Numerical exploration of solid boundary conditions

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1 Introduction

Continents acts as solid boundaries in the world ocean, and this is the case almost everywhere except at the poles. The equator also acts as a solid boundary in both the ocean and atmosphere. It is therefore central to understand the effect of these boundaries on the dynamics. To study solid boundaries, we will solve the shallow water equations (SWEs) for an initial disturbance with a solid boundary numerically and analytically.

In Section 2 we present the SWEs and describe Kelvin wave solutions, while the numerical method and details are described in Section 3. A few central results are presented in Section 4 showing comparisons with theory and numerical scenarios. Attempts at explaining the results is done in Section 5, concluding the report.

2 Theory

The presence of solid boundaries in a fluid has a large impact on the dynamics. Such boundaries may be actual physical boundaries such as the continents in the oceans and vorticity boundaries such as the equator in both atmosphere and ocean. A perturbation interacting with these boundaries produce Kelvin waves. To study this, we will consider the linearized shallow water equations:

$$\partial_t u - f_0 v = -g\partial_x h \tag{1}$$
$$\partial_t v + f_0 u = -g\partial_y h \tag{2}$$
$$\partial_t h + D_0(\partial_x u + \partial_y v) = 0, \tag{3}$$

where ∂_{q_i} denotes partial differentiation or ordinary differentiation given the context. u and v are the zonal and meridional velocities, respectively, while h is the sea-surface height. The planetary vorticity is f_0, while g is the gravitational acceleration, and D_0 is the depth of the system.

2.1 Southern boundary

We will first consider the case of a southern boundary, i.e. the equator. The meridional velocity of the flow must be zero at the boundary, and in the absence of friction, a reasonable assumption is then that the meridional velocity is zero everywhere. A general solution to the system of equations (eqs. (1), (2), and (3)) is then

$$h = \frac{D_0}{g} G_0(x - ct)\exp(-y/R) \tag{4}$$
$$u = G_0 \exp(-y/R) \tag{5}$$
$$v = 0. \tag{6}$$

Here G_0 is a general wave solution propagating eastwards, while $c \equiv \sqrt{gH}$ is the phase speed of the wave. The sea surface height and zonal velocity is maximum at the boundary and decays northwards.

2.2 Equatorial wave

Kelvin waves are observed travelling in the same direction on both sides of the equator. To study this, we will use the equatorial beta-plane approximation where $f0$ is small, and so $f \approx \beta y$, where β is a constant. In this scenario, the system of equations takes the form

$$\partial_t u - \beta y v = -g\partial_x h \tag{7}$$
$$\partial_t v + \beta y u = -g\partial_y h \tag{8}$$
$$\partial_t h + D_0(\partial_x u + \partial_y v) = 0. \tag{9}$$

Assuming again that $v = 0$, a general solution can be found:

$$h = \frac{D_0}{g} G_0(x - ct) \exp\left(-\frac{y^2}{2R_e^2}\right) \tag{10}$$
$$u = G_0(x - ct) \exp\left(-\frac{y^2}{2R_e^2}\right) \tag{11}$$
$$v = 0, \tag{12}$$

where $R_e \equiv \left(\sqrt{gD_0}/\beta\right)^{1/2}$ is the equatorial barotropic Rossby radius. The ocean can roughly be divided into two layers in the vertical due to the presence of a thermocline. Assuming a bottom layer with essentially no flow, we get the reduced gravity equations:

$$\partial_t u - \beta y v = -\partial_x \phi \tag{13}$$
$$\partial_t v + \beta y u = -\partial_y \phi \tag{14}$$
$$\partial_t \phi + c^2(\partial_x u + \partial_y v) = 0, \tag{15}$$

Here $\phi \equiv g'h$ is the streamfunction, $g' \equiv \delta\rho/\rho$ is the reduced gravity, and $c \equiv \sqrt{g'D_0'}$ is the propagation velocity where D_0 is the depth of the thermocline.

3 Method

We will follow the same dicretization scheme and method as done in a previous paper (Kallmyr, 2021), with some further implementations of reduced gravity and the beta plane.

