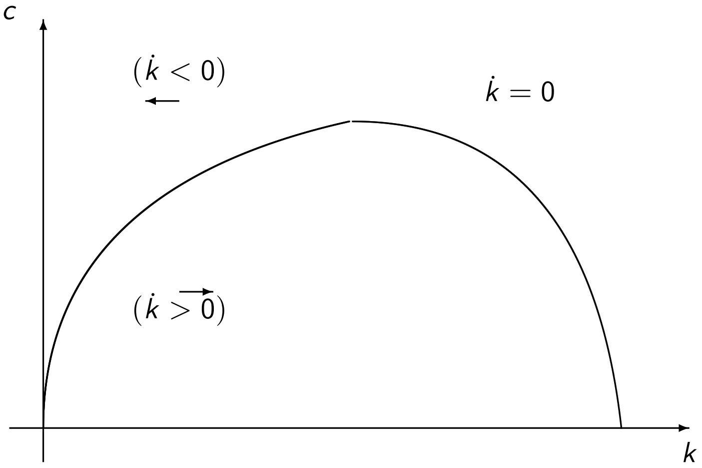
$$\dot{c}(t) = c(t)\frac{f'(k(t)) - \rho - \theta g}{\theta}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

define two important loci:

- $\dot{c}(t) = 0$: Consumption growth is 0 when $f'(k(t)) = \rho + \theta g$. This corresponds to a constant return to capital and, therefore, also to a constant capital stock k^*.
- $\dot{k}(t) = 0$: The capital stock remains constant, if the whole production is consumed (including an indirect "consumption" by productivity and population growth), $c(t) = f(k(t)) - (n+g)k(t)$.


c
$\dot{c} = 0$
$(\dot{c} > 0)$
$(\dot{c} < 0)$
k

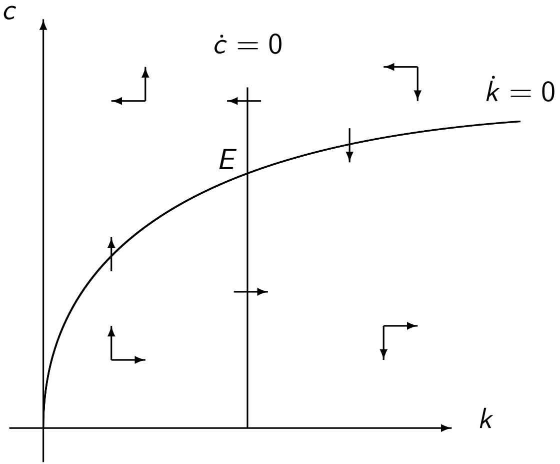

c
$(\dot{k} < 0)$
$\dot{k} = 0$
$(\dot{k} > 0)$
k

Phase Diagram (ctd.)

These two loci define 4 different regimes in our diagram:

- NW: $\dot{c}(t) > 0$, $\dot{k}(t) < 0$
- NE: $\dot{c}(t) < 0$, $\dot{k}(t) < 0$
- SE: $\dot{c}(t) < 0$, $\dot{k}(t) > 0$
- SW: $\dot{c}(t) > 0$, $\dot{k}(t) > 0$

In the north, the capital stock is decreasing (as consumption is too high).
In the east, consumption is falling (as the return to capital is too low).


c
$\dot{c} = 0$
$\dot{k} = 0$
E
k

Dynamics and Initial Value of Consumption

We now know the evolution of consumption and capital for each potential combination of these two variables $c(t)$ and $k(t)$.

But we do not yet know whether each of these combinations is feasible:

- *Important*: The initial capital stock at $t = 0$ is a pre-determined variable (determined by past actions).
- But we still have to find $c(0)$. Let us check the plausibility of potential initial values of consumption.

Initial Consumption (2)

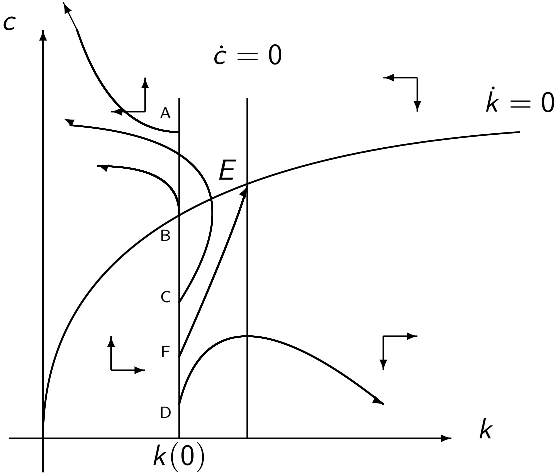
Assumption: $k(0) < k^*$
the current capital stock is smaller than the steady state capital stock.

