CHAPTER 6

The Random Sea

In the last two chapters we have considered the fluid motions accompanying water waves. This analysis has assumed that waves in deep water have a sinusoidal shape and that one wave is much like another, with wave height and period essentially the same. Such a depiction of the water surface is shown schematically in Figure 6.1a, but it is not an accurate portrait. A brief visit to the seashore will convince anyone that no two waves are exactly alike, and that successive waves can be quite different in both their height and period. A more typical picture of the ocean surface is given in Figure 6.1b, which is drawn from a record of the water-surface level off the California coast. Both the wave height and period are variable, and at times it is difficult to tell where one wave ends and the next begins. This, then, is the real world, and it leaves us with two large questions to answer: Why does the real ocean surface have these characteristics? And how can we reconcile the ideas we have derived for sinusoidal waves with this kind of ocean surface?

Wind-Generated Waves

In general, the waves that we see on the surface of the ocean are produced by the wind. As wind blows across a water surface, some of its energy is transferred to the fluid, producing waves. There are several mechanisms by which this energy transfer is accomplished. For example, localized variations in air pressure associated with turbulence in the wind may push and pull on the water's surface, forming small waves. As the wind blows over the crests of preexisting waves, its velocity increases, resulting in a decrease in air pressure above and downwind of the wave crest (see the discussion of lift in Chapter 11). The resulting force acts over a distance as the wave propagates, adding to the wave's energy. Various other mechanisms of energy transfer have been proposed, and their relative importance is a matter of some controversy. Regardless of the precise mechanism by which wind raises waves, the net result is the production of waves with a large range of heights and lengths, traveling in all directions. This wind-generated, chaotic ocean surface (often produced well out at sea in the midst of a storm) is the source of the waves that break on shore.

The size of waves produced by the wind depends on three factors: the wind speed, the length of time the wind blows, and the *fetch*, the length of the stretch of water affected by the wind. For a given wind speed, the wave height can be limited either by the length of time the wind blows, the

Figure 6.1. Sinusoidal vs. actual surface waves. (a) A monochromatic, sinusoidal wave train; wave height, length, and period are well defined. (b) A record of sea surface elevations; wave height, length, and period vary (redrawn from U.S. Army Corps of Engineers 1984).

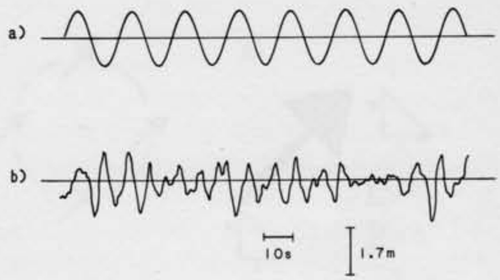


Table 6.1
Conditions in Fully Developed Seas
(from Bascom 1980)

Wind Velocity (m/s)	Fetch (km)	Time (hours)	Average Wave Height (m)	Sig. Wave Height (m)
5.1	18	2.4	0.3	0.4
7.7	63	6	0.8	1.1
10.3	139	10	1.5	2.4
12.9	297	16	2.7	4.3
15.4	519	23	4.3	6.7
20.6	1315	42	8.5	13.4
25.7	2630	69	14.6	23.8

fetch, or both. If neither the time nor the fetch limits the wave height, the waves reach an equilibrium height, and the sea is said to be "fully developed." Table 6.1 gives empirical measurements for the wave heights observed in fully developed seas as a function of the wind speed. These are the largest waves that the wind can produce; for the winds found in hurricanes and typhoons (> 60 knots, or about 30 m/s), the waves can reach monstrous proportions. For instance, instruments mounted on offshore oil rigs have recorded waves 24 m high. The highest wave reliably recorded was sighted by the watch officer of the HMS *Ramapo* during a storm in the South Pacific on February 6, 1933. By sighting from the bridge to the crow's-nest and on to the wave crest, Lt. F. C. Margraff estimated the wave to be 34 m high (Bascom 1980). Fortunately, these huge waves are rare. This is largely a result of the long time and fetch required for the sea to become fully developed at these high wind velocities. Hurricane winds need nearly three days to raise waves to their equilibrium height and require a fetch of more than 2250 km. Storms of this duration and size are seldom seen.

Several semiempirical methods are available for predicting wave heights from meteorological data. These are somewhat complex and will not be dealt with here. Readers interested in wave generation and forecasting should consult Pierson et al. (1955), Kinsman (1965), or the U.S. Army Corps of Engineers' *Shore Protection Manual* (1984).

Wave Propagation

Waves produced by storms often outlive the disturbance that produced them and propagate to a different section of the ocean. It is these propagated waves that we often see, and their size and shape are different from those in storms where waves are actively being produced.

Why should waves be any different after they have propagated? The most obvious reason is the dispersive nature of water waves; waves of different wave lengths travel with different celerities (eq. 4.9). Although a localized storm produces waves with a range of lengths, these waves are spatially sorted out as they travel. The longer waves (which travel fastest) arrive at a distant location first, spending the least time in transit. Conversely, the shorter waves arrive last and spend the greatest time in transit. In calculating the time it takes a wave to travel a certain distance, we must remember that wave energy is transmitted at the group velocity rather than at the celerity of individual waves. In deep water the group velocity is half the celerity (Table 5.2).

As waves travel they lose some of their energy to viscous processes. The loss depends not so much on the distance traveled as on the length of travel time. The height of a deep-water wave at time t is

$$H(t) = H_i \exp\left\{\frac{-32\pi^4 \nu t}{g^2 T^4}\right\}$$

$$H(t) = H_i \exp\left\{\frac{-8\pi^2 \nu t}{L^2}\right\}, \qquad (6.1)$$

where H_i is the initial wave height at time $t = 0$ (Komar 1976). As before, T is the wave period, L the wave length, and ν the kinematic viscosity of water. The factors T^4 and L^2 in the denominators of these equations mean that short period, short wave-length waves lose energy to viscosity much faster than do waves of longer periods and wave lengths, and shorter wave-length waves are more attenuated when they reach the shore. Waves with periods above 10 s lose relatively little height in propagation. For example a wave with $T = 1$ s would lose half its height in traveling 16 km, while a wave with $T = 10$ s would have to travel about 16 million km to lose half its initial height. These easily propagated waves of long wave length and long period are the *swell* described earlier. Swell more closely approximate our idealized sinusoidal wave train of Figure 6.1a than do the waves shown in Figure 6.1b.

