The Madsen–Weiss Theorem

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Abstract

For Σ_g a surface of genus g, we give a proof of the Madsen–Weiss Theorem, which roughly states that we can identify the stable homology of the mapping class group $\Gamma_g = \pi_0 \mathrm{Diff}(\Sigma_g)$ for g tending to infinity with the homology of a certain infinite loop space. Moreover, we discuss how to use this result to obtain a positive answer to the famous Mumford Conjecture, which claims that the rational cohomology ring $H^*(B\Gamma_\infty; \mathbb{Q})$ can be computed explicitly, and is simply given by a free polynomial ring on countably many generators.

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Contents

1. Introduction

Around 1857 Riemann published his influential paper [Rie57] on moduli spaces $\mathcal{M}_g$ of complex curves of genus g, which was the birth of the study of moduli spaces. Only much later, with the rigorous development of (co)homology in the early 1980s, were results published about the groups $H^*(\mathcal{M}_g)$, for instance through work by Mumford [Mum83]. Importantly, it was realised that this moduli space shares the same rational cohomology groups as the classifying space $B\Gamma_g$ of the mapping class group of Σ_g, as long as $g \geq 2$. It is for this reason that suddenly a lot of effort was put into studying this latter space, yielding many results, e.g. the fact that the cohomology $H^*(\Gamma_g)$ is independent of g for "small" degrees in a range, which is due to Harer [Har85]. However, by far the most significant result is the Madsen–Weiss Theorem, first proven by Ib Madsen and Michael Weiss in [MW04], which forms the central focus of this thesis. Below, we give a rough outline and explain the main results in more detail.

1.1. Structure and Statement of the Main Results

Before giving any precise definitions and proofs, let us heuristically state the main results of the thesis and discuss the structure of the document, which is largely inspired by the Park City lecture notes written up by Søren Galatius in [Gal12].

In sec:manifolds we will mostly concern ourselves with a set of surfaces denoted by $\Psi(\mathbb{R}^n)$. It is defined as the set of oriented smooth 2-manifolds without boundary which are topologically closed in $\mathbb{R}^n$, so in particular it includes all affine planes in $\mathbb{R}^n$. It turns out that we can endow this set with a suitable topology, so that many constructions in arbitrary dimensions become very geometric in the sense that we shall be able to argue certain proofs by simply "drawing pictures" in $\mathbb{R}^3$. However, a priori it is not entirely clear what properties this topology should carry and, in fact, the construction is quite tedious, which is why all details necessary to gain an understanding of the topology are summarized in this chapter.

Interestingly, if we restrict to the subset $B_n := \{W \in \Psi(\mathbb{R}^n) \mid W \subseteq (0,1)^n\}$, then in the infinite case $n = \infty$ there is a weak homotopy equivalence

$$B_\infty \simeq \coprod_W B\mathrm{Diff}(W), \tag{1}$$

where $\mathrm{Diff}(W)$ denotes the orientation-preserving diffeomorphisms of W, and where the disjoint union runs over oriented closed surfaces, one of each diffeomorphism class. Using the space B_n (in the case $n < \infty$), we will define an explicit map $\alpha\colon B_n \to \Omega^n\Psi(\mathbb{R}^n)$ in Chapter 3, which in turn gives rise to a well-defined map in the colimit

$$\alpha\colon B_\infty \to \Omega^\infty\mathbf{\Psi} := \operatorname*{colim}_n \Omega^n\Psi(\mathbb{R}^n),$$

into an infinite loop space. This map by itself is, in a sense, still too wildly behaved, for instance because neither the domain nor the target are path connected. However, if we restrict the left-hand side to a single path component $B\mathrm{Diff}(W)$ for $W = \Sigma_g$ being a surface of genus g, using the identification (1), we find that the restricted map $\alpha^g := \alpha|_{B\mathrm{Diff}(\Sigma_g)}$ is an $H_*(-;\mathbb{Z})$-isomorphism in a specific range. More precisely, the main theorem states the following:

Theorem 1.1 (Madsen–Weiss; [Gal12], Theorem 1.8)**.** *For Σ_g a surface of genus g, the map $\alpha^g\colon B\mathrm{Diff}(\Sigma_g) \to \Omega^\infty_{[\Sigma_g]}\mathbf{\Psi}$ is an $H_*(-;\mathbb{Z})$-isomorphism for $* \leq \frac{2}{3}(g-1)$.*

The extra subscript on the right-hand side simply indicates the path-component in which the image of α^g is contained. Moreover, the above theorem is stated in a slightly non-standard form. In the literature one usually finds the theorem stated in terms of the Thom spectrum $\mathrm{MTSO}(2)$, which we will briefly discuss as well.

In Chapter 4 we will argue precisely how the Madsen–Weiss Theorem resolved the long-standing Mumford-Conjecture, roughly stating that, rationally, the cohomology ring structure of the infinite-genus mapping class group is given by a free polynomial algebra on infinitely many generators, i.e. $H^*(B\Gamma_\infty;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\ldots]$, where $|\kappa_i| = 2i$. It was shown by Miller in [Mil86] that the ring $\mathbb{Q}[\kappa_1,\kappa_2,\ldots]$ includes into the cohomology ring, but the full conjecture was only settled when Madsen and Weiss published their paper [MW04] in 2004. This result is intriguing, since the cohomology of mapping class groups is relatively hard to compute in most cases.

The final two chapters are dedicated to a much more thorough analysis and many technical proofs. For instance, we argue that the map $\alpha\colon B_n \to \Omega^n\Psi(\mathbb{R}^n)$ can be factored through n different maps, the majority of which turn out to be weak equivalences, which one sees by studying a certain family of topological monoids. This already finishes the bulk of the proof and drastically reduces the problem to essentially just a single map which needs to be studied. Herein, however, lies the crux of the proof. After dealing with some subtleties and more technical lemmata, the proof is finished in Chapter 6, where we conclude by discussing how all arguments should be put together.

1.2. Notation and Conventions

- Define $\mathbb{N} := \{0,1,2,3,\ldots\}$. The space I denotes the standard unit interval $[0,1]$.
- For any two smooth manifolds V, W, the embedding space $\mathrm{Emb}(V,W) \subseteq C^\infty(V,W)$ carries the subspace topology coming from the *Whitney C^∞-topology*, most effectively described as a subspace of the space of sections of jet bundles in Chapter 2 of [GG73].
- Unless otherwise specified, we always view the standard n-sphere S^n as the one-point compactification $(\mathbb{R}^n)^+$.
- For a smooth manifold W, $\mathrm{Diff}(W)$ denotes the topological group of orientation-preserving diffeomorphisms which restrict to the identity near the boundary of W, if applicable.

2. Spaces of Manifolds and Their Topology

In this chapter we mainly focus on constructing a topology on an important set of manifolds which will be one of the main ingredients in the Madsen–Weiss Theorem. We mostly follow Section 2 of [GR10] and try to give as many intuitive explanations as possible along the way. Due to the technical nature of many constructions in the above paper, we take the liberty to omit some of the details in hopes of creating a clearer picture.

2.1. Definitions and Constructions

Fix some natural number n. We start with the most important definition:

Definition 2.1. Let $U \subseteq \mathbb{R}^n$ be an open set and define the following set of submanifolds of U:

$$\Psi_d(U) \coloneqq \{(M, o_M) \mid M \subseteq_{\mathrm{tc}} U \text{ smooth } d\text{-submanifold, } \partial M = \emptyset,\ o_M \text{ an orientation}\},$$

where the subscript tc means that M is topologically closed in U. Usually, we will omit the orientation o_M for brevity.

Remark 2.2. Notice that the set $\Psi_d(U)$ contains more than just compact manifolds, for instance all affine planes in the case of $\Psi_2(\mathbb{R}^3)$. Throughout this chapter we will fix a dimension d and hence we write $\Psi(U) \coloneqq \Psi_d(U)$ for the sake of clarity. It is convenient for the reader to keep in mind that the use-case for the Madsen–Weiss Theorem will be $d = 2$ and $U = \mathbb{R}^n$ later on.

Example 2.3. As sets, we find that $\Psi_0(\mathbb{R}^n) = \coprod_{k \geq 0} C_k(\mathbb{R}^n) \amalg C_\infty(\mathbb{R}^n)$, where $C_k(\mathbb{R}^n)$ denotes the ordered configuration space of k points in $\mathbb{R}^n$.

The task of defining a suitable topology on $\Psi(U)$ is not as straightforward as one would hope. In particular, we must first construct two intermediate topologies, whose definitions are somewhat delicate, before we can give the full definition of a topology on $\Psi(U)$ that is useful for our purposes.

Step 1. *(Compactly Supported Topology)* Fix a manifold $M \in \Psi(U)$ and let $C_c^\infty(M)$ denote the set of smooth functions $M \to \mathbb{R}$ with compact support. Given some function $\varepsilon : M \to (0, \infty)$ and a finite collection of vector fields $X = (X_1, \ldots, X_\ell)$ on M, define the "ε-ball"

$$B(\varepsilon, X) \coloneqq \{f \in C_c^\infty(M) \mid |(X_1 X_2 \cdots X_\ell f)(x)| < \varepsilon(x) \text{ for all } x \in M\}.$$

We now declare the family of sets $\{f + B(\varepsilon, X)\}$ to be the generating subbasis of a topology on $C_c^\infty(M)$, where f ranges over $C_c^\infty(M)$, ε over functions $M \to (0, \infty)$, X over ℓ-tuples of vector fields and ℓ over non-negative integers.

Next, consider the vector space $\Gamma_c(NM)$ of compactly supported sections $s\colon M \to NM$ of the normal bundle NM, defined to be the subbundle of the trivial bundle $\underline{\mathbb{R}^n}$ which is the orthogonal complement to the tangent bundle TM in $\underline{\mathbb{R}^n}$. Using the standard map $\mathrm{std}\colon NM \to \mathbb{R}^n$ given on each fiber by $(m, v) \mapsto m + v$, and the fact that there is an isomorphism of vector spaces $\Gamma_c(\underline{\mathbb{R}^n}) \cong \Gamma_c(\underline{\mathbb{R}})^{\oplus n}$, we can identify $\Gamma_c(NM)$ with a linear subspace of $C_c^\infty(M)^{\oplus n}$. Topologise $\Gamma_c(NM)$ as a subspace with respect to the topology defined before.

The crucial observation in this step is to notice that the Tubular Neighborhood Theorem implies that the map $\mathrm{std}\colon NM \to \mathbb{R}^n$ restricts to an embedding of a neighborhood of the zero-section $0\colon M \to NM$. With this in mind, the assignment $s \mapsto s(M)$ gives rise to a partially defined injective map

$$c_M\colon \Gamma_c(NM) \dashrightarrow \Psi(U), \tag{2}$$

whose domain $\Gamma_c^0(NM) \subseteq \Gamma_c(NM)$ is an open set. Lastly, topologise $\Psi(U)$ by declaring the family of sets

$$\left\{c_M(O)\right\}_{M\in\Psi(U),\, O\subseteq\Gamma_c^0(NM)\text{ open}}$$

to be a basis. We write $\Psi(U)^{\mathrm{cs}}$ for the set $\Psi(U)$ endowed with this topology. Notice that $\Psi(U)^{\mathrm{cs}}$ is an infinite dimensional manifold, with the underlying model being the topological vector spaces $\Gamma_c(NM)$.

Remark 2.4. Instead of considering the partially defined map c_M, we could in theory consider a suitable neighborhood of the zero-section so the standard map restricts to an embedding. However, for each manifold M we would have to make a choice about precisely which neighborhood to pick. For the sake of simplicity, we stick to the general case at the cost of not being well-defined in full generality.

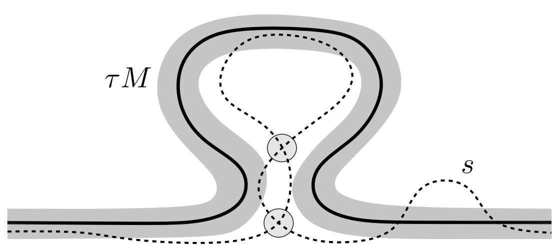

FIGURE 1. A 1-dimensional analogue of a section $s\colon M \to NM$ which (partially) lies outside the tubular neighborhood τM of the zero section. Since $s(M)$ has two self-intersections, it is not a manifold and hence $s \notin \Gamma_c^0(NM)$.

Remark 2.5. Morally, the construction of the cs-topology on $\Psi(U)^{\mathrm{cs}}$, and in particular the explicit map $s \mapsto s(M)$, allows us to interpret a neighborhood of $M \in \Psi(U)^{\mathrm{cs}}$ as being homeomorphic to a neighborhood of the zero-section in $\Gamma_c(NM)$, which is good to keep in mind.

Step 2. (*K-Topology*) Next, fix some compact set $K \subseteq U$ and let $\pi_K\colon \Psi(U)^{\mathrm{cs}} \twoheadrightarrow \Psi(K \subseteq U)$ denote the quotient map identifying two manifolds M and M' if there exists a neighborhood

$N_K \supseteq K$ such that $M \cap N_K = M' \cap N_K$. Endow $\Psi(K \subseteq U)$ with the quotient topology and define the space $\Psi(U)^K$ as being the set $\Psi(U)^{\mathrm{cs}}$ endowed with the pullback-topology under the quotient map π_K, which we shall call the *K-topology*. Notice that the K-topology is coarser than the cs-topology. By the universal properties of initial and quotient topologies, we find that the identity map $\Psi(U)^L \to \Psi(U)^K$ is continuous if $K \subseteq L$ are two compact sets.

Step 3. Lastly, we define $\Psi(U)$ to have the coarsest topology finer than all K-topologies. More precisely, choose some exhaustive tower of inclusions $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$ of compact sets in U, with $U = \bigcup_{i\geq 0} K_i$, and define

$$\Psi(U) \coloneqq \varprojlim \left(\cdots \to \Psi(U)^{K_2} \to \Psi(U)^{K_1} \to \Psi(U)^{K_0}\right),$$

where all structure maps are just the identity. By convention, we define the empty manifold $\emptyset \in \Psi(U)$ to be the preferred base point.

Example 2.6. A straightforward example to get a feeling for the notion of convergence is the following: let $\gamma : \mathbb{R} \to \Psi_2(\mathbb{R}^3)$ be the map $t \mapsto \{t\} \times \mathbb{R}^2$. For any compact $K \subseteq \mathbb{R}^3$ we know that there exists a neighborhood $N_K \supseteq K$ such that $\gamma(t) \cap N_K = \emptyset$, provided t is sufficiently large. This means that $\gamma(t)$ converges to $\emptyset$ in the K-topology as t grows indefinitely. Since K was arbitrary, $\gamma(t)$ converges to $\emptyset$ in $\Psi_2(\mathbb{R}^3)$ as well. This phenomenon, i.e. sending a manifold "off to infinity", will be used frequently in later constructions, so it is worthwhile for the reader to keep this in mind. Note that the continuity of γ will be dealt with later, as it is not yet clear how we should approach this problem.

Remark 2.7. In practice, it suffices for our purpose to consider the much easier case where $K_i \coloneqq B(0, i)$, i.e. we restrict to compact n-balls of radius i centered at the origin. Then we can simply define

$$\Psi(\mathbb{R}^n) \coloneqq \varprojlim \left(\cdots \to \Psi(\mathbb{R}^n)^{B(0,2)} \to \Psi(\mathbb{R}^n)^{B(0,1)}\right),$$

as is done in [Kup19, p. 279], without ever considering the more general case to begin with.

2.2. Elementary Properties

Fix an open set $V \subseteq U$ and consider the restriction map $r\colon \Psi(U) \to \Psi(V)$ given by sending $M \mapsto M \cap V$. This turns $\Psi(-)$ into a sheaf of spaces, and not just sets, as we shall soon see.

Example 2.8. To begin with, notice that $r\colon \Psi(U)^{\mathrm{cs}} \to \Psi(V)^{\mathrm{cs}}$ is *not* continuous in full generality. For a counterexample, consider the inclusion $(0, 1) \subset (0, 2)$ and a sequence of 0-manifolds given by points $\{1 - 1/n\}_{n\geq 2}$. By construction of the cs-topology, the following set N_M is an open neighborhood of some fixed 0-manifold $M \in \Psi_0((0, 1))$, provided that $\#\pi_0 M < \infty$:

$$N_M \coloneqq \left\{M' \in \Psi_0((0, 1)) \mid M = M' \text{ outside some compact set}, \#\pi_0 M = \#\pi_0 M'\right\}.$$

So roughly speaking, if $r\colon \Psi((0,2))^{\mathrm{cs}} \to \Psi((0,1))^{\mathrm{cs}}$ were continuous, then we would find that the sequence of points converged to $\emptyset$. But this is a contradiction, since $\emptyset$ is isolated in $\Psi((0,1))^{\mathrm{cs}}$ whereas the sequence of 0-manifolds is never empty.

A priori this is bad news, but fortunately we will be able to recover different sufficiently nice properties about the restriction map, starting with the next lemma:

Lemma 2.9. *For $V \subseteq U$ open, the restriction $r\colon \Psi(U)^{\mathrm{cs}} \to \Psi(V)^{\mathrm{cs}}$ is an open map.*

Proof. Let $\mathcal{U} \subseteq \Psi(U)^{\mathrm{cs}}$ be an open set and fix some $M \in \mathcal{U}$. We have the following (partially defined) commutative diagram

$$\begin{tikzcd}
\Gamma_c(NM) \arrow[r, dashed, "c_M"] & \Psi(U)^{\mathrm{cs}} \arrow[d, "r"] \\
\Gamma_c(N(M \cap V)) \arrow[u, "z"] \arrow[r, dashed, "c_{M\cap V}"] & \Psi(V)^{\mathrm{cs}}
\end{tikzcd}$$

where z denotes the trivial extension of the zero-section. By choosing an open neighborhood $\mathcal{U}' \subseteq \operatorname{im}(c_{M\cap V})$, we can assume that there exists an open set $\mathcal{U}'_V \subseteq \Gamma_c(N(M\cap V))$ such that $c_{M\cap V}(\mathcal{U}'_V) = \mathcal{U}'$. By the commutativity of the square, we see that

$$\widetilde{\mathcal{U}} \coloneqq c_M(z(\mathcal{U}'_V))$$

is contained in $\mathcal{U}$, with $M \in \widetilde{\mathcal{U}}$ by definition, and this set is open in $\Psi(U)^{\mathrm{cs}}$. Consequently, we have that $r(\widetilde{\mathcal{U}}) = \mathcal{U}'$, showing that $M \cap V$ is interior in $r(\mathcal{U})$, which proves the claim. □

Lemma 2.10 ([GR10], Lemma 2.5)**.** *Let $K \subseteq U$ be a compact subset. Then quotient map $\pi_K\colon \Psi(U)^{cs} \twoheadrightarrow \Psi(K \subseteq U)$ is open.*

Lemma 2.11. *Let $V \subseteq U$ be open. Then the map $\rho\colon \Psi(K \subseteq U) \to \Psi(K \subseteq V)$, induced by the restriction map $r\colon \Psi(U)^{cs} \to \Psi(V)^{cs}$, is injective.*

Proof. One easily sees that the map $[M] \mapsto [M\cap V]$ is well-defined, using the definition. Next, suppose that $[M\cap V] = [M'\cap V]$, i.e. there exists a neighborhood N_K of K such that $(M\cap V)\cap N_K = (M'\cap V)\cap N_K$. Then we can simply define a new neighborhood $\overline{N}_K \coloneqq V \cap N_K$ of K and we see that $M \cap \overline{N}_K = M' \cap \overline{N}_K$, and thus $[M] = [M']$, showing that ρ is injective. □

Lemma 2.12. *Let $V \subseteq U$ be open. Then the injection $\rho\colon \Psi(K \subseteq U) \hookrightarrow \Psi(K \subseteq V)$ is a homeomorphism onto an open subset.*

Proof. Let $\varepsilon > 0$ be such that $0 < 3\varepsilon < \operatorname{dist}(K, \mathbb{R}^n \setminus U)$. By the technical Lemma 2.4 in [GR10] we know there exists a suitable compactly-supported bump-function $\lambda\colon \mathbb{R}^n \to [0,1]$ such that $\lambda(x) = 1$ if $\operatorname{dist}(x,K) \leq \varepsilon$, $\lambda(x) = 0$ if $\operatorname{dist}(x,K) \geq 2\varepsilon$ and such that $\operatorname{supp}(\lambda) \subseteq V$. This function induces a continuous multiplication map $\overline{\lambda}\colon \Gamma_c(NM) \to \Gamma_c(N(M\cap V))$ on the fibers, and we obtain a diagram

$$\begin{tikzcd}
\Gamma_c(NM) \arrow[r, dashed, "c_M"] \arrow[d, "\bar{\lambda}"'] & \Psi(U)^{\mathrm{cs}} \arrow[r, "\pi_K"] \arrow[d, "r"] & \Psi(K \subseteq U) \arrow[d, "\rho"] \\
\Gamma_c(N(M \cap V)) \arrow[r, dashed, "c_{M\cap V}"] & \Psi(V)^{\mathrm{cs}} \arrow[r, "\pi_K"] & \Psi(K \subseteq V)
\end{tikzcd}$$

where the outer rectangle and right-hand square commute (but not the left-hand square!). This shows that the composite $\rho \circ \pi_K \circ c_M$ is continuous, and moreover that ρ is continuous, since π_K is a quotient map and c_M is a local homeomorphism by definition.

If $A \subseteq \Psi(K \subseteq U)$ denotes an open set, then we have that $\rho(A) = (\pi_K \circ r)(B)$ for $B := \pi_K^{-1}(A)$. Since both π_K and r are open maps by Lemmata 2.10 and 2.9 respectively, the result follows. $\square$

Now we are in a place to resolve the problem that the restriction map is not in general continuous if we merely work in the cs-topology. We obtain the following useful statement:

Theorem 2.13. *For $V \subseteq U$ open, the restriction map $r\colon \Psi(U) \to \Psi(V)$ is continuous.*

Proof. By definition of the K-topology and limit-topology, we need to show that the composite $\pi_K \circ r\colon \Psi(U)^K \to \Psi(V)^K \to \Psi(K \subseteq V)$ is continuous for all compact $K \subseteq V$. But by Lemma 2.12 this is equal to $\rho \circ \pi_K$, which is continuous by the previous lemma. $\square$

We need one more intermediate lemma:

Lemma 2.14 ([GR10], Lemma 2.10)**.** *Let $\mathrm{Emb}(U,V)$ denote the space of embeddings of open subsets of $\mathbb{R}^n$ and fix some $j_0 \in \mathrm{Emb}(U,V)$. Then there exists a compact space $K \subseteq U$ and a partially defined map*

$$\varphi\colon \mathrm{Emb}(U,V) \dashrightarrow \mathrm{Diff}_c(U),$$

defined in an open neighborhood of j_0 such that $j(x) = (j_0 \circ \varphi(j))(x)$ for x in a neighborhood of K, where Diff_c denotes compactly-supported diffeomorphisms.

The above lemma motivates the following important result, which will be used frequently throughout the thesis:

Theorem 2.15. *In the above setting, the map*

$$\mathrm{Emb}(U,V) \times \Psi(V) \to \Psi(U), \quad (j, M) \mapsto j^{-1}(M)$$

is continuous.

Proof. If suffices to check that the composite $\mathrm{Emb}(U,V) \times \Psi(V) \to \Psi(U) \to \Psi(K \subseteq U)$ is continuous in a neighborhood of each $\{j_0\} \times \Psi(V)$ for each compact $K \subseteq U$. Let N_0 be the open neighborhood of j_0 from Lemma 2.14 on which $\varphi\colon \mathrm{Emb}(U,V) \dashrightarrow \mathrm{Diff}_c(U)$ is defined. By the same lemma we get a factorization

$$\begin{tikzcd} N_0 \times \Psi(V) \arrow[rr, "(j,M)\mapsto j^{-1}(M)"] \arrow[d, "\varphi\times j_0^{-1}"'] & & \Psi(U) \\ \mathrm{Diff}_c(U)\times\Psi(U) \arrow[urr] & & \end{tikzcd}$$

Notice that the unmarked map is given by the standard continuous $\mathrm{Diff}(U)$-action on $\Psi(U)$. Moreover, the left-hand map is continuous in the Whitney C^∞-topology essentially by its construction in the aforementioned lemma, and hence the claim follows. □

Remark 2.16. Since the structure of the topology on $\Psi(U)$ is rather involved, as we have seen by now, the importance of Theorem 2.15 cannot be overstated, as it comprises the main tool to determine whether maps involving the space $\Psi(U)$ are continuous. In practice, we will be concerned with the case $d = 2$, $V = U = \mathbb{R}^n$ and the subgroup $\mathrm{Diff}(\mathbb{R}^n) \subseteq \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$.

Now we are in a position to show that the map $\gamma(t) = \{t\}\times\mathbb{R}^2$ from Example 2.6 is continuous: first, rewrite γ as $\gamma(t) = \mathbb{R}^2 + te_1$, where e_1 denotes the first canonical basis vector of $\mathbb{R}^3$, hence we can view $\gamma(t)$ as being the preimage under the diffeomorphism $x \mapsto x - te_1$ in $\mathbb{R}^3$. Therefore, it follows from Theorem 2.15 that γ is continuous, as claimed.

We can use the same theorem to show another favourable property about our space of manifolds:

Proposition 2.17. *For $n \geq 3$ and $d < n$, the space $\Psi_d(\mathbb{R}^n)$ is path-connected.*

Proof. Fix some $W \in \Psi_d(\mathbb{R}^n)$ and pick a point $p \in \mathbb{R}^n \setminus W$. For $t \in I$, denote by $W - tp$ the manifold defined by the preimage of W under the diffeomorphism $x \mapsto x + tp$ in $\mathbb{R}^n$. Scaling by $(1-t)^{-1}$ gives a path $\gamma(t) = (1-t)^{-1}(W - tp)$, interpreted as $\emptyset$ for $t = 1$, using that the point p lies in the complement, to make sure that W is *not* translation-invariant. By virtue of Theorem 2.15, we find that this map is a continuous path in $\Psi_d(\mathbb{R}^n)$ from W to $\emptyset$, thus giving a path from any manifold to the basepoint. □

Remark 2.18. Notice that the topology constructed on $\Psi(\mathbb{R}^n)$ exhibits some rather unusual phenomena. For instance, the proof of Proposition 2.17 shows that two points in the image of a continuous map $X \to \Psi(\mathbb{R}^n)$ (in this case $X = [0, 1]$) need *not* share the same homotopy type, which is useful to keep in mind for many further constructions that will be discussed later.

2.3. Smooth Maps

We briefly discuss the notion of *smooth* maps in our setting. Broadly speaking, this is a useful property of maps between smooth manifolds and the space $\Psi(\mathbb{R}^n)$ that we will make use of several times throughout the thesis.

Definition 2.19 (Smooth Maps)**.** For a smooth manifold X, let $f\colon X \to \Psi(\mathbb{R}^n)$ be a continuous map and write $f(x) = M_x$. Define the *graph* of f to be the space of pairs

$$\Gamma(f) \coloneqq \bigcup_{x \in X} \{x\} \times M_x \subseteq X \times \mathbb{R}^n.$$

We call the map $f\colon X \to \Psi(\mathbb{R}^n)$ *smooth* if $\Gamma(f) \subseteq X \times \mathbb{R}^n$ is a smooth submanifold such that the projection $\pi : \Gamma(f) \to X$ is a submersion and the orientations of $f(x)$ assemble to an orientation of $\ker(D\pi\colon T\Gamma(f) \to TX)$, the *vertical tangent bundle*, i.e. the orientations vary "continuously".

Remark 2.20. Even though we really only view $\Psi(\mathbb{R}^n)$ as a topological space, the previous definition supplies us with a heuristic idea of how one might think of $\Psi(\mathbb{R}^n)$ as having a smooth structure, even though this is not in the usual sense.

Remark 2.21. The reason for requiring the projection $\pi\colon \Gamma(f) \to X$ to be a submersion is simply to guarantee that the fibers of π are *smooth* manifolds, see Theorem 3.2 in [Hir76], and are hence contained in $\Psi(\mathbb{R}^n)$.

