The Lyndon-Hochschild-Serre Spectral Sequence

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Abstract

This thesis is dedicated to studying the homological Lyndon-Hochschild-Serre spectral sequence, first introduced by Roger C. Lyndon in 1948, and rigorously characterized by Gerhard Hochschild and Jean-Pierre Serre in the 1950s. More precisely, our goal is to develop a thorough understanding of group homology and spectral sequences before tackling the proof of the main theorem and discussing various concrete examples and interesting algebraic applications.

Zusammenfassung

Diese Bachelorarbeit beschäftigt sich mit der homologischen Lyndon-Hochschild-Serre Spektralsequenz, die erstmals 1948 von Roger C. Lyndon eingeführt und in den 1950er Jahren von Gerhard Hochschild und Jean-Pierre Serre auf ein rigoroses Fundament gestellt wurde. Insbesondere ist das Ziel dieser Arbeit, ein gutes Verständnis über Gruppenhomologie und Spektralsequenzen zu entwickeln, bevor wir den Beweis des Hauptresultats in Angriff nehmen. Abschließend werden noch einige konkrete Beispiele und interessante algebraische Anwendungen diskutiert.

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Introduction

Historically, the starting point of this thesis is the development of group (co-)homology in the 1930s by Witold Hurewicz. Among other things, Hurewicz studied *aspherical spaces* and showed that the homotopy type of such spaces X is fully characterized by their fundamental group $\pi = \pi_1(X)$. In particular, the singular homology groups of X depend only on π, allowing for an interpretation of the homology of X as the homology of the group π, denoted by $H_*(\pi)$. Therefore, it seemed reasonable to develop a rigorous definition for the (co-)homology of an arbitrary group. By the mid 1940s, a purely algebraic definition of group (co-)homology had been established and papers published by Hopf and Eilenberg-MacLane had made substantial advances in the field. Soon after, Lyndon realized in his dissertation, that it was possible to use spectral sequences to compute group (co-)homology. In 1953, Hochschild and Serre published their paper [HS53], clarifying the ideas of Lyndon and giving rise to what is now known as the Lyndon-Hochschild-Serre (LHS) spectral sequence, which constitutes the main subject of study in this thesis.

The main reason for studying the LHS spectral sequence is that we are interested in thoroughly understanding the (co-)homology of groups, which is a concept that is still relevant in today's research. For instance, group (co-)homology allows for a characterization of group extensions, see [Bro82, Chapter IV], and it can be used to give a characterization of finite groups that admit free actions on spheres, as in [Löh19, Chapter 4.3]. Of course, group (co-)homology can also simply be used to distinguish groups from one another. For these reasons, having a tool at hand—the LHS spectral sequence—with which to compute group (co-)homology, is highly convenient.

The thesis is structured as follows: After some preliminary results, Chapter 2 introduces group homology and aims to state some basic results and notions that will be relevant later on. Chapter 3 surveys the strong connection between the strictly algebraic definition of group homology and the usual definition of singular homology.

Chapter 4 is concerned with spectral sequences—a powerful tool widely used in topology and algebra, and the main tool necessary to understand the main theorem. As the theory behind spectral sequences is not the main focus of this thesis, the goal of this chapter is to give a rough overview, leaving out details where we deem appropriate. Chapter 5 is concerned with the formulation and proof of the main theorem: The LHS spectral sequence. Finally, the last two chapters deal with direct consequences of the LHS spectral sequence. More precisely, Chapter 6 illustrates two explicit non-trivial computations of group homology, while Chapter 7 aims to use the LHS spectral sequence to obtain purely algebraic and group theoretic results, which would have been harder to derive without the framework surrounding the main theorem.

An important note is that this thesis is restricted to the case of group *homology*. However, as expected, most statements do in fact have their usual "dual" cohomological version.

1 Preliminaries

We begin by giving a variety of definitions and basic results in order to establish a foundation on which to work throughout the thesis. Most proofs will be omitted.

1.1 Notation and Conventions

We set a few conventions, which will be used implicitly.

- Groups G are always assumed to be discrete, allowing us to identify the classifying space BG and the Eilenberg-MacLane space $K(G,1)$. This is a consequence of the long exact sequence of homotopy groups associated to the fibration $G \hookrightarrow EG \to BG$, using that G is discrete, EG is contractible and the fact that Eilenberg-MacLane spaces are unique up to homotopy equivalence, see [Hat10, Proposition 4.30].
- All maps between topological spaces are assumed to be continuous.

1.2 Group Rings

One of the most important algebraic structures which is used throughout the entire thesis is the concept of a group ring. Essentially, we extend a group G in some way to obtain a ring structure, allowing us to give a well-defined definition of group homology later on.

Definition 1.1 (Group Ring)**.** Let G be a (multiplicative) group. Then the **group ring** $\mathbb{Z}G := \mathbb{Z}[G]$ is defined to be the free $\mathbb{Z}$-module generated by elements of G. To be precise, an element in $\mathbb{Z}G$ can be written as a formal sum $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{Z}$ is non-zero for only finitely many $g \in G$. A priori, it is not clear that $\mathbb{Z}G$ is indeed a ring. However, we have two operations as follows: Addition in $\mathbb{Z}G$ is defined point-wise, and multiplication is explicitly given by the convolution

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) := \sum_{g,h\in G} (a_g b_h) gh.$$

This provides $\mathbb{Z}G$ with a well-defined ring structure.

Example 1.2. The following are a few straightforward examples of group rings:

- Let $\{1\}$ denote the trivial group. Then we have a ring isomorphism $\mathbb{Z}[\{1\}] \cong \mathbb{Z}$.
- If G is infinite cyclic with generator t, $\mathbb{Z}G$ has a $\mathbb{Z}$-basis given by $\{t^i\}_{i\in\mathbb{Z}}$, and hence we can identify $\mathbb{Z}G$ with the Laurent polynomials $\mathbb{Z}[t,t^{-1}]$.
- If $G = \mathbb{Z}/n$ for some $n \geq 2$, then $\mathbb{Z}[\mathbb{Z}/n]$ has a $\mathbb{Z}$-basis given by $\{1, t, \ldots, t^{n-1}\}$, and hence $\mathbb{Z}[\mathbb{Z}/n] \cong \mathbb{Z}[t]/(t^n - 1)$.

Definition 1.3 (Augmentation Ideal)**.** For a group G, define the **augmentation map** $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ by sending a generator $g \in G$ to 1. We define the **augmentation ideal** IG as $\ker(\varepsilon) = \{x \in \mathbb{Z}G \mid \varepsilon(x) = 0\}$, sometimes just written as I.

Lemma 1.4. *Let G be the free group on some set S. Then IG is a free $\mathbb{Z}G$-module and we have $IG = \langle s - 1 \mid s \in S\rangle$, i.e. IG is free with basis $\{s - 1 \mid s \in S\}$.*

Proof. See [Wei94, Proposition 6.2.6] for a proof. □

1.3 $\mathbb{Z}G$-Modules, Tensor Products and Coinvariants

Definition 1.5 ($\mathbb{Z}G$-Module)**.** Let M be an abelian group. Then M is a **(left) $\mathbb{Z}G$-module**, if there exists a map $\mathbb{Z}G \times M \to M$, $(r, m) \mapsto r \cdot m$, with $e \cdot m = m$, $r \cdot (m + m') = r \cdot m + r \cdot m'$, $(r + r') \cdot m = r \cdot m + r' \cdot m$ and $(rr') \cdot m = r \cdot (r' \cdot m)$ for all $r, r' \in G$ and $m, m' \in M$, where $e \in \mathbb{Z}G$ is the identity.

In practice, it suffices to consider a map $G \times M \to M$, $(g, m) \mapsto g \cdot m$, with the same properties as above. Then one can extend this map uniquely to a map $\mathbb{Z}G \times M \to M$, since G generates the ring $\mathbb{Z}G$. For this reason, $\mathbb{Z}G$-modules are sometimes simply referred to as "G-modules".

Notice that M is not automatically a *right* $\mathbb{Z}G$-module, since group rings need not be commutative, and hence we must be careful when taking tensor products. However, there is an easy solution to this inconvenience, namely the involution $g \mapsto g^{-1}$ that comes with G. In this way, we can regard any left $\mathbb{Z}G$-module as a right $\mathbb{Z}G$-module by setting $m \cdot g := g^{-1} \cdot m$ for any $m \in M$ and $g \in G$. Now the expression $M \otimes_{\mathbb{Z}G} N$ is well-defined, if M and N are $\mathbb{Z}G$-modules. If there is no confusion, we simply write $M \otimes_G N := M \otimes_{\mathbb{Z}G} N$ and $M \otimes N := M \otimes_{\mathbb{Z}} N$.

Definition 1.6. Let G be a group and let M be a $\mathbb{Z}G$-module. We define two groups:

- $M^G := \{m \in M \mid g \cdot m = m \text{ for all } g \in G\}$ is the group of **invariants** of M.
- $M_G := M/\langle gm - m \mid g \in G,\ m \in M\rangle$ is the group of **coinvariants** of M. Essentially, we "divide out" the G-action on M.

Note that taking invariants $(-)^G$ and coinvariants $(-)_G$ are functors $\mathbb{Z}G$-**Mod** $\to \mathbb{Z}$-**Mod**.

Proposition 1.7. *Let G be a group and M be a $\mathbb{Z}G$-module. Then there exists a group isomorphism $M_G \cong \mathbb{Z} \otimes_G M$.*

Proof. A short proof can be found in [Bro82, p. 34]. □

Another useful observation is the following: $M \otimes_G N$ is obtained from $M \otimes N$ by dividing out the relations $g^{-1}m \otimes n = m \otimes gn$. Now if we replace m by gm, these relations take the form $m \otimes n = gm \otimes gn$, which allows us to write

$$M \otimes_G N = (M \otimes N)_G, \tag{1.1}$$

where the action of G on $M \otimes N$ is the diagonal action, given by $g \cdot (m \otimes n) := gm \otimes gn$. Moreover, this shows that tensoring over $\mathbb{Z}G$ is commutative, i.e.

$$M \otimes_G N \cong N \otimes_G M.$$

The important takeaway from the above considerations is that we need not distinguish between left and right $\mathbb{Z}G$-modules when tensoring over $\mathbb{Z}G$, which will save some work in later chapters.

The next lemma establishes two fundamental properties of coinvariants, which will be relevant in the proof of the main theorem.

Lemma 1.8. *Let G be a group with normal subgroup H and let M be a $\mathbb{Z}G$-module.*

(1) The action $G \curvearrowright M$ induces an action $G/H \curvearrowright M_H$.

(2) There is an isomorphism $M_G \cong (M_H)_{G/H}$ of groups.

Proof. Let $I := IH = \ker(\varepsilon : \mathbb{Z}H \to \mathbb{Z})$. First, we claim that $I = \langle h - 1 \mid h \in H \rangle$. The set $S := \{h - 1 \mid h \in H,\ h \neq 1\}$ is linearly independent, because from the expression $\sum_{h \in H} a_h(h-1) = 0$ it follows that $\sum_{h\in H} a_h h = \sum_{h\in H} a_h = \sum_{h \in H} a_h \cdot 1 + \sum_{h\in H} 0 \cdot h$. Since $\mathbb{Z}H$ is a free $\mathbb{Z}$-module we get that $a_h = 0$ for all $h \in H$ and $h \neq 1$. Next, we show that $I = \langle S \rangle$. To this end, consider some $\sum_{h\in H} a_h h \in I$, i.e. $\sum_{h \in H} a_h = 0$. Then we write

$$\sum_{h\in H} a_h h = \sum_{h \in H} a_h h - \sum_{h\in H} a_h = \sum_{h\in H} a_h(h-1),$$

and hence S generates I. This proves the claim. Now we proceed with (1) and (2).

(1) By the above we have an isomorphism $M_H \cong M/IM$ of groups, where $IM := I \cdot M$ is not to be confused with IH. Then one defines an action $G/H \curvearrowright M/IM$ via

$$\begin{aligned} G/H \times M/IM &\to M/IM \\ (gH, m + IM) &\mapsto (gm) + IM. \end{aligned}$$

Consider an element $g' = gh$ for some $h \in H$. Then we can write

$$(g'm) + IM = (ghm - gm + gm) + IM = ((h-1)gm + gm) + IM = (gm) + IM,$$

since $(h-1)gm \in IM$, and hence the action is well-defined.

(2) Define a map $\Phi : M_G \to (M_H)_{G/H}$ via $[m] \mapsto [m + IM]$, where square brackets denote an equivalence class in the group of coinvariants. One checks that Φ is indeed a well-defined group homomorphism. Similarly, we can define a well-defined group homomorphism $\Psi : (M_H)_{G/H} \to M_G$ via $[m + IM] \mapsto [m]$. Then one checks that Φ and Ψ are mutually inverse and hence isomorphisms. $\square$

1.4 The Bar Resolution

It turns out that the definition of the homology of some group G requires the concept of a *projective resolution of* $\mathbb{Z}$ *over the group ring* $\mathbb{Z}G$. In this short chapter we roughly follow [Bro82, p. 19] and [Knu01, pp. 151] to argue that such a resolution, called the *bar resolution*, always exists. Historically, it was Eilenberg and MacLane in their paper [EM45] who first discovered this specific resolution.

Definition 1.9 (Bar Resolution)**.** Let G be a group. We define the **bar resolution** $\overline{C_\bullet}(G)$ as the following $\mathbb{Z}G$-chain complex:

- For any $n \in \mathbb{N}$, set

$$\overline{C_n}(G) \coloneqq \mathbb{Z}G\big[[g_1|\dots|g_n] \mid g_1,\dots,g_n \in G\big],$$

  i.e. $\overline{C_n}(G)$ is the free $\mathbb{Z}G$-module with basis the $(n+1)$-tuples $[g_1|\dots|g_n] \coloneqq (1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n)$. If $n = 0$, then we write $[] = 1$ for the unique element of the generating set of $\overline{C_0}(G)$ and we identify $\overline{C_0}(G) \cong \mathbb{Z}G$.

- For any $n \in \mathbb{N}$, define the action $\mathbb{Z}G \curvearrowright \overline{C_n}(G)$ to be the one induced by the diagonal action

$$G \times G^{n+1} \to G^{n+1}; \quad \big(g, (g_0,\dots,g_n)\big) \mapsto g\cdot(g_0,\dots,g_n) \coloneqq (gg_0,\dots,gg_n).$$

- For any $n \geq 1$, define the differential $\overline{\partial_\bullet}$ as the $\mathbb{Z}G$-linear map

$$\overline{\partial_n} : \overline{C_n}(G) \to \overline{C_{n-1}}(G); \quad [g_1|\dots|g_n] \mapsto \sum_{j=0}^{n} (-1)^j d_j([g_1|\dots|g_n]),$$

  where d_j is defined as follows:

$$d_j([g_1|\dots|g_n]) \coloneqq \begin{cases} g_1\cdot[g_2|\dots|g_n] & \text{if } j=0,\\ [g_1|\dots|g_{j-1}|g_jg_{j+1}|g_{j+2}|\dots|g_n] & \text{if } 0<j<n,\\ [g_1|\dots|g_{n-1}] & \text{if } j=n. \end{cases}$$

One can show that $\overline{\partial_n} \circ \overline{\partial_{n+1}} = 0$ for all $n \geq 1$, so $(\overline{C_\bullet}(G), \overline{\partial_\bullet})$ is indeed a chain complex.

The next proposition provides the desired result, which will later allow us to consider suitable resolutions for computations.

Proposition 1.10. *Let G be a group and let $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ be the augmentation map. Then $\overline{C_\bullet}(G) \xrightarrow{\varepsilon} \mathbb{Z}$ is a free (and hence projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ with the trivial G-action.*

Proof. See [Löh19, Proposition 1.6.5] for a proof. □

Remark 1.11. One important observation is that we obtain a more or less explicit description of the lower terms of the bar resolution, which will be useful in the proof of Proposition 2.6, for instance. Explicitly, we have the following:

$$\cdots \longrightarrow \overline{C_2}(G) \xrightarrow{\overline{\partial_2}} \overline{C_1}(G) \xrightarrow{\overline{\partial_1}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where one computes the differentials to be $\overline{\partial_1}([g]) = g[] - [] = g - 1$ and $\overline{\partial_2}([g|h]) = g[h] - [gh] + [g]$.

2 Definition of Group Homology

Definition 2.1 (Group Homology)**.** Let $F_\bullet$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and let M be a $\mathbb{Z}G$-module. Then we define the homology of G with coefficients in M as the $\mathbb{Z}$-module

$$H_n(G; M) \coloneqq H_n(F_\bullet \otimes_G M)$$

for any $n \in \mathbb{N}$. This definition is well-defined, for if $F'_\bullet$ is another projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, we get a chain homotopy equivalence $F_\bullet \simeq F'_\bullet$ from the fundamental lemma of homological algebra, see [Bro82, Lemma I.7.4], and hence a chain homotopy equivalence $F_\bullet \otimes_G M \simeq F'_\bullet \otimes_G M$. Therefore, their homology groups are isomorphic.

Remark 2.2. If we take $M = \mathbb{Z}$ with trivial G-action, then for any $n \geq 0$ we get

$$H_n(G; \mathbb{Z}) = H_n(F_\bullet \otimes_G \mathbb{Z}) \cong H_n((F_\bullet \otimes \mathbb{Z})_G) \cong H_n((F_\bullet)_G),$$

where we used equation (1.1). This is precisely how group homology with trivial coefficients is usually defined, see [Bro82, Chapter II.3]. Henceforth, we simply write $H_n(G)$ for $H_n(G; \mathbb{Z})$ if there is no confusion.

Remark 2.3. Recalling the definition of the Tor-functor, we obtain an alternative description of group homology as

$$H_n(G; M) \cong \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M).$$

This will be useful when discussing functoriality of group homology in a later chapter.

A natural question to ask is whether the definition of group homology is well-behaved in the sense that, if two groups G, G' are "similar", then the group homology of these groups should be the same. Indeed, we have the following:

Proposition 2.4. *If $G \cong G'$ is an isomorphism of groups, then $H_n(G; M) \cong H_n(G'; M)$ for all $n \in \mathbb{N}$ and any $\mathbb{Z}G$-module M.*

Proof. Let $F_\bullet$ and $F'_\bullet$ be projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively. Then $F'_\bullet$ can be regarded as a projective $\mathbb{Z}G$-resolution via the isomorphism $G \cong G'$, and similarly M can be viewed as a $\mathbb{Z}G'$-module. The fundamental lemma gives a chain homotopy equivalence $F_\bullet \simeq F'_\bullet$, and hence $H_n(G; M) = H_n(F_\bullet \otimes_G M) \cong H_n(F'_\bullet \otimes_{G'} M) = H_n(G'; M)$ for any n. □

For now, we are interested in the explicit computation of the homology of some specific groups, in order to obtain a feeling for the usefulness of our definitions so far. A first step is computing the homology of the trivial group:

Proposition 2.5. *Let M be a $\mathbb{Z}$-module. Then there are isomorphisms*

$$H_n(\{1\}; M) \cong \begin{cases} M & \text{if } n = 0, \\ 0 & \text{if } n \geq 1. \end{cases}$$

Proof. Since $\mathbb{Z}[\{1\}] \cong \mathbb{Z}$ as rings, we have a projective resolution

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}_{\mathbb{Z}}} & \mathbb{Z} & \longrightarrow & 0 \\ & & [2] & & [1] & & [0] & & [-1] & & \end{array}$$

of $\mathbb{Z}$ as a $\mathbb{Z}$-module, where the degrees are included for clarity. Applying $- \otimes M$ yields a chain complex of the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow M.$$

Now simply taking homology yields the result. □

In the next chapter, we start working towards closed form expressions of the zeroth homology of some arbitrary group G, and the homology of cyclic groups. We will notice that it is rather cumbersome to compute homology explicitly, using only the definition.