- $c(0)$ above the $\dot{k}(t) = 0$ *locus (NW)*:
  consumption increasing, capital decreasing (and finally negative): not sustainable, violates the transversality condition.
- $c(0)$ on the $\dot{k}(t) = 0$ *locus (W)*:
  consumption increasing, capital constant in the beginning, then decreasing and negative: not sustainable, violates the transversality condition.
- $c(0)$ below the $\dot{k}(t) = 0$ *locus (SW)*:
  consumption and capital stock increasing.

Initial Consumption (3)

$c(0)$ below the $\dot{k}(t) = 0$ *locus (SW)*:

1. $\{c(t), k(t)\}$ reaches the $\dot{k}(t) = 0$ line after some time:
   The capital stock begins to fall.
   The initial value of consumption is <u>too high</u>, the entire capital stock is eaten away.
2. $\{c(t), k(t)\}$ reaches the $\dot{c}(t) = 0$ line after some time:
   Consumption begins to fall, the capital stock keeps on increasing (until ∞). This cannot be optimal for the individuals.
   The initial value of consumption is <u>too low</u>!
3. $\{c(t), k(t)\}$ reaches the steady state, $\dot{k}(t) = \dot{c}(t) = 0$, after some time:
   Consumption and capital stock remain constant.
   The path associated with this particular value of initial consumption is the only one to satisfy the transversality condition.


c
$\dot{c} = 0$
$\dot{k} = 0$
A
E
B
C
F
D
k
$k(0)$

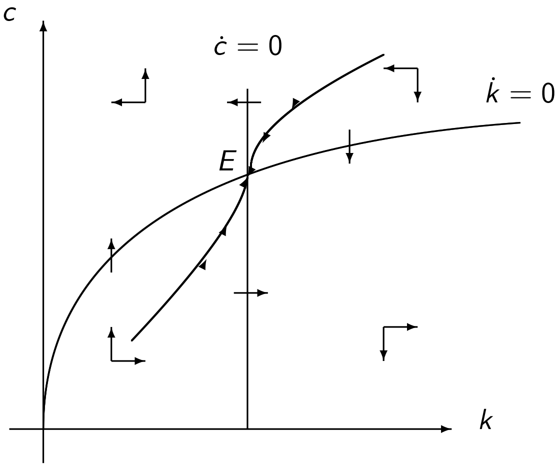
Saddle path of the optimization problem

- The only path compatible with the transversality condition is called the saddle path of the maximization problem.
- *The transversality condition thus determines the initial value of consumption (as a function of the capital stock).*
  The transversality condition and the Euler equation fully determine the entire consumption path. Note that in this class of problems we need the transversality condition to derive the intertemporal budget constraint. The level of optimal consumption is derived in a similiar way as in a microeconomic problem: derive the MRS (here: Euler equation) then plug into the budget constraint (here: use transversality condition pinning down the saddle path)

Saddle path of the optimization problem

- The saddle path always passes through the steady state $\dot{k}(t) = \dot{c}(t) = 0$. The saddle path for values of the capital stock $k(t) > k^*$ can be found in an analogous way. It is to be found in the NE region of the diagram.
- The optimal consumption path is *time consistent*!

Saddle path of the optimization problem



Comparative Statics/Dynamics

The graphical representation of the Ramsey model allows for a simple analysis of policy actions and parameter changes.

To analyze this, two points must be fulfilled:

- 1. As $\lambda(t)$ is continuous in t, marginal utility per price unit must be continuous as well.
  This follows from

$\frac{\partial \mathcal{H}}{\partial c(t)} = 0 = u'(A(t)c(t))A(t) - \lambda(t)p(t) = A(t)^{1-\theta}c(t)^{-\theta} - \lambda(t)p(t)$
with CRRA.
$\rightarrow$ consumption smoothing if $p(t)$ does not jump!