The swell produced by a distant storm can interact with the waves created locally by wind. These local waves (the *seas* referred to in Chapter 4) are simply added onto the swell. This combination of locally produced seas and propagated swell is perhaps the most typical condition for the ocean surface in coastal waters.

Specifying the Random Sea

Having arrived at a reasonable, if cursory, explanation for the origin of waves and the complex wave pattern we typically observe, we are now faced with devising some means of reconciling this observed pattern with our theoretical understanding of wave motions. The crux of the problem is that each wave is different from the rest. One cannot speak of *the* wave height or *the* wave length. Instead, we are forced to deal with what appears to be a continuous distribution of wave heights and lengths. The reconciliation between wave theories and the random sea requires the use of a specialized set of statistics.

The Mean Square Amplitude

Recall from Chapter 3 that the amplitude of a wave is its maximum vertical deviation from mean sea level. Each wave has a positive amplitude (the crest) and a negative amplitude (the trough). A linear wave has half its wave form above sea level and half below, and the amplitude is simply one-half of the wave height. We make the assumption that this is true for real ocean waves as well.

To describe the "typical" sea surface shown in Figure 6.1b, we must have some means of measuring the average wave amplitude of a complex wave form. The first step is to measure the surface elevation at a large number of evenly spaced points. The average of these elevations is, by definition, the mean sea level. Each elevation is then expressed as a deviation from mean sea level, A_i, (a positive number for elevations above still water level, negative for points below). Each deviation is then squared (thereby automatically ending up with nothing but positive values), and squared deviations are averaged,

$$\text{mean square deviation} = \frac{1}{N}\sum_{i=1}^{N} A_i^2, \quad (6.2)$$

where N is the number of surface elevations measured. This statistic is formally known as the *mean square amplitude*, $\bar{A}^2$; it is the same as the *variance* of the surface elevation. Because it is the sum of squared lengths, it has the units m^2. To return to units of amplitude, we simply take the square root of $\bar{A}^2$ to arrive at the *root mean square* (*rms*) *wave amplitude*, A_{rms}. The rms amplitude is the same as the standard deviation of the sea surface about mean sea level. By doubling A_{rms} we get a measure of the rms wave height, H_{rms}.

The Rayleigh Distribution

In itself, the root mean square wave amplitude is not very useful. It gives us a general measure of how "wavy" a section of ocean is, but it is certainly not true that every wave in that section of ocean has the rms amplitude. In general, we are more concerned with the probability of encountering a wave of at least a certain amplitude than with knowing the amplitude of some hypothetical "average wave." Fortunately, a knowledge of the rms wave height allows for the calculation of these sorts of probabilities.

The amplitudes of ocean waves have been shown to follow a Rayleigh distribution (Longuet-Higgins 1952; Fig. 6.2a). There are few waves with small or very large amplitudes, the most probable wave amplitude being somewhat less than the rms amplitude. This distribution is described by the following equation:

$$P(A) = \left\{\frac{2A}{A_{\rm rms}^2}\right\} \exp - \left\{\frac{A}{A_{\rm rms}}\right\}^2, \quad (6.3)$$

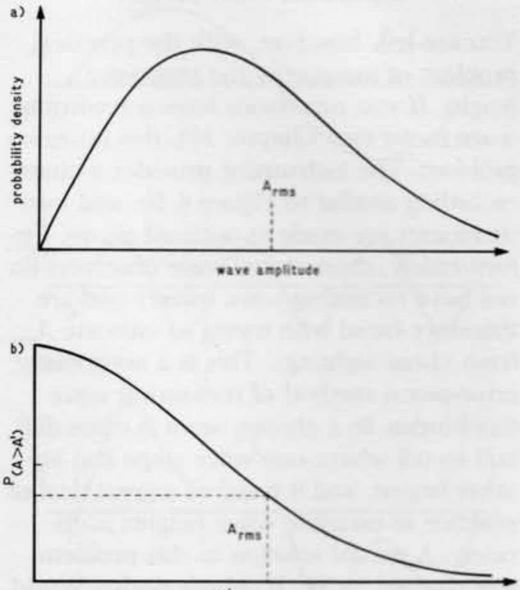
where $P(A)$ is the probability density for amplitude A. In other words, the probability that a wave has an amplitude within range $A \pm dA/2$ is $P(A)\,dA$. The total area under the probability density curve, $\int_0^\infty P(A)\,dA$, is the probability that a wave has a height between zero and infinity, and is therefore equal to 1.

The probability that a wave chosen at random has an amplitude greater than a certain value A' is

$$P_{(A>A')} = 1 - \int_0^{A'} P\,dA = \exp - \left\{\frac{A'}{A_{\rm rms}}\right\}^2. \quad (6.4)$$

This cumulative probability curve is shown in Figure 6.2b. Even on the calmest of days there are always some waves present on the ocean, thus the probability that there are waves with amplitude greater than zero is 1. The probability that a particular wave chosen at random is greater than the rms amplitude is $\exp - (1) = 0.37$. Thus 37% of waves have amplitudes greater than $A_{\rm rms}$. The higher the wave we specify, the smaller

Figure 6.2. The Rayleigh distribution. (a) The Rayleigh probability density distribution. (b) The Rayleigh exceedance distribution.



the probability that a wave chosen at random exceeds that height.

The Rayleigh distribution is satisfyingly simple to use. Unlike the Gaussian (normal) distribution where we must know both an arithmetic mean and a standard deviation, the Rayleigh distribution is scaled by a single factor, $A_{\rm rms}$.

