Homotopy Groups of Wedges of Spheres

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Abstract

These are notes for the seminar "Advanced Topics in Homotopy Theory" given by Prof. Stefan Schwede and Dr. Jack Davies in Bonn during the WS2023/24. Our goal is to present the main result of Hilton's paper "On the Homotopy Groups of the Union of Spheres" [Hil54]. Additionally, we discuss a selection of applications and exemplary computations.

1 Motivation

Wedge sums of spheres appear frequently throughout constructions in homotopy theory, for instance as the n-skeleton of an Eilenberg-MacLane space $K(A, n)$. It is therefore natural to ask questions about homotopical properties of these wedges and in particular about their homotopy groups. More precisely, given a finite wedge sum of spheres $T \coloneqq S^{m_1} \vee \ldots \vee S^{m_k}$, where $m_i \geq 0$, we are looking for a systematic way to compute $\pi_n(T)$. On the surface this might not look like a hard problem, since we merely deal with spheres and we know that "similar" homotopy invariants, such as singular homology, behave well with respect to wedge sums. However, already in low dimensions/degrees we run into trouble: by passing to universal covers, we know that $\pi_2(S^1 \vee S^2) \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}$, which is not even finitely generated, so it seems like there is not much hope for a general formula enabling us to compute $\pi_n(T)$ for arbitrary numbers n. However, by imposing a restriction on the dimensions of the spheres we get a favourable result—Hilton's Theorem:

Theorem 1 (Hilton, 1954)**.** *Let $T \coloneqq S_1 \vee \ldots \vee S_k$ be a finite wedge sum of spheres, where S_i is of dimension $r_i + 1$, with $r_i \geq 1$. Then for all $n \geq 2$ there is an isomorphism*

$$\pi_n(T) \cong \bigoplus_{i=1}^{\infty} \pi_n(S^{q_i+1}),$$

where the summand $\pi_n(S^{q_i+1})$ embeds in $\pi_n(T)$ by composition with a basic product $p_i \in \pi_{q_i+1}(T)$.

Remark 2. Note the following:

- Basic products and the numbers q_i are discussed in definition 4.
- We will see that $q_i \to \infty$ as $i \to \infty$. Thus, for fixed n, the group $\pi_n(T)$ is always finitely generated, since homotopy groups of spheres are always finitely generated.
- The restriction on the dimension says that we shall only consider spheres of dimension ≥ 2. For this reason we excluded the case $n = 1$ in Theorem 1.

Remark 3. Since homotopy groups commute with filtered colimits, the isomorphism

$$\pi_n\left(\bigvee_{i=1}^{\infty} S_i\right) \cong \operatorname*{colim}_{m\geq 1} \pi_n\left(\bigvee_{i=1}^{m} S_i\right)$$

constitutes a method for computing homotopy groups of infinite wedge sums of spheres.

2 Definitions and Preliminaries

Definition 4 (Basic Products)**.** The basic products of weight 1 are the positive generators $\iota_1, \ldots, \iota_k$ of the groups $\pi_{r_1+1}(S_1), \ldots, \pi_{r_k+1}(S_k)$ respectively with an ordering defined as $\iota_1 < \ldots < \iota_k$. Now define arbitrary basic products inductively: if all basic products of weights $1, \ldots, w-1$ have been defined and ordered, define a basic product of weight $w > 1$ as a Whitehead product $[a, b]$, where a is a basic product of weight u, b is a basic product of weight v, $u + v = w$, $a < b$ and if $b = [c, d]$ then $a \geq c$. Then, order the basic products of weight w arbitrarily amongst themselves and set them to be greater than any basic product of lesser weight.

Thus, basic products are just expressions with the ι_i's suitably bracketed. If ι_i occurs w_i times in a basic product, we call the number $q = \sum_{i=1}^{k} r_i w_i$ the height of the basic product. Summarising, we have constructed an ordered list of basic products $p_1 < p_2 < p_3 < \ldots$ from which we obtain a list of corresponding heights $q_1, q_2, q_3, \ldots$.

Remark 5. It is a result of Witt in [Wit37] that we can order the basic products of a given weight w arbitrarily amongst themselves, i.e. that the main result does not depend on this ordering.

Example 6. Consider $T \coloneqq S^2 \vee S^3$, i.e. $r_1 = 1$ and $r_2 = 2$. We get the following table:

i	1	2	3	4	5	$\cdots$
p_i	ι_1	ι_2	$[\iota_1, \iota_2]$	$[\iota_2, [\iota_1, \iota_2]]$	$[\iota_1, [\iota_1, \iota_2]]$	$\cdots$
q_i	1	2	3	5	4	$\cdots$
weight	1	1	2	3	3	$\cdots$

In particular, note that the sequence $\{q_i\}_{i\geq 1}$ is not monotonically increasing. However, by the following easy lemma, it does in fact become arbitrarily large.

Lemma 7. *Let q denote the height of a basic product of weight w. Then $q \geq w$.*

We continue with some definitions and conventions:

Definition 8. We always write $H_\bullet(-)$ for $H_\bullet^{\mathrm{sing}}(-;\mathbb{Z})$. Let X be a path connected space and let $\Omega \coloneqq \Omega X$ be the loop space of X. We will write $\eta\colon \pi_{p+1}(X) \xrightarrow{\cong} \pi_p(\Omega)$ for the natural loop space isomorphism and $h\colon \pi_p(\Omega) \to H_p(\Omega)$ for the Hurewicz homomorphism. Define a map $\rho \coloneqq h \circ \eta\colon \pi_{p+1}(X) \to H_p(\Omega)$. Moreover, recall the Pontryagin product $x \cdot y \in H_{p+q}(\Omega)$ for $x \in H_p(\Omega)$, $y \in H_q(\Omega)$, given by the composite $H_p(\Omega) \otimes H_q(\Omega) \xrightarrow{\times} H_{p+q}(\Omega \times \Omega) \xrightarrow{\mu_*} H_{p+q}(\Omega)$, where $\mu\colon \Omega \times \Omega \to \Omega$ is the H-space structure given by composition of loops.

