Single Source Shortest Path Algorithms

Introduction

- Consider the problem of finding shortest paths between all pairs of vertices in a graph
- Problem might arise in making a table of distances between all pairs of cities for a road atlas
- given a weighted, directed graph $G = (V, E)$ with a weight function $w: E \rightarrow R$ that maps edges to real-valued weights

Introduction

- We wish to find, for every pair of vertices u, v ∈ V , a shortest (least–weight) path from u to v, where the weight of a path is the sum of the weights of its constituent edges
- We typically want the output in tabular form:
- the entry in u's row and v's column should be the weight of a shortest path from u to v

Introduction

- We can solve an all–pairs shortest–paths problem by running a single–source shortest–paths algorithm |V| times, once for each vertex as the source.
- If all edge weights are nonnegative, we can use Dijkstra's algorithm

Introduction

- If the graph has negative–weight edges, we cannot use Dijkstra’s algorithm
- Instead, we must run the slower Bellman–Ford algorithm once from each vertex
- The resulting running time is $O(V^2E)$, which on a dense graph is $O(V^4)$

Dijkstra's algorithm

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

Dijkstra's algorithm

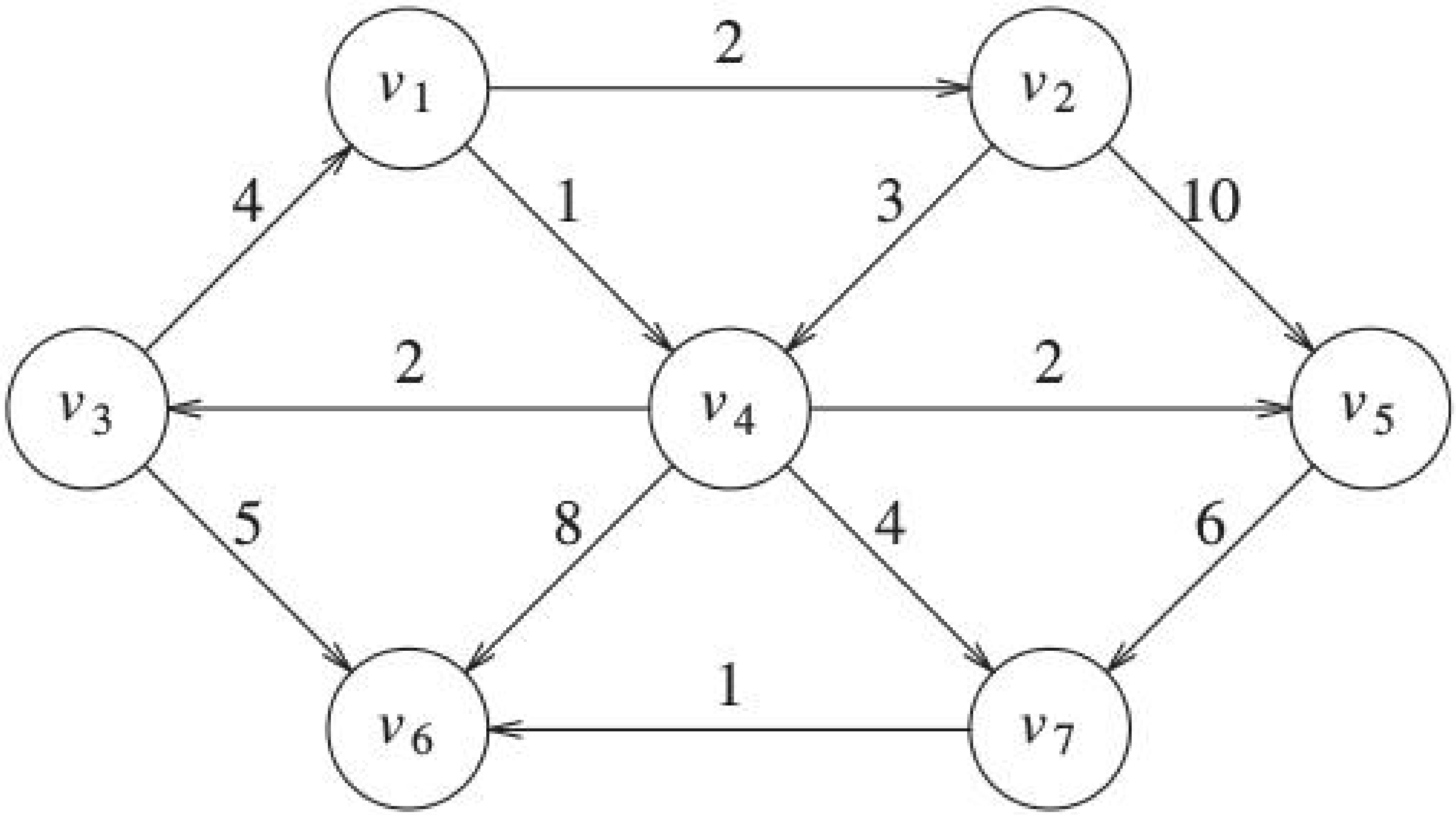


Figure 9.20 The directed graph G (again)

Dijkstra's algorithm

v	*known*	d_v	p_v
v_1	F	0	0
v_2	F	∞	0
v_3	F	∞	0
v_4	F	∞	0
v_5	F	∞	0
v_6	F	∞	0
v_7	F	∞	0

Figure 9.21 Initial configuration of table used in Dijkstra's algorithm

Dijkstra's algorithm

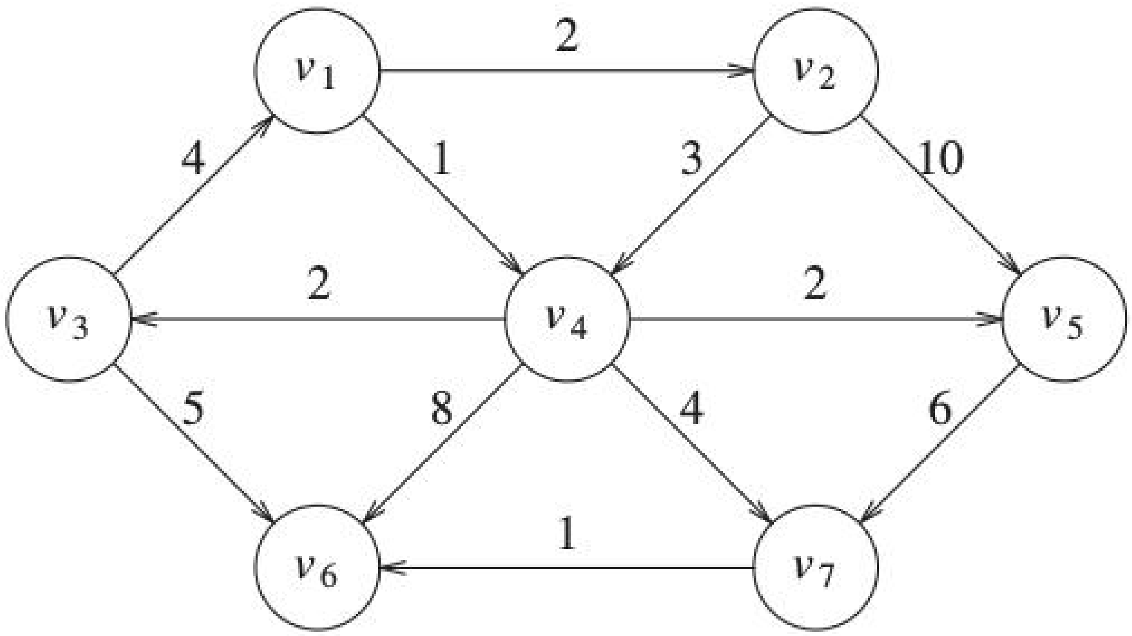


Figure 9.20 The directed graph G (again)

v	*known*	d_v	p_v
v_1	T	0	0
v_2	F	2	v_1
v_3	F	∞	0
v_4	F	1	v_1
v_5	F	∞	0
v_6	F	∞	0
v_7	F	∞	0

Figure 9.22 After v_1 is declared *known*

Dijkstra's algorithm

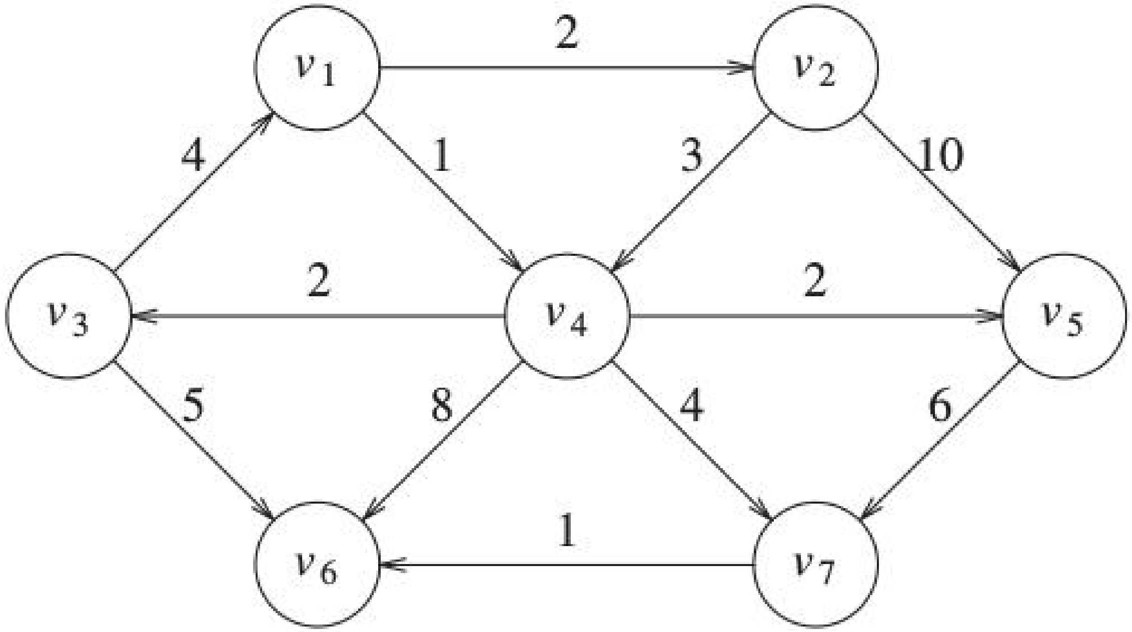


Figure 9.20 The directed graph G (again)

v	*known*	d_v	p_v
v_1	T	0	0
v_2	F	2	v_1
v_3	F	3	v_4
v_4	T	1	v_1
v_5	F	3	v_4
v_6	F	9	v_4
v_7	F	5	v_4

Figure 9.23 After v_4 is declared *known*

Dijkstra's algorithm

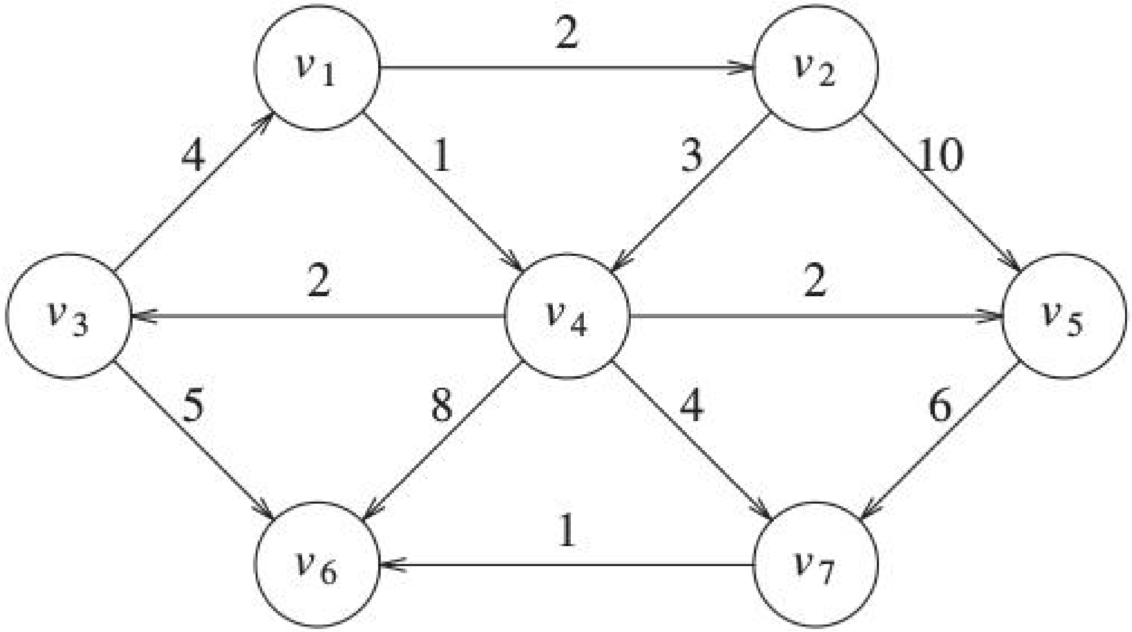


Figure 9.20 The directed graph G (again)

v	*known*	d_v	p_v
v_1	T	0	0
v_2	T	2	v_1
v_3	F	3	v_4
v_4	T	1	v_1
v_5	F	3	v_4
v_6	F	9	v_4
v_7	F	5	v_4

Figure 9.24 After v_2 is declared *known*

Dijkstra's algorithm

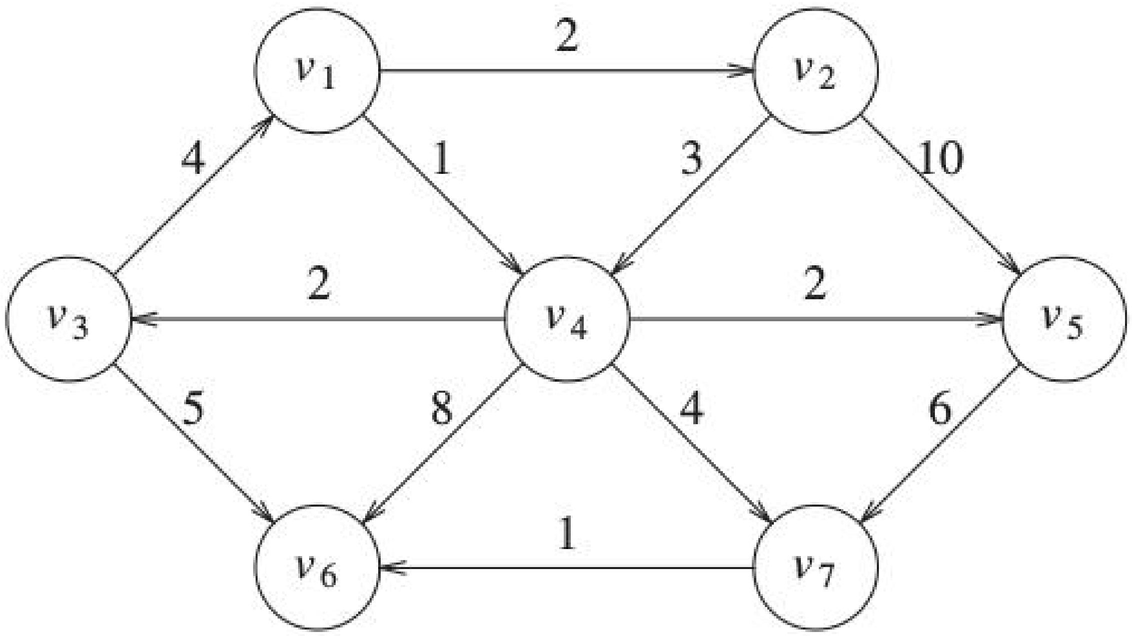


Figure 9.20 The directed graph G (again)

v	*known*	d_v	p_v
v_1	T	0	0
v_2	T	2	v_1
v_3	T	3	v_4
v_4	T	1	v_1
v_5	T	3	v_4
v_6	F	8	v_3
v_7	F	5	v_4

Figure 9.25 After v_5 and then v_3 are declared *known*

Dijkstra’s algorithm

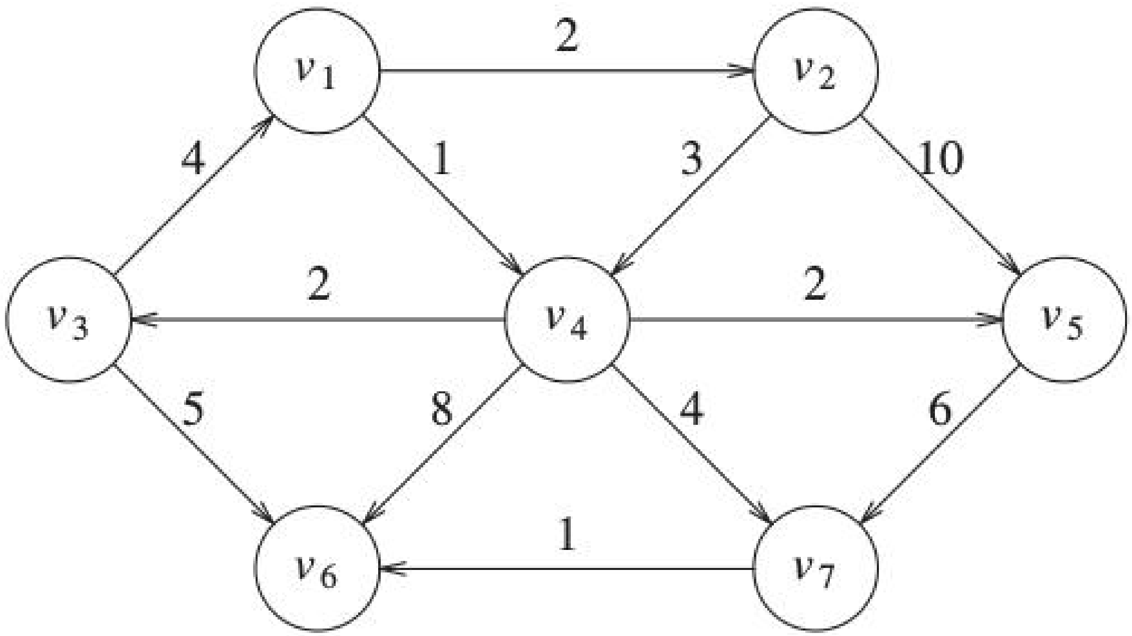


Figure 9.20 The directed graph G (again)

v	*known*	d_v	p_v
v_1	T	0	0
v_2	T	2	v_1
v_3	T	3	v_4
v_4	T	1	v_1
v_5	T	3	v_4
v_6	F	6	v_7
v_7	T	5	v_4