Knowing that wave amplitudes follow a Rayleigh distribution, we can specify all sorts of interesting probabilities by simply measuring the root mean square wave height. Take a tangible example: You have established an experimental site on the shore and know from past experience that when a wave exceeds a certain height, H, the surge reaches your site. As you sit on the rock, counting snails in your quadrat, you might reasonably ask the question, "Will the next wave get me wet?" By specifying the wave conditions using $A_{\rm rms}$, this question can be answered:

$$P_{(A>H/2)} = \exp - \left\{\frac{H/2}{A_{rms}}\right\}^2.$$

Significant Wave Height

You are left, however, with the practical problem of measuring the rms wave height. If you perchance have a recording wave meter (see Chapter 19), this poses no problem. The instrument provides a chart recording similar to Figure 6.1b, and measurements are made as outlined above. Unfortunately, most casual wave observers do not have recording wave meters and are therefore faced with trying to estimate A_{rms} from visual sightings. This is a notoriously error-prone method of measuring wave amplitudes. In a choppy sea it is often difficult to tell where one wave stops and another begins, and it requires a great deal of practice to estimate wave heights accurately. A partial solution to this problem was devised by W. H. Munk during World War II. In those days oceanographers were assigned the problem of forecasting surf conditions for the beaches where amphibious assaults were to be made. Providing these forecasts involved, in part, knowing the sea conditions near the beach, and the only reliable way to get this information was to send someone out to look. The observer came back with an estimate along the lines of, "Well, the waves looked to be about 6 feet high." What precisely did this mean? By comparing visual observations with accurate wave records, Munk (1944) determined that the wave height as estimated by a skilled observer was approximately equal to the average height of the one-third-highest waves. In other words, when you look at waves, in an attempt to make some order out of the visual chaos your brain ignores two-thirds of the waves present (those with the smallest amplitudes) and averages what is left. This visually estimated wave height is called the *significant wave height*, H_s or $H_{1/3}$. For our purposes we define a significant wave amplitude, A_s, equal to half the significant wave height. It has been shown (Longuet-Higgins 1952) that

$$A_s = 1.412\, A_{rms},$$
$$A_{rms} = 0.71 A_s. \qquad (6.5)$$

Thus, from a visual estimation of wave height, it is possible to calculate a rough estimate of A_{rms}, and thereby to gain access to the Rayleigh distribution. This conclusion should be treated with caution. Upon first observing waves, we have a strong tendency to overestimate their height. When we are standing at sea level, or, worse yet, swimming, a 1 m wave appears quite large and a 2 m wave looks as big as a house. Only when we have carefully watched a number of waves pass a fixed object marked with a meter scale can we train our eyes to be reasonably accurate.

We must also use caution in applying the Rayleigh distribution to extreme situations. The U.S. Army Corps of Engineers (1984) has shown that the Rayleigh distribution overestimates by about 5% the height of waves occurring with probabilities of 0.01, and overestimates by about 15% those with probabilities of 0.0001 or less. There can also be problems with assuming a Rayleigh distribution in shallow water. Thornton and Guza (1983) measured wave heights at various places in the surf zone on a sandy beach and found that as waves steepen and break, the nonlinearity of their behavior can cause their height distribution to deviate from the Rayleigh distribution. However, once waves have broken, they again become Rayleigh-distributed. This is a surprising result, one not easily accounted for by theory, and we will use it cautiously here. In summary, we can use the Rayleigh distribution with some confidence for waves outside the surf zone and, with some reservations, for broken waves in the surf zone, but we must be careful in using this distribution to describe breaking wave heights.

The Rayleigh distribution will be used

extensively in Chapter 16 in the discussion of structural wave exposure.

The Wave Spectrum

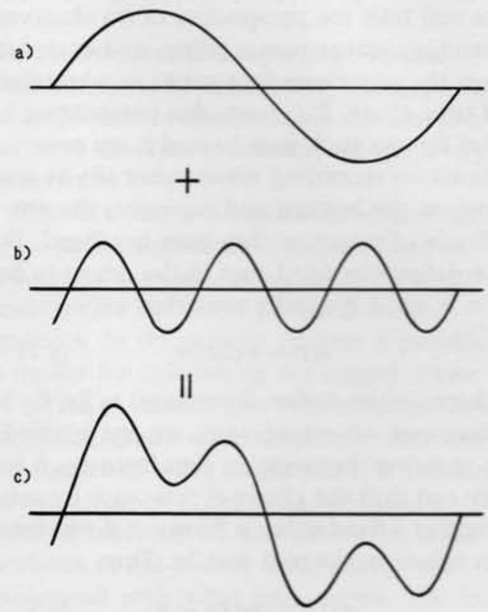
Useful as the above approach may be as a means for coping with distributed wave amplitudes, it is only a partial fulfillment of our needs. An examination of Figure 6.1b (our typical wave record) clearly shows that not only do wave amplitudes vary, but wave periods vary as well. The use of an rms amplitude and the probabilities calculated from the Rayleigh distribution tell us nothing about which wave heights are associated with which wave lengths. However, most calculations regarding wave-induced water motions require knowing both the wave amplitude and the wave period. Solitary waves (which are aperiodic) are an important exception to this rule, but this exception is not sufficient to alleviate the general need to know something about both the amplitude and the period of real ocean waves.

Information about the relationship between amplitude and period is best conveyed by the *wave power spectrum* or, more simply, the *wave spectrum*.

We begin by examining the additive properties of linear waves in a laboratory wave tank. Recall that waves are produced in this tank by sinusoidally moving a paddle. We can vary the frequency with which the paddle oscillates, but only one frequency can be produced at a time. We now replace the paddle with a more sophisticated device, one capable of oscillating at several different frequencies simultaneously. Devices appropriate for this task are described in Chapter 19.

We first use our wave maker to produce waves of a single frequency, causing the surface of the tank to look like Figure 6.3a. This is the familiar sinusoidal (also known as monochromatic) wave train discussed in Chapter 4. Alternatively, we can produce waves of a different, higher frequency, as shown in Figure 6.3b. If we

Figure 6.3. The surface elevations of water waves are additive. The sum of the wave forms (a) and (b) is the wave form shown in (c).



produce both wave forms simultaneously, we find that the surface of the tank at one instant in time looks like Figure 6.3c. At points where the peaks of the two wave forms coincide, the amplitude of the combined wave is equal to the sum of the amplitudes of the component waves. Similarly, where troughs coincide, the negative amplitude is greater than before. At points where the trough of one wave coincides with the crest of another, the resulting amplitude is again the sum of the two, and the surface deviation is less than that present in either wave form alone. The surface resulting from the superposition of two wave trains of different frequencies is far more complex than that of a single frequency wave train, and looks more like Figure 6.1b.