The initial condition is a gaussian centered at the equator

$$h_i = h_0 \exp\left[-((x - 0.5 * L_x)/L_w)^2 - (y/L_w)^2\right], \tag{16}$$

where L_i denotes the domain size and L_w is a scaling factor.

For studying the southern border, we will assume periodic boundary conditions in the zonal direction and a open northern border. We will look at two different cases, that of the Atlantic ocean and the Baltic sea with basin parameters shown in Table 1.

Region	**L**	$\mathbf{D_0'}$
Atlantic ocean	10^7 m	1000 m
Baltic sea	10^6 m	30 m

Table 1: Basin parameters for the Atlantic ocean and Baltic sea. L denotes the domain size, while D_0' is the thermocline depth.

When looking at the equatorial wave, we use solid boundaries in all directions with domain size $L = 2 \cdot 10^7$ m and thermocline depth $D_0' = 1000$ m.

Global parameters for the simulations are presented in Table 2

Parameter	**Value**
g'	$0.04\,\mathrm{ms}^{-2}$
f_0	$10^{-4}\,\mathrm{s}^{-1}$
β	$2.287 \cdot 10^{-11}\ (\mathrm{sm})^{-1}$

Table 2: Model parameters.

4 Results

Figure 1 shows a comparison between numerical and analytical meridional profile of a wave travelling along a southern boundary in the Atlantic ocean (upper) and Baltic sea (lower). In both scenarios, the numerical solution underestimate the sea surface height, rapidly decaying away from the boundary.

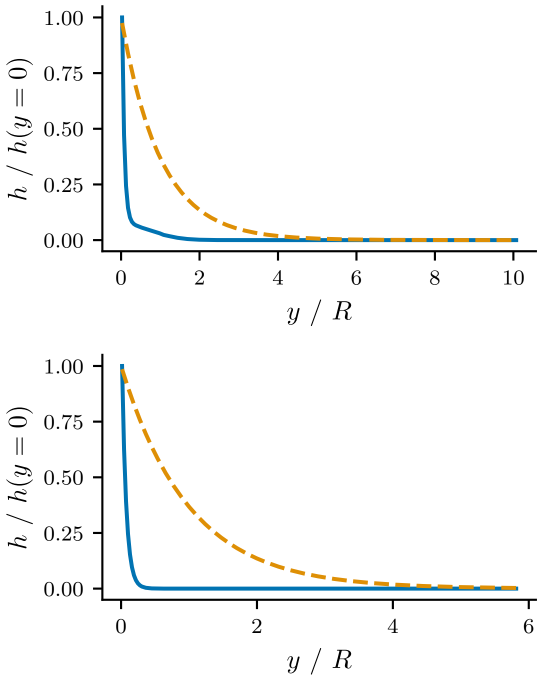

Figure 1: Meridional component of coastal Kelvin wave. Upper - Atlantic ocean: $L = 10^7$ m and $D_0 = 1000$ m. Lower - Baltic sea: $L = 10^6 m$ and $D_0 = 30$ m. The dashed line is an analytical solution, while the solid line is numerical.

The time evolution of the waves at the southern boundary are shown in Figures 2 (Atlantic ocean) and 3 (Baltic sea). A red linear line denotes a visually placed linear regression which slope is the velocity of the wave. The phase velocities are tabulated in Table 3 alongside the phase velocity of the equatorial wave. We can also see that the wave travels faster in the Baltic sea, completing almost two transects compared to the one transect in the Atlantic ocean.

Initial and final states of an gaussian perturbation centered at the equator for a beta plane is shown in Figure 4. We see that the ini-

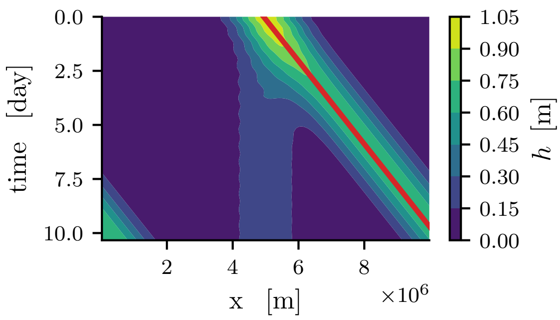

Figure 2: Hovmoller diagram at the equator (y=0) showing time evolution of a coastal Kelvin wave in the Atlantic ocean over 10 days. Red line represents a visually placed slope $x =$ $5.2 \cdot 10^5(t + 9.5)$.