2.1 Zeroth Homology and Cyclic Groups

In general, there is no closed form expression for the homology $H_n(G; M)$. However, there are exceptions for $n = 0, 1, 2$. In the case $n = 0$, we can compute the homology group of any group G without too much trouble, using the definition. The cases $n = 1, 2$ are discussed in later chapters.

Proposition 2.6. *Let G be a group and M be a $\mathbb{Z}G$-module. Then there is a (natural) isomorphism $H_0(G; M) \cong M_G$.*

Proof. We follow [Knu01, Proposition A.1.6]. Consider the bar resolution $\overline{C_\bullet}(G) \xrightarrow{\varepsilon} \mathbb{Z}$. We apply the functor $- \otimes_G M$ to obtain a chain complex

$$\cdots \longrightarrow \overline{C_2}(G) \otimes_G M \xrightarrow{\overline{\partial_2} \otimes_G \mathrm{id}_M} \overline{C_1}(G) \otimes_G M \xrightarrow{\overline{\partial_1} \otimes_G \mathrm{id}_M} \mathbb{Z}G \otimes_G M.$$

We have that $H_0(G; M) \cong \mathrm{coker}(\overline{\partial_1} \otimes_G \mathrm{id}_M)$, where $(\overline{\partial_1} \otimes_G \mathrm{id}_M)([g] \otimes m) = (g - 1) \otimes m$, which can be seen from Remark 1.11. If we identify $\mathbb{Z}G \otimes_G M$ with M, then under this isomorphism the element $(g - 1) \otimes_G m$ maps to $(g - 1)m = gm - m$, and hence we get

$$H_0(G; M) = \mathrm{coker}(\overline{\partial_1} \otimes_G \mathrm{id}_M) \cong (\mathbb{Z}G \otimes_G M) / \mathrm{im}(\overline{\partial_1} \otimes_G \mathrm{id}_M) \cong M / \langle gm - m \rangle = M_G.$$

Naturality will briefly be shown in Lemma 7.4, after discussing functorial properties of group homology, as it is used in the proof of Corollary 7.3. □

Lemma 2.7. *Let $n > 0$, $G := \mathbb{Z}/n = \langle t \mid t^n \rangle$ and let $N = \sum_{j=0}^{n-1} t^j \in \mathbb{Z}G$. Then the sequence*

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a projective resolution of $\mathbb{Z}$ (with the trivial G-action) over $\mathbb{Z}G$.

Proof. First, we notice that all chain modules in the sequence are free $\mathbb{Z}G$-modules and hence projective. Next, observe that $N \cdot (t-1) = (t-1) \cdot N = 0$, which shows that the sequence is a chain complex. In order to show the non-trivial inclusions $\ker(N) \subseteq \operatorname{im}(t-1)$ and $\ker(t-1) \subseteq \operatorname{im}(N)$ one can explicitly compute these subgroups and use arguments from linear algebra. This is rather tedious and not very insightful, so we omit this part. □

Proposition 2.8. *Let $n \geq 1$ and set $G \coloneqq \mathbb{Z}/n = \langle t \mid t^n \rangle$. Moreover, let $N \coloneqq \sum_{j=0}^{n-1} t^j \in \mathbb{Z}G$ and let M be a $\mathbb{Z}G$-module. Then, for all $k \in \mathbb{N}$ we have*

$$H_k(G; M) \cong \begin{cases} M_G & \text{if } k = 0, \\ M^G / N \cdot M & \text{if } k \text{ is odd}, \\ \ker(N : M \to M) / (t-1) \cdot M & \text{if } k > 0 \text{ is even}. \end{cases}$$

In particular, this means that $H_{k+2}(G; M) \cong H_k(G; M)$ for all $k > 0$. Moreover, if M is the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, we get

$$H_k(\mathbb{Z}/n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}/n & \text{if } k \text{ is odd}, \\ 0 & \text{if } k > 0 \text{ is even}. \end{cases}$$

Proof. Consider the resolution from Lemma 2.7. Now we apply the functor $- \otimes_G M$ to obtain a chain complex

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z}G \otimes_G M \arrow[r, "N \otimes \mathrm{id}_M"] \arrow[d, "\cong"] & \mathbb{Z}G \otimes_G M \arrow[r, "(t-1)\otimes \mathrm{id}_M"] \arrow[d, "\cong"] & \mathbb{Z}G \otimes_G M \arrow[d, "\cong"] \\ \cdots \arrow[r] & M \arrow[r, "N"] & M \arrow[r, "t-1"] & M \end{tikzcd}$$

where one easily checks that all squares indeed commute, using that the vertical maps are given by $g \otimes m \mapsto gm$. Then, we simply take homology to find that

$$H_k(G; M) = \ker(t-1) / \operatorname{im}(N) \cong M^G / N \cdot M,$$

if k is odd and

$$H_k(G; M) = \ker(N) / \operatorname{im}(t-1) \cong \ker(N : M \to M) / (t-1) \cdot M,$$

if k is even. Together with $H_0(G; M) \cong M_G$ the result follows. For the special case that $M = \mathbb{Z}$, notice that

$$(t-1) \otimes \mathrm{id}_\mathbb{Z} = t \otimes \mathrm{id}_\mathbb{Z} - 1 \otimes \mathrm{id}_\mathbb{Z} = 1 \otimes \mathrm{id}_\mathbb{Z} - 1 \otimes \mathrm{id}_\mathbb{Z} = 0$$

and similarly

$$N \otimes \mathrm{id}_\mathbb{Z} = \left(\sum_{j=0}^{n-1} t^j \right) \otimes \mathrm{id}_\mathbb{Z} = \sum_{j=0}^{n-1} (t^j \otimes \mathrm{id}_\mathbb{Z}) = \sum_{j=0}^{n-1} (1 \otimes \mathrm{id}_\mathbb{Z}) = n \otimes \mathrm{id}_\mathbb{Z},$$

where we used that $G \curvearrowright \mathbb{Z}$ was trivial. Taking homology now gives the desired result. □

2.2 Functoriality

As with ordinary homology theories, group homology is a functor in several ways. We roughly follow the descriptions of [Bro82, Chapter II.6 and III.6]. The claim is that, for all $n \geq 0$, we have two functors as follows:

- $H_n(-;\mathbb{Z}) : \mathbf{Grp} \to \mathbb{Z}\text{-}\mathbf{Mod}$.
- $H_n(G;-) : \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbb{Z}\text{-}\mathbf{Mod}$, for a fixed group G.

For both items it is clear what happens on objects. It remains to check for morphisms: For the first claim, consider a group homomorphism $\alpha : G \to G'$ and let $F_\bullet$, $F'_\bullet$ be projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively. We can regard $F'_\bullet$ as an acyclic $\mathbb{Z}G$-module resolution via α, allowing us to use the fundamental lemma to choose an augmentation-preserving G-chain map $\tau^\alpha_\bullet : F_\bullet \to F'_\bullet$, well-defined up to chain homotopy. The fact that $\tau^\alpha_\bullet$ is compatible with G is expressed in the formula

$$\tau^\alpha_\bullet(gx) = \alpha(g)\tau^\alpha_\bullet(x)$$

for any $g \in G$ and $x \in F_\bullet$. Notice that $\tau^\alpha_\bullet$ induces a map

$$\tau^\alpha_\bullet \otimes \mathrm{id}_\mathbb{Z} : (F_\bullet)_G \cong F_\bullet \otimes_G \mathbb{Z} \longrightarrow F'_\bullet \otimes_{G'} \mathbb{Z} \cong (F'_\bullet)_{G'}$$

which is well-defined up to chain homotopy. Therefore, we are left with a well-defined map

$$\begin{aligned} \alpha_* \coloneqq (\tau^\alpha_\bullet \otimes \mathrm{id}_\mathbb{Z})_* : H_*(G;\mathbb{Z}) &\to H_*(G';\mathbb{Z}) \\ [gx \otimes_G z] &\mapsto [\alpha(g)\tau^\alpha_\bullet(x) \otimes_{G'} z], \end{aligned}$$

where $g \in G$, $x \in F_\bullet$ and $z \in \mathbb{Z}$. This finishes the first claim.

For the second claim, simply notice that $F_\bullet \otimes_G -$ is a covariant functor, immediately giving a covariant functor $H_*(G;-)$. Explicitly, we have

$$\begin{aligned} f_* \coloneqq H_n(G;f) : H_n(G;M) &\to H_n(G;M') \\ [x \otimes_G m] &\mapsto [x \otimes_G f(m)], \end{aligned}$$

where $M, M' \in \mathbb{Z}G\text{-}\mathbf{Mod}$ and $f : M \to M'$ is a $\mathbb{Z}G$-module homomorphism.

More generally, we actually have a functor of the form $H_*(-;-)$, which is discussed in more detail in Definition 5.3. Viewing group homology in this way will allow us to provide a rigorous foundation upon which to state the main theorem. For now, let us continue with some basic properties of the functor $H_*(G;-)$:

Proposition 2.9. *Let G be a group.*

(1) (Projectives) *If M is a projective $\mathbb{Z}G$-module and $n > 0$, we have $H_n(G;M) \cong 0$.*

(2) (Long exact sequence) *Let $0 \to M' \xrightarrow{i} M \xrightarrow{\pi} M'' \to 0$ be a short exact sequence of $\mathbb{Z}G$-modules. Then there exists an associated (natural) long exact sequence*

$$\cdots \longrightarrow H_n(G;M') \xrightarrow{i_*} H_n(G;M) \xrightarrow{\pi_*} H_n(G;M'') \xrightarrow{\partial_n} H_{n-1}(G;M') \longrightarrow \cdots .$$

(3) (Dimension shifting) *Let $0 \to K \xrightarrow{i} P \xrightarrow{\pi} M \to 0$ be a short exact sequence of $\mathbb{Z}G$-modules and let P be projective. Then the connecting homomorphism ∂ induces isomorphisms*

$$\partial_* : H_{n+1}(G;M) \xrightarrow{\cong} H_n(G;K)$$

for all $n > 0$ and $H_1(G;M) \cong \ker\big(i_* : H_0(G;K) \to H_0(G;P)\big)$.

Proof. The first item follows from the isomorphism $H_n(G;M) \cong \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M)$ (see Remark 2.3) and then using the fact that M is, in particular, a flat $\mathbb{Z}G$-module and hence $\operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M) \cong 0$ for all $n > 0$ by [DF04, Proposition 17.1.16].

The second item is a direct consequence of the associated Tor-long exact sequence, as in [DF04, Theorem 17.1.15]. The final item follows from the previous two, using exactness. □

We give one application of the above proposition:

Example 2.10 ([Löh19, Example 3.1.14])**.** Let G be any finite group. Consider the group $S^1 \cong \mathbb{R}/\mathbb{Z}$ with trivial G-action, i.e. we have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1 \longrightarrow 0$$

of trivial $\mathbb{Z}G$-modules. Our goal is to give a description of $H_n(G;S^1)$. As we will see in Corollary 2.21, we have $H_n(G;\mathbb{R}) \cong 0$ for all $n \geq 1$, since G is finite. Using the long exact sequence from Proposition 2.9, we find that

$$H_n(G;S^1) \cong H_{n-1}(G;\mathbb{Z}) = H_{n-1}(G)$$

for all $n \geq 2$. In particular, we get $H_1(G;S^1) \cong \mathbb{Z}$ and $H_2(G;S^1) \cong H_1(G;\mathbb{Z}) \cong G^{\mathrm{ab}}$, which follows from Corollary 3.4.

2.3 Extension, Restriction and Induction

Fix some ring homomorphism $\alpha : R \to S$. We can regard any S-module as an R-module via α, namely as $R \times M \to M$, $(r,m) \mapsto \alpha(r) \cdot m$. We thus get a functor S-**Mod** $\to R$-**Mod** which we call **restriction of scalars**. Conversely, we would like to define a functor in the other direction, R-**Mod** $\to S$-**Mod**. To this end, take some (left) R-module M and consider $S \otimes_R M$, where S is understood to be a right R-module via $s \cdot r \coloneqq s \cdot \alpha(r)$. Since the natural left action of S on itself commutes with the above right action of R on S, we can view $S \otimes_R M$ as a (left) S-module via $s \cdot (s' \otimes_R m) \coloneqq (ss') \otimes_R m$. This S-module is obtained from M by what we call **extension of scalars** from R to S.

Now we apply the above construction to the ring homomorphism $\mathbb{Z}H \hookrightarrow \mathbb{Z}G$, which is induced by the inclusion $H \hookrightarrow G$ of groups.

Definition 2.11 (Restriction)**.** Let G be a group and H some subgroup. Let $i : H \hookrightarrow G$ denote the inclusion. Then we write

$$\operatorname{Res}_H^G(-) \coloneqq i^*(-) : \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbb{Z}H\text{-}\mathbf{Mod}$$

for the **restriction functor**. Essentially, we "forget" the action of elements lying in the set $G \setminus H$.

The next proposition will be convenient for computations of group homology, namely if we have a group H within an ambient group G, then we can essentially use the projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ for both the homology of G and H.

Proposition 2.12. *Let G be a group and H a subgroup. Then we have:*

(1) If P is a projective $\mathbb{Z}G$-module, then $\operatorname{Res}_H^G(P)$ is a projective $\mathbb{Z}H$-module.

(2) If $\varepsilon : P_\bullet \to \mathbb{Z}$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then $\operatorname{Res}_H^G(\varepsilon) : \operatorname{Res}_H^G(P_\bullet) \to \operatorname{Res}_H^G(\mathbb{Z}) = \mathbb{Z}$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$.

Proof. See [Löh19, Proposition 1.7.2] for a proof. □

Definition 2.13 (Induction)**.** Let G be a group and H a subgroup. Define the **induction functor** as

$$\operatorname{Ind}_H^G(-) \coloneqq \mathbb{Z}G \otimes_H - : \mathbb{Z}H\text{-}\mathbf{Mod} \to \mathbb{Z}G\text{-}\mathbf{Mod},$$

where the action is given as in the above construction of extension of scalars.

Remark 2.14. A module of the form $\operatorname{Ind}_{\{1\}}^G(M) = \mathbb{Z}G \otimes M$ is called an **induced module**.

A natural question one might ask is the following: Given an inclusion of groups $H \hookrightarrow G$, is it possible to describe the homology of H in terms of the homology of the ambient group G? And if so, does the coefficient module change in any way? Shapiro's Lemma gives a relatively satisfying answer to these questions:

Lemma 2.15 (Shapiro)**.** *Let G be a group, H a subgroup and let M be a $\mathbb{Z}H$-module. Then, for all $n \in \mathbb{N}$, there is an isomorphism*

$$H_n(H; M) \cong H_n(G; \operatorname{Ind}_H^G(M)). \tag{2.1}$$

Proof. Let $F_\bullet$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. By Proposition 2.12 we can interpret $F_\bullet$ as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$, so we have an isomorphism $H_n(H; M) \cong H_n(\operatorname{Res}_H^G(F_\bullet) \otimes_H M)$ for all $n \in \mathbb{N}$. Notice that there are isomorphisms

$$\operatorname{Res}_H^G(F_\bullet) \otimes_H M \cong F_\bullet \otimes_G (\mathbb{Z}G \otimes_H M) \cong F_\bullet \otimes_G \operatorname{Ind}_H^G(M), \tag{2.2}$$

and hence the result follows. □

Remark 2.16. After discussing further functorial properties of group homology in Chapter 5.1 we will come to find in Lemma 5.4 that the isomorphism (2.1) is in fact induced by the inclusions $\alpha : H \hookrightarrow G$ and $i : M \hookrightarrow \operatorname{Ind}_H^G(M)$, $m \mapsto 1 \otimes m$, thus giving an explicit description.

Remark 2.17. Since Shapiro's Lemma uses the functor $\operatorname{Ind}_H^G(-)$, one might expect that we have a similar isomorphism of the form $H_n(G; M) \cong H_n(H; \operatorname{Res}_H^G(M))$ for a $\mathbb{Z}G$-module M. However, this does not hold, which can be seen by setting $H := \{1\}$, $G := \mathbb{Z}/2$, $M := \mathbb{Z}$ and $n = 1$, for instance.

Corollary 2.18. *Let G be a group and M a $\mathbb{Z}G$-module. Then the induced module $\mathbb{Z}G \otimes M$ is **H_*-acyclic**, i.e. $H_n(G; \mathbb{Z}G \otimes M) \cong 0$ for all $n \geq 1$.*

Proof. Take $H := \{1\}$. Using Shapiro's Lemma, we write

$$H_n(G; \mathbb{Z}G \otimes M) = H_n(G; \operatorname{Ind}_{\{1\}}^G(M)) \cong H_n(\{1\}; M),$$

which vanishes if $n \geq 1$. □

2.4 The Transfer Map

Given an inclusion of groups $i : H \hookrightarrow G$ and a $\mathbb{Z}G$-module M we have seen, using functoriality, that we get an induced map $i_* : H_n(H; \operatorname{Res}_H^G(M)) \to H_n(G; M)$ for all n, which is often denoted cor_H^G (corestriction map). The goal of this section is to establish a map in the opposite direction in the case that the index $(G : H)$ is finite, i.e. some map $\operatorname{tr}_H^G : H_n(G; M) \to H_n(H; \operatorname{Res}_H^G(M))$ which roughly acts as an inverse to cor_H^G. We follow [Löh19, Section 1.7.3] in order to give a more or less explicit description of the maps involved.

Definition 2.19 (Transfer)**.** Let G be a group, H a subgroup with $(G : H) < \infty$ and let M be a $\mathbb{Z}G$-module. We define the (homological) **transfer** tr_H^G as the map on homology induced by the composite

$$\begin{aligned} C_\bullet(G) \otimes_G M &\to \operatorname{Res}_H^G(C_\bullet(G)) \otimes_H \operatorname{Res}_H^G(M) \xleftarrow[\simeq]{C_\bullet(i) \otimes_H \operatorname{id}_M} C_\bullet(H) \otimes_H \operatorname{Res}_H^G(M) \\ x \otimes_G m &\mapsto \sum_{gH \in G/H} (g^{-1}x) \otimes_H (g^{-1}m), \end{aligned} \tag{2.3}$$

where $i : H \hookrightarrow G$ is the inclusion. Here the map $C_\bullet(i) : C_\bullet(H) \to \operatorname{Res}_H^G(C_\bullet(G))$ is a chain map extending the map $\operatorname{id}_{\mathbb{Z}}$. Since both chain complexes are projective resolutions, $C_\bullet(i)$ is a $\mathbb{Z}H$-chain homotopy equivalence by [Bro82, Theorem I.7.5], and hence $C_\bullet(i) \otimes_H \operatorname{id}_M$ is indeed a chain homotopy equivalence. Notice that we consider elements g^{-1} in (2.3), since

$$(gh)^{-1}x \otimes_H (gh)^{-1}m = h^{-1}g^{-1}x \otimes_H h^{-1}g^{-1}m = g^{-1}x \otimes_H g^{-1}m,$$

and hence the first map is well-defined. Simply writing $gx \otimes_H gm$ would not work, since the groups H and G are not necessarily abelian.