We can state these findings in a more formal manner. A sinusoidal wave form can be expressed in both time and space by the equation

$$\eta(x, t) = \alpha \cos\left\{\frac{2\pi x}{L} - \frac{2\pi t}{T}\right\}, \qquad (6.6)$$

where $\eta(x, t)$ is the surface elevation at position x at time t, and α is the wave amplitude. For the present discussion, we need not worry about both time and space; instead, we will take the perspective of an observer watching waves pass a piling, and examine how the water's surface varies as a function of time alone. It is from this perspective that figures such as 6.1a and b are produced—a recording wave meter sits at one spot on the bottom and measures the amplitude of waves as they pass overhead. If we define our fixed spot in the ocean to be $x = 0$, eq. 6.6 can be rewritten as

$$\eta(t) = \alpha \cos \omega t, \qquad (6.7)$$

where ω (the radian frequency) $= 2\pi/T$. Since $\cos(\omega t) = \cos(-\omega t)$, we are justified in ignoring the negative sign from eq. 6.6. We can shift the phase of this wave by subtracting a fixed value ϕ from ωt; ϕ can take on values between 0 and 2π. Thus

$$\eta(t) = \alpha \cos(\omega t - \phi). \qquad (6.8)$$

The surface elevation produced by two waves acting simultaneously is simply the sum of the surface elevations for the individual waves:

$$\eta(t) = \alpha_1 \cos(\omega_1 t - \phi_1) + \alpha_2 \cos(\omega_2 t - \phi_2). \qquad (6.9)$$

Any arbitrary surface-elevation pattern can be represented by the appropriate sum of several component wave forms,

$$\eta(t) = \sum_{i=1}^{N} \alpha_i \cos(\omega_i t - \phi_i), \qquad (6.10)$$

where N is the number of wave forms of different frequency that we add. For example, the wave form shown in Figure 6.4a, which is very reminiscent of our typical wave record, is formed from the component waves shown beneath it.

It would be possible to continue the analysis of wave patterns using eq. 6.10 as it stands, but it will be useful to restate this equation in a more conventional form. Consider Figure 6.5. Here a sine wave and a cosine wave of equal amplitudes and the

Figure 6.4. Complex surface elevations. The wave form shown in (a) is the sum of the component waves shown in (b)–(g).

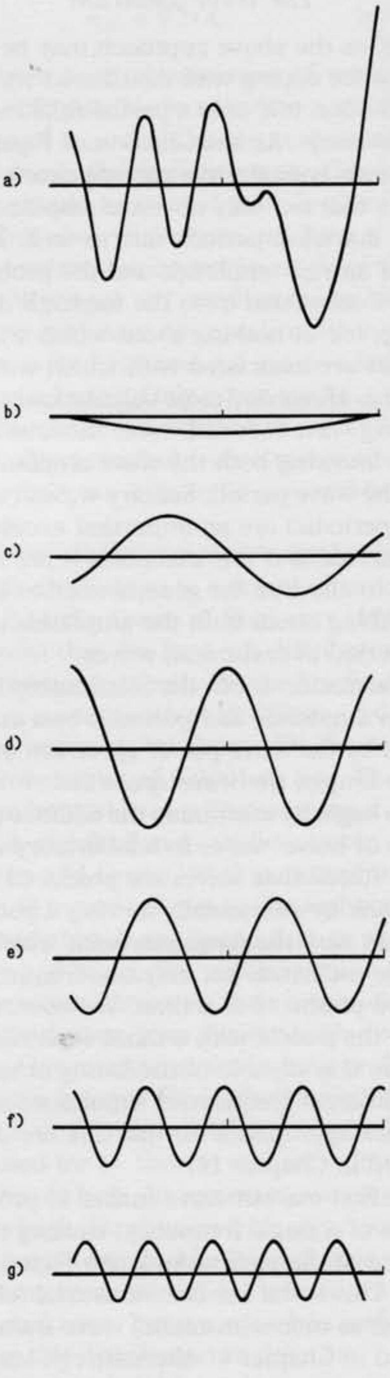
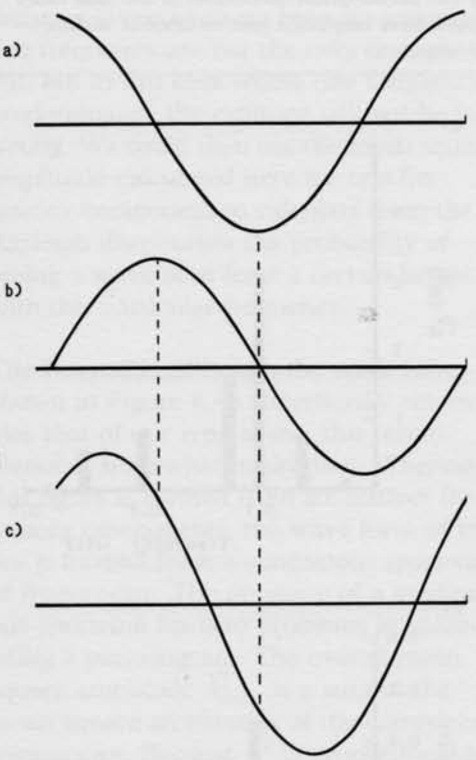


Figure 6.5. The addition of a sine wave (a) and a cosine wave (b) of the same frequency results in a wave with the same frequency but a different phase (c).