Figure 9.26 After v_7 is declared *known*

Dijkstra's algorithm

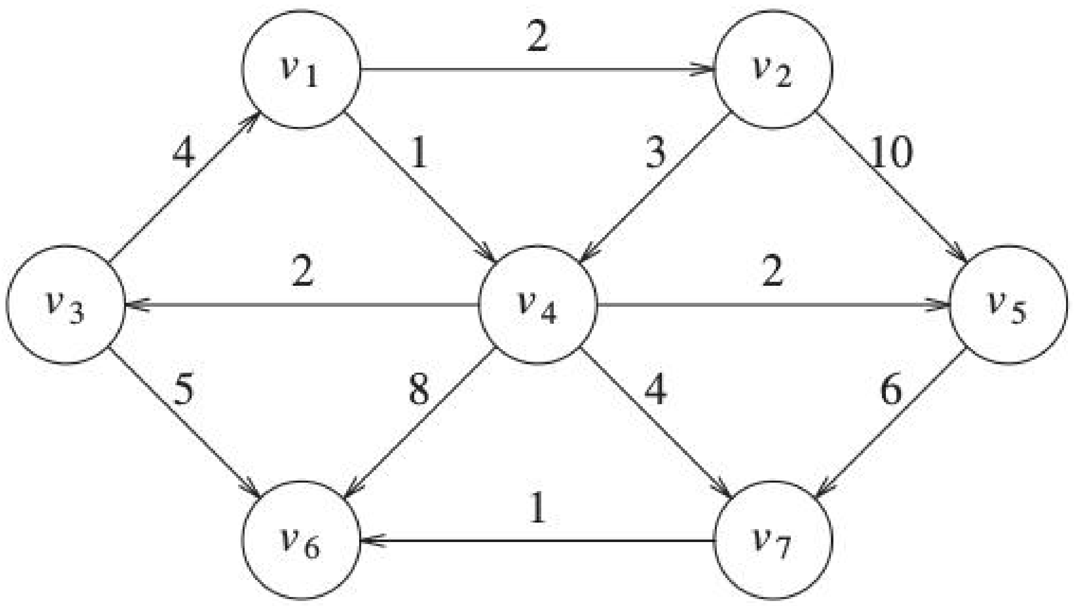
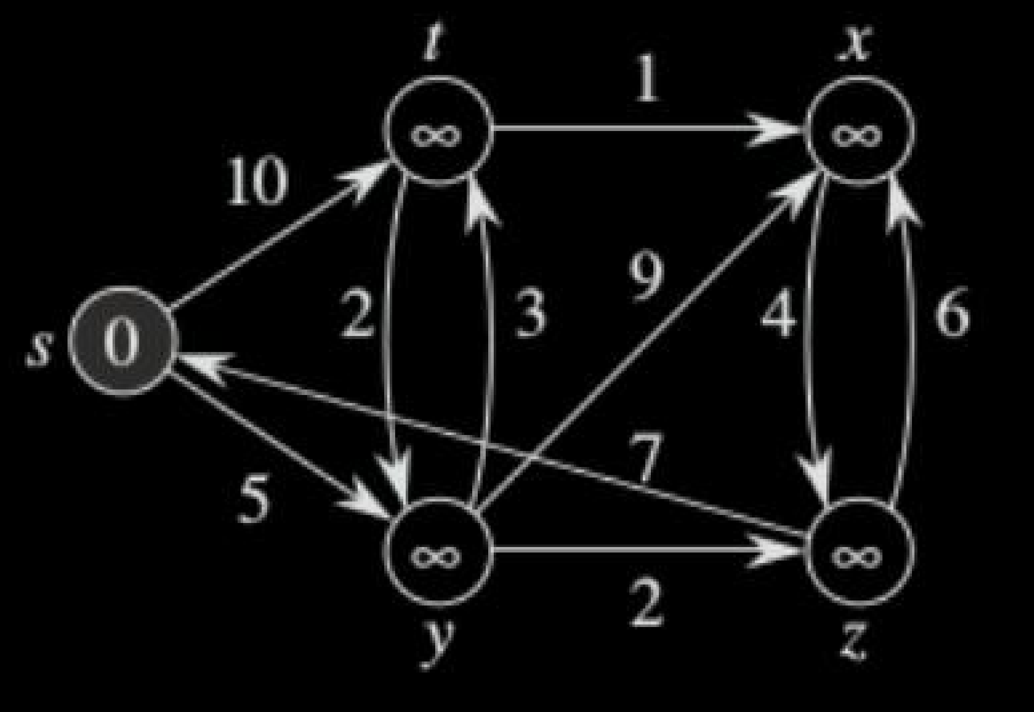


Figure 9.20 The directed graph G (again)

v	*known*	d_v	p_v
v_1	T	0	0
v_2	T	2	v_1
v_3	T	3	v_4
v_4	T	1	v_1
v_5	T	3	v_4
v_6	T	6	v_7
v_7	T	5	v_4

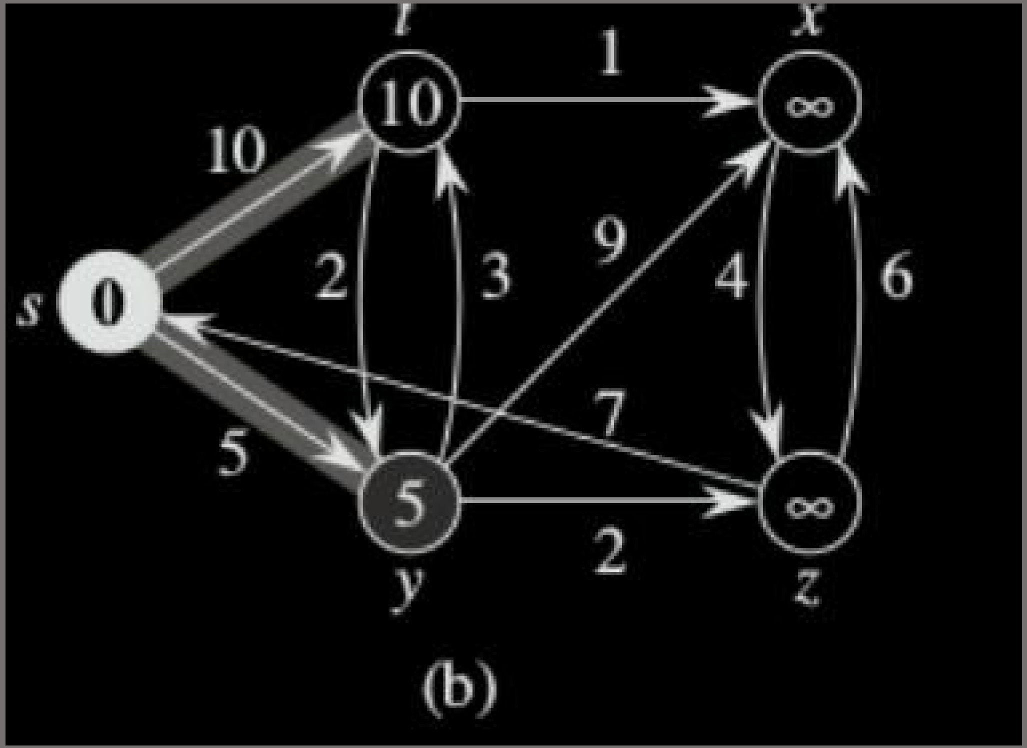
Figure 9.27 After v_6 is declared *known* and algorithm terminates

Dijkstra's algorithm

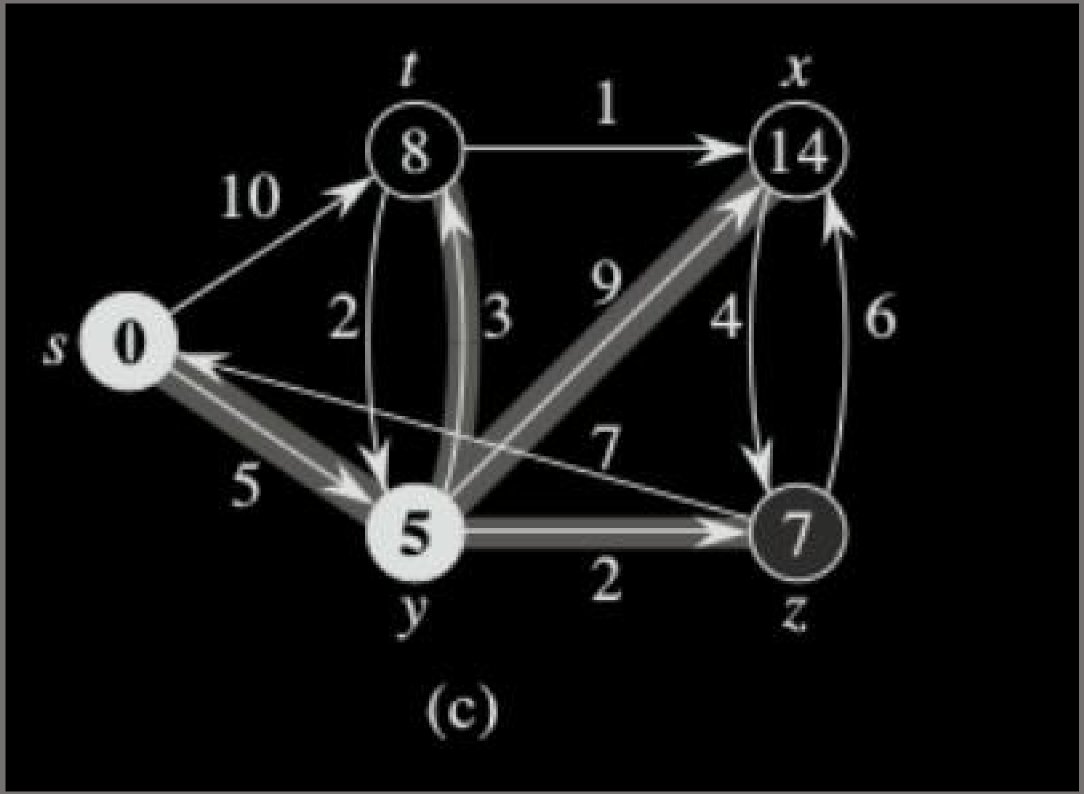


(a)

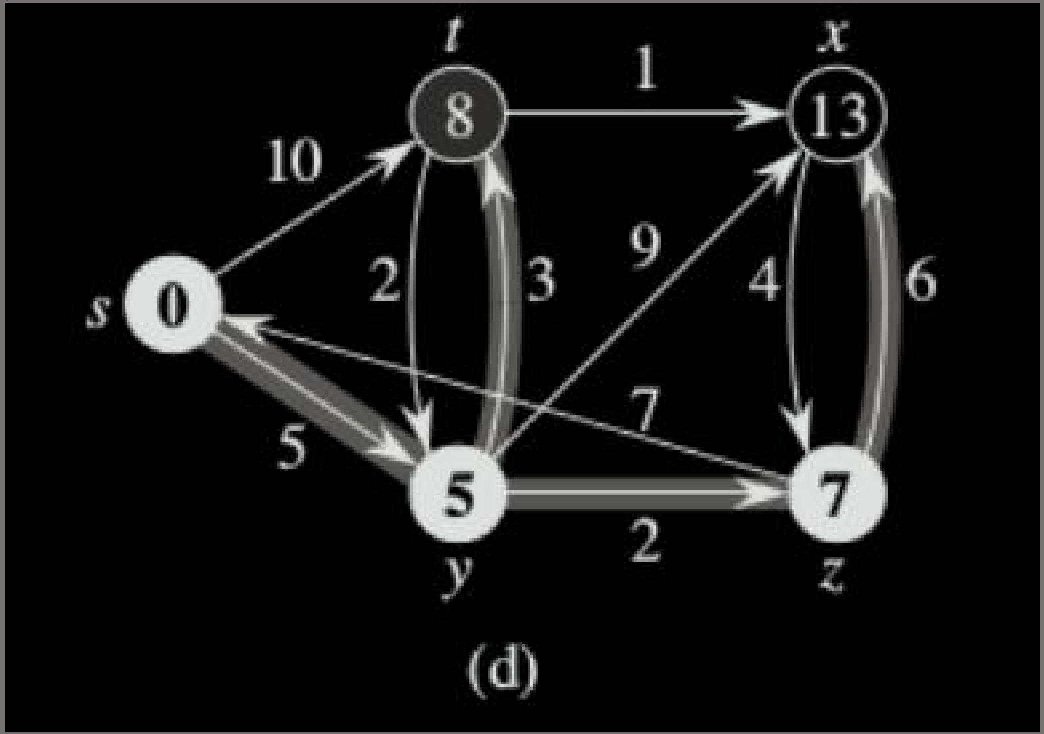
Dijkstra’s algorithm



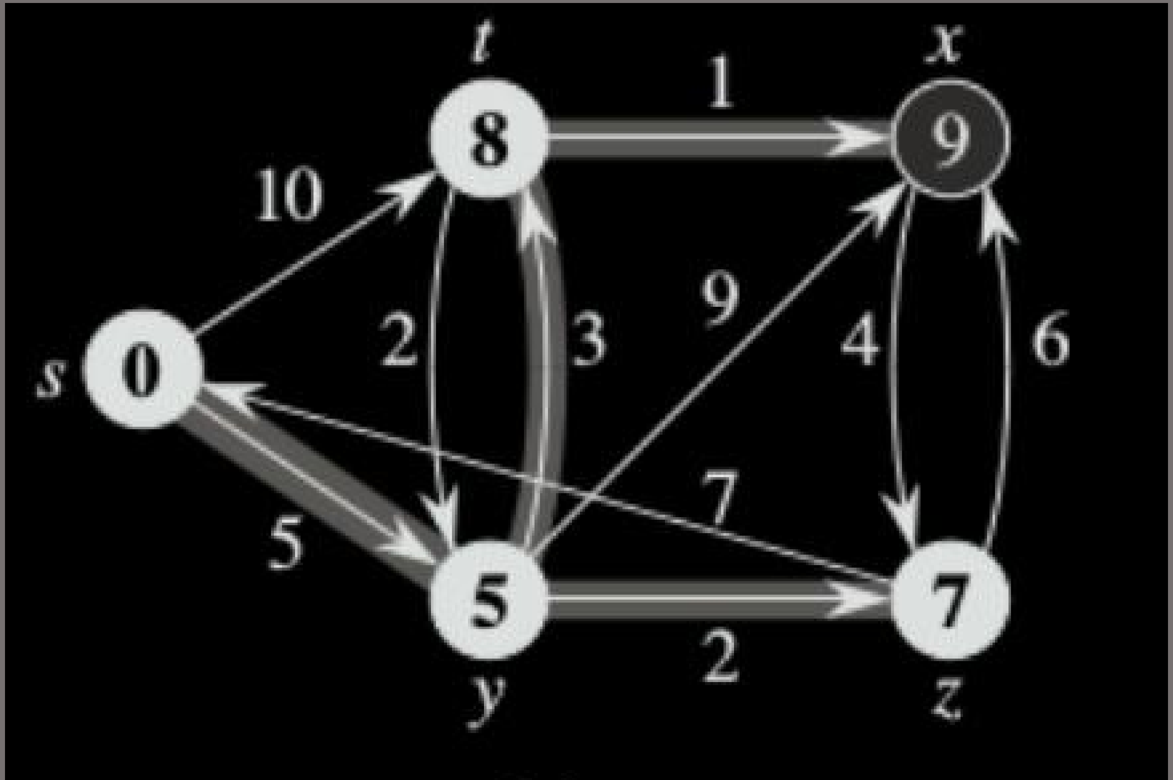
Dijkstra's algorithm



Dijkstra's algorithm

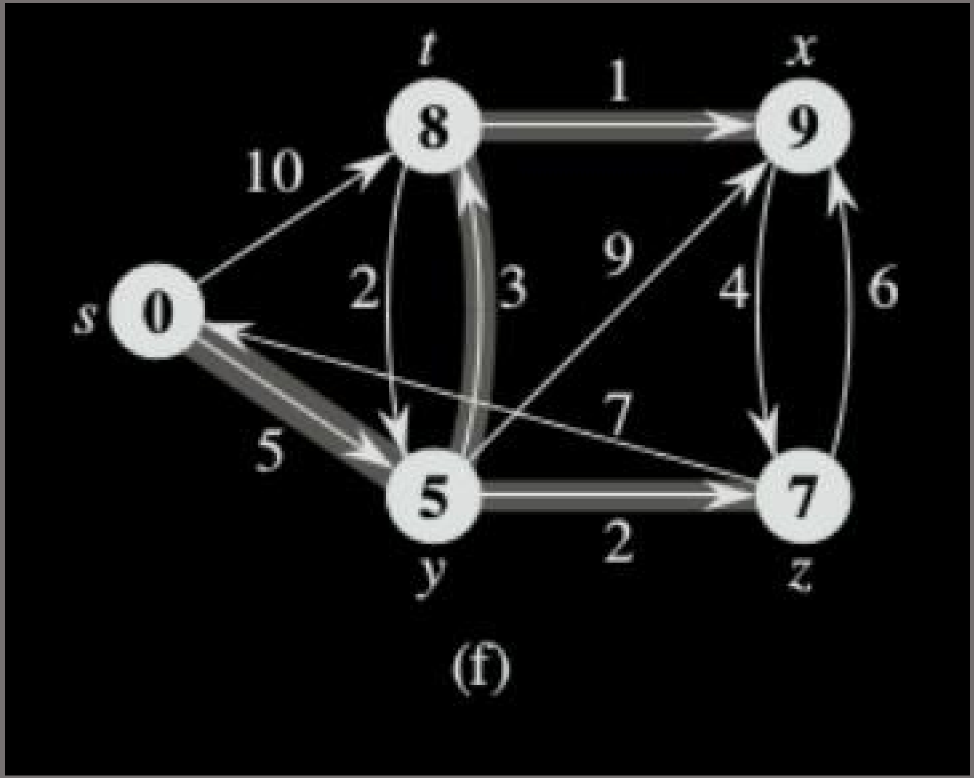


Dijkstra's algorithm



(e)

Dijkstra’s algorithm



Analysis

- Because each vertex $u \in V$ is added to set S exactly once, each edge in the adjacency list Adj[u] is examined in the for loop of lines 7 – 8 exactly once during the course of the algorithm.
- Since the total number of edges in all the adjacency lists is $|E|$, this for loop iterates a total of $|E|$ times, and thus the algorithm calls DECREASE–K EY at most $|E|$ times overall

Analysis

- running time of Dijkstra's algorithm depends on how we implement the min–priority queue
- Case I – simply store v.d in the v^{th} entry of an array
- Each INSERT and DECREASE–K EY operation takes O(1) time, and each E XTRACT–M IN operation takes O(V) time
- total time of $O(V^2 + E) = O(V^2)$

Analysis

- Case II – we can improve the algorithm by implementing the min–priority queue with a binary min–heap
- Each EXTRACT–MIN operation then takes time O(lg V)
- time to build the binary min–heap is O(V)
- Each DECREASE –KEY operation takes time O(lg V), and there are still at most |E| such operations
- total running time is therefore O(V + E) lg V) which is O.E lg V / if all vertices are reachable from the source

Analysis

- Case III – achieve a running time of O(V lg V + E) by implementing the min–priority queue with a Fibonacci heap
- The amortized cost of each of the |V| EXTRACT –M IN operations is O(lg V), and each DECREASE –K EY call, of which there are at most |E|, takes only O(1) amortized time

Dijkstra’s algorithm

- We allow negative–weight edges, but we assume for the time being that the input graph contains no negative–weight cycles.
- Time Complexity of Dijkstra's Algorithm is $O(V^2)$ but with min–priority queue it drops down to $O(V + E \log V)$

Bellman–Ford algorithm

- Like other Dynamic Programming Problems, the algorithm calculates shortest paths in a bottom–up manner
- It first calculates the shortest distances which have at–most one edge in the path.
- Then, it calculates the shortest paths with at–most 2 edges, and so on

Bellman–Ford algorithm

- After the i–th iteration, the shortest paths with at most i edges are calculated
- There can be maximum |V| – 1 edges in any simple path, that is why the outer loop runs |v| – 1 times

Bellman–Ford algorithm

- Solves the single–source shortest–paths problem in which edge weights may be negative
- Given a weighted, directed graph $G = (V, E)$ with source s and weight function $w : E \rightarrow R$, the Bellman–Ford algorithm returns a boolean value indicating whether or not there is a negative–weight cycle that is reachable from the source.
- If there is such a cycle, the algorithm indicates that no solution exists.