Lemma 9. *Let $\Omega \coloneqq \Omega T$ for T as in Theorem 1 and let $e_i \coloneqq \rho(\iota_i) \in H_{r_i}(\Omega)$ for $\iota_i \in \pi_{r_i+1}(T)$, $i = 1, \ldots, k$. Then $H_*(\Omega)$ is a free associative ring generated by $e_1, \ldots, e_k$.*

Proof sketch. Let $\tilde{T} \coloneqq \bigvee_{i=1}^{k} S^{r_i}$. Thus, $T \cong \Sigma\tilde{T}$. By the Bott-Samelson Theorem we get

$$H_*(\Omega) \cong H_*(\Omega\Sigma\tilde{T}) \cong T(\widetilde{H}_*(\tilde{T})) \cong T\left(\bigoplus_{i=1}^{k} \widetilde{H}_*(S^{r_i})\right),$$

where $T(M)$ denotes the tensor algebra of a graded module M. The proof is finished by using the universal property of tensor algebras. $\square$

Remark 10. Note that in Lemma 9 we identified $\iota_i \in \pi_{r_i+1}(S_i)$ with its image in $\pi_{r_i+1}(T)$ under the map induced by the canonical inclusion $S_i \hookrightarrow T$. This can be done since the inclusion comes with a retraction $T \to S_i$, given by collapsing all spheres that are not S_i to a point.

3 Proof of the Main Theorem

Let q_i be the height of a basic product p_i. We will often times identify $p_i \in \pi_{q_i+1}(T)$ with a representative $p_i \colon S^{q_i+1} \to T$. By definition of Whitehead products one sees that p_i really is an element in $\pi_{q_i+1}(T)$. Define $\Omega_i \coloneqq \Omega S^{q_i+1}$. Then p_i induces a map $f_i : \Omega_i \to \Omega$, given by $\varphi \mapsto p_i \circ \varphi$. Moreover, define $b_i \coloneqq \rho(p_i)$ and $b_i' \coloneqq (h_i \circ \eta_i)(\iota)$, where $\iota \in \pi_{q_i+1}(S^{q_i+1}) \cong \mathbb{Z}$ is the positive generator. The maps η_i and h_i are just η and h for suitable degrees of homotopy groups. Note that $h_i \colon \pi_{q_i}(\Omega_i) \to H_{q_i}(\Omega_i)$ is an isomorphism by the Hurewicz Theorem, since Ω_i is $(q_i - 1)$-connected. By Lemma 9, $H_*(\Omega_i) \cong \mathbb{Z}[b_i']$, with $|b_i'| = q_i$.

Lemma 11. *The map $(f_i)_* \colon H_\bullet(\Omega_i) \to H_\bullet(\Omega)$ is a ring homomorphism, and $(f_i)_*(b_i') = b_i$.*

Proof. The first claim follows more generally, i.e. any map of spaces $X \to Y$ induces a ring homomorphism $H_\bullet(\Omega X) \to H_\bullet(\Omega Y)$. For the second claim, we look at the following commutative diagram:

$$\begin{tikzcd}
\pi_{q_i+1}(S^{q_i+1}) \arrow[r, "\eta_i", "\cong"'] \arrow[d, "(p_i)_*"'] & \pi_{q_i}(\Omega_i) \arrow[r, "h_i", "\cong"'] \arrow[d, "(f_i)_*"'] & H_{q_i}(\Omega_i) \arrow[d, "(f_i)_*"'] \\
\pi_{q_i+1}(T) \arrow[r, "\eta", "\cong"'] \arrow[rr, bend right, "\rho"'] & \pi_{q_i}(\Omega) \arrow[r, "h"] & H_{q_i}(\Omega)
\end{tikzcd} \tag{1}$$

Both squares commute by naturality of η and h. Note that we can identify $(p_i)_*(\iota)$ with p_i, since a representative of ι is just the identity map. The claim follows by computing $(f_i)_*(b_i') = (f_i)_*(h_i(\eta_i(\iota))) = h(\eta((p_i)_*(\iota))) = \rho(p_i) = b_i$. $\square$

Next, we define for $m \geq 1$ a family of maps ${}_m f \colon \Omega_1 \times \ldots \times \Omega_m \to \Omega$, given by $(\omega_1, \ldots, \omega_m) \mapsto f_1(\omega_1) \cdot \ldots \cdot f_m(\omega_m)$, where the product in Ω is given by composition of loops. If we define $\Omega^* \coloneqq \prod_{i=1}^{\infty} \Omega_i$, then the set of maps ${}_m f$ induces homomorphisms

$$\phi \colon H_\bullet(\Omega^*) \to H_\bullet(\Omega),$$

which follows from the fact that the natural map $\operatorname{colim}_{m\geq 1} \prod_{i=1}^{m} \Omega_i \to \Omega^*$ is a weak equivalence and hence $H_\bullet(\Omega^*) \cong \operatorname{colim}_{m\geq 1} H_\bullet(\prod_{i=1}^{m} \Omega_i)$. Recall that $H_*(\Omega_i) \cong \mathbb{Z}[b_i']$ with $|b_i'| = q_i$, that Ω_i is $(q_i - 1)$-connected and that $q_i \to \infty$. This means that there exists some $N \gg 0$ such that $\prod_{j=N+1}^{\infty} \Omega_j$ is K-connected for K depending on N. In particular, by the Hurewicz Theorem and Künneth, we see that

$$H_\bullet(\Omega^*) \cong H_\bullet\left(\prod_{i=1}^{N} \Omega_i \times \prod_{j=N+1}^{\infty} \Omega_j\right) \cong H_\bullet\left(\prod_{i=1}^{N} \Omega_i\right),$$

for $\bullet \leq N$, and hence $H_\bullet(\Omega^*)$ is additively generated by finite tensor products of the form $b_1'^{n_1} \otimes b_2'^{n_2} \otimes \ldots$. Naturally, we are interested in an explicit description of the map ϕ, given that we now know what the generators of $H_\bullet(\Omega^*)$ look like.

