Numerical exploration of geostrophic adjustment

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1 Introduction

Geostrophic adjustment is the process where an unbalanced pressure perturbation adjusts to achieve geostrophic balance. A central concept of the large-scale general circulation, we will explore geostrophic adjustment by numerically solving the shallow water equations (SWEs). In particular, we will look at the flow characteristics and energetics under a set of scenarios.

In Section 2 we present the SWEs and central analytical results, while the numerical method is described in Section 3. A few central results are shown in Section 4 and briefly discussed in Section 5, wrapping up the report.

2 Theory

The linearized shallow water equations are as follows:

$$\partial_t u - f_0 v = -g\partial_x h \tag{1}$$
$$\partial_t v + f_0 u = -g\partial_y h \tag{2}$$
$$\partial_t h + D_0(\partial_x u + \partial_y v) = 0, \tag{3}$$

where ∂_{q_i} denotes partial differentiation or ordinary differentiation given the context. u and v are the zonal and meridional velocities, respectively, while h is the sea-surface height. The planetary vorticity is f_0, while g is the gravitational acceleration, and D_0 is the depth of the system.

In this report, we will study the case of a stepwise sea surface height perturbation of $\pm h_0$

$$h_i = h_0 \begin{cases} h_0, & x > 0 \\ -h_0, & x < 0 \end{cases}, \tag{4}$$

which can be solved analytically.

2.1 Dynamics

The resulting sea surface height after geostrophic adjustment is

$$h_f = h_0 \begin{cases} -1 + \exp(-x/R), & x > 0 \\ 1 - \exp(x/R), & x < 0 \end{cases}, \tag{5}$$

where $R \equiv \sqrt{gD_0}/f_0$ is the Rossby radius. In geostrophic balance, the meridional velocity is just $v = -\partial_x h$, giving

$$v = -\frac{gh_0}{f_0 R}\exp(-|x|/R) \tag{6}$$

for the final state.

2.2 Energetics

The total energy of a system with density ρ can be decomposed into potential and kinetic components

$$V = \frac{1}{2}\rho g\big(h^2 - D_0^2\big) \tag{7}$$
$$K = \frac{1}{2}\rho D_0\big(u^2 + v^2\big), \tag{8}$$

and the available potential energy is obtained by subtracting the constant term. For an initial height perturbation eq. 4 the resulting change in available potential and kinetic energy is

$$\Delta V = -\frac{3}{2}\rho g h_0^2 R \tag{9}$$

$$\Delta K = \frac{1}{2}\rho g h_0^2 R. \tag{10}$$

Initially, the potential energy constitutes the total energy, and so from eqs. 9 and 10 the final kinetic energy should be a 1/3 of the total energy. Likewise, the potential energy should be a 2/3 of the total energy.

3 Method

The shallow water equations (eqs. 1, 2, 3) are solved numerically in the FORTRAN programming language using finite difference schemes. An Arakawa C-grid is used for representing spatial coordinates, while the temporal integration is calculated using the leapfrog scheme. Discretization then yields

$$\begin{aligned} u_{i,j}^{n+1} &= u_{i,j}^{n-1} + 2\Delta t\left(-g\frac{h_{i+1,j}^n - h_{i,j}^n}{\Delta x}\right) \\ &+ \frac{f_0}{4}\left(v_{i,j}^n + v_{i+1,j}^n + v_{i+1,j-1}^n + v_{i,j-1}^n\right), \end{aligned} \tag{11}$$

$$\begin{aligned} v_{i,j}^{n+1} &= v_{i,j}^{n-1} + 2\Delta t\left(-g\frac{h_{i,j+1}^n - h_{i,j}^n}{\Delta y}\right) \\ &- \frac{f_0}{4}\left(u_{i,j}^n + u_{i,j+1}^n + u_{i-1,j+1}^n + u_{i-1,j}^n\right), \end{aligned} \tag{12}$$

$$\begin{aligned} h_{i,j}^{n+1} &= h_{i,j}^n \\ &- 2\Delta t\left(\frac{u_{i,j}^n - u_{i-1,j}^n}{\Delta x} + \frac{v_{i,j}^n - v_{i,j-1}^n}{\Delta y}\right). \end{aligned} \tag{13}$$