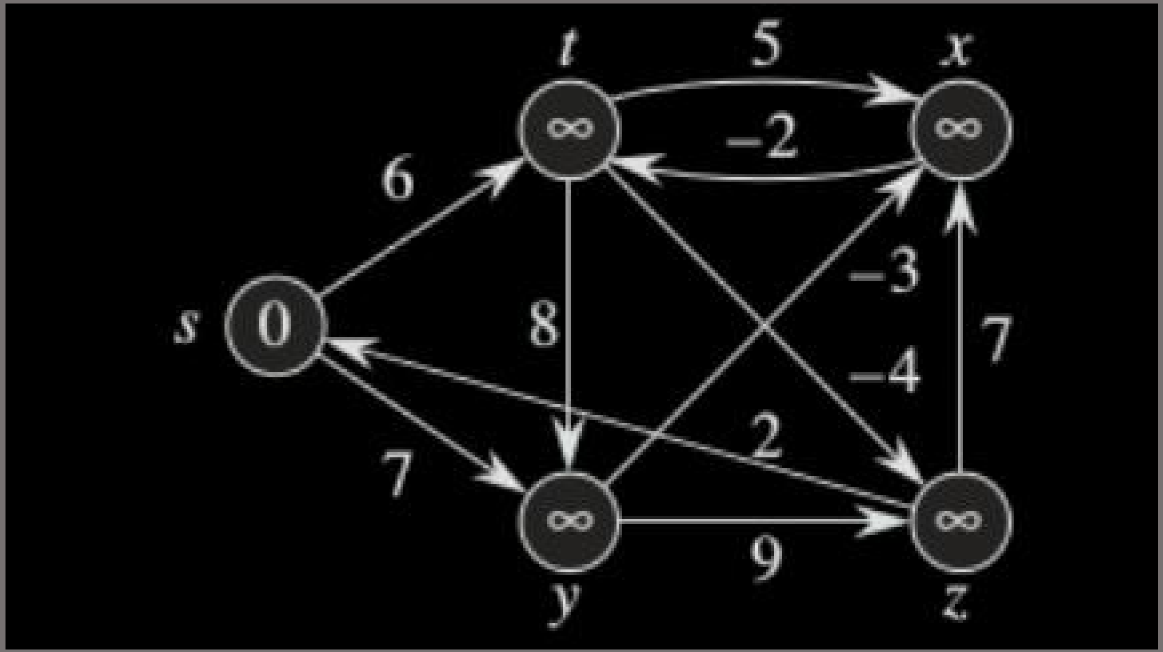
Bellman–Ford algorithm

- No such cycle, the algorithm produces shortest paths and their weights
- The algorithm relaxes edges, progressively decreasing an estimate $v.d$ on the weight of a shortest path from the source s to each vertex $v \in V$ until it achieves the actual shortest–path weight $\delta(s,v)$
- The algorithm returns TRUE if and only if the graph contains no negative–weight cycles that are reachable from the source.

Bellman–Ford algorithm

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| − 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

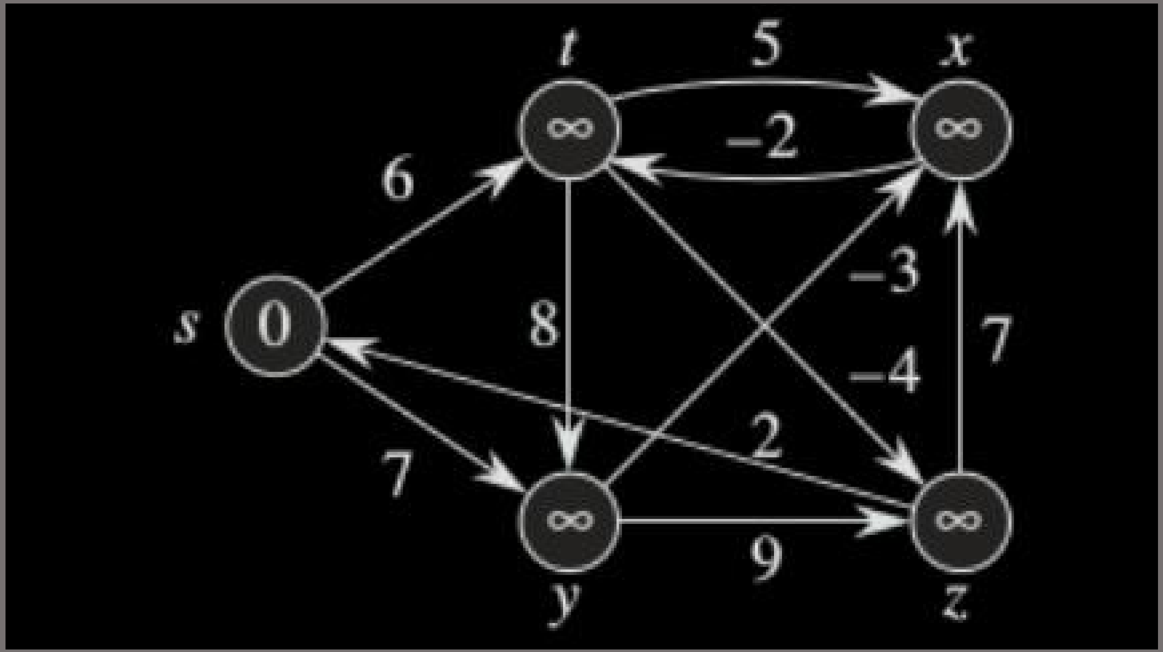
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	
x	∞	∞
y	∞	
z	∞	

(t, x)									

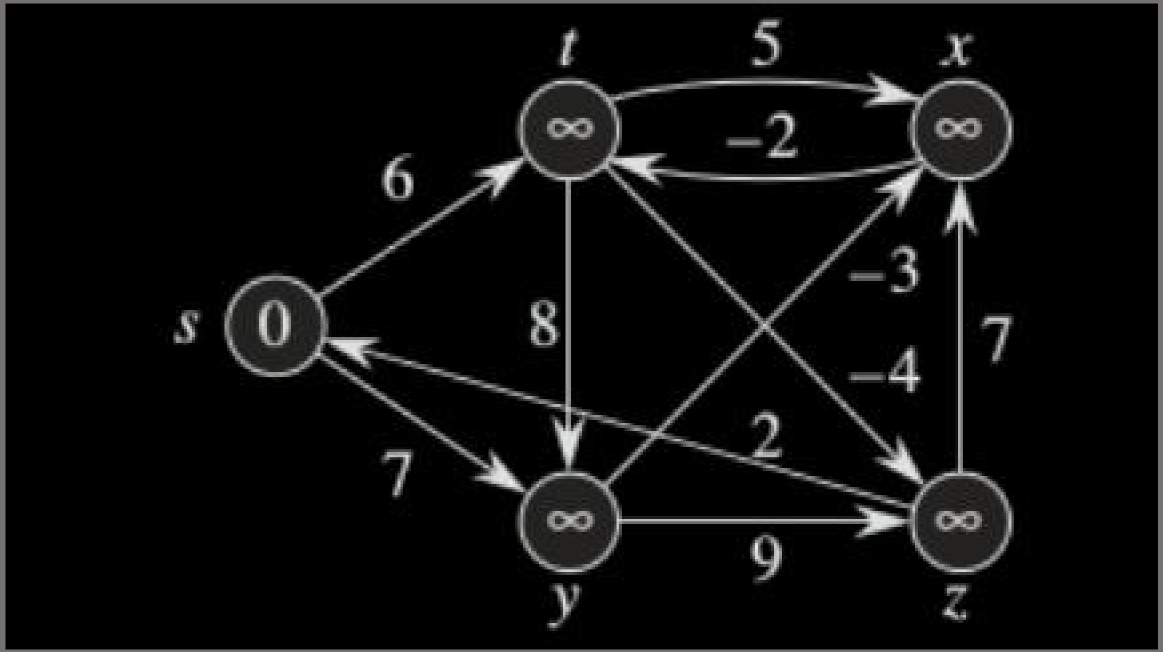
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	
x	∞	∞
y	∞	∞
z	∞	

(t, x)	(t, y)								

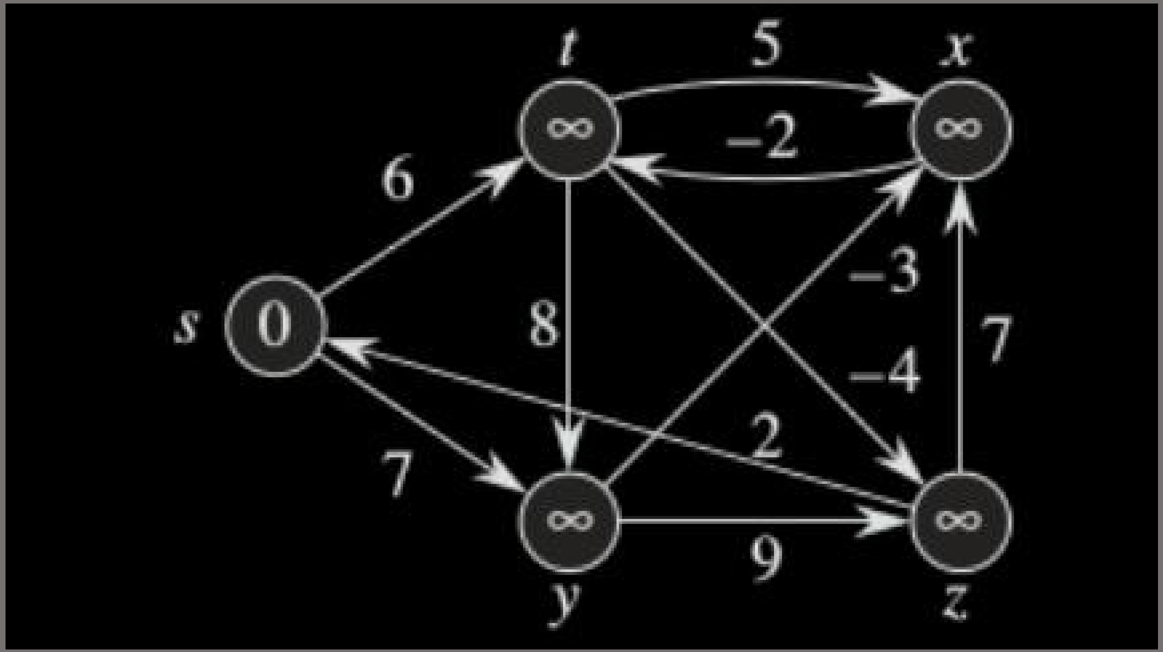
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	
x	∞	∞
y	∞	∞
z	∞	∞

(t, x)	(t, y)	(t, z)							

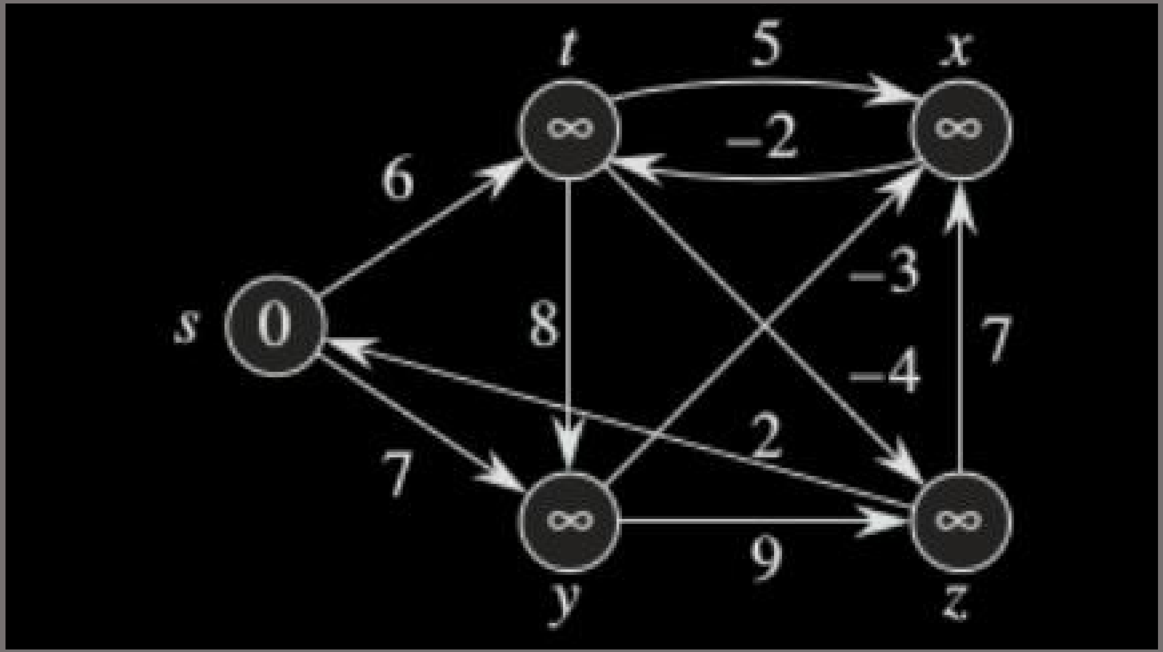
Bellman–Ford algorithm



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	∞
x	∞	∞
y	∞	∞
z	∞	∞

(t, x)	(t, y)	(t, z)	(x, t)						

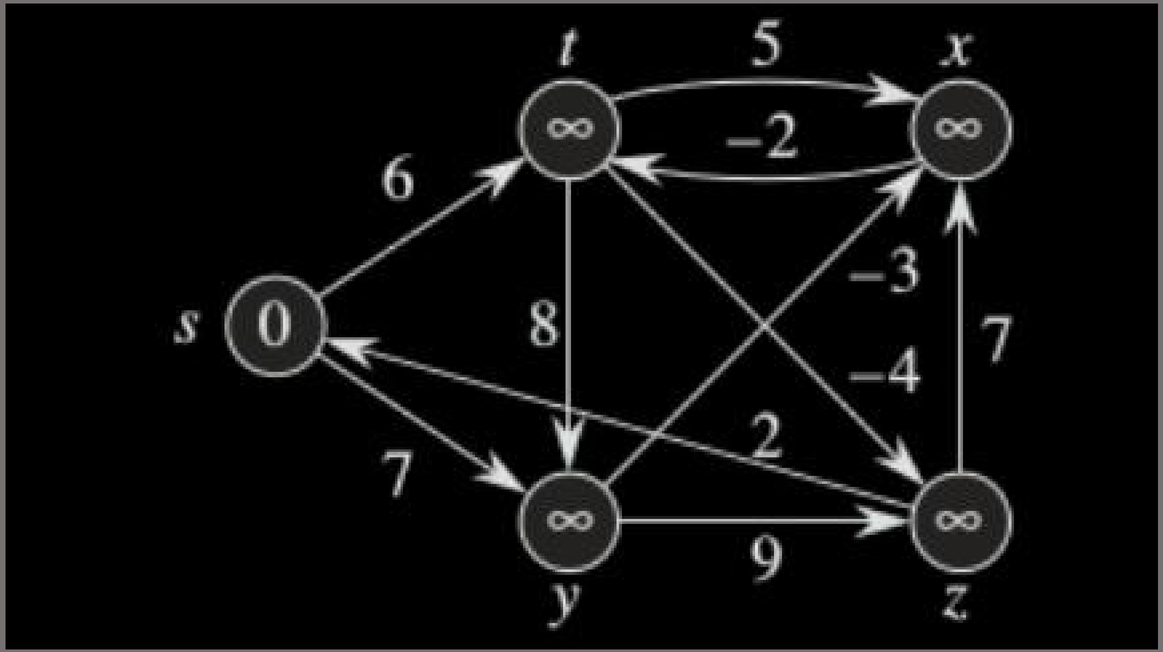
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	∞
x	∞	∞
y	∞	∞
z	∞	∞

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)					

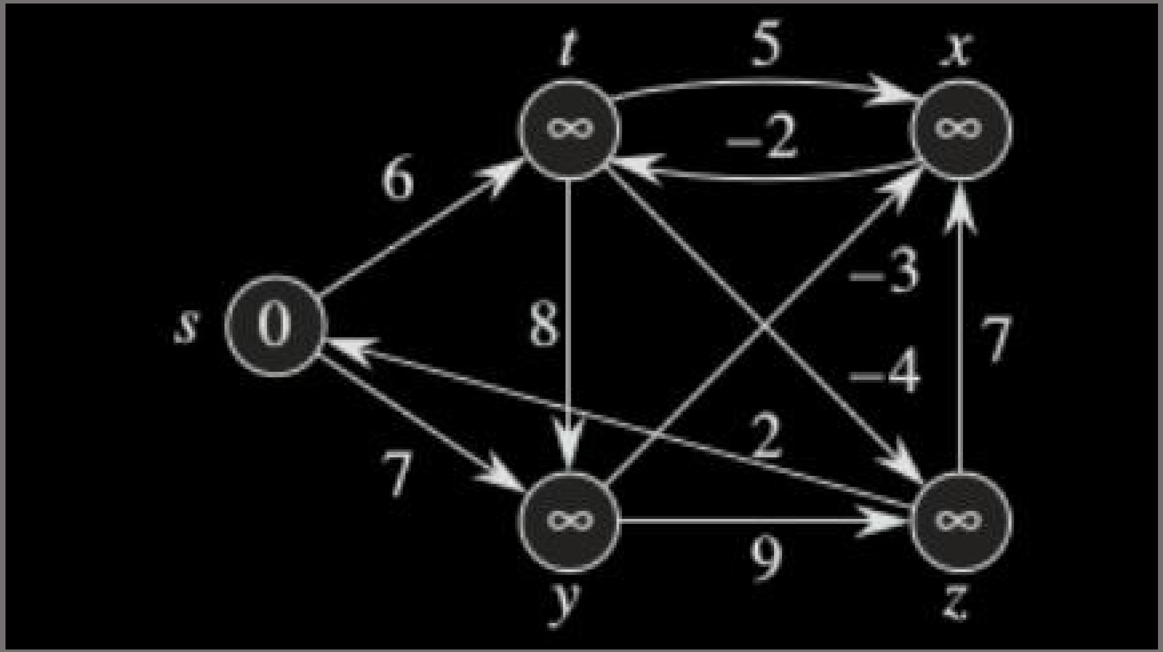
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	∞
x	∞	∞
y	∞	∞
z	∞	∞

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)				

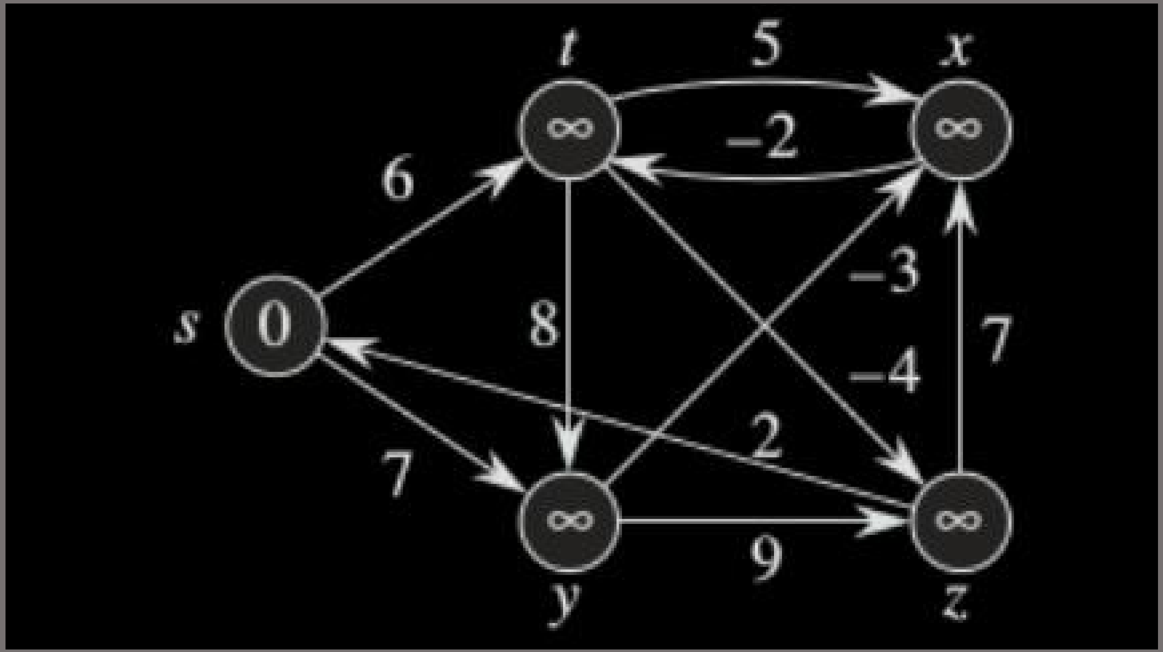
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	∞
x	∞	∞
y	∞	∞
z	∞	∞

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)			