Lemma 12. *Let $\gamma_i \in H_\bullet(\Omega_i)$, $\gamma_j \in H_\bullet(\Omega_j)$ and consider the map $f_{ij}\colon \Omega_i \times \Omega_j \to \Omega$ given by composition of loops. Then, on homology, we have $(f_{ij})_*(\gamma_i \otimes \gamma_j) = (f_i)_*(\gamma_i) \cdot (f_j)_*(\gamma_j)$, where the multiplication on the right-hand side is the Pontryagin product.*

Proof. This is Lemma 4.2 in [Hil54]. □

By Lemma 11 and an extension of Lemma 12, we get that $\phi(b_1'^{n_1} \otimes b_2'^{n_2} \otimes \ldots) = b_1^{n_1} \cdot b_2^{n_2} \cdot \ldots$, where again the right-hand side is the Pontryagin product. The purely algebraic Theorem 3.2 in [Hil54] now tells us that precisely the expressions of the form $b_1^{n_1} \cdot b_2^{n_2} \cdot \ldots$ constitute a free additive basis of $H_*(\Omega)$. Thus, it is clear that ϕ is surjective. Moreover, ϕ is injective, which can be seen as follows: assume that $\phi(x) = \phi(y)$ for $x, y \in H_n(\Omega^*)$, i.e. $x = b_1'^{n_1} \otimes b_2'^{n_2} \otimes \ldots$ and $y = b_1'^{\tilde{n}_1} \otimes b_2'^{\tilde{n}_2} \otimes \ldots$. Then there exists minimal i such that $n_i \neq \tilde{n}_i$. Define elements $\tilde{x} \coloneqq b_1'^{n_1} \otimes b_2'^{n_2} \otimes \ldots \otimes b_i'^{n_i}$ and $\tilde{y} \coloneqq b_1'^{n_1} \otimes b_2'^{n_2} \otimes \ldots \otimes b_i'^{\tilde{n}_i}$. Since $\phi(x) = \phi(y)$, we also have $\phi(\tilde{x}) = \phi(\tilde{y})$ and hence

$$0 = \phi(\tilde{x}) - \phi(\tilde{y}) = b_1^{n_1} \cdot \ldots \cdot b_i^{n_i} - b_1^{n_1} \cdot \ldots \cdot b_i^{\tilde{n}_i} = b_1^{n_1} \cdot \ldots \cdot b_{i-1}^{n_{i-1}} \cdot (b_i^{n_i} - b_i^{\tilde{n}_i}).$$

Since $b_1^{n_1} \cdot \ldots \cdot b_{i-1}^{n_{i-1}}$ is an element in the base of $H_*(\Omega)$, we necessarily have $b_i^{n_i} - b_i^{\tilde{n}_i} = 0$ and therefore $n_i = \tilde{n}_i$. Finally, we use the fact that the expressions x, y are of finite length, i.e. we can repeat the above argument to obtain $x = y$. We have shown that $\phi\colon H_\bullet(\Omega^*) \to H_\bullet(\Omega)$ is an additive isomorphism.

Remark 13. Despite its appearance, the map ϕ is not a ring-homomorphism. Identify $b_1' \in H_\bullet(\Omega_1)$ with $b_1' \otimes u_2 \in H_\bullet(\Omega_1 \times \Omega_2)$ and $b_2' \in H_\bullet(\Omega_2)$ with $u_1' \otimes b_2' \in H_\bullet(\Omega_1 \times \Omega_2)$. Then $b_1' \cdot b_2' = (b_1' \otimes u_2) \cdot (u_1 \otimes b_2') = b_1' \otimes b_2'$ and $b_2' \cdot b_1' = (u_1 \otimes b_2') \cdot (b_1' \otimes u_2) = \pm(b_1' \otimes b_2')$. However, then we get

$$b_1 b_2 = \phi(b_1 \otimes b_2) = \phi(\pm(b_2 \cdot b_1)) = \pm b_2 b_1,$$

which does not hold in general, since $H_*(\Omega)$ is non-commutative.

If the spaces Ω^* and Ω were simply-connected, we could already conclude Theorem 1 by using the Homology Whitehead Theorem. However, this is generally not the case, as can be checked for $T = S^2 \vee S^3$. To circumvent this problem, we pass to universal covers in the next part of the proof.

Assume that the first t of the spheres in T are of dimension 2. We claim that the set of maps ${}_m f$ also induces an isomorphism on fundamental groups, i.e. $\pi_1(\Omega^*) \xrightarrow{\cong} \pi_1(\Omega)$, sending $\eta_i(\iota_i) \mapsto \eta(\iota_i)$ for $i = 1, \ldots, t$. Using that Ω^* is an H-space and the fact that H-spaces have abelian fundamental groups, we get the isomorphism as follows:

$$\pi_1(\Omega^*) \cong H_1(\Omega^*) \xrightarrow{\phi, \cong} H_1(\Omega) \cong \pi_1(\Omega)$$