The energies are calculated as follows

$$V = \frac{1}{2}\rho g \sum_i \sum_j (h_{i,j})^2 \Delta x \Delta y \tag{14}$$

$$\begin{aligned} K =& \frac{1}{2}\rho \sum_i \sum_j \\ & \frac{1}{2}\left(u_{i-1,j}^2 + u_{i,j}^2 + v_{i-1,j}^2 + v_{i,j}^2\right) h_{i,j}. \end{aligned} \tag{15}$$

Unless otherwise mentioned, the following parameter values are used

$$\begin{aligned} g &= 9.81\,\mathrm{m/s^2} \\ f_0 &= 10^4\,\mathrm{m/s^2} \\ h_0 &= 0.5\,\mathrm{m} \\ D_0 &= 4000\,\mathrm{m} \\ \rho &= 1027\,\mathrm{kg/m^3}. \end{aligned}$$

The model and scripts used for analysis are found in a github repository [1].

4 Results

Figure 1 shows the initial sea surface height (solid black) and compares the adjusted height and velocity profiles between numerical (solid) and analytical (dashed) solutions. Overall, there is good agreement in shape and magnitude. There are small discrepancies close to the boundaries where the numerical solution underestimates sea surface height and meridional velocity.

The time evolution of relative potential (blue) and kinetic energy (orange) of the system is shown in Figure 2. The black lines marks a 1/3 and 2/3. We see that V and K fluctuate initially before converging to 2/3 for the potential energy and 1/3 for the kinetic energy, in agreement with theory.

5 Conclusion

[1] https://github.com/janadr/MO8004/tree/master/L1

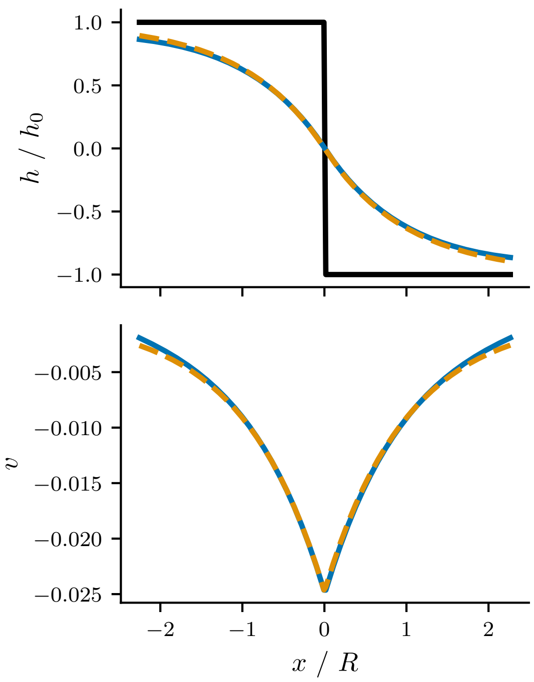

Figure 1: Upper: relative sea surface height. Lower: meridional velocity. Solid line shows numerical result, while dashed line shows analytical results.

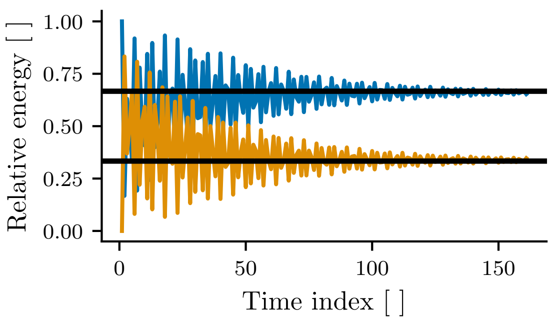

Figure 2: Relative energies where blue line shows potential energy and orange line shows kinetic energy. The horizontal black lines marks a 1/3 and 2/3.