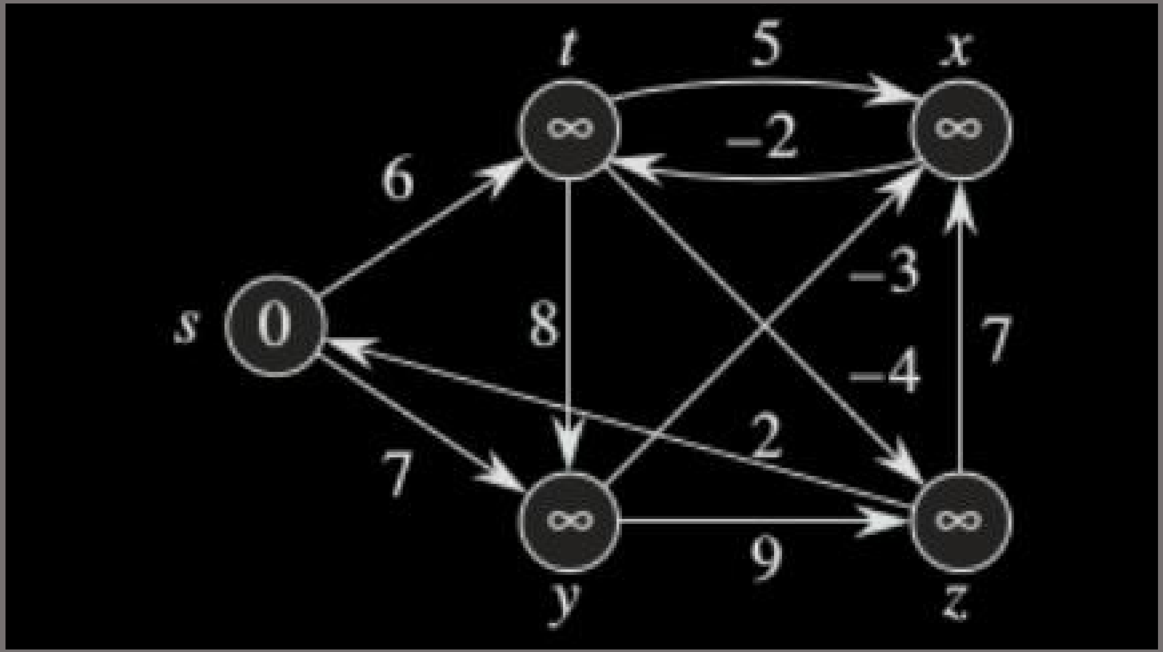
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	∞
x	∞	∞
y	∞	∞
z	∞	∞

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)		

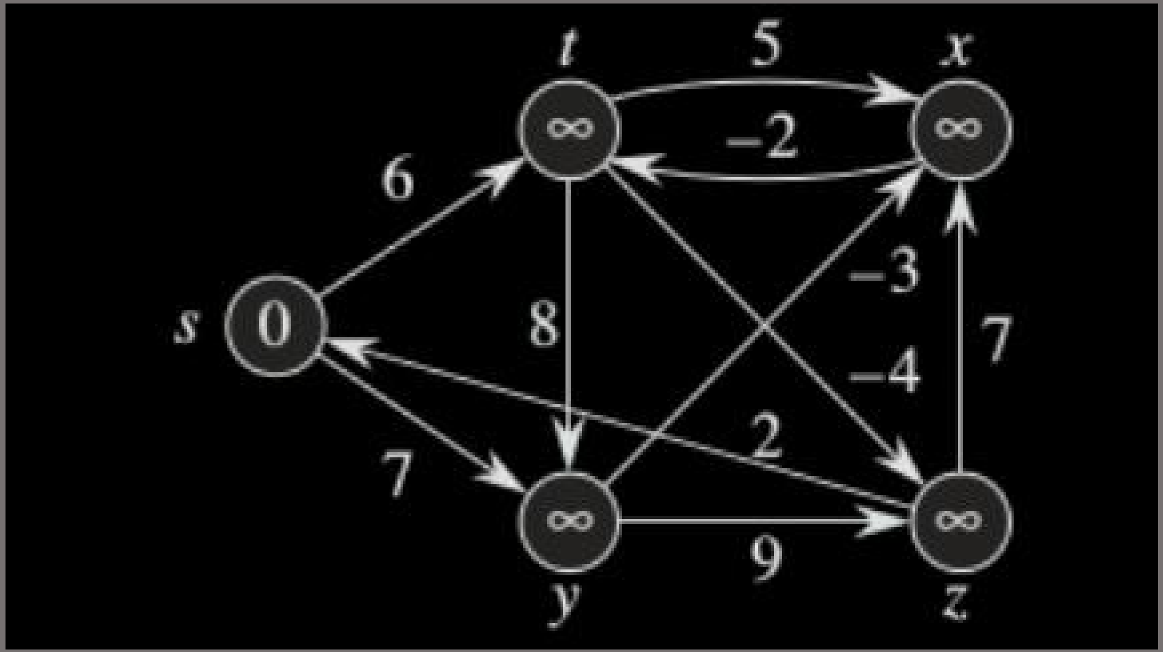
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	6
x	∞	∞
y	∞	∞
z	∞	∞

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	

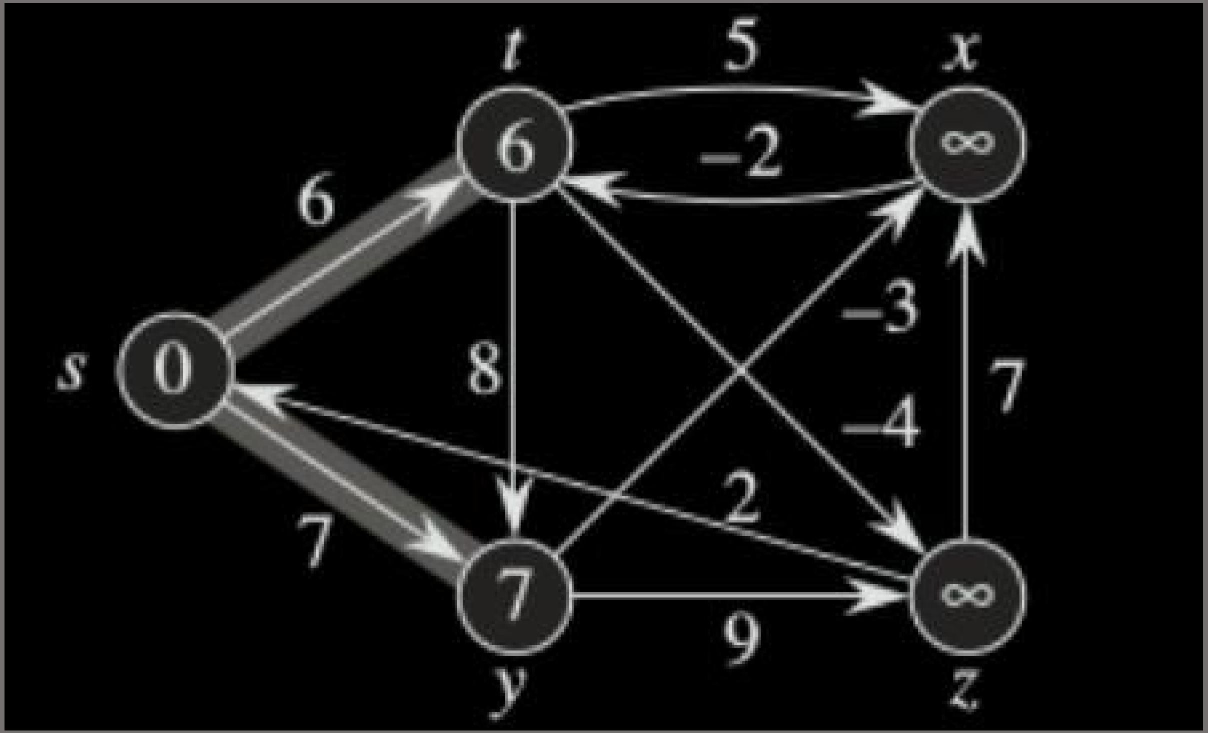
Bellman–Ford algorithm – Iteration 1



Edge Name	Old Cost	Updated Cost
s	0	0
t	∞	6
x	∞	∞
y	∞	7
z	∞	∞

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	(s,y)

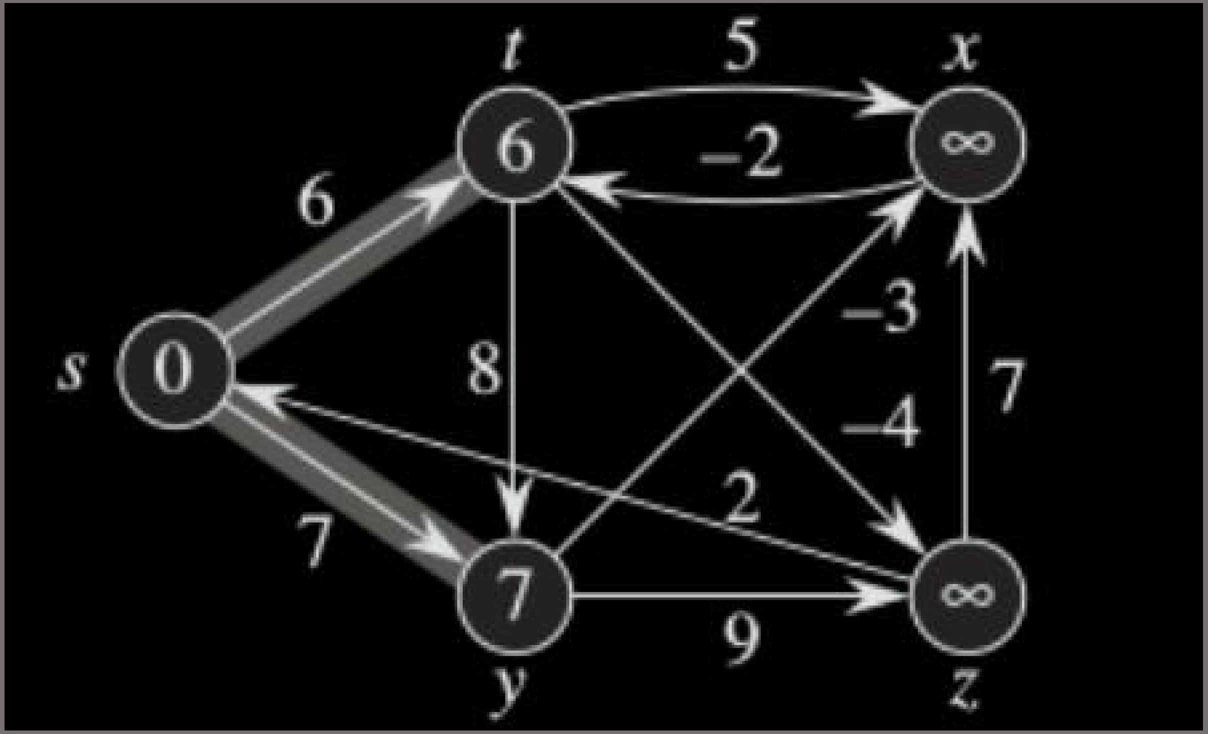
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	
x	∞	11
y	7	
z	∞	

(t, x)									

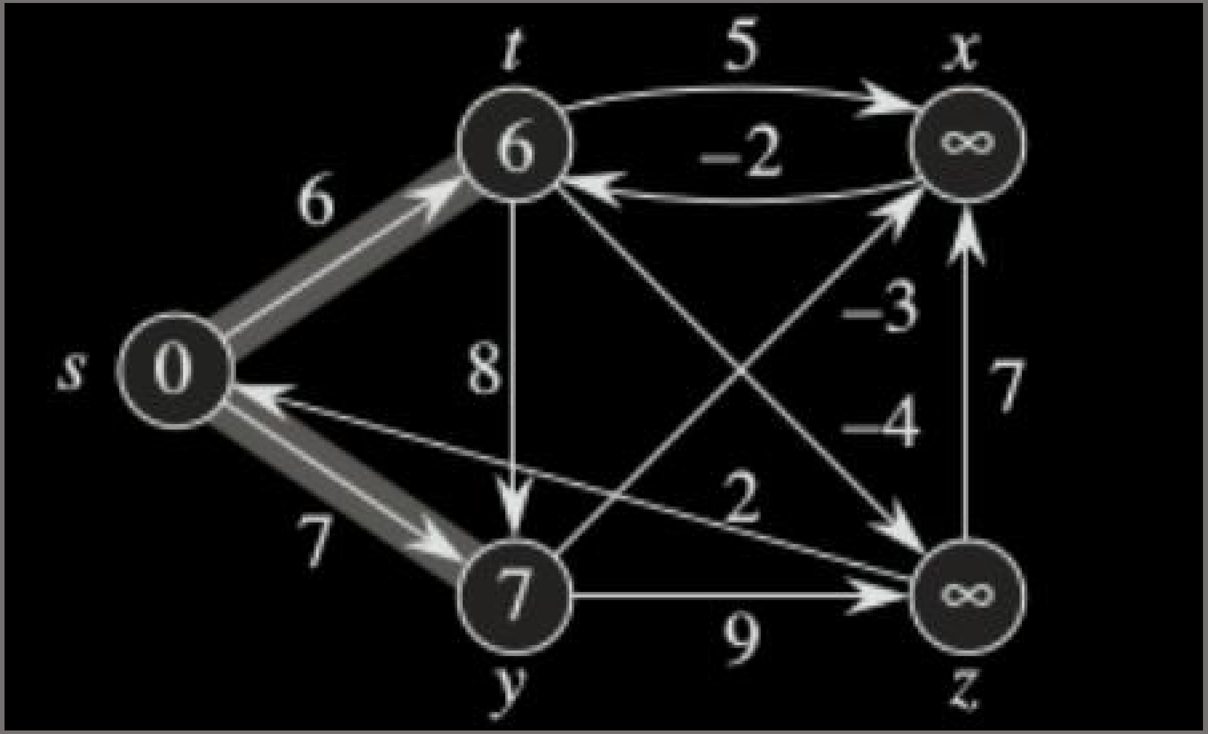
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	
x	∞	11
y	7	~~14~~ 7
z	∞	

(t, x)	(t, y)								

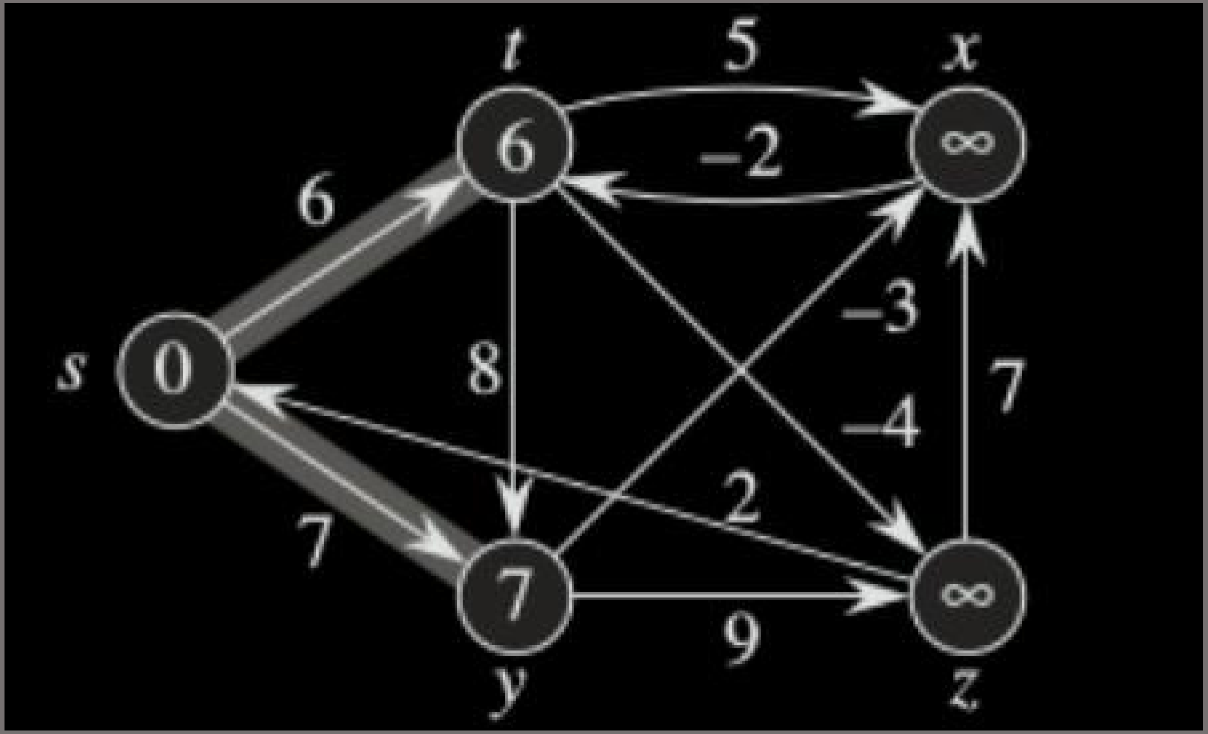
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	
x	∞	11
y	7	~~14~~ 7
z	∞	2

(t, x)	(t, y)	(t, z)							

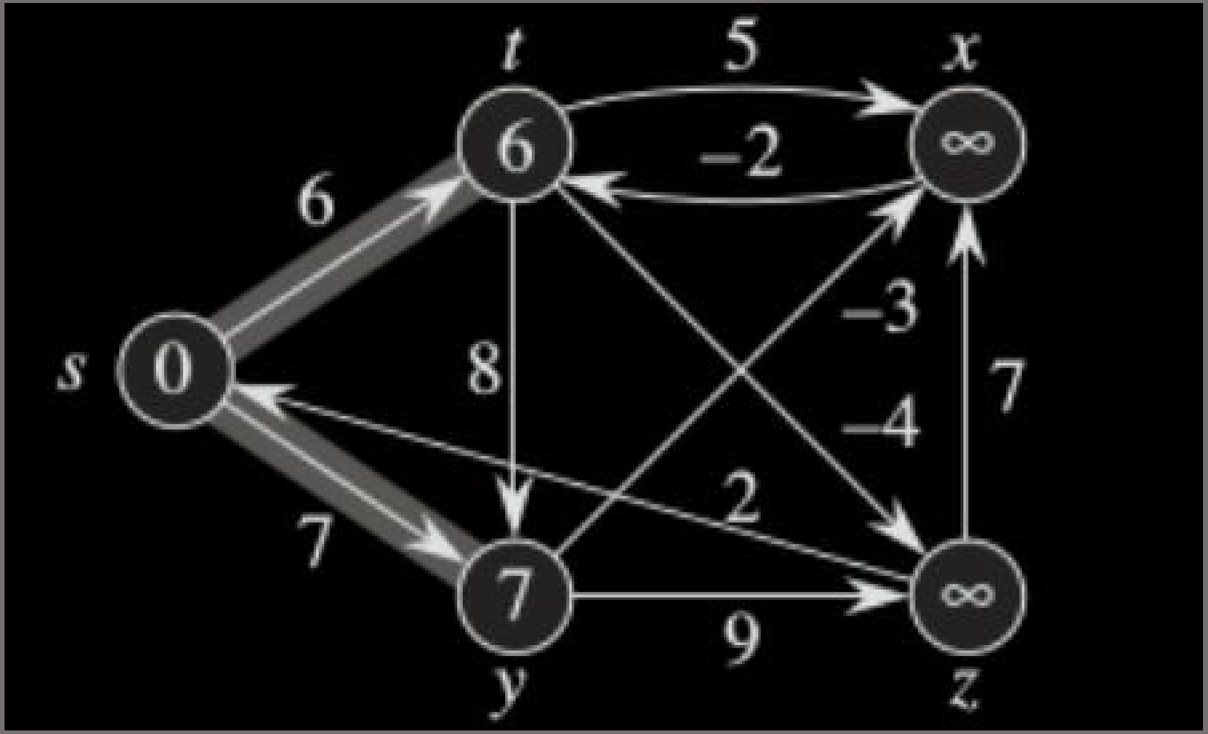
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	~~9~~ 6
x	∞	11
y	7	~~14~~ 7
z	∞	2

(t, x)	(t, y)	(t, z)	(x, t)						

