Synchronizing Expanders Notes

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1 Notes

- Each node in the model corresponds to an oscillator.
- $\theta_i(t) : \mathbb{R} \to \mathbb{S}^1$ - phase of oscillator i at time t.
- The Kuramoto model is modelled by the following system

$$\frac{d\theta_x}{dt} = \omega_x - \sum_{y \in V} A_{x,y} \sin(\theta_x - \theta_y), \qquad \text{for all } x \in V. \tag{1}$$

- We assume that all oscillators have same intrinsic frequency.
- We rotate the frame $\theta(t) \leftarrow \theta(t) - \omega t$.

$$\frac{d\theta_x}{dt} + \omega = \omega - \sum_{y \in V} A_{x,y} \sin(\theta_x + \omega - (\theta_y + \omega))$$

$$\frac{d\theta_x}{dt} = - \sum_{y \in V} A_{x,y} \sin(\theta_x - \theta_y) \tag{2}$$

- The Kuramoto model correspond to the gradient flow, i.e. $\frac{d\theta}{dt} = -\nabla \mathcal{E}_G(\theta)$.
- The energy function $\mathcal{E}_G : (\mathbb{S}^1)^{|V|} \to \mathbb{R}$ and the derivatives are given bellow:

$$\mathcal{E}_G(\theta) = \frac{1}{2} \sum_{x,y \in V} A_{x,y}(1 - \cos(\theta_x - \theta_y)) \tag{3}$$

$$(\nabla \mathcal{E}_G(\theta))_x = \sum_{z \in V} A_{x,z} \sin(\theta_x - \theta_z) \tag{4}$$

$$(\nabla^2 \mathcal{E}_G(\theta))_{x,y} = \begin{cases} -A_{x,y} \cos(\theta_x - \theta_y) & \text{if } x \neq y, \\ \sum_{z \in V \setminus \{x\}} A_{x,z} \cos(\theta_x - \theta_z) & \text{if } x = y. \end{cases} \tag{5}$$

- The equilibrium condition is when for all $y \in V$

$$\sum_{x \in V} A_{x,y} \sin(\theta_x - \theta_y) = 0 \tag{6}$$

Which is equivalent to the following: For all y and some $r_y \in \mathbb{R}$

$$\sum_{x \in V} A_{x,y} e^{i\theta_x} = r_y e^{i\theta_y} \tag{7}$$

Proof. We can rewrite equation (7) as

$$\sum_{x \in V} A_{x,y}(\cos(\theta_x) + i \sin(\theta_x)) = r_y(\cos(\theta_y) + i \sin(\theta_y)) \tag{8}$$

By comparing the real and imaginary parts we get the following set of equations

$$\sum_{x\in V} A_{x,y}\cos(\theta_x) = r_y\cos(\theta_y)$$

$$\sum_{x\in V} A_{x,y}\sin(\theta_x) = r_y\sin(\theta_y)$$

Multiplying the first equation with $(-\sin(\theta_y))$ and multiplying the second equation by $\cos(\theta_y)$ and adding both equations together we get

$$\left(\sum_{x\in V} A_{x,y}\cos(\theta_x)(-\sin(\theta_y))\right) + \left(\sum_{x\in V} A_{x,y}\sin(\theta_x)\cos(\theta_y)\right) = r_y(\cos(\theta_y)(-\sin(\theta_y)) + \sin(\theta_y)\cos(\theta_y))$$

$$\implies \sum_{x\in V} A_{x,y}\sin(\theta_x)\cos(\theta_y) - \cos(\theta_x)\sin(\theta_y) = 0$$

$$\implies \sum_{x\in V} A_{x,y}\sin(\theta_x - \theta_y) = 0$$

Proving the other way around we can rewrite equation (6) as

$$\sum_{x\in V} A_{x,y}\sin(\theta_x)\cos(\theta_y) - \cos(\theta_x)\sin(\theta_y) = 0$$

$$\implies \sum_{x\in V} A_{x,y}\sin(\theta_x)\cos(\theta_y) = \sum_{x\in V} A_{x,y}\cos(\theta_x)\sin(\theta_y)$$

$$\implies \cos(\theta_y)\sum_{x\in V} A_{x,y}\sin(\theta_x) = \sin(\theta_y)\sum_{x\in V} A_{x,y}\cos(\theta_x)$$