By definition of t, we have $q_i = 1$ for all $i = 1, \ldots, t$ and hence the left square in the commutative diagram (1) becomes

$$\begin{tikzcd} \pi_2(S^2) \arrow[r, "\eta_i", "\cong"'] \arrow[d, "(p_i)_* = \iota_i"'] & \pi_1(\Omega_i) \arrow[d, "(f_i)_*"] \\ \pi_2(T) \arrow[r, "\eta", "\cong"'] & \pi_1(\Omega) \end{tikzcd}$$

Notice, again by definition of t, that the p_i can be identified with a basic product of weight 1, namely ι_i, and hence this map is certainly injective. By commutativity, we indeed see that $(f_i)_*(\eta_i(\iota_i)) = \eta(\iota_i)$. Since $\pi_1(\Omega_i) \cong 0$ for $i \geq t+1$, we have

$$\pi_1(\Omega^*) \cong \pi_1(\Omega) \cong \mathbb{Z}\langle \eta(\iota_1), \ldots, \eta(\iota_t)\rangle. \tag{2}$$

Let $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ be the universal covering spaces of Ω and Ω^* respectively. These exist since the latter two loop spaces have homotopy type of a CW-complex. More precisely, we have $\widetilde{\Omega}^* \cong \widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_t \times \Omega_{t+1} \times \ldots$, since the right-hand side is certainly a covering space of Ω^*, and since it is simply-connected by the computation

$$\pi_1\left(\widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_t \times \Omega_{t+1} \times \ldots\right) \cong \prod_{i=t+1}^{\infty} \pi_1(\Omega_i) \cong \prod_{i=t+1}^{\infty} \pi_2(S^{q_i+1}) \cong 0,$$

which follows from the easy fact that $q_i \geq 2$ for $i \geq t+1$ by definition of t. Now we may lift the maps ${}_m f$ to maps ${}_m g \colon \widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_t \times \Omega_{t+1} \times \ldots \times \Omega_m \to \widetilde{\Omega}$ for $m \geq t$ by the lifting criterion, since the domain is still simply-connected. As before, the set of maps ${}_m g$ induces a well-defined homomorphism

$$\widetilde{\phi} \colon H_\bullet(\widetilde{\Omega}^*) \to H_\bullet(\widetilde{\Omega}).$$

Lemma 14. *The map $\widetilde{\phi}$ is an isomorphism.*

Note that this lemma will allow for an application of the Homology Whitehead Theorem from which the main result will follow.

Proof. We start by defining a few spaces. For $u \leq t$, let

$$\Omega^{*(0)} := \Omega^*, \qquad \Omega^{*(u)} := \widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_u \times \Omega_{u+1} \times \ldots, \qquad \Omega^{(0)} := \Omega$$

and denote by $\Omega^{(u)}$ the covering space of Ω such that $\pi_1(\Omega^{(u)}) \cong \mathbb{Z}\langle \eta(\iota_{u+1}), \ldots, \eta(\iota_t)\rangle$. Note that $\Omega^{(u)}$ exists by Galois correspondence, since $\mathbb{Z}\langle \eta(\iota_{u+1}), \ldots, \eta(\iota_t)\rangle$ is always a normal subgroup of $\pi_1(\Omega)$ by (2). Then $\Omega^{*(t)} = \widetilde{\Omega}^*$, $\Omega^{(t)} \cong \widetilde{\Omega}$ (since $\pi_1(\Omega^{(t)}) \cong 0$) and $\overline{p}_u \colon \Omega^{*(u)} \to \Omega^{*(u-1)}$ is a covering with deck transformation group

$$\begin{aligned} \mathrm{Deck}(\overline{p}_u) &\cong \pi_1(\Omega^{*(u-1)}) \big/ (\overline{p}_u)_*(\pi_1(\Omega^{*(u)})) \\ &\cong \mathbb{Z}\langle \eta_u(\iota_u), \ldots, \eta_t(\iota_t)\rangle \big/ \mathbb{Z}\langle \eta_{u+1}(\iota_{u+1}), \ldots, \eta_t(\iota_t)\rangle \\ &\cong \mathbb{Z}\langle \eta_u(\iota_u)\rangle, \end{aligned} \tag{3}$$

which follows from the fact that $\overline{p}_u$ is given by the map $(\mathrm{pr}_1, \ldots, \mathrm{pr}_u, \mathrm{id}_{\Omega_{u+1}}, \mathrm{id}_{\Omega_{u+2}}, \ldots)$, where $\mathrm{pr}_j \colon \widetilde{\Omega}_j \to \Omega_j$ denotes the universal covering projection. Similarly, notice that

$\pi_1(\Omega^{(u)})$ is always a normal subgroup of $\pi_1(\Omega^{(u-1)})$, thus yielding a covering map $q_u\colon \Omega^{(u)} \to \Omega^{(u-1)}$ with deck transformation group

$$\mathrm{Deck}(q_u) \cong \pi_1(\Omega^{(u-1)}) \big/ (q_u)_*(\pi_1(\Omega^{(u)})) \cong \mathbb{Z}\langle \eta(\iota_u)\rangle. \tag{4}$$