Comparative Statics/Dynamics

- 2. $(c(t), k(t))$ must be on the saddle path when no further shock occurs. $\rightarrow$ solve problem by graphical backwards induction
- Example: *Smaller rate of time preference ρ:*
  - $\dot{c}(t) = 0$ locus moves to the right: Individuals get more patient and save more (at a given interest rate).
  - $\dot{k}(t) = 0$ locus unchanged, production possibilities are unaffected.
  - The saddle path moves to the right: Current consumption falls.
- Further examples, study temporary changes as well: change in g, capital and consumption taxation

Interest Rate in Steady State

Steady state interest rate is the marginal product of capital in the steady state where $\dot{c} = 0$

$$f'(k^*) = \rho + \theta g$$

Why do we see very low or even negative nominal interest rates today?

- $i = r + \pi^e$, nominal interest rate equals real interest rate plus expected inflation. Low inflationary expectations
- Lower (expected) productivity growth g
- Demographics lower real interest rates (saving for retirement $\rightarrow$ OLG model)

What about the Golden Rule?

Solow-Model: Golden Rule maximizes long-run consumption

$$k^*_{GR}: \ f'(k^*_{GR}) = n + g$$

(with $\delta = 0$)

Ramsey-Model: Modified Golden Rule
$\dot{c}(t) = 0$ locus lies to the left of maximum of the $\dot{k}(t) = 0$ locus because of the transversality condition. In equilibrium (to compare it, we have positive population growth here, the relevant discount rate for the household is $\rho - n$ in that case)

$$\lim_{t \to \infty} \lambda(t) exp(-[\rho - n]\, t) k(t) = 0$$

What about the Golden Rule?

Recall $\lambda(t) = A(t)^{1-\theta} c(t)^{-\theta}$, $A(t)^{1-\theta}$ grows at rate $g(1-\theta)$ and in steady state $c(t)$ and $k(t)$ are constant. Hence, the limes is zero only if

$$g(1-\theta) - \rho + n < 0$$

Compare now the marginal product of capital in the Ramsey steady state and at the Golden Rule level (top of $\dot{k} = 0$ -locus)

$$\begin{aligned} f'(k^*) &= \rho + \theta g > n + g = f'(k^*_{GR}) \\ k^* &< k^*_{GR} \end{aligned}$$

Intuition: Discounting.

Finite Lifetime Utility

An alternative way to see why the condition $g(1-\theta)-\rho+n<0$ has to hold, is by considering lifetime utility. More precisely, using (aggr.) consumption, and that, e.g., $A(t)=A(0)e^{gt}$, we can rewrite U as

$$
\begin{aligned}
& \int_0^\infty e^{-\rho t}\frac{[c(t)A(t)]^{1-\theta}}{1-\theta}L(t)dt \\
= & \int_0^\infty e^{-\rho t}\frac{[c(t)A(0)e^{gt}]^{1-\theta}}{1-\theta}L(0)e^{nt}dt \\
= & A(0)^{1-\theta}L(0)\int_0^\infty e^{-(\rho-n-g(1-\theta))t}\frac{c(t)^{1-\theta}}{1-\theta}dt,
\end{aligned}
$$

where we neglect the -1 for notational simplicity.
If $g(1-\theta)-\rho+n>0$, infinite lifetime utility can be attained, and the maximization problem has no well-defined solution.

Slope of the saddle path

Recall: The saddle path represents equilibrium (optimal) consumption as a function of the capital stock $k(t)$. This relation is called the policy function of the problem.

The saddle path

- is monotonically increasing
- passes through the origin
- passes through the steady state
- its precise form depends on parameters, in particular θ:

Slope of the saddle path (ctd.)

- θ **large** $\Rightarrow \frac{1}{\theta}$ small:
  * Individuals have a high preference for a smooth consumption path and dislike large variations in consumption (small intertemporal elasticity of substitution).
  * As individuals know that consumption will be increasing in the future, they try to consume a share of this increase today. As a consequence, investment is low.
  * The saddle path is "close" to the $\dot{k}(t) = 0$ locus.
  * $\Rightarrow$ the speed of convergence is low.

Slope of the saddle path (ctd.)

- θ **small** $\Rightarrow \frac{1}{\theta}$ large:
  * Individuals have a small preference for a smooth consumption path and are willing to bear larger variation in consumption to get a higher average consumption level (high intertemporal elasticity of substitution).
  * Individuals know that a small consumption today will increase average consumption in the future. They are willing to invest more of their income.
  * The saddle path is "far away" from the $\dot{k}(t) = 0$ locus.
  * $\Rightarrow$ the speed of convergence is high.