From the last equation dividing by $\cos(\theta_y)$ we get

$$\sum_{x\in V} A_{x,y}\sin(\theta_x) = \sin(\theta_y)\left(\sum_{x\in V} A_{x,y}\cos(\theta_x)\right) / \cos(\theta_y)$$

We let $r_y = \left(\sum_{x\in V} A_{x,y}\cos(\theta_x)\right) / \cos(\theta_y)$ to obtain $\sum_{x\in V} A_{x,y}\sin(\theta_x) = r_y\sin(\theta_y)$. Similarly by dividing the last equation by $\sin(\theta_y)$ we obtain $\sum_{x\in V} A_{x,y}\cos(\theta_x) = r_y\cos(\theta_y)$. Adding both equations and having sin being the imaginary part we obtain equation (8) which is equivalent to equation (7). □

- The stability condition could be written as

$$\langle f, \nabla^2\mathcal{E}_G(\theta)f\rangle = \frac{1}{2}\sum_{x,y\in V} A_{x,y}\cos(\theta_x - \theta_y)|f(x) - f(y)|^2 \geq 0 \tag{9}$$

Taking $f(x) = exp(i\theta_x)$ we obtain

$$\sum_{x,y\in V} A_{x,y}\cos(\theta_x - \theta_y)(1 - \cos(\theta_x - \theta_y)) \geq 0 \tag{10}$$

- The energy function is always non-negative and the state where all $\theta_x = \theta_y$ for all $x, y \in V$ achieves global minimum. These states are called *synchronized states.* The other local minima are called *spurious local minima.*

Definition 1.1. *A graph is said to be globally synchronising if the energy function $\mathcal{E}_G$ has no spurious local minima.*

Claim 1.0.1. *A graph consisting of two cliques connected by an edge is globally synchronising.*

Proof. First I show that if two graphs H and H' are globally synchronising and share a vertex v, then the graph G obtained by joining H and H' is also globally synchronising. The energy function $\mathcal{E}_G$ would factor as a sum of the energy functions $\mathcal{E}_G = \mathcal{E}_H + \mathcal{E}'_H$. This is because

$$\mathcal{E}_G(\theta) = \frac{1}{2}\sum_{x,y\in(V_H\cup V_{H'})} A_{x,y}(1 - \cos(\theta_x - \theta_y))$$

$$\mathcal{E}_G(\theta) = \left(\frac{1}{2}\sum_{x,y\in V_H} A_{x,y}(1-\cos(\theta_x-\theta_y))\right) + \left(\frac{1}{2}\sum_{x,y\in V_{H'}} A_{x,y}(1-\cos(\theta_x-\theta_y))\right)$$

$$\mathcal{E}_G(\theta) = \mathcal{E}_H + \mathcal{E}'_H$$

Assume $\theta_{x'} - \theta_{y'}$ is constant for all $x', y' \in V_{H'}$, as H is globally synchronising, any initial state can be deformed to a state where $\theta_x = \theta_y$ for all $x, y \in V_H$ without increasing $\mathcal{E}_G$ and without changing $\theta_{x'} - \theta_{y'}$ for all $x', y' \in V_{H'}$. Repeating the process and exchanging the roles of H and H' will synchronize G. Since complete graphs are synchronizing, adding an edge to another vertex will still be synchronizing from argument above. Adding a second clique containing the new vertex will be synchronizing again by the argument above. □

- Trees are globally synchronizing because we can repeatedly use Claim 1.0.1.