Instead of lifting ${}_mf$ to the map ${}_mg$ right away, we now lift ${}_mf$ towards ${}_mg$ in smaller steps as follows: inductively, we claim that there exist sets of maps

$${}_mf^{(u)}\colon \widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_u \times \Omega_{u+1} \times \ldots \times \Omega_t \times \ldots \times \Omega_m \to \Omega^{(u)},$$

such that ${}_mf^{(u)}$ covers ${}_mf^{(u-1)}$, i.e. if τ^m, π^m denote the inclusion and projection of the first m factors in the product $\Omega^{*(u)}$ respectively, and $\tilde{p}_u := \pi^m \circ \overline{p}_u \circ \tau^m$, we have $q_u \circ {}_mf^{(u)} = {}_mf^{(u-1)} \circ \tilde{p}_u$. To see this, we verify the assumption for the lifting criterion. Inductively, we have that the set of maps ${}_mf^{(u)}$ induce homomorphisms $\phi^{(u)}\colon H_\bullet(\Omega^{*(u)}) \to H_\bullet(\Omega^{(u)})$ and an isomorphism $\pi_1(\Omega^{*(u)}) \xrightarrow{\cong} \pi_1(\Omega^{(u)})$, sending $\eta_s(\iota_s) \mapsto \eta(\iota_s)$ for $u+1 \le s \le t$. Using this and writing ${}_m\Omega^{*(u)} := \widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_u \times \Omega_{u+1} \times \ldots \times \Omega_t \times \ldots \times \Omega_m$, we compute

$$\begin{aligned}
({}_mf^{(u)} \circ \tilde{p}_{u+1})_*(\pi_1({}_m\Omega^{*(u+1)})) &= ({}_mf^{(u)})_*\Big((\tilde{p}_{u+1})_*(0 \times \ldots \times 0 \times \pi_1(\Omega_{u+2}) \times \ldots \times \pi_1(\Omega_m))\Big) \\
&= ({}_mf^{(u)})_*\Big(\pi_1(\Omega_{u+2}) \times \ldots \times \pi_1(\Omega_t)\Big) \\
&= ({}_mf^{(u)})_*\Big(\mathbb{Z}\langle \eta_{u+2}(\iota_{u+2})\rangle \times \ldots \times \mathbb{Z}\langle \eta_t(\iota_t)\rangle\Big) \\
&\cong \mathbb{Z}\langle \eta(\iota_{u+2})\rangle \times \ldots \times \mathbb{Z}\langle \eta(\iota_t)\rangle.
\end{aligned} \tag{5}$$

On the other hand, we see that

$$(q_{u+1})_*(\pi_1(\Omega^{(u+1)})) \cong \mathbb{Z}\langle \eta(\iota_{u+2}), \ldots, \eta(\iota_t)\rangle, \tag{6}$$

and hence in particular (5) is contained in (6), so we can lift ${}_mf^{(u)}$ to a map ${}_mf^{(u+1)}$, as shown in the following diagram:

$$\begin{tikzcd}
 & \Omega^{(u+1)} \arrow[d, "q_{u+1}"] \\
\widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_u \times \Omega_{u+1} \times \ldots \times \Omega_t \times \ldots \times \Omega_m \arrow[r, "{}_mf^{(u)} \circ \tilde{p}_{u+1}"'] \arrow[ur, dashed, bend left, "{}_mf^{(u+1)}"] & \Omega^{(u)}
\end{tikzcd}$$

Moreover, the maps ${}_mf^{(u+1)}$ induce homomorphisms $\phi^{(u+1)}\colon H_\bullet(\Omega^{*(u+1)}) \to H_\bullet(\Omega^{(u+1)})$ and an isomorphism $\pi_1(\Omega^{*(u+1)}) \xrightarrow{\cong} \pi_1(\Omega^{(u+1)})$, sending $\eta_s(\iota_s) \mapsto \eta(\iota_s)$ for $u+2 \le s \le t$, which essentially follows from injectivity of $(\tilde{p}_{u+1})_*$ and $(q_{u+1})_*$ on fundamental groups. To summarise, we have constructed a tower of covering spaces with commuting squares:

$$\begin{tikzcd}
\widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_t \times \Omega_{t+1} \times \ldots \times \Omega_m \arrow[r, "{}_mf^{(t)} = {}_mg"] \arrow[d, "\tilde{p}_t"'] & \Omega^{(t)} = \Omega \arrow[d, "q_t"] \\
\widetilde{\Omega}_1 \times \ldots \times \widetilde{\Omega}_{t-1} \times \Omega_t \times \ldots \times \Omega_m \arrow[r, "{}_mf^{(t-1)}"] \arrow[d] & \Omega^{(t-1)} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\widetilde{\Omega}_1 \times \Omega_2 \times \ldots \times \Omega_m \arrow[r, "{}_mf^{(1)}"] \arrow[d, "\tilde{p}_1"'] & \Omega^{(1)} \arrow[d, "q_1"] \\
\Omega_1 \times \ldots \times \Omega_m \arrow[r, "{}_mf^{(0)} = {}_mf"] & \Omega^{(0)} = \Omega
\end{tikzcd} \tag{7}$$

Next, we check that the above deck transformation groups act trivially on the homology groups of $\Omega^{*(u)}$ and $\Omega^{(u)}$, as this will be needed later. This result follows from:

Lemma 15. *Let X be an H-space and $p\colon Y \to X$ a covering. Then Y is an H-space and $\pi_1(X)/p_*(\pi_1(Y)) \cong \mathrm{Deck}(p)$ acts trivially on $H_\bullet(Y)$.*

Proof. A constructive proof can be found in [Hil54] pp. 160. □

The last step towards the proof of Lemma 14 requires one more technical Lemma.