same frequency have been added together. The result is a wave with an amplitude slightly larger than either component and, more importantly, with a phase that has been shifted from either component. Thus, one effect of adding sine and cosine waves of the same frequency is a shift in the phase of the resulting wave. By varying the relative amplitudes of the component sine and cosine waves, we can produce phase shifts from 0 to 2π. Given this fact, we can rewrite eq. 6.10 as

$$\eta(t) = \sum_{i=1}^{N} \{\alpha_i \cos \omega_i t + \beta_i \sin \omega_i t\}, \qquad (6.11)$$

where the term containing $\sin(\omega t)$ substitutes for ϕ in eq. 6.10. Here η is measured relative to mean sea level. If η is measured relative to some other reference point, we need to add a term for the average elevation, $\bar{\eta}$, to eq. 6.11 to account for the shift in reference. Thus, in general,

$$\eta(t) = \bar{\eta} + \sum_{i=1}^{N} \{\alpha_i \cos \omega_i t + \beta_i \sin \omega_i t\}. \qquad (6.12)$$

This is the standard expression for the Fourier series. By appropriate manipulation of amplitude coefficients, we can use this series to describe practically any arbitrary wave form.[1]

The fact that a complicated surface pattern can be the result of the addition of a series of component waves of different frequencies has many practical consequences. In the present context it provides a means for calculating the contributions of waves of various frequencies to the overall mean square wave amplitude. Because the square of wave amplitude is a measure of wave energy (Table 5.2), we have a method whereby we can determine what portions of the overall wave energy are associated with what frequencies. The frequency of a wave is the inverse of its period. Thus our knowledge of the energy associated with a given frequency gives us the energy (and power; see Chapter 7) associated with a certain wave period.

How do we calculate the energy contributions of different wave-frequency components? We begin by calculating the mean square amplitude, $\overline{A_f^2}$, for a single frequency, f. By definition, the mean square amplitude is

$$\begin{aligned}\overline{A_f^2} &= \frac{1}{2\pi} \int_0^{2\pi} \alpha_f^2 \cos^2\theta \, d\theta \\ &= \frac{\alpha_f^2}{2\pi} \left\{\frac{2\pi}{2} + \frac{\sin 4\pi}{4}\right\} \\ &= \frac{\alpha_f^2}{2}, \end{aligned} \qquad (6.13)$$

[1] To be described by a Fourier series, a function, $f(t)$, must fulfill Dirichlet's conditions: the integral $\int f(t)\, dt$ must be finite over the period of interest and $f(t)$ must be piecewise continuous and piecewise monotonic. We will not concern overselves with these technicalities. By their physical nature, water waves fulfill Dirichlet's conditions.

where α_f is the amplitude of the wave form. The same result can be obtained for the sine component of a wave form—the mean square amplitude is one-half the square of the overall amplitude. By applying this fact, we see that the mean square amplitude of one frequency component of a complex wave form is

$$\overline{A_f^2} = \frac{\alpha_f^2}{2} + \frac{\beta_f^2}{2} = (\tfrac{1}{2})(\alpha_f^2 + \beta_f^2). \quad (6.14)$$

The overall mean square amplitude for a complex wave formed of N equally spaced frequency components is

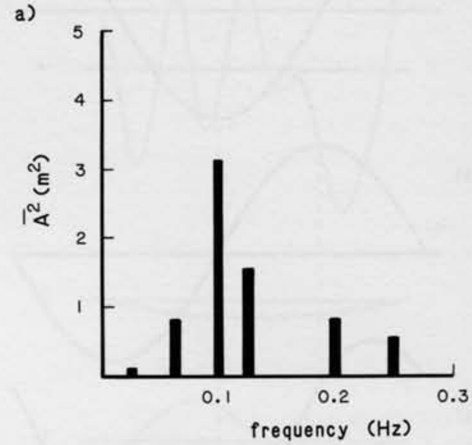
$$\overline{A_{\text{total}}^2} = (\tfrac{1}{2}) \sum_{i=1}^{N-1} (\alpha_{f_i}^2 + \beta_{f_i}^2) + \alpha_{f_N}^2, \quad (6.15)$$

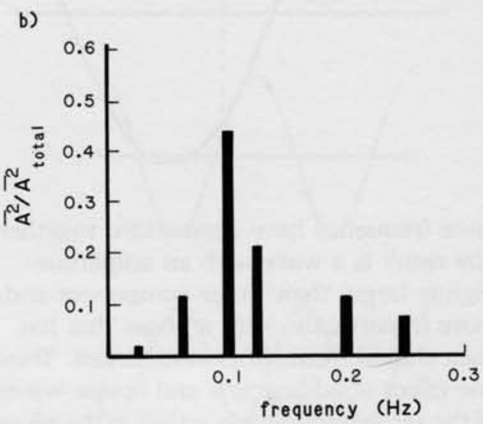
where the subscript, f_i, is used to denote the different frequencies.[2] Note that the highest frequency (f_N) is a special case in what is otherwise a simple summation of the contributions from individual frequencies.

The Periodogram. The additive nature of mean squares allows one to construct a periodogram. In (Fig. 6.6a) the individual mean squares are plotted for the various component frequencies of the complex wave form shown in Figure 6.5a. Because the overall mean square wave height is the sum of these individual mean square amplitudes, the height of the bars is an unbiased representation of the relative contribution of various frequencies to the overall mean square amplitude. This is even more clearly seen if the ordinate is normalized by dividing by the total mean square amplitude (Fig. 6.6b). The normalized contributions made by various frequencies sum to 1, and it is easy to see that the component with a frequency of 0.1 Hz contributes 44% of the overall mean square amplitude, while the component with frequency 0.2 Hz contributes only 11%.

[2] This result (a consequence of Parseval's theorem) is less intuitive and less easily proved. The interested reader should consult Jenkins and Watts (1968) or Chatfield (1984).

Figure 6.6. The periodogram quantifies the contribution of component waves of different frequencies to the overall mean square wave amplitude. (a) The periodogram for the wave form shown in Figure 7.5a. (b) The periodogram normalized to the total mean square wave amplitude (the variance of surface elevation).





This sort of analysis is useful in a different way from that of the Rayleigh distribution. For example, from the periodogram shown in Fig. 6.6a, we know that waves with a frequency of 0.1 Hz ($T = 10$ s) account for nearly half the wave energy in this case and have a mean square amplitudes of about 3 m. Thus, for this component frequency, the rms amplitude (A_{rms}) is about $\sqrt{3} = 1.7$ m, and the rms wave height is $2\ A_{\text{rms}} = 3.4$ m. In deep water these waves have a length of 156 m (eq.