We can use the transfer map to obtain the following useful result:

Lemma 2.20. *Let G, H and M be as in Definition 2.19. Then for any $\alpha \in H_n(G; M)$ and any $n \in \mathbb{N}$ we have the identity*

$$(\mathrm{cor}_H^G \circ \mathrm{tr}_H^G)(\alpha) = (G : H) \cdot \alpha.$$

Proof. Using the definitions (and some abuse of notation), on the chain level we have

$$\begin{aligned}(\mathrm{cor}_H^G \circ \mathrm{tr}_H^G)(x \otimes_G m) &= \sum_{gH \in G/H} (g^{-1}x) \otimes_G (g^{-1}m) \\ &= \sum_{gH \in G/H} x \otimes_G m \\ &= (G : H) \cdot (x \otimes_G m),\end{aligned}$$

which shows that the claim holds in homology as well. □

We can use the transfer map to show that all finite groups are indistinguishable from the trivial group if we require the coefficient module M to have certain properties.

Corollary 2.21. *Let G be a finite group and R a commutative ring with unit. If $|G|$ is invertible in R, then $H_n(G; R) \cong 0$ for all $n \geq 1$. In particular, $H_n(G; \mathbb{Q}) \cong 0$.*

Proof. Take $H := \{1\} \leq G$. Then by Lemma 2.20 the map $(G : H) \cdot \mathrm{id}_{H_n(G;R)}$ factors through $H_n(H; \mathrm{Res}_H^G(R)) = H_n(\{1\}; R)$, which vanishes since $n \geq 1$. But $(G : H)$ was invertible in R, and consequently $H_n(G; R) \cong 0$. □

Another direct consequence of the transfer map is a torsion result which will prove to be useful in later computations, when using spectral sequences:

Corollary 2.22. *Let D_∞ denote the infinite dihedral group (see Example 6.1). Then we have that $2 \cdot H_n(D_\infty; \mathbb{Z}) \cong 0$ for all $n \geq 2$.*

Proof. From the short exact sequence $1 \to \mathbb{Z} \to D_\infty \to \mathbb{Z}/2 \to 1$, as can be seen in Example 6.1, we know that D_∞ contains an index 2 infinite cyclic subgroup, namely $\mathbb{Z}$. The map $2 \cdot \mathrm{id}_{H_n(D_\infty;\mathbb{Z})}$ therefore factors through $H_n(\mathbb{Z}; \mathbb{Z})$ by Lemma 2.20. However, $H_n(\mathbb{Z}; \mathbb{Z}) \cong 0$ for $n \geq 2$, which follows from Example 3.6, and hence $2 \cdot \alpha = 0$ for any $\alpha \in H_n(D_\infty; \mathbb{Z})$. □

3 A Topological Viewpoint

As one might expect, there is a strong link between the purely algebraic definition of group homology and the homology defined for topological spaces. In this chapter, we establish an important connection between the homology of a group G and the (singular) homology of its classifying space BG. This result proves to be useful in the computation of some homology groups, as we will see. Throughout, we follow [Löh19, Chapter 4.1].

Definition 3.1 (Classifying space)**.** Let G be a discrete group. A **classifying space** of G is a path-connected pointed CW-complex (X, x_0) with contractible universal cover, and such that there exists an isomorphism $G \cong \pi_1(X, x_0)$. We typically omit the basepoint x_0 and oftentimes we simply write $BG = X$ and denote by EG the universal cover of BG.

For assertions regarding existence and uniqueness (up to homotopy equivalence) of the spaces EG and BG, we refer to the literature mentioned at the start, where it is also shown that these constructions are functors **Grp** $\to$ **Top**, which will be useful in Corollary 3.3.

Theorem 3.2. *Let G be a group and let X be a classifying space for G. Moreover, let $p : Y \to X$ be the universal covering of X. Then $C_\bullet(Y) \xrightarrow{\varepsilon} \mathbb{Z}$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Here, the singular chain complex $C_\bullet(Y)$ carries the G-action induced from the deck transformation action, and $\varepsilon : C_0(Y) \to \mathbb{Z}$ is defined via $\varepsilon(\sigma) = 1$ for some $\sigma : \Delta^0 \to Y$.*

Proof. We know that $H_n(C_\bullet(Y); \mathbb{Z}) = H_n(Y; \mathbb{Z}) \cong H_n(\mathrm{pt}; \mathbb{Z}) \cong 0$, since X was a classifying space. It remains to show that $\ker(\varepsilon)/\operatorname{im}(\partial_1) \cong 0$, i.e. $\ker(\varepsilon) = \operatorname{im}(\partial_1)$, where $\partial_1 : C_1(Y) \to C_0(Y)$ is the differential in the chain complex $C_\bullet(Y)$.

By definition, $\operatorname{im}(\partial_1)$ is the free abelian subgroup generated by elements $\sigma_0 - \sigma_1 \in C_0(Y)$ such that there exists a path $\gamma : \Delta^1 \to Y$ with $\gamma(0,1) = \sigma_0(\mathrm{pt})$ and $\gamma(1,0) = \sigma_1(\mathrm{pt})$. Then we get $\varepsilon(\sigma_0 - \sigma_1) = \varepsilon(\sigma_0) - \varepsilon(\sigma_1) = 1 - 1 = 0$. This shows $\operatorname{im}(\partial_1) \subseteq \ker(\varepsilon)$.

For the other inclusion, take some 0-cycle $c = \sum_{i=1}^m a_i\sigma_i \in C_0(Y)$ with $a_i \in \mathbb{Z}$ and $\sigma_i \in Y$ (this is abuse of notation, but we can identify σ_i with $\operatorname{im}(\sigma_i)$), since Δ^0 is just a point. Because Y is path-connected, there is some path $\gamma_i : \Delta^1 \to Y$ for every $i = 1, \ldots, m$ such that $\delta^{1,0}(\gamma_i) = \sigma_i$ and $\delta^{1,1}(\gamma_i) = \sigma'$, where $\delta^{n,j} : \Delta^{n-1} \to \Delta^n$ denotes the inclusion of the j-th face of Δ^n and where $\sigma' \in Y$ is some fixed point. Then we can write

$$\partial_1\left(\sum_{i=1}^m a_i\gamma_i\right) = \sum_{i=1}^m a_i(\delta^{1,0}(\gamma_i) - \delta^{1,1}(\gamma_i)) = \sum_{i=1}^m a_i(\sigma_i - \sigma') = \sum_{i=1}^m a_i\sigma_i = c,$$

where we used that $\sum_{i=1}^m a_i = 0$, which holds because c was an ε-cocycle. Finally, the chain modules $C_n(Y)$ consist of free $\mathbb{Z}$-modules that admit $\mathbb{Z}$-bases, given by the singular simplices, on which G acts freely, and hence by [Bro82, Proposition I.4.1], the chain modules are free $\mathbb{Z}G$-modules. This shows that $C_\bullet(Y) \xrightarrow{\varepsilon} \mathbb{Z}$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. $\square$

Corollary 3.3. *Let G be a group and let M be a $\mathbb{Z}$-module with trivial action $G \curvearrowright M$ and let $n \in \mathbb{N}$. Then there exists a natural isomorphism*

$$H_n(G; M) \cong H_n(BG; M),$$

where the right-hand side is singular/cellular homology.

Proof. By Theorem 3.2 we know that $C_\bullet(EG) \xrightarrow{\varepsilon} \mathbb{Z}$ is a free and hence projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. By the fundamental lemma we thus have an isomorphism

$$H_n(G; M) \cong H_n(C_\bullet(EG) \otimes_G M). \tag{3.1}$$

By lifting properties of covering maps, we get that the chain map $p_* : C_\bullet(EG) \to C_\bullet(BG)$ induced by the covering projection $p : EG \to BG$ induces a chain isomorphism

$$C_\bullet(EG) \otimes_G M \cong C_\bullet(BG) \otimes M,$$

since $G \curvearrowright M$ was trivial. To be more precise, consider some $\sigma : \Delta^n \to BG$. Then, by the lifting theorem, there exists some $\tilde{\sigma} : \Delta^n \to EG$ such that $\sigma = p \circ \tilde{\sigma}$, since $\Delta^n \simeq \mathrm{pt}$. Then we can write

$$(p_* \otimes_G \mathrm{id}_M)(g \cdot \tilde{\sigma} \otimes_G m) = (p_* \otimes_G \mathrm{id}_M)(\tilde{\sigma} \otimes_G g \cdot m) = p_*(\tilde{\sigma}) \otimes m = \sigma \otimes m.$$

This calculation shows that the map $p_* \otimes_G \mathrm{id}_M : C_\bullet(EG) \otimes_G M \to C_\bullet(BG) \otimes M$ is well-defined and surjective. Injectivity follows from the fact that the lift $\tilde{\sigma}$ is unique. Using (3.1), we deduce that

$$H_n(G; M) \cong H_n(C_\bullet(EG) \otimes_G M) \cong H_n(C_\bullet(BG) \otimes M) = H_n(BG; M), \tag{3.2}$$

which shows that we have the desired isomorphism. To show naturality, consider a group homomorphism $f : G \to G'$. One needs to check that the following diagram commutes:

$$\begin{tikzcd}
H_n(G; M) \arrow[r, "f_*"] \arrow[d, "\cong"] & H_n(G'; M) \arrow[d, "\cong"'] \\
H_n(C_\bullet(EG) \otimes_G M) \arrow[r, "(C_\bullet(Ef)\otimes \mathrm{id})_*"] \arrow[d, "\cong", "(C_\bullet(p)\otimes \mathrm{id})_*"'] & H_n(C_\bullet(EG') \otimes_{G'} M) \arrow[d, "\cong"', "(C_\bullet(p')\otimes \mathrm{id})_*"] \\
H_n(C_\bullet(BG) \otimes M) \arrow[r, "(C_\bullet(Bf)\otimes \mathrm{id})_*"] \arrow[d, equal] & H_n(C_\bullet(BG') \otimes M) \arrow[d, equal] \\
H_n(BG; M) \arrow[r, "(Bf)_*"] & H_n(BG'; M)
\end{tikzcd}$$

Notice that the vertical compositions are precisely the isomorphisms in (3.2). The upper square commutes by definition, since $C_\bullet(EG)$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and then by using the fundamental lemma. The middle square commutes because, already on spaces, we have $Bf \circ p = p' \circ Ef$, and then one uses functoriality of $C_\bullet$ and H_*. The lower square commutes by definition of singular homology with coefficients. $\square$

Corollary 3.4. *Let G be a group and M a $\mathbb{Z}$-module with trivial G-action. Then there is an isomorphism of groups*

$$H_1(G;M) \cong G^{\mathrm{ab}} \otimes M.$$

Proof. We can write, using Theorem 3.3 and the Hurewicz Theorem:

$$H_1(G;M) \cong H_1(BG;M) \cong H_1(BG) \otimes M \cong \pi_1(BG)^{\mathrm{ab}} \otimes M \cong G^{\mathrm{ab}} \otimes M,$$

where we used the universal coefficient theorem for the second isomorphism. □

Remark 3.5. Notice that, by setting M to be the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, we have an isomorphism $H_1(G) \cong G^{\mathrm{ab}}$ for any group G.

With Theorem 3.3 established, we can compute group homology explicitly for a few distinct groups, by using the fact that we know quite a few homology groups of topological spaces:

Example 3.6. Since $B\mathbb{Z} \simeq S^1$ we have $H_n(\mathbb{Z}) \cong H_n(S^1;\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}, 0, 0, \ldots)$.

Example 3.7. Since $B\mathbb{Z}/2 \simeq \mathbb{R}\mathrm{P}^\infty$, we have

$$H_n(\mathbb{Z}/2) \cong H_n(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \ldots).$$

Example 3.8. Note that $B(G \times H) \simeq BG \times BH$ for groups G and H. For instance, we have $B(\mathbb{Z} \oplus \mathbb{Z}) \simeq S^1 \times S^1$. Using this homotopy equivalence, we can derive a Künneth theorem for group homology: If R is a principal ideal domain with trivial G-action, there is an isomorphism

$$H_n(G \times H;R) \cong \bigoplus_{p+q=n} \big(H_p(G;R) \otimes_R H_q(H;R)\big) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R\big(H_p(G;R), H_q(H;R)\big).$$

Corollary 3.9. *Let G_1 and G_2 be groups and let A be a subgroup of both G_1 and G_2. Consider the following pushout of groups:*

$$\begin{tikzcd} A \arrow[r, hook, "i^2"] \arrow[d, hook, "i^1"'] & G_2 \arrow[d, "j^2"] \\ G_1 \arrow[r, "j^1"'] & G \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{3.3}$$

Then there exists a (natural) "Mayer-Vietoris" long exact sequence of the form

$$\cdots \longrightarrow H_n(A) \xrightarrow{i^1_* \oplus i^2_*} H_n(G_1) \oplus H_n(G_2) \xrightarrow{j^1_* - j^2_*} H_n(G) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots.$$

Proof. By a result due to Whitehead (see [Bro82, Theorem II.7.3]), the diagram (3.3) can be realized from a pushout of classifying spaces via taking fundamental groups as follows:

$$\begin{tikzcd} BA \arrow[r, hook, "Bi^2"] \arrow[d, hook, "Bi^1"'] & BG_2 \arrow[d, "Bj^2"] \\ BG_1 \arrow[r, "Bj^1"'] & BG \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

In particular, we can write $BG \cong BG_1 \cup BG_2$ and $BA \cong BG_1 \cap BG_2$. This allows us to use the usual Mayer-Vietoris sequence for spaces. We obtain a diagram with an exact upper row, where we define $(Bi)_* := (Bi^1)_* \oplus (Bi^2)_*$ and $(Bj)_* := (Bj^1)_* - (Bj^2)_*$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(BA) \arrow[r, "(Bi)_*"] & H_n(BG_1) \oplus H_n(BG_2) \arrow[r, "(Bj)_*"] & H_n(BG) \arrow[r, "\partial_n"] & H_{n-1}(BA) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r, "i^1_* \oplus i^2_*"] \arrow[u, "\cong"] & H_n(G_1) \oplus H_n(G_2) \arrow[r, "j^1_* - j^2_*"] \arrow[u, "\cong"] & H_n(G) \arrow[r, "\partial_n"] \arrow[u, "\cong"] & H_{n-1}(A) \arrow[r] \arrow[u, "\cong"] & \cdots
\end{tikzcd}$$

By applying Corollary 3.3 (in particular the naturality statement), we see that all squares in the diagram commute, finishing the proof. Naturality follows from the naturality of the topological Mayer-Vietoris sequence. $\square$

Using that $H_n(\{1\};\mathbb{Z}) \cong 0$ for all $n \geq 1$, a direct consequence of Corollary 3.9 is the following:

Corollary 3.10. *Let $n \geq 1$ and let G, H be groups. Then there is an isomorphism $H_n(G * H) \cong H_n(G) \oplus H_n(H)$.*

Example 3.11. With Corollary 3.10 one can compute the homology of the *infinite dihedral group* $D_\infty \cong \langle x, y \mid x^2, y^2 \rangle$. Notice that $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$, and hence (using Example 3.7), we obtain

$$H_n(D_\infty) \cong H_n(\mathbb{Z}/2) \oplus H_n(\mathbb{Z}/2) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \text{ is odd}, \\ 0 & \text{else}. \end{cases}$$

In Example 6.1 we will see how one could have computed these homology groups using spectral sequences, instead of relying on any topological interpretation.

Another direct application of Corollary 3.3 shows that we can compute the homology of all free groups without much effort.

Corollary 3.12. *Let $\mathfrak{F}_n$ denote the free group on n generators. Then*

$$H_k(\mathfrak{F}_n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}^n & \text{if } k = 1, \\ 0 & \text{else}. \end{cases}$$

Proof. We know that the 1-dimensional CW-complex $X := \bigvee_{i=1}^n S^1$ is a classifying space of $\mathfrak{F}_n$, since X is connected, $\pi_1(X) \cong \mathfrak{F}_n$ and $H_k(\widetilde{X}) \cong 0$ for $k \geq 2$, where $\widetilde{X}$ is the universal covering space of X. Hence, $H_0(\mathfrak{F}_n) \cong \mathbb{Z}$ and

$$H_k(\mathfrak{F}_n) \cong H_k(X) \cong \bigoplus_{i=1}^n H_k(S^1) \cong \begin{cases} \mathbb{Z}^n & \text{if } k = 1, \\ 0 & \text{else}, \end{cases}$$

for any $k \geq 1$. Alternatively, one could have used Corollary 3.10, by noting that $\mathfrak{F}_n \cong \mathbb{Z} * \ldots * \mathbb{Z}$ n-times, and $B\mathbb{Z} \simeq S^1$. $\square$

4 Spectral Sequences: An Overview

In this chapter, we give a brief summary of the theory surrounding spectral sequences. In short, they constitute a powerful tool which one often finds to be useful in the computation of certain (co-)homology groups of spaces or groups that were not easily computable before. So even though the subject might not be easy to grasp, understanding spectral sequences is worth the effort. Historically, spectral sequences were developed in the 1940s independently by Lyndon and Jean Leray. While Lyndon published his work in his dissertation, Leray had worked under much more dire circumstances. The following is taken from [Rot09, p. 622]:

> "Leray was a German prisoner of war from 1940 through 1945, during World War II. He was an applied mathematician, but, because he did not want the Nazis to exploit his expertise, he did abstract work in Algebraic Topology there".

We discuss basic notions, the existence of spectral sequences and convergence results but we mostly omit proofs. We follow [Rot09, Chapter 10]. Throughout, R is an arbitrary ring.

4.1 Filtrations

Definition 4.1 (Filtration)**.** A **filtration** of an R-module M is a family $(M_p)_{p\in\mathbb{Z}}$ of submodules of M such that there is a chain of inclusions $\ldots \subseteq M_{p-1} \subseteq M_p \subseteq M_{p+1} \subseteq \ldots$ with $M = \bigcup_{p\in\mathbb{Z}} M_p$.

Suppose we are working over some abelian category, which allows us to define a filtration for any object in this category. In particular, suppose we are given some R-chain complex $C_\bullet$, i.e. an object in the abelian category **Ch**. A filtration of $C_\bullet$ is a family of sub-complexes $(F^pC_\bullet)_{p\in\mathbb{Z}}$, often denoted by $(F^pC_\bullet)$, such that we have inclusions $\ldots F^{p-1}C_\bullet \subseteq F^pC_\bullet \subseteq F^{p+1}C_\bullet \subseteq \ldots$.

Definition 4.2 (Total Complex)**.** Let $(M_{\bullet,\bullet}, d', d'')$ be a double complex, i.e. a bigraded R-module $M = (M_{p,q})$ together with differentials $d', d'' : M \to M$ of bidegree $(-1, 0)$ and $(0, -1)$ respectively, such that $d'd' = 0$, $d''d'' = 0$ and $d'd'' + d''d' = 0$. We define its **total complex** $\mathrm{Tot}(M) \coloneqq \mathrm{Tot}(M_{\bullet,\bullet})$ by

$$\mathrm{Tot}(M)_n \coloneqq \bigoplus_{p+q=n} M_{p,q}$$

with differential $\partial_n : \mathrm{Tot}(M)_n \to \mathrm{Tot}(M)_{n-1}$ given by $\partial_n|_{M_{p,q}} = d'_{p,q} \oplus d''_{p,q}$. One can check that $\partial_n \circ \partial_{n+1} = 0$, and hence $\mathrm{Tot}(M)$ is an R-chain complex.