Lemma 16. *Let $f\colon X_1 \to X_2$ be a map inducing isomorphisms*

$$\phi\colon H_\bullet(X_1) \xrightarrow{\cong} H_\bullet(X_2) \qquad \text{and} \qquad \pi_1(X_1) \xrightarrow{\cong} \pi_1(X_2).$$

Let π be a normal subgroup of $\pi_1(X_1)$ such that $\pi^1 := \pi_1(X_1)/\pi \cong \mathbb{Z}$ and let $\pi^2 := \pi_1(X_2)/\phi(\pi)$. Let Y_1, Y_2 be covering spaces of X_1, X_2 respectively with deck transformation groups π^1, π^2 which act trivially on $H_\bullet(Y_1)$ and $H_\bullet(Y_2)$ and let $g\colon Y_1 \to Y_2$ be the unique lift of f sending the class of trivial loops to the class of trivial loops. Then g induces isomorphisms

$$\psi\colon H_\bullet(Y_1) \xrightarrow{\cong} H_\bullet(Y_2) \qquad \text{and} \qquad \pi_1(Y_1) \xrightarrow{\cong} \pi_1(Y_2).$$

Proof. It is clear that g induces an isomorphism $\pi_1(Y_1) \xrightarrow{\cong} \pi_1(Y_2)$. Let $\pi^i \cong \mathbb{Z}\langle\sigma_i\rangle$ and let $\alpha_i\colon Y_i \to X_i$ be the covering projection for $i = 1, 2$. Serre shows in [Ser51] that

$$0 \longrightarrow C_\bullet(Y_i) \xrightarrow{1-\sigma_i} C_\bullet(Y_i) \xrightarrow{\alpha_i} C_\bullet(X_i) \longrightarrow 0 \tag{8}$$

is a short exact sequence of chain complexes. Moreover, we have a commutative diagram of chain complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(Y_1) \arrow[r, "1-\sigma_1"] \arrow[d, "g"'] & C_\bullet(Y_1) \arrow[r, "\alpha_1"] \arrow[d, "g"'] & C_\bullet(X_1) \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & C_\bullet(Y_2) \arrow[r, "1-\sigma_2"] & C_\bullet(Y_2) \arrow[r, "\alpha_2"] & C_\bullet(X_2) \arrow[r] & 0
\end{tikzcd} \tag{9}$$

For arbitrary $n \geq 0$ we obtain an exact sequence

$$H_n(Y_i) \xrightarrow{(1-\sigma_i)_*} H_n(Y_i) \longrightarrow H_n(X_i) \longrightarrow H_{n-1}(Y_i) \xrightarrow{(1-\sigma_i)_*} H_{n-1}(Y_i),$$

which is just a fragment of the long exact sequence in homology associated to (8). In particular, this yields a short exact sequence

$$0 \longrightarrow H_n(Y_i)_{\pi^i} \longrightarrow H_n(X_i) \longrightarrow H_{n-1}(Y_i)^{\pi^i} \longrightarrow 0, \tag{10}$$

where $(-)_{\pi^i}$ denotes coinvariants and $(-)^{\pi^i}$ the invariants with respect to π^i. Since π^i acts trivially on homology by assumption, we see that $H_n(Y_i)_{\pi^i} = H_n(Y_i) = H_n(Y_i)^{\pi^i}$. From (9) and (10), we obtain another commutative diagram with exact rows, namely

$$\begin{tikzcd}
0 \arrow[r] & H_n(Y_1) \arrow[r, "(\alpha_1)_*"] \arrow[d, "\psi"'] & H_n(X_1) \arrow[r, "\partial"] \arrow[d, "\phi"', "\cong"] & H_{n-1}(Y_1) \arrow[r] \arrow[d, "\psi"', "\cong"] & 0 \\
0 \arrow[r] & H_n(Y_2) \arrow[r, "(\alpha_2)_*"] & H_n(X_2) \arrow[r, "\partial"] & H_{n-1}(Y_2) \arrow[r] & 0
\end{tikzcd}$$

By induction on n the map $\psi\colon H_{n-1}(Y_1) \to H_{n-1}(Y_2)$ is assumed to be an isomorphism, and hence the result follows by the Five-Lemma. $\square$

We apply Lemma 16 as follows: set $X_1 = \Omega^{*(0)}$, $X_2 = \Omega^{(0)}$. We know from before that $\phi = \phi^{(0)}\colon H_\bullet(\Omega^{*(0)}) \to H_\bullet(\Omega^{(0)})$ is an isomorphism induced by the set of maps ${}_mf$, which also induce an isomorphism on fundamental groups. Take $\pi = \pi_1(\Omega^{*(1)})$ and $Y_1 = \Omega^{*(1)}$, $Y_2 = \Omega^{(1)}$, so that they indeed have the correct infinite cyclic deck transformation groups, as computed in equations (3) and (4). Since X_1, X_2 are H-spaces, it follows from Lemma 15 that these groups act trivially on $H_\bullet(Y_1)$ and $H_\bullet(Y_2)$. Lastly, the map g is given by the set of maps ${}_mf^{(1)}$, the unique lifts of ${}_mf^{(0)}$, sending trivial loops to trivial loops.

Now it follows from Lemma 16 that the set of maps ${}_mf^{(1)}$ induces isomorphisms

$$\phi^{(1)}\colon H_\bullet(\Omega^{*(1)}) \xrightarrow{\cong} H_\bullet(\Omega^{(1)}) \qquad \text{and} \qquad \pi_1(\Omega^{*(1)}) \xrightarrow{\cong} \pi_1(\Omega^{(1)}).$$

More importantly, the deck transformation groups of the next covering maps are still infinite cyclic and act trivially on homology by Lemma 15. Consequently, we repeat the argument t times and work our way up the tower in diagram (7). We finally get that the set of maps ${}_mf^{(t)} = {}_mg$ induces isomorphisms on homology, which finishes the proof of Lemma 14. $\square$