Claim 1.0.2. *Cycles of length greater than or equal to five are not globally synchronizing.*

Proof. We can show that cycles of length larger than or equal to five will have at least one a spurious local minimum. Consider a cycle with n nodes. We perform a global rotation on all nodes so that WLOG the first oscillator has phase $\theta_0^* = 0$. One equilibrium of the energy function $\mathcal{E}_G$ would be the roots of unity $\theta_i^* = \frac{2i\pi}{n}$ for $i = 0, 1, ..., n$. This certainly satisfies $(\nabla\mathcal{E}_G(\theta^*))_x = 0$. We also see that θ^* satisfies the stability condition given by equation (10). Thus θ^* is a spurious local minimum. □

- For the following Theorem 1.1 and proof, I use the following notation:
  - Zero equilibrium refers to an equilibrium where all phases are equal to each other.
  - I represent each θ_j as $z_j = e^{i\theta_j}$.
  - Let $\Delta_{ij} = \theta_i - \theta_j$.
  - The stability condition for the equilibrium point θ^* is:
    For any non-empty subset S,

$$\sum_{(i,j)\in(S,S^c)} A_{ij} cos(\Delta_{ji}^*) \geq 0. \tag{11}$$

Theorem 1.1. *Any complete graph has no non-zero stable equilibrium.*

Proof. Let θ^* be any non-zero stable equilibrium point solution. Let

$$p = \sum_{j=1}^{n} z_j = \sum_{j=1}^{n} \cos(\theta_j^*) + i\sum_{j=1}^{n} \sin(\theta_j^*).$$

Then for any m we have

$$\overline{z_m}p = \overline{z_m}\sum_{j=1}^{n} z_j = \sum_{j=1}^{n} \overline{z_m}z_j = 1 + \sum_{j\neq m} \overline{z_m}z_j = 1 + \sum_{j\neq m} \cos(\Delta_{jm}^*) + i\sum_{j\neq m} \sin(\Delta_{jm}^*) = 1 + \sum_{j\neq m} \cos(\Delta_{jm}^*) \tag{12}$$

Where in the last equality the imaginary part disappears because of the equilibrium condition of the complete graph. Now we have the following cases for p.
Case 1: $p = 0$
From equation (12) we get $\sum_{j\neq m} \cos(\Delta_{jm}^*) = -1$ which is unstable by the stability condition (11). **Case 2:** $p \neq 0$
For some angle θ^p

$$p = \|p\|(\cos(\theta^p) + i\sin(\theta^p))$$

$$\implies \overline{z_m}p = \|p\|(\cos(\theta^p - \theta_m^*) + i\sin(\theta^p - \theta_m^*)) \tag{13}$$

When we compare equation (1.1) with equation (13), then we know that the imaginary part in equation (13) must be zero. Therefore we know that for any m in the vertex set we have that $\theta_m^* = \theta^p$ or $\theta_m^* = \theta^p + \pi$. Split the vertices into two sets.

$$S = \{x \in V \mid \theta_x^* = \theta^p\}$$

$$S^C = \{y \in V \mid \theta_y^* = \theta^p\}$$

We know that both sets are non-empty otherwise we θ^* would be a zero stable equilibrium. Then we get that

$$\sum_{(i,j)\in(S,S^C)} A_{ij}\cos(\Delta_{ji}^*) = -|S||S^C| \leq -1$$

By the stability condition this is unstable.
We proved for all possible cases that there is no non-zero stable equilibrium. □

- **Notation:**

$$\mathbf{d} \text{ is the expected average degree}$$
$$\Delta_A := A - (\mathbf{d}/n)J$$
$$\Delta_D := D - \mathbf{d}I$$
$$\Delta_L := \Delta_D - \Delta_A = L - \mathbf{d}I + (\mathbf{d}/n)J$$
$$\|H\| = \sup_{f\neq 0}\|Hf\|/\|f\|$$

The notation $A \preceq B$ means that $B - A$ is a positive semi definite matrix.