Lastly, we present an important technical lemma which roughly states that any continuous map $X \to \Psi(U)$ can be "perturbed" to a smooth map, thus allowing us to assume that any such map is smooth in general. More precisely, we have:

Lemma 2.22 ([GR10], Lemma 2.17)**.** *Let X be a smooth manifold and $f\colon X \to \Psi(U)$ be a continuous map. Let $V \subseteq X \times U$ be open and $W \subseteq X \times U$ be such that $\overline{V} \subseteq \operatorname{int}(W)$. Then there exists a homotopy $F\colon X \times I \to \Psi(U)$ such that $F(-, 0) = f$, which is smooth on $V \times (0, 1] \subseteq X \times U \times I$ and is the constant homotopy outside W. Moreover, if f is already smooth on an open set $A \subseteq V$, then the homotopy can be chosen smooth on $A \times I$.*

Proof sketch. We give a very rough outline of the proof. First, consider the case where $W \subseteq X \times U$ satisfies the following property: there exist closed sets $K \subseteq X$ and $L \subseteq U$ such that $W \subseteq K \times L$ and the composite $K \hookrightarrow X \xrightarrow{f} \Psi(U) \xrightarrow{r} \Psi(L)$ factors through a continuous map $K \to \Psi(U)^{\mathrm{cs}}$. One calls W *small* if this property holds. Then, one can use the manifold structure on $\Psi(U)^{\mathrm{cs}}$ to construct a homotopy $h\colon I \times K \to \Psi(U)^{\mathrm{cs}}$ which is smooth on $(0, 1]$ already. One composes with an appropriate bump-function, so h can be assumed to be constant on $(K \times U) \setminus W$. Thus, h extends to a homotopy $\overline{h}\colon I \times X \to \Psi(U)$ by definition of its topology.

The second case is when W is given by a disjoint union of *small* open sets. Then, we can simply superimpose the respective homotopies.

Lastly, for general W, we first triangulate the space $X \times U$ such that every simplex is contained in a *small* open set. An analysis of the open stars $\mathrm{st}(\sigma_p)$ of p-simplices, together with the two cases from before and an inductive argument, concludes the proof. ∎

3. Towards the Madsen–Weiss Theorem

In this chapter we develop the ideas leading up to the Madsen–Weiss Theorem. We largely follow Section 1 of [Gal12] and begin with some recollections. Then we state rigorously how we can essentially relate the space $\Psi_2(\mathbb{R}^n)$ to the mapping class group of a surface and formulate the main result. Finally, we explore some basic properties and an important connection of $\Psi_2(\mathbb{R}^n)$ to a certain Thom space.

3.1. The Main Theorem

Fix the dimension $d = 2$ henceforth. Recall that $\Psi(\mathbb{R}^n)$ consists precisely of those oriented surfaces W in $\mathbb{R}^n$ that are smooth, have no boundary and are topologically closed in $\mathbb{R}^n$.

Definition 3.1. Define the subspace $B_n \coloneqq \{W \in \Psi(\mathbb{R}^n) \mid W \subseteq (0,1)^n\} \subseteq \Psi(\mathbb{R}^n)$, i.e. B_n consists of those manifolds in $\Psi(\mathbb{R}^n)$ that are compact in the standard open n-cube.

Roughly speaking, the reason for restricting to the unit cube is to have more control over each dimension, which will be useful for an application of the Earle–Eells Theorem below and later on in the proof for concrete constructions. For now, let's observe that we have a different way to interpret each space B_n:

Proposition 3.2. *Let $\mathcal{S}$ be the set containing one representative of each diffeomorphism class of oriented, closed surfaces in $(0,1)^n$. Then there exists a homeomorphism*

$$\Phi\colon \coprod_{W\in\mathcal{S}} \operatorname{Emb}(W,(0,1)^n)/\operatorname{Diff}(W) \xrightarrow{\cong} B_n,$$

where the left-hand side carries the standard Whitney C^∞-topology.

Proof sketch. First, define maps $\Phi_n\colon \operatorname{Emb}(W,(0,1)^n) \to \Psi(\mathbb{R}^n)$ by the assignment $f \mapsto f(W)$ and notice that these factor through the quotient $\operatorname{Emb}(W,(0,1)^n)/\operatorname{Diff}(W)$. Hence, they assemble to a map

$$\Phi\colon \coprod_{W\in\mathcal{S}} \operatorname{Emb}(W,(0,1)^n)/\operatorname{Diff}(W) \to \Psi(\mathbb{R}^n).$$

To see that Φ is continuous, we will show that, for any neighborhood U of $f(W)$, the set $\Phi^{-1}(U)$ contains a neighborhood N of f such that $\Phi(N) \subseteq U$. To this end, we consider a typical neighborhood $U \ni f(W)$, i.e. by definition $U = \{V \in \Psi(\mathbb{R}^n) \mid V \cap (0,1)^n \text{ is a section of } \tau f(W)\}$, where $\tau f(W)$ is a fixed tubular neighborhood of $f(W)$ in $(0,1)^n$. Recall that, roughly speaking, a typical neighborhood of $f \in \operatorname{Emb}(W,(0,1)^n) \subseteq C^\infty(W,\mathbb{R}^n)$ is given by some

$$U_k = \big\{f'\colon W \hookrightarrow (0,1)^n \;\big|\; \|f - f'\| \text{ is “small” up to the } k\text{-th derivative}\big\},$$

where this notion is made precise in e.g. Chapter II.3 of [GG73]. For the neighborhood N of f we can then simply choose U_0, meaning that we only require embeddings $f' \in N$ to satisfy $\|f - f'\| < \varepsilon$, i.e. the absolute difference is small in the usual Euclidean sense.

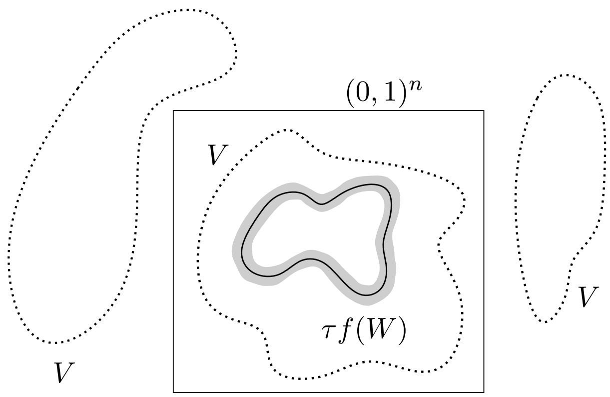

FIGURE 2. An element V of a typical neighborhood of $f(W)$.

For bijectivity, notice that $U_W \coloneqq \{V \in B_n \mid V \text{ is a section of } \tau W\}$, for some fixed tubular neighborhood τW, is a typical neighborhood of W in B_n. In a geometric sense, this allows us to think of a neighborhood of W to consist of "small perturbations" of W. For varying τW these sets U_W constitute a system of neighborhoods, and since tubular neighborhoods preserve local properties of manifolds, we find that we can write

$$B_n = \coprod_{g \geq 0} B_n^g,$$

where $B_n^g \coloneqq \{W \in B_n \mid W \cong \Sigma_g\}$, using the standard Classification Theorem of compact surfaces. Since we have a bijection $\mathcal{S} \to \mathbb{N}$, given by mapping $[W]$ to its genus $g(W)$, we see that it suffices to show that the restricted map $\Phi|\colon \mathrm{Emb}(W, (0,1)^n)/\mathrm{Diff}(W) \to B_n^g$ on each component is bijective, but this is immediate.

Lastly, we briefly argue that Φ is an open map. For each neighborhood $U \subseteq \mathrm{Emb}(W, (0,1)^n)$ of f we want to find a neighborhood U' of $f(W)$ such that $U' \subseteq \Phi(U)$. The rough idea is to notice that, by definition, U has some constraints on derivatives being "small", and U' is constructed from $U_{f(W)}$ by adding precisely these constraints to its definition. Thus, U' is automatically contained in $\Phi(U)$. By definition of the quotient topology we have shown that Φ is an open, continuous bijection. $\square$

Since the standard action of $\mathrm{Diff}(W)$ on $\mathrm{Emb}(W, (0,1)^n)$ is a free and continuous action (with respect to the C^∞-topology), a result due to Palais [Pal61, p. 315], known as the Slice Theorem, states that the quotient map

$$\pi\colon \mathrm{Emb}(W, (0,1)^n) \twoheadrightarrow \mathrm{Emb}(W, (0,1)^n)/\,\mathrm{Diff}(W)$$

is a principal $\mathrm{Diff}(W)$-bundle for all $n \leq \infty$. Moreover, it is not too hard to see that in the infinite case, the space $\mathrm{Emb}(W,(0,1)^\infty)$ is contractible for every W: we can push each embedding $W \hookrightarrow (0,1)^\infty$ into the odd coordinates by composing with a linear isotopy of $(0,1)^\infty$ into odd coordinates and afterwards take a linear isotopy to a fixed embedding of W into the even coordinates. Thus, we find that $\mathrm{Emb}(W,(0,1)^\infty)$ is a model for $E\mathrm{Diff}(W)$.

Theorem 3.3. (Earle–Eells, [Hat14, p. 28]). *Let W be a compact, connected and oriented surface, not necessarily without boundary. If W is not diffeomorphic to either S^2 or $S^1 \times S^1$, then the components of* $\mathrm{Diff}(W)$ *are contractible. In particular, the quotient map* $\mathrm{Diff}(W) \twoheadrightarrow \pi_0 \mathrm{Diff}(W)$ *is a homotopy equivalence.*

Since all manifolds W are compact in our current setting, we can use the isomorphism

$$\pi_i \mathrm{Diff}(W) \cong \pi_{i+1}(\mathrm{Emb}(W,(0,1)^\infty)/\mathrm{Diff}(W)),$$

in order to see that all these groups are trivial except for the case $i = 0$, meaning that the quotient $\mathrm{Emb}(W,(0,1)^\infty)/\mathrm{Diff}(W)$ is a model for the classifying space $B\mathrm{Diff}(W)$. It follows that, for $n = \infty$, we have an equivalence

$$B_\infty \simeq \coprod_{W \in \mathcal{S}} B\mathrm{Diff}(W).$$

By Theorem 3.3, this becomes an equivalence

$$B_\infty \simeq \left(\coprod_{W \in \mathcal{S}'} B\pi_0\mathrm{Diff}(W) \right) \amalg B\mathrm{Diff}(S^2) \amalg B\mathrm{Diff}(S^1 \times S^1), \tag{3}$$

where $\mathcal{S}' := \mathcal{S} \setminus \{[S^2], [S^1 \times S^1]\}$. These two additional disjoint summands look cumbersome, but do not pose any problem, since we restrict to surfaces of genus $g \geq 2$ in the main theorem later. Notice that $\pi_0\mathrm{Diff}(W)$ is the *mapping class group* of W, which we define below in Definition 3.8. This motivates the study of these mapping class groups, since they fully determine the homotopy type of $\mathrm{Diff}(W)$, except for the two cases $W = S^2$ and $W = S^1 \times S^1$.

Remark 3.4. In fact, the homotopy type of the two right-most disjoint summands in (3) is fully known as well: we have a weak equivalence $\mathrm{Diff}(S^2) \simeq O(3)$, which is due to Smale [Sma59], and it is known that $\mathrm{Diff}(S^1 \times S^1) \simeq \mathbb{R}^2/\mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$.

Now that we have a fairly good understanding of the space B_∞, we will relate it back to $\Psi(\mathbb{R}^n)$, by defining an important map α, which will be the main focus of study for the rest of the thesis. Recall that we always consider $S^n \cong (\mathbb{R}^n)^+$ as the one-point compactification.

Definition 3.5 (Scanning Map)**.** Define a map $\alpha_{(n)} \colon B_n \to \Omega^n \Psi(\mathbb{R}^n)$ by the formula

$$\alpha_{(n)}(W)(v) = \begin{cases} W + v & \text{if } v \in \mathbb{R}^n, \\ \emptyset & \text{if } v = \infty. \end{cases}$$

It will follow later from Proposition 5.3 that $\alpha_{(n)}$ is in fact continuous. One can check that $\alpha_{(n)}$ is compatible with passing to the infinite case, i.e. each diagram

$$\begin{tikzcd}
B_n \arrow[r, "\alpha_{(n)}"] \arrow[d, hook] & \Omega^n\Psi(\mathbb{R}^n) \arrow[d, hook, "\Omega^n i_n"] \\
B_{n+1} \arrow[r, "\alpha_{(n+1)}"] & \Omega^{n+1}\Psi(\mathbb{R}^{n+1})
\end{tikzcd}$$

commutes, where $B_n \hookrightarrow B_{n+1}$ is the canonical inclusion and the right-hand map is given by $i_n(W) = (t \mapsto W \times \{t\})$ before applying the functor $\Omega^n(-)$. Thus, we obtain a well-defined continuous map

$$\alpha\colon B_\infty \to \Omega^\infty \mathbf{\Psi} := \operatorname*{colim}_n \Omega^n\Psi(\mathbb{R}^n).$$

By the identification (3) above, each path component $B\mathrm{Diff}(W)$ is sent to a path component of $\Omega^\infty\mathbf{\Psi}$, which we will denote by $\Omega^\infty_{[W]}\mathbf{\Psi}$ for now.

Remark 3.6. The term “scanning” is likely due to the geometric nature of the map. Through translation by any vector $v \in \mathbb{R}^n$, one might imagine W as “scanning” through all of $\mathbb{R}^n$.

Now we are finally in a position to state the main theorem of the thesis. By restricting the scanning map to a single path component of (3) (and disregarding the cases $g = 0$ and $g = 1$), we have the following major result:

Theorem 3.7 (Madsen–Weiss)**.** *Let Σ_g be a surface of genus $g \geq 2$ with empty boundary. Then the restricted map*

$$\alpha^g := \alpha|_{B\mathrm{Diff}(\Sigma_g)}\colon B\mathrm{Diff}(\Sigma_g) \to \Omega^\infty_{[\Sigma_g]}\mathbf{\Psi}$$

is an $H_*(-;\mathbb{Z})$-isomorphism for $* \leq \frac{2}{3}(g-1)$. In particular, the map is a homology-isomorphism in the infinite-genus case $g = \infty$.

There is a neat way to rephrase the main theorem in terms of mapping class groups, an object studied mostly in lower-dimensional topology and topological field theories.

Definition 3.8 (Mapping Class Group)**.** Let $\Sigma_{g,r}$ be an oriented surface of genus $g \geq 2$ with $r \geq 0$ boundary components. We define its mapping class group as $\Gamma_{g,r} := \pi_0\mathrm{Diff}(\Sigma_{g,r})$, i.e. isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,r}$ fixing a neighborhood of each boundary component.

However, notice that the above definition allows for an arbitrary number of boundary components, whereas Theorem 3.7 calls for precisely one component. To bridge this gap, we use *homological stability* theorems for mapping class groups. Define a collection of stabilisation maps as follows: gluing a pair of pants along either one or two boundary components of $\Sigma_{g,r}$ respectively gives maps

$$\vartheta^1_{g,r}\colon \Sigma_{g,r} \hookrightarrow \Sigma_{g,r+1} \qquad \text{and} \qquad \vartheta^2_{g,r}\colon \Sigma_{g,r} \hookrightarrow \Sigma_{g+1,r-1}.$$

Denote by $\vartheta = \vartheta_{g,r}$ the composite $\vartheta^1_{g+1,r-1} \circ \vartheta^2_{g,r} \colon \Sigma_{g,r} \hookrightarrow \Sigma_{g+1,r}$. By extending diffeomorphisms to be the identity on the added pairs of pants, the maps ϑ^1 and ϑ^2 induce well-defined maps

$$\vartheta^1 \colon \Gamma_{g,r} \to \Gamma_{g,r+1} \qquad \text{and} \qquad \vartheta^2 \colon \Gamma_{g,r} \to \Gamma_{g+1,r-1}$$

on the respective mapping class groups.

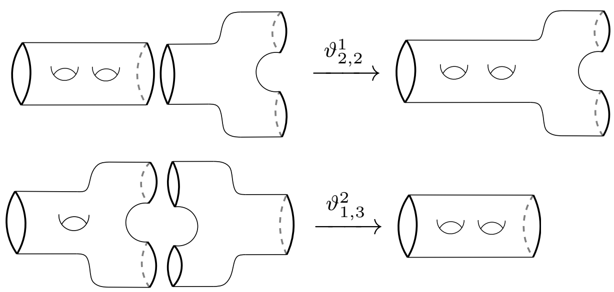

FIGURE 3. Examples of stabilisation maps, gluing on pairs of pants in two distinct ways.

The following major result, taken from [Wah11], is a combined effort of Harer, Ivanov, Boldsen and Randal-Williams, but is best known as Harer's Stability Theorem.

Theorem 3.9 (Homological Stability for Mapping Class Groups of Surfaces)**.** *Let $g \geq 0$ and $r \geq 1$. In group homology, the induced map*

$$H_*(\vartheta^1) \colon H_*(\Gamma_{g,r}; \mathbb{Z}) \to H_*(\Gamma_{g,r+1}; \mathbb{Z})$$

is always injective and an isomorphism for $* \leq \frac{2}{3}g$. Similarly, the map

$$H_*(\vartheta^2) \colon H_*(\Gamma_{g,r}; \mathbb{Z}) \to H_*(\Gamma_{g+1,r-1}; \mathbb{Z})$$

is surjective for $* \leq \frac{2}{3}g + \frac{1}{3}$ and injective for $* \leq \frac{2}{3}(g-1)$. Moreover, the map $\delta_g \colon \Gamma_{g,1} \to \Gamma_g$, induced by gluing a disc onto the one boundary component via the identity map, induces a map

$$H_*(\delta_g) \colon H_*(\Gamma_{g,1}; \mathbb{Z}) \to H_*(\Gamma_g; \mathbb{Z}),$$

which is an isomorphism for $* \leq \frac{2}{3}g$.

Now it is easy to see that we can use the maps $H_*(\vartheta^1)$ and $H_*(\delta_g)$, together with Theorem 3.3, to restate the main theorem in terms of mapping class groups.

Theorem 3.10. *For $g \geq 2$ and $r \geq 0$, the diagram*

$$B\Gamma_{g,r} \xleftarrow{B\vartheta^1} \cdots \xleftarrow{B\vartheta^1} B\Gamma_{g,2} \xleftarrow{B\vartheta^1} B\Gamma_{g,1} \xrightarrow{B\delta_g} B\Gamma_g \xleftarrow{B\pi_0} B\mathrm{Diff}(\Sigma_g) \xrightarrow{\alpha^g} \Omega^\infty_{[\Sigma_g]}\mathbf{\Psi}$$

induces a homology isomorphism

$$H_*(\Gamma_{g,r}; \mathbb{Z}) \xrightarrow{\cong} H_*\big(\Omega^\infty_{[\Sigma_g]}\mathbf{\Psi}; \mathbb{Z}\big),$$

provided that $* \leq \frac{2}{3}(g-1)$.

Notice that the above two theorems are stated in terms of group homology, the fundamentals of which can be found in [Bro82]. Most importantly, for any group G we can simply define $H_*(G;\mathbb{Z}) \coloneqq H_*(BG;\mathbb{Z})$. Moreover, note that, curiously, the isomorphism only depends on the genus g and *not* the number r of boundary components. As we will see towards the end of our proof of the Madsen–Weiss Theorem, a fruitful approach is to consider the case $r = 1$, as this allows for some more structure to work with, while simultaneously keeping the complexity of any geometric constructions with the boundary component at a minimum.

3.2. The Homotopy Type of $\Psi(\mathbb{R}^n)$

For the remainder of this section we will discuss the homotopy type of the space $\Psi(\mathbb{R}^n)$ and explain how we can possibly hope to compute the (co)homology of the space $\Omega^\infty_\bullet \mathbf{\Psi}$, where $\Omega^\infty_\bullet(-)$ simply denotes the path component containing the basepoint. Each component will have the same homotopy type, so we suppress the fact that we consider exactly $\Omega^\infty_{[\Sigma_g]}(-)$.

Consider the Grassmannian $\mathrm{Gr}_2^+(\mathbb{R}^n) \subseteq \Psi(\mathbb{R}^n)$ of *oriented* 2-planes in $\mathbb{R}^n$, i.e. the orientation double-cover of the tautological bundle over $\mathrm{Gr}_2(\mathbb{R}^n)$, and let

$$\gamma_n^\perp = \{(V,v) \mid v \in V^\perp\} \subseteq \mathrm{Gr}_2^+(\mathbb{R}^n) \times \mathbb{R}^n$$

denote the orthogonal complement bundle of the tautological bundle $\gamma_n \to \mathrm{Gr}_2^+(\mathbb{R}^n)$, i.e. we have a vector bundle of the form

$$\mathbb{R}^{n-2} \to \gamma_n^\perp \to \mathrm{Gr}_2^+(\mathbb{R}^n).$$

There exists a canonical inclusion map $\gamma_n^\perp \hookrightarrow \Psi(\mathbb{R}^n)$ given by $(V,v) \mapsto V + v$, which can be extended to a continuous injective map from the Thom space, namely

$$q\colon \mathrm{Th}(\gamma_n^\perp) \hookrightarrow \Psi(\mathbb{R}^n), \qquad (V,v) \mapsto V + v,\ \infty \mapsto \emptyset,$$

since every manifold $V + v$ eventually has empty intersection with any compact set $K \subset \mathbb{R}^n$ as $|v| \to \infty$. Importantly, we used the fact that $\mathrm{Th}(\gamma_n^\perp) \cong (\gamma_n^\perp)^+$ in this case, since the Grassmannian is compact, which can easily be seen by the following short lemma:

Lemma 3.11. *The space* $\mathrm{Gr}_2^+(\mathbb{R}^n)$ *is compact.*

Proof. Consider the map $f\colon S^2 \times S^2 \to \mathbb{R}$ given by $(v,w) \mapsto (v \cdot w)$. Let $C \coloneqq f^{-1}(0)$, i.e. C is the set of tuples of orthogonal vectors in S^2. By definition C is closed, and since $S^2 \times S^2$ is compact, so is C. Lastly, we have a quotient map $C \twoheadrightarrow \mathrm{Gr}_2^+(\mathbb{R}^n)$ where we identify $x, y \in C$ if $\mathrm{span}(x) = \mathrm{span}(y)$ and $\det(A_{xy}) > 0$, where A_{xy} denotes the unique linear transformation mapping x to y. Thus, as a quotient, $\mathrm{Gr}_2^+(\mathbb{R}^n)$ is compact. $\square$

The Thom space $\mathrm{Th}(\gamma_n^\perp)$ is interesting, because it contains the same homotopical information as the much larger space $\Psi(\mathbb{R}^n)$, but will allow for some more concrete cohomological computations in the next chapter. This motivates the next theorem:

Theorem 3.12. *The map* $q\colon \mathrm{Th}(\gamma_n^{\perp}) \hookrightarrow \Psi(\mathbb{R}^n)$ *is a weak equivalence.*

Proof. We follow the proof presented in [Gal11, p. 765]. Define two open sets in $\Psi(\mathbb{R}^n)$:

$$U_0 \coloneqq \{W \in \Psi(\mathbb{R}^n) \mid 0 \notin W\},$$
$$U_1 \coloneqq \{W \in \Psi(\mathbb{R}^n) \mid p \mapsto |p|^2 \text{ has a unique non-degenerate minimum for all } p \in W\}.$$

To see that U_0 is open, note that for all $W \in U_0$ we can construct a neighborhood $N(W) \subseteq \Psi(\mathbb{R}^n)$ fully contained in U_0, i.e. avoiding the origin. To do so, fix a tubular neighborhood τW such that $0 \notin \tau W$ and fix a large ball $B(0,r)$ of radius r. Then observe that the set

$$N(W) \coloneqq \{V \in \Psi(\mathbb{R}^n) \mid V \cap B(0,r) \subseteq \tau W\}$$

is a typical neighborhood of W and fully avoids the origin. A similar argument shows that U_1 is open as well. Morally, the second condition says that any such manifold W has a unique point $p_0 \in W$ that is closest the origin 0, i.e. near that point the surface "curves away" from 0. Let $U_{01} \coloneqq U_0 \cap U_1$ and notice that $\Psi(\mathbb{R}^n) = \mathrm{colim}(U_0 \hookleftarrow U_{01} \hookrightarrow U_1)$. Similarly, notice that

$$\mathrm{Th}(\gamma_n^{\perp}) = \mathrm{colim}\left(q^{-1}(U_0) \hookleftarrow q^{-1}(U_{01}) \hookrightarrow q^{-1}(U_1)\right).$$

We briefly mention an important general lemma that we will use for the further proof.

Lemma 3.13 ([Kup19], Lemma 32.1.5)**.** *If* $U_0 \cup U_1 = X$ *is an open cover of* X *by two subsets, then the pushout* $\mathrm{colim}(U_0 \hookleftarrow U_0 \cap U_1 \hookrightarrow U_1)$ *is weakly equivalent to its homotopy pushout.*

Thus, we can replace both pushout diagrams for $\Psi(\mathbb{R}^n)$ and $\mathrm{Th}(\gamma_n^{\perp})$ by their homotopy pushouts. We will then show that the three restricted maps $q^{-1}(U_0) \to U_0$, $q^{-1}(U_1) \to U_1$ and $q^{-1}(U_{01}) \to U_{01}$ are weak equivalences and hence we can write

$$\begin{aligned} \mathrm{Th}(\gamma_n^{\perp}) &= \mathrm{hocolim}\left(q^{-1}(U_0) \hookleftarrow q^{-1}(U_{01}) \hookrightarrow q^{-1}(U_1)\right) \\ &\simeq \mathrm{hocolim}(U_0 \hookleftarrow U_{01} \hookrightarrow U_1) \\ &= \Psi(\mathbb{R}^n), \end{aligned}$$

where the weak equivalence is induced by q. To this end, notice that U_0 is contractible by radially pushing manifolds to infinity, for instance by the map γ from Proposition 2.17. Since any $(V,v) \in q^{-1}(U_0)$ must have $v \neq 0$, we can use essentially the same map to show that $q^{-1}(U_0)$ is contractible as well. To see that $q^{-1}(U_1) \to U_1$ is a weak equivalence, let $W \in U_1$ have p_0 as such a unique minimum and consider the map

$$\varphi\colon \mathbb{R}^n \times I \to \mathbb{R}^n, \qquad (x,t) \mapsto (1-t)(x-p_0)+p_0.$$

For $t<1$, $\varphi(-,t)$ is a diffeomorphism of $\mathbb{R}^n$. We can then define a path $\gamma\colon I \to U_1$ from W to a point in $\mathrm{im}(q)$ by the formula

$$\gamma(t) \coloneqq \begin{cases} \varphi^{-1}(W,t) & \text{if } t<1, \\ T_{p_0}W + p_0 & \text{if } t=1. \end{cases}$$

By Theorem 2.15, this path is continuous, and since q was injective, this shows that $q^{-1}(U_1) \to U_1$ is a deformation retraction. By requiring that $p_0 \neq 0$, this restricts to a deformation retraction $q^{-1}(U_{01}) \to U_{01}$ as well. □

Remark 3.14. Pictorially, q being a weak equivalence comes down to the fact that each surface is locally diffeomorphic to a plane. This is roughly what the above path $\gamma\colon I \to U_1$ achieves, i.e. "zooming in" on the point p_0 of a surface until only a tangential plane is left.

Example 3.15. For a very geometrically-minded example of the above path $\gamma\colon I \to U_1$, consider the paraboloid $W \coloneqq \{(x_1, x_2, x_3) \in \mathbb{R}^n \mid x_3 = x_1^2 + x_2^2 + 1\} \in U_1 \subseteq \Psi(\mathbb{R}^3)$. In this case, we have that $p_0 = (0, 0, 1)$, and hence for $x = (x_1, x_2, x_3)$ we get

$$\varphi(x, t) = \big((1-t)x_1, (1-t)x_2, (1-t)x_3 + t\big).$$

For $0 \leq t < 1$, an easy computation shows that

$$\varphi^{-1}(W, t) = \left\{x \in \mathbb{R}^3 \mid x_3 = (1-t)x_1^2 + (1-t)x_2^2 + 1\right\},$$

and thus letting $t \to 1$ we indeed see that this "wider" paraboloid converges to the tangential plane $T_{(0,0,1)}W + (0,0,1)$, as expected.