4.8). With this information regarding wave height and length, we can use linear wave theory to provide a rough estimate of what the water motion is at any depth. This estimate will not be exact because waves of this frequency are not the only ones present, but in this case where one frequency predominates, the estimate will not be far wrong. We could then use the mean square amplitude calculated here for one frequency component to calculate from the Rayleigh distribution the probability of seeing a wave of at least a certain height with this particular frequency.

The Intensity. Although the wave form shown in Figure 6.4a superficially resembles that of our typical sea, this resemblance is somewhat misleading. Whereas this figure is formed from six distinct frequency components, the wave form of the sea is formed from a continuous spectrum of frequencies. The presence of a continuous spectrum leads to problems in calculating a periodogram. The overall mean square amplitude $\overline{A^2_{\text{total}}}$, is a sum of the mean square amplitudes of the component frequencies. Because $\overline{A^2}$ is proportional to wave energy, we know a priori that it must have a finite value. No wave has an infinite energy. This poses no problem when only a few frequencies are present; each contributes a measurable portion of the overall mean square amplitude. However, as the number of frequency components increases, $\overline{A^2_{\text{total}}}$ does not (lest the total wave energy tend toward infinity), and the contribution of each component must decrease. For a continuous spectrum of frequencies that, by definition, is made up of an infinite number of components, each component frequency contributes an infinitesimal amount to the overall mean square amplitude. A periodogram for a sea surface with a continuous spectrum of frequencies would look like a bare set of axes, a totally uninformative picture.

How can we avoid this problem? The simplest procedure is to define a new term for the contribution of each frequency to the overall "waviness" of the surface. At each frequency the *intensity* is

$$I(f_i) = \frac{N}{2}(\alpha^2_{f_i} + \beta^2_{f_i}) \qquad i = 1, 2, \ldots, N-1 \tag{6.16}$$

$$I(f_N) = N(\alpha^2_{f_N} + \beta^2_{f_N}),$$

where N is the total number of frequency components contributing to the surface elevation. As N increases, $(\alpha^2_f + \beta^2_f)$ at each frequency decreases in a manner so that the intensity remains finite. By using this simple stratagem we can plot an equivalent to the periodogram for as many frequencies as we care to examine. Given this definition of the intensity,

$$\overline{A^2_{\text{total}}} = \sum_{i=1}^{N} I(f_i)\frac{1}{N}. \tag{6.17}$$

$(1/N)$ is a measure of how small the difference in frequency, Δf, is between the component waves contributing to the surface elevation. In the limit as N goes to infinity, $1/N$ becomes df, and we can rewrite eq. 6.17 in integral form:

$$\overline{A^2_{\text{total}}} = \int_0^\infty I(f)\,df. \tag{6.18}$$

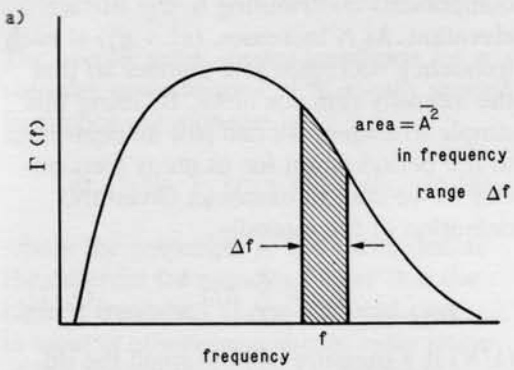
In this form $I(f)$ is given a new symbol, $\Gamma(f)$, and is called the *power spectral density function,* or more simply, the *power spectrum.* Thus

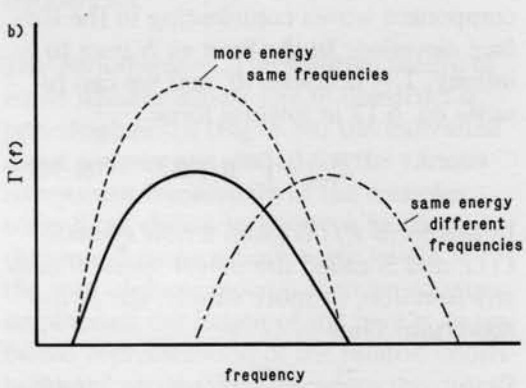
$$\overline{A^2_{\text{total}}} = \int_0^\infty \Gamma(f)\,df. \tag{6.19}$$

In other words, $\Gamma(f)$ is defined to be a function of frequency such that the *area* under the function [equal to the integral of $\Gamma(f)$ over its entire range] is equal to the overall mean square amplitude (Fig. 6.7a). In essence we have recalculated the periodogram so that the contribution of each frequency to the $\overline{A^2_{\text{total}}}$ is proportional to the area of the graph beneath $\Gamma(f)$ at that frequency, rather than to the value of $\overline{A^2}$ corresponding to that frequency. Defined in this manner, $\Gamma(f)$ has the units m^2 s.

In one sense this new calculation does

Figure 6.7. The power spectrum. (a) The area under the spectrum is the total mean square amplitude (the variance of surface elevation). The area under one portion of the curve is the contribution of that frequency range to the overall mean square wave amplitude. (b) A shift in the height of the wave spectrum reflects a change in wave amplitude. Shifts along the frequency axis reflect changes in the frequency of component waves contributing to a complex surface elevation.





not change anything. Although a particular frequency may have a finite value for $\Gamma(f)$, because it is a discrete point on the abscissa no measurable area is associated with this single frequency. Thus we arrive at the same conclusion as before: each individual frequency in a continuous spectrum contributes an infinitesimal amount to the overall mean square amplitude. But with this new function we are not confined to looking at individual frequencies. From Figure 6.7a it is clear that a range of frequencies does correspond to an area on the graph. Thus if we know the shape of the power spectrum, we can quantify the contribution of any range of frequencies to the overall mean square amplitude. For example, the range of frequencies (Δf) shown in Figure 6.7a accounts for about 15% of the area under the spectrum, and therefore for 15% of $\overline{A^2_{\text{total}}}$. From eq. 6.13 we can then write

$$0.15\overline{A^2_{\text{total}}} = (\tfrac{1}{2})A_f^2$$
$$A_f = \sqrt{0.3\overline{A^2_{\text{total}}}},$$

where A_f is the average amplitude of the waves in the frequency range, Δf. More generally, $\Gamma(f)$, the value of the spectrum at a point, when multiplied by Δf, a small range of frequencies, yields an area such that

$$\Gamma(f)\,\Delta f = (\tfrac{1}{2})A_f^2$$
$$A_f = \sqrt{2\Gamma(f)\,\Delta f}. \qquad (6.20)$$