Appendix: Analytical Treatment of the Dynamic System

Consider the dynamic system in its general form (with $g = 0$ and $n \geq 0$)

$$\dot{k}(t) = f(k(t)) - c(t) - nk(t) \quad (3)$$

$$\dot{c}(t) = -\frac{U'(c(t))}{U''(c(t))}\left(f'(k(t)) - \rho\right) \quad (4)$$

How can we characterize the solution to this system of differential equations?

Appendix: Analytical Treatment of the Dynamic System (ctd.)

Technically, the solution of the system of differential equations can be obtained in different ways. However, only in special cases explicit solutions are available. We therefore make a 1st order Taylor approximation around the steady state, which allows to find the solution via eigenvalues:

We look for a solution of the system.

Rewrite the linearized approximation of (3) and (4) in matrix notation as

$$\dot{\mathbf{x}} \equiv \begin{pmatrix} \dot{k} \\ \dot{c} \end{pmatrix} = \mathbf{A} \begin{pmatrix} k \\ c \end{pmatrix} = \mathbf{Ax}$$

Now suppose **A** has two eigenvalues r_1, r_2 and two eigenvectors v_1 and v_2, then

$$\begin{aligned} Av_i &= r_i v_i \\ & \text{and let } \; P = [v_1, v_2] \\ & \Leftrightarrow AP = PD \; \text{ where } \; D = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \end{aligned}$$

Appendix: Analytical Treatment of the Dynamic System (ctd.)

Since eigenvectors are linearly independent, **P** is invertible

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$$

Then we can rewrite our initial system using $y \equiv P^{-1}x$ as

$$\begin{aligned} \dot{y} &= P^{-1}\dot{x} \\ &= P^{-1}Ax \\ &= P^{-1}APy \\ &= Dy \end{aligned}$$

Appendix: Analytical Treatment of the Dynamic System (ctd.)

Since D is diagonal, the system $\dot{y} = Dy$ can be written as $\dot{k}_v = r_1 k_v$, $\dot{c}_v = r_2 c_v$ with solution

$$\begin{aligned} k &= c_1 e^{r_1 t} \\ c &= c_2 e^{r_2 t} \end{aligned}$$

substituting back gives

$$\begin{aligned} x &= Py \\ &= (v_1, v_2) \begin{pmatrix} c_1 e^{r_1 t} \\ c_2 e^{r_2 t} \end{pmatrix} \\ &= c_1 e^{r_1 t} v_1 + c_2 e^{r_2 t} v_2 \end{aligned}$$

Appendix: Analytical Treatment of the Dynamic System (ctd.)

Stability of the System:
The solution of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ has the form

$$\text{constant} \cdot e^{\text{eigenvalue} \cdot t} \cdot \text{eigenvector}$$

which (for real eigenvalues) implies that

$$\lim_{t \to \infty} x(t) = 0 \quad \Leftrightarrow \quad \text{each } e^{r_i t} \to 0 \quad \Leftrightarrow \quad \text{each } r_i < 0$$

If one $r_i > 0$ then $e^{r_i t}$ goes to infinity.
Saddle-path stability implies that the two eigenvalues go in opposite directions.
Note that the trace of D and the determinant of A have the same sign, which allows us to check the saddle path stability on A.

Appendix: Analytical Treatment of the Dynamic System (ctd.)

To check, linearize (Taylor-approximate) the dynamic system around the steady state (i.e., form the Jacobian matrix of the system evaluated at the steady state (k^*, c^*)):

$$J_E = \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial c} \\ \frac{\partial \dot{c}}{\partial k} & \frac{\partial \dot{c}}{\partial c} \end{bmatrix}_{(k^*, c^*)} = \begin{bmatrix} f'(k^*) - n & -1 \\ -\frac{U'(c^*)}{U''(c^*)} f''(k^*) & 0 \end{bmatrix}$$

where we know from the discussion on the Modified Golden Rule that $f'(k^*) - n = \rho > 0$.

$$r_1 r_2 = |J_E| = -\frac{U'(c^*)}{U''(c^*)} f''(k^*) < 0$$

which implies that the two roots have opposite signs, and that the system exhibits a saddle-point.