Two essential filtrations of $\mathrm{Tot}(M)$, which will be used later on, are the following:

Definition 4.3 (First/Second Filtration)**.** Let $(M_{\bullet,\bullet}, d', d'')$ be a double complex. Then the **first** and **second filtration** of $\mathrm{Tot}(M)$ are given by

$$\left({}^{I}F^p\,\mathrm{Tot}(M_{\bullet,\bullet})\right)_n \coloneqq \bigoplus_{i\le p} M_{i,n-i} \qquad \text{and} \qquad \left({}^{II}F^p\,\mathrm{Tot}(M_{\bullet,\bullet})\right)_n \coloneqq \bigoplus_{j\le p} M_{n-j,j}.$$

Next, suppose we have a filtration $(F^pC_\bullet)$ of some R-chain complex $C_\bullet$ with inclusions $i^p : F^pC_\bullet \hookrightarrow C_\bullet$. Taking homology yields a map $i^p_* : H_*(F^pC_\bullet) \to H_*(C_\bullet)$. Notice that $\operatorname{im}(i^p_*) \subseteq \operatorname{im}(i^{p+1}_*)$. This gives rise to the following definition:

Definition 4.4 (Induced Filtration)**.** If $(F^pC_\bullet)$ is a filtration of a chain complex $C_\bullet$ and if $i^p : F^pC_\bullet \hookrightarrow C_\bullet$ is the inclusion, we define $(\Phi^pH_n(C_\bullet))_{p\in\mathbb{Z}}$ to be the **induced filtration** of $H_n(C_\bullet)$, where

$$\Phi^pH_n(C_\bullet) \coloneqq \operatorname{im}(i^p_*).$$

We need one more definition in order to get reasonable convergence results later on:

Definition 4.5 (Bounded Filtration)**.** A filtration (F^pM) of some graded R-module $M = (M_n)$ is **bounded** if, for each $n \in \mathbb{N}$, there exist integers $s = s(n)$ and $t = t(n)$ such that $F^sM_n = \{0\}$ and $F^tM_n = M_n$.

An important observation is that, if $(F^pC_\bullet)$ is a bounded filtration of an R-chain complex, then the induced filtration $(\Phi^pH_*(C_\bullet))$ is bounded with *the same* bounds. More precisely, if we write $H_n \coloneqq H_n(C_\bullet)$, then the homology is filtered by a chain of inclusions

$$\{0\} \subseteq \Phi^sH_n \subseteq \Phi^{s+1}H_n \subseteq \ldots \subseteq \Phi^{t-1}H_n \subseteq \Phi^tH_n = H_n,$$

where s, t are the bounds. For convenience's sake, we oftentimes set $\Phi^{-1}H_n = \{0\}$ and $\Phi^nH_n = H_n$. This condition is usually automatically satisfied, for instance when dealing with first quadrant double complexes, as we shall see when discussing convergence.

4.2 Exact Couples and Spectral Sequences

Now we give a rough argument for the existence of spectral sequences, by using the notion of exact couples, first introduced by W. S. Massey in 1952.

Definition 4.6 (Exact Couple)**.** An **exact couple** is a 5-tuple $(D, E, \alpha, \beta, \gamma)$, where D, E are bigraded R-modules, α, β, γ are bigraded maps and we have a diagram

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

which is exact at each vertex.

Proposition 4.7. *Every filtration $(F^pC_\bullet)_{p\in\mathbb{Z}}$ of an R-chain complex determines an exact couple as in Definition 4.6, where α, β, γ have bidegrees $(1, -1), (0, 0), (-1, 0)$ respectively.*

Proof. Set $F^p \coloneqq F^pC_\bullet$. If p is fixed, we have a short exact sequence of R-chain complexes

$$0 \longrightarrow F^{p-1} \xrightarrow{j^{p-1}} F^p \xrightarrow{\pi^p} F^p/F^{p-1} \longrightarrow 0,$$

from which we obtain a natural long exact sequence of homology groups:

$$\cdots \longrightarrow H_{p+q}(F^{p-1}) \xrightarrow{\alpha} H_{p+q}(F^p) \xrightarrow{\beta} H_{p+q}(F^p/F^{p-1}) \xrightarrow{\gamma} H_{p+q-1}(F^{p-1}) \longrightarrow \cdots,$$

where $\alpha := j_*^{p-1}$, $\beta := \pi_*^p$ and γ is the connecting homomorphism ∂. Then we simply define bigraded R-modules $D := (D_{p,q})$ with $D_{p,q} := H_{p+q}(F^p)$ and $E := (E_{p,q})$ with $E_{p,q} := H_{p+q}(F^p/F^{p-1})$. The above long exact sequences reduces to

$$\cdots \longrightarrow D_{p-1,q+1} \xrightarrow{\alpha} D_{p,q} \xrightarrow{\beta} E_{p,q} \xrightarrow{\gamma} D_{p-1,q} \longrightarrow \cdots$$

and the maps have the desired bidegrees. Hence $(D, E, \alpha, \beta, \gamma)$ is an exact couple. □

Proposition 4.8. *If $(D, E, \alpha, \beta, \gamma)$ is an exact couple, then $d^1 := \beta \circ \gamma$ is a differential $d^1 : E \to E$, and there exists an exact couple $(D^2, E^2, \alpha^2, \beta^2, \gamma^2)$, called the* ***derived couple****, such that $E^2 = H_*(E, d^1) = \ker(d^1)/\operatorname{im}(d^1)$.*

Proof. A detailed proof can be found in [Rot09, Proposition 10.9]. □

Roughly speaking, the important idea is that we simply repeat the step made in Proposition 4.8. This will essentially allow us to define spectral sequences.

Definition 4.9 (Derived Couples)**.** Let $(D, E, \alpha, \beta, \gamma)$ be an exact couple. Define its **$(r+1)$-st derived couple** inductively as the derived couple of $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$, the r-th derived couple.

Remark 4.10. In the setting of the previous definition, one can use induction arguments to verify the following two important claims, see [Rot09, Corollary 10.10]:

- The differential $d^r : E^r \to E^r$ has bidegree $(-r, r-1)$. Thus, for all p, q, we can write the differentials as maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$.
- For all p, q we have $E^{r+1}_{p,q} = \ker(d^r_{p,q})/\operatorname{im}(d^r_{p+r,q-r+1})$.

Finally, we are able to define spectral sequences.

Definition 4.11 (Spectral Sequence)**.** A (homological) **spectral sequence** is a sequence $(E^r, d^r)_{r\geq 1}$ of bigraded R-modules and bigraded homomorphisms $d^r : E^r \to E^r$ such that:

- For all $r \geq 1$, the maps d^r have bidegree $(-r, r-1)$.
- For all p, q and $r \geq 1$ we have $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$, i.e. d^r is a differential.
- For all p, q and $r \geq 1$ there are isomorphisms $E^{r+1}_{p,q} \cong H_*(E^r_{p,q}, d^r_{p,q})$.

Remark 4.12. Intuitively, a nice way to think of spectral sequences is that each bigraded R-module E^r is a "page" with entries $E^r_{p,q}$. Each differential d^r can be visualized as an arrow with slope $(1-r)/r$ in the E^r-page. The isomorphism $E^{r+1} \cong H_*(E^r, d^r)$ is usually referred to as the "page-turning isomorphism", since we move from the E^r-page to the E^{r+1}-page.

Remark 4.13. In view of Proposition 4.7 and Remark 4.10, we obtain the desired existence result as follows: Every filtration of an R-chain complex $C_\bullet$ yields a spectral sequence as constructed above, namely by Proposition 4.7 the filtration yields an exact couple and by Remark 4.10 we obtain a spectral sequence $(E^r, d^r)_{r\geq 1}$.

Going forward, the next step is to introduce the notion of *convergence* for spectral sequences, which will allow for the recovery of some homological information later on.

4.3 Convergence

So far, we claimed that understanding spectral sequences will offer a new way to compute homology, without actually establishing any useful results. The next goal is to explain the connection between spectral sequences and homology. We begin with some preparations:

Definition 4.14 (Subquotient)**.** Let M be an R-module. A **subquotient** of M is an R-module of the form M'/M'', where $M'' \subseteq M' \subseteq M$ are submodules.

Theorem 4.15 (Correspondence Theorem)**.** *If T is a submodule of a left R-module M, then the map $\varphi : S \mapsto S/T$ is a bijection as follows:*

$$\varphi : \{\textit{intermediate submodules } T \subseteq S \subseteq M\} \to \{\textit{submodules of } M/T\}.$$

Moreover, $T \subseteq S \subseteq S'$ if and only if $S/T \subseteq S'/T$ in M/T.

Proof. A proof can be found in [Rot09, Theorem 2.14]. □

Next, consider a spectral sequence (E^r, d^r). Then, by definition, $E^2 \cong H_*(E^1, d^1)$ is a subquotient of E^1, and hence we can write $E^2 = Z^2/B^2$, where we understand Z^2 to be the cycles $\ker(d^1)$ and B^2 to be the boundaries $\operatorname{im}(d^1)$. Moreover, we have inclusions $B^2 \subseteq Z^2 \subseteq E^1$. Now we use the Correspondence Theorem to infer that every submodule of E^2 is equal to S/B^2 for a unique submodule S of Z^2 with $B^2 \subseteq S$. In particular, the cycles $Z^3 \subseteq E^2$ and boundaries $B^3 \subseteq Z^3 \subseteq E^2$ may be viewed as quotients

$$B^3/B^2 \subseteq Z^3/B^2 \subseteq Z^2/B^2 = E^2.$$

From this observation it follows that we have inclusions

$$B^2 \subseteq B^3 \subseteq Z^3 \subseteq Z^2 \subseteq E^1.$$

Iterating the above steps, we get inclusions of the form

$$B^2 \subseteq \ldots \subseteq B^r \subseteq Z^r \subseteq \ldots \subseteq Z^2 \subseteq E^1,$$

for every $r \geq 2$.

Definition 4.16 (Limit Term)**.** Let $(E^r, d^r)_{r\geq 1}$ be a spectral sequence. Define R-modules $Z^\infty := \bigcap_r Z^r$ and $B^\infty := \bigcup_r B^r$. Then we have $B^\infty \subseteq Z^\infty$ and we define the **limit term** of the spectral sequence as the bigraded R-module E^∞, where $E^\infty_{p,q} := Z^\infty_{p,q}/B^\infty_{p,q}$.

Informally, we say that Z^∞ consists of those elements that "live forever", and B^∞ consists of those that are "eventually bounded". In this way, one can think of the E^r-terms as being "approximations" of the limit term E^∞. We are now finally in a position to define convergence of a spectral sequence.

Definition 4.17 (Convergence)**.** Let $(E^r, d^r)_{r\geq 1}$ be a spectral sequence. We say that $(E^r, d^r)_{r\geq 1}$ **converges** to a graded R-module H if there exists a bounded filtration $(\Phi^p H_n)$ of H such that

$$E^\infty_{p,q} \cong \Phi^p H_n / \Phi^{p-1} H_n \tag{4.1}$$

for all n, where, by convention, we write $n = p + q$. We typically denote the above convergence by the expression

$$E^2_{p,q} \implies H_n.$$

Remark 4.18. If a spectral sequence $(E^r, d^r)_{r\geq 1}$ converges to a graded R-module H, it is possible to recover some information about H. Using (4.1) and the fact that the given filtration of H_n is bounded, we can summarize convergence by a finite number of extension problems as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 = \Phi^{-1}H_n & \longrightarrow & \Phi^0 H_n & \xrightarrow{\cong} & E^\infty_{0,n} & \longrightarrow & 0 \\
0 & \longrightarrow & \Phi^0 H_n & \longrightarrow & \Phi^1 H_n & \longrightarrow & E^\infty_{1,n-1} & \longrightarrow & 0 \\
0 & \longrightarrow & \Phi^1 H_n & \longrightarrow & \Phi^2 H_n & \longrightarrow & E^\infty_{2,n-2} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \Phi^{n-1}H_n & \longrightarrow & \Phi^n H_n = H_n & \longrightarrow & E^\infty_{n,0} & \longrightarrow & 0
\end{array}$$

Essentially, this means that the R-module H_n is isomorphic to some extension of the modules $E^\infty_{p,q}$ that lie on the "diagonal" given by $p + q = n$. In particular, H_n is not necessarily uniquely determined by these extension problems. However, in practice, oftentimes the majority of R-modules $E^\infty_{p,q}$ turn out to be trivial, which makes the extension problems significantly easier, and sometimes even solvable. If there is only one non-trivial R-module $E^\infty_{p,q}$ on the line $p+q = n$, we call the corresponding extension problem **trivial**.

One last observation is that, by the definition of induced filtrations, these extension problems are natural in chain complexes.

The following theorem is at the heart of everything that will have to do with converging spectral sequences. A direct consequence of this theorem will be a similar theorem dealing with converging spectral sequences and total complexes, which we use to prove the main theorem in the next chapter.

Theorem 4.19 (Filtration Spectral Sequence)**.** *Let $(F^p C_\bullet)$ be a bounded filtration of an R-chain complex $C_\bullet$ and let $(E^r, d^r)_{r\geq 1}$ be the spectral sequence arising from this filtration, as discussed in Remark 4.13. Then:*

(1) For each p, q we have $E^\infty_{p,q} \cong E^r_{p,q}$ for sufficiently large r, depending on p and q.

(2) We have a naturally converging spectral sequence $E^2_{p,q} \implies H_n(C_\bullet)$, where $n = p+q$.

Proof. A thorough proof can be found in [Rot09, Theorem 10.14]. An alternative proof, which does not rely on exact couples in any way, can be found in [Gru23, Theorem 2.1]. □

Remark 4.20. For Theorem 4.19, naturality is to be understood as follows: Suppose we have a filtration-preserving map $f : (F^p C_\bullet) \to (F^p D_\bullet)$ of filtered R-chain complexes. Then the maps induced by f on the E^r-pages are compatible with the differentials d^r for all r. Moreover, the map on the E^{r+1}-pages, induced by f, is induced by the map on the E^r-pages.

Definition 4.21 (Transpose)**.** Let $(M_{\bullet,\bullet}, d', d'')$ be a double complex. Then the **transpose** $M^\top_{\bullet,\bullet}$ is defined by $M^\top_{p,q} := M_{q,p}$ and hence $\bigoplus_{i \leq p} M^\top_{i,n-i} = \bigoplus_{i \leq p} M_{n-i,i}$.

Using the notion of the transpose, an important observation is that the second filtration ${}^{II}F^p$ of $\mathrm{Tot}(M)$ is nothing but the first filtration ${}^{I}F^p$ of $\mathrm{Tot}(M^\top)$, where $\mathrm{Tot}(M^\top)$ is the total complex associated with the transposed double complex $(M^\top_{\bullet,\bullet}, d'', d')$.

Corollary 4.22 (Double Complex Spectral Sequence)**.** *Let $(M_{\bullet,\bullet}, d', d'')$ be a first quadrant double complex, and let ${}^{I}E^r$ and ${}^{II}E^r$ be the spectral sequences determined by the first and second filtrations of* $\mathrm{Tot}(M_{\bullet,\bullet})$*. Then:*

(1) The first and second filtration are bounded.

(2) For all p, q we have ${}^{I}E^\infty_{p,q} \cong {}^{I}E^r_{p,q}$ and ${}^{II}E^\infty_{p,q} \cong {}^{II}E^r_{p,q}$ for sufficiently large r, depending on p and q.

(3) ${}^{I}E^2_{p,q} \implies H_n(\mathrm{Tot}(M_{\bullet,\bullet}))$ and ${}^{II}E^2_{p,q} \implies H_n(\mathrm{Tot}(M_{\bullet,\bullet}))$. Both converge naturally.

Proof. The bounds of the first and second filtrations are given by $s(n) = -1$ and $t(n) = n$, since M was a first quadrant double complex. This means that (2) and (3) for ${}^{I}E$ follow directly from Theorem 4.19. Next, we use that $\mathrm{Tot}(M) = \mathrm{Tot}(M^\top)$ and that the second filtration of $\mathrm{Tot}(M)$ equals the first filtration of $\mathrm{Tot}(M^\top)$ to deduce that ${}^{II}E^\infty_{p,q} \cong {}^{II}E^r_{p,q}$ for sufficiently large r and that ${}^{II}E^2_{p,q} \implies H_n(\mathrm{Tot}(M^\top)) = H_n(\mathrm{Tot}(M))$, as claimed. □

Remark 4.23. For Corollary 4.22, naturality is to be understood as follows: Suppose we have a morphism $\tau : M_{\bullet,\bullet} \to N_{\bullet,\bullet}$ of double complexes, i.e. in particular it preserves the first and second filtrations of the total complexes. Then the maps induced by τ on the E^r-pages are compatible with the differentials d^r for all r. Moreover, the map on the E^{r+1}-pages, induced by τ, is induced by the map on the E^r-pages.

In the setting of Corollary 4.22 we now have, in theory, two converging spectral sequences, which allow us to recover information about the homology of the total complex. However, computing the modules ${}^{I}E^r_{p,q}$ and ${}^{II}E^r_{p,q}$ may not be very straightforward, which is why we give an alternative description, allowing for easier calculations.

Let us focus only on the first case and write $E^r := {}^I E^r$ and let F^p denote the first filtration ${}^I F^p$ of $\operatorname{Tot}(M)$, where we simply write M for $M_{\bullet,\bullet}$. By definition (as seen in the proof of Proposition 4.7), we have $E^1_{p,q} = H_n(F^p/F^{p-1})$. Consider the R-modules $(F^p)_n$ and $(F^{p-1})_n$ as in Definition 4.3. Then, taking quotients kills everything but $(F^p/F^{p-1})_n = M_{p,q}$, showing that the n-th term of the quotient complex F^p/F^{p-1} is given by $M_{p,q}$. We have an induced differential

$$\overline{\partial}_n : (F^p/F^{p-1})_n \to (F^p/F^{p-1})_{n-1}, \qquad x + (F^{p-1})_n \mapsto \partial_n(x) + (F^{p-1})_{n-1},$$

where ∂_n is the differential from the total complex and $x \in (F^p)_n$. In particular, we may assume that $x \in M_{p,q}$, as we have seen. By definition of ∂_n one sees that $\overline{\partial}_n$ is well-defined for all n. We know that

$$\partial_n(x) = (d'_{p,q} \oplus d''_{p,q})(x) \in M_{p-1,q} \oplus M_{p,q-1}.$$

However, notice that $M_{p-1,q} \subseteq (F^{p-1})_{n-1}$. Consequently, the first part of ∂_n is killed and we are left with $\partial_n(x) \equiv d''_{p,q} \mod (F^{p-1})_{n-1}$. Moreover, we find that

$$E^1_{p,q} = H_n(F^p/F^{p-1}) = \ker(\overline{\partial}_n)/\operatorname{im}(\overline{\partial}_{n+1}) \cong \ker(d''_{p,q})/\operatorname{im}(d''_{p,q+1}) = H_q(M_{p,\bullet}),$$

which is arguably much easier to compute than $H_n(F^p/F^{p-1})$ itself. Here, we view $M_{p,\bullet}$ as the p-th column of M with differential $\overline{d}''$ induced by d''. Thus, we have constructed a new bigraded R-module with (p,q)-term $H_q(M_{p,\bullet}, \overline{d}'')$. Now, for each fixed q, consider the q-th row, denoted by $H''(M)_{\bullet,q}$, given by R-modules

$$\ldots H_q(M_{p+1,\bullet}),\ H_q(M_{p,\bullet}),\ H_q(M_{p-1,\bullet}), \ldots. \tag{4.2}$$

This sequence of R-modules can be turned into an R-chain complex by defining a differential $\overline{d}'_p : H_q(M_{p,\bullet}) \to H_q(M_{p-1,\bullet})$ via $[z] \mapsto [d'_{p,q}(z)]$, where $z \in \ker(d''_{p,q})$ by definition of $H_q(M_{p,\bullet})$. This differential is well-defined in view of the identity $d'd'' = -d''d'$. We can define a new bigraded R-module with (p,q)-term being the p-th homology of the q-th row (4.2), denoted by $H'_pH''_q(M)$.