Let us note that the way we stated the Madsen–Weiss Theorem 3.7 is slightly unorthodox. A more conventional way is via the Madsen–Tillmann–Weiss spectrum $\mathrm{MTSO}(2)$, which is defined as follows: with the above notation, there is always a pullback square

$$\begin{tikzcd} \gamma_n^\perp \oplus \mathbb{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_{n+1}^\perp \arrow[d] \\ \mathrm{Gr}_2^+(\mathbb{R}^n) \arrow[r, hook] & \mathrm{Gr}_2^+(\mathbb{R}^{n+1}) \end{tikzcd}$$

which induces a map on Thom spaces $j_n\colon \Sigma\mathrm{Th}(\gamma_n^\perp) \to \mathrm{Th}(\gamma_{n+1}^\perp)$. These spaces, together with these structure maps, constitute the spectrum $\mathrm{MTSO}(2)$. The classical Madsen–Weiss Theorem can then be stated as follows:

Theorem 3.16 ([Wah13], Theorem 1.4)**.** *There is a homology isomorphism*

$$B\Gamma_\infty \to \Omega^\infty_\bullet \mathrm{MTSO}(2),$$

where the target is the basepoint component of the infinite loop space of the Madsen-Tillmann-Weiss spectrum, and $\Omega^\infty\mathrm{MTSO}(2) = \operatorname{colim} \Omega^n \,\mathrm{Th}(\gamma_n^\perp)$.

We can then relate Theorem 3.16 back to our original statement in Theorem 3.7 with a short lemma:

Lemma 3.17. *There is a weak equivalence* $\Omega^\infty\mathrm{MTSO}(2) \to \Omega^\infty \boldsymbol{\Psi}$.

Proof sketch. By Theorem 3.12 we know that $\Omega^n(q)\colon \Omega^n\mathrm{Th}(\gamma_n^\perp) \hookrightarrow \Omega^n\Psi(\mathbb{R}^n)$ is a weak equivalence. One checks that these maps are compatible with the (injective) structure maps of both directed systems, i.e. the maps $\Omega^n \mathrm{Th}(\gamma_n^\perp) \hookrightarrow \Omega^{n+1}\mathrm{Th}(\gamma_{n+1}^\perp)$ and $\Omega^n\Psi(\mathbb{R}^n) \hookrightarrow \Omega^{n+1}\Psi(\mathbb{R}^{n+1})$, both given by the map $\Omega^n i_n$ from the definition of the scanning map. To be a little more precise, we give the full form of the map $i_n\colon \mathrm{Th}(\gamma_n^\perp) \hookrightarrow \Omega\,\mathrm{Th}(\gamma_{n+1}^\perp)$, as this map will play an important role later: we can write the map as

$$\begin{cases} (V,v) & \longmapsto \left(t \mapsto \left(\begin{cases} V + te_{k+1}, & t \in \mathbb{R} \\ \emptyset, & t = \infty \end{cases}, v \right) \right), \\ \infty & \longmapsto (t \mapsto \emptyset), \end{cases}$$

where e_{k+1} is the $(k+1)$-st canonical basis vector of $\mathbb{R}^n$. This map then makes the diagram

$$\begin{tikzcd} \mathrm{Th}(\gamma_n^\perp) \arrow[r, hook, "i_n"] \arrow[d, hook, "q"'] & \Omega\mathrm{Th}(\gamma_{n+1}^\perp) \arrow[d, hook, "\Omega q"] \\ \Psi(\mathbb{R}^n) \arrow[r, hook, "i_n"'] & \Omega\Psi(\mathbb{R}^{n+1}) \end{tikzcd}$$

commute. Notice that weak equivalences are homology isomorphisms by [Hat02, p. 356], and therefore we recover the original result (at least in the infinite-genus case, but the finite case is similar). $\square$

Next, we discuss some further basic properties of the space $\Psi(\mathbb{R}^n)$ in order to develop a more concrete understanding of its (weak) homotopy type, especially in lower dimensions. First, we find that $\Psi(\mathbb{R}^n)$ is somewhat highly connected:

Proposition 3.18. *We have $\pi_k\Psi(\mathbb{R}^n) \cong 0$ for all $k < n-2$. In particular, we have $\Psi(\mathbb{R}^\infty) \simeq *$.*

Before discussing its proof, we need an additional lemma to gain a better picture of the cell structure on Thom spaces:

Lemma 3.19 ([MS74], Lemma 18.1)**.** *Let ξ be an n-dimensional vector bundle over some CW-complex B. Then the Thom space $\mathrm{Th}(\xi)$ is an $(n-1)$-connected CW-complex having (in addition to the basepoint/0-cell ∞) one $(k+n)$-cell for each k-cell of the base B.*

Proof of Proposition 3.18. Replacing $\Psi(\mathbb{R}^n)$ by the weakly equivalent space $\mathrm{Th}(\gamma_n^\perp)$, we use Lemma 3.19 for $\xi = \gamma_n^\perp$ and $B = \mathrm{Gr}_2^+(\mathbb{R}^n)$. Since $\gamma_n^\perp$ is an $(n-2)$-bundle, we thus find that $\mathrm{Th}(\gamma_n^\perp)$ is $(n-2)-1 = (n-3)$-connected. $\square$

Proposition 3.20. *For $n \geq 2$ we have $\pi_{n-2}\Psi(\mathbb{R}^n) \cong \mathbb{Z}$. In particular, Proposition 3.18 gives the optimal connectivity result.*

Proof. This quickly follows by the chain of isomorphisms

$$\pi_{n-2}\Psi(\mathbb{R}^n) \cong \pi_{n-2}\,\mathrm{Th}(\gamma_n^\perp) \cong H_{n-2}(\mathrm{Th}(\gamma_n^\perp)) \cong H_0(\mathrm{Gr}_2^+(\mathbb{R}^n)) \cong \mathbb{Z},$$

where we used Theorem 3.12, the Hurewicz Theorem, the Thom Isomorphism Theorem and the fact that $\mathrm{Gr}_2^+(\mathbb{R}^n)$ is path-connected. $\square$

Proposition 3.21. *In the case $n = 3$, there is a weak equivalence $\Psi(\mathbb{R}^3) \simeq S^1 \vee S^3$. In particular, we have $\pi_1\Psi(\mathbb{R}^3) \cong \pi_3\Psi(\mathbb{R}^3) \cong \mathbb{Z}$, $\pi_2\Psi(\mathbb{R}^3) \cong 0$ and $\pi_k\Psi(\mathbb{R}^3) \cong \pi_k(\bigvee_{i\in\mathbb{Z}} S^3)$ for $k \geq 4$.*

Proof. Recall that for general spaces X we have that $\Sigma X_+ \cong \mathrm{Th}(X \times \mathbb{R})$, where $X_+ := X \amalg \{\mathrm{pt}\}$ denotes X with a disjoint basepoint, and $X \times \mathbb{R}$ is the trivial 1-dimensional real vector bundle over X. Then we can write

$$S^1 \vee S^3 \cong \Sigma(S^0 \vee S^2) \cong \Sigma(\{\mathrm{pt}\} \amalg S^2) = \Sigma(S^2_+) \cong \mathrm{Th}(S^2 \times \mathbb{R}).$$

Next, notice that the Gauss-map $T\colon S^2 \to \mathrm{Gr}_2^+(\mathbb{R}^3)$ given by $x \mapsto T_xS^2$ is bijective. Since S^2 is compact and $\mathrm{Gr}_2^+(\mathbb{R}^3)$ is Hausdorff, it follows from a standard point-set topological result [Mun00, p. 167] that the map T is a homeomorphism, provided that T is continuous, which we will show below. Then we can define a bundle isomorphism $f\colon S^2 \times \mathbb{R} \to \gamma_3^\perp$ over S^2 via the map $(x, v) \mapsto (T_xS^2, v)$, and hence we have

$$\mathrm{Th}(S^2 \times \mathbb{R}) \cong \mathrm{Th}(\gamma_3^\perp).$$

Lastly, we post-compose with q to get a weak equivalence $S^1 \vee S^3 \to \Psi(\mathbb{R}^3)$, as claimed.

To see that T is continuous, it suffices to choose some typical neighborhood U of $T(x) = T_xS^2$ and some neighborhood V of x such that $T(V) \subseteq U$. By definition of the subspace topology on $\mathrm{Gr}_2^+(\mathbb{R}^n) \subseteq \Psi(\mathbb{R}^n)$, we can write

$$U = \{W \in \mathrm{Gr}_2^+(\mathbb{R}^n) \mid W \text{ is a section of } N(T_xS^2)\},$$

where $N(T_xS^2)$ denotes the normal bundle. But this means that $T^{-1}(U) = S^2 \setminus \{y_1, y_2\}$, where $y_1, y_2 \in S^2$ are the unique two points such that $T(y_1) \perp T_xS^2$ and $T(y_2) \perp T_xS^2$, since all other tangent spaces constitute a section of T_xS^2 in its normal bundle. Thus, we can easily choose some small neighborhood $V \subseteq S^2 \setminus \{y_1, y_2\}$ of x such that $T(V) \subseteq U$, as needed. $\square$

Remark 3.22. Notice that only the result $\pi_1\Psi(\mathbb{R}^3) \cong \mathbb{Z}$ follows from Proposition 3.20, so the above Proposition is stronger at least for $n = 3$.

Remark 3.23. In Proposition 3.21, we derived the weak equivalence $\Psi(\mathbb{R}^3) \simeq S^1 \vee S^3$. However, since we are ultimately interested in the space $\Omega^3\Psi(\mathbb{R}^3)$ (this being the codomain of a finite stage in the scanning map), it is natural to ask about its weak homotopy type. Unfortunately, the Hilton-Milnor Theorem [Mil72] tells us that

$$\Omega\Psi(\mathbb{R}^3) \simeq \Omega(S^1 \vee S^3) \simeq \Omega S^1 \times \Omega S^3 \times \Omega(S^3 \vee S^4 \vee S^5 \vee \ldots),$$

i.e. already just applying $\Omega(-)$ once yields a much more cumbersome space than before, so there is not much hope for $\Omega^3\Psi(\mathbb{R}^3)$ having some "nice" weak homotopy type in general.

4. Deriving Mumford's Conjecture

Naturally, one may ask about any significant consequences of the Madsen–Weiss Theorem, now that we have stated the Theorem (and a variant) in full detail. In this chapter we discuss one of its most important corollaries: the Mumford Conjecture, a striking result about the rational cohomology ring structure of the mapping class group of an infinite genus surface.

After giving some motivation and discussing why we should study certain characteristic classes, we move to the statement and proof of the Mumford Conjecture. Roughly speaking, it states that the cohomology ring structure is given by a free polynomial algebra on countably many generators, which is surprisingly simple, considering that the cohomology of mapping class groups is, generally speaking, relatively complicated and hard to compute.

4.1. Characteristic Classes of Surface Bundles

Recall the space $B_n = \{W \in \Psi(\mathbb{R}^n) \mid W \subseteq (0,1)^n\}$. We follow [Gal12, p. 3] to show how exactly we can relate these spaces to surface bundles, i.e. bundles whose fibers are given by surfaces.

Proposition 4.1. *Let X be a smooth k-dimensional manifold. For $n > 2k+4$, there is a bijection*

$$[X, B_n] \overset{1:1}{\longleftrightarrow} \big\{E \to X \text{ surface bundle}\big\}/\,\text{isom}.$$

Proof. By Lemma 2.22 any map $f\colon X \to B_n$ is homotopic to a smooth map and is given by a smooth and topologically closed subspace $\Gamma(f) \subseteq X \times (0,1)^n$. Moreover, the projection $\pi\colon \Gamma(f) \to X$ is a surface bundle, and hence we define the map in the one direction as $f \mapsto (\Gamma(f) \to X)$. To see that it is well-defined, consider two maps $f_0, f_1\colon X \to B_n$ that are smoothly homotopic. Then, by Theorem 2.1 in [Coh05], we find that the bundles $\Gamma(f_0) \to X$ and $\Gamma(f_1) \to X$ are isomorphic, since $\Gamma(f_i)$ is given by the pullback bundle $f_i^*(B_n)$ for $i \in \{0,1\}$ by definition, i.e. $\Gamma(f_i) = \lim(X \xrightarrow{f_i} B_n \xleftarrow{\text{id}} B_n)$.

For a map in the other direction, consider a surface bundle $E \to X$ with fiber Σ. Since $n > 2k+4 = 2\dim(E)$, we can choose an embedding $j\colon E \hookrightarrow X \times (0,1)^n$ by Whitney's Embedding Theorem [Lee12, p. 134] and then define a smooth map $f\colon X \to B_n$ by the formula $j(E_x) = \{x\} \times f(x)$, where E_x is the fiber over $x \in X$. Equivalently, we can define f to be the map $x \mapsto \text{proj}_2(j(E_x))$ to be a bit more precise. If j' is another such embedding, then there exists an isotopy $i\colon E \times I \to X \times (0,1)^n$ with $i(-,0) = j$ and $i(-,1) = j'$, again by the Whitney Embedding Theorem. Then the resulting maps f, f' are homotopic via the map $h\colon X \times I \to B_n$ given by $(x,t) \mapsto \text{proj}_2(i(E_x,t))$. Finally, one checks that the maps we constructed are mutually inverse to each other. $\square$

If we pick a cohomology class $c \in H^*(B_\infty)$ in the infinite case, this gives rise to a characteristic class of surface bundles: given some surface bundle $E \to X$, we may pick an embedding $j\colon E \to X \times (0,1)^n$ and let $f\colon X \to B_n \subseteq B_\infty$ be the corresponding map under the bijection in Proposition 4.1, i.e. $f(x) = \mathrm{proj}_2(j(E_x))$. Then we can pull the class c back to a class $f^*(c) \in H^*(X)$, and this only depends on the isomorphism class of the surface bundle $E \to X$. We say that $f^*(c)$ is the characteristic class associated to c, evaluated on the bundle $E \to X$.

By passing to the colimit $B_\infty = \operatorname{colim} B_n$ it follows immediately from the previous Proposition that there is a bijection

$$[X, B_\infty] \xleftrightarrow{1:1} \big\{E \to X \text{ surface bundle}\big\}/\text{ isom.}$$

Moreover, we can essentially compute the cohomology of the space B_∞.

Corollary 4.2. *The cohomology ring $H^*(B_\infty;\mathbb{Z})$ is given by the ring of characteristic classes of surface bundles.*

Proof. We modify the definition of surface bundles slightly, to allow the identity map $B_\infty \to B_\infty$ to classify a surface bundle over B_∞. Then we essentially use a standard Yoneda-argument. Let

$$S(X) := \{E \to X \text{ surface bundle}\}/\text{isom.}$$

Then the functor $X \mapsto S(X)$ is represented by B_∞, since we had $S(X) \cong [X, B_\infty]$, and hence by the Yoneda Lemma we find that

$$\mathrm{Nat}(S(-), H^*(-;\mathbb{Z})) \cong H^*(B_\infty;\mathbb{Z}),$$

i.e. the cohomology ring of B_∞ corresponds precisely to all characteristic classes coming from surface bundles. $\square$

4.2. Cohomology of the Infinite Loop Space $\Omega^\infty\Psi$

Recall the mapping class group $\Gamma_{\infty,1} := \operatorname{colim} \Gamma_{g,1}$, where we include $\Gamma_{g,1}$ into $\Gamma_{g+1,1}$ by extending any diffeomorphism via the identity. Even if possible, computing any of these groups more explicitly is usually quite tedious. For instance, we have that $\Gamma_{1,1} \cong \mathrm{SL}(2,\mathbb{Z})$, $\Gamma_{2,1}$ is generated by five Dehn twists (with some relations) and for general $g \geq 3$ the group $\Gamma_{g,1}$ fits into the *Birman exact sequence* [CS18, p. 5]

$$1 \to \pi_1(\mathrm{UT}\Sigma_g) \to \Gamma_{g,1} \to \Gamma_g \to 1,$$

where $\mathrm{UT}\Sigma_g$ denotes the unit tangent bundle over Σ_g, so we can expect the middle term to be hard to calculate explicitly. However, here is where the surprising Mumford Conjecture comes into frame, as it provides us with a relatively elegant answer to a seemingly hard computation, involving a polynomial ring on so-called κ-classes. For the time being, we will give a purely algebraic description and only later will we give the more geometric definition of these classes.

Theorem 4.3 (Mumford Conjecture)**.** *There is an isomorphism of rings*

$$H^*(B\Gamma_{\infty,1};\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\kappa_3,\ldots],$$

where $|\kappa_i| = 2i$ *for all* $i \geq 1$*.*

Proof sketch. First, notice that we have a homotopy equivalence $B\mathrm{Diff}(\Sigma_{\infty,1}) \to B\Gamma_{\infty,1}$ by the Earle–Eells Theorem. The Madsen–Weiss Theorem, together with Harer's Stability result, states that the scanning map $\alpha\colon B\mathrm{Diff}(\Sigma_{\infty,1}) \to \Omega^\infty_\bullet \mathbf{\Psi}$ is an $H_*(-;\mathbb{Z})$-isomorphism, and hence an $H^*(-;\mathbb{Q})$-isomorphism by the Universal-Coefficient Theorem. Lastly, there is an isomorphism of rings $\mathbb{Q}[\kappa_1,\kappa_2,\ldots] \to H^*(\Omega^\infty_\bullet\mathbf{\Psi};\mathbb{Q})$, which we will show in the remainder of this section. □

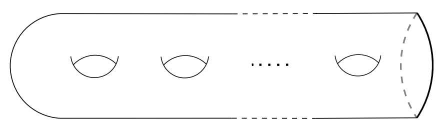

FIGURE 4. The infinite genus surface $\Sigma_{\infty,1}$ underlying the mapping class group $\Gamma_{\infty,1}$.

Remark 4.4. Ulrike Tillmann gives some historical insight about the Mumford Conjecture in [Til07]. Most importantly, due to work of Bödigheimer it was known that the homology of each $B\Gamma_{g,1}$ is finitely generated, and hence by Harer's Stability Theorem the homology of $B\Gamma_{\infty,1}$ is of finite type. Rationally, finite type commutative and co-commutative Hopf algebras are polynomial algebras (on even degree generators) tensored with exterior algebras (on odd degree generators), which is a result due to Milnor and Moore [MM65], known as the Structure Theorem. In 1983, Mumford first conjectured his famous result in his paper [Mum83]:

> "[...] it seems reasonable to guess, in view of the results of Harer and Miller, that in low [degrees] $H^i(\mathcal{M}_g)\otimes\mathbb{Q}$ is a polynomial ring in the κ_i."

Later, Miller showed in [Mil86] that the κ-classes $\kappa_1,\kappa_2,\ldots$ include into the cohomology of $B\Gamma_{g,1}$ in a range, and hence $\mathbb{Q}[\kappa_1,\kappa_2,\kappa_3,\ldots] \hookrightarrow H^*(B\Gamma_{\infty,1};\mathbb{Q})$ was at least a monomorphism of rings. Only about 20 years later, in 2004, was the conjecture finally resolved in [MW04].

Remark 4.5. The reason for discussing characteristic classes of surface bundles at the start of this section is the following: Notice that $B\Gamma_{\infty,1}$ is a path component in B_∞ under the identification (3), so as a first step it makes sense to compute the cohomology $H^*(B_\infty)$ instead of $H^*(B\Gamma_{\infty,1})$ right away. But by Corollary 4.2 we know that this is precisely given by characteristic classes of *surface* bundles. Consequently, it suffices to search *only* for specific classes coming from surface bundles, in hopes of these being exactly the generators of $H^*(B\Gamma_{\infty,1})$ later on.

For the rest of this chapter we will roughly follow Section 2 of [Gal12] to explain how we construct the ring isomorphism $\mathbb{Q}[\kappa_1,\kappa_2,\ldots] \to H^*(\Omega^\infty_\bullet\mathbf{\Psi};\mathbb{Q})$. Before tackling the cohomology of $\Omega^\infty_\bullet\mathbf{\Psi}$, let's try to understand the cohomology of $\Psi(\mathbb{R}^n) \simeq \mathrm{Th}(\gamma_n^\perp)$. To begin with, notice that the inclusion $\iota_n\colon \mathrm{Gr}_2^+(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_2^+(\mathbb{R}^\infty) = BSO(2) \cong BU(1) \simeq \mathbb{C}P^\infty$, given by $x \mapsto x \times \{0\}$, induces an isomorphism

$$\mathbb{Z}[e] \cong H^*(\mathrm{Gr}_2^+(\mathbb{R}^\infty);\mathbb{Z}) \xrightarrow{\iota_n^*} H^*(\mathrm{Gr}_2^+(\mathbb{R}^n);\mathbb{Z}) \tag{4}$$

for all degrees $* < n-2$, where e denotes the Euler class in degree 2. To see this, it suffices to show the following more general lemma for the Grassmannian of oriented k-planes in $\mathbb{R}^n$, denoted by $\mathrm{Gr}_k^+(\mathbb{R}^n)$:

Lemma 4.6. *The inclusion $\iota_n\colon \mathrm{Gr}_k^+(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_k^+(\mathbb{R}^\infty)$ is $(n-k)$-connected.*

Proof. Consider the standard principal $SO(k)$-bundle $V_k(\mathbb{R}^n) \to \mathrm{Gr}_k^+(\mathbb{R}^n)$, given by $x \mapsto \mathrm{span}(x)$, where $V_k(\mathbb{R}^n)$ denotes the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. For some (isometric) embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$ we get a commutative square

$$\begin{tikzcd}
V_k(\mathbb{R}^n) \arrow[r, hook] \arrow[d] & V_k(\mathbb{R}^m) \arrow[d] \\
\mathrm{Gr}_k^+(\mathbb{R}^n) \arrow[r, hook] & \mathrm{Gr}_k^+(\mathbb{R}^m)
\end{tikzcd}$$

which induces maps between the long exact sequences of homotopy groups of each bundle. It is well-known that $V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected, a computation of which can be found in [Whi78, p. 202] for instance. Thus, by a comparative analysis of the respective long exact sequences on homotopy groups, we find that the inclusion $\mathrm{Gr}_k^+(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_k^+(\mathbb{R}^m)$ is $(n-k)$-connected. Using the technical fact that any map $X \to \mathrm{Gr}_k^+(\mathbb{R}^\infty)$ for compact X has image in some finite stage $\mathrm{Gr}_k^+(\mathbb{R}^N)$, we find that the original map $\iota_n\colon \mathrm{Gr}_k^+(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_k^+(\mathbb{R}^\infty)$ is $(n-k)$-connected, as claimed. $\square$

For $k=2$ it follows by a general result [Bre93, p. 164] that the induced map on cohomology groups is an isomorphism in the desired range. Next, we invoke the Thom Isomorphism Theorem for the bundle $\gamma_n^\perp \to \mathrm{Gr}_2^+(\mathbb{R}^n)$ to obtain an isomorphism

$$H^k(\mathrm{Gr}_2^+(\mathbb{R}^n);\mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{k+(n-2)}(\mathrm{Th}(\gamma_n^\perp);\mathbb{Z}), \quad x \mapsto x \cup u_n,$$

where $u_n \in H^{n-2}(\mathrm{Th}(\gamma_n^\perp);\mathbb{Z})$ denotes the Thom class. This means we have a full understanding of the cohomology groups $H^k(\mathrm{Th}(\gamma_n^\perp);\mathbb{Z})$ for $k < 2n-4$, which is a good start. For completeness, let us write

$$\widetilde{H}^k(\mathrm{Th}(\gamma_n^\perp);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}[\iota_n^*(e)^{i+1}] & \text{if } k = n+2i < 2n-4 \text{ for some } i \geq -1, \\ 0 & \text{else} \end{cases}$$

in the range $0 \leq k < 2n-4$. Notice that we do not claim any results about cohomology groups above this bound.

However, the above computations were only for $\mathrm{Th}(\gamma_n^\perp)$, so now we must study what happens when we apply the functor $\Omega^n(-)$. To this end, we need the *suspension homomorphism* on cohomology, which (for an arbitrary space X) is a natural homomorphism

$$\sigma\colon \widetilde{H}^{k+1}(X;\mathbb{Z}) \to \widetilde{H}^{k}(\Omega X;\mathbb{Z}),$$

defined by the composite map

$$\widetilde{H}^{k+1}(X;\mathbb{Z}) \xrightarrow{\mathrm{ev}^*} \widetilde{H}^{k+1}(\Sigma\Omega X;\mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^{k}(\Omega X;\mathbb{Z}),$$

where $\mathrm{ev}\colon \Sigma\Omega X \to X$, $(t,\gamma)\mapsto \gamma(t)$, is the evaluation map and the isomorphism afterwards is the standard suspension isomorphism. Letting

$$\sigma^n\colon \widetilde{H}^{k+n}(X;\mathbb{Z}) \to \widetilde{H}^{k}(\Omega^n X;\mathbb{Z})$$

denote the n-fold iteration of the suspension, we can now give the algebraic definition of the κ-classes, also referred to as *MMM-classes*:

Definition 4.7 ([Gal12], Miller-Morita-Mumford Classes)**.** Let $\kappa_i^n \in \widetilde{H}^{2i}(\Omega^n \operatorname{Th}(\gamma_n^\perp);\mathbb{Z})$ denote the cohomology class defined by

$$\kappa_i^n := \sigma^n\big(\iota_n^*(e)^{i+1} \cup u_n\big) \in \widetilde{H}^{2i}(\Omega^n \operatorname{Th}(\gamma_n^\perp);\mathbb{Z}) \cong \widetilde{H}^{2i}(\Omega^n \Psi(\mathbb{R}^n);\mathbb{Z}),$$

where e was the generator of $H^*(\mathbb{C}P^\infty;\mathbb{Z})$ and $\iota_n^*(e)^{i+1} \in H^{2i+2}(\mathrm{Gr}_2^+(\mathbb{R}^n);\mathbb{Z})$ is the $(i+1)$-fold cup-product of $\iota_n^*(e)$ with itself. Moreover, we assume that $n \geq 5$ to make sure that the map ι_n^* in (4) is an isomorphism.

The next proposition explains in some detail how we can interpret the above κ-classes as living in the the colimit, i.e. in $\widetilde{H}^{2i}(\Omega^\infty \mathbf{\Psi};\mathbb{Z})$, as this is the cohomology that we are ultimately interested in. First, however, we need a short general lemma.