Definition 1.2. *We say that a graph G is an $(n, \boldsymbol{d}, \boldsymbol{\alpha})$-expander if it has n vertices and*

$$\|\Delta_A\| := \|A - (\boldsymbol{d}/n)J\| \leq \boldsymbol{\alpha}\boldsymbol{d}.$$

If in addition the graph G is $\boldsymbol{d}$-regular then the condition above is equivalent to $\max_{i\neq 1}|\lambda_i(A)| \leq \boldsymbol{\alpha}\boldsymbol{d}$.

Proof. Let v_1 be the normalized all ones vector, then

$$\|(A - (\mathbf{d}/n)J)v_1\| = \|dv_1 - dv_1\| = 0.$$

Let v_j for $2 \leq j \leq n$ be normalized eigenvectors corresponding to $\lambda_j \neq d$ which means they are orthogonal to all ones vector. Therefore, we have

$$\|(A - (\mathbf{d}/n)J)v_j\| = \|\lambda_j v_j\| = |\lambda_j|.$$

Now let $u = \sum_{i=1}^n a_i v_i$ and let $\|u\|^2 = \sum_{i=1}^n a_i^2 = 1$, and let $\lambda = \max_{i=2}^n(|\lambda_i|)$, then we have

$$\|(A - (\mathbf{d}/n)J)u\|^2 = \left\|\sum_{i=2}^n \lambda_i a_i v_i\right\|^2 = \sum_{i=2}^n \lambda_i^2 a_i^2 \leq \lambda^2 \sum_{i=2}^n a_i^2 \leq \lambda^2.$$

Taking the square root we get that $\|(A - (\mathbf{d}/n)J)u\| \leq \lambda$. Therefore, if G is $\mathbf{d}$-regular then Definition 1.2 is equivalent to $\max_{i\neq 1}|\lambda_i(A)| \leq \boldsymbol{\alpha}\mathbf{d}$. ☐

Theorem 1.2. *All $\boldsymbol{d}$-regular $(n, \boldsymbol{d}, \boldsymbol{\alpha})$-expander graphs with $\boldsymbol{\alpha} \leq 0.0816$ are globally synchronising.*

- Ramanujan graphs have $\boldsymbol{\alpha} = 2(\sqrt{d-1})/d$.

Theorem 1.3. *Any Ramanujan graph is globally synchronising as long as $d \geq 600$. Moreover, a random d-regular graph is globally synchronising with high probability in the same range of d.*

Proof. Plug $\boldsymbol{\alpha} = 2(\sqrt{d-1})/d$ into 1.2. ☐

Definition 1.3. (Spectral expander graph). *A graph G is an $(n, \boldsymbol{d}, \boldsymbol{\alpha}, \boldsymbol{c}^-, \boldsymbol{c}^+)$-expander if it is an $(n, \boldsymbol{d}, \boldsymbol{\alpha})$-expander and it holds that*

$$\boldsymbol{c}^- \boldsymbol{d}I \preceq \Delta_L \preceq \boldsymbol{c}^+ \boldsymbol{d}I.$$

Theorem 1.4. *If G is an $(n, \boldsymbol{d}, \boldsymbol{\alpha}, \boldsymbol{c}^-, \boldsymbol{c}^+)$-expander graph with $\boldsymbol{c}^- > -1$, $\boldsymbol{\alpha} \leq 1/5$ and*

$$\max\left\{\frac{64\boldsymbol{\alpha}(1 + 2\boldsymbol{c}^+ - \boldsymbol{c}^-)}{(1 + \boldsymbol{c}^-)^2}, \frac{64\boldsymbol{\alpha}(1 + \boldsymbol{c}^+)\log(\frac{1+\boldsymbol{c}^+ + \boldsymbol{\alpha}}{2\boldsymbol{\alpha}}}{(1 + \boldsymbol{c}^-)(1 + 5\boldsymbol{c}^+ - 4\boldsymbol{c}^-}\right\} < 1$$

- Note: Theorem 1.2 has a higher upper bound on $\boldsymbol{\alpha}$ for d-regular graphs than Theorem 1.4.