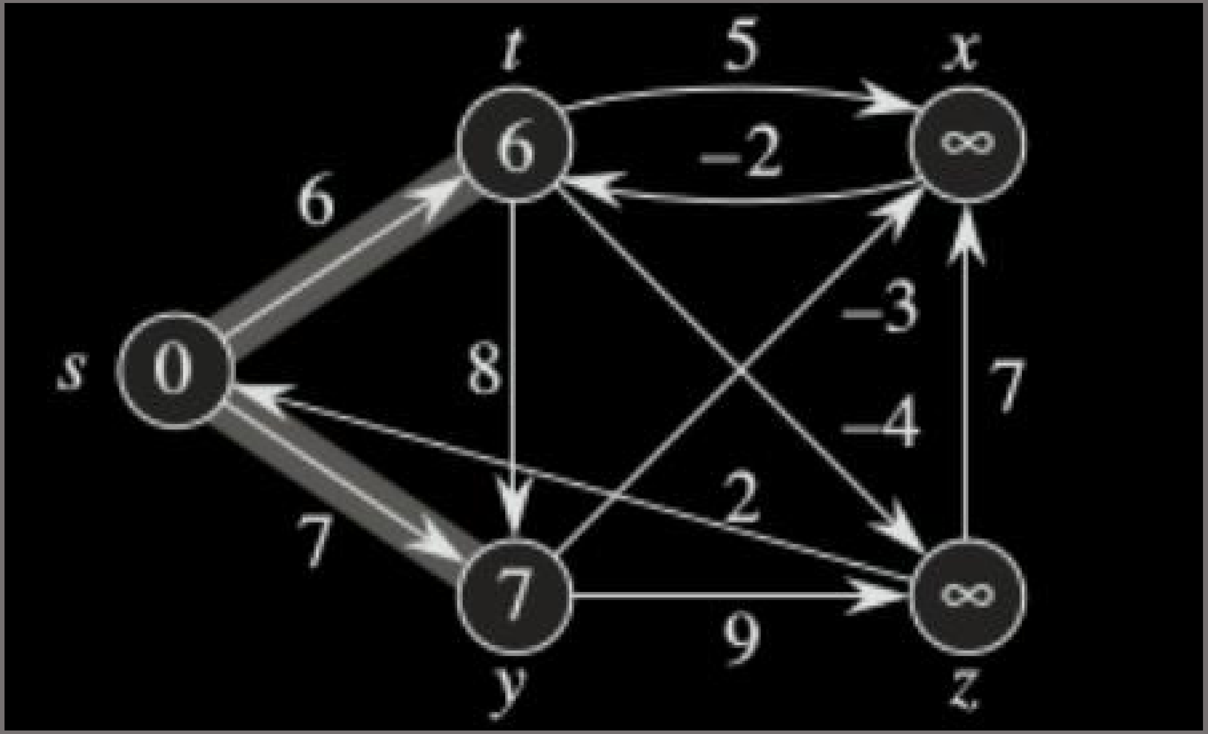
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	~~9~~ 6
x	11	4
y	7	~~14~~ 7
z	∞	2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)					

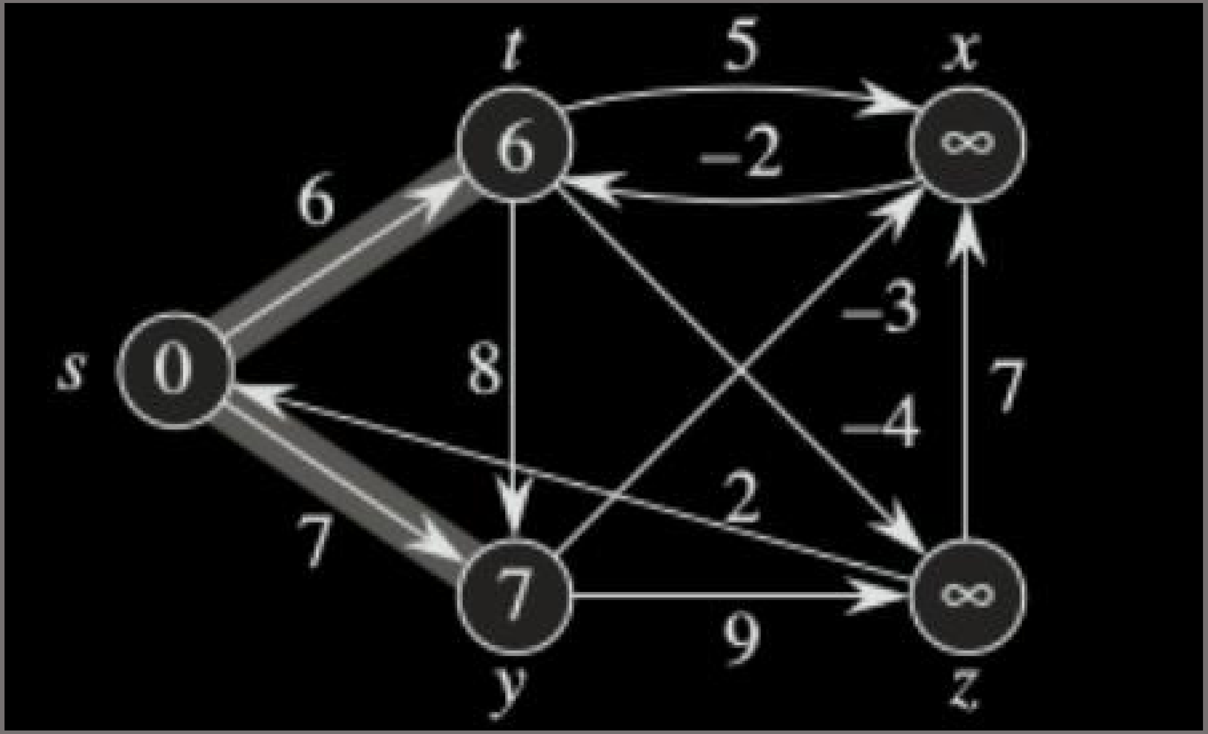
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	~~9~~ 6
x	11	4
y	7	~~14~~ 7
z	∞	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)				

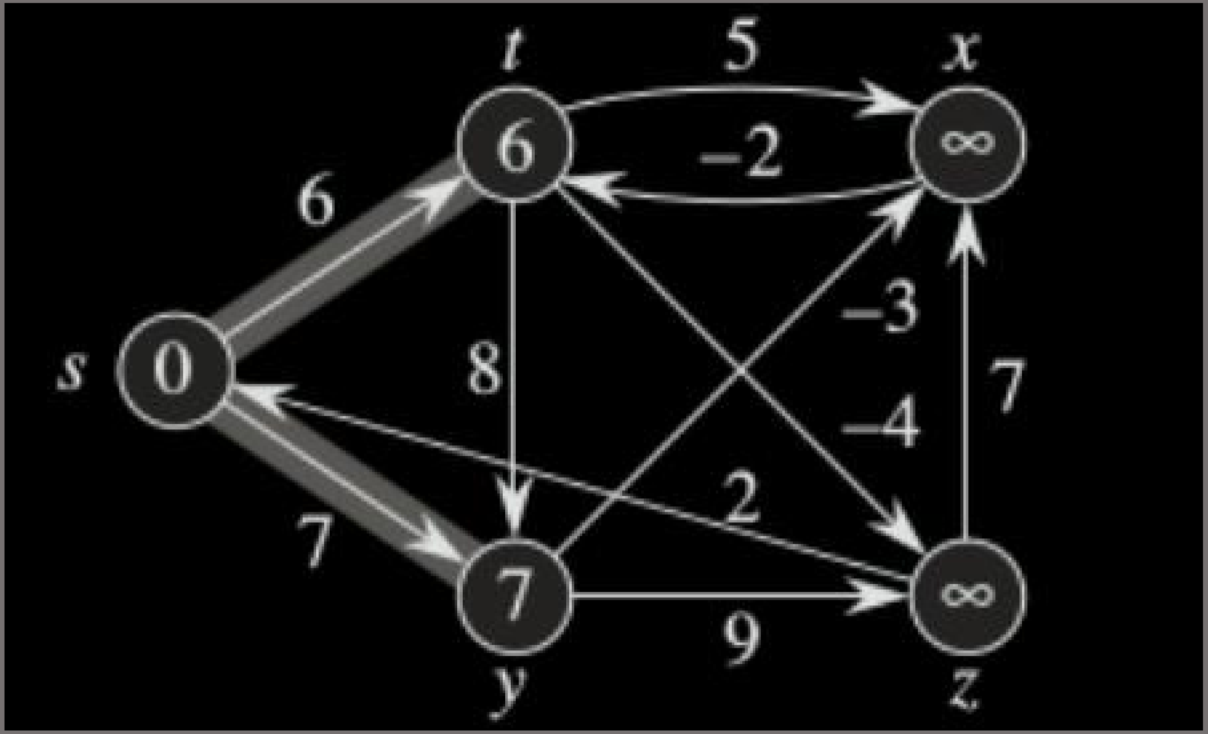
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	~~9~~ 6
x	11	~~9~~ 4
y	7	~~14~~ 7
z	∞	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)			

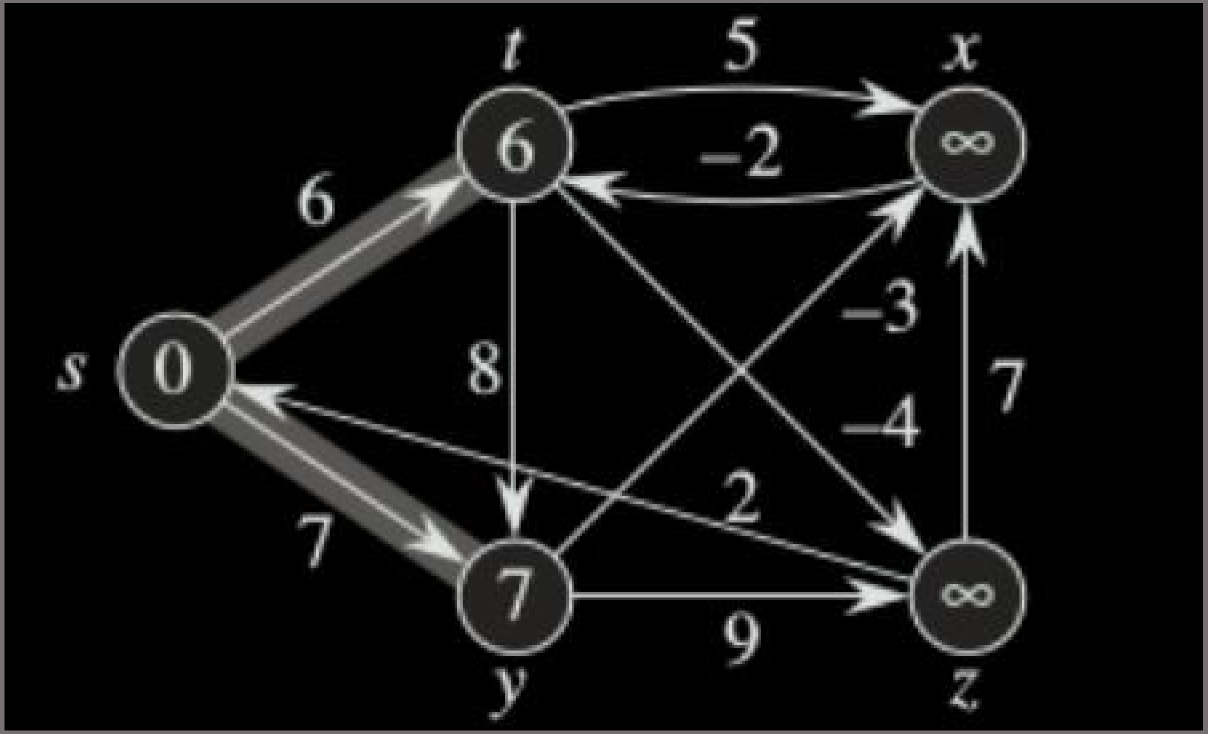
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	~~9~~ 6
x	11	~~9~~ 4
y	7	~~14~~ 7
z	∞	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)		

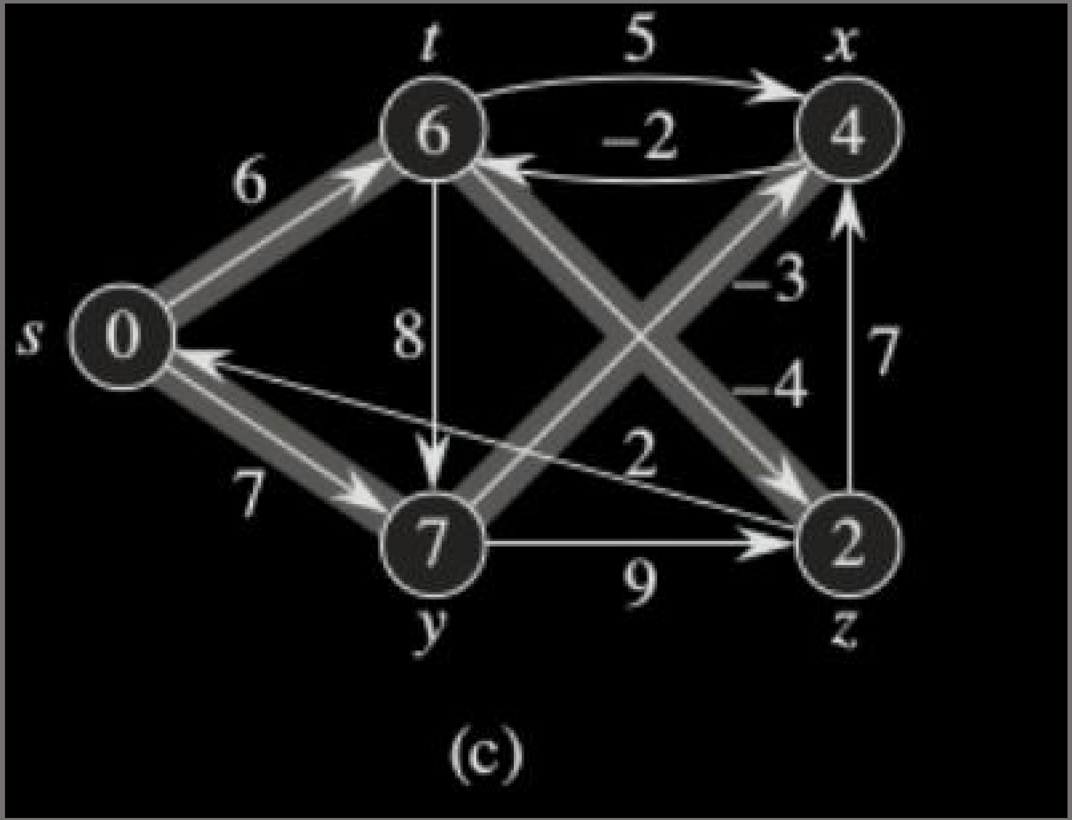
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	~~9~~ 6
x	11	~~9~~ 4
y	7	~~14~~ 7
z	∞	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	

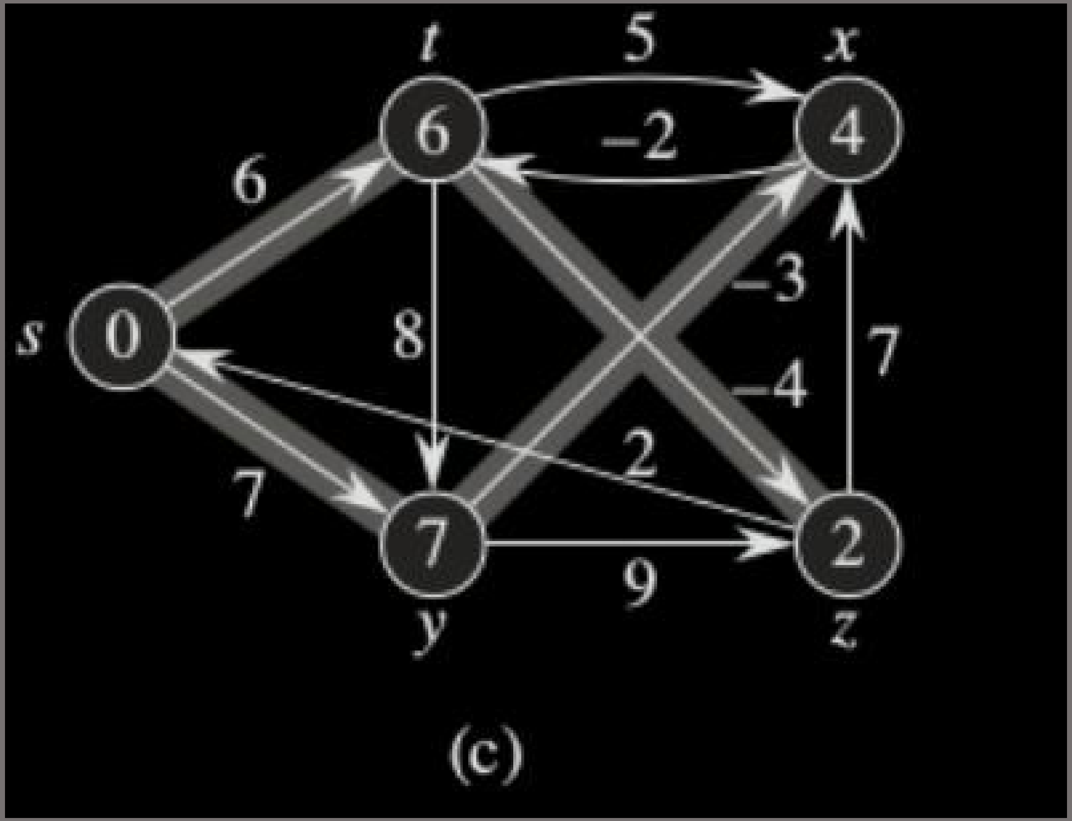
Bellman–Ford algorithm – Iteration 2



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	~~9~~ 6
x	11	~~9~~ 4
y	7	~~14~~ 7
z	∞	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	(s,y)

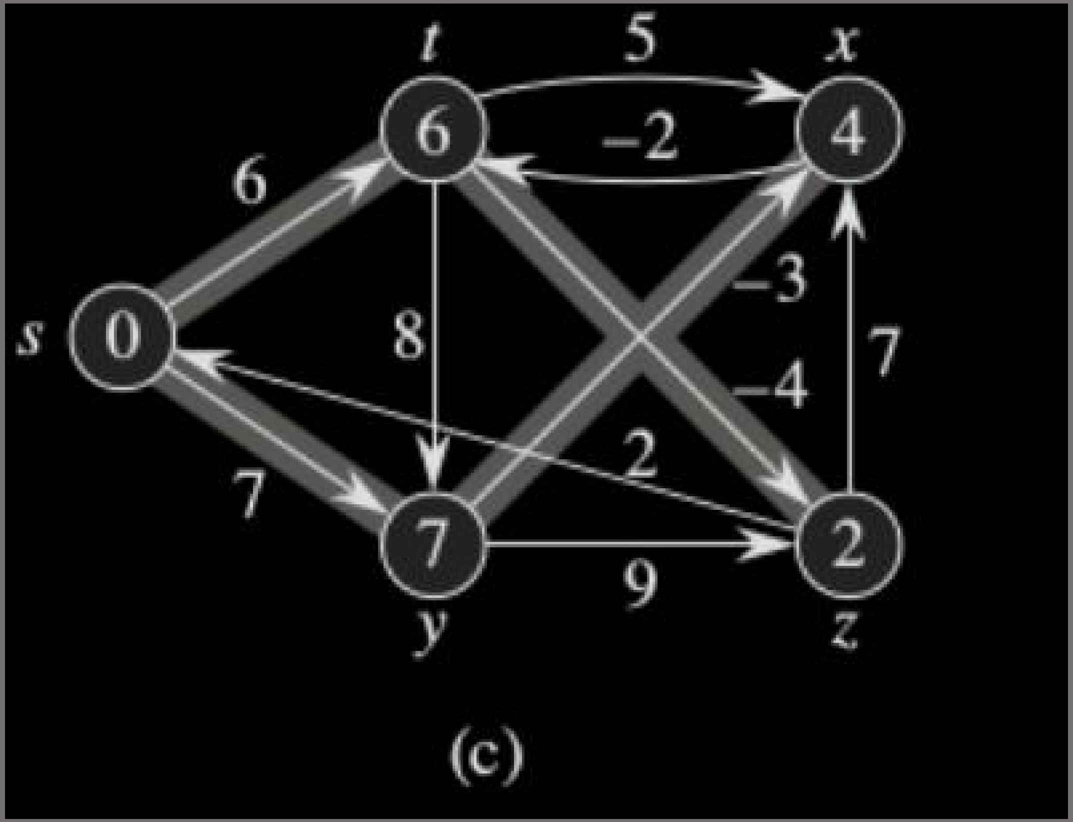
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	6
x	4	~~11~~ 4
y	7	7
z	2	2