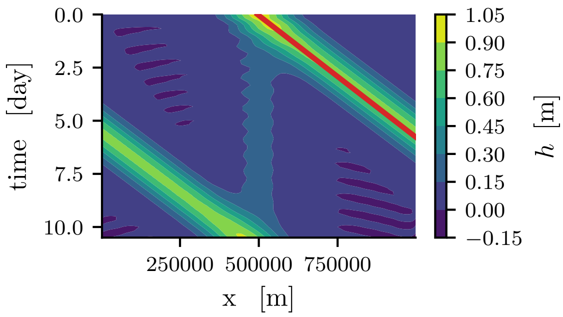

Figure 3: Hovmoller diagram at the equator (y=0) showing time evolution of a coastal Kelvin wave in the Baltic sea over 10 days. Red line represents a visually placed slope $x =$ $8.7 \cdot 10^4(t + 5.7)$.

tial state evolves into at least two waves travelling in opposite directions. The wave travelling eastward is propagating faster and reachers the eastern boundary, splitting in two. We identify this as an equatorial Kelvin wave, agreeing with theory in propagation direction and velocity (Table 3) . In an animation, it is possible to see that some of the wave reflects, taking on a similar shape to the large westward wave. The westward waves are slower equatorial Rossby waves.

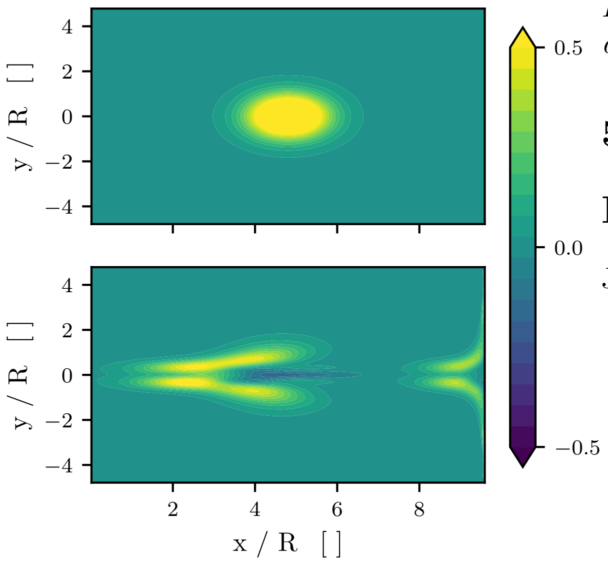

Figure 4: Upper - initial sea state of an equatorial perturbation. Lower - final sea state of an equatorial perturbation.

Scenario	Numerical [ms^{-1}]	Analytical [ms^{-1}]
Atlantic	6.0	6.3
Baltic	1.0	1.1
Equatorial	5.8	6.3

Table 3: Numerical and analytical values of phase velocities for three Kelvin wave scenarios.

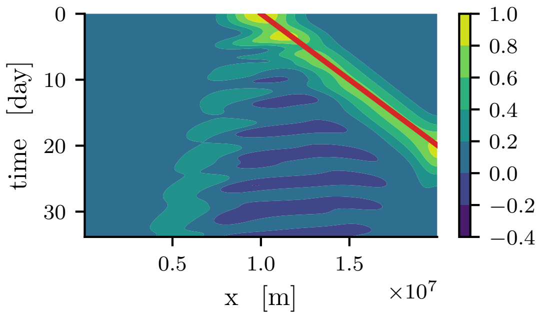

Figure 5: Hovmoller diagram at the equator (y=0) showing time evolution of an equatorial Kelvin wave over 35 days. Red line represents a visually placed slope $5 \cdot 10^5(t + 20)$.

5 Discussion

References

Jan-Adrian Kallmyr. Numerical exploration of geostrophic adjustment. 2021.