Definition 4.24 (First Iterated Homology)**.** If (M, d', d'') is a double complex, we define its **first iterated homology** to be the bigraded R-module with (p,q)-term $H'_pH''_q(M)$.

Note that the above derivation of the first iterated homology works in a similar fashion with ${}^{II}E^r$, from which one obtains the **second iterated homology**.

Finally, by the next proposition it follows that taking first iterated homology $H'_pH''_q(M)$ really is the same as computing ${}^IE^r_{p,q}$ and hence both converge to $H_n(\operatorname{Tot}(M))$. This is necessary because a priori it is not clear that the map $d^1 = \beta \circ \gamma$ from Proposition 4.8 maps $[z] \in E^1_{p,q}$ to $[\overline{d'_{p,q}}(z)] \in E^1_{p-1,q}$. Being able to compute ${}^IE^r_{p,q}$ in this way will be useful in the proof of the main theorem in Chapter 5.2.

Proposition 4.25. *If $(M_{\bullet,\bullet}, d', d'')$ is a first quadrant double complex, then ${}^IE^1_{p,q} = H_q(M_{p,\bullet})$ and we have a converging spectral sequence of the form*

$${}^IE^2_{p,q} = H'_pH''_q(M_{\bullet,\bullet}) \implies H_n(\operatorname{Tot}(M_{\bullet,\bullet})).$$

Proof. A proof is found in [Rot09, Proposition 10.17]. ∎

5 The Main Theorem

The main result of this thesis will be stated in terms of group extensions, which is why we start this chapter by giving its definition. Moreover, we discuss certain group actions necessary for developing a good understanding of the main theorem and for concrete computations in later chapters.

5.1 Group Action on Group Homology

Definition 5.1 (Group Extension)**.** A group G is an **extension** of groups Q by H if there exists a short exact sequence of the form

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1.$$

Remark 5.2 (Quotient Acts on Kernel)**.** Consider a group extension

$$0 \longrightarrow H \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$$

with an abelian kernel (which is why we write 0 at the start). We claim that this extension induces an action of Q on the group H. To this end, notice that G acts on its normal subgroup $i(H)$ via conjugation, and hence also on $H \cong i(H)$, which we denote by $G \overset{c}{\curvearrowright} H$. But since H was assumed to be abelian, the restricted action $H \overset{c}{\curvearrowright} H$ is just the trivial action and hence we obtain a well defined action of $Q \cong G/H$ on the group H, given by $i(q \cdot x) = gi(x)g^{-1}$, where $x \in H$ and $\pi(g) = q$. This remark will later be useful for concrete computations using the main theorem, e.g. in Chapter 6.

For a precise description of our main theorem, we will come to find that we need a group action $G/H \curvearrowright H_n(H; M)$ for all $n \geq 0$, where G is a group, H a normal subgroup and M a $\mathbb{Z}G$-module. Luckily, there is a way in which we can state this action rather explicitly. Let us start by defining a category in which we work throughout this chapter. We loosely follow [Löh19, Chapters 1.1.3, 1.2.2] and [Bro82, Chapter III.8]:

Definition 5.3. Let **GrpMod** be the category consisting of:

- Objects: Pairs (G, M), where G is a group and M is a $\mathbb{Z}G$-module.
- Morphisms: Maps $(\alpha, f) : (G, M) \to (G', M')$, where $\alpha : G \to G'$ is a group homomorphism and $f : M \to M'$ is a map of abelian groups compatible with α, i.e. $f(gm) = \alpha(g)f(m)$, for $g \in G$ and $m \in M$.

Now, given such a pair (α, f), let $F_\bullet$ and $F'_\bullet$ be projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively. Moreover, let $\tau^\alpha_\bullet : F_\bullet \to F'_\bullet$ be a chain map compatible with α as in Chapter 2.2, i.e. $\tau^\alpha_\bullet(gx) = \alpha(g)\tau^\alpha_\bullet(x)$ for any $g \in G$ and $x \in F_\bullet$. Then there exists a chain map of the form

$$\tau^\alpha_\bullet \otimes f : F_\bullet \otimes_G M \to F'_\bullet \otimes_{G'} M',$$

which in turn induces a well-defined map

$$(\alpha, f)_* : H_n(G; M) \to H_n(G'; M')$$

for any $n \geq 0$. In this way, we get a covariant functor $H_n(-;-) : \mathbf{GrpMod} \to \mathbb{Z}\text{-}\mathbf{Mod}$ for all $n \geq 0$. By the above, we get an explicit description of this map as follows:

$$(\alpha, f)_* ([gx \otimes_G hm]) = [\alpha(g)\tau_n^\alpha(x) \otimes_{G'} \alpha(h)f(m)] \,,$$

where $g, h \in G$, $x \in F_n$ and $m \in M$. In the special case that $M = M'$ and $f = \mathrm{id}$, we simply write $\alpha_* = (\alpha, f)_*$. Note that we can always write $(\alpha, f)_*$ as the composite $\alpha_* \circ H_*(G; f) : H_*(G; M) \to H_*(G; M') \to H_*(G'; M')$, which can be shown by considering the explicit descriptions of these two maps.

Before continuing, we briefly turn back towards Shapiro's Lemma.

Lemma 5.4. *Let G be a group, H a subgroup and M a $\mathbb{Z}H$-module. Then the isomorphism $H_n(H; M) \cong H_n(G; \mathrm{Ind}_H^G(M))$ from Lemma 2.15 is induced by the inclusion $(\alpha, i) : (H, M) \hookrightarrow (G, \mathrm{Ind}_H^G(M))$, $(h, m) \mapsto (h, 1 \otimes m)$, in* **GrpMod**.

Proof. Recall that we had a projective resolution $F_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}G$. We can take $\tau_\bullet^\alpha : F_\bullet \to F_\bullet$ to be the identity, since $\tau_\bullet(hx) = hx = \alpha(h)\tau_\bullet(x)$ for $h \in H$ and $x \in F_\bullet$. Moreover, $\mathrm{Res}_H^G(F_\bullet)$ was a free resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. Notice that $(\alpha, i)_*$ on homology is induced by the map $\mathrm{Res}_H^G(F_\bullet) \otimes_H M \to F_\bullet \otimes_G \mathrm{Ind}_H^G(M)$, given by $x \otimes m \mapsto x \otimes (1 \otimes m)$, which is precisely the map (2.2) in the proof of Shapiro's Lemma. Finally, we see that this map is a chain homotopy equivalence, since we can define an inverse via the map $x \otimes (g \otimes m) \mapsto xg \otimes m$, and hence the isomorphism (2.1) and $(\alpha, i)_*$ agree. □

Let us continue with an important example of a morphism in **GrpMod**. Consider for groups $H \leq G$, M a $\mathbb{Z}G$-module and $g \in G$ the conjugation map

$$\begin{aligned} c(g) : (H, M) &\to (gHg^{-1}, M) \\ (h, m) &\mapsto (ghg^{-1}, gm), \end{aligned}$$

where we write $\alpha(h) = ghg^{-1}$ and $f(m) = gm$. In order to compute $c(g)_* : H_n(H; M) \to H_n(gHg^{-1}; M)$, we analyse the map on the chain level: Choose again a projective resolution $F_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}G$. Then we use this resolution to compute the homology of H as well as of gHg^{-1}, which works because restricting a resolution gives a resolution once more, see Proposition 2.12. As in [Bro82, Proposition II.6.2] we can choose $\tau_\bullet : F_\bullet \to F_\bullet$ to simply be multiplication by g, since, in this case, τ commutes with the boundary operator $\partial_\bullet$ of $F_\bullet$ and is compatible with α, which can be checked by hand. Tensoring with M, we get an induced map

$$\tau_\bullet \otimes (g\cdot\,) : F_\bullet \otimes_H M \to F_\bullet \otimes_{gHg^{-1}} M \tag{5.1}$$

between chain complexes, and we see that

$$c(g)_*([hx \otimes m]) = [\alpha(h)\tau_n(x) \otimes gm] = [ghg^{-1} \cdot gx \otimes gm] = [g(hx) \otimes gm],$$

with $h \in H$, $x \in F_\bullet$ and $m \in M$, i.e. $c(g)_*$ can be viewed as the map induced by the diagonal action map. Now, for some $z \in H_n(H; M)$, we set

$$g \cdot z \coloneqq c(g)_*(z) \in H_n(gHg^{-1}; M).$$

Proposition 5.5. *If $h \in H$, then $h \cdot z = z$ for all $z \in H_n(H; M)$ and $n \geq 0$.*

Proof. Fix some n and let $z = [x \otimes m]$ with $x \in F_n$ and $m \in M$. Then we compute $h \cdot z = c(h)_*(z) = [hx \otimes hm] = [x \otimes m] = z$, where we used that $hHh^{-1} \cong H$ and hence $F_n \otimes_{hHh^{-1}} M \cong F_n \otimes_H M \cong (F_n \otimes M)_H$, so we kill the diagonal H-action. □

The above proposition is used to show well-definedness in the following corollary:

Corollary 5.6. *If $n \geq 0$, G is a group, H is a normal subgroup and M is a $\mathbb{Z}G$-module, then the conjugation action $c(g)$ induces an action of G/H on $H_n(H; M)$ as follows:*

$$\begin{aligned} G/H \times H_n(H; M) &\to H_n(H; M) \\ ([g], z) &\mapsto c(g)_*(z). \end{aligned}$$

From now on, whenever we mention an action of the form $G/H \curvearrowright H_n(H; M)$, we will always mean the action from Corollary 5.6. We end this chapter with a lemma on central groups, which will prove to be useful for later results.

Lemma 5.7. *Let G be a group, let $H \leq G$ be a central subgroup and let M be an abelian group with trivial G-action. Then the action $G/H \curvearrowright H_n(H; M)$ is trivial for all $n \geq 0$.*

Proof. In this setting we have $ghg^{-1} = h$ and $gm = m$, and hence the isomorphism $c(g) : (H, M) \to (gHg^{-1}, M) = (H, M)$ becomes the identity. By Corollary 5.6 there is an action $G/H \curvearrowright H_n(H; M)$. Denote by α the homomorphism $H \to gHg^{-1} = H$ in $c(g)$. If we take some projective resolution $F_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}G$, then we can choose the chain map $\tau_\bullet : F_\bullet \to F_\bullet$ to be the identity, since it still satisfies the condition $\tau_\bullet(hx) = \alpha(h)\tau_\bullet(x)$ for $h \in H$ and $x \in F_\bullet$. Next, consider the chain map $F_\bullet \otimes_H M \to F_\bullet \otimes_H M$ as in (5.1). We see that it is given by $x \otimes m \mapsto x \otimes gm = x \otimes m$, i.e. the identity. Therefore, the map $c(g)_*$ on homology is the identity as well, giving the trivial action $G/H \curvearrowright H_n(H; M)$. □

5.2 Proof of the Main Theorem

With all necessary tools at hand, we begin by stating the main result of the thesis:

Theorem 5.8 (Lyndon-Hochschild-Serre, [Bro82, Theorem VII.6.3])**.** *If M is a $\mathbb{Z}G$-module, then any group extension $1 \to H \to G \to Q \to 1$ admits a natural spectral sequence of the form*

$$E^2_{p,q} = H_p(Q; H_q(H; M)) \implies H_{p+q}(G; M).$$

Remark 5.9. To be more precise, we should write $\mathrm{Res}^G_H(M)$ instead of just M in the homology group $H_q(H; M)$ in Theorem 5.8, since we want a $\mathbb{Z}H$-module in the second entry of $H_q(H; -)$. We slightly abuse notation to declutter the expression.

Remark 5.10. For the term $H_p(Q; H_q(H; M))$ to be well-defined, we must make sure that $H_q(H; M)$ is a $\mathbb{Z}Q$-module. However, this is guaranteed by Corollary 5.6, using that $Q \cong G/H$.

Remark 5.11 ([Löh19, Remark 3.2.14])**.** Naturality of the LHS spectral sequence is to be understood as follows: Consider a commutative diagram of groups with exact rows

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r] \arrow[d, "f"'] & G \arrow[r] \arrow[d, "g"'] & Q \arrow[r] \arrow[d, "h"'] & 1 \\ 1 \arrow[r] & H' \arrow[r] & G' \arrow[r] & Q' \arrow[r] & 1 \end{tikzcd}$$

and a $\mathbb{Z}G'$-module M, which can be viewed as a $\mathbb{Z}G$-module via the map $g : G \to G'$. Then we have a morphism of spectral sequences

$$\begin{tikzcd} E^2_{p,q} = H_p(Q; H_q(H; M)) \arrow[r, Rightarrow] \arrow[d, "H_p(h; H_q(f; M))"'] & H_{p+q}(G; M) \arrow[d, "H_{p+q}(g; M)"] \\ E^2_{p,q} = H_p(Q'; H_q(H'; M)) \arrow[r, Rightarrow] & H_{p+q}(G'; M) \end{tikzcd}$$

that satisfy the conditions from Remark 4.23.

Proof of Theorem 5.8. Recall that the homology $H_n(G; M)$ of a group G was defined as $H_n(F_\bullet \otimes_G M)$ for some projective resolution $F_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}G$. Now we replace the $\mathbb{Z}G$-module M by a non-negative chain complex $C_\bullet = (C_n)_{n \geq 0}$ of $\mathbb{Z}G$-modules and set

$$H_n(G; C_\bullet) := H_n(F_\bullet \otimes_G C_\bullet),$$

for any $n \geq 0$. Notice that $F_\bullet \otimes_G C_\bullet$ is, by definition, nothing but the total complex of the double complex given by the groups $F_p \otimes_G C_q$ for $p, q \in \mathbb{N}$. By this observation, we get from Theorem 4.22 that there exist two spectral sequences converging to $H_*(G; C_\bullet)$.

For the first of these two, we have that

$$E^1_{p,q} = H_q(F_p \otimes_G C_\bullet) \cong F_p \otimes_G H_q(C_\bullet),$$

where we use the universal coefficient theorem for chain complexes and the fact that F_p is a projective $\mathbb{Z}G$-module. Taking the first iterated homology, see Definition 4.24, yields a converging spectral sequence of the form

$$E^2_{p,q} = H_p(F_\bullet \otimes_G H_q(C_\bullet)) = H_p(G; H_q(C_\bullet)) \implies H_{p+q}(G; C_\bullet). \tag{5.2}$$

The second one of these spectral sequences then gives

$$E^1_{p,q} = H_q(F_\bullet \otimes_G C_p) = H_q(G; C_p) \implies H_{p+q}(G; C_\bullet). \tag{5.3}$$

Now suppose that C_p is H_*-acyclic for all $p \geq 0$, i.e. $H_n(G; C_p) \cong 0$ for all $n \geq 1$. Then the E^1-term of (5.3) is non-trivial only on the line $q = 0$ with $E^1_{p,0} = H_0(G; C_p) \cong (C_p)_G$ and, in particular, this spectral sequence degenerates at the E^2-term, meaning that $E^r \cong E^2$ for all $r \geq 2$, since all differentials are trivial for $r \geq 2$. In particular, this means that $E^\infty \cong E^2$. Taking second iterated homology then gives

$$H_n(G; C_\bullet) \cong H_n((C_\bullet)_G), \tag{5.4}$$

since all extension problems are trivial. By the isomorphism (5.4) we can then write the spectral sequence (5.2) simply as

$$E^2_{p,q} = H_p(G; H_q(C_\bullet)) \implies H_{p+q}((C_\bullet)_G).$$

For the next step, consider a group extension $1 \to H \to G \to Q \to 1$ and let $F_\bullet$ again be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and M a $\mathbb{Z}G$-module and let $\tilde{F}_\bullet$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}Q$. Now the group Q takes the role of the group G above, and we consider the total complex $\tilde{F}_\bullet \otimes_Q (F_\bullet \otimes_H M)$. This is the total complex that will allow us to compute the homology of G, since we have isomorphisms

$$\begin{aligned} H_n(G; M) &= H_n(F_\bullet \otimes_G M) \\ &\cong H_n((F_\bullet \otimes_H M)_Q) \\ &\cong H_n(Q; F_\bullet \otimes_H M) \\ &= H_n(\tilde{F}_\bullet \otimes_Q (F_\bullet \otimes_H M)), \end{aligned} \tag{5.5}$$

where we used that $F_\bullet \otimes_G M \cong (F_\bullet \otimes_H M)_Q$ (Lemma 1.8) and (5.4). Once again, we technically consider $\mathrm{Res}^G_H(F_\bullet)$ and $\mathrm{Res}^G_H(M)$ in the isomorphism $F_\bullet \otimes_G M \cong (F_\bullet \otimes_H M)_Q$, but we omit this notation for brevity. As before, the first spectral sequence gives terms

$$E^1_{p,q} \cong H_q(\tilde{F}_p \otimes_Q (F_\bullet \otimes_H M)) \cong \tilde{F}_p \otimes_Q H_q(F_\bullet \otimes_H M) \cong \tilde{F}_p \otimes_Q H_q(H; M)$$

and this yields a chain complex $\tilde{F}_\bullet \otimes_Q H_q(H; M)$ with differentials $\tilde{d}_\bullet \otimes_Q \mathrm{id}$, where the maps $\tilde{d}_p : \tilde{F}_p \to \tilde{F}_{p-1}$ come from the projective resolution $\tilde{F}_\bullet$. Then, taking first iterated homology, we get a converging spectral sequence

$$E^2_{p,q} = H_p(\tilde{F}_\bullet \otimes_Q H_q(F_\bullet \otimes_H M)) \implies H_{p+q}(\tilde{F}_\bullet \otimes_Q (F_\bullet \otimes_H M)).$$

Finally, using (5.5) and the fact that $H_q(F_\bullet \otimes_H M) \cong H_q(H; M)$, we get a converging spectral sequence

$$E^2_{p,q} = H_p(Q; H_q(H; M)) \implies H_{p+q}(G; M),$$

as claimed. Before turning to the proof of naturality, it remains to show that the $\mathbb{Z}Q$-modules $C_p \coloneqq F_p \otimes_H M$ are indeed H_*-acyclic, as was assumed at the start of the proof, in order to obtain the isomorphism (5.4):

Lemma 5.12. *Let $C_\bullet$, $F_\bullet$ and M be as above. Then the $\mathbb{Z}Q$-modules $C_p = F_p \otimes_H M$ are H_*-acyclic for all $p \geq 0$.*