(t, x)									

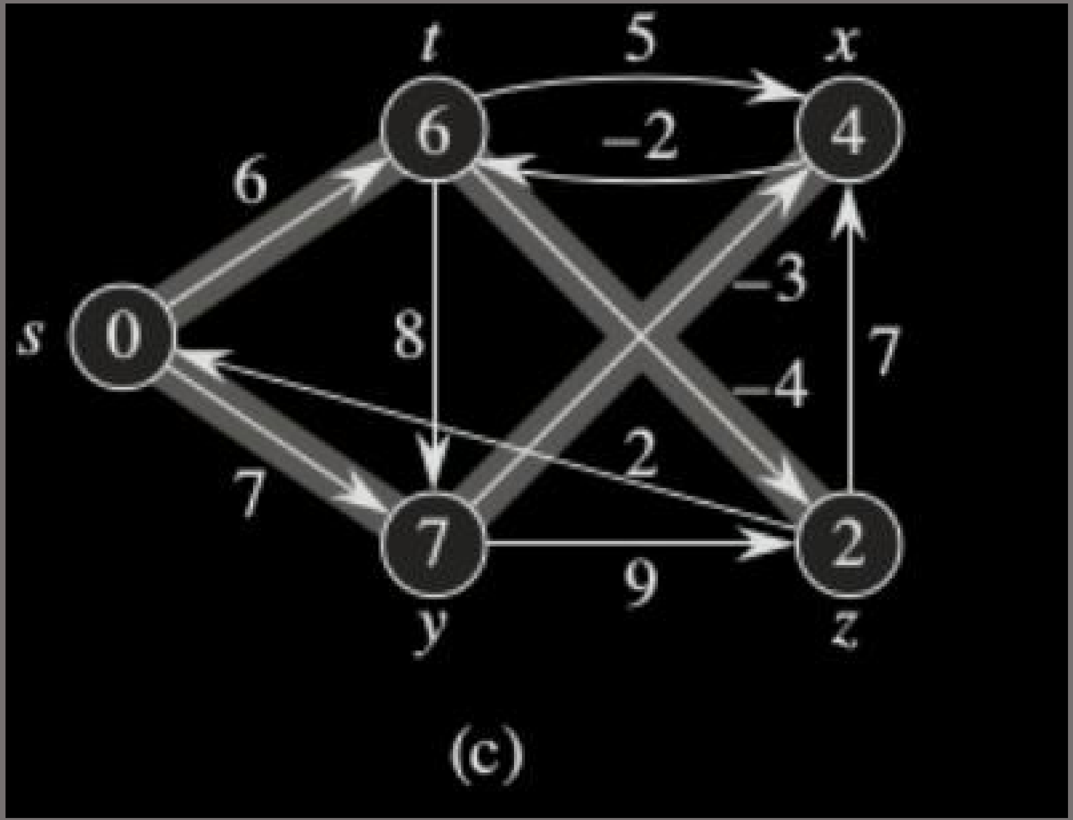
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	6
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	2

(t, x)	(t, y)								

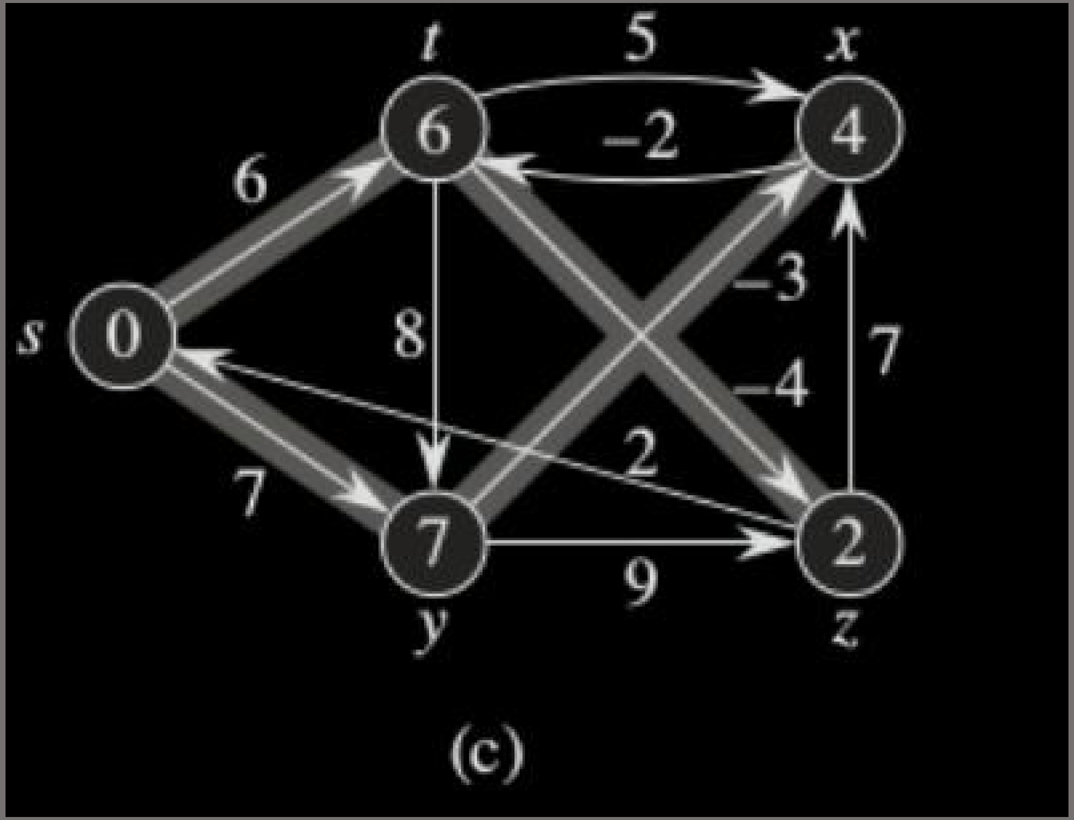
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	6
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	2

(t, x)	(t, y)	(t, z)							

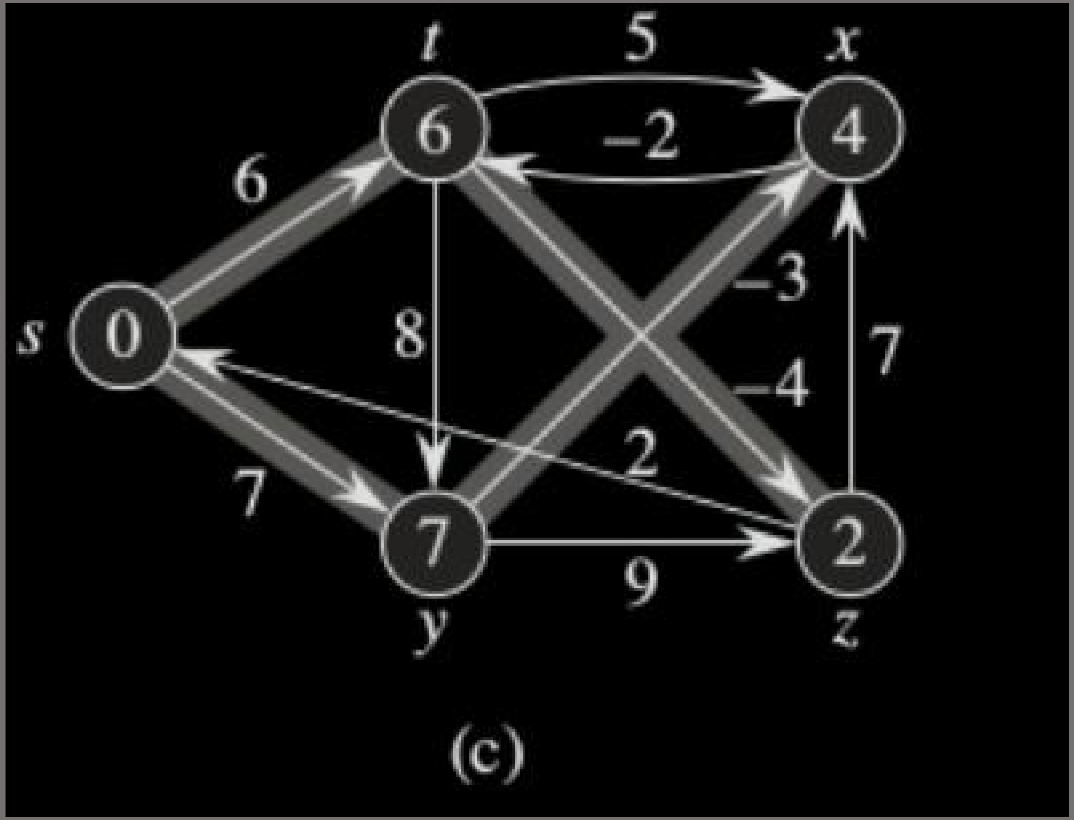
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	2
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	2

(t, x)	(t, y)	(t, z)	(x, t)						

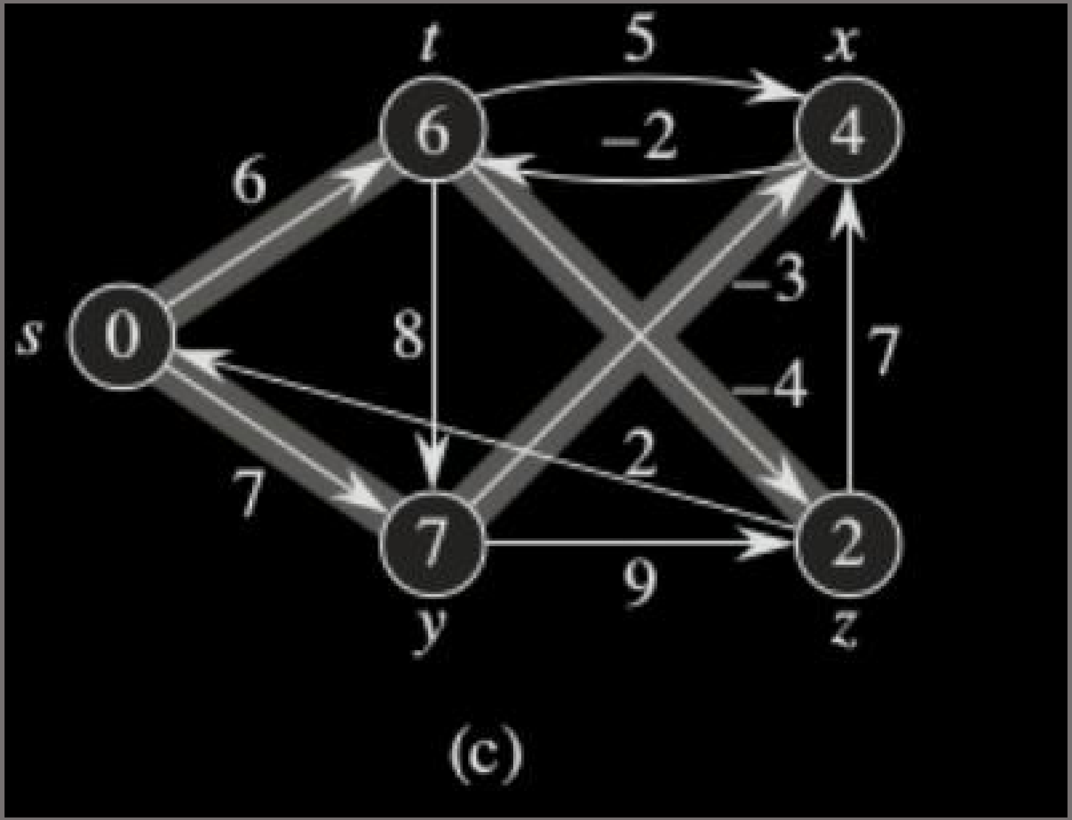
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	2
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)					

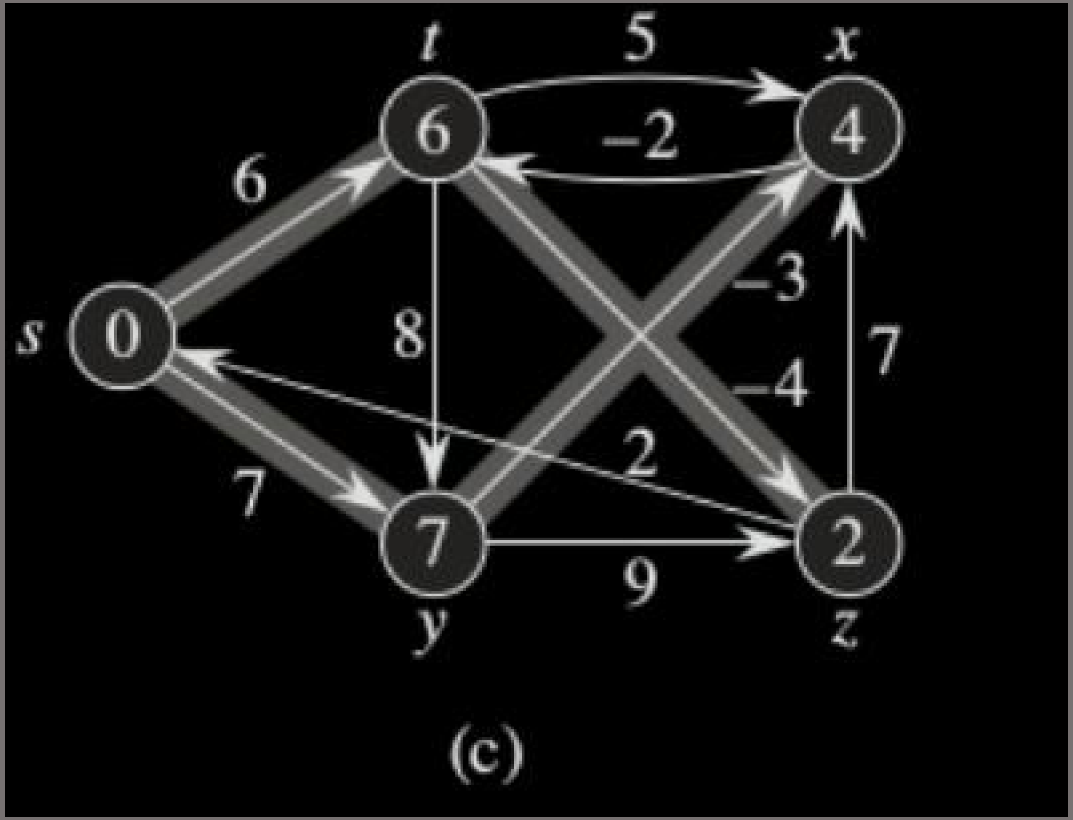
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	2
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)				

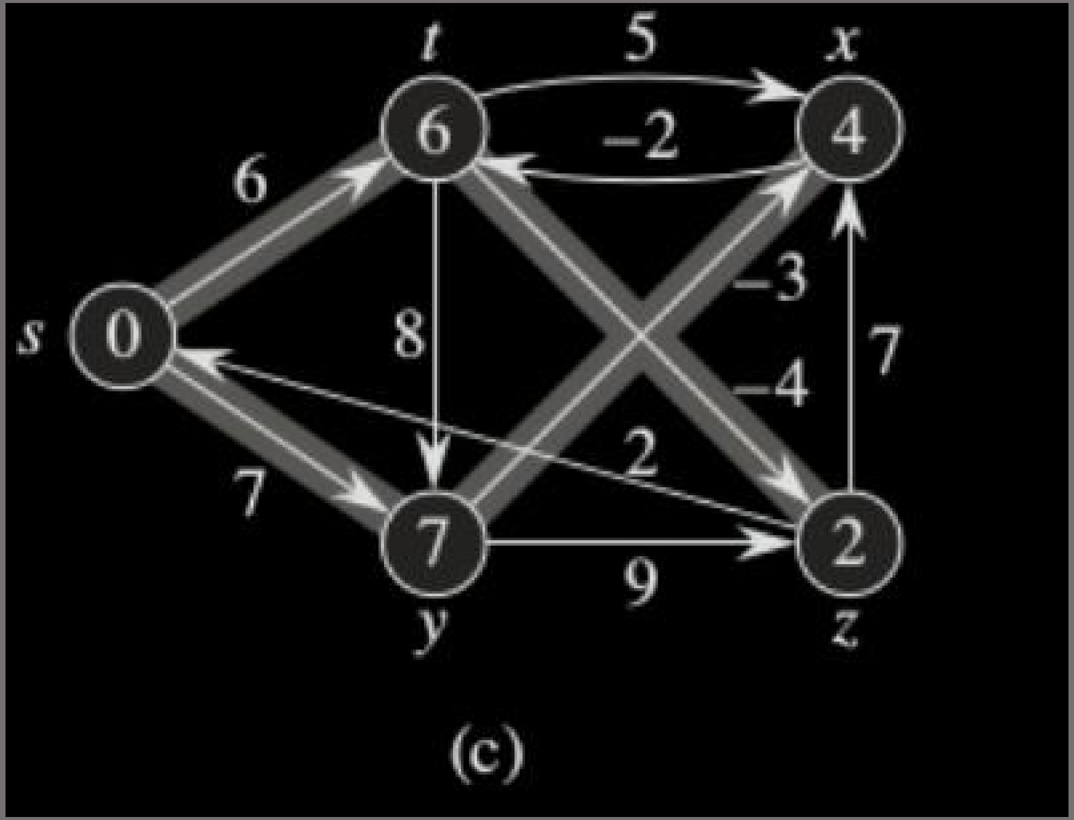
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	2
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)			

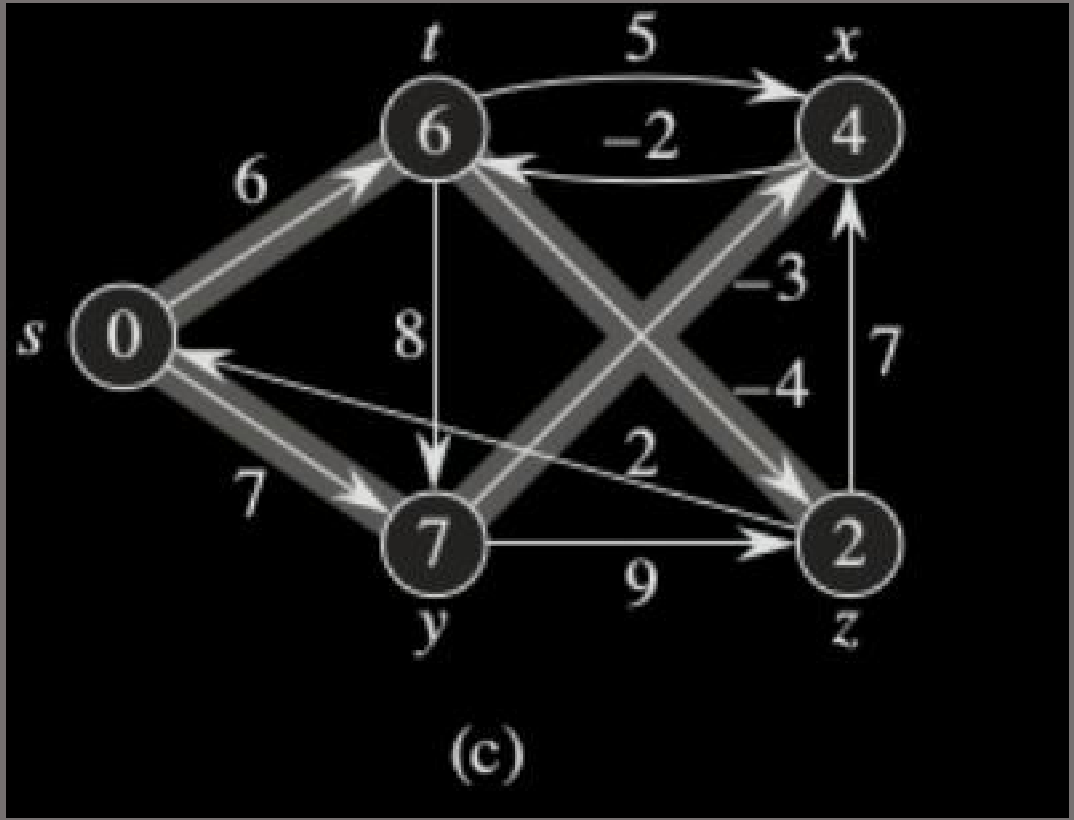
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	2
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)		