Lemma 4.8. *Let $f\colon X \to Y$ be a k-connected map, $E_Y \to Y$ a real vector bundle of rank l and let E_X denote the pullback-bundle $f^*(E_Y)$ over X. Then the induced map $\operatorname{Th}(E_X) \to \operatorname{Th}(E_Y)$ is $(k+l)$-connected.*

Proof. We follow the argument presented in [Gol16, p. 3]. By the assumption on f we find that $f_*\colon H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ is an isomorphism for $i < k$ and an epimorphism for $i = k$. Then we have a commutative diagram

$$\begin{array}{ccc}
\widetilde{H}_i(\operatorname{Th}(E_X);\mathbb{Z}) & \longrightarrow & \widetilde{H}_i(\operatorname{Th}(E_Y);\mathbb{Z}) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
H_{i-l}(X;\mathbb{Z}) & \xrightarrow{\quad f_* \quad} & H_{i-l}(Y;\mathbb{Z})
\end{array}$$

where the vertical maps are homological versions of the Thom isomorphism, see Theorem 3.31 in [Lüc04]. Thus, the lower map is an isomorphism for all $i < k+l$ and an epimorphism for $i = k+l$, so the same holds true for the upper map. Lastly, it follows from Whitehead's Theorem [Swi75, p. 187] that the map $\operatorname{Th}(E_X) \to \operatorname{Th}(E_Y)$ is $(k+l)$-connected. $\square$

Proposition 4.9. *For each $i \in \mathbb{N}$ there exists a unique class κ_i which restricts to the κ-classes from Definition 4.7 for each n.*

Proof sketch. This follows from two facts, both of which will be roughly sketched:

- **Claim 1.** *The map $\Omega^n i_n\colon \Omega^n \mathrm{Th}(\gamma_n^\perp) \hookrightarrow \Omega^{n+1}\mathrm{Th}(\gamma_{n+1}^\perp)$ is $(n-5)$-connected. In particular, the induced map*
$$\widetilde{H}^{2i}(\Omega^\infty \mathbf{\Psi};\mathbb{Z}) \to \widetilde{H}^{2i}(\Omega^n\Psi(\mathbb{R}^n);\mathbb{Z})$$
is an isomorphism for $n > 2i+5$.
- **Claim 2.** *The κ-classes $\kappa_i^n \in \widetilde{H}^{2i}(\Omega^n\Psi(\mathbb{R}^n);\mathbb{Z})$ are compatible with increasing dimension n, i.e. with the map $\Omega^n i_n\colon \Omega^n\Psi(\mathbb{R}^n) \hookrightarrow \Omega^{n+1}\Psi(\mathbb{R}^{n+1})$.*

To show the first claim, it suffices to show that the map $i_n\colon \mathrm{Th}(\gamma_n^\perp) \hookrightarrow \Omega\mathrm{Th}(\gamma_{n+1}^\perp)$ is $(2n-5)$-connected. To this end, notice that we have a factorization of i_n as follows

$$\begin{tikzcd}
\mathrm{Th}(\gamma_n^\perp) \arrow[rr, "i_n"] \arrow[dr, "\eta_n"'] & & \Omega\mathrm{Th}(\gamma_{n+1}^\perp) \\
& \Omega\Sigma\mathrm{Th}(\gamma_n^\perp) \arrow[ur, "\Omega j_n"'] &
\end{tikzcd}$$

where η_n is the unit of the suspension-loop adjunction and j_n was the map adjoint to i_n. Thus, it suffices to show that both η_n and Ωj_n are at least $(2n-5)$-connected. By Proposition 3.18 we know that $\mathrm{Th}(\gamma_n^\perp)$ is $(n-3)$-connected. By the Freudenthal Theorem, it immediately follows that
$$\eta_n\colon \mathrm{Th}(\gamma_n^\perp) \to \Omega\Sigma\mathrm{Th}(\gamma_n^\perp)$$
is $2\cdot(n-3)+1 = (2n-5)$-connected. For the second map, we apply Lemma 4.8 to the $(n-2)$-connected map $\mathrm{Gr}_2^+(\mathbb{R}^n) \hookrightarrow \mathrm{Gr}_2^+(\mathbb{R}^{n+1})$ (see Lemma 4.6) and we use the fact that the $(n-1)$-bundle $\gamma_{n+1}^\perp \to \mathrm{Gr}_2^+(\mathbb{R}^{n+1})$ pulls back to an $(n-1)$-bundle $\gamma_n^\perp \oplus \underline{\mathbb{R}} \to \mathrm{Gr}_2^+(\mathbb{R}^n)$. Thus, we get that the map
$$\Sigma\mathrm{Th}(\gamma_n^\perp) \cong \mathrm{Th}(\gamma_n^\perp \oplus \underline{\mathbb{R}}) \to \mathrm{Th}(\gamma_{n+1}^\perp)$$
is $(n-2)+(n-1) = (2n-3)$-connected, and this finishes the first claim.

For the second claim, one shows that
$$(\Omega^n i_n)^*\colon \widetilde{H}^{2i}(\Omega^{n+1}\Psi(\mathbb{R}^{n+1});\mathbb{Z}) \to \widetilde{H}^{2i}(\Omega^n\Psi(\mathbb{R}^n);\mathbb{Z})$$
maps the class κ_i^{n+1} to the class κ_i^n. This is discussed on p. 104 of [Mor13] in some detail, using Claim 1 along the way. □

Remark 4.10. This remark explains in some detail the relationship between our Definition 4.7 of the κ-classes and a more well-known geometric definition.[1] As such, it is to be understood only as a lengthy proof sketch, and not a fully rigorous description. More details can be found in Section 3 of [GMT06] and Sections 7.1 and 7.2 of [Til07]. Notice that, by Proposition 4.9, we have

[1]We thank Prof. Johannes Ebert for an extended discussion and for explaining some details of this construction.

defined, for each i, a class $\kappa_i \in H^{2i}(\Omega^\infty \mathbf{\Psi}; \mathbb{Z})$, and this class can evidently be pulled back along the scanning map $\alpha\colon B_\infty \to \Omega^\infty \mathbf{\Psi}$ to a cohomology class $\alpha^*(\kappa_i) \in H^{2i}(B_\infty; \mathbb{Z})$, corresponding to a surface bundle by Corollary 4.2. A priori, it is not clear which surface bundle these κ-classes are exactly related to. The following result provides an answer to this question:

Fact 4.11. *Let $\pi\colon E = E\mathrm{Diff}(\Sigma_g) \times_{\mathrm{Diff}(\Sigma_g)} \Sigma_g \to B\mathrm{Diff}(\Sigma_g) = B$ denote the universal surface bundle, and let $T_\pi E$ be its vertical tangent bundle, i.e. the oriented 2-dimensional bundle given by $\ker(D\pi\colon TE \to TB)$, where a recollection of this construction can be found in Section 1.3 of [Wah13]. We denote by $e(T_\pi E) \in H^2(E; \mathbb{Z})$ the Euler class. Then, one can identify the MMM-classes from Definition 4.7 as*

$$\alpha^*(\kappa_i) = (-1)^{i+1} \pi_!(e(T_\pi E)^{i+1}),$$

where $\pi_!\colon H^{2i+2}(E) \to H^{2i}(B)$ is the Gysin homomorphism*, also commonly known as* integration along the fibers.

We briefly explain the definition of the Gysin map: let $F \to E \to B$ be an arbitrary bundle with F an oriented, d-dimensional, closed and connected manifold. Consider its Leray–Serre spectral sequence

$$E_2^{p,q} \cong H^p(B; H^q(F)) \Rightarrow H^{p+q}(E).$$

Note that $E_2^{k-d,d} \cong H^{k-d}(B)$, by assumption on F. Moreover, there is a canonical injection $E_\infty^{k,k-d} \hookrightarrow E_2^{k,k-d}$, since there are no incoming differentials due to the fact that $H^l(F) \cong 0$ for $l > d$. Then we define $\pi_!$ as the composite

$$H^k(E) \twoheadrightarrow E_\infty^{k,k-d} \hookrightarrow E_2^{k,k-d} \cong H^{k-d}(B).$$

Next, we give a rough proof sketch of Fact 4.11 for a more general case, using the Pontrjagin–Thom construction: let $\pi\colon E \to B$ be a bundle as in the definition of the Gysin map, where we impose the additional assumption that B is a closed, oriented manifold. We will later give an argument how one can discard this assumption. First, embed $E \hookrightarrow B \times \mathbb{R}^n$ fiberwise over B, using Whitney's Embedding Theorem, such that the composite $E \hookrightarrow B \times \mathbb{R}^n \xrightarrow{\mathrm{pr}} B$ agrees with π. We can identify $T_\pi E$ with a subbundle of $E \times \mathbb{R}^n$, and after choosing a tubular neighborhood of E in $B \times \mathbb{R}^n$ we obtain a "collapse" map on Thom spaces

$$c\colon \mathrm{Th}(B \times \mathbb{R}^n) \to \mathrm{Th}(T_\pi E^\perp),$$

which simply collapses everything outside the chosen tubular neighborhood. Since we have a homeomorphism $\mathrm{Th}(B \times \mathbb{R}^n) \cong B_+ \wedge S^n$, c has an adjoint map

$$c^{\mathrm{ad}}\colon B \to \Omega^n \,\mathrm{Th}(T_\pi E^\perp).$$

Moreover, we have the standard Gauss-map $G\colon E \to \mathrm{Gr}_d^+(\mathbb{R}^n)$, and above this sits a bundle map $\overline{G}\colon T_\pi E^\perp \to \gamma_{d,n}^\perp$ into the tautological bundle, i.e. $T_\pi E^\perp \cong G^*(\gamma_{d,n}^\perp)$. This map induces a map on Thom spaces $\mathrm{Th}(\overline{G})\colon \mathrm{Th}(T_\pi E^\perp) \to \mathrm{Th}(\gamma_{d,n}^\perp)$ and we see that the map

$$\alpha \coloneqq (\mathrm{Th}(\overline{G}) \circ c)^{\mathrm{ad}} = \Omega^n \mathrm{Th}(\overline{G}) \circ c^{\mathrm{ad}} \colon B \to \Omega^n \mathrm{Th}(\gamma_{d,n}^\perp)$$

is precisely the scanning map, provided that we consider the case $d = 2$ and $B = B\mathrm{Diff}(\Gamma_g)$. The codomain then continues into $\operatorname{colim} \Omega^n \mathrm{Th}(\gamma_{d,n}^\perp) \simeq \Omega^\infty \Psi$. In order to relate the class $\alpha^*(\kappa_i)$ to any other definition, we must study the composite

$$\begin{array}{ccccc} H^k(BSO(d)) \xrightarrow{\mathrm{incl}^*} H^k(\mathrm{Gr}_d^+(\mathbb{R}^n)) \xrightarrow[\cong]{\mathrm{Thom}} & \widetilde{H}^{k+n-d}(\mathrm{Th}(\gamma_{d,n}^\perp)) & & \\ & \big\downarrow \sigma^n & & \\ & \widetilde{H}^{k-d}(\Omega^n \mathrm{Th}(\gamma_{d,n}^\perp)) & \xrightarrow{\alpha^*} & H^{k-d}(B). \end{array}$$

Notice that we recover our definition for $d = 2$, $k = 2i + 2$ and $e^{i+1} \in H^{2i+2}(BSO(2))$. Now we argue that we can drastically simplify the composite, so a comparison becomes much easier later. First, since the Thom-isomorphism is natural, and because $\overline{G}$ lies over G, it suffices to consider the composite

$$H^k(E) \xrightarrow[\cong]{\mathrm{Thom}} \widetilde{H}^{k+n-d}(\mathrm{Th}(T_\pi E^\perp)) \xrightarrow{\sigma^n} \widetilde{H}^{k-d}(\Omega^n \mathrm{Th}(T_\pi E^\perp)) \xrightarrow{(c^{\mathrm{ad}})^*} H^{k-d}(B), \tag{5}$$

i.e. we can safely ignore classifying spaces and can pass to the orthogonal complement of $T_\pi E$, instead of working with tautological bundles and Grassmannians. Next, notice that, for any map $f\colon X_+ \wedge S^n \to Y$ with adjoint $f^{\mathrm{ad}}\colon X \to \Omega^n Y$, we have a commutative square

$$\begin{array}{ccc} \widetilde{H}^k(Y) & \xrightarrow{\sigma^n} & \widetilde{H}^{k-n}(\Omega^n Y) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle (f^{\mathrm{ad}})^*} \\ \widetilde{H}^k(\Sigma^n X) & \xrightarrow{\cong} & \widetilde{H}^{k-n}(X) \end{array}$$

where the lower map is the standard suspension isomorphism in cohomology. Setting $f = c$, we thus find that (5) is equivalent to the composite

$$H^k(E) \xrightarrow[\cong]{\mathrm{Thom}} \widetilde{H}^{k+n-d}(\mathrm{Th}(T_\pi E^\perp)) \xrightarrow{c^*} \widetilde{H}^{k+n-d}(B_+ \wedge S^n) \xrightarrow{\cong} H^{k-d}(B) \tag{6}$$

The composite (6) is now in its most "reduced" form. In fact, it agrees precisely with the map $\pi_{!PT}$ from Appendix B in [Gri17]. Grigoriev shows in Proposition B.1 that $\pi_{!PT}$ agrees with the classical definition of the Gysin map $\pi_!$, i.e. the one using the Leray–Serre spectral sequence. Finally, Lemma B.2 in [Gri17] gives an argument for why we can replace the assumption on B, using that we care about rational coefficients in the end. This concludes the short interlude into the more geometric nature of the MMM-classes.

Recall that, so far, we have an understanding of the cohomology groups $H^k(\mathrm{Th}(\gamma_n^\perp); \mathbb{Z})$ in degrees $k < 2n - 4$. However, we aim to obtain a full understanding of the cohomology of the much "larger" space $\Omega^\infty_\bullet \boldsymbol{\Psi}$. Luckily, we can use some powerful general tools to reach this goal, albeit at a cost, namely we shall only be able to compute said cohomology *rationally*.

We begin with some notation and conventions. For a graded vector space $V = \bigoplus_{n\geq 1} V_n$ denote by $\mathbb{Q}[V]$ the free graded-commutative $\mathbb{Q}$-algebra generated by V. If X is a based space and $\phi\colon V \to H^*(X;\mathbb{Q})$ is a homomorphism, i.e. $\mathbb{Q}$-linear and grading-preserving, then the cup-product gives rise to a unique extension to a $\mathbb{Q}$-linear map

$$\mathbb{Q}[V] \to H^*(X;\mathbb{Q}). \tag{7}$$

Define the graded vector space $s^{-1}V$ by declaring $(s^{-1}V)_n = V_{n+1}$ and $(s^{-1}V)_0 = 0$. Then if we compose ϕ with the cohomological suspension map σ, we get a "shifted" map

$$\sigma \circ \phi\colon V_{n+1} \to H^{n+1}(X;\mathbb{Q}) \to H^n(\Omega X;\mathbb{Q}),$$

and similarly the cup-product in ΩX extends this map to a $\mathbb{Q}$-algebra homomorphism

$$\mathbb{Q}[s^{-1}V] \to H^*(\Omega X;\mathbb{Q}). \tag{8}$$

Before discussing why the above consideration is useful, we need a general technical lemma from rational homotopy theory.

Lemma 4.12. *Let $f\colon X \to Y$ be a map of connected CW-complexes. If the induced map $H^*(f;\mathbb{Q})\colon H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is an isomorphism in degrees $* \leq n$, then the map*

$$\pi_*(f) \otimes \mathbb{Q}\colon \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$$

on rational homotopy groups is an isomorphism in degrees $* \leq n$ as well.

Proof sketch. First, consider the case where both X, Y are Eilenberg–MacLane spaces, i.e. write $X = K(G_1, n_1)$ and $Y = K(G_2, n_2)$ for $n_1, n_2 \geq 1$. Notice that we only need to study the map $\pi_k(f) \otimes \mathbb{Q}$ for $n_1 = n_2$ and $k \leq n$. Then, by the Universal Coefficient Theorem, we can rewrite $H^*(f;\mathbb{Q})$ as an isomorphism $\mathrm{Hom}(H_*(Y;\mathbb{Q}),\mathbb{Q}) \to \mathrm{Hom}(H_*(X;\mathbb{Q}),\mathbb{Q})$, and by applying the functor $\mathrm{Hom}(-,\mathbb{Q})$ to both sides we find that $H_*(f;\mathbb{Q})\colon H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$ is an isomorphism in degrees $* \leq n$, where we used that the cohomology was of finite type in those degrees, allowing us to use general linear algebra about dual vector spaces. Equivalently, this is an isomorphism $H_*(X) \otimes \mathbb{Q} \to H_*(Y) \otimes \mathbb{Q}$ in these degrees. If $k \neq n_1 = n_2$, we are left with the trival isomorphism $\pi_k(f) \otimes \mathbb{Q}\colon 0 \to 0$. Otherwise, if $k = n_1 = n_2$, we get from the Hurewicz Theorem that $\pi_k(f) \otimes \mathbb{Q}\colon G_1 \otimes \mathbb{Q} \to G_2 \otimes \mathbb{Q}$ is an isomorphism, which was to be shown.

For the general case, consider the commutative diagram of fibrations

$$\begin{tikzcd}
K(\pi_k X, k) \arrow[r] \arrow[d, "\overline{f}_k"'] & \tau_{\leq k} X \arrow[r] \arrow[d, "f_k"'] & \tau_{\leq k-1} X \arrow[d, "f_{k-1}"'] \\
K(\pi_k Y, k) \arrow[r] & \tau_{\leq k} Y \arrow[r] & \tau_{\leq k-1} Y
\end{tikzcd}$$

coming from the respective Postnikov towers of X and Y, with the Eilenberg–MacLane spaces being the homotopy fibers. The vertical maps are induced by $f\colon X \to Y$, using naturality of Postnikov towers. Notice that $\pi_*(\overline{f}_k) \otimes \mathbb{Q}$ is an isomorphism for $* \leq n$ by the first part, and that we can assume the same for f_{k-1} by an inductive argument, using that $\tau_1 X$ and $\tau_1 Y$ are again just Eilenberg–MacLane spaces. Thus, a comparison of long exact sequences and the Five-Lemma shows that f_k is a $\pi_*(-)\otimes\mathbb{Q}$-isomorphism for $*\leq n$, and finally one inducts up to $k=n$ to finish the proof. □

The above construction of the cohomological shifting map motivates the following theorem, which constitutes one of the more important ingredients for the concrete computation of the cohomology of $\Omega^\infty_\bullet \mathbf{\Psi}$.

Theorem 4.13. *Let X be a space with rational cohomology of finite type up to degree n and let $V=\bigoplus_{n\geq 1} V_n$ be a graded vector space with a homomorphism $V\to H^*(X;\mathbb{Q})$ such that its extension (7) is an isomorphism in degrees $\leq n$. Then the extended "shifted" map (8) restricts to an isomorphism in degrees $\leq n-1$ on the path component $\Omega_\bullet X\subseteq \Omega X$ containing the basepoint, i.e. the composite*

$$\mathbb{Q}[s^{-1}V]\to H^*(\Omega X;\mathbb{Q})\to H^*(\Omega_\bullet X;\mathbb{Q})$$

is an isomorphism in degrees $\leq n-1$.

Proof. If we fix a basis for $\bigoplus_{i=1}^n V_i$, then, using classical Brown representability, this corresponds via the map $V\to H^*(X;\mathbb{Q})$ to a map $f\colon X\to\prod_{i=1}^n K(\mathbb{Q},i)^{\dim(V_i)}$. More precisely, for each basis element of each V_i we get a map $f_{i_j}\colon X\to K(\mathbb{Q},i)$ for $1\leq j\leq d_i:=\dim(V_i)$. These maps assemble to the map $f=(f_{i_j})_{1\leq i\leq n,\,1\leq j\leq d_i}$. The induced map

$$\mathbb{Q}\Big[\bigoplus_{i=1}^n V_i\Big]\cong H^*\Big(\prod_{i=1}^n K(\mathbb{Q},i)^{d_i};\mathbb{Q}\Big)\to H^*(X;\mathbb{Q}) \tag{9}$$

is an isomorphism up to degree n by assumption, and hence the same is true on rational *homology*, by the same argument as presented in the proof of Lemma 4.12. Notice that the first isomorphism in (9) comes from the Künneth Theorem, using that $\mathbb{Q}$ is a field, and the fact that each summand $K(\mathbb{Q},i)^{d_i}$ in rational cohomology admits d_i many independent generators in degree i. By Lemma 4.12, it follows that the map $\pi_*(f)\otimes\mathbb{Q}$ on rational homotopy groups is an isomorphism up to degree n. Looping once gives a map $\Omega f\colon \Omega X\to\prod_{i=1}^n K(\mathbb{Q},i-1)^{d_i}$, which now induces isomorphisms on $\pi_*(-)\otimes\mathbb{Q}$ only for $*\leq n-1$. In particular, this map restricts to a map

$$\Omega_\bullet f\colon \Omega_\bullet X\to\prod_{i=2}^n K(\mathbb{Q},i-1)^{d_i}, \tag{10}$$

giving isomorphisms on rational homotopy groups in degrees $*\leq n-1$. Note that the product in (10) starts at $i=2$, since this is the path-component of $\prod_{i=1}^n K(\mathbb{Q},i-1)^{d_i}$ which contains $\Omega f(\Omega_\bullet X)$, also using the fact that $K(\mathbb{Q},i-1)^{d_i}$ is a discrete group for $i=1$. Now we essentially reverse the above argument and omit only a few details: we are left with the map

$$H^*(\Omega_\bullet f;\mathbb{Q})\colon H^*\Big(\prod_{i=2}^n K(\mathbb{Q},i-1)^{d_i};\mathbb{Q}\Big)\to H^*(\Omega_\bullet X;\mathbb{Q})$$

which is an isomorphism in degrees $* \leq n-1$, and as before we can identify the left-hand side with the free $\mathbb{Q}$-algebra generated by d_i many elements in degree $i-1$. But this is just $\mathbb{Q}[\bigoplus_{i=1}^{n-1}(s^{-1}V)_i] \subseteq \mathbb{Q}[V]$, so indeed we find that the map $\mathbb{Q}[s^{-1}V] \to H^*(\Omega_\bullet X;\mathbb{Q})$ is an isomorphism in the desired range. $\square$

Finally, we explain how to use our findings to compute the rational cohomology of $\Omega^n \mathrm{Th}(\gamma_n^\perp)$. Set $X = \mathrm{Th}(\gamma_n^\perp)$ and let V denote the graded rational vector space with basis $\{\iota_n^*(e)^{i+1} \cup u_n\}_{i \geq -1}$. Define a homomorphism of rational vector spaces $V \to H^*(\mathrm{Th}(\gamma_n^\perp);\mathbb{Q})$ by the identity. By our earlier computation of the cohomology of $\mathrm{Th}(\gamma_n^\perp)$ this map extends to a $\mathbb{Q}$-algebra isomorphism $\mathbb{Q}[V] \to H^*(\mathrm{Th}(\gamma_n^\perp);\mathbb{Q})$ in degrees $< 2n-4$, where the extra class in $H^0(\mathrm{Th}(\gamma_n^\perp);\mathbb{Q})$ corresponds to the unit element $1 \in \mathbb{Q}[V]$. Now we simply apply Theorem 4.13 n times. More precisely, using that $\mathrm{Th}(\gamma_n^\perp)$ carries a CW-structure by Lemma 3.19 and that we know its rational cohomology up to a degree, we arrive at the space $\Omega_\bullet \mathrm{Th}(\gamma_n^\perp)$ and we can apply the theorem again to the spaces $\Omega_\bullet^2 \mathrm{Th}(\gamma_n^\perp), \Omega_\bullet^3 \mathrm{Th}(\gamma_n^\perp)$ etc. After n iterations, we are left with an isomorphism

$$\begin{aligned} \mathbb{Q}[s^{-n}V] &\longrightarrow H^*(\Omega_\bullet^n \mathrm{Th}(\gamma_n^\perp);\mathbb{Q}) \\ \iota_n^*(e)^{i+1} \cup u_n &\longmapsto \sigma^n(\iota_n^*(e)^{i+1} \cup u_n) = \kappa_i^n \end{aligned}$$

in degrees $* < n-4$ and we can factorize this map as

$$\begin{tikzcd} \mathbb{Q}[s^{-n}V] \arrow[r, "\sigma^n"] \arrow[d, "\sigma^n"'] & H^*(\Omega_\bullet^n \mathrm{Th}(\gamma_n^\perp);\mathbb{Q}) \\ \mathbb{Q}[\kappa_1^n, \kappa_2^n, \kappa_3^n, \ldots] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

with the vertical map being an isomorphism in degrees $* < n-4$. In particular, by Proposition 4.9, we conclude that, in the colimit, the map $\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots] \to H^*(\Omega_\bullet^\infty \mathbf{\Psi};\mathbb{Q})$ is an isomorphism in all degrees.

This computation then finishes the proof of the Mumford Conjecture, Theorem 4.3 above, and we continue with the main focus of this thesis: the proof of the Madsen–Weiss Theorem itself.

5. Proof of the Main Theorem

In this section we turn towards the proof of the Madsen–Weiss Theorem. We begin with a rough outline, which is necessary since the proof is both lengthy and technically involved in parts. We then review some general theory concerning topological monoids. For increased legibility and because some case distinctions are unavoidable, the main proof itself will be split up into smaller steps and intermediate results. As before, we roughly follow Chapters 3 and 4 of [Gal12] and try to give additional details wherever possible.

5.1. Outline

As discussed earlier, recall that the scanning map $\alpha_{(n)}\colon B_n \to \Omega^n\Psi(\mathbb{R}^n)$, obtained by moving compact manifolds in arbitrary directions in $\mathbb{R}^n$, is at the center of the Madsen–Weiss Theorem. The goal is to study properties of this map in thorough detail until we have sufficient information which lets us conclude the theorem. However, instead of directly studying the map $\alpha_{(n)}$ itself, let's try to break the problem down into smaller ones. The next definition makes this precise.

Definition 5.1. We define a family of spaces with corresponding maps:

(1) Define the subspace $\psi(n,k) \subseteq \Psi(\mathbb{R}^n)$ by $\psi(n,k) \coloneqq \{W \in \Psi(\mathbb{R}^n) \mid W \subseteq \mathbb{R}^k \times (0,1)^{n-k}\}$.

(2) Let $\alpha_k\colon \psi(n,k) \to \Omega\psi(n,k+1)$ be the map given by the assignment

$$W \mapsto \left(t \mapsto \begin{cases} W + te_{k+1} & \text{for } t \in \mathbb{R} \\ \emptyset & \text{for } t = \infty \end{cases} \right),$$

where e_{k+1} again denotes the canonical $(k+1)$-st basis vector of $\mathbb{R}^n$.

Remark 5.2. An important observation is that, roughly speaking, the map $\alpha_{(n)}$ factors through all spaces $\psi(n,k)$, i.e. we can write $\alpha_{(n)}$ as

$$B_n = \psi(n,0) \xrightarrow{\alpha_0} \Omega\psi(n,1) \xrightarrow{\Omega\alpha_1} \Omega^2\psi(n,2) \xrightarrow{\Omega^2\alpha_2} \cdots \xrightarrow{\Omega^{n-1}\alpha_{n-1}} \Omega^n\psi(n,n) = \Omega^n\Psi(\mathbb{R}^n)$$

Therefore, it suffices to study all constituents α_k one at a time, instead of the entire map at once.

Proposition 5.3. *For all $0 \le k \le n-1$ the map $\alpha_k\colon \psi(n,k) \to \Omega\psi(n,k+1)$ is continuous.*

Proof. We follow the argument presented on pp. 20 in [GR10]. Consider the map

$$\hat{\alpha}_k\colon \mathbb{R} \times \psi(n,k) \to \psi(n,k+1), \quad (t,W) \mapsto W - te_{k+1},$$

where the element $W - te_{k+1}$ denotes the preimage of W under the diffeomorphism $x \mapsto x + te_{k+1}$ of $\mathbb{R}$. The map $\hat{\alpha}_k$ is continuous by Theorem 2.15, since it is induced by the above action of the subgroup $\mathbb{R} \subseteq \mathrm{Diff}(\mathbb{R}^n)$. With $\emptyset \in \psi(n,k+1)$ being the basepoint, $\hat{\alpha}_k$ extends uniquely to a continuous map $\Sigma\psi(n,k) \to \psi(n,k+1)$ with continuous adjoint $\alpha_k\colon \psi(n,k) \to \Omega\psi(n,k+1)$. $\square$

Using the factorization $\alpha_{(n)} = \Omega^{n-1}\alpha_{n-1} \circ \ldots \circ \Omega\alpha_1 \circ \alpha_0$ we pass to the colimit to obtain the following quick corollary:

Corollary 5.4. *The scanning map* $\alpha\colon B_\infty \to \Omega^\infty_\bullet \mathbf{\Psi}$ *is continuous.*

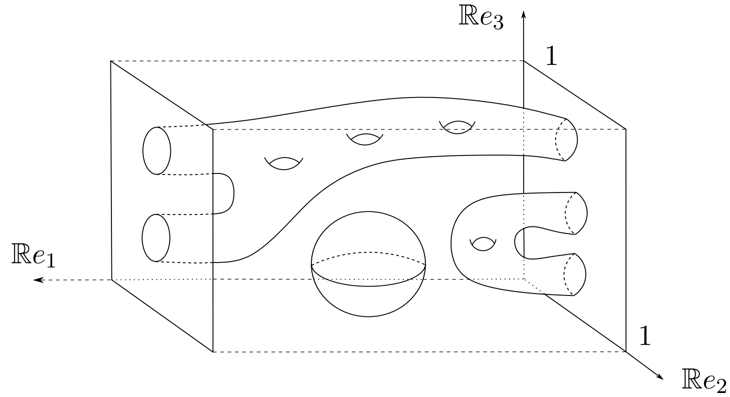

FIGURE 5. Part of an element of the space $\psi(3,1) \subseteq \Psi(\mathbb{R}^3)$. This particular example is inspired by a similar sketch in [Til07, p. 4]

Now we explain the rough a proof strategy for the main theorem: the list below outlines some of the important steps that we will discuss to arrive at a proof of the Madsen–Weiss Theorem.

(1) For each $k > 0$, the map $\alpha_k\colon \psi(n,k) \to \Omega\psi(n,k+1)$ factors through a special topological monoid, with all involved maps being (at least) weak equivalences. Thus, α_k is a weak equivalence.