Proof. First, notice that we can take $F_\bullet$ to be the standard resolution, i.e. $F_p \cong \bigoplus_{I_p} \mathbb{Z}G$ with I_p an index set, and hence

$$F_p \otimes_H M \cong \bigoplus_{I_p} (\mathbb{Z}G \otimes_H M).$$

It suffices to show that the modules $\mathbb{Z}G \otimes_H M$ are H_*-acyclic, since

$$\begin{aligned} H_n(Q; C_p) &\cong H_n(F'_\bullet \otimes_Q (F_p \otimes_H M)) \\ &\cong H_n\Big(F'_\bullet \otimes_Q \bigoplus\nolimits_{I_p} (\mathbb{Z}G \otimes_H M)\Big) \\ &\cong H_n\Big(\bigoplus\nolimits_{I_p} (F'_\bullet \otimes_Q \mathbb{Z}G \otimes_H M)\Big) \\ &\cong \bigoplus\nolimits_{I_p} H_n(F'_\bullet \otimes_Q (\mathbb{Z}G \otimes_H M)) \\ &\cong \bigoplus\nolimits_{I_p} H_n(Q; \mathbb{Z}G \otimes_H M), \end{aligned}$$

where $F'_\bullet$ is some projective resolution of $\mathbb{Z}$ over $\mathbb{Z}Q$ and $n \geq 1$. In order to show that $H_n(Q; \mathbb{Z}G \otimes_H M) \cong 0$ for all $n \geq 1$, define $\mathbb{Z}Q$-module homomorphisms

$$\begin{aligned} \varphi : \mathbb{Z}G \otimes_H M &\to \mathbb{Z}Q \otimes M \\ (g \otimes_H m) &\mapsto \pi(g) \otimes g^{-1}m, \end{aligned} \qquad \begin{aligned} \psi : \mathbb{Z}Q \otimes M &\to \mathbb{Z}G \otimes_H M \\ q \otimes m &\mapsto g \otimes_H gm, \end{aligned}$$

where $\pi : G \to Q$ is the canonical projection and $\pi(g) = q$ in the definition of ψ. Then one checks that both maps are well-defined and mutually inverse, so we have shown that

$$\mathbb{Z}G \otimes_H M \cong \mathbb{Z}Q \otimes M = \mathrm{Ind}_{\{1\}}^Q(M),$$

and from Corollary 2.18 we know that induced modules are H_*-acyclic, which finishes the proof of the lemma. □

To finish the main proof, we show naturality of the LHS spectral sequence. Consider a commutative diagram as in Remark 5.11. Moreover, let $(\tilde{F}_\bullet, \tilde{d}_\bullet)$, $(F_\bullet, d_\bullet)$, $(\tilde{F}'_\bullet, \tilde{d}'_\bullet)$ and $(F'_\bullet, d'_\bullet)$ be projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}Q$, $\mathbb{Z}G$, $\mathbb{Z}Q'$ and $\mathbb{Z}G'$ respectively. Let $\tau^h_\bullet : \tilde{F}_\bullet \to \tilde{F}'_\bullet$ and $\tau^f_\bullet : F_\bullet \to F'_\bullet$ be the chain maps that we get from the fundamental lemma, as in Chapter 2.2, and define a map

$$\tau_{\bullet,\bullet} := \tau^h_\bullet \otimes_Q (\tau^f_\bullet \otimes_H \mathrm{id}_M) : \underbrace{\tilde{F}_\bullet \otimes_Q (F_\bullet \otimes_H M)}_{C_{\bullet,\bullet}} \to \underbrace{\tilde{F}'_\bullet \otimes_{Q'} (F'_\bullet \otimes_{H'} M)}_{C'_{\bullet,\bullet}},$$

between double complexes, where we take M to be a $\mathbb{Z}G'$-module, which can be viewed as a $\mathbb{Z}G$-module via the map g. Lastly, consider the following differentials, which are part of the data of these double complexes:

$$\begin{aligned} \partial^H_{p,q} : \tilde{F}_p \otimes_Q (F_q \otimes_H M) &\to \tilde{F}_{p-1} \otimes_Q (F_q \otimes_H M) \\ \tilde{x} \otimes (x \otimes m) &\mapsto \tilde{d}_p(\tilde{x}) \otimes (x \otimes m), \end{aligned}$$

$$\begin{aligned} \partial^V_{p,q} : \tilde{F}_p \otimes_Q (F_q \otimes_H M) &\to \tilde{F}_p \otimes_Q (F_{q-1} \otimes_H M) \\ \tilde{x} \otimes (x \otimes m) &\mapsto (-1)^p \tilde{x} \otimes (d_q(x) \otimes m), \end{aligned}$$

and similarly for $\tilde{F}'_\bullet$ and $F'_\bullet$. The goal is to show that, for all p, q, we have two commutative diagrams as follows:

$$\begin{tikzcd} C_{p,q} \arrow[r, "\tau_{p,q}"] \arrow[d, "\partial^V_{p,q}"'] & C'_{p,q} \arrow[d, "\partial'^V_{p,q}"] \\ C_{p,q-1} \arrow[r, "\tau_{p,q-1}"'] & C'_{p,q-1} \end{tikzcd} \qquad \begin{tikzcd} C_{p,q} \arrow[r, "\tau_{p,q}"] \arrow[d, "\partial^H_{p,q}"'] & C'_{p,q} \arrow[d, "\partial'^H_{p,q}"] \\ C_{p-1,q} \arrow[r, "\tau_{p-1,q}"'] & C'_{p-1,q} \end{tikzcd}$$

We will show commutativity of the first square, which can be checked directly by using the above definitions. For any $\tilde{x} \in \tilde{F}_p$, $x \in F_q$ and $m \in M$ we have

$$\begin{aligned} (\partial'^V_{p,q} \circ \tau_{p,q})(\tilde{x} \otimes (x \otimes m)) &= \partial'^V_{p,q} \left(\tau^h_p(\tilde{x}) \otimes (\tau^f_q(x) \otimes m)\right) \\ &= (-1)^p \tau^h_p(\tilde{x}) \otimes \left(d'_q\big(\tau^f_q(x)\big) \otimes m\right) \\ &= \tau^h_p\big((-1)^p \tilde{x}\big) \otimes \left(\tau^f_{q-1}\big(d_q(x)\big) \otimes m\right) \\ &= \tau_{p,q-1}\big((-1)^p \tilde{x} \otimes (d_q(x) \otimes m)\big) \\ &= (\tau_{p,q-1} \circ \partial^V_{p,q})(\tilde{x} \otimes (x \otimes m)), \end{aligned}$$

where we used the fact that $\tau^f_\bullet$ is a chain map $F_\bullet \to F'_\bullet$, i.e. it is compatible with the differentials $d_\bullet$ and $d'_\bullet$. A similar calculation shows that the second square commutes as well. It follows that we have a morphism $\tau_{\bullet,\bullet}$ of double complexes. By Theorem 4.22, we get that the LHS spectral sequence is natural with respect to maps induced by $\tau_{\bullet,\bullet}$, and in particular, by definition of this map and by the construction of the spectral sequence, it is given precisely by $H_p(h; H_q(f; M)) = (\tau^h_\bullet \otimes_Q (\tau^f_\bullet \otimes_H \mathrm{id}_M)_*)_*$ on the E^2-page, as claimed. This finishes the proof of the main theorem. $\square$

6 Explicit Computations

In this chapter, we use the LHS spectral sequence in two concrete examples, the first of which is a computation that we have actually already encountered in a different context. However, it serves as a nice illustration of the general approach to such problems. The second example is more involved, but it shows that we do in fact obtain new results, which were not easily computable before.

Example 6.1. The goal is to compute the homology of the *infinite dihedral group*

$$D_\infty = \langle d, s \mid s^2, sdsd \rangle, \tag{6.1}$$

with trivial $\mathbb{Z}$-coefficients, as seen in Example 3.11. However, this time we use spectral sequences to do so, completely ignoring the isomorphism $H_n(G) \cong H_n(BG)$. We follow [Löh19, Example 3.2.17]. Notice that D_∞ contains the normal subgroup $\mathbb{Z} \cong \langle d \rangle$, so we have a group extension

$$1 \longrightarrow \mathbb{Z} \xrightarrow{i} D_\infty \xrightarrow{\pi} \mathbb{Z}/2 \longrightarrow 1.$$

The action $\mathbb{Z}/2 \curvearrowright \mathbb{Z}$ is given by taking inverses, i.e. $[1] \cdot n = -n$. To see this, we use the definition from Remark 5.2: Choose some $\tilde{g} \in D_\infty$ such that $\pi(\tilde{g}) = [1] \in \mathbb{Z}/2$, i.e. we can choose $\tilde{g} = s$. Then $\mathbb{Z}/2 \curvearrowright \mathbb{Z}$ is fully characterized by

$$i([1] \cdot a) = si(a)s^{-1} = si(a)ss^{-1}s^{-1} = si(a)s = i(a)^{-1} = i(-a)$$

where $a \in \mathbb{Z}$. Since i is injective, we get $[1] \cdot a = -a$, as claimed. The LHS spectral sequence takes the form

$$E^2_{p,q} = H_p(\mathbb{Z}/2; H_q(\mathbb{Z}; \mathbb{Z})) \implies H_{p+q}(D_\infty; \mathbb{Z}),$$

where $\mathbb{Z}/2 \curvearrowright H_q(\mathbb{Z}; \mathbb{Z})$ is given by the map induced by the above action $\mathbb{Z}/2 \curvearrowright \mathbb{Z}$. For $q \geq 2$ we have $H_q(\mathbb{Z}; \mathbb{Z}) \cong 0$, since $B\mathbb{Z} \simeq S^1$, so we only need to worry about the cases $q = 0, 1$. The following fact is shown in Remark A.1 in the appendix:

$\mathbb{Z}/2 \curvearrowright H_0(\mathbb{Z}; \mathbb{Z})$ is trivial and $\mathbb{Z}/2 \curvearrowright H_1(\mathbb{Z}; \mathbb{Z})$ is multiplication by (-1).

Right away, we find that for $q = 0$ we have $H_p(\mathbb{Z}/2; \mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \ldots)$. For $q = 1$ we want to compute $H_p(\mathbb{Z}/2; \mathbb{Z})$ with the *non-trivial* action. To this end, we use Proposition 2.8 with $t := [1] \in \mathbb{Z}/2$ and $N = \sum_{j=0}^{n-1} t^j = 1 + t$, to obtain

$$\begin{aligned} H_0(\mathbb{Z}/2; \mathbb{Z}) &\cong \mathbb{Z}/\langle g \cdot n - n \mid g \in \mathbb{Z}/2, n \in \mathbb{Z} \rangle \\ &\cong \mathbb{Z}/\langle [1] \cdot n - n \mid n \in \mathbb{Z} \rangle \\ &\cong \mathbb{Z}/\langle -2n \mid n \in \mathbb{Z} \rangle \\ &\cong \mathbb{Z}/2 \end{aligned}$$

and

$$\begin{aligned} H_1(\mathbb{Z}/2;\mathbb{Z}) &\cong \mathbb{Z}^{\mathbb{Z}/2} \Big/ (1+t)\mathbb{Z} \\ &\cong \{n \in \mathbb{Z} \mid g \cdot n = n,\ g \in \mathbb{Z}/2, n \in \mathbb{Z}\} \Big/ (1+t)\mathbb{Z} \\ &\cong \{n \in \mathbb{Z} \mid -n = n\} \Big/ (1+t)\mathbb{Z} \\ &\cong 0. \end{aligned}$$

One more calculation yields

$$H_2(\mathbb{Z}/2;\mathbb{Z}) \cong \ker(1+t : \mathbb{Z} \to \mathbb{Z}) \Big/ (t-1)\mathbb{Z} \cong \mathbb{Z}/2,$$

since $(1+t)(n) = n + t \cdot n = n - n = 0$ and $(1-t)(n) = n - t \cdot n = n - (-n) = 2n$. Now we computed the entire row $q = 1$ in the E^2-term, since $H_{k+2}(\mathbb{Z}/2;\mathbb{Z}) \cong H_k(\mathbb{Z}/2;\mathbb{Z})$ for $k \geq 1$. To summarize, the E^2-term looks as follows:

	4	0	0	0	0	0
	3	0	0	0	0	0
q	2	0	0	0	0	0
	1	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
	$\boldsymbol{E^2}$	0	1	2	3	4
				p		

By inspection, all differentials d^2 and all higher differentials are trivial, so we get $E^\infty \cong E^2$. We conclude that $H_0(D_\infty;\mathbb{Z}) \cong \mathbb{Z}$ and $H_k(D_\infty;\mathbb{Z}) \cong 0$ if k is even. For odd k, we have the non-trivial extension problems

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow H_k(D_\infty;\mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

so the middle group is either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. To decide which group is the right one, we first consider $H_1(D_\infty;\mathbb{Z})$. Notice that we have a presentation $\langle x, y \mid x^2, y^2 \rangle$ of D_∞, which is seen by setting $x \coloneqq s$ and $y \coloneqq sd$ in the original definition (6.1). Thus we can write $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$, and hence $H_1(D_\infty, \mathbb{Z}) \cong D_\infty^{\mathrm{ab}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. If $k \geq 2$ we can use Corollary 2.22 to show that $2 \cdot H_k(D_\infty;\mathbb{Z}) \cong 0$. This holds only if $H_k(D_\infty;\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, since $2 \cdot \mathbb{Z}/4 \neq 0$. Finally, we conclude that

$$H_k(D_\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } k \text{ is odd}, \\ 0 & \text{else.} \end{cases}$$

Example 6.2. Let $n \geq 3$ be odd. We aim to compute the homology of the *dihedral group*

$$D_n = \langle s, d \mid s^2, d^n, sdsd \rangle$$

with trivial $\mathbb{Z}$-coefficients using te LHS spectral sequence. We roughly follow [Rup18]. Let C_n denote the cyclic group $\langle d \mid d^n \rangle$. Then we have a group extension

$$1 \longrightarrow C_n \xrightarrow{i} D_n \xrightarrow{\pi} C_2 \longrightarrow 1,$$

where i is the inclusion and π the canonical projection. The associated LHS spectral sequence looks as follows:

$$E^2_{p,q} = H_p(C_2; H_q(C_n; \mathbb{Z})) \implies H_{p+q}(D_n; \mathbb{Z}).$$

If the action $C_2 \curvearrowright H_q(C_n; \mathbb{Z})$ were trivial, we would save ourselves lots of work later on. However, notice that the above group extension is not central, i.e. $i(C_n) \cong C_n$ is not central in D_n. To see this, assume the contrary. Then we have

$$sd = d^{-1}s^{-1} = ds \iff dsds = 1 = ddss \iff 1 = d^2s^2 \iff 1 = d^2,$$

but this is a contradiction, since $d^n = 1$ only for some $n \geq 3$. Hence, we need to be careful about the action $C_2 \curvearrowright H_q(C_n; \mathbb{Z})$.

Using that $C_m \cong \mathbb{Z}/m$, recall from Proposition 2.8 the explicit form of $H_q(C_m; M)$ for M a $\mathbb{Z}C_m$-module. Now we are in a position to start computing some of the E^2-terms. To begin with, it follows that

$$E^2_{p,0} = H_p(C_2; H_0(C_n; \mathbb{Z})) \cong H_p(C_2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ \mathbb{Z}/2 & \text{if } p \text{ is odd}, \\ 0 & \text{if } p > 0 \text{ is even}. \end{cases}$$

Moreover, if $q > 0$ is even, then $H_q(C_n; \mathbb{Z}) \cong 0$ and so $E^2_{p,q} \cong 0$, and if q is odd, then $H_q(C_n; \mathbb{Z}) \cong \mathbb{Z}/n$ and we get $E^2_{p,q} \cong H_p(C_2; \mathbb{Z}/n)$. Computing these groups will be the crux of this example. We will need the following:

The action $C_2 \curvearrowright H_{2i-1}(C_n; \mathbb{Z})$ is given by multiplication by $(-1)^i$.

This important fact is proved in detail in the appendix and will allow us to interpret $H_{2i-1}(C_n; \mathbb{Z})$ as a $\mathbb{Z}C_2$-module, so we can use Proposition 2.8 to compute further E^2-terms. Notice that, by Proposition 2.6, it follows that $E^2_{0,q} \cong H_q(C_n; \mathbb{Z})_{C_2}$.

We start with $q = 1$ and have $E^2_{0,1} \cong H_1(C_n; \mathbb{Z})_{C_2} \cong (\mathbb{Z}/n)_{C_2}$. Since $1 = q = 2i - 1$ we have $i = 1$, and hence $C_2 \curvearrowright \mathbb{Z}/n$ is given by multiplication by (-1). Then we write

$$\begin{aligned} (\mathbb{Z}/n)_{C_2} &= (\mathbb{Z}/n) \Big/ \langle g \cdot \overline{k} - \overline{k} \mid g \in C_2, \overline{k} \in \mathbb{Z}/n \rangle \\ &\cong (\mathbb{Z}/n) \Big/ \langle -\overline{k} - \overline{k} \mid \overline{k} \in \mathbb{Z}/n \rangle \\ &\cong (\mathbb{Z}/n) \Big/ \langle \overline{2k} \mid \overline{k} \in \mathbb{Z}/n \rangle \\ &\cong 0, \end{aligned}$$

where we used that 2 is invertible in $\mathbb{Z}/n$ for the third isomorphism, which follows from $\gcd(2,n)=1$, since n was odd. This shows that the group generated by $\overline{2k}$ is already $\mathbb{Z}/n$. Thus, we have shown that $E^2_{0,1}\cong 0$. Next, we compute $E^2_{0,3}$. Again, we have $E^2_{0,3}\cong(\mathbb{Z}/n)_{C_2}$, but in this case we have $3=q=2i-1$, i.e. $i=2$, and hence $C_2 \curvearrowright \mathbb{Z}/n$ is trivial. Therefore, $E^2_{0,3}\cong\mathbb{Z}/n$. Repeating these arguments for $q=5,7,9,\dots$ shows that

$$E^2_{0,q}\cong\begin{cases}\mathbb{Z} & \text{if } q=0,\\ \mathbb{Z}/n & \text{if } q\equiv 3 \pmod 4,\\ 0 & \text{else.}\end{cases}$$

The next step will be to argue that $E^2_{p,q}\cong 0$ if both $p>0$ and $q>0$. Again, using Proposition 2.8, we find that

$$\begin{aligned}E^2_{1,1}&\cong (\mathbb{Z}/n)^{C_2}\Big/N\cdot(\mathbb{Z}/n)\\ &\cong\{\overline{k}\in\mathbb{Z}/n\mid -\overline{k}=\overline{k}\}\Big/N\cdot(\mathbb{Z}/n)\\ &\cong\{\overline{k}\in\mathbb{Z}/n\mid \overline{2k}=0\}\Big/N\cdot(\mathbb{Z}/n)\\ &\cong 0,\end{aligned}$$

where we used that 2 is invertible in $\mathbb{Z}/n$, and where $N=1+s$, since we write $C_2\cong\langle s\mid s^2\rangle$. From Proposition 2.8 it then follows that $0\cong E^2_{1,1}\cong E^2_{3,1}\cong E^2_{5,1}\cong\dots$. Now consider $(p,q)=(2,1)$. Still using that $C_2\curvearrowright\mathbb{Z}/n$ is multiplication by (-1), we obtain an expression

$$E^2_{2,1}\cong H_2(C_2;\mathbb{Z}/n)\overset{2.8}{\cong}\{\overline{k}\in\mathbb{Z}/n\mid N(\overline{k})=0\}\Big/(s-1)\cdot\mathbb{Z}/n.$$