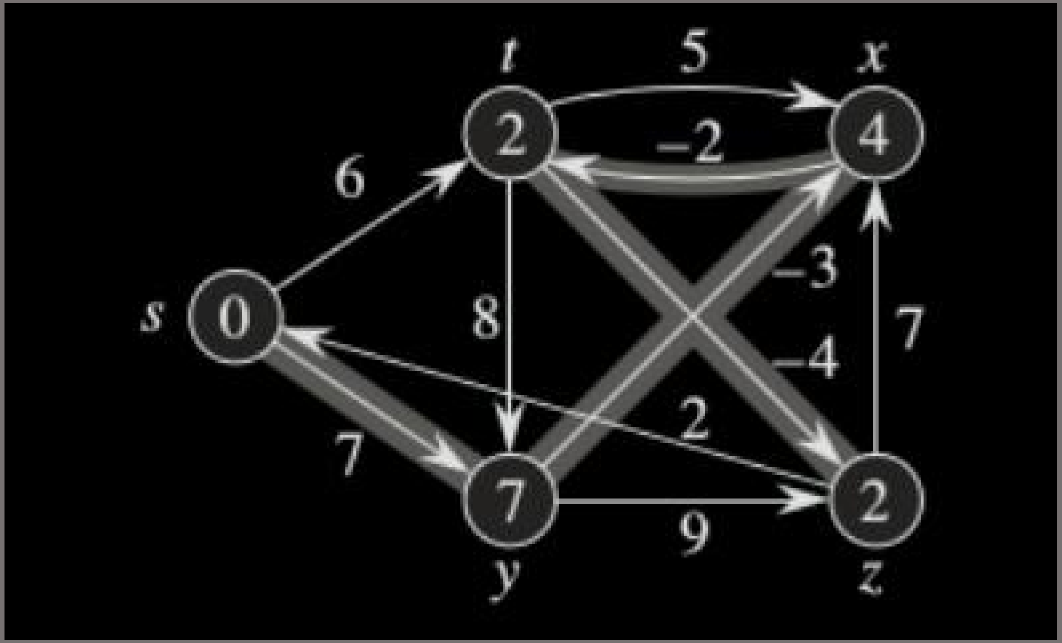
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	2
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	

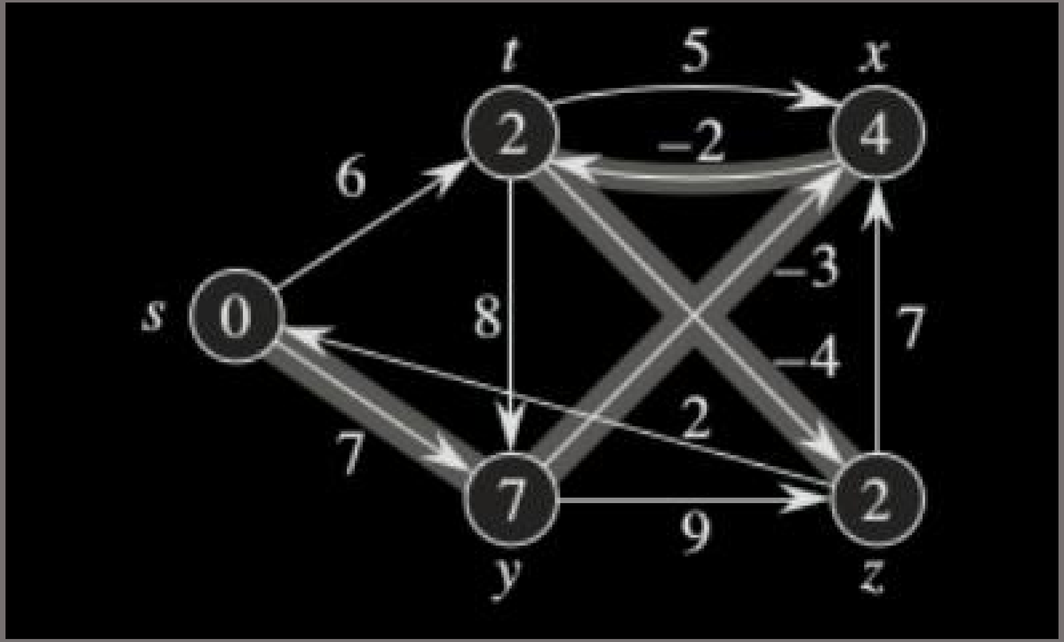
Bellman–Ford algorithm – Iteration 3



Edge Name	Old Cost	Updated Cost
s	0	0
t	6	2
x	4	~~11~~ 4
y	7	~~14~~ 7
z	2	~~16~~ 2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	(s,y)

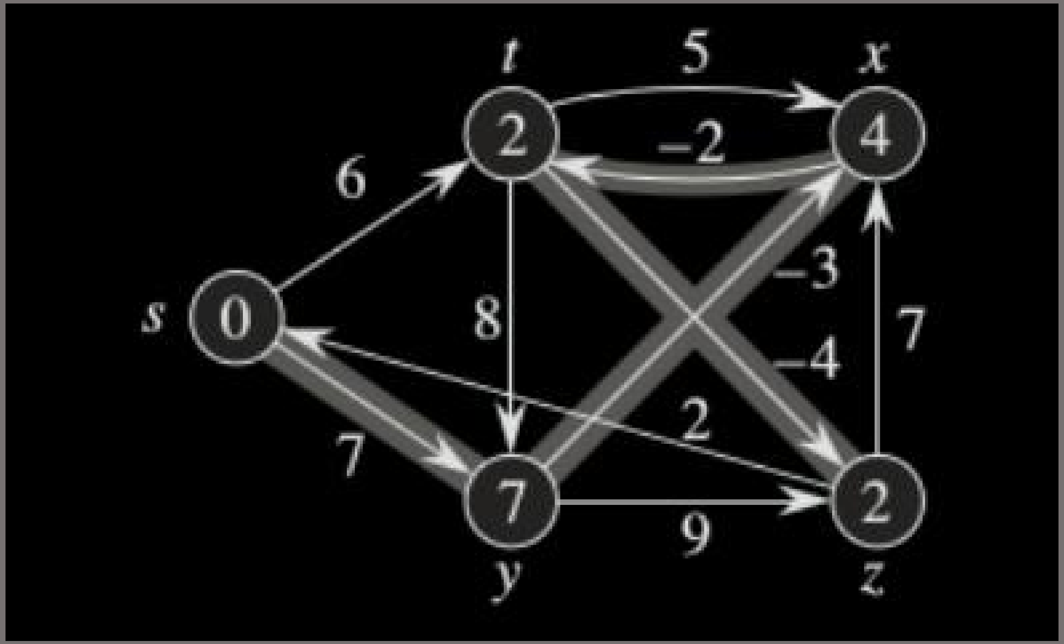
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	
x	4	~~7~~ 4
y	7	
z	2	

(t, x)									

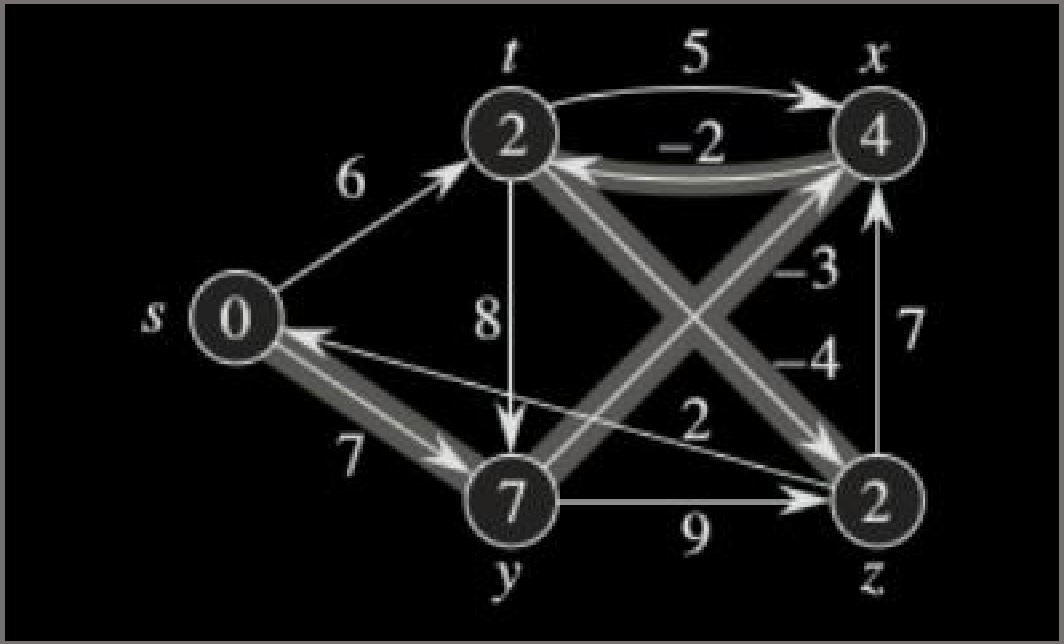
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	

(t, x)	(t, y)								

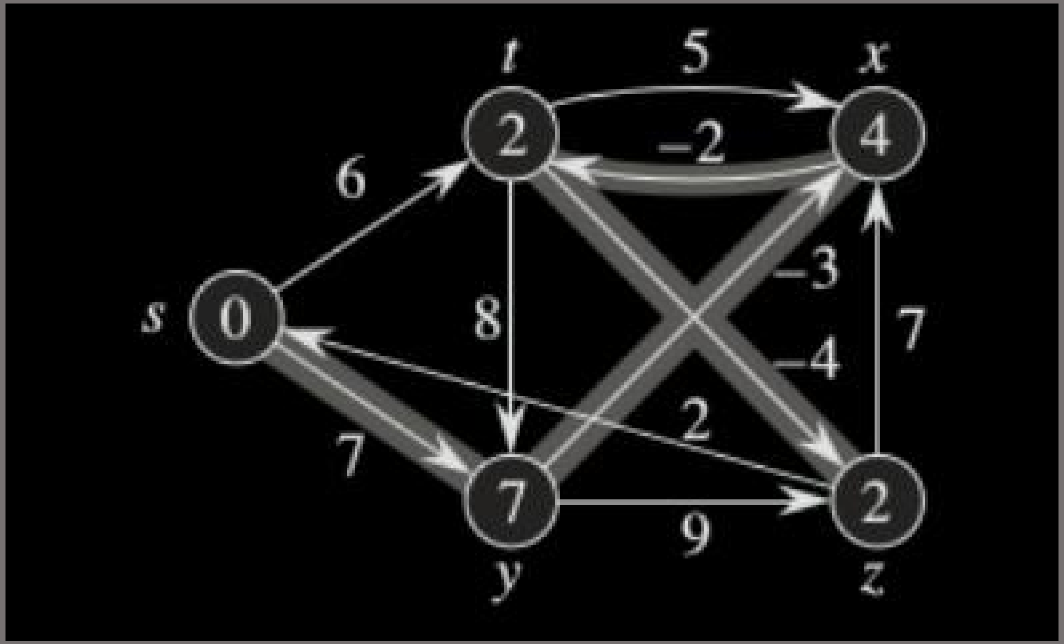
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	–2

(t, x)	(t, y)	(t, z)							

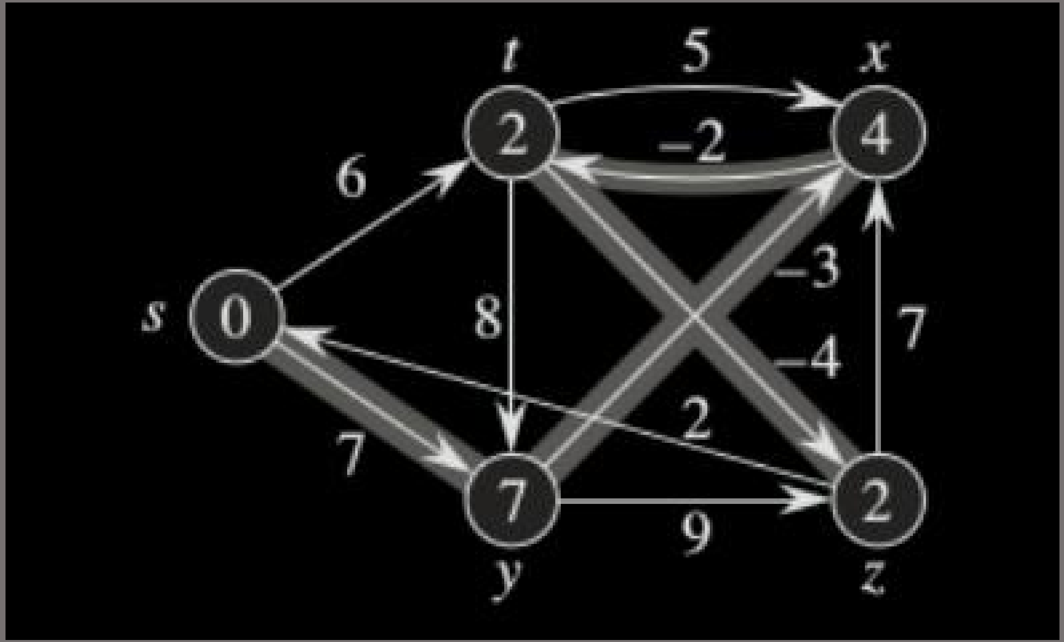
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	–2

(t, x)	(t, y)	(t, z)	(x, t)						

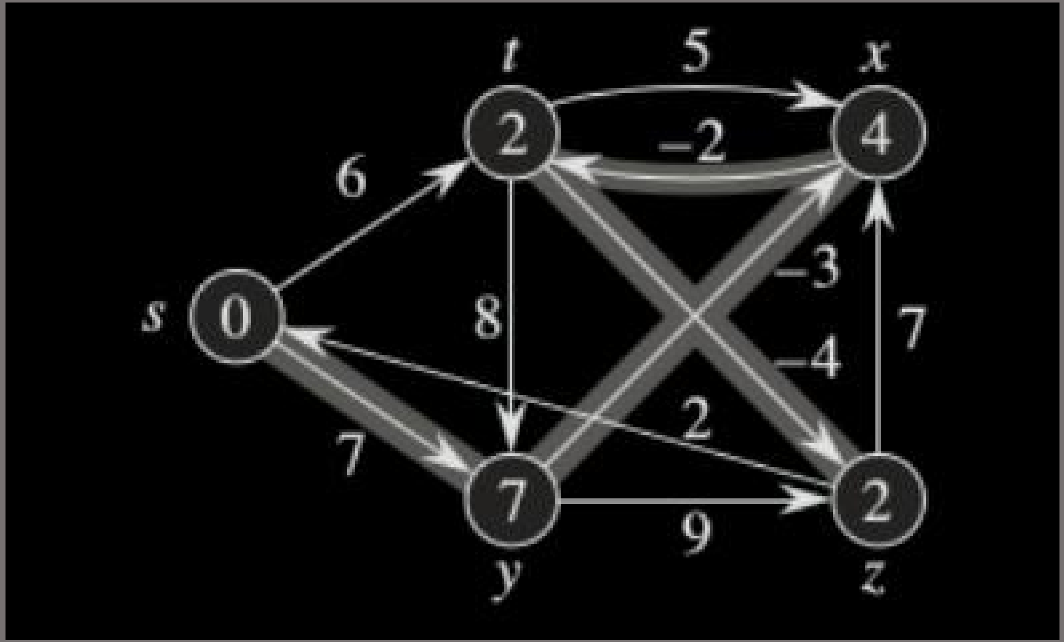
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	−2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)					

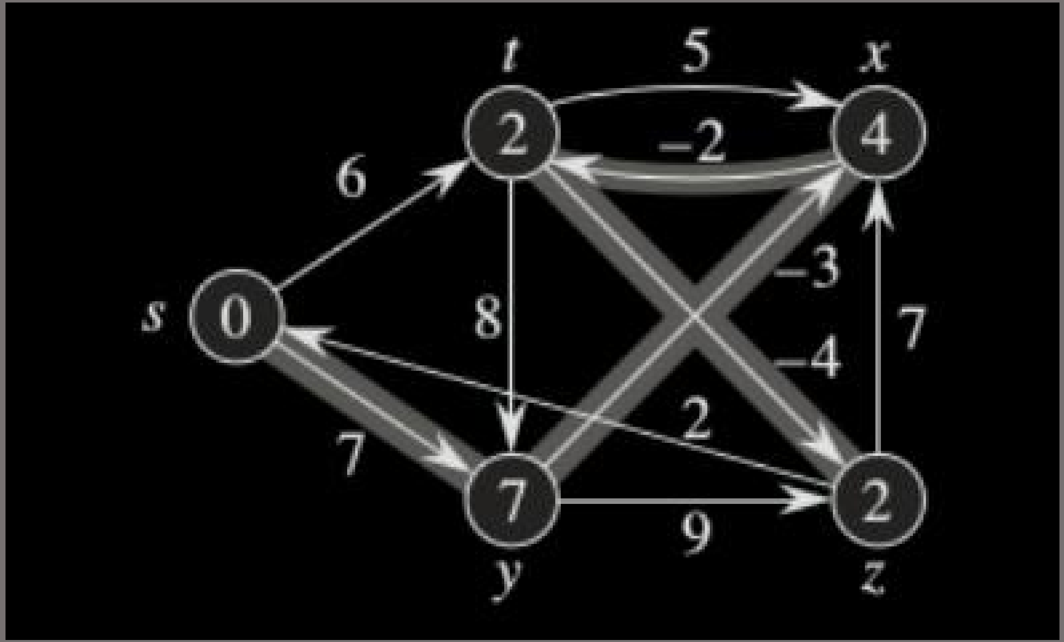
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)				

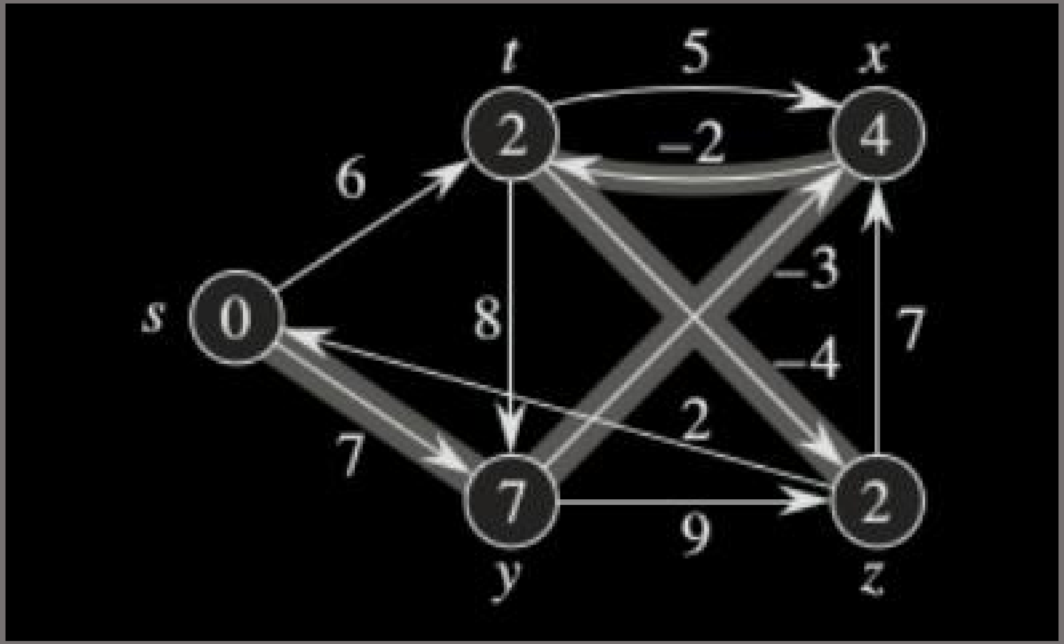
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)			

