Disorientation attacks on CSIDH

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CSIDH group action

CSIDH group action [2]: Fix a prime $p = 4 \cdot \ell_1 \dots \ell_n - 1$ with ℓ_i odd primes. Get a regular action of $\mathrm{cl}(\mathbb{Z}[\sqrt{-p}])$ on $\mathcal{E} = \{E/\mathbb{F}_p$ supersingular and in Montgomery form $y^2 = x^3 + Ax^2 + x\}$. Write

$$(\mathfrak{a}, E) \mapsto \mathfrak{a} \star E.$$

Easy to act with ideals $\mathfrak{l}_i = (\ell_i, \sqrt{-p} - 1)$, for efficiency work with $\mathfrak{a} = \prod_{i=1}^{n} \mathfrak{l}_i^{e_i}$ for $(e_1, \dots, e_n) \in \mathcal{K} \subset \mathbb{Z}^n$ for some explicit choice of *keyspace* $\mathcal{K}$.

Note: in CSIDH-like cryptosystems, secrets are which isogenies we compute, and how many times we do so.

Computing one ℓ-isogeny

Find a point of order ℓ in $E(\mathbb{F}_{p^2})$:

- to act by $\mathfrak{a} = \mathfrak{l}$: find $P = (x, y)$ with $x, y \in \mathbb{F}_p$,
- to act by $\mathfrak{a} = \mathfrak{l}^{-1}$: find $P = (x, y)$ with $x \in \mathbb{F}_p$ and $y \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$,

and compute the isogeny $E \to E/\langle P \rangle =: \mathfrak{a} \star E$.

We usually compute with x-coordinates only.

Orientation

Point $P = (x, y) \in E(\mathbb{F}_{p^2})$ with $x \in \mathbb{F}_p$ is oriented

- *positively* if $x^3 + Ax^2 + x$ is a square in $\mathbb{F}_p$,
- *negatively* if $x^3 + Ax^2 + x$ is a non-square in $\mathbb{F}_p$.

Denote the orientation of the point by s.

Note: positively-oriented points will allow steps in positive direction $\mathfrak{l}$, negatively in negative directions $\mathfrak{l}^{-1}$.

Computing group action

Sampling points of order ℓ is expensive; for efficiency, we always evaluate multiple isogeny steps (with the same orientation).

Call one iteration of the while loop a *round*.

Algorithm 1: Evaluation of CSIDH group action

Input: $A \in \mathbb{F}_p$ and a list of integers $(e_1, \dots, e_n)$.
Output: $B \in \mathbb{F}_p$ such that $\prod [\mathfrak{l}_i]^{e_i} * E_A = E_B$

```
1:  while some e_i ≠ 0 do
2:      Sample a random x ∈ F_p, defining a point P.
3:      Set s ← IsSquare(x^3 + Ax^2 + x).
4:      Let S = {i | e_i ≠ 0, sign(e_i) = s}. Restart
        with new x if S is empty.
5:      Let k ← ∏_{i∈S} ℓ_i and compute Q ← [(p+1)/k]P.
6:      for each i ∈ S do
7:          Compute R ← [k/ℓ_i]Q. If R = ∞, skip this i.
8:          Compute φ : E_A → E_B with kernel ⟨R⟩.
9:          Set A ← B and k ← k/ℓ_i and Q ← φ(Q)
            and e_i ← e_i − s.
10: return A.
```

(Fine print: specific implementations impose their own conditions on the set of indices in S, but always choose steps with the same orientation.)

General disorientation

What if we disorient the point P used in **Algorithm 1**? Assume that we disoriented in round r. If P had full order and orientation s, then

$$E^{r,s} = \prod_{i \in S} \mathfrak{l}_i^{-2s} \star E_B.$$

If P did not have full order, we obtain a different curve

$$E_t = \prod_{\ell_i \nmid \mathrm{ord}(P)} \mathfrak{l}_i^{2s} \star E^{r,s}.$$

Suppose we keep disorienting points at exactly the same point in the evaluation of **Algorithm 1**.

Observations:

1. $\ell_i \mid \mathrm{ord}(P)$ is more likely than $\ell_i \nmid \mathrm{ord}(P)$ and so the curve $E^{r,s}$ will be the most common one;
2. all other curves E_t are connected to $E^{r,s}$ by a short isogeny walk:
   (a) this walk only includes degres ℓ_i for $i \in S$,
   (b) direction of these walks reveals the orientation of P (and hence all ℓ_i for $i \in S$).

(Fine print: more 'torsion behavior' is possible.)

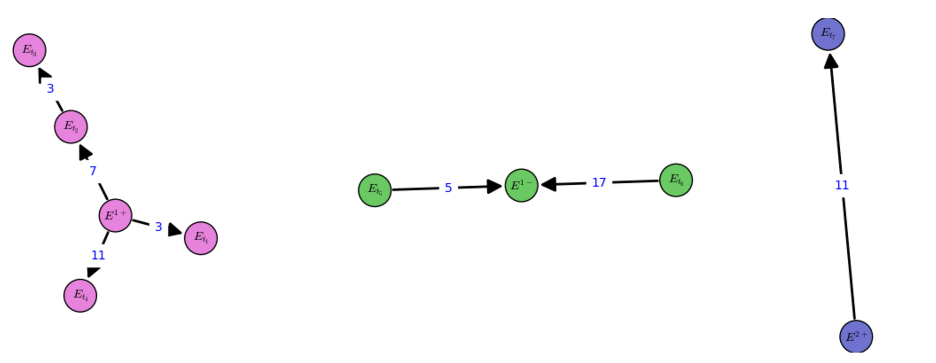
Toy example

We will illustrate the general algorithm to recover secret keys on a very toy example with 10 different primes ℓ_i and $-1 \leq e_i \leq 2$. For simplicity, assume $e_i \neq 0$, and let us examine what happens for the secret key (e_i) :