We see that $N(\overline{k})=(1+s)(\overline{k})=\overline{k}+s\cdot\overline{k}=\overline{k}-\overline{k}=0$ and similarly that $(s-1)\cdot\mathbb{Z}/n\cong\mathbb{Z}/n$. Therefore, we get that

$$\dots\cong E^2_{6,1}\cong E^2_{4,1}\cong E^2_{2,1}\cong(\mathbb{Z}/n)\Big/(\mathbb{Z}/n)\cong 0.$$

It remains to check the row $q=3$ and then we will essentially be done. As seen above, the action $C_2\curvearrowright\mathbb{Z}/n$ is now the trivial one. Using Corollary 3.4, we have

$$E^2_{1,3}\cong H_1(C_2;\mathbb{Z}/n)\cong C_2^{\mathrm{ab}}\otimes\mathbb{Z}/n\cong\mathbb{Z}/\gcd(2,n)\cong 0,$$

where we use that n is odd. Hence $0\cong E^2_{1,3}\cong E^2_{3,3}\cong E^2_{5,3}\cong\dots$. Finally, we compute

$$\begin{aligned}E^2_{2,3}&\cong H_2(C_2;\mathbb{Z}/n)\\ &\cong\{\overline{k}\in\mathbb{Z}/n\mid N(\overline{k})=0\}\Big/(s-1)\cdot\mathbb{Z}/n\\ &\cong\{\overline{k}\in\mathbb{Z}/n\mid \overline{2k}=0\}\Big/(s-1)\cdot\mathbb{Z}/n\\ &\cong\{\overline{0}\}\Big/(s-1)\cdot\mathbb{Z}/n\\ &\cong 0,\end{aligned}$$

and so $0\cong E^2_{2,3}\cong E^2_{4,3}\cong E^2_{6,3}\cong\dots$. For $q=5$ and $q=7$ we repeat the arguments made for $q=1$ and $q=3$ respectively. We can summarize our findings in the following table:

q						
	4	0	0	0	0	0
	3	$\mathbb{Z}/n$	0	0	0	0
q	2	0	0	0 d^4	0	0
	1	0	0	0	0	0
	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
	$\boldsymbol{E^4}$	0	1	2	3	4
				p		

By inspection, all differentials are zero. The only case one has to be careful about is if $q \equiv 3 \pmod 4$, since then $E^2_{0,q} \cong \mathbb{Z}/n$. Assume $q = 4k+3$ for some $k \in \mathbb{N}$. Then the differential d^4 going into $E^4_{0,4k+3} \cong E^2_{0,4k+3}$ comes from the term $E^4_{4k+4,0} \cong E^2_{4k+4,0} \cong 0$, since $4k+4$ is even, so in particular $d^4 = 0$, as can be seen in the E^4-page above. It follows that $E^\infty \cong E^4 \cong E^2$, and it only remains to solve the extension problems. By considering the above table, we find that

$$H_k(D_n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z}/2 & \text{if } k \equiv 1 \ (\mathrm{mod} 4), \\ \mathbb{Z}/2n & \text{if } k \equiv 3 \ (\mathrm{mod} 4), \\ 0 & \text{else,} \end{cases}$$

since $\mathbb{Z}/n \oplus \mathbb{Z}/2 \cong \mathbb{Z}/2n$, which follows from $\gcd(2,n) = 1$.

7 Applications of the LHS Spectral Sequence

In this chapter, we discuss some consequences of the LHS spectral sequence. More precisely, the goal is to derive a number of interesting corollaries which one can use to obtain purely algebraic and group theoretic results.

7.1 Sylow p-Subgroups

A classical application of the LHS spectral sequence is a formula for computations involving Sylow p-subgroups. We begin by recalling the definition:

Definition 7.1 (Sylow p-Subgroup)**.** Let G be a finite group and p a prime number.

(1) G is a **p-group** if $\mathrm{ord}(G) = p^k$ for some $k \in \mathbb{N}$.

(2) A subgroup $H \leq G$ is called a **Sylow p-subgroup** if H is a p-subgroup and if p does not divide the index $(G : H)$, i.e. there exist $k, m \in \mathbb{N}$ such that $\mathrm{ord}(H) = p^k$, $\mathrm{ord}(G) = p^k m$ and $p \nmid m$.

Proposition 7.2. *Let G be a finite group. If P is a normal Sylow p-subgroup of G, then for all $n \geq 0$ there is an isomorphism*

$$H_n(G; \mathbb{F}_p) \cong H_n(P; \mathbb{F}_p)_Q,$$

where $\mathbb{F}_p \coloneqq \mathbb{Z}/p$ is a trivial $\mathbb{Z}G$-module and $Q \coloneqq G/P$.

Proof. We roughly follow [Ric21, Proposition V.5.5]. By definition of Q we have a group extension $1 \to P \to G \to Q \to 1$. The associated LHS spectral sequence takes the form

$$E^2_{s,t} \cong H_s(Q; H_t(P; \mathbb{F}_p)) \implies H_{s+t}(G; \mathbb{F}_p).$$

Since the action on $\mathbb{F}_p$ was trivial, we have $H_t(P; \mathbb{F}_p) \cong H_t(BP; \mathbb{F}_p)$ by Corollary 3.3, and hence we can consider the cellular chain complex $C^{\mathrm{cell}}_\bullet(BP; \mathbb{F}_p) \cong \bigoplus_{I_\bullet} \mathbb{F}_p$ in the computation of the right-hand side, where I_i denotes the set of i-cells of BP. However, since the only subgroups of $\mathbb{F}_p$ are $\{0\}$ and $\mathbb{F}_p$ itself, it follows that

$$H_t(P; \mathbb{F}_p) \cong \mathbb{F}_p^k$$

for some $k = k(t) \in \mathbb{N} \cup \{\infty\}$. Recalling that group homology is just a Tor-functor, see Remark 2.3, we can thus write

$$E^2_{s,t} \cong H_s(Q; H_t(P; \mathbb{F}_p)) \cong H_s(Q; \mathbb{F}_p^k) \cong \bigoplus_{i=1}^{k} H_s(Q; \mathbb{F}_p).$$

Next, notice that $\gcd(|Q|, p) = 1$ by definition of Sylow p-subgroups. Consequently, $|Q|$ is invertible in $\mathbb{F}_p$ and hence, by Corollary 2.21, we have $H_s(Q; \mathbb{F}_p) \cong 0$ for all $s > 0$. This means that the E^2-page has non-trivial entries only on the line $s = 0$. In particular, we see that $E^2 \cong E^\infty$ with $E^2_{0,t} \cong H_0(Q; H_t(P; \mathbb{F}_p)) \cong H_t(P; \mathbb{F}_p)_Q$. Therefore, all extension problems are trivial, giving the desired result. $\square$

7.2 A Five-Term Exact Sequence

The next useful result, first stated and proved by John Stallings in 1964, allows for algebraic characterizations of the low-dimensional homology groups associated to some given group extension.

Corollary 7.3 (Five-Term Exact Sequence)**.** *Let $1 \to H \to G \to Q \to 1$ be a group extension and let M be a $\mathbb{Z}G$-module. Then there exists a (natural) five-term exact sequence*

$$H_2(G;M) \to H_2(Q;M_H) \to H_1(H;M)_Q \xrightarrow{i_*} H_1(G;M) \to H_1(Q;M_H) \to 0,$$

where the map i_* is induced by the inclusion $i : H \hookrightarrow G$.

We roughly follow the proofs in [Bro82, Corollary VII.6.4] and [McC01, Theorem 8^{bis}.14].

Proof. Since the LHS spectral sequence converges to $H_*(G;M)$, we straight away have an extension problem of the form

$$0 \longrightarrow E^\infty_{0,1} \longrightarrow H_1(G;M) \longrightarrow E^\infty_{1,0} \longrightarrow 0. \tag{7.1}$$

Moreover, notice that any differential involving the group $E^r_{1,0}$ is trivial if $r \geq 2$, so we immediately get that $E^\infty_{1,0} \cong E^2_{1,0}$, and the only differential involving $E^r_{0,1}$ or $E^r_{2,0}$, which is potentially non-trivial, is the map $d^2 : E^2_{2,0} \to E^2_{0,1}$. From this last observation we obtain an exact sequence of groups

$$0 \longrightarrow E^\infty_{2,0} \longrightarrow E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \longrightarrow E^\infty_{0,1} \longrightarrow 0, \tag{7.2}$$

since $E^\infty_{2,0} \cong \ker(d^2_{2,0})$ and $E^\infty_{0,1} \cong E^2_{0,1}/\operatorname{im}(d^2_{2,0})$, which one sees by inspection of the differentials. Now we splice the two sequences (7.1) and (7.2) together to form a new exact sequence, as in the following diagram:

$$\begin{tikzcd}
& H_2(G;M) \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & \\
0 \arrow[r] & E^\infty_{2,0} \arrow[r] \arrow[d] & E^2_{2,0} \arrow[r, "d^2_{2,0}"] & E^2_{0,1} \arrow[r] \arrow[dr, dashed] & E^\infty_{0,1} \arrow[r] \arrow[d] & 0 \\
& 0 & & & H_1(G;M) \arrow[d] & \\
& & & & E^\infty_{1,0} \arrow[d] & \\
& & & & 0 &
\end{tikzcd} \tag{7.3}$$

where the map $H_2(G;M) \to E^\infty_{2,0}$ is the edge homomorphism, i.e. the quotient map $H_2 \to H_2/\Phi^1 H_2 = \Phi^2 H_2/\Phi^1 H_2 \cong E^\infty_{2,0}$, where we write $H_2 := H_2(G;M)$. Here the dashed arrows are defined as the composition of the two other maps respectively. Next, notice that we have three natural isomorphisms

$$\begin{aligned}
E^2_{2,0} &= H_2(Q; H_0(H;M)) \cong H_2(Q;M_H), \\
E^2_{0,1} &= H_0(Q; H_1(H;M)) \cong H_1(H;M)_Q, \\
E^2_{1,0} &= H_1(Q; H_0(H;M)) \cong H_1(Q;M_H).
\end{aligned}$$

Plugging these expressions into the exact sequence (7.3) yields our result. Naturality follows from the fact that all respective maps that were used are natural: The first map was the edge homomorphism and the second one just an inclusion. Then, $d^2_{2,0}$ is natural by Theorem 5.8, followed by a canonical projection and two maps that arise in the extension problem when computing $H_1(G;M)$, which are natural as seen in Remark 4.18.

Finally, in order to show that $H_1(H;M)_Q \to H_1(G;M)$ is induced by $i : H \hookrightarrow G$, we exploit naturality of the LHS spectral sequence: Consider a commutative diagram of groups with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "i"] \arrow[d, "i"'] & G \arrow[r] \arrow[d, "\mathrm{id}"'] & Q \arrow[r] \arrow[d, "c"'] & 1 \\
1 \arrow[r] & G \arrow[r, "\mathrm{id}"'] & G \arrow[r] & 1 \arrow[r] & 1
\end{tikzcd}$$

Then, from the naturality of the LHS spectral sequence it follows that, in particular, the maps must be compatible with maps on the E^∞-page and hence we get a commutative diagram

$$\begin{tikzcd}
H_1(H;M)_Q \cong H_0(Q;H_1(H;M)) \arrow[r, "\varphi"] \arrow[d, "i_* := H_0(c;H_1(i;M))"'] & H_1(G;M) \arrow[d, "\mathrm{id}"] \\
H_1(G;M) \cong H_0(\{1\};H_1(G;M)) \arrow[r, "\mathrm{id}"'] & H_1(G;M)
\end{tikzcd}$$

where the map φ is precisely the map $E^2_{0,1} \to E^\infty_{0,1} \to H_1(G;M)$ that we are interested in. The two vertical maps are obtained by functoriality and the lower map is the identity, which follows from the construction of the spectral sequence corresponding to the short exact sequence $1 \to G \overset{\mathrm{id}}{\to} G \to 1 \to 1$. More explicitly, this spectral sequence has non-trivial entries only on the line $p = 0$ and hence all extension problems are trivial. Consequently, $\varphi = i_*$ and this finishes the proof. ☐

In the above proof we used the fact that the isomorphism $H_0(G;M) \cong M_G$ from Proposition 2.6 is natural in G and M. We briefly prove this in the following lemma:

Lemma 7.4. *Consider a tuple $(G,M) \in$ **GrpMod**. Then the isomorphism $H_0(G;M) \cong M_G$ is natural both in G and M.*

Proof. Let $(\alpha, f) : (G,M) \to (G',M')$ be a morphism in **GrpMod** and let $[m]_G$ denote an equivalence class in M_G. Then, using the proof of Proposition 2.6, we see that we have a commutative diagram

$$\begin{tikzcd}
H_0(G;M) \arrow[r, "\cong"] \arrow[d, "(\alpha,f)_*"'] & M_G \arrow[d] \\
H_0(G';M') \arrow[r, "\cong"'] & M'_{G'}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[g \otimes m]} \arrow[r, mapsto] \arrow[d, mapsto] & {[gm]_G} \arrow[d, mapsto] \\
{[\alpha(g) \otimes f(m)]} \arrow[r, mapsto] & {[\alpha(g)f(m)]_{G'} = [f(gm)]_{G'}}
\end{tikzcd}$$

where we used that f is compatible with α. This shows the claim. ☐

We continue with a special case of the five-term exact sequence:

Corollary 7.5. *Let $1 \to H \xrightarrow{i} G \to Q \to 1$ be a group extension. Then there exists a (natural) five-term exact sequence*

$$H_2(G) \to H_2(Q) \to H_1(H)_Q \cong H/[G,H] \xrightarrow{i_*} H_1(G) \to H_1(Q) \to 0.$$

Proof. We take M to be the trivial module $\mathbb{Z}$ and use Corollary 7.3. We compute the middle term, using Corollary 3.4:

$$\begin{aligned} H_1(H)_Q &\cong H^{\mathrm{ab}}/\langle q \cdot \overline{h} - \overline{h} \mid h \in H,\ q \in Q \rangle \\ &= H^{\mathrm{ab}}/\langle \overline{ghg^{-1}} - \overline{h} \mid h \in H,\ g \in G \rangle \\ &= H^{\mathrm{ab}}/\langle \overline{ghg^{-1}h^{-1}} \mid h \in H,\ g \in G \rangle \\ &= H/[G,H], \end{aligned}$$

where $\overline{h}$ is an equivalence class in H^{ab}. Naturality follows from Corollary 7.3. □

Corollary 7.6 (Hopf's Formula)**.** *Consider a group $G \cong F/R$ with F a free group and $R \leq F$ a normal subgroup. Then there is an isomorphism of groups*

$$H_2(G) \cong \big(R \cap [F,F]\big)/[F,R],$$

where $[A,B]$ denotes the commutator subgroup of groups A and B. In particular, the right-hand side is independent of the group presentation of G.

Proof. Consider the group extension $1 \to R \xrightarrow{i} F \to G \to 1$ and take $M = \mathbb{Z}$ with the trivial action. Then, by Corollaries 7.5 and 3.12 the five-term exact sequence reduces to

$$0 \longrightarrow H_2(G) \longrightarrow R/[F,R] \xrightarrow{i_*} F/[F,F] \longrightarrow G/[G,G] \longrightarrow 0,$$

and hence $H_2(G) \cong \ker(i_* : R/[F,R] \to F/[F,F]) \cong (R \cap [F,F])/[F,R]$, as claimed. □

Example 7.7 ([Knu01, Example A.2.4])**.** To highlight the usefulness of our results thus far, we show how one can use the LHS spectral sequence and the five-term exact sequence to explicitly compute the homology (with trivial coefficients $\mathbb{Z}$) of the *Heisenberg group*, a group which is of interest in quantum mechanics, introduced by Hermann Weyl. It is defined as

$$\mathcal{H} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle|\ x, y, z \in \mathbb{Z} \right\} \subset \mathrm{SL}(3, \mathbb{Z}),$$

together with standard multiplication of matrices. To begin with, notice that we have a group extension of the form

$$1 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{H} \xrightarrow{\pi} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 1,$$

where the maps i and π are respectively defined as

$$t \mapsto \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y).$$

A short calculation demonstrates that $i(t) \cdot A = A \cdot i(t)$ for all $A \in \mathcal{H}$ and $t \in \mathbb{Z}$, showing that $\mathbb{Z}$ is central in $\mathcal{H}$, and hence by Lemma 5.7 the action $\mathbb{Z} \oplus \mathbb{Z} \curvearrowright H_q(\mathbb{Z}; \mathbb{Z})$ is trivial for all q. The LHS spectral sequence takes the form

$$E^2_{p,q} \cong H_p(\mathbb{Z} \oplus \mathbb{Z}; H_q(\mathbb{Z}; \mathbb{Z})) \implies H_{p+q}(\mathcal{H}; \mathbb{Z}).$$

Since $H_q(\mathbb{Z}) \cong 0$ for $q > 1$, we only have non-trivial entries in the rows $q = 0$ and $q = 1$. By using the group homology version of the Künneth Theorem (Example 3.8), we find that the E^2-page looks like the left-hand table below:

q					
	2	0	0	0	0
	1	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	0
	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	0
	$\boldsymbol{E^2}$	0	1	2	3
			p		

q					
	2	0	0	0	0
	1	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	0
	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0
	$\boldsymbol{E^3}$	0	1	2	3
			p		

The only differential that is potentially non-trivial is $d^2_{2,0} : \mathbb{Z} \cong E^2_{2,0} \to E^2_{0,1} \cong \mathbb{Z}$. We claim that $d^2_{2,0}$ is an isomorphism. To see this, we start by observing that the five-term exact sequence from Corollary 7.5, associated to the above group extension, reduces to the following:

$$H_2(\mathcal{H}) \longrightarrow \mathbb{Z} \xrightarrow{d^2_{2,0}} \mathbb{Z} \xrightarrow{i_*} \mathcal{H}^{\mathrm{ab}} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0,$$

where we used that $B(\mathbb{Z} \oplus \mathbb{Z}) \simeq S^1 \times S^1$ to compute the homology of $\mathbb{Z} \oplus \mathbb{Z}$. If we can show that $i_* = 0$, then we know that $d^2_{2,0}$ is surjective and hence an isomorphism. To this end, it is fairly straightforward to see that we have an equality

$$[\mathcal{H}, \mathcal{H}] = \left\langle \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, t \in \mathbb{Z} \right\rangle, \tag{7.4}$$

where the right-hand side is just $\mathrm{im}(i)$. This allows us to write $\mathcal{H}^{\mathrm{ab}} = \mathcal{H}/[\mathcal{H}, \mathcal{H}] = \mathcal{H}/\mathrm{im}(i)$. In particular, the map $i_* : \mathbb{Z} \to \mathcal{H}^{\mathrm{ab}}$ is identically zero, which lets us compute the E^3-page, as seen in the right-hand table above. By inspection of the differentials, we find that $E^3 \cong E^\infty$, and hence all extension problems are trivial, yielding the final result

$$H_n(\mathcal{H}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, 3, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1, 2, \\ 0 & \text{if } n > 3, \end{cases}$$

which concludes the computation of the homology of the Heisenberg group.

Next, we use the five-term exact sequence to establish some links between purely group theoretic statements and our theory about group homology and the LHS spectral sequence. The following corollary is not taken from any literature, so it constitutes some original work by the author, albeit on a small scale.