(2) A priori, only the map α_0 is *not* a weak equivalence. However, if we restrict the map to the path-component $B\mathrm{Diff}(\Sigma_{g,1})$ in the case $n = \infty$, i.e. $\alpha_0\colon B_\infty \to \Omega\psi(\infty,1)$, it turns out to be an $H_*(-;\mathbb{Z})$-isomorphism in a range.

(3) Said range is given by the bound $* \leq \frac{2}{3}(g-1)$, coming from Harer's Stability Theorem. Arguing step (2) involves more work, however, including a factorization through another similar (but different) topological monoid and using the Group-Completion Theorem.

(4) α_0 being a homology isomorphism in a range, and all other maps α_k being weak equivalences then allows to conclude that the scanning map α in the colimit is a homology isomorphism in the stated range, keeping in mind the factorization from Remark 5.2.

Remark 5.5. An alternative, more categorical approach to proving that the constituents α_k of the scanning map are weak equivalences (for $k > 0$) was developed by Randal-Williams in [Ran10]. He shows that the sheaf $\Psi_d(-)$ is *micro-flexible*, and hence satisfies Gromov's h-principle. From this, it follows that α_k is a weak equivalence. For this thesis, however, we choose the approach via topological monoids, as described above.

5.2. Factoring Through Topological Monoids

A significant tool in the main proof are *topological monoids* and their classifying spaces. Generally speaking, we will try to relate the space $\psi(n,k)$ to a certain topological monoid M, whose precise definition will be given later. This allows for using more general theory about topological monoids and will ultimately be an important step in showing that the map $\alpha_k\colon \psi(n,k) \to \Omega\psi(n,k+1)$ is a weak equivalence.

Recall that a topological monoid is a space M endowed with an associative (but not necessarily commutative) multiplication $M \times M \to M$. We do not assume M to have a unit element, even though in practice all monoids in our constructions will at least have homotopy units. To M we can associate its *classifying space* BM, typically defined as the geometric realization of the nerve of M. For our purposes we will give an alternative definition, with which it will be easier to work later on.

Definition 5.6. Define the set BM by

$$BM \coloneqq \{(A,f) \mid A \subseteq \mathbb{R} \text{ is a finite subset, } f\colon A \to M \text{ is a function}\}$$

and topologise it as follows: let $K \subseteq \mathbb{R}$ be compact, $V \subseteq M$ an open set and fix two values $a < b \in \mathbb{R}$. Furthermore, denote by $\mathcal{U}(K) \subseteq BM$ the set $\{(A,f) \in BM \mid A \cap K = \emptyset\}$. Then define the sets $\mathcal{U}(a,b,V)$ by the equivalence

$$(A,f) \in \mathcal{U}(a,b,V) :\Longleftrightarrow \begin{cases} \text{If } A \cap (a,b) = (a_1 < \ldots < a_k), \\ \text{then } k \geq 1 \text{ and } f(a_1)f(a_2)\cdots f(a_k) \in V. \end{cases}$$

Finally, declare the collection of subsets $\{\mathcal{U}(K)\}_K \cup \{\mathcal{U}(a,b,V)\}_{a,b,V}$, running over the obvious choices for K, a, b, V, to be a sub-basis for the topology on BM.

Remark 5.7. Morally, we can think of points in BM as being depicted by finitely many points in $\mathbb{R}$, each labeled by an element of M. This allows for an intuitive interpretation of the topology as well: we allow labels to move continuously in M and the points themselves to move continuously in $\mathbb{R}$, and moreover we allow points to "collide", in which case we multiply their labels in the canonical order they appear on the real line. In case the labels go to $\pm\infty$ we simply forget the labels altogether.

Naturally, one may ask how the space BM relates to the classical bar construction of the monoid M. This motivates the following short proposition:

Proposition 5.8. *The space BM agrees with the usual definition, i.e. there is a homeomorphism $BM \simeq |B_\bullet M|$, where $B_\bullet M$ is the classical bar construction of M (a detailed construction of which can be found in Chapter 10.2 of [BB24]) and $|-|$ denotes the geometric realization.*

Proof sketch. Fix some homeomorphism $\lambda\colon (0,1)^+ \to \mathbb{R}^+$. For $t = (t_0, \ldots, t_p) \in \Delta^p$ we define p-many points $a_r \coloneqq \lambda\left(\sum_{i=0}^{r-1} t_i\right) \in \mathbb{R}^+$ for $1 \leq r \leq p$. Given some tuple $m = (m_1, \ldots, m_p) \in M^p$, we can label each a_r by m_r for $1 \leq r \leq p$, and thus we define a map

$$\phi_p \colon \Delta^p \times M^p \to BM, \quad (t, m) \mapsto (\{a_1, \dots, a_p\}, (a_r \mapsto m_r)_{1 \leq r \leq p}).$$

We need to be careful, as this map is technically not well-defined. However, it is understood that one multiplies labels if any two consecutive a_r's coincide and we simply forget the label if any a_r is infinite. One checks that all maps ϕ_p are continuous and that they glue to a map from the realization $|B_\bullet M|$ to BM with continuous inverse, which can be seen by an analysis of respective subbases. In order to construct an inverse map somewhat more explicitly, one uses the fact that any element in the semi-simplicial realization

$$|B_\bullet M| = \coprod_p (\Delta^p \times M^p)/\sim$$

has a unique "minimal" representative, namely with $t_i \neq 0$ for all i in the tuple, for if we have $t_i = 0$ in the quotient we remove it and multiply m_i with m_{i+1} or simply remove m_1 or m_p, by definition of the bar construction. Then one sees more easily that the ϕ_p induce a bijection. □

Next, we introduce a map relating a general topological monoid M to ΩBM, the loop space of its classifying space, that will play an important role:

Definition 5.9. There is a natural map $\beta \colon M \to \Omega BM$, given by the assignment

$$m \mapsto \left(t \mapsto \begin{cases} (\{t\}, (t \mapsto m)) & \text{for } t \in \mathbb{R} \\ \emptyset & \text{for } t = \infty \end{cases} \right)$$

Lemma 5.10. *The above map $\beta \colon M \to \Omega BM$ is continuous.*

Proof sketch. We will describe how to construct a map $i_M \colon M \to \Omega|B_\bullet M|$ as in Section 10.2 [BB24]. The claim then follows by noting that β factors as

$$M \xrightarrow{i_M} \Omega|B_\bullet M| \xrightarrow{\cong} \Omega BM.$$

To this end, consider the standard homotopy $H \colon |E_\bullet M| \times I \to |E_\bullet M|$ from the 0-simplex to the identity, from which one sees that $|E_\bullet M| \simeq *$. If $p \colon |E_\bullet M| \to |B_\bullet M|$ denotes the quotient map (dividing out the free and proper action of M on $|E_\bullet M|$), then the map $p \circ H$ corresponds to a map

$$\overline{p \circ H} \colon |E_\bullet M| \to \mathrm{Map}(I, |B_\bullet M|),$$

via the standard adjunction. In fact, this map has image in the path space $P_*|B_\bullet M|$ (with basepoint being the 0-simplex), since H is the contracting homotopy. If we identify $M \cong E_0 M$, we define the map i_M as the composite

$$M \hookrightarrow |E_\bullet M| \xrightarrow{\overline{p \circ H}} P_*|B_\bullet M|.$$

We claim that $\mathrm{im}(i_M)$ is contained in $\Omega|B_\bullet M|$. To check this, we evaluate the map

$$\overline{p \circ H}(m_0|; t_0, \dots, t_n) = (p \circ H)((m_0|; t_0, \dots, t_n), -)$$

at both 0 and 1, where $m_0 \in M$, $\sum_{i=0}^{n} t_i = 1$ and $(m_0|; t_0, \dots, t_n)$ is a general element in the image of the inclusion $M \hookrightarrow |E_\bullet M|$. We find that in both cases this equates to the class of the 0-simplex, and hence $i_M(m_0)$ is indeed a loop in $|B_\bullet M|$. □

Theorem 5.11. *Let M be a topological monoid. The map $\beta \colon M \to \Omega BM$ is a weak equivalence if and only if M is group-like, i.e. M has a homotopy unit and the monoid $\pi_0 M$ is a group.*

Proof. We follow the proof presented in [BB24], Theorem 10.10. Assume M is group-like, and pick for each $m \in M$ an element $m' \in M$ in the path-component of M such that $[mm'] = e \in \pi_0 M$ is the identity. Thus, if we fix a path $\gamma \colon I \to M$ from mm' to e, we obtain a homotopy $h \colon M \times I \to M$ from $(-) \cdot mm' \colon M \to M$ to $\mathrm{id}_M = (-) \cdot e \colon M \to M$. Notice that we can write $(-) \cdot mm' = (-) \cdot m' \circ (-) \cdot m$, which shows that $(-) \cdot m$ is injective on homotopy groups. If we replace mm' by $m'm$, the same argument shows that $(-) \cdot m$ is surjective on homotopy groups, and hence the map is a weak equivalence. From Lemma 10.8 in [BB24] we get that the induced map on realizations $\overline{m} \coloneqq (-) \cdot m \colon |E_\bullet M| \to |B_\bullet M|$ is a quasi-fibration with fiber M. The above lecture notes construct a map $|E_\bullet M| \to P_*|B_\bullet M|$ into the path space, where the basepoint is given by the 0-simplex $e \in M$, and this map fits into a commutative diagram of quasifibrations

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\beta"'] & {|E_\bullet M|} \arrow[r, "\overline{m}"] \arrow[d] & {|B_\bullet M|} \arrow[d, equal] \\
\Omega|B_\bullet M| \arrow[r] & P_*|B_\bullet M| \arrow[r] & {|B_\bullet M|}
\end{tikzcd}$$

where the lower row is the standard path-space fibration and we use Proposition 5.8 throughout. By considering the respective long exact sequences in $\pi_*(-)$ and using the Five-Lemma, we see that β is a weak equivalence, since both total spaces are contractible. The other direction of the theorem is immediate: if β is a weak equivalence, we can write $\pi_0 M \cong \pi_0(\Omega BM) \cong \pi_1 BM$, and this is always a group. □

Now we finally explain the strategy to showing that the maps $\alpha_k \colon \psi(n,k) \to \Omega\psi(n,k+1)$ are a weak equivalences: first, define a specific group-like topological monoid $M = M_{n,k}$. Then define maps $\iota \colon \psi(n,k) \hookrightarrow M$ and $\mu \colon BM \to \psi(n,k+1)$ and show that both are at least weak equivalences. Lastly, show that the diagram

$$\begin{tikzcd}
M \arrow[r, "\beta"] & \Omega BM \arrow[d, "\Omega\mu"] \\
\psi(n,k) \arrow[u, hook, "\iota"] \arrow[r, "\alpha_k"] & \Omega\psi(n,k+1)
\end{tikzcd} \tag{11}$$

commutes. Then it follows by Theorem 5.11 that α_k is a weak equivalence, as claimed. In the next step, we give the precise definition of the aforementioned monoid M that we want to study, and afterwards we define the above maps ι and μ.

Definition 5.12. For $0 \leq k < n$, define the space

$$M \coloneqq M_{n,k} \coloneqq \left\{ (t, W) \in (0, \infty) \times \psi(n, k+1) \mid W \subseteq \mathbb{R}^k \times (0, t) \times (0,1)^{n-k-1} \right\}.$$

We endow this space with the monoidal multiplication given essentially by juxtaposition, i.e.

$$(t, W) \cdot (t', W') \coloneqq (t + t', W \cup (W' + te_{k+1})),$$

where the monoidal unit is given by the element $(0, \emptyset)$. Heuristically, the product simply places W and W' next to each other after forcing them to be disjoint by translation along the $(k+1)$-axis.

Lemma 5.13. *The inclusion $\iota\colon \psi(n,k) \hookrightarrow M$, given by $W \mapsto (1, W)$, is a homotopy equivalence.*

Proof. Define a homotopy

$$h\colon M \times I \to M, \quad ((t, W), s) \mapsto ((1-s)t + s, W).$$

One checks that $h((t,W), 0) = (t, W)$, $h((t,W),1) = (1,W) \in \mathrm{im}(\iota)$ and $h((1,W),s) = (1,W) \in \mathrm{im}(\iota)$, so h is a strong deformation retraction, and hence ι is a homotopy equivalence. $\square$

Lemma 5.14. *The topological monoid $M = M_{n,k}$ is group-like.*

Proof. Fix an element $m = (t, W) \in M$ with $W \subseteq \mathbb{R}^k \times (0,t) \times (0,1)^{n-k-1}$. The goal is to construct some $m' \in M$ which descends to an inverse element of m in $\pi_0 M$.

To this end, consider the map $R\colon \mathbb{R}^n \to \mathbb{R}^n$ which rotates W in the $e_k e_{k+1}$-plane around the point $(0, \frac{t}{2})$, i.e. R is given by the composite $T \circ R_{k,k+1} \circ T^{-1}$, where the translation T is given by $T(x_{k+1}) = x_{k+1} + \frac{t}{2}$ and $R_{k,k+1}$ is the rotation map in $SO(n)$ sending $x_i \mapsto -x_i$ for $i \in \{k, k+1\}$ and fixing all other canonical basis vectors. Let $W' = R(W)$ and define the element $m' \coloneqq (t, W') \in M$. We want to show that both mm' and $m'm$ are contained in the

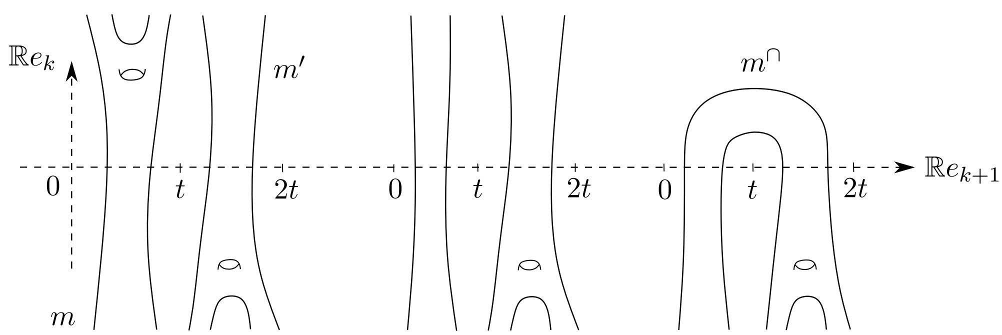

FIGURE 6. Three frames describing a path from mm' to the empty manifold.

path component of the empty set. To see this, we refer to Figure 6. The first frame shows the product $mm' \in \mathbb{R}^k \times (0, 2t) \times (0,1)^{n-k-1}$, the disjoint union of W and W'. Next, we stretch the first disjoint summand, i.e. we consider the manifold $f_t(mm')$, where f_t stretches $\mathbb{R}^k$ in the first "half" $\mathbb{R}^k \times (0,t) \times (0,1)^{n-k-1}$, for $t \in I$, using a fixed monotonically increasing homeomorphism $\nu\colon I \to [1, \infty]$, so we can write $f_t(x, \tilde{t}, \tilde{s}) = (\nu(t)x, \tilde{t}, \tilde{s})$ for $\tilde{t} \in (0,t)$ and $f_t = \mathrm{id}$ otherwise, where

$\tilde{s} \in (0,1)^{n-k-1}$. Next, notice that, by definition of the topology on $\Psi(\mathbb{R}^n)$, we can pull back $f_1(mm')$ (frame 2) from infinity to the "bent" manifold $m^\cap$ depicted in frame 3. More precisely, $m^\cap$ is obtained by bending m in the $e_k e_{k+1}$-plane in a way such that

$$mm' \cap \mathbb{R}^{k-1} \times (-\infty, 0) \times (0,1)^{n-k-1} = m^\cap \cap \mathbb{R}^{k-1} \times (-\infty, 0) \times (0,1)^{n-k-1},$$

i.e. the first and third frame agree for sufficiently negative k-th coordinates. By a diffeomorphism of $\mathbb{R}^{k-1} \times [0,\infty) \times (0,1)^{n-k-1}$ stretching the k-th coordinate to infinity, we define a path from $m^\cap$ to $f(mm')$. By a similar argument we can "push" $m^\cap$ off to infinity in the negative e_k-direction to arrive at the empty manifold. Concatenating all defined paths, we thus obtain a path from mm' to $\emptyset$, meaning that $[mm'] = [\emptyset] \in \pi_0 M$, i.e. m' is an inverse element for m. The proof for $m'm$ is identical. $\square$

The previous two lemmas tell us that the first two maps in the factorization (11) of α_k are weak equivalences. The next objective is to show that the remaining map $\mu \colon BM \to \psi(n, k+1)$ is a weak equivalence, which involves more work and an additional factorization of the map. To start, let's define the map μ making the diagram (11) commute. Let $(A, m) \in BM$ be a point with $A = (a_1 < \ldots < a_p) \subset \mathbb{R}$ and labels $m_1 = (t_1, W_1), \ldots, m_p = (t_p, W_p)$. Moreover, define inductively the numbers $b_1, \ldots, b_p$ by the rule

$$b_1 \coloneqq a_1, \quad b_{i+1} \coloneqq \max\{a_{i+1}, b_i + t_i\},$$

i.e. the b_i's are the smallest numbers making the intervals $(b_i, b_i + t_i)$ disjoint. For $k \geq 2$, define the map

$$\begin{aligned} \mu \coloneqq \mu_{n,k} \colon BM &\longrightarrow \psi(n, k+1) \\ \big((a_1 < \ldots < a_p); (t_1, W_1), \ldots, (t_p, W_p)\big) &\longmapsto \bigcup_{i=1}^{p} W_i + b_i e_{k+1}. \end{aligned}$$

By construction, this means that each $\mu(A, m)$ is given as the disjoint union of elements in $\psi(n, k+1)$. One also quickly checks that this map really makes (11) commute, by writing

$$\begin{aligned} \Omega\mu(\beta(1, W))(t) = \mu(\beta(1, W)(t)) &= \begin{cases} \mu(\{t\}, (t \mapsto (1, W))) & \text{for } t \in \mathbb{R} \\ \mu(\emptyset) & \text{for } t = \infty \end{cases} \\ &= \begin{cases} W + te_{k+1} & \text{for } t \in \mathbb{R} \\ \emptyset & \text{for } t = \infty \end{cases}, \end{aligned}$$

and this is precisely $\alpha_k(W)(t)$, where $t \in \mathbb{R}^+$. Before carrying on, we sketch a short continuity-argument for μ.

Lemma 5.15. *The manifold $\mu(A, m)$ depends continuously on $(A, m) \in BM$.*

Proof sketch. Notice that we have to check three cases. The first is to see what happens when a_i "collides" with a_{i+1}. It follows that $W \coloneqq \mu(A, m)$ is independent of a_{i+1}, as long as $a_{i+1} \leq$

$a_i + t_i$, since in this case we have $b_{i+1} = \max\{a_{i+1}, b_i + t_i\} = b_i + t_i$, using that $b_i \geq a_i$. The following calculation shows that, in this case, we have that W is given by $\mu(A', m')$, where we set $A' \coloneqq (a_1 < \ldots < \widehat{a_{i+1}} < \ldots < a_p)$ and $m' \coloneqq (m_1, \ldots, m_i m_{i+1}, \ldots, m_p)$, with $m_i m_{i+1} = W_i \cup (W_{i+1} + t_i e_{k+1})$:

$$\begin{aligned}
\mu(A', m') &= \bigcup_{j \in \{1,\ldots,p\} \setminus \{i,i+1\}} (W_j + b_j e_{k+1}) \cup \big(W_i \cup (W_{i+1} + t_i e_{k+1}) + b_i e_{k+1}\big) \\
&= \bigcup_{j \in \{1,\ldots,p\} \setminus \{i,i+1\}} (W_j + b_j e_{k+1}) \cup \big(W_i + b_i e_{k+1}\big) \cup \big(W_{i+1} + (b_i + t_i) e_{k+1}\big) \\
&= \bigcup_{j \in \{1,\ldots,p\} \setminus \{i,i+1\}} (W_j + b_j e_{k+1}) \cup \big(W_i + b_i e_{k+1}\big) \cup \big(W_{i+1} + b_{i+1} e_{k+1}\big) \\
&= \mu(A, m).
\end{aligned}$$

The second case is when the smallest point $a_1 \to -\infty$. Then we readily see that $\mu(A, m)$ is eventually constant near any compact subset of $\mathbb{R}^n$ and indeed converges to $\mu(A', m')$ for $A' \coloneqq (a_2 < \ldots < a_p)$ and $m' \coloneqq (m_2 < \ldots < m_p)$, as expected. The third case is when $a_p \to \infty$, and this is similar to the second case. $\square$

Apart from a small caveat in the case $k = 1$ we can finally state the desired result.

Proposition 5.16. *For $k \geq 2$ the map $\mu\colon BM \to \psi(n, k+1)$ is a weak equivalence. For $k = 1$ the map is a weak equivalence onto the path-component containing $\emptyset \in \psi(n, 2)$.*

We tackle the proof of the above proposition throughout the next subsection, as it is quite technical and requires some additional motivation and new definitions.

5.3. μ is a Weak Equivalence – a Technical Endeavour

At first glance, it seems as though the proof of Proposition 5.16 should be straightforward: for surjectivity on $\pi_d(-)$, say, we only need to show that an arbitrary map $f\colon S^d \to \psi(n, k+1)$ is homotopic to a map which lifts to a map $\tilde{f}\colon S^d \to BM$. In particular, we need for each point $x \in S^d$ a path from $W = f(x) \in \psi(n, k+1)$ to a point in the image of μ. This is because a homotopy $h\colon S^d \times I \to \psi(n, k+1)$ with $h_0 = f$ and $h_1 = f'$ (with f' being the map homotopic to f lifting to BM) can be viewed as a path $h(x, -)\colon I \to \psi(n, k+1)$ with the properties that $h(x, 0) = f(x) = W$ and $h(x, 1) = f'(x) = \mu(\tilde{f}(x))$.

Constructing such a path is not too hard, provided W satisfies the following property:

$$\exists\, a \in \mathbb{R} \text{ such that } \big(\mathbb{R}^k \times \{a\} \times \mathbb{R}^{n-k-1}\big) \cap W = \emptyset. \tag{12}$$

Geometrically speaking, this property tells us that we can think of W as being "separable" in the sense that we can slice W into two disjoint pieces, one contained in $\mathbb{R}^k \times (-\infty, a) \times \mathbb{R}^{n-k-1}$ and the other in $\mathbb{R}^k \times (a, \infty) \times \mathbb{R}^{n-k-1}$. If W satisfies this property for a finite set of such real numbers $(a_1 < \ldots < a_{p+1})$, each satisfying property (12), define numbers $t_i \coloneqq a_{i+1} - a_i$ and manifolds

$$W_i \coloneqq \Big(W \cap (\mathbb{R}^k \times (a_i, a_{i+1}) \times \mathbb{R}^{n-k-1})\Big) - a_i e_{k+1}, \tag{13}$$

i.e. W_i is the part of W contained within the interval (a_i, a_{i+1}) and then shifted to be contained in the interval $(0, t_i)$. Next, setting $A \coloneqq (a_1 < \ldots < a_p)$ and $m \coloneqq (m_1, \ldots, m_p)$ with $m_i = (t_i, W_i)$, we are left with an element $(A, m) \in BM$. Now there is a path from $W \in \psi(n, k+1)$ to $\mu(A, m)$, which we can see as follows: we want the path to shift $W \cap (\mathbb{R}^k \times (a_{p+1}, \infty) \times \mathbb{R}^{n-k-1})$ by an increasing positive multiple of e_{k+1}, i.e. we want to "push off" that part of W to infinity on the right, and similarly we want to push off $W \cap (\mathbb{R}^k \times (-\infty, a_1) \times \mathbb{R}^{n-k-1})$ to $-\infty$ in the e_{k+1}-direction. To do so rigorously, fix again a monotonically increasing homeomorphism $\nu \colon I \to [1, \infty]$ and define the paths

$$\nu_t^1(x) \coloneqq \nu(t)(x - a_1) + a_1 \qquad \text{and} \qquad \nu_t^{p+1}(x) \coloneqq \nu(t)(x - a_{p+1}) + a_{p+1}.$$

The reason for introducing these paths is that we do not assume that $a_1 < 0 < a_{p+1}$, i.e. we cannot just multiply by $\nu(t)$, but need to shift before and after as well. Then, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define the homotopy $\gamma^{k+1} \colon \mathbb{R}^n \times I \to \mathbb{R}^n$ by the formula

$$\gamma_t^{k+1}(x) \coloneqq \begin{cases} \big(x_1, \ldots, x_k, \nu_t^1(x_{k+1}), x_{k+2}, \ldots, x_n\big) & \text{for } x_{k+1} < a_1 \\ (x_1, \ldots, x_{k+1}, \ldots, x_n) & \text{for } a_1 \leq x_{k+1} \leq a_{p+1} \\ \big(x_1, \ldots, x_k, \nu_t^{p+1}(x_{k+1}), x_{k+2}, \ldots, x_n\big) & \text{for } x_{k+1} > a_{p+1} \end{cases}.$$

Finally, we obtain the desired path $t \mapsto \gamma_t^{k+1}(W)$ and one checks that $\gamma_0^{k+1}(W) = W$ and $\gamma_1^{k+1}(W) = \mu(A, m)$, as claimed. We will discuss later in the proof of Lemma 5.21 how this process can be generalised in order to end up with a continuous lift (up to homotopy) of each manifold $W = f(x)$ for all $x \in S^d$ simultaneously, as opposed to one x at a time.

Injectivity of $\pi_*(\mu) \colon \pi_d(BM) \to \pi_d(\psi(n, k+1))$ is similar. The goal here is to lift a map $f \colon S^d \times I \to \psi(n, k+1)$ with fixed lifts f_0, f_1 over $S^d \times \{0, 1\}$ at the start- and end-point. More precisely, we have a lifting problem

$$\begin{tikzcd} S^d \times \{0,1\} \arrow[r, "{f_0, f_1}"] \arrow[d, hook] & BM \arrow[d, "\mu"] \\ S^d \times I \arrow[r, "f"] \arrow[ur, dashed] & \psi(n, k+1) \end{tikzcd}$$

i.e. given f_0 and f_1 with $\mu \circ f_0$ homotopic to $\mu \circ f_1$ via f, we want to construct the diagonal arrow, which gives us that f_0 is already homotopic to f_1, showing injectivity. Note that by the same argument as in the surjectivity proof, we can assume that $\mathrm{im}(f) \subseteq \mathrm{im}(\mu)$.

To begin with, fix some $x \in S^d$ and write $f_0(x) = (A, m) \in BM$ and $f_1(x) = (A', m') \in BM$. Similar to the surjectivity part earlier, the aim is to lift the path $f(x, -) \colon I \to \psi(n, k+1)$, from

$W := \mu(A,m)$ to $W' := \mu(A',m')$, to a path $\tilde{f}(x,-)\colon I \to BM$ from (A,m) to (A',m'). By virtue of property (12) we write

- $A = (a_1 < \ldots < a_p)$ and $A' = (a'_1 < \ldots < a'_p)$
- $m = (m_1, \ldots, m_p)$ and $m'_i = (t'_i, W'_i)$
- $m_i = (t_i, W_i)$ and $m'_i = (t'_i, W'_i)$, with W_i, W'_i as in (13).
- $t_i = a_{i+1} - a_i$ and $t'_i = a'_{i+1} - a'_i$

Notice that both elements in BM have the same number p of labels, which is due to $\mu \circ f_0$ being homotopic to $\mu \circ f_1$, i.e. both values necessarily have the same number of path components. We can then define a path ℓ from A to A' given by a collection of linear paths ℓ_i from a_i to a'_i for each i, for instance

$$\ell_i(t) := ta'_i + (1-t)a_i,$$

and then set $\ell(t) := (\ell_1(t), \ldots, \ell_n(t))$. Similarly, we have a path $f|_i\colon I \to \psi(n, k+1)$ from $W_i + b_i e_{k+1}$ to $W'_i + b'_i e_{k+1}$, given essentially by the restricted map $f(x,-)\colon I \to \psi(n,k+1)$. If we denote by ε_i the path with $\varepsilon_i(0) = W_i$ and $\varepsilon_i(1) = W_i + b_i e_{k+1}$, then we can finally define the lift by the formula

$$\tilde{f}(x,t) := \left(\ell(t), \left(tt'_i + (1-t)t_i,\ \varepsilon_i * f|_i * (\varepsilon'_i)^{-1}(t)\right)_{i=1,\ldots,p}\right),$$

where $\varepsilon_i * f|_i * (\varepsilon'_i)^{-1}$ is the concatenation of paths yielding the desired path from W_i to W'_i. By assumption on f this makes the lifting diagram commute and hence we have shown that f_0 is homotopic to f_1.