Now we can finally apply the Homology Whitehead Theorem, since $\widetilde{\Omega}^*$ and $\widetilde{\Omega}$ are simply-connected. Thus, we obtain that the set of maps ${}_mg$ induce isomorphisms on all homotopy groups $\pi_n(\widetilde{\Omega}^*) \xrightarrow{\cong} \pi_n(\widetilde{\Omega})$. We get the following commutative diagram:

$$\begin{tikzcd}
\pi_{n-1}(\widetilde{\Omega}^*) \arrow[r, "\cong"] \arrow[d, "(\{{}_mg\})_*"', "\cong"] & \pi_{n-1}(\Omega^*) \arrow[d, "(\{{}_mf\})_*"', "\cong"] & \bigoplus_{i=1}^{\infty} \pi_{n-1}(\Omega_i) \arrow[l, "\cong"'] \arrow[d, "\bigoplus (f_i)_*"', "\cong"] & \bigoplus_{i=1}^{\infty} \pi_n(S^{q_i+1}) \arrow[l, "\bigoplus \eta_i"', "\cong"] \arrow[d, "\bigoplus p_i"'] \\
\pi_{n-1}(\widetilde{\Omega}) \arrow[r, "\cong"'] & \pi_{n-1}(\Omega) \arrow[r, equal] & \pi_{n-1}(\Omega) & \pi_n(T) \arrow[l, "\eta", "\cong"']
\end{tikzcd}$$

The left square commutes already on spaces, the right square commutes by naturality of η and the commutativity of the middle square can be checked by hand, using that we have an explicit description of the natural isomorphism $\prod_{i\in I} \pi_n(X_i) \xrightarrow{\cong} \pi_n(\prod_{i\in I} X_i)$ for an arbitrary family of spaces $(X_i)_{i\in I}$. Note that we can indeed replace the product by the direct sum since Ω_i is $(q_i - 1)$-connected and $q_i \to \infty$. Moreover, the horizontal maps in the left square are isomorphisms by elementary covering space theory. Thus, we get that $\bigoplus (f_i)_*$ is an isomorphism, giving the desired result that

$$\bigoplus_{i=1}^{\infty} p_i\colon \bigoplus_{i=1}^{\infty} \pi_n(S^{q_i+1}) \to \pi_n(T)$$

is an isomorphism. This concludes the proof of the main theorem.

4 Applications and Examples

A first consequence of the main result follows from considering a special case for the wedge sum T, namely where all spheres in T have identical dimensions.

Corollary 17. *Let $T \coloneqq \bigvee_{i=1}^{k} S^{r+1}$, $r \geq 1$. Then for all $n \geq 2$ there is an isomorphism*

$$\pi_n(T) \cong \bigoplus_{w=1}^{\infty} \pi_n(S^{wr+1})^{\oplus Q(w,k)},$$

where $Q(w,k)$ denotes the number of basic products of weight w for the wedge sum T.

Proof. Recall the formula for the height of the j-th basic product $q_j = \sum_{i=1}^{k} r_i w_{ij}$, where w_{ij} denotes the number of times the symbol ι_i appears in the j-th basic product. Since $r_i = r$ for all $i = 1, \ldots, k$, we obtain $q_j = r \sum_{i=1}^{k} w_{ij}$. Now instead of running over natural numbers i we run over the weights w, thus effectively replacing $\sum_{i=1}^{k} w_{ij}$ by w. Since it is possible for two different basic product to have the same height, we account for these cases by taking $Q(w,k)$ many copies of all homotopy groups, thus giving the result. $\square$

Remark 18. An explicit formula for $Q(w,k)$ is derived in [Wit37], namely $Q(w,k) = \frac{1}{w} \sum_{d|w} \mu(d) k^{w/d}$, where μ denotes the Möbius function.

Example 19. It is a well known result that there is an isomorphism $\pi_n(\bigvee_{i=1}^{k} S^n) \cong \bigoplus_{i=1}^{k} \pi_n(S^n)$ induced by inclusions of factors. We briefly check that we recover this result through Hilton's Theorem. Using Corollary 17, we find that

$$\begin{aligned}
\pi_n\left(\bigvee_{i=1}^{k} S^n\right) &= \bigoplus_{w=1}^{\infty} \pi_n(S^{(n-1)w+1})^{\oplus Q(w,k)} \\
&= \pi_n(S^n)^{\oplus Q(1,k)} \oplus \bigoplus_{w=2}^{\infty} \pi_n(S^{(n-1)w+1})^{\oplus Q(w,k)} \\
&= \pi_n(S^n)^{\oplus k},
\end{aligned}$$

where we used that $(n-1)w + 1 > n$ for $w \geq 2$ and that $Q(1,k) = \mu(1)k = k$. Moreover, this isomorphism is induced by the first k basic products $\iota_1, \ldots, \iota_k$ by the main theorem, but these are just the basic products of weight 1 and hence are represented by the inclusion maps $S^n \hookrightarrow \bigvee_{i=1}^{k} S^n$, as claimed.

Next, let us compute some explicit homotopy groups of a specific wedge sum of spheres:

Example 20. Recall Example 6 in which we computed some basic products and corresponding heights for $T = S^2 \vee S^3$. Using the main Theorem, we obtain the following groups for $n = 2, 3, 4, 5$:

- $\pi_2(T) \cong \pi_2(S^2) \cong \mathbb{Z}$.
- $\pi_3(T) \cong \pi_3(S^2) \oplus \pi_3(S^3) \cong \mathbb{Z}^{\oplus 2}$.
- $\pi_4(T) \cong \pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_4(S^4) \cong (\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}$.
- $\pi_5(T) \cong \pi_5(S^2) \oplus \pi_5(S^3) \oplus \pi_5(S^4) \oplus \pi_5(S^5) \cong (\mathbb{Z}/2)^{\oplus 3} \oplus \mathbb{Z}$.

Note that the first two computations are not particularly interesting, since one could likely have guessed these results. The third computation is more interesting, however this could alternatively have been obtained using the Blakers-Massey Theorem and the homotopy long exact sequence for the pair $(S^2 \times S^3, S^2 \vee S^3)$. The fourth calculation could not have been done using this alternative method, and thus constitutes a "new" result.