Suppose we are given a surjective group homomorphism $H \to G$. Then it is clear that the induced map $H^{\mathrm{ab}} \to G^{\mathrm{ab}}$ is surjective as well. However, if the map $H \to G$ were injective, then the resulting map $H^{\mathrm{ab}} \to G^{\mathrm{ab}}$ is not necessarily injective as well, as can be seen by taking H and G to be the free groups $\mathfrak{F}_n$ and $\mathfrak{F}_2$ respectively, where $n \geq 3$. The first part of the next corollary constitutes a compromise, giving a positive answer to our question under certain conditions.

Corollary 7.8. *Let $1 \to H \to G \to Q \to 1$ be a short exact sequence of groups such that $H_2(Q) \cong 0$. Then we have the following:*

(1) If the action $Q \curvearrowright H_1(H)$ is trivial, then the map $H^{\mathrm{ab}} \to G^{\mathrm{ab}}$ induced by the inclusion $H \hookrightarrow G$ is injective.

(2) If Q^{ab} is free, there is an isomorphism of groups

$$G^{\mathrm{ab}} \cong H/[G, H] \oplus Q^{\mathrm{ab}}.$$

If, in addition, the action $Q \curvearrowright H_1(H)$ is trivial, there is an isomorphism of groups

$$G^{\mathrm{ab}} \cong H^{\mathrm{ab}} \oplus Q^{\mathrm{ab}}. \tag{7.5}$$

Proof. (1) Since the action $Q \curvearrowright H_1(H)$ is trivial, it follows that $H_1(H)_Q \cong H_1(H)$. By exactness of the five-term exact sequence, the map $\varphi : H_1(H) \to H_1(G)$, induced by the inclusion, is injective if $H_2(Q)$ is trivial.
(2) This is a direct consequence of Corollary 7.5, using that Q^{ab} is free abelian together with the splitting lemma and the fact that $H_1(H)_Q \cong H/[G, H]$, as seen in the proof of Corollary 7.5. If the action $Q \curvearrowright H_1(H)$ is trivial, we have $H_1(H)_Q \cong H^{\mathrm{ab}}$. □

Remark 7.9. The second part of Corollary 7.8 establishes a connection between the abelianization of the middle group and the outer two groups of the group extension. Note that this result is not trivial, since the functor $(-)^{\mathrm{ab}} : \mathbf{Grp} \to \mathbf{Ab}$ is merely *right-exact*, i.e. a short exact sequence $1 \to H \to G \to Q \to 1$ only yields an exact sequence $H^{\mathrm{ab}} \to G^{\mathrm{ab}} \to Q^{\mathrm{ab}} \to 0$ and hence the isomorphism (7.5) does not hold in general, even if Q^{ab} is assumed to be free abelian.

The condition in Corollary 7.8 that $H_2(Q)$ must be trivial is, strictly speaking, not a necessary condition. To be more precise, it suffices if the map $H_2(Q) \to H_1(H)_Q$ is identically zero. However, requiring that $H_2(Q)$ vanishes is more straightforward, and we conclude this chapter by giving some conditions for when this holds:

Proposition 7.10. *Let $1 \to H \to G \to Q \to 1$ be a group extension and let F/R be a group presentation of Q, i.e. F is free and $R \leq F$ is a normal subgroup.*

(1) $H_2(Q) \cong 0$ if and only if $R \cap [F,F] = [F,R]$.

(2) If there are inclusions $R \hookrightarrow H$ and $F \hookrightarrow G$ and if $H \cap [G,G] = \{1\}$, then $H_2(Q) \cong 0$.

Proof. (1) follows immediately from Hopf's Formula, see Corollary 7.6. For (2), the two inclusions yield an inclusion $R \cap [F,F] \hookrightarrow H \cap [G,G] \cong \{1\}$, so in particular we have $R \cap [F,F] \cong \{1\}$. By Hopf's Formula we have $H_2(Q) \cong 0$. □

Proposition 7.11. *Let Q be a group which admits a finite group presentation F/R such that $|F| = |R|$. If, in addition, $\operatorname{rk}_{\mathbb{Z}}(Q^{\mathrm{ab}}) = 0$, then $H_2(Q) \cong 0$.*

Proof. This is essentially [Bro82, Exercise II.5.5(b)]. Let $Q \cong \langle f_1, \dots, f_n \mid r_1, \dots, r_n \rangle$ be a presentation with the same number n of generators and relations. We have a 2-dimensional CW-complex $X \coloneqq \left(\bigvee_{j=1}^n S^1\right) \cup_{r_1} e^2 \cup_{r_2} \dots \cup_{r_n} e^2$ with $\pi_1(X) \cong Q$. Then we use that we can compute the Euler characteristic in two different ways, namely

$$\sum_{i \geq 0} (-1)^i \operatorname{rk}_{\mathbb{Z}}(H_i(X)) = \sum_{i \geq 0} (-1)^i |K_i(X)|, \tag{7.6}$$

where $K_i(X)$ denotes the set of i-cells of X. Since X is 2-dimensional, (7.6) reduces to

$$1 - \operatorname{rk}_{\mathbb{Z}}(Q^{\mathrm{ab}}) + \operatorname{rk}_{\mathbb{Z}}(H_2(X)) = 1 - n + n \iff \operatorname{rk}_{\mathbb{Z}}(H_2(X)) = 0,$$

where we used that $H_1(X) \cong \pi_1(X)^{\mathrm{ab}} \cong Q^{\mathrm{ab}}$. Note that $H_2(X) \cong \ker(\partial_2)$, where ∂_2 is the differential in the cellular chain complex, and hence $H_2(X)$ is free abelian, showing that $H_2(X) \cong 0$. By [Bro82, Theorem II.5.2] we have an epimorphism $H_2(X) \to H_2(Q)$ and thus $H_2(Q)$ must be trivial as well. □

7.3 Lower Central Series

Following [HS97, pp. 204], we discuss further consequences of the five-term exact sequence. The main goal is to derive a purely group-theoretic result concerning *nilpotent* groups, using the theory around group homology that we have developed so far.

Definition 7.12 (Lower Central Series)**.** Given some group G, define a series of subgroups $G_n \leq G$ for all $n \geq 0$ as follows:

$$G_0 \coloneqq G, \qquad G_{n+1} \coloneqq [G, G_n].$$

We call this series the **lower central series** of G. A group G with $G_n = \{1\}$ for the minimal such $n \geq 0$ is called **nilpotent** of class n.

Example 7.13. The Heisenberg group $\mathcal{H}$ from Example 7.7 is nilpotent of class 2, since $G_0 = \mathcal{H}$ and $G_1 = [\mathcal{H}, \mathcal{H}]$, which was just the group generated by matrices $\left(\begin{smallmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, see equation (7.4). In particular, G_1 is central in $\mathcal{H}$, which means that $G_2 = [\mathcal{H}, G_1] = \{1\}$.

Proposition 7.14. *Let G and G_n be as above. Then G_n is normal in G for all $n \geq 0$.*

Proof. We proceed by induction on n. For $n = 0$ we trivially have that G is normal in G. So assume that $G_n = [G, G_{n-1}]$ is normal in G, i.e. if $u \in G_n$ then $gug^{-1} \in G_n$ for any $g \in G$. Take some $x \in G_{n+1} = [G, G_n]$, hence we can write the element x as $hg_nh^{-1}g_n^{-1}$ for some $h \in G$ and $g_n \in G_n$. Let g be an arbitrary element from G. Then we write

$$\begin{aligned} gxg^{-1} &= g(hg_nh^{-1}g_n^{-1})g^{-1} \\ &= g(hg_nh^{-1})g_n^{-1}g^{-1} \\ &= gg_n'g_n^{-1}g^{-1} \\ &= gg_n''g^{-1} \\ &= (gg_n''g^{-1}(g_n'')^{-1})g_n'' \end{aligned}$$

and we claim this lies in $[G, G_n]$. We have $g_n' := hg_nh^{-1} \in G_n$ by hypothesis, and $g_n'' := g_n'g_n^{-1} \in G_n$ by closure of multiplication, so $gg_n''g^{-1}(g_n'')^{-1} = [g, g_n''] \in [G, G_n]$. Moreover, note that $g_n'' = g_n'g_n^{-1} = hg_nh^{-1}g_n^{-1} = x \in G_{n+1} = [G, G_n]$. In particular, $(gg_n''g^{-1}(g_n'')^{-1})g_n'' \in [G, G_n] = G_{n+1}$, as claimed, proving that G_{n+1} is normal in G. $\square$

Note that for all n the groups G_n/G_{n+1} are abelian, for if we write $G_{n+1} = [G, G_n] = \langle gg_ng^{-1}g_n^{-1} \mid g \in G,\, g_n \in G_n \rangle$ we have in particular elements of the form $gg_ng^{-1}g_n^{-1}$ with $g \in G_n$ that lie in G_{n+1}, and hence $[G_n, G_n] \leq [G, G_n]$ which shows that $G_n/G_{n+1} \leq G_n/[G_n, G_n] = G_n^{\mathrm{ab}}$. Therefore, G_n/G_{n+1} is abelian.

One more property of the lower central series is that it behaves well under group homomorphisms. More precisely, if $f : G \to H$ is a homomorphism, then $f|_{G_n}$ is a map $G_n \to H_n$ for every $n \geq 0$. To see this, take some $[g, g_{n-1}] \in [G, G_{n-1}] = G_n$. Then $f([g, g_{n-1}]) = [f(g), f(g_{n-1})] \in [H, H_{n-1}] = H_n$ if we assume by induction that $f(g_{n-1})$ lies in H_{n-1}. This fact is used in the following theorem to see that the induced maps are well-defined.

Theorem 7.15. *Let $f : G \to H$ be a group homomorphism such that the induced map $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is an isomorphism and such that $f_* : H_2(G; \mathbb{Z}) \to H_2(H; \mathbb{Z})$ is surjective. Then, for all $n \geq 0$, f induces isomorphisms*

$$f_n : G/G_n \to H/H_n.$$

Proof. We proceed via induction on n. First, the cases $n = 0$ and $n = 1$ are trivial by assumption. For $n \geq 2$, consider the two short exact sequences

$$1 \longrightarrow G_{n-1} \longrightarrow G \longrightarrow G/G_{n-1} \longrightarrow 1,$$

$$1 \longrightarrow H_{n-1} \longrightarrow H \longrightarrow H/H_{n-1} \longrightarrow 1.$$

From these two sequences we obtain, using the five-term exact sequence, a commutative diagram with exact rows as follows:

$$\begin{tikzcd} H_2(G) \arrow[r] \arrow[d, "\alpha_1"'] & H_2(G/G_{n-1}) \arrow[r] \arrow[d, "\alpha_2"'] & G_{n-1}/G_n \arrow[r] \arrow[d, "\alpha_3"'] & G^{\mathrm{ab}} \arrow[r] \arrow[d, "\alpha_4"'] & (G/G_{n-1})^{\mathrm{ab}} \arrow[r] \arrow[d, "\alpha_5"'] & 0 \\ H_2(H) \arrow[r] & H_2(H/H_{n-1}) \arrow[r] & H_{n-1}/H_n \arrow[r] & H^{\mathrm{ab}} \arrow[r] & (H/H_{n-1})^{\mathrm{ab}} \arrow[r] & 0 \end{tikzcd}$$

Here we used the fact that $G_n = [G, G_{n-1}]$, $H_n = [H, H_{n-1}]$ and $H_1(G_{n-1}/G_n) \cong (G_{n-1}/G_n)^{\mathrm{ab}} \cong G_{n-1}/G_n$. The vertical homomorphisms $\alpha_1, \ldots, \alpha_5$ arise from the naturality of the five-term exact sequence, i.e. these maps are induced by $f : G \to H$. By hypothesis, α_1 is surjective and α_4 is bijective and moreover, by the induction hypothesis, it follows that α_2 and α_5 are isomorphisms. Consequently, by the five-lemma, $\alpha_3 : G_{n-1}/G_n \to H_{n-1}/H_n$ is an isomorphism.

Finally, consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G_{n-1}/G_n \arrow[r] \arrow[d, "\alpha_3"'] & G/G_n \arrow[r] \arrow[d, "f_n"'] & G/G_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & 0 \\
0 \arrow[r] & H_{n-1}/H_n \arrow[r] & H/H_n \arrow[r] & H/H_{n-1} \arrow[r] & 0
\end{tikzcd}$$

where again all maps are induced by f. Since f_{n-1} is an isomorphism by the induction hypothesis, it follows by another application of the five-lemma that the map f_n is an isomorphism, as claimed. □

We can apply Theorem 7.15 to the special case that the groups G and H are nilpotent. Say, for instance, that G and H are nilpotent of class n_1 and n_2 respectively. Then, if we define $n \coloneqq \max\{n_1, n_2\}$, we have $G_n = \{1\} = H_n$ and we obtain the following result:

Corollary 7.16. *Let $f : G \to H$ be a group homomorphism such that the induced map $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is an isomorphism and such that $f_* : H_2(G;\mathbb{Z}) \to H_2(H;\mathbb{Z})$ is surjective. If, additionally, G and H are nilpotent groups, then f is already an isomorphism $G \xrightarrow{\cong} H$.*

A Computing Group Actions

We show that the action $C_2 \curvearrowright H_{2i-1}(C_n;\mathbb{Z})$ from Example 6.2 is given by multiplication by $(-1)^i$, as claimed. We follow [Wei94, Example 6.7.10]. Recall that $C_n = \langle d \mid d^n\rangle$ and $C_2 = \langle s \mid s^2\rangle$. Let $\alpha : C_n \to C_n$ be the map $d \mapsto d^{-1}$. If we view C_n as a subgroup of $D_n = \langle s, d \mid s^2,\ d^n,\ sdsd\rangle$, this map is precisely the map $d \mapsto sds^{-1}$, since

$$sds^{-1} = sds^{-1}s^2 = sds = d^{-1}.$$

To compute the action $C_2 \curvearrowright H_{2i-1}(C_n;\mathbb{Z})$, it suffices to see how s acts on this homology group. However, by the definition of induced group actions on homology in Chapter 5.1, this is precisely the same as asking what the map $\alpha_* : H_{2i-1}(C_n;\mathbb{Z}) \to H_{2i-1}(C_n;\mathbb{Z})$ does, since α and and the conjugation by s are the same. To compute α_* explicitly, consider the projective resolutions $P_\bullet$ and $\alpha_*P_\bullet$, where $P_\bullet$ is the $(N, 1-d)$-resolution from Lemma 2.7. We get the following diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r, "N"] & \mathbb{Z}C_n \arrow[r, "1-d"] \arrow[d, "d^2"'] & \mathbb{Z}C_n \arrow[r, "N"] \arrow[d, "-d"'] & \mathbb{Z}C_n \arrow[r, "1-d"] \arrow[d, "-d"'] & \mathbb{Z}C_n \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{id}"'] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r, "N"] & \mathbb{Z}C_n \arrow[r, "1-d^{-1}"] & \mathbb{Z}C_n \arrow[r, "N"] & \mathbb{Z}C_n \arrow[r, "1-d^{-1}"] & \mathbb{Z}C_n \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Since $\alpha_*\mathbb{Z}C_n = \mathbb{Z}C_n$, only the maps $1-d$ change to $1-d^{-1}$. The map N stays unchanged, because $\alpha_*(N) = \alpha_*(1 + d + \ldots + d^{n-1}) = 1 + d^{-1} + \ldots + d^{1-n} = 1 + d^{n-1} + \ldots + d = N$. Using that the upper resolution is projective and the lower resolution exact, it follows from the fundamental lemma that we can lift the map $\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ to a map $\mathbb{Z}C_n \to \mathbb{Z}C_n$ such that the first square commutes. We repeat this argument for all the following squares to obtain vertical maps in the diagram, making the entire diagram commute. These maps are unique up to chain homotopy, and since the maps $-d, d^2, -d^3, \ldots$ are isomorphisms and make this diagram commute, we can take as vertical maps precisely these. Thus we obtain an augmentation-preserving map $\tau : P_\bullet \to \alpha_*P_\bullet$.

Now τ induces a map $\tau_* : (P_\bullet)_{C_n} \to (\alpha_*P_\bullet)_{C_n}$, i.e. maps between the complexes $P_\bullet$ and $\alpha_*P_\bullet$ with the functor $- \otimes_{C_n} \mathbb{Z}$ applied. The above diagram becomes

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "(-1)"', "\circledast"] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "1"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "1"', "\circledast"] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "(-1)"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "(-1)"', "\circledast"] & \mathbb{Z} \arrow[d, equal] \\
\cdots \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "n"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z}
\end{tikzcd}$$

since $(1-d) \otimes_{C_n} \mathrm{id}_{\mathbb{Z}} = 0$ and $N \otimes_{C_n} \mathrm{id}_{\mathbb{Z}} = n$. Moreover, one can check by direct computations that $-d \otimes_{C_n} \mathrm{id}_{\mathbb{Z}} = -1$, $d^2 \otimes_{C_n} \mathrm{id}_{\mathbb{Z}} = 1$ and so on. The chain map τ_* is well-defined up to chain homotopy, and hence we see that we get a well-defined map $\alpha_* := (\tau_*)_* : H_k(C_n) \to H_k(C_n)$ for all $k \geq 0$. The above diagram shows that

$$\alpha_* = \begin{cases} \mathrm{id}_{H_k(C_n)} & \text{if } k = 2i, \\ -\mathrm{id}_{H_k(C_n)} & \text{if } k = 2i-1,\ i \geq 1, \end{cases}$$

so in particular we get our result, indicated by the arrows marked with the symbol $\circledast$.

Remark A.1. Let $\alpha : \mathbb{Z} \to \mathbb{Z}$ be the map $n \mapsto -n$. In order to show that the action $\mathbb{Z}/2 \curvearrowright H_0(\mathbb{Z};\mathbb{Z})$ is trivial and $\mathbb{Z}/2 \curvearrowright H_1(\mathbb{Z};\mathbb{Z})$ is given by multiplication by (-1), as was claimed in Example 6.1, we use [Löh19, Proposition 1.6.21] to obtain a commutative diagram of free resolutions

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}[\mathbb{Z}] \arrow[r, "\partial"] \arrow[d, "(-1)"'] & \mathbb{Z}[\mathbb{Z}] \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{id}_{\mathbb{Z}[\mathbb{Z}]}"'] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}[\mathbb{Z}] \arrow[r, "-\partial"] & \mathbb{Z}[\mathbb{Z}] \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where the lower resolution is just α_* applied to the upper resolution and where the differential $\partial : \mathbb{Z}[\mathbb{Z}] \to \mathbb{Z}[\mathbb{Z}]$ is given by sending a generator $z \in \mathbb{Z}$ of $\mathbb{Z}[\mathbb{Z}]$ to $z - 1$. By Lemma 1.4 it follows that $\ker(\varepsilon) = \mathrm{im}(\partial) = \mathrm{im}(-\partial)$, and an argument for the injectivity of ∂ is given in [Löh19, pp. 55]. Now we proceed as in the beginning of Appendix A and apply the functor $- \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}$. We get a map $H_n(\mathbb{Z};\mathbb{Z}) \to H_n(\mathbb{Z};\mathbb{Z})$ induced by α which is the identity for $n = 0$, multiplication with (-1) for $n = 1$ and identically zero for all $n > 1$, as claimed.

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