Despite some work, having property (12) readily available significantly simplified the proof of Proposition 5.16, so it makes sense to study this property further. We will see that we can essentially assume this property to hold for all manifolds, at least up to some homotopy-constraints.

Definition 5.17. Let $f\colon X \to \psi(n, k+1)$ be a map and define the set

$$X_a := \left\{ x \in X \;\middle|\; \left(\mathbb{R}^k \times \{a\} \times \mathbb{R}^{n-k-1}\right) \cap f(x) = \emptyset \right\},$$

i.e. X_a is the set of all $x \in X$ such that $f(x)$ satisfies (12). Additionally, we declare a smooth map $f\colon X \to \psi(n,k+1)$ to be *good*, if we can write $X = \bigcup_{a\in\mathbb{R}} \operatorname{int}(X_a)$.

The next technical result constitutes the main ingredient which we need for the proof of Proposition 5.16.

Lemma 5.18. *For $1 \le k < n$ and X a compact manifold (possibly with boundary), any map $f\colon X \to \psi(n,k+1)$ with image in the component containing $\emptyset \in \psi(n,k+1)$ is homotopic to a good map. Moreover, any such homotopy can be taken to be constant in a neighborhood of any closed set on which f is already good.*

Proof. We first check the case where $f\colon X \to \psi(n, k+1)$ is a smooth map such that for all $x \in X$, the restriction of the projection $\mathbb{R}^n \twoheadrightarrow \mathbb{R}^{k+1}$ to the submanifold $f(x)$, i.e. the composite

$$\pi_x\colon f(x) \hookrightarrow \mathbb{R}^n \twoheadrightarrow \mathbb{R}^{k+1}$$

is *not* surjective. Notice that this assumption is tautological for $k \geq 2$, since $f(x)$ is a 2-manifold. In this particular case, for each $x \in X$, pick a point $q_x \in \mathbb{R}^{k+1} \setminus \operatorname{im}(\pi_x)$ and an $\varepsilon_x > 0$ such that $B(q_x, \varepsilon_x)$, the closed ε_x-ball around q_x, satisfies $\operatorname{im}(\pi_x) \cap B(q_x, \varepsilon_x) = \emptyset$. Note that these choices would still work just fine if we had considered a neighborhood U_x of x, since another point $x' \in U_x$ yields a manifold $f(x')$ which is "close" to $f(x)$ in the $\Psi(\mathbb{R}^n)$-topology, by continuity of f. By compactness of X, we can therefore find a "global" $\varepsilon > 0$ and finitely many open sets $U_i \subseteq X$, $i = 1, \dots, m$, each corresponding to a point $q_i \in \mathbb{R}^{k+1}$, such that for each i we have $B(q_i, \varepsilon) \cap \operatorname{im}(\pi_x) = \emptyset$ for all $x \in U_i$. To be more precise, we simply choose $\varepsilon \coloneqq \min\{\varepsilon_1, \dots, \varepsilon_m\}$. If we write $q_i = (p_i, t_i) \in \mathbb{R}^k \times \mathbb{R}$, we can assume without loss of generality that all numbers t_i are distinct and that for any two different $i, j \in \{1, \dots, m\}$ we have $[t_i - \varepsilon, t_i + \varepsilon] \cap [t_j - \varepsilon, t_j + \varepsilon] = \emptyset$, potentially after replacing ε by some smaller $0 < \varepsilon' < \varepsilon$. Then there exists an isotopy of diffeomorphisms $\xi\colon \mathbb{R}^k \times \mathbb{R} \times I \to \mathbb{R}^k \times \mathbb{R}$ such that $\xi(-,-,0) = \mathrm{id}_{\mathbb{R}^k \times \mathbb{R}^k}$ and $\xi(p_i, t_i, 1) = (0, t_i)$ for each i and such that

$$\operatorname{supp}(\xi(-,-,,t)) \subseteq \bigcup_{i=1}^{m} \mathbb{R}^k \times (t_i - \varepsilon, t_i + \varepsilon).$$

Note that we do *not* have $\xi(x, y, 1) = (0, y)$ for every $y \in \mathbb{R}$, since this map is not a diffeomorphism. Thus, heuristically, we use this isotopy to deform $\mathbb{R}^{k+1}$ in such a way that all p_i's equate to 0, so from now on we simply assume $p_i = 0$ for all i, and hence that $f(x) \cap B(0, \varepsilon) \times \{t_i\} \times \mathbb{R}^{n-k-1} = \emptyset$, where the ball is contained in $\mathbb{R}^k$. Lastly, we fix an isotopy of embeddings $e : \mathbb{R}^k \times I \to \mathbb{R}^k$ such that $e_0 = \mathrm{id}_{\mathbb{R}^k}$ and $e_1(\mathbb{R}^k) \subseteq B(0, \varepsilon)$, i.e. we simply "shrink" $\mathbb{R}^k$ until it is contained in the ball. Then the isotopy defined by $\phi_s \coloneqq e_s \times \mathrm{id}_{\mathbb{R}^{n-k}} : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ yields a homotopy

$$f_{(-)}(-)\colon I \times X \to \psi(n, k+1), \qquad f_s(x) \coloneqq \phi_s^{-1}(f(x)),$$

where continuity follows from Theorem 2.15. We see that $f_1(x) \cap \mathbb{R}^k \times \{t_i\} \times \mathbb{R}^{n-k-1} = \emptyset$ and $f_0(x) = f(x)$ for all i, since we have

$$\phi_1\left(f_1(x) \cap \mathbb{R}^k \times \{t_i\} \times \mathbb{R}^{n-k-1}\right) \subseteq f(x) \cap B(0, \varepsilon) \times \{t_i\} \times \mathbb{R}^{n-k-1} = \emptyset,$$

and hence f is homotopic to a good map, because we can write $X = \bigcup_{i=1}^m U_i$, as needed.

Next, we tackle the case when $k = 1$: the goal is to show that f is homotopic to a map f' satisfying the above "non-surjectivity" assumption, i.e. that $\pi_x\colon f'(x) \hookrightarrow \mathbb{R}^n \twoheadrightarrow \mathbb{R}^2$ is not surjective.

In order to give a clearer idea of the following proof strategy, we first only consider the case $X = \{\mathrm{pt}\}$, and only later will we consider general compact manifolds X. The manifold $W \coloneqq f(\mathrm{pt}) \in \psi(n, 2)$ is in the path-component of the basepoint $\emptyset \in \psi(n, 2)$, and recall that W is thus a topologically closed 2-manifold in $\mathbb{R}^2 \times (0, 1)^{n-2}$. Let $\gamma\colon I \to \psi(n, 2)$ denote a path from

W to $\emptyset$. Then we can construct a smooth path $\mathbb{R} \to \psi(n,2)$, constant on $(-\infty,0)$ and $(1,\infty)$, which is given by a 3-manifold $E \subseteq \mathbb{R}\times\mathbb{R}^n$ such that $E \cap \{0\}\times\mathbb{R}^n = W$ and $E\cap\{1\}\times\mathbb{R}^n = \emptyset$.

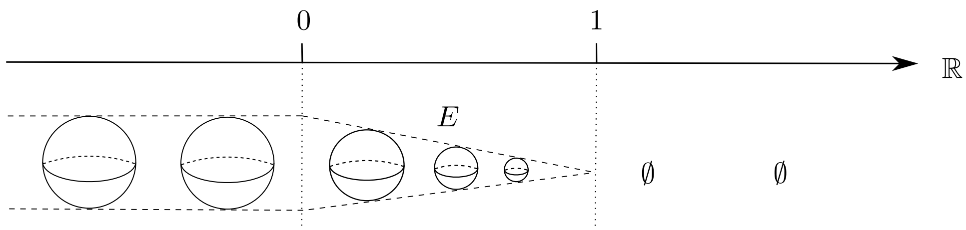

FIGURE 7. A schematic of the manifold E for the case $W = S^2$.

Pictorially, we essentially define a sort of null-cobordism starting at W, except that we elongate the start and end indefinitely in the positive/negative direction. Explicitly, E is given by

$$E \coloneqq (-\infty,0]\times W \cup \bigcup_{t\in I}\big(\{t\}\times\gamma(t)\big) \cup [1,\infty)\times\emptyset.$$

Then we project onto the second and third coordinate of $\mathbb{R}^n$ to obtain a proper smooth map $W \to \mathbb{R}^2$. By Sard's Theorem [Hir76, p. 69], we may pick a regular value $(p,t)\in\mathbb{R}^2$ in the image. Then $(0,p,t)\in\mathbb{R}^3$ is a regular value for the projection $E\to\mathbb{R}^3$ and, similar to the case before, we can assume that $p=0$, possibly after applying an isotopy of self-diffeomorphisms of $\mathbb{R}^2$. Using Sard's Theorem once more, i.e. the fact that critical values are not dense, we can choose a sufficiently small $\varepsilon>0$ such that the entire cube $[-\varepsilon,\varepsilon]^3+(0,0,t)$ surrounding $(0,0,t)$ contains *only* regular values of the map $E\to\mathbb{R}^3$. Then we fix an intermediate smooth function $\lambda\colon\mathbb{R}\to I$ with $\lambda(x)=1$ for x in a neighborhood of 0 and with $\operatorname{supp}(\lambda)\subseteq(-\varepsilon,\varepsilon)$, thus we think of λ as a classical bump-function. Finally, we define the most important (and most cluttered) map in this proof, namely for $r\in[0,1]$, let

$$\begin{aligned}\phi_r\colon \mathbb{R}^n &\longrightarrow \mathbb{R}\times\mathbb{R}^n\\ (x_1,\dots,x_n) &\longmapsto \Big(x_1\sin\Big(\frac{\pi r}{2}\lambda(x_2-t)\Big), \Big(x_1\cos\Big(\frac{\pi r}{2}\lambda(x_2-t)\Big), x_2, x_3,\dots,x_n\Big)\Big).\end{aligned}$$

In order to obtain a better intuitive understanding of the map ϕ_r, notice that for $r=0$, the map ϕ_0 is simply given by the inclusion $(x_1,\dots,x_n)\mapsto(0,x_1,\dots,x_n)$. However, as r increases, ϕ_r rotates in the first two coordinates, while being "dampened" by λ, i.e. the rotation happens only in a neighborhood of $x_2=t$. Moreover, ϕ_r is independent of r unless $x_2\in(t-\varepsilon,t+\varepsilon)$, since otherwise ϕ_r is again just the inclusion map.

Define subsets $W(r)\coloneqq\phi_r^{-1}(E)\subseteq\mathbb{R}^n$ and note that $W(0)=E\cap\{0\}\times\mathbb{R}^n=W$ by the previous observation about ϕ_0. Each $W(r)$ is a closed subset in $\mathbb{R}^2\times(0,1)^{n-2}$ by the fact that $E\subseteq\mathbb{R}\times\mathbb{R}^n$ is closed and by inspecting the map ϕ_r and using that $W\subseteq\mathbb{R}^2\times(0,1)^{n-2}$. Unfortunately, a caveat is that $W(r)$ is not necessarily a smooth manifold, since ϕ_r may not be *transverse* to E, see [Hir76, p. 22] for the general theorem. Roughly speaking, the upshot is that ϕ_r is transverse at least on a specific subset of $\mathbb{R}^n$, namely if we define the set

$$T \coloneqq \big(\{0\} \times \mathbb{R} \cup \mathbb{R} \times \{t\}\big) \times \mathbb{R}^{n-2} \subseteq \mathbb{R}^2 \times \mathbb{R}^{n-2},$$

we have that the restricted map $\overline{\phi}_r \coloneqq \phi_r|_{\nu T} \colon \nu T \to \mathbb{R} \times \mathbb{R}^n$, where νT denotes a neighborhood of T, is transverse to E. Hence, $W(r)$ is smooth near the set T for all $r \in [0,1]$. Now we are able to (smoothly) "push singularities to infinity" in the x_1-direction by replacing each $W(r)$ by the manifold $\widetilde{W}(r) \coloneqq (e \times \mathrm{id}_{\mathbb{R}^{n-2}})^{-1}(W(r))$, where $e\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an embedding which is isotopic to $\mathrm{id}_{\mathbb{R}^2}$ and satisfies $e|_{\{0\}\times\mathbb{R}\cup\mathbb{R}\times\{t\}} = \mathrm{id}_{\{0\}\times\mathbb{R}\cup\mathbb{R}\times\{t\}}$. Notice that $\widetilde{W}(r) \in \psi(n,2)$ and that $\widetilde{W}(0) \cong W$.

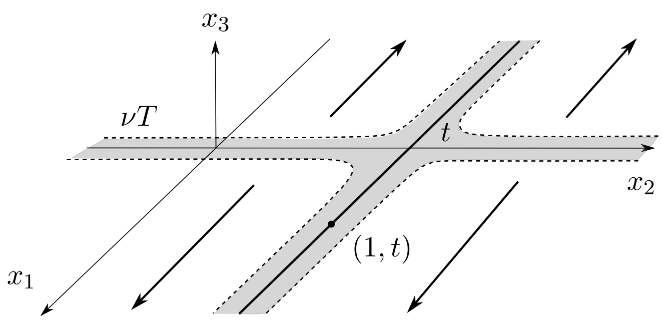

FIGURE 8. A possible choice for the embedding $e\colon \mathbb{R}^2 \to \mathbb{R}^2$, fixing the subspace T. Essentially, we push any point not in T to $\pm\infty$ along the x_1-axis.

Technically, we are almost done, since it only remains to see that the "end" $\widetilde{W}(1)$ is disjoint from some set of the form $\mathbb{R} \times \{-\} \times \mathbb{R}^{n-k-1}$, but the easiest way to see this requires a little more work. First, assume without loss of generality that $t \neq 0$. Then we can show that the projection $W(1) \twoheadrightarrow \mathbb{R}^2$ is not surjective as follows: we can compute $\phi_1(1, t, x_3, \ldots, x_n) = (1, 0, t, x_3, \ldots, x_n)$, but this means that this value does *not* lie in E, since $E \cap \{1\} \times \mathbb{R}^n = \emptyset$, and hence $(1, t, x_3, \ldots, x_n) \notin W(1)$, showing that $(1,t)$ is not in the image of the projection. Now if we consider the replacement $\widetilde{W}(r)$ as described above, we still have that the projection $\widetilde{W}(1) \twoheadrightarrow \mathbb{R}^2$ does not hit $(1,t)$, since the map $e \times \mathrm{id}_{\mathbb{R}^{n-2}}$ fixes T by assumption, so in particular it fixes the point $(1, t, x_3, \ldots, x_n)$. Then there exists some small $\delta > 0$ such that the ball $B \coloneqq B((1, t, x_3, \ldots, x_n), \delta)$ is fixed by this map. Finally, we "blow up" this ball in the first two coordinates radially outwards from the point $(1,t)$ via a map $R\colon S^2 \to S^2$, where we view $S^2 \cong (\mathbb{R}^2)^+$. This blown-up version of $\widetilde{W}(1)$ is simply the empty manifold, so in particular it is disjoint from $\mathbb{R} \times \{t\} \times \mathbb{R}^{n-k-1}$, showing that we can indeed homotope f to a map which is good.

For the general case we give a proof sketch: since $\mathrm{im}(f)$ is contained in the component of $\emptyset \in \psi(n,2)$, we can pick for each $x \in X$ an open, contractible neighborhood U_x of x and a null-homotopy of $f|_{U_x}\colon U_x \to \psi(n,2)$ given by a smooth map $h_x\colon U_x \times \mathbb{R} \to \psi(n,2)$. Then, for any $y \in U_x$ we identify the map $h_x(y,-)\colon \mathbb{R} \to \psi(n,2)$ with its graph $E_x(y) \coloneqq \{(t, h_x(y,t)) \mid t \in \mathbb{R}\}$, which is a smooth 3-manifold in $\mathbb{R} \times \mathbb{R}^n$. Thus, each $E_x(y)$ can be thought of as being a version of the null-cobordism E from the earlier part of the proof, where we assumed $X = \{\mathrm{pt}\}$. By convention, we will write $(s, x_1, \dots, x_n)$ for an element in $\mathbb{R} \times \mathbb{R}^n$. As before, consider the projection $E_x(y) \twoheadrightarrow \mathbb{R}^3$ onto the first three coordinates. Again, we may pick a regular value $(0, p_x, t_x)$ of this map, along with an $\varepsilon_x > 0$ such that the shifted "cube"

$$\big([-\varepsilon_x, \varepsilon_x]^3 + (0, p_x, t_x)\big) \times \mathbb{R}^{n-2}$$

contains *only* regular values of the projection, by Sard's Theorem, where we may replace U_x by a smaller neighborhood if necessary. By compactness of X we may choose a finite subset $\{U_1, \dots, U_m\}$ of the U_x's which cover X, where we denote the corresponding regular values by $(0, p_i, t_i)$ and we set $\varepsilon \coloneqq \min\{\varepsilon_1, \dots, \varepsilon_m\}$. Analogously to the case before, we can assume that all t_i are distinct (possibly by choosing a different regular value), that all intervals $[t_i - \varepsilon, t_i + \varepsilon]$ are pairwise disjoint (possibly after replacing ε by some $0 < \varepsilon' < \varepsilon$) and that $p_i = 0$ for all $i = 1, \dots, m$. Let $\lambda\colon \mathbb{R} \to I$ be the same bump-function as before and choose a partition of unity $\rho_i\colon X \to I$ with $\operatorname{supp}(\rho_i) \subseteq U_i$ and such that $X = \bigcup_{i=1}^m U_i'$, where $U_i' \coloneqq \operatorname{int}(\rho_i^{-1}(1))$, for each i. Notice that this last assumption equivalently says that we assume at least some of the functions ρ_i to attain the value 1 in some neighborhood, because otherwise we would be left with $U_i' = \emptyset$ for all i. Then, consider the analogue of the map ϕ_r in this setting, namely the map

$$\begin{aligned} \phi_{y,i,r}\colon \mathbb{R}^n &\longrightarrow \mathbb{R} \times \mathbb{R}^n \\ (x_1, \dots, x_n) &\longmapsto \Big(x_1 \sin\Big(\frac{\pi r}{2}\rho_i(y)\lambda(x_2 - t_i)\Big), \Big(x_1 \cos\Big(\frac{\pi r}{2}\rho_i(y)\lambda(x_2 - t_i)\Big), x_2, x_3, \dots, x_n\Big)\Big) \end{aligned}$$

and define subsets $W_i(y, r) \coloneqq \phi_{y,i,r}^{-1}(E_i(y)) \subseteq \mathbb{R}^n$. As was the case earlier, $W_i(y, r)$ is closed in $\mathbb{R}^2 \times (0,1)^{n-2}$ and one sees that

$$W_i(y,r) \cap \Big(\mathbb{R} \times \big(\mathbb{R} \setminus (t_i - \varepsilon, t_i + \varepsilon)\big) \times \mathbb{R}^{n-2}\Big) = f(y) \cap \Big(\mathbb{R} \times \big(\mathbb{R} \setminus (t_i - \varepsilon, t_i + \varepsilon)\big) \times \mathbb{R}^{n-2}\Big),$$

since we have $\lambda(x_2 - t_i) = 0$ in this case, which implies that $\phi_{y,i,r}$ is given by the inclusion $(x_1, \dots, x_n) \mapsto (0, x_1, \dots, x_n)$, and hence the preimage of $E_i(y)$ is just $f(y)$ here. Thus, we can define a more amalgamated variant of $W_i(y, r)$, namely the space $W(y, r)$ which agrees with $W_i(y, r)$ inside $\mathbb{R} \times (t_i - \varepsilon, t_i + \varepsilon) \times \mathbb{R}^{n-2}$ and with $f(y)$ on the complement of these sets. By construction, the space $W(y, r)$ is still closed in $\mathbb{R}^2 \times (0,1)^{n-2}$, but it is not necessarily smooth, unless $y \in U_i'$, in which case $W(y, r)$ is smooth on a neighborhood of $\{0\} \times \mathbb{R}^{n-1} \cup \mathbb{R} \times \{t_i\} \times \mathbb{R}^{n-2}$, by the same reasoning as earlier. By an embedding $e\colon \mathbb{R}^2 \to \mathbb{R}^2$, similar to the one from above, we "push singularities to infinity" in the x_1-direction. The difference is that we need to be slightly more careful, because now we're not just "stretching" a neighborhood of $(\{0\} \times \mathbb{R} \cup \{t_i\}) \times \mathbb{R}^{n-2}$, but rather a neighborhood of the collection of these individual spaces, i.e. of

$$\Big(\bigcup_{i=1}^m \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{t_i\}\Big) \times \mathbb{R}^{n-2}.$$

We finally obtain smooth manifolds $\widetilde{W}(y, r) \in \psi(n, 2)$ and now we can, roughly speaking, simultaneously "blow-up" around all points $(1, t_i) \in \mathbb{R}^2$, thus showing that $\widetilde{W}(y, 1)$ is disjoint from each $\mathbb{R} \times \{t_i\} \times \mathbb{R}^{n-2}$. $\square$

After this rather lengthy proof, we finally discuss how exactly we use Lemma 5.18 to show that $\mu\colon BM \to \psi(n,k+1)$ is a weak equivalence. We introduce yet another space, similar in spirit to BM, but containing slightly more information.

Definition 5.19. Let BP be the space of triples (A, C, W), where $A = (a_1 < \ldots < a_p)$ and $C = (c_0 < c_1 < \ldots < c_p)$ are finite sets of points in $\mathbb{R}$ and $W \in \psi(n,k+1)$ satisfies the property

$$W \cap (\mathbb{R}^k \times C \times \mathbb{R}^{n-k-1}) = \emptyset.$$

By convention, we define $a_0 = -\infty$ and $a_{p+1} = \infty$. Similar to the definition of BM, we topologize BP by simply allowing points a_i to "collide" and go to $\pm\infty$, in both of which cases the number of intervals in $\mathbb{R} \setminus A$ decreases, by forgetting the corresponding c_i. In practice, it is useful to think of each c_i as "labelling" the interval (a_i, a_{i+1}).

Remark 5.20. The purpose of introducing BP is that we can factorize the map μ through this space, namely as a commutative diagram

$$\begin{tikzcd}
BM \arrow[rr, "\mu"] \arrow[dr, "\overline{\mu}"'] & & \psi(n,k+1) \\
& BP \arrow[ur, "\mathrm{ft}"'] &
\end{tikzcd}$$

where the map $\overline{\mu}$ is given by sending

$$\big(A,\ (t_i, W_i)_{i=1,\ldots,p}\big) \mapsto \Big(A, \big(b_1 < \ldots < b_p < b_p + t_p\big), \bigcup_{i=1}^{p} W_i + b_i e_{k+1}\Big)$$

and $\mathrm{ft}\colon BP \to \psi(n,k+1)$ is the forgetful map, simply disregarding the sets A and C in the triple. We will then show in two steps that both $\overline{\mu}$ and ft are at least weak equivalences, thus showing that μ has the same property.

Lemma 5.21. *The forgetful map* $\mathrm{ft}\colon BP \to \psi(n,k+1)$ *is a weak equivalence.*

Proof sketch. We first prove surjectivity on $\pi_d(-)$: by Lemma 5.18 we can assume that a fixed element in $\pi_d(\psi(n,k+1))$ is represented by a good map $f\colon S^d \to \psi(n,k+1)$. Since S^d is compact, there exists a finite set $\mathcal{C} \coloneqq \{c_1, \ldots, c_m\}$ such that the open sets U_{c_i}, defined as

$$U_{c_i} \coloneqq \operatorname{int}\left\{x \in S^d \;\middle|\; f(x) \cap (\mathbb{R}^k \times \{c_i\} \times \mathbb{R}^{n-k-1}) = \emptyset\right\},$$

cover the sphere S^d. Then by a partition of unity argument we get a collection of maps $\lambda_1, \ldots, \lambda_m\colon S^d \to I$, where $\lambda_i \coloneqq \lambda_{c_i}$, which are smooth, locally finite and subordinate to the open sets U_{c_i} respectively, such that $\sum_{c\in\mathcal{C}} \lambda_c(x) = 1$ for every $x \in S^d$. Before defining a lift for the map f, we introduce two variants of the set $\mathcal{C}$: define two positive integers

$$\begin{aligned}
i_x^0 &\coloneqq \max\{i \mid \lambda_1(x) = \lambda_2(x) = \ldots = \lambda_i(x) = 0\} \\
i_x^1 &\coloneqq \min\{i \mid \lambda_i(x) = \lambda_{i+1}(x) = \ldots = \lambda_m(x) = 0\},
\end{aligned}$$

i.e. these numbers characterise the maximum number of points c_i at the lower/upper end of $\mathcal{C}$ that evaluate to 0 on x. Then define variants of $\mathcal{C}$ as

$$\mathcal{C}'_x \coloneqq \mathcal{C} \setminus \{c_1, \dots, c_{i_x^0}, c_{i_x^1}, \dots, c_m\} \qquad \mathcal{C}''_x \coloneqq \mathcal{C}'_x \setminus \{c_{i_x^1 - 1}\},$$

i.e. the set $\mathcal{C}'_x$ simply disregards all points at the start and end that evaluate to zero on x and $\mathcal{C}''_x$ additionally forgets the largest point $c_{i_x^1-1}$ that does *not* evaluate to zero, which means that the sum over $\mathcal{C}''_x$ of evaluations will always be strictly smaller than 1. For some fixed $x \in S^d$, define sets

$$C_x \coloneqq \left\{c \in \mathcal{C}'_x \mid \lambda_c(x) > 0\right\} \qquad A_x \coloneqq \left\{\rho\Bigg(\sum_{c \in \mathcal{C}''_{x,t}} \lambda_c(x)\Bigg) \;\Bigg|\; t \in \mathbb{R}\right\}$$

where $\mathcal{C}''_{x,t} \coloneqq (-\infty, t] \cap \mathcal{C}''_x$ and $\rho\colon (0,1) \to \mathbb{R}$ is an increasing homeomorphism, for instance $\rho(y) = \tan(\pi y - \pi/2)$. Finally, define a lift of f by the assignment

$$\overline{f}\colon S^d \to BP, \quad x \mapsto (A_x, C_x, f(x)).$$

Clearly, we have $f = \mathrm{ft} \circ \overline{f}$, and since ft is an open map and by the fact that $\overline{f}^{-1}(z) \subseteq f^{-1}(\mathrm{ft}(z))$ for any $z \in BP$, we see that $\overline{f}$ is continuous.

To show injectivity on $\pi_d(-)$, suppose we have two good maps $f, f'\colon S^d \to \psi(n, k+1)$ which are homotopic through $h\colon S^d \times I \to \psi(n, k+1)$. Since $S^d \times I$ is still a compact manifold we can invoke Lemma 5.18 once more to see that the homotopy h itself is homotopic to a good map. Similar to the surjectivity case, we lift this good homotopy to a map $\overline{h}\colon S^d \times I \to BP$, using affine linear homotopies between the first two factors and the fact that f was homotopic to f' for the third factor. □

Lemma 5.22. *The map $\overline{\mu}\colon BM \to BP$ is a homotopy equivalence.*

Proof. As a reminder, recall Remark 5.20 for the definition of $\overline{\mu}$. The first important observation towards a proof is that BM deformation retracts onto the subspace $B'M \subseteq BM$ given by

$$B'M \coloneqq \left\{\Big((a_1 < \ldots < a_p), \big((t_1, W_1), \ldots, (t_p, W_p)\big)\Big) \in BM \;\Big|\; t_i \geq a_{i+1} - a_i\right\},$$

where the inequality is understood to hold for $i = 1, \ldots, p-1$ and for $i = p$ there is no condition, since we have $a_{p+1} = \infty$ by convention, which would contradict the inequality. Loosely phrased, the deformation is given by linearly increasing the numbers t_i until the inequalities hold. To be a little more precise, we define the deformation $H\colon BM \times I \to BM$ by

$$\Big((a_1 < \ldots < a_p), (t_i, W_i)_{i=1,\ldots,p}, t\Big) \mapsto \Big((a_1 < \ldots < a_p), (H_t(t_i), W_i)_{i=1,\ldots,p}\Big),$$

where the maps H_t are defined such that $H_t(t_i) = t_i$ if $t_i \geq a_{i+1} - a_i$, $H_0(t_i) = t_i$ and $H_1(t_i) = a_{i+1} - a_i$ if $t_i < a_{i+1} - a_i$ and $H_t(t_p) = t_p$ for all $t \in I$. Then one checks that, indeed, $H((A, m), 0) = (A, m)$, $H((A, m), 1) \in B'M$ and $H((A, m), 1) = (A, m)$ if $(A, m) \in B'M$, thus showing that $B'M \hookrightarrow BM$ is a deformation retraction.