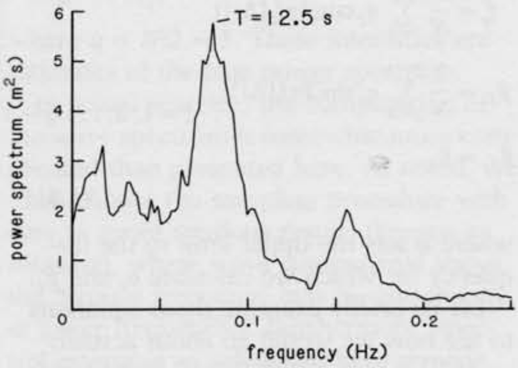
Be warned that this is an approximation; it gets less and less exact as Δf is allowed to get wider.

Figure 6.7b shows a spectrum with roughly the same shape as Figure 6.7a, but with double the area under the curve. This doubling of area means that $\overline{A^2_{\text{total}}}$ is doubled, corresponding to a 1.4-fold increase in the rms wave height. Thus a shift of the spectrum up or down on the ordinate corresponds to a difference in the "waviness" of the sea surface. Shifting of spectral peaks to the left or right (Fig. 6.7b) corresponds to a change in the frequency pattern of the sea surface.

It is possible to normalize this kind of graph in exactly the same fashion as for the periodogram. If each value of $\Gamma(f)$ is divided by $\overline{A^2_{\text{total}}}$, the area under the normalized curve must equal 1. In this case the curve is known as the *normalized power spectral density function.*[3] Figure 6.8 is

[3] There is some variation among authors in the names used for these functions. For example, Jenkins and Watts (1968) refer to what we have here termed the "power spectral density function" as simply the

Figure 6.8. The wave spectrum for broken waves (turbulent bores), recorded along the rocky coastline of Hopkins Marine Station, Pacific Grove, California. The major spectral peak indicates that the bores have a period of 12.5 s. The bores with this period have an average amplitude of 0.55 m (as calculated using eq. 6.20), corresponding to an average height of 1.1 m. Spectral estimates were averaged in groups of ten, i.e., there are twenty degrees of freedom.

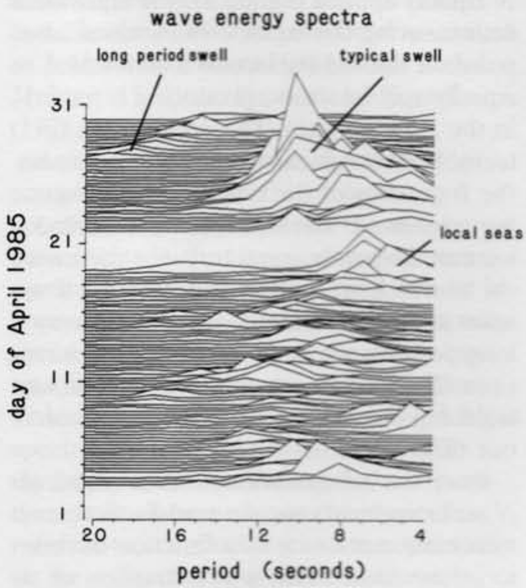


the normalized spectral density curve for waves breaking on a rocky shore during a storm in December 1985. The spectrum tells us that the peak energy is associated with waves with frequencies around 0.1 Hz ($T = 10$ s). Thus the waves produced by this storm behave (for the most part) as if they have wave lengths of approximately 150 m.

Figure 6.9 is taken from the records of the U.S. Army Corps of Engineers' Coastal Data Information Program for the month of April 1985. Four spectra are calculated for each day, and the compilation clearly shows how the wave spectrum changes from day to day. On April 18 and 19 local seas with periods of 4–8 s accounted for most of the wave energy. By April 24 these seas had been replaced by swell with periods centered around 10 s. On April 27–28 the spectrum was dominated by very long period swell (17–19 s), presumably from a distant storm. Note that the

"power spectrum," and the "normalized power spectral density function" as the "power spectral density." Here we follow the convention used by Chatfield (1984).

Figure 6.9. Daily wave spectra recorded in Monterey Bay, April 1985. Reproduced from the Coastal Data Information Program, U.S. Army Corps of Engineers and the California Department of Boating and Waterways.



abscissa in this graph is slightly different from what we have examined so far—it is linear in period rather than in frequency, tending to expand the low-frequency end of the spectral curve while compressing the high-frequency end.

The mathematical process we have used here to examine the frequency components of the random sea is useful in other contexts as well. In general, this process is known as "spectral analysis," and it is a standard tool for exploring the predictability of a series of events.

Measuring the Wave Spectrum

Knowing that a complex sea surface is composed of a spectrum of waves of different frequencies does not make it readily apparent how, given a wave record, we should go about deciding which ranges of frequencies contributed which amplitudes. In practice, we arrive at the wave spectrum through a simple, but somewhat laborious, process. We start with a record of the sea-surface elevations, such as that in Figure 6.1b, and measure the derivations from mean sea level at a series of N equally spaced points. This is equivalent to measuring the surface elevation at one point on the sea surface at a series of equally spaced times, producing a record in the *time domain*. The appropriate interval between measurements depends on the frequency of the waves in which we are interested. The shortest wave period we can discern is equal to twice the interval at which we sample. Conversely, if we want information about waves with very long periods, we need to sample for a long time. The longest wave period we can say anything about is equal to the length of our time series.