Example 21. As a final note, we return to the proof of Lemma 16. Observe that in the short exact sequence (10) the middle term $H_n(X_i)$ is free abelian if and only if the outer two terms $H_n(Y_i)$ and $H_{n-1}(Y_i)$ are free abelian, in which case we have $H_n(X_i) \cong H_n(Y_i) \oplus H_{n-1}(Y_i)$. This allows for concrete computations of $H_\bullet(\Omega^{(u)})$ for $1 \leq u \leq t$, which is interesting since the spaces $\Omega^{(u)}$ are defined rather abstractly. For $X_2 = \Omega^{(0)} = \Omega$ and $Y_2 = \Omega^{(1)}$ we know by Lemma 9 that $H_\bullet(\Omega)$ is free abelian, so we have a splitting

$$H_n(\Omega) \cong H_n(\Omega^{(1)}) \oplus H_{n-1}(\Omega^{(1)}). \tag{11}$$

We compute only the first four homology groups of $\Omega^{(1)}$, as the calculations become cumbersome quite quickly. For $n = 0$, we have $H_0(\Omega^{(1)}) \cong H_0(\Omega) \cong \mathbb{Z}$, since Ω is path connected. For $n = 1$, we have $H_1(\Omega) \cong H_1(\Omega^{(1)}) \oplus H_0(\Omega^{(1)})$, i.e. $\mathbb{Z}^t \cong H_1(\Omega^{(1)}) \oplus \mathbb{Z}$ by (2), where t again denotes the number of 2-spheres in T. Therefore, $H_1(\Omega^{(1)}) \cong \mathbb{Z}^{t-1}$. Notice that we assumed that $t \geq 1$, since otherwise the construction of $\Omega^{(1)}$ is ill-defined. In the case $t = 0$ we simply have $\Omega \cong \widetilde{\Omega}$ and there is nothing to be done. It turns out that the case $n = 2$ is already much less straightforward, as we will see now.

Assume without loss of generality that $r_1 \leq \ldots \leq r_k$ in the definition of T. By the isomorphism ϕ and Künneth, we have for sufficiently large $\alpha \in \mathbb{N}$ that

$$H_2(\Omega) \cong H_2(\Omega^*) \cong \bigoplus_{i_1+\ldots+i_\alpha=2} H_{i_1}(\Omega_1) \otimes \ldots \otimes H_{i_\alpha}(\Omega_\alpha), \tag{12}$$

without any Tor-terms, since all homology groups involved are free by Lemma 9. Again, this argument uses the fact that Ω_i is $(q_i - 1)$-connected and that q_i is at least as large as the weight of the corresponding basic product by Lemma 7. In particular, we want $q_i - 1 \geq 2$, i.e. $q_i \geq 3$. A possible choice for α is thus given by

$$\alpha = Q(1,k) + Q(2,k) = k + (1 + \ldots + (k-1)) = \frac{k(k+1)}{2},$$

i.e. α is given by the sum of number of basic products of weight 1 and 2. For this specific choice the spaces $\Omega_{\alpha+1}, \Omega_{\alpha+2}, \ldots$ are indeed $\geq$ 2-connected, since $q_{\alpha+\ell} - 1 \geq 3 - 1 = 2$ for all $\ell \geq 1$ by definition of α. By definition of t we have $q_1 = \ldots = q_t = 1$ and moreover we know that, for $i = 1, \ldots, t$, $H_n(\Omega_i) \cong \mathbb{Z}$ for every n by Lemma 9. Therefore, these homology groups never contribute to the tensor product in the decomposition (12) and can be ignored, namely

$$H_2(\Omega) \cong \bigoplus_{i_{t+1}+\ldots+i_\alpha=2} H_{i_{t+1}}(\Omega_{t+1}) \otimes \ldots \otimes H_{r_\alpha}(\Omega_\alpha). \tag{13}$$

By definition of t we know that $H_1(\Omega_{t+\ell}) \cong 0$ for all $\ell \geq 0$, by the Hurewicz Theorem and the fact that $\Omega_{t+\ell}$ is simply-connected. This means that if one of the numbers $i_{t+1}, \ldots, i_\alpha$ is 1, the whole tensor product in (13) vanishes and we are left only with the cases $i_{t+1} = 2, \ldots, i_\alpha = 2$ and all others equal to zero, since $i_{t+1} + \ldots + i_\alpha = 2$. Thus, (13) reduces to an isomorphism

$$H_2(\Omega) \cong \bigoplus_{j=t+1}^{\alpha} H_2(\Omega_j) \cong \mathbb{Z}^N,$$

where $N := \#\{j = t+1, \ldots, \alpha \mid q_j = 2\}$. We only have $q_j = 2$ for basic products of weight ≤ 2, so for basic products of weight 1 which are generators of $\pi_3(S^3)$ and for basic products $[\iota_a, \iota_b]$ of weight 2 with $a < b \leq t$. Therefore, if t' denotes the number of 3-spheres in T, we can compute N to be $N = t' + \frac{t(t-1)}{2}$.

Returning to the computation of $H_2(\Omega^{(1)})$, we know from the splitting (11) that $\mathbb{Z}^N \cong H_2(\Omega^{(1)}) \oplus \mathbb{Z}^{t-1}$, and hence we obtain the final result

$$H_2(\Omega^{(1)}) \cong \mathbb{Z}^{t' + \frac{1}{2}(t-1)(t-2)}.$$

In particular, $H_2(\Omega^{(1)})$ depends only on the number of 3-spheres in T if T involves only one or two 2-spheres. A similar argument as before shows that in the case $n = 3$ we get

$$H_3(\Omega^{(1)}) \cong \mathbb{Z}^{t'' + (t-1)(t' + (t - \frac{1}{2})t + 1)},$$

where t'' denotes the number of 4-spheres in T. Note that $H_3(\Omega^{(1)})$ only depends on t'' if $t = 1$.

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