The second, similar observation, is that BP deformation retracts to a subspace $B'P$, given by

$$B'P \coloneqq \big\{(A, C, W) \in BP \;\big|\; a_1 = c_0,\ a_i \leq c_{i-1},\ W \subseteq \mathbb{R}^k \times (c_0, c_p) \times \mathbb{R}^{n-k-1}\big\}.$$

Seeing this deformation explicitly is less straightforward, but still manageable: first, write $t_i \coloneqq c_i - c_{i-1}$ and define $c'_{i-1} \geq a_i$ to be the smallest number such that $c'_i - c'_{i-1} \geq t_i$. The idea is to map each c_i to c'_i linearly, while simultaneously mapping the component of W that is contained in $\mathbb{R}^k \times (c_{i-1}, c_i) \times \mathbb{R}^{n-k-1}$ to $\mathbb{R}^k \times (c'_{i-1}, c'_{i-1} + t_i) \times \mathbb{R}^{n-k-1}$, also using only linear maps. Heuristically, the space $B'P$ can be thought of as a convenient "rescaling" of BP, since the deformation will push both components

$$W^0 \coloneqq W \cap \mathbb{R}^k \times (-\infty, c_0) \times \mathbb{R}^{n-k-1} \quad \text{and} \quad W^{p+1} \coloneqq W \cap \mathbb{R}^k \times (c_p, \infty) \times \mathbb{R}^{n-k-1}$$

to $\emptyset$ in the negative/positive e_{k+1}-direction respectively, and leaves the components inside $\mathbb{R}^k \times (c_0, c_p) \times \mathbb{R}^{n-k-1}$ largely unchanged. Notice, for instance, that the intervals (c_{i-1}, c_i) and $(c'_{i-1}, c'_{i-1} + t_i)$ have the same length, i.e. any deformations will only consist of transformations, and no stretching.

We make an effort to be as explicit about this deformation as possible. To this end, let ℓ_i be the unique linear transformation mapping (c_{i-1}, c_i) to the interval $(c'_{i-1}, c'_{i-1} + t_i)$, i.e. the linear homeomorphism $\mathbb{R} \to \mathbb{R}$ determined by $c_{i-1} \mapsto c'_{i-1}$, and let $\varepsilon^0, \varepsilon^1$ be the transformations sending W^0, W^1 to $\emptyset$, e.g. for the e_{k+1}-direction we can spell out a parametrised version of ε^1 as

$$\mathbb{R} \times I \to \mathbb{R}, \quad (x, t) \mapsto x + \rho(t) e_{k+1}$$

for a fixed homeomorphism $\rho\colon I \to [0, \infty]$ such as $\rho(t) = \tan(\pi t/2 - \pi)$. Denote by $\overline{\varepsilon}^0, \overline{\varepsilon}^1$ and $\overline{\ell}_i$ the extensions of the above maps to $\mathbb{R}^k \times (-) \times \mathbb{R}^{n-k-1}$ via the identity. Then, at last, we can define the entire transformation T as the collection of these maps, namely

$$T\colon \coprod_{i=0}^{p+1} \Big(\mathbb{R}^k \times (c_{i-1}, c_i) \times \mathbb{R}^{n-k-1}\Big) \to \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}, \qquad T = \overline{\varepsilon}^0 \amalg \coprod_{i=1}^{p} \overline{\ell}_i \amalg \overline{\varepsilon}^1,$$

where we set $c_{-1} = -\infty$ and $c_{p+1} = \infty$ by convention. By parametrising the maps ℓ_i as well, we can thus define the deformation $H\colon BP \times I \to BP$ by mapping $c_i \mapsto c'_i$ in C using ℓ_i and mapping $W \mapsto T(W)$. Again, one checks the defining properties to confirm that H really is a deformation retraction $B'P \hookrightarrow BP$.

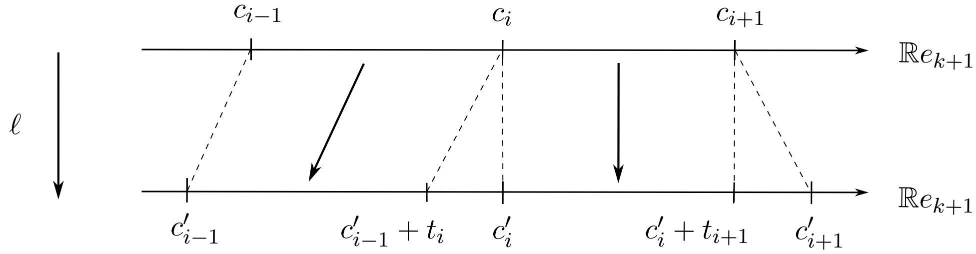

Figure 9. The maps ℓ_i mapping intervals to intervals, giving a "rescaling" of $\mathbb{R}e_{k+1}$.

Next, we claim that the map $\overline{\mu}\colon BM \to BP$ restricts to a homeomorphism $\overline{\mu}\colon B'M \to B'P$. To see this, consider some $(A, m) \in B'M$, i.e. $A = (a_1 < \ldots < a_p)$ and $m = (t_i, W_i)_{i=1,\ldots,p}$ with $a_i + t_i \geq a_{i+1}$ for all $i = 1, \ldots, p-1$. Then we have $\overline{\mu}(A, m) = (A, C, W)$, where $C = (b_1 < \ldots < b_p < b_p + t_p)$ and $W = \bigcup_{i=1}^{p} W_i + b_i e_{k+1}$. Using the definition of the numbers b_i and the assumption $a_i + t_i \geq a_{i+1}$, an inductive argument shows that

$$c_{i-1} = b_i = a_1 + \sum_{j=1}^{i-1} t_j,$$

so in particular W is contained in the space

$$\mathbb{R}^k \times (a_1, a_1 + t_1 + \ldots + t_p) \times \mathbb{R}^{n-k-1} = \mathbb{R}^k \times (c_0, c_p) \times \mathbb{R}^{n-k-1},$$

which shows that $\overline{\mu}(A, m) \in B'P$. One checks that an inverse map to $\overline{\mu}$ is given by setting $t_i \coloneqq c_i - c_{i-1}$ for $i = 1, \ldots, p$ and

$$W_i \coloneqq W \cap (\mathbb{R}^k \times (c_{i-1}, c_i) \times \mathbb{R}^{n-k-1}).$$

Consequently, we have shown that there exists a commutative diagram

$$\begin{tikzcd}
B'M \arrow[r, "\overline{\mu}", "\cong"'] \arrow[d, hook, "\simeq_{\mathrm{dr}}"'] & B'P \arrow[d, hook, "\simeq_{\mathrm{dr}}"] \\
BM \arrow[r, "\overline{\mu}"] & BP
\end{tikzcd}$$

where the decorations $\simeq_{\mathrm{dr}}$ denote the deformation retractions, and hence the proof is finished. $\square$

We conclude this section by reiterating the result that we obtained, and giving a final, clear proof to put all pieces in the correct order:

Corollary 5.23. *The map $\alpha_k\colon \psi(n, k) \to \Omega\psi(n, k+1)$ is a weak equivalence for $k \geq 1$. Thus, we have a weak equivalence $\Omega\psi(n, 1) \simeq \Omega^n\psi(n, n) = \Omega^n\Psi(\mathbb{R}^n)$ and in particular an equivalence $\Omega\psi(\infty, 1) \simeq \Omega^\infty\mathbf{\Psi}$.*

Proof. Combining Remark 5.2 with Lemmas 5.21 and 5.22 yields the following commutative diagram

$$\begin{tikzcd}
 & \Omega B'M \arrow[r, "\Omega\overline{\mu}", "\cong"'] \arrow[d, hook, "\simeq"'] & \Omega B'P \arrow[d, hook, "\simeq"] \\
M \arrow[r, "\beta", "\simeq"'] & \Omega BM \arrow[r, "\Omega\overline{\mu}"] \arrow[d, "\Omega\mu"'] & \Omega BP \arrow[dl, bend left, "\simeq"', "\Omega(\mathrm{ft})"] \\
\psi(n, k) \arrow[u, "\iota", "\simeq"'] \arrow[r, "\alpha_k"] & \Omega\psi(n, k+1) &
\end{tikzcd}$$

which still constitutes a factorization of α_k, with all factors being at least weak equivalences. For the second claim, one checks that the following diagram commutes

$$\begin{tikzcd} \psi(n,k) \arrow[r, hook, "\alpha_k"] \arrow[d, hook, "\hat{\imath}"'] & \Omega\psi(n,k) \arrow[d, hook, "\Omega\hat{\imath}"] \\ \psi(n+1,k) \arrow[r, hook, "\alpha_k"] & \Omega\psi(n+1,k) \end{tikzcd}$$

where the injective structure maps $\hat{\imath}$ are given by $W \mapsto W + \frac{1}{2}e_{n+1} \subseteq \mathbb{R}^k \times (0,1)^{n-k+1}$. □

This finishes the larger part of the proof: we have shown that all maps α_k, save for one, are weak equivalences. In particular, we can infer that these α_k's are homology isomorphisms, which is an important step towards the Madsen–Weiss Theorem. Now it only remains to study properties of the map $\alpha_0 \colon \psi(n,0) \to \Omega\psi(n,1)$ in the next chapter, and show that it restricts to the desired homology-isomorphism (depending on the genus g) in the case $n = \infty$, as stated earlier.

6. Finishing the Proof

Before continuing with all the technical details that remain to be discussed, we give some basic motivation to see why exactly it remains to study the map $\alpha_0\colon \psi(n,0) \to \Omega\psi(n,1)$ by itself in the first place, since one might wonder why we don't simply use the same proof strategies as with the maps $\alpha_1, \alpha_2, \alpha_3$, etc.

6.1. Replacing the Monoid M

Notice that we can take the same monoid $M = M_{n,0}$ from Definition 5.12 and Lemma 5.13 to see that $\psi(n,0)$ is homotopy equivalent to M, i.e. so far we don't find any contradiction. However, if we try to apply Theorem 5.11 in this setting, we see that it is not clear that M is group-like, since we cannot use the same proof as in the one from Lemma 5.14. We need at least one coordinate in which to push our manifolds off to infinity, and for any $W \in M$ we have $W \subseteq (0,t) \times (0,1)^{n-1}$, i.e. every dimension is bounded.

Another issue is that $\psi(n,0)$ is likely not quite the correct space to consider: the Madsen–Weiss Theorem concerns itself with *connected* surfaces and "high" genus, whereas the space $\psi(n,0)$ contains *all* surfaces. Thus, the logical next step would be to restrict to the subspace $\psi_{\mathrm{conn}}(n,0) \subseteq \psi(n,0)$ of connected surfaces, but this again leads to a problem, namely that this subspace is not homotopy equivalent to a monoid with our multiplication being given essentially by disjoint union. The solution to this issue is to consider surfaces with boundary, together with a monoidal multiplication mimicking juxtaposition, except that we glue along boundaries, so everything stays path-connected. Again, we roughly follow the outline given in [Gal12] and supply as many additional details and intuition as possible. We make the replacement for M precise in the following definition.

Definition 6.1. Let $L_t \coloneqq [0,t] \times [0,1] \subseteq \mathbb{R}^2$ and define the subspace

$$\Psi^c(L_t) \coloneqq \left\{ W \in \Psi(\mathbb{R}^n) \;\middle|\; W \subseteq L_t \times (-1,1)^{n-2} \text{ is connected} \right\} \subseteq \Psi(\mathbb{R}^n)$$

Next, denote by M the subspace

$$M \coloneqq M^n \coloneqq \left\{ (t,W) \in (0,\infty) \times \Psi^c(L_t) \;\middle|\; W \cap (L_t \times \{0\}) = W \cap ((\partial L_t) \times \mathbb{R}^{n-2}) \right\}$$

Define a multiplication by the following gluing construction:

$$(t,W) \cdot (t',W') \coloneqq (t+t', W \cup (W' + te_1)).$$

Pictorially, this means we place two elements next to each other along the x_1-axis and glue along this part of the boundary.

Remark 6.2. Technically speaking, we need to be more precise about the above definition of M. In order for the product $(t,W)\cdot(t',W')$ to again be a *smooth* manifold, we require the condition

$W \cap (L_t \times \{0\}) = W \cap ((\partial L_t) \times \mathbb{R}^{n-2})$ to hold in a *neighborhood*, i.e. if N_t denotes a neighborhood (or "open thickening") of ∂L_t, the condition in Definition 6.1 should be replaced by

$$\exists \,\text{neighborhood } N_t \supseteq \partial L_t \text{ such that } W \cap (L_t \times \{0\}) = W \cap (N_t \times \mathbb{R}^{n-2}).$$

Intuitively, this ensures that (t, W) is "flat" near its boundary.

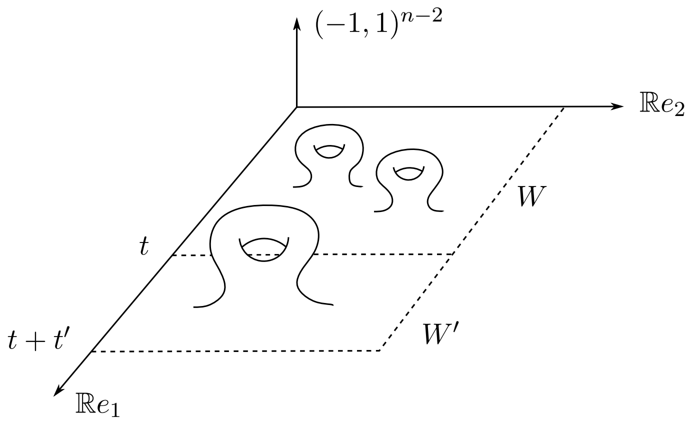

Figure 10. Schematic of the product $(t, W) \cdot (t', W')$.

Next, we argue that this new monoid is a good candidate for replacing the old monoid M.

Lemma 6.3. *The topological monoid M fom Definition 6.1 is homotopy-commutative. Moreover, for $n \geq 5$, we have $\pi_0 M = \mathbb{N}$.*

Proof. To see that the multiplication is homotopy-commutative, one uses the exact same argument as one uses to show that the standard additive group structure on $\pi_2(-)$ is abelian, i.e. one shrinks the interior of each $L_t \times \mathbb{R}^{n-2}$ slightly and then passes them along each other, before enlarging them again. A standard picture can be found in e.g. [Hat02, p. 340]. For the second claim, denote by $g(W)$ the genus of the surface W and consider the map $M \to \mathbb{N}$ given by $W \mapsto g(W)$. Since any surface of genus 1 can be embedded in $L_t \times \mathbb{R}$, we have that the monoidal map $\pi_0 M \to \mathbb{N}$, $[W] \mapsto g(W)$, is surjective for $n \geq 3$, as we can consider arbitrary products of genus 1 surfaces. By the Whitney–Wu Theorem [Wu58], we see that the map $\pi_0 M \to \mathbb{N}$ is injective for $n \geq 5$, since any two embeddings of a surface become isotopic in these dimensions, and hence we have $\pi_0 M = \mathbb{N}$. $\square$

Notice that there is still a problem with Lemma 6.3: the fact that $\pi_0 M = \mathbb{N}$ tells us that M is *not* group-like, and thus the map $\beta \colon M \to \Omega BM$ will not be a weak equivalence, so it seems as though this new definition of M did not bring us any closer to proving the Madsen–Weiss Theorem. However, we will soon find that we can invoke the Group-Completion Theorem to finish the proof. First, we offer an alternative interpretation of the new monoid M, establishing a connection with the classifying spaces of the form $B\mathrm{Diff}(\Sigma_{g,1})$.

Proposition 6.4. *For $M^\infty \coloneqq \operatorname{colim} M^n$, there is a weak equivalence $M^\infty \simeq \coprod_{g\geq 0} B\mathrm{Diff}(\Sigma_{g,1})$.*

Proof. Throughout the proof, we keep track of the dimension $n \leq \infty$. Note that there is a weak equivalence $M^n \simeq M^n_{t=1} = \coprod_{g\geq 0} M^n_{g,t=1}$, where $M^n_{t=1}$ is obtained from M^n by deforming L_t such that each $t = 1$ and the monoidal multiplication is juxtaposition followed by rescaling by a factor $\frac{1}{2}$, and $M^n_{g,t=1}$ is the subset of $M^n_{t=1}$ solely consisting of genus g surfaces. The goal is to find a weak equivalence

$$B\mathrm{Diff}(\Sigma_{g,1}) \to M^\infty_{g,t=1}.$$

To this end, we use that $M^n \simeq \widetilde{M}^n_{g,t=1}/\mathrm{Diff}(\Sigma_{g,1})$, where $\widetilde{M}^n_{g,t=1} \coloneqq \mathrm{Emb}(\Sigma_{g,1}, L_1 \times (-1,1)^{n-2})$. Thus, we will instead construct a $\mathrm{Diff}(\Sigma_{g,1})$-equivariant map $|E_\bullet \mathrm{Diff}(\Sigma_{g,1})| \to \widetilde{M}^n_{g,t=1}$, since $|B_\bullet \mathrm{Diff}(\Sigma_{g,1})| = |E_\bullet \mathrm{Diff}(\Sigma_{g,1})|/\mathrm{Diff}(\Sigma_{g,1})$, and more precisely we will do this inductively for each skeleton. Since we want to construct equivariant maps, it suffices to specify where the identity element is sent.

For convenience, let us write $G \coloneqq \mathrm{Diff}(\Sigma_{g,1})$. The classical construction of $BG = |B_\bullet G|$ says that we have $B_p G = G^p$ and $E_p G = G^p \times G$ with the right G-action, omitting face maps and degeneracies. We can view $G = E_0 G \subseteq EG$ and define a map from the 0-skeleton to $\widetilde{M}^n_{g,t=1}$, by sending $\mathrm{id}_{\Sigma_{g,1}}$ to a fixed embedding in $\mathrm{Emb}(\Sigma_{g,1}, L_1 \times (-1,1)^\infty)$, and then passing to the realization $|-|$. This choice does not matter, since $\mathrm{Emb}(\Sigma_{g,1}, L_1 \times (-1,1)^\infty) \simeq *$. Then, if we have defined the map on the $(p-1)$-skeleton, we can use the definition of $|E_\bullet G|$ to see that there is a unique G-equivariant extension

$$\begin{tikzcd}
E_pG \times \Delta^p \arrow[r, dashed] & \widetilde{M}^n_{g,t=1} \\
E_pG \times \partial\Delta^p \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

again using that $\widetilde{M}^n_{g,t=1} \simeq *$. This induces a map on the realization once more and by induction we have defined a map $EG \to \widetilde{M}^n_{g,t=1}$. Then consider the diagram of principal G-bundles

$$\begin{tikzcd}
G \arrow[r] \arrow[d, equal] & {|E_\bullet G|} \arrow[r] \arrow[d] & {|E_\bullet G|/G} \arrow[d] \\
G \arrow[r] & \widetilde{M}^\infty_{g,t=1} \arrow[r] & \widetilde{M}^\infty_{g,t=1}/G
\end{tikzcd}$$

From the homotopy long exact sequence and the Five-Lemma, we get that $B\mathrm{Diff}(\Sigma_{g,1}) \to M^\infty_{g,t=1}$ is a weak equivalence, and hence $M^\infty \simeq \coprod_{g\geq 0} M^n_{g,t=1} \xleftarrow{\simeq} \coprod_{g\geq 0} B\mathrm{Diff}(\Sigma_{g,1})$, as claimed. □

Before discussing how the replacement M relates to the map $\alpha_0\colon \psi(n,0) \to \Omega\psi(n,1)$, we need some additional definitions and intermediate technical lemmas, since we must adapt some of our older definitions to align with the new monoid. We start with a variant of the space $\psi(n,1)$.

Definition 6.5. Let $\psi'(n,1) \subseteq \Psi(\mathbb{R}^n)$ denote the space of topologically closed submanifolds $W \subseteq \mathbb{R} \times [0,1] \times (-1,1)^{n-2}$ such that there exists some $\varepsilon > 0$ with

$$W \cap (\mathbb{R} \times [0,1] \times \{0\}) = W \cap (\mathbb{R} \times [\varepsilon, 1-\varepsilon] \times \{0\}),$$

i.e. W agrees with the plane $\mathbb{R} \times [0,1] \times \{0\}$ in a neighborhood of its boundary.

Lemma 6.6. *The map $u\colon \psi(n,1) \to \psi'(n,1)$, given by $W \mapsto W \cup (\mathbb{R} \times [0,1] \times \{0\})$, is a homotopy equivalence for $n \geq 3$.*

Proof. Notice that, pictorially, the map u essentially places a disjoint plane $\mathbb{R} \times [0,1] \times \{0\}$ "below" the given manifold W. The goal is to construct a homotopy inverse v to the map u. First, fix some compact 1-manifold $S \subseteq ([-2,2]^2) \setminus ((0,1) \times (-1,1))$ with $\partial S = (\partial[0,1]) \times \{0\}$. Moreover, assume that S contains the interval $[-1,1] \times \{-2\}$ and that S is collared near its boundary.

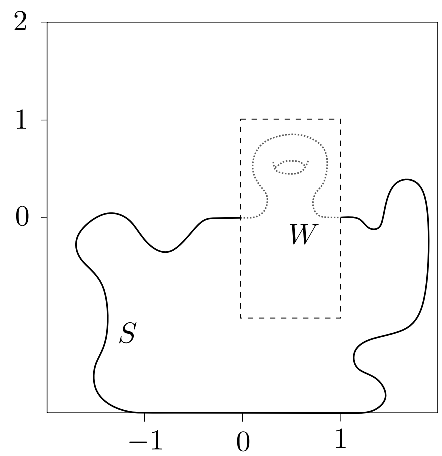

FIGURE 11. A possible choice for the fixed 1-manifold S. We later glue W into the middle rectangle of $\mathbb{R} \times S$, as indicated within the above subspace $(0,1) \times (-1,1)$.

Consider some $W \in \psi'(n,1)$. Then $W \cup (\mathbb{R} \times S \times \{0\}) \subseteq \mathbb{R}^n$ is again a smooth manifold, provided that the collar of S is contained in $[-2,2] \times \{0\}$, simply to ensure that the gluing point of $\mathbb{R} \times S \times \{0\}$ and W is smooth. Without loss of generality we can safely assume that $W \cup (\mathbb{R} \times S \times \{0\}) \subseteq \mathbb{R} \times (-3,3)^{n-1}$. Denote by $\lambda\colon (-3,3) \to (0,1)$ the fixed affine diffeomorphism $x \mapsto \frac{x+3}{6}$. Then we define the map v via

$$v\colon \psi'(n,1) \to \psi(n,1), \qquad W \mapsto (\mathrm{id} \times \lambda^{n-1})\big(W \cup (\mathbb{R} \times S \times \{0\})\big).$$

Morally, we thus think of v as gluing on a strip to the manifold W and then appropriately rescaling along the $e_2, \ldots, e_n$-axes, so the result is contained in $\mathbb{R} \times (0,1)^{n-1}$, as desired. Now consider the composite $u \circ v\colon \psi'(n,1) \to \psi(n,1) \to \psi'(n,1)$. An element in its image is sketched in the first frame of Figure 12 below:

We explain pictorially how to construct a deformation from $(u \circ v)(W)$ to W, as an explicit description would be very lengthy, dense with additional notation and would serve no actual

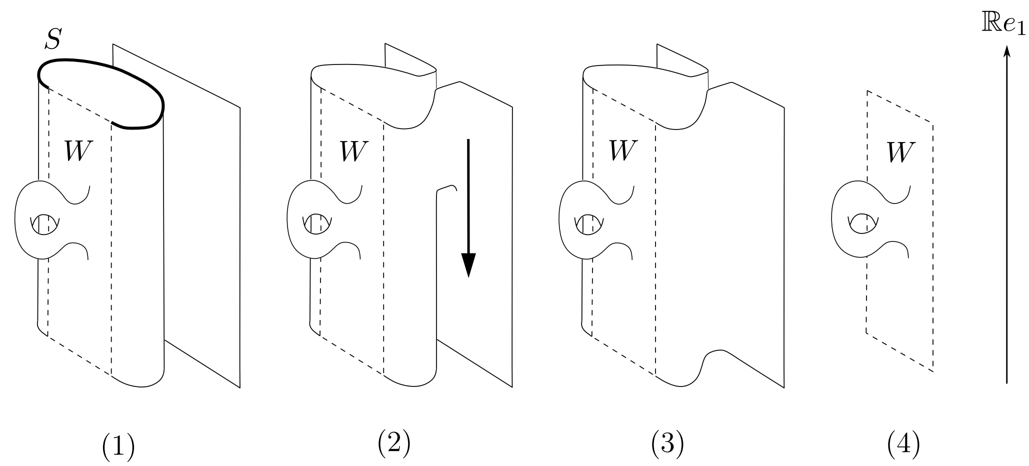

FIGURE 12. A path from $(u \circ v)(W)$ to W. For added simplicity, the arc S is roughly chosen to be a semi-circle. The second frame depicts the "saddle point"-move.

benefit to our understanding. Galatius describes the process in [Gal12, p. 24] as follows: "The image of W contains a piece which, up to scaling of coordinates, is the disjoint union of $\mathbb{R} \times S \times \{0\}$ and a plane. Then slide down a 'saddle point' to get a manifold which can be canonically deformed to the original W", namely by retracting the "flaps" on either side of W. Notice that we used that S contains the interval $[-1, 1] \times \{-2\}$ in order to perform the saddle-point trick in a standard way.

Similar to the first case, we consider the composite $v \circ u$ and find that we can easily define a canonical retraction from $(v \circ u)(W)$ to W, by capping off the "cylinder" below W and pushing the result to $-\infty$ in the e_1-direction, thus only leaving W. □

Recall that our definition of the map $\mu \colon BM \to \psi(n, k)$ included the case $k = 0$. However, an important concept was that of a *good* map, the definition of which we will slightly modify to suit this case better.

Definition 6.7. Let X be a smooth manifold and $f \colon X \to \psi'(n, 1)$ a smooth map. For $a \in \mathbb{R}$, define a subspace

$$X_a \coloneqq \left\{ x \in X \;\middle|\; f(x) \cap (\{a\} \times \mathbb{R}^{n-1}) = \{a\} \times [0, 1] \times \{0\} \right\}.$$

Moreover, denote by $X^{\mathrm{nc}} \subseteq X$ the "non-compact" subspace consisting precisely of all $x \in X$ such that no path component of $f(x) \subseteq \mathbb{R} \times [0, 1] \times \mathbb{R}^{n-2}$ is compact. We declare f to be a *good* map if $X = X^{\mathrm{nc}} = \bigcup_{a \in \mathbb{R}} \mathrm{int}(X_a)$.

Remark 6.8. We can easily give the condition in the definition of X_a an intuitive geometric meaning. The condition simply states that if we slice $f(x)$ at the hyperplane $\{a\} \times \mathbb{R}^{n-1}$, we only see the standard interval $[0, 1]$, and we do not see anything more interesting like genus, even though $f(x)$ can have much more structure outside of the hyperplane.