ℓ_i	3	5	7	11	13	17	19	23	29	31
e_i	1	−1	1	2	2	−1	1	1	−1	2

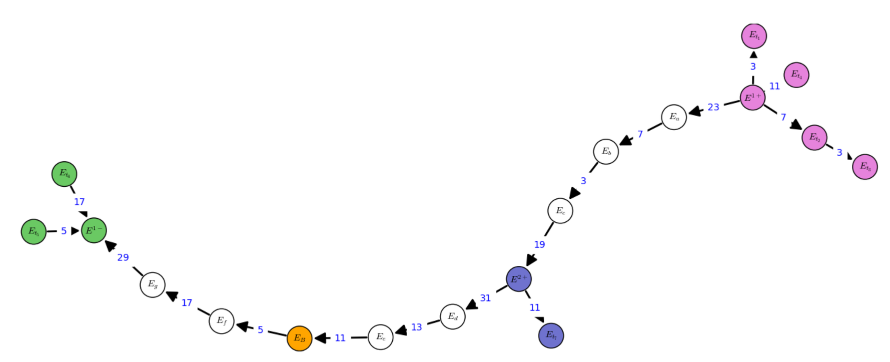
Assume we fault in round 1 and 2 of **Algorithm 1** repeatedly, generating faulty curves E_t. We also know the correct public key $E_B = \prod \mathfrak{l}_i^{e_i} \star E$.

1. From the faulty curves from round 1, pick the two most common ones, say $E^{1,a}$ and $E^{1,b}$ (we do not know the orientation yet). From the curves faulted in round 2, pick the most common curve not yet seen - this is most likely $E^{2,+}$.
2. Perform a small neighbourhood walk around the three curves and see if we see any of the faulty curves. We obtain three disjoint trees with the three curves as *roots*:



Note that the edges with labels ℓ_i actually correspond to two steps in the isogeny graph.

3. This allows us to determine the orientation of the curves, and signs of e_i for some of the primes ℓ_i:
   - 3, 7, 11 point away from the root: positive;
   - 5, 17 point towards the root: negative.
4. Finally, we run `pubcrawl` to find the paths connecting the positive and negative curves: $E^{1,+} \to E^{2,+} \to E_B$ and $E_B \to E^{1,-}$. Note that no negatively oriented prime will ever occur in a positive path, and vice versa, which significantly speeds up the search.



5. Read off the secret key from the labels of the path!

Examples

CSIDH-512 [2] uses 74 primes $3, 5, \dots, 373, 587$, and the keyspace $\mathcal{K} = [-5, \dots, 5]^{74}$, so each $|e_i| \leq 5$.

Fault-injection attacks

Think of a device computing with secret data. Now consider the following magic power:

↯ force a mistake at one point in the computation.

For instance, you can replace a value by a random value, or even skip an instruction (line) in the algorithm.

Disorientation

What if we want to compute $\mathfrak{l} \star E$ and generate a point P with the wrong orientation?

- What we wanted: $E_B = \mathfrak{l} \star E$,
- What we obtained: $E_B^{↯} = \mathfrak{l}^{-1} \star E$.

These two curves are related: $E_B = \mathfrak{l}^2 \star E_B^{↯}$.

How to force disorientations?

In **Algorithm 1**, we attack **Step 3**: `IsSquare` check is usually implemented as exponentiation $z \mapsto z^{\frac{p-1}{2}}$. Forcing a fault anywhere in this computation replaces the orientation of the point P with a random orientation, which is different from the orientation of P about half of the time. Another way to sample points is the Elligator 2 map, which can be attacked similarly.

Curves in different rounds

Notice that **Algorithm 1** is randomized: we will generate different points and orientations every time. Moreover, the computation in round r depends on what was computed in rounds $1, \dots, r-1$.

Faulty curves from different rounds are again related by paths that reveal information on the secret key.

`pubcrawl`

Our main subroutine is finding a path in the isogeny graph between either the public key curve and a faulty curve, or between two faulty curves.

For this, we developed an optimized meet-in-the-middle brute-force search tool called `pubcrawl`. To find a path between E_1 to E_2:

- specify primes ℓ_i to use as isogeny steps,
- specify orientation from E_1 to E_2,

and let `pubcrawl` do the work!

Results [1]

- We define a new class of fault attacks on CSIDH-like schemes we call *disorientation attacks*;
- We show that almost all current implementations are susceptible. In particular, batching techniques like SIMBA or CTIDH seem easier to attack because fewer isogenies are computed at each step;
- We argue these attacks are inherent to the way we compute group actions via isogenies, and so every cryptographic implementation needs to be strengthened. We propose lightweight countermeasures.
- We develop a tool `pubcrawl` for finding isogeny paths between (faulty) curves, optimized for the Meet-in-the-middle approach and for specifying the set of degrees among which we want to search;
- We consider a cryptographically more realistic scenario of not obtaining faulty curves E_t directly but only a *derived* value: think only seeing a *hash* $h(E_t)$.

References

[1] G. Banegas, J. Krämer, T. Lange, M. Meyer, L. Panny, K. Reijnders, J. Sotáková, and M. Trimoska. "Disorientation attacks on CSIDH". In: eprint soon!, 2022.

[2] W. Castryck, T. Lange, C. Martindale, L. Panny, and J. Renes. "CSIDH: An Efficient Post-Quantum Commutative Group Action". In: *ASIACRYPT 2018*. Vol. 11274. LNCS. Springer, 2018, pp. 395–427.

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