Once we have obtained a time series of N surface elevations, we transform this information about η as a function of time to information about η as a function of frequency. In other words, we transform the data from the time domain to the *frequency domain* via the Fourier series. We know (eq. 6.12) that by specifying the amplitudes of the sine and cosine waves at each of many frequencies we can tailor the Fourier series to reproduce any arbitrary wave form. The task, then, is to correctly specify α_i and β_i in eq. 6.12 to best fit a given time series.

The time period for which we have information is equal to $N\Delta$, where N is the total number of data points and Δ is the time between samples. Noting the fact that frequency = 1/period, we define a fundamental frequency, $f = 1/(N\Delta)$. We will examine the surface elevation at even multiples (harmonics) of this fundamental frequency. In other words, we define α_i and β_i at each frequency f_i, where f_i is $i/(N\Delta)$ and i is any integer between 1 and q, the highest harmonic that can be discerned. As noted before, the shortest period we can discern is 2Δ, so the highest frequency we can deal with is $1/(2\Delta)$, known as the Nyquist frequency. Consequently, $q = N/2$.

By a derivation that we will not delve into here (see Jenkins and Watts 1968; Chatfield 1984), we can show that the appropriate values for α_{f_i} and β_{f_i} are

$$\alpha_{f_i} = \frac{2}{N}\sum_{j=0}^{N-1} \eta_j \cos(2\pi f_i[\Delta j]). \qquad \ldots i = 1, 2, \ldots q-1$$

$$f_q = \frac{1}{N}\sum_{j=0}^{N-1} \eta_j \cos(2\pi f_q[\Delta j])$$

$$\beta_{f_i} = \frac{2}{N}\sum_{j=0}^{N-1} \eta_j \sin(2\pi f_i[\Delta j]) \qquad \ldots i = 1, 2, \ldots q-1$$

$$\beta_{f_q} = 0, \tag{6.21}$$

where q sets the upper limit to the frequency for which we calculate α_i and β_i.

Let us briefly examine these equations to see how we would go about actually carrying out the calculation. Consider the equation defining α_{f_2}, the second harmonic of the fundamental frequency, for a time series consisting of $N = 10$ points taken at intervals of $\Delta = 1$ s:

$$\alpha_{f_2} = \frac{2}{N}\sum_{j=0}^{N-1} \eta_j \cos(2\pi f_2[\Delta j]).$$

For the moment, consider only the cosine term in this series. In this example, $f_2 = 2/(N\Delta) = 0.2$ Hz. Now, 2π times frequency (in cycles per second) is the radian frequency, ω (eq. 6.7). The variable j in the expression for α_{f_i} is just a tool for counting through the sample; Δj is equal to the time t (from the beginning of our time series) at which a certain value of surface elevation (η_j) was taken. For instance, when $j = 5$, we are dealing with the sample taken 5 s after the beginning of the series. Thus the term $\cos(2\pi f_2 \Delta j)$ is equivalent to $\cos(\omega t)$, the same sort of expression encountered in eq. 6.12. For each time in the series, we multiply the measured surface elevation (η_j) by the appropriate term for $\cos(\omega t)$. Adding all N of these values together and multiplying by $(2/N)$ yields α_{f_2} for frequency f_2 (in this example 0.2 Hz). To calculate α_{f_i} for a different frequency, we repeat the whole procedure using a different i.

Once the various α_{f_i} and β_{f_i} have been calculated, it is a simple procedure to calculate the intensity at each frequency:

$$I(f_i) = \frac{N}{2}(\alpha_{f_i}^2 + \beta_{f_i}^2) \qquad i = 1, 2, \ldots, q-1$$

$$I(f_q) = N\alpha_{f_q}^2, \qquad (6.22)$$

where $q = N/2 = 5$. These intensities are estimates of the true power spectrum.

In actual practice, the computation of the wave spectrum is somewhat more complicated than presented here. As noted, we must choose the sampling procedure with care to avoid spurious results (known as *aliasing*), where wave components above the Nyquist frequency bias measurements at lower frequencies. Furthermore, spectral estimates as calculated here provide an imperfect picture of the true spectrum. In particular, when we examine frequencies near the fundamental frequency, only a few cycles are present in a time series, and there is a substantial possibility for error in estimating the true spectrum. As a result, it is common to find that spectral estimates fluctuate wildly at low frequencies. This problem is circumvented by "smoothing" the spectral estimates in a process known as *apodizing* or *applying a spectral window*. There are many methods for apodizing; the simplest is to average each spectral estimate with estimates at several adjacent frequencies. Thus the smoothed estimate at frequency f_i is

$$I(f_i) = \frac{1}{(2j+1)} \sum_{m=i-j}^{i+j} I(f_m), \qquad (6.23)$$

where $J = 2j + 1$ is the size of the group averaged at each frequency. The larger the group of estimates used in this running average, the smoother the spectrum. Herein lies the art of spectral estimation. If J is chosen too small, the detail shown by the spectral estimate may be spurious.[4] If J is chosen too large, the spectrum is smoothed to the point where important detail is lost. Readers interested in developing expertise in spectral analysis should consult the excellent introductions by Denman (1975) and Chatfield (1984) or standard texts such as Jenkins and Watts (1968), Box and Jenkins (1970), or Bendat and Piersol (1971).

As one might imagine, the number of computations involved in calculating a spectrum rises rapidly with the number of data points in the time series. For oceanographical time series that may consist of tens of thousands of points, the computational load is immense. To ease this load, an efficient algorithm (the Fast Fourier Transform, or FFT) has been devised. Most spectra are now calculated using FFT methods rather than using the computation outlined here. Again the reader is referred to standard texts on spectral analysis for a discussion of this technique.

Summary

Although the ocean's surface is exceedingly complex, the statistical techniques outlined in this chapter allow us to derive some order from this complexity. The Rayleigh distribution provides a tool for predicting the height distribution of waves. In Chapter 16 we use these predictions to study the structural exposure of wave-swept organisms. The power spectrum allows us to calculate equivalent wave heights and lengths for a complex ocean surface, and thereby provides a method by which we can apply wave theory to real ocean waves.

[4] The parameter $2J$, twice the group size, is the degrees of freedom of the spectral estimate. This parameter can be used in a chi-squared calculation of confidence interval size. See Chatfield (1984) for details.

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