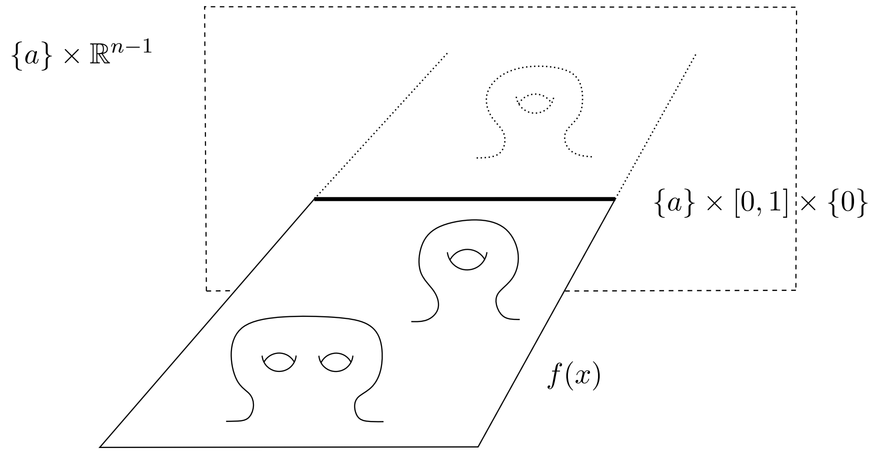

FIGURE 13. The geometric intuition behind the subspace X_a.

The following proposition is an adaptation of Lemma 5.18:

Proposition 6.9. *Let $f\colon X \to \psi'(n,1)$ be a smooth map for some compact manifold X and let $n \geq 5$. Then f is smoothly homotopic to a good map in the sense of Definition 6.7, and the homotopy can be taken constant on a neighborhood of a closed set on which f is already good.*

Proof sketch. The first step is to homotope f in such a way that for any $f(x) \subseteq \mathbb{R}\times(-1,1)^{n-1}$ we have $f(x) \cap (\{0\} \times (-1,1)^{n-1}) = \emptyset$. To be more precise, fix a diffeomorphism $\varphi\colon \mathbb{R} \to (-\infty, 0)$, e.g. $\varphi(s) \coloneqq -\exp(-s)$. Then define a homotopy

$$h\colon X \times I \to \psi'(n,1), \qquad (x,t) \mapsto \big(((1-t)\,\mathrm{id}_{\mathbb{R}} + t\varphi) \times \mathrm{id}_{\mathbb{R}^{n-1}}\big)(f(x)).$$

One briefly checks that, indeed, $h(-,0) = f$ and $h(-,1) = (\varphi \times \mathrm{id}_{\mathbb{R}^{n-1}})(f(x))$ does not intersect $\{0\} \times \mathbb{R}^{n-1}$. Thus, we can assume without loss of generality that $f(x)$ does not intersect this hyperplane at the origin. Moreover, for each $x \in X$ we can choose some $\varepsilon_x > 0$ such that $f(x) \cap \mathrm{pr}_1^{-1}(-\varepsilon_x, \varepsilon_x) = \emptyset$, where $\mathrm{pr}_1\colon \mathbb{R}^n \to \mathbb{R}$ is the projection onto the first coordinate, and there is an open neighborhood $U_x \ni x$ for which this still holds. The set $\{U_x\}_{x\in X}$ is an open covering of X, so we can choose a finite subcover $\{U_i\}_i$ and set $\varepsilon \coloneqq \min\{\varepsilon_i\}$. Then there is still no path component of $f(x)$ contained in $(-\varepsilon, \varepsilon) \times \mathbb{R}^{n-1}$ for any $x \in X$. Next, pick an isotopy of embeddings $e\colon \mathbb{R}\times I \to \mathbb{R}$ such that $e(-,0) = \mathrm{id}_{\mathbb{R}}$ and $\mathrm{im}(e(-,1)) \subseteq (-\varepsilon, \varepsilon)$ for some fixed $\varepsilon > 0$. An explicit such isotopy is given, for instance, by the formula $e(x,s) \coloneqq (1-s)x + s\frac{2\varepsilon}{\pi}\arctan(x)$. Define another homotopy by

$$H\colon X \times I \to \psi'(n,1), \qquad (x,t) \mapsto (e(-,t) \times \mathrm{id}_{\mathbb{R}^{n-1}})^{-1}(f(x)),$$

and notice that H (and the earlier homotopy h) are continuous by appealing to Theorem 2.15. More importantly, observe that $H(x,1)$ has no compact path-components, as the deformation "pushes everything away" to infinity by definition.

The next goal is to homotope f such that for all $x \in X$ there exists a $t \in \mathbb{R}$ with $f(x) \cap \mathrm{pr}_1^{-1}(t)$ being diffeomorphic to a closed interval. If we pick an arbitrary $t \in \mathbb{R}$, then the 1-manifold $f(x) \cap \mathrm{pr}_1^{-1}(t)$ will be a disjoint union of an interval and a finite number of circles $C_i \cong S^1$, by the Classification Theorem. We consider two cases: if $f(x) \cap \mathrm{pr}_1^{-1}(t)$ has no circle-components, we are done. Otherwise, we proceed by fixing a point y_i in each circle-component C_i and connecting it to a point in the standard boundary via a "tube", i.e. to a point in a neighborhood of the intersection $W \cap (\mathbb{R} \times \{0,1\} \times \{0\})$, mimicking a connected sum operation.

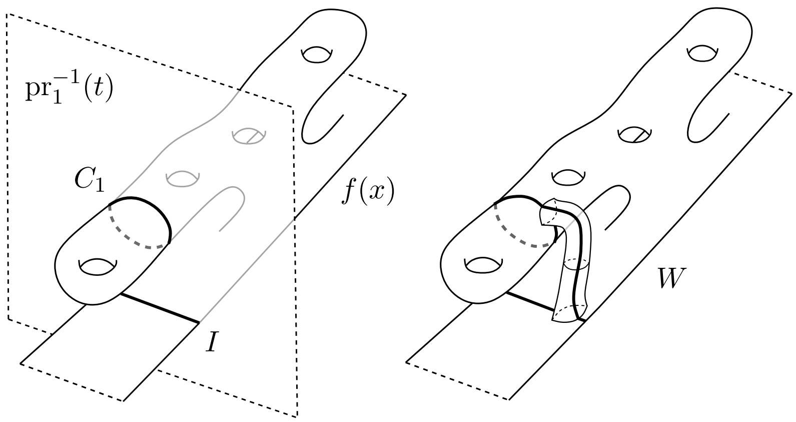

FIGURE 14. Connecting each circle component C_i to the interval I by gluing a small cylinder.

After this gluing construction, the 1-manifold $f(x) \cap \mathrm{pr}_1^{-1}(t)$ is diffeomorphic to a simple interval, provided that $n \geq 4$, because this allows enough space to choose the tube to be contained in the surrounding Euclidean space. Let W denote the manifold resulting from this process, i.e. it satisfies that $W \cap \mathrm{pr}_1^{-1}(t) \cong [0,1]$. Lastly, we need to describe a path from $f(x)$ to W in $\psi'(n,1)$. This is done by the entire process described above, except that we "slide" the tube to infinity inside $\mathbb{R} \times [0,1] \times (-1,1)^{n-2}$, which is possible, since $f(x)$ had no compact components. Thus, $W \cap \mathrm{pr}_1^{-1}(t)$ is diffeomorphic to an interval, as needed, and we can additionally deform this interval to agree with $\{t\} \times [0,1] \times \{0\}$ via an isotopy of embeddings $W \times I \to \mathbb{R}^n$, since the space $\mathrm{Emb}(W, \mathbb{R}^n)$ is path-connected for $n \geq 5$ by the Whitney–Wu Theorem [Wu58], and hence f can be pointwise homotoped to a good map. $\square$

Remark 6.10. Some caution is required at the end of the above proof. In general, we need a *homotopy* of f to a good map. The issue is that the above proof only concerns a pointwise construction from $f(x)$ to a good version W, for a fixed x. Essentially, we want to run through the above process simultaneously for each $x \in X$, but this is more involved and is discussed in detail in Corollary 4.17 of [GR10].

The natural next step is to discuss how the replacement M relates back to the constituent $\alpha_0 \colon \psi(n,0) \to \Omega\psi(n,1)$ of the scanning map, since this was ultimately the map that we are interested in. A first step is to introduce a new map $c \colon M \to \psi(n,0)$, which may be roughly thought of as an "inverse" to the map $\iota \colon \psi(n,k) \to M$ from Lemma 5.13:

Definition 6.11. Consider the monoid $M_{t=1} \simeq M$, obtained by deforming M to have $t = 1$ for each manifold, i.e. each element $(1, W) \in M_{t=1}$ can be thought of as having $\partial W = \partial([0,1]^2)$. Moreover, the monoidal multiplication includes a rescaling by a factor of $\frac{1}{2}$ to ensure well-definedness. For each $W \in M_{t=1}$ we consider a canonical capping-off by a manifold C diffeomorphic to a disc D^2. We choose this operation in a standard way such that the result W_C is contained in $(-1,2)^2 \times (-2,1)^{n-2}$ and C only intersects W at its boundary. Consider affine linear diffeomorphisms $f_1 \colon (-1,2) \to (0,1)$ and $f_2 \colon (-2,1) \to (0,1)$ given by $x \mapsto \frac{x+1}{3}$ and $x \mapsto \frac{x+2}{3}$ respectively. Then define the map c by

$$c \colon M_{t=1} \to \psi(n,0), \qquad W \mapsto \big(f_1^2 \times f_2^{n-2}\big)(W_C),$$

Pictorially, we think of c as capping off W (in order to close the boundary) and rescaling appropriately to be contained in $(0,1)^n$.

With this definition, we can finally discuss the connection between the monoid M and the map α_0. Essentially, we obtain a weaker version of the commutative diagram (11) above.

Remark 6.12. The map $\overline{c} \colon M_{t=1} \to \Psi(\mathbb{R}^n)$, given by $W \mapsto W_C$, is continuous. To see this, consider the typical neighborhood $U = \{V \in \Psi(\mathbb{R}^n) \mid V \text{ is a section of } \tau W_C\}$. Then the set

$$N(W) \coloneqq \big\{V' \in M_{t=1} \mid V' \text{ is a section of } \tau W \text{ and } V' \cap L_1 \times \{0\} = V' \cap \partial L_1 \times \mathbb{R}^{n-2}\big\}$$

is a neighborhood of W such that $\overline{c}(N(W)) \subseteq U$. Notice that $c = (f_1^2 \times f_2^{n-2}) \circ \overline{c}$, so by Theorem 2.15 the above "capping-off" map c is continuous.

Theorem 6.13. *With $v \colon \psi'(n,1) \to \psi(n,1)$ as in the proof of Lemma 6.6, the diagram*

$$\begin{tikzcd} M \arrow[r, "\beta"] \arrow[d, "c"'] & \Omega BM \arrow[d, "\Omega v \circ \Omega\mu"] \\ \psi(n,0) \arrow[r, "\alpha_0"] & \Omega\psi(n,1) \end{tikzcd}$$

is homotopy commutative for any $n \geq 3$. Moreover, the map $v \circ \mu \colon BM \to \psi(n,1)$ is a weak equivalence for $n \geq 5$.

Proof. First, replace M by the homotopy equivalent space $M_{t=1}$, and notice that homotopy commutativity is equivalent to showing that $\Omega\mu \circ \beta$ is canonically homotopic to $\Omega u \circ \alpha_0 \circ c$, using the homotopy equivalence from Lemma 6.6. Evaluating on elements $W \in M_{t=1}$ and $s \in \mathbb{R}^+$, a direct pointwise computation shows that

$$(\Omega\mu \circ \beta)(W)(s) = \begin{cases} W + se_1, & \text{for } s \in \mathbb{R} \\ \emptyset, & \text{for } s = \infty \end{cases}$$

$$(\Omega u \circ \alpha_0 \circ c)(W)(s) = \begin{cases} \Big(\big(f_1^2 \times f_2^{n-2}\big)(W_C) + se_1\Big) \cup \mathbb{R} \times [0,1] \times \{0\}, & \text{for } s \in \mathbb{R} \\ \emptyset, & \text{for } s = \infty \end{cases}$$

Thus, we must show that we can canonically deform W into $\big(f_1^2\times f_2^{n-2}\big)(W_C)\cup\mathbb{R}\times[0,1]\times\{0\}$. The following sequence of frames serves as an illustration of each step in this standard deformation:

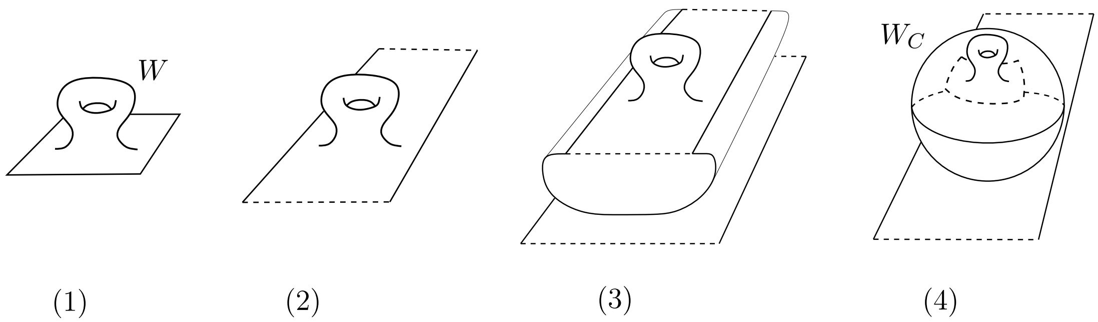

FIGURE 15. A canonical deformation from W to $\big(f_1^2 \times f_2^{n-2}\big)(W_C) \cup \mathbb{R} \times [0,1] \times \{0\}$.

First, we "elongate" the square $[0,1]^2$ in the positive/negative e_1-direction, using that there is a homotopy equivalence $\{0,1\} \times [0,1] \simeq ((-\infty,0] \cup [1,\infty)) \times [0,1]$, to obtain a manifold W'. Next, we use the canonical path from W' to $(u \circ v)(W')$ from the proof of Lemma 6.6 to arrive at the third frame, i.e. W' with some $\mathbb{R} \times S$ attached and with some disjoint strip below. Lastly, we pull back a capped-off version of the "cylinder" from $\pm\infty$ until we are left with the shrunk version of W_C with a disjoint strip, which is precisely $\big(f_1^2 \times f_2^{n-2}\big)(W_C) \cup \mathbb{R} \times [0,1] \times \{0\}$.

For the second claim, we use Proposition 6.9 to assume that the map $\mu\colon BM \to \psi'(n,1)$ is homotopic to a good map in the sense of Definition 6.7 for $n \geq 5$. Then, we continue analogously to Proposition 5.16, i.e. we use the fact that μ factors as $\mathrm{ft} \circ \overline{\mu}\colon BM \to BP \to \psi'(n,1)$ to see that μ is a weak equivalence by Lemmas 5.21 and 5.22, and afterwards we post-compose with the homotopy equivalence v from Lemma 6.6, which finishes the proof. $\square$

Remark 6.14. Perhaps the reader has noticed that the map c is *not*, in fact, a map of monoids. The issue is that the multiplication in $M_{t=1}$ is essentially the boundary connected sum operation, whereas the multiplication in $\psi(n,0)$ is given by disjoint union, which entails that $c(mm')$ consists of a single path component, while $c(m)c(m')$ consists of two components!

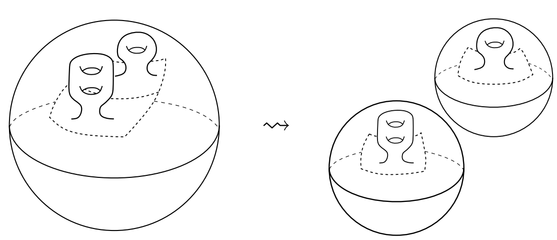

FIGURE 16. Comparing the two manifolds $c(mm')$ and $c(m)c(m')$.

To remedy this, we simply pass to the space $\Omega\psi(n,1)$ in the lower right corner of the diagram. The fact that $\psi(n,1)$ contains one dimension which is unbounded allows us to construct a deformation from $c(m)c(m')$ to $c(mm')$ by using a variant of the saddle-point trick from Lemma 6.6 above.

6.2. Group-Completion – Putting the Pieces Together

The remainder of this section is concerned with finishing the proof of the Madsen–Weiss Theorem. We almost have all ingredients necessary to conclude the Theorem—only one important observation needs to be made. Notice that, as mentioned earlier, the monoid M is *not* group-like by Lemma 6.3, which means that $\beta\colon M \to \Omega BM$ is not a weak equivalence. Moreover, we do not know, a priori, whether the map $c\colon M \to \psi(n,0)$ is a weak equivalence or has any similar "nice" properties, and since its definition was rather cluttered, we will focus on β. We can invoke (a version of) the Group-Completion Theorem to extract some homological information about the map β, and it will turn out later that this suffices to conclude the main proof.

Definition 6.15. Let M be as above and let $m_1 \in M$ denote a surface of genus 1. Denote by $M_\infty \coloneqq \mathrm{Tel}\big(M \xrightarrow{\cdot m_1} M \xrightarrow{\cdot m_1} \cdots\big) \cong (M \times \mathbb{N} \times [0,1])/((m,n,1) \sim (mm_1, n+1, 0))$ the mapping telescope of the direct system given by multiplying by m_1 in each step. We view $M \subseteq M_\infty$ by identifying M with the subspace $M \times \{0\} \times \{0\}$.

Theorem 6.16 (Group-Completion; [Hat14], Appendix D)**.** *Let M be a homotopy commutative topological monoid with $\pi_0 M = \mathbb{N}$. Then the standard map $\beta\colon M \to \Omega BM$ extends to an $H_*(-;\mathbb{Z})$-isomorphism $\beta_\infty\colon M_\infty \to \Omega BM$.*

To finish the proof of the Madsen–Weiss Theorem, we shall use the Group-Completion Theorem for the monoid $M = M^\infty \simeq \coprod_{g\geq 0} B\mathrm{Diff}(\Sigma_{g,1})$, from Proposition 6.4, not to be confused with the mapping telescope $M_\infty = (M^\infty)_\infty$, where we admit to cumbersome notation. Next, we follow the same argument as in section 12.4 of [BB24]: let $M^g \coloneqq B\mathrm{Diff}(\Sigma_{g,1})$, so that $M \simeq \coprod_{g\geq 0} M^g$. In general, we can write $M_\infty = \coprod_{g\in\mathbb{Z}} M_{\infty,g}$, where

- $M_{\infty,g} \coloneqq \mathrm{Tel}\big(M^g \xrightarrow{\cdot m_1} M^{g+1} \xrightarrow{\cdot m_1} M^{g+2} \xrightarrow{\cdot m_1} \cdots\big)$ for $g \geq 0$, and
- $M_{\infty,g} \coloneqq \mathrm{Tel}\big(\emptyset \to \ldots \to \emptyset \to M^0 \xrightarrow{\cdot m_1} M^1 \xrightarrow{\cdot m_1} \cdots\big)$ for $g < 0$, where the space M^0 appears in the $\max\{-g, 0\}$-th place.

Note that for any two $g < g' \in \mathbb{Z}$, the canonical inclusion $M_{\infty,g'} \hookrightarrow M_{\infty,g}$ is a deformation retraction by collapsing the first part of right-hand mapping telescope, and thus we can identify $M_\infty \simeq \mathbb{Z} \times M_{\infty,0}$. Moreover, in our case, we can compute $M_{\infty,0}$ to be

$$\begin{aligned} M_{\infty,0} &= \mathrm{Tel}\big(M^0 \xrightarrow{\cdot m_1} M^1 \xrightarrow{\cdot m_1} M^2 \xrightarrow{\cdot m_1} \cdots\big) \\ &= \mathrm{Tel}\big(B\mathrm{Diff}(\Sigma_{0,1}) \xrightarrow{\cdot m_1} B\mathrm{Diff}(\Sigma_{1,1}) \xrightarrow{\cdot m_1} B\mathrm{Diff}(\Sigma_{2,1}) \xrightarrow{\cdot m_1} \cdots\big) \eqqcolon B\mathrm{Diff}_\infty\,. \end{aligned}$$

Then we consider the following (homotopy) commutative diagram in the case $n = \infty$

$$\begin{tikzcd} B\mathrm{Diff}_\infty \arrow[d, hook, "\mathrm{incl}_0"'] & \coprod_{g\geq 0} B\mathrm{Diff}(\Sigma_{g,1}) \arrow[l, "(\star)"'] \arrow[r, "\simeq"] \arrow[dd] & M \arrow[r, "c"] \arrow[d, "\beta"'] & \psi(\infty, 0) \arrow[d, "\alpha_0"] \\ \mathbb{Z} \times B\mathrm{Diff}_\infty \arrow[r, "\simeq"'] & M_\infty \arrow[r, "\beta_\infty", "H_*(-;\mathbb{Z})\text{-isom.}"'] & \Omega BM \arrow[r, "\Omega v \circ \Omega \mu", "\simeq"'] & \Omega\psi(\infty, 1) \end{tikzcd} \tag{14}$$

where the second vertical map from the left is essentially defined as the extension $M \hookrightarrow M_\infty$ from the Group-Completion Theorem, so the middle square commutes by definition. The map decorated by $(\star)$ is given by the unique map into the mapping telescope for each component. By the fact that $B\mathrm{Diff}_\infty$ and M are embedded in their telescopes in the 0-th position, it follows that the left-hand square commutes, and the right-hand square is simply the one from Theorem 6.13 above.

In the final important step, we now restrict to a single path-component $B\mathrm{Diff}(\Sigma_{g,1})$ of the space $\coprod_{g\geq 0} B\mathrm{Diff}(\Sigma_{g,1})$ (with slight abuse of notation). It follows from Harer's Stability Theorem 3.9 that the map $(\star)\colon B\mathrm{Diff}(\Sigma_{g,1}) \to B\mathrm{Diff}_\infty$ is an isomorphism in $H_*(-;\mathbb{Z})$ for $* \leq \frac{2}{3}(g-1)$, since $B\mathrm{Diff}_\infty$ was precisely defined as the colimit of the direct system consisting of multiplying by m_1, which corresponds to the map $\vartheta\colon \Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1}$ attaching a surface of genus 1 at the boundary. Next, we map via incl_0, so we land in the basepoint-component and hence stay in any following basepoint-component. Consequently, the composite

$$B\mathrm{Diff}(\Sigma_{g,1}) \xrightarrow{\mathrm{incl}_0 \circ (\star)} (M_\infty)_\bullet \xrightarrow{\beta_\infty} \Omega_\bullet BM \xrightarrow{\Omega v \circ \Omega \mu} \Omega_\bullet \psi(\infty, 1)$$

is an $H_*(-;\mathbb{Z})$-isomorphism in the range $* \leq \frac{2}{3}(g-1)$. Combining this with Corollary 5.23, this essentially concludes the proof of the Madsen–Weiss Theorem. Technically, we need to be a little more careful, since we claimed that the restricted scanning map $\alpha^g\colon B\mathrm{Diff}(\Sigma_g) \to \Omega^\infty_\bullet \mathbf{\Psi}$, i.e. with $r = 0$ many boundary components, is the map inducing the homology-isomorphism in this range. But notice that we have a commutative diagram

$$\begin{tikzcd} B\mathrm{Diff}(\Sigma_g) \arrow[r, hook] & \psi(\infty, 0) \arrow[r, "\alpha_0"] & \Omega\psi(\infty, 1) \arrow[r, "\simeq"] & \Omega^\infty \mathbf{\Psi} \\ B\mathrm{Diff}(\Sigma_{g,1}) \arrow[u, "B\delta_g"] \arrow[r, hook, "\mathrm{incl}"] & \coprod_{g\geq 0} B\mathrm{Diff}(\Sigma_{g,1}) \arrow[u, "c"'] & & \end{tikzcd}$$

where the square commutes by (the very geometric) definition of c, and the entire upper composite is precisely the map $\alpha^g\colon B\mathrm{Diff}(\Sigma_g) \to \Omega^\infty_\bullet \mathbf{\Psi}$, using the disjoint-union-description (3) for the space $\psi(\infty, 0) = B_\infty$. Thus, in $H_*(-;\mathbb{Z})$, the homomorphism $H_*(\alpha^g)$ factors through $H_*(\delta_g)^{-1}$ (which is an isomorphism by Harer's Stability Theorem, since $* \leq \frac{2}{3}(g-1) \leq \frac{2}{3}g$), the canonical inclusion $H_*(\mathrm{incl})$ and the homomorphism $H_*(c)$. By the discussion on diagram (14) above, this composite (in particular the part $H_*(\alpha_0) \circ H_*(c) \circ H_*(\mathrm{incl})$) is an $H_*(-;\mathbb{Z})$ isomorphism, provided that $* \leq \frac{2}{3}(g-1)$.

To conclude this chapter, we present a large diagram that serves as a roadmap for the entire proof of the Madsen–Weiss Theorem. One traces the diagram in $H_*(-;\mathbb{Z})$, beginning at $B\mathrm{Diff}(\Sigma_g)$ and ending at $\Omega^n\Psi(\mathbb{R}^n)$, following the dashed arrows along the way. Since the diagram is not well-defined at every stage, it should be regarded only as a schematic summary to gain a bird's-eye perspective, rather than a precise statement.

$$
\begin{tikzcd}
 & \mathbb{Z}\times B\mathrm{Diff}_\infty \arrow[d, "\simeq"'] & B\mathrm{Diff}_\infty \arrow[l, hook', "\mathrm{incl}_0"'] & \\
 & M_\infty \arrow[d, "\beta_\infty"'] & \coprod_{g\geq 0} B\mathrm{Diff}(\Sigma_{g,1}) \arrow[l] \arrow[u, "(\star)"'] \arrow[d, "\simeq"] & B\mathrm{Diff}(\Sigma_{g,1}) \arrow[l, hook', "\mathrm{incl}"'] \arrow[dd, "B\delta_g"] \\
 & \Omega BM \arrow[d, "\Omega v\circ\Omega\mu"', "\simeq"] & M \arrow[l, "\beta"'] \arrow[d, "c"] & \\
\Omega M^{\mathrm{old}} \arrow[d, "\Omega\beta"', "\simeq"] & \Omega\psi(n,1) \arrow[l, "\Omega\iota"', "\simeq"] \arrow[d, dashed, "\Omega\alpha_1"] & \psi(n,0) \arrow[l, dashed, "\alpha_0"'] & B\mathrm{Diff}(\Sigma_g) \arrow[l, dashed, hook'] \\
\Omega^2 BM^{\mathrm{old}} \arrow[r, "\Omega^2\mu", "\simeq_\bullet"'] & \Omega^2\psi(n,2) \arrow[r, "\Omega^2\iota", "\simeq"'] \arrow[d, dashed, "\Omega^2\alpha_2"] & \Omega^2 M^{\mathrm{old}} \arrow[d, "\simeq"', "\Omega^2\beta"] & \\
\Omega^3 M^{\mathrm{old}} \arrow[d, "\Omega^3\beta"', "\simeq"] & \Omega^3\psi(n,3) \arrow[l, "\Omega^3\iota"', "\simeq"] \arrow[d, dashed, "\Omega^3\alpha_3"] & \Omega^3 BM^{\mathrm{old}} \arrow[l, "\Omega^3\mu"', "\simeq"] & \\
\Omega^4 BM^{\mathrm{old}} \arrow[r, "\Omega^4\mu", "\simeq"'] & \Omega^4\psi(n,4) \arrow[r, "\Omega^4\iota", "\simeq"'] \arrow[d, dashed] & \Omega^4 M^{\mathrm{old}} \arrow[d] & \\
 & \vdots \arrow[d, dashed] & \vdots \arrow[d] & \\
 & \Omega^n\Psi(\mathbb{R}^n) & \Omega^n BM^{\mathrm{old}} \arrow[l, "\Omega^n\mu"', "\simeq"] &
\end{tikzcd}
$$

To give a few more details, notice that the diagram encapsulates almost all the relevant maps from the proof of the Madsen–Weiss Theorem that were considered throughout the thesis.

We only change some notation to denote by M^{old} the "old" monoid from Definition 5.12. It is important to note, however, that the Madsen–Weiss Theorem is concerned specifically with the case $n = \infty$, and hence one may have noticed that the map $B\mathrm{Diff}(\Sigma_g) \hookrightarrow \psi(n, 0)$, as it stands, is not well-defined in the finite case and only serves as a placeholder. We simply write n to emphasize that we factored the map α through all loop spaces $\Omega^k\psi(n, k)$ and through the diagrams (11), which contained the bulk of the technical work. Moreover, we decorate the map $\Omega^2\mu$ with the symbol $\simeq_\bullet$, since by Proposition 5.16 it is a weak equivalence only onto the basepoint component of $\psi(n, 2)$.

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