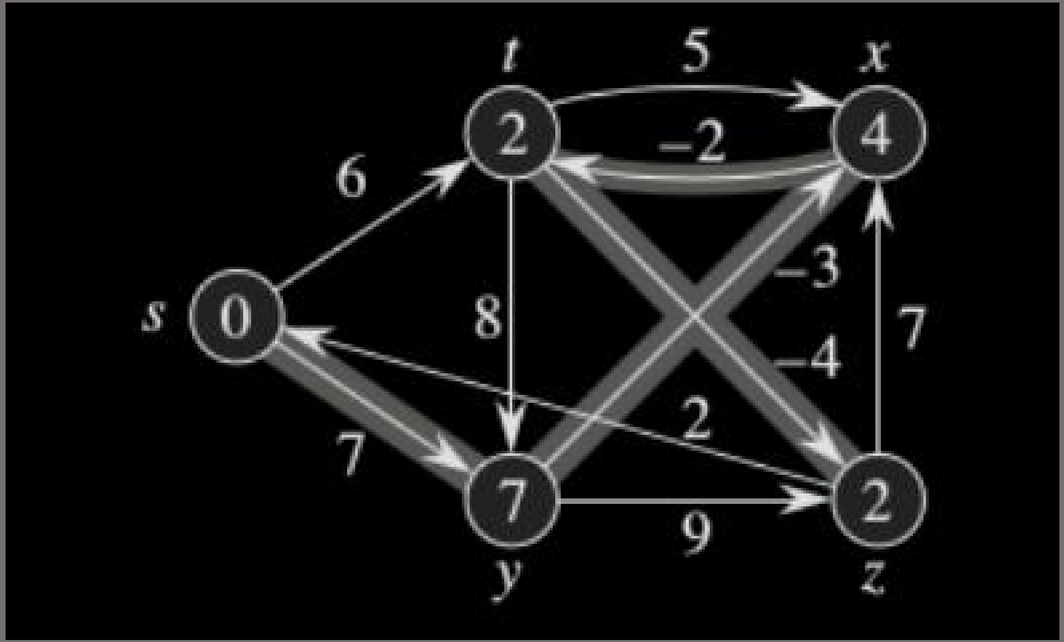
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)		

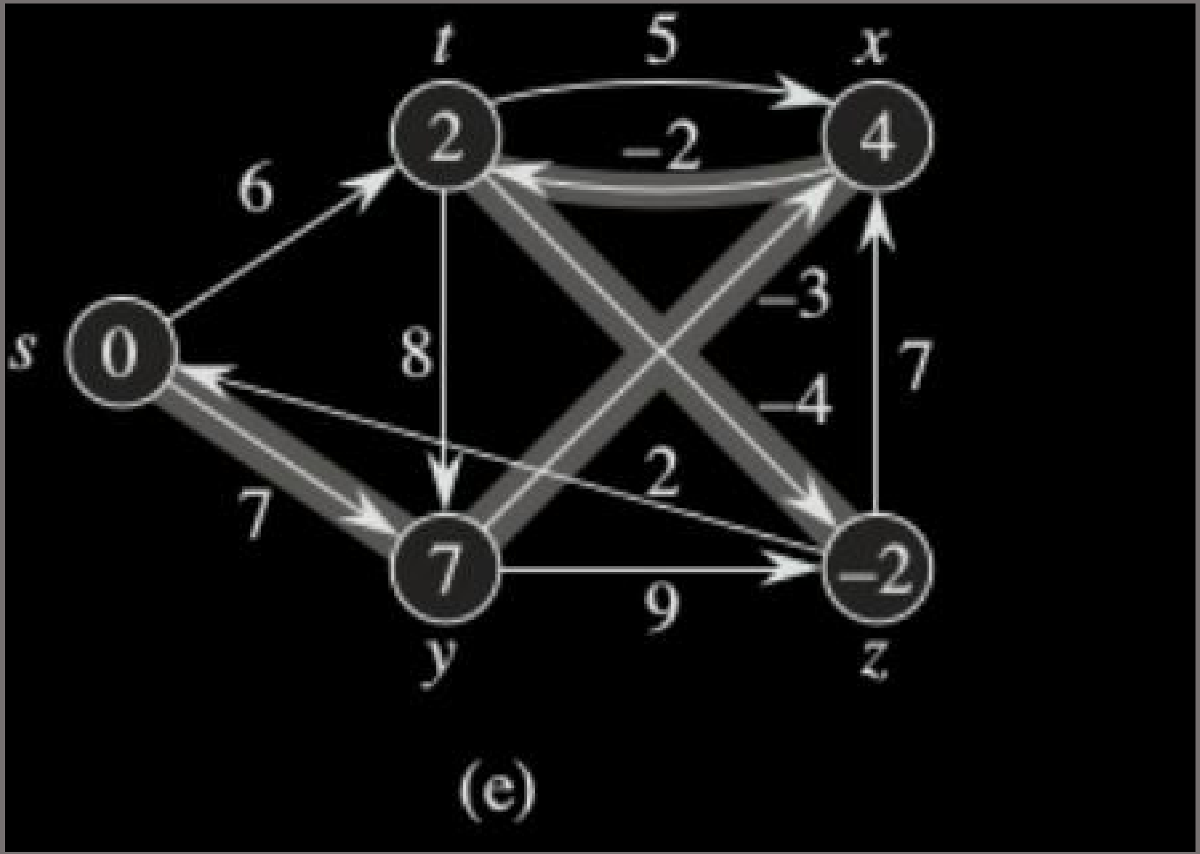
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	

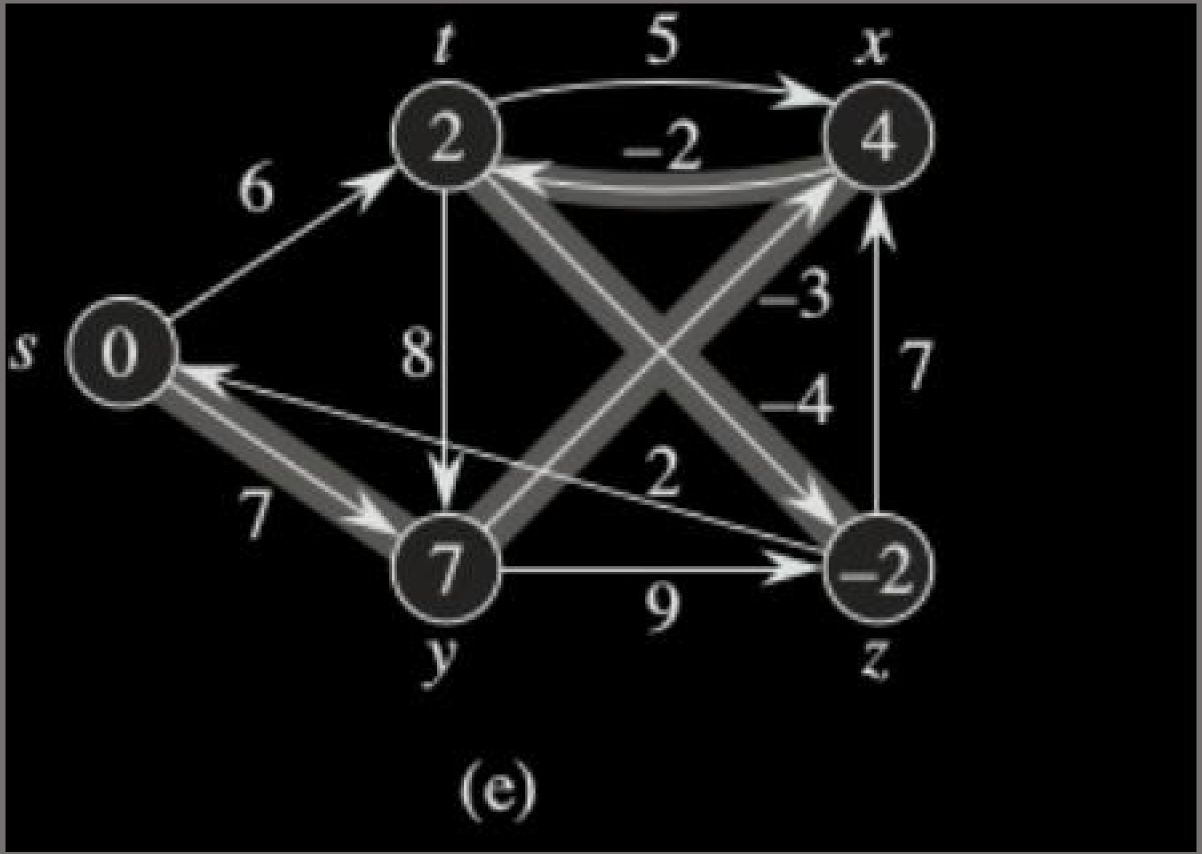
Bellman–Ford algorithm – Iteration 4



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	(s,y)

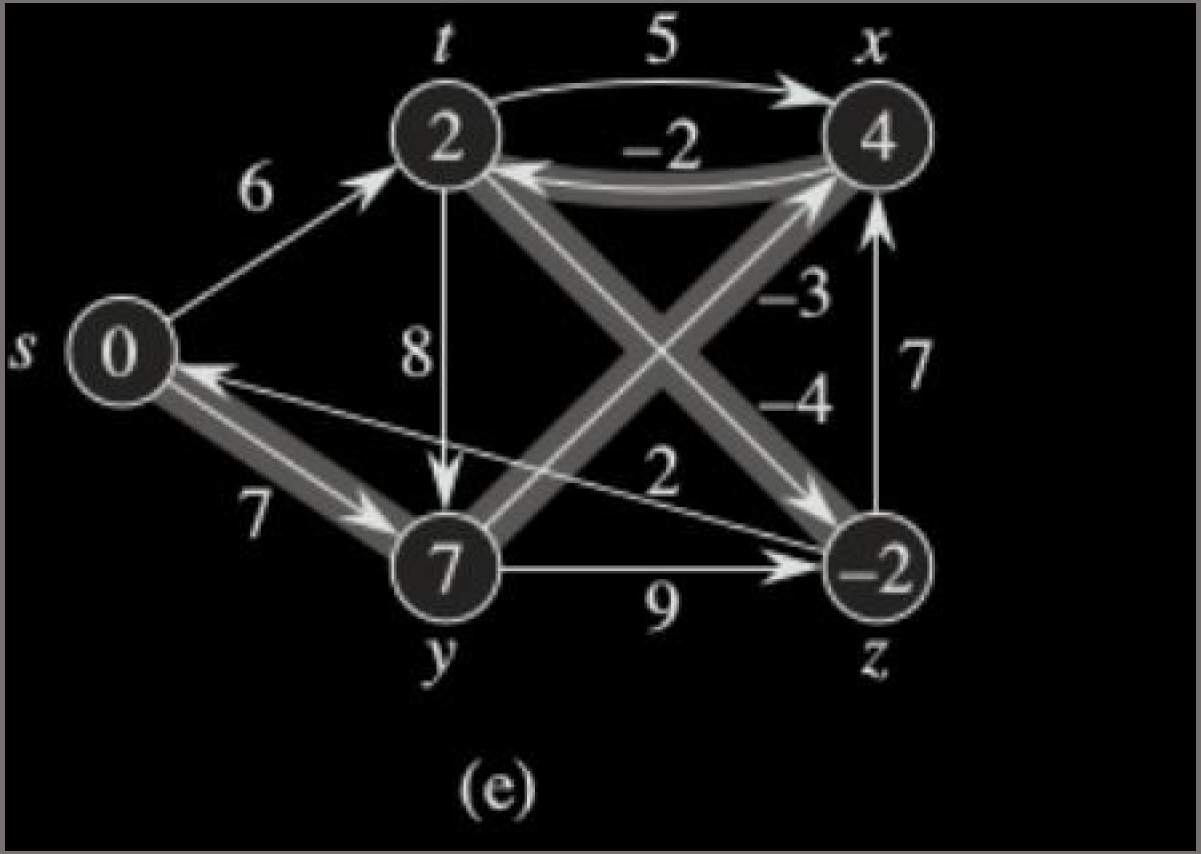
Bellman–Ford algorithm – Check for Cycles



Edge Name	Old Cost	Updated Cost
s	0	
t	2	
x	4	~~7~~ 4
y	7	
z	−2	

(t, x)									

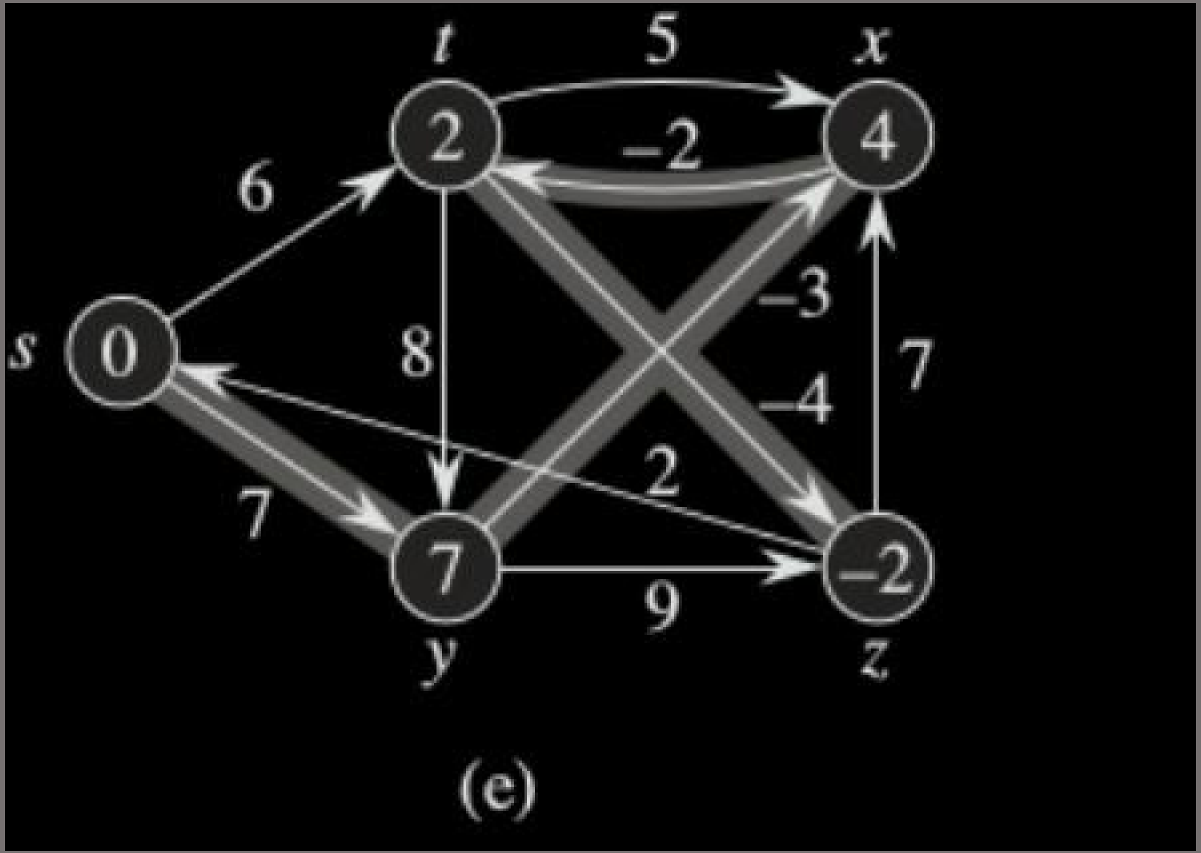
Bellman–Ford algorithm – Check for Cycles



Edge Name	Old Cost	Updated Cost
s	0	
t	2	
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	

(t, x)	(t, y)								

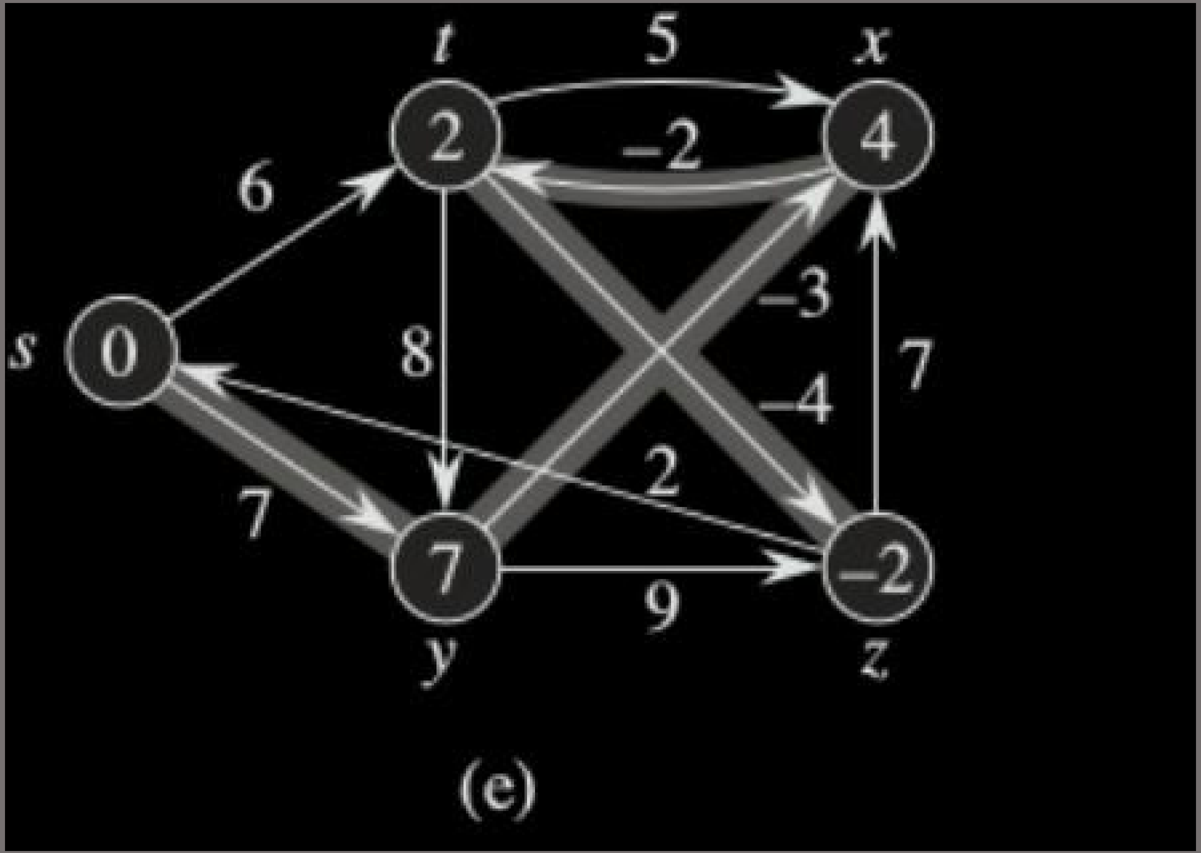
Bellman–Ford algorithm – Check for Cycles



Edge Name	Old Cost	Updated Cost
s	0	
t	2	
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	–2

(t, x)	(t, y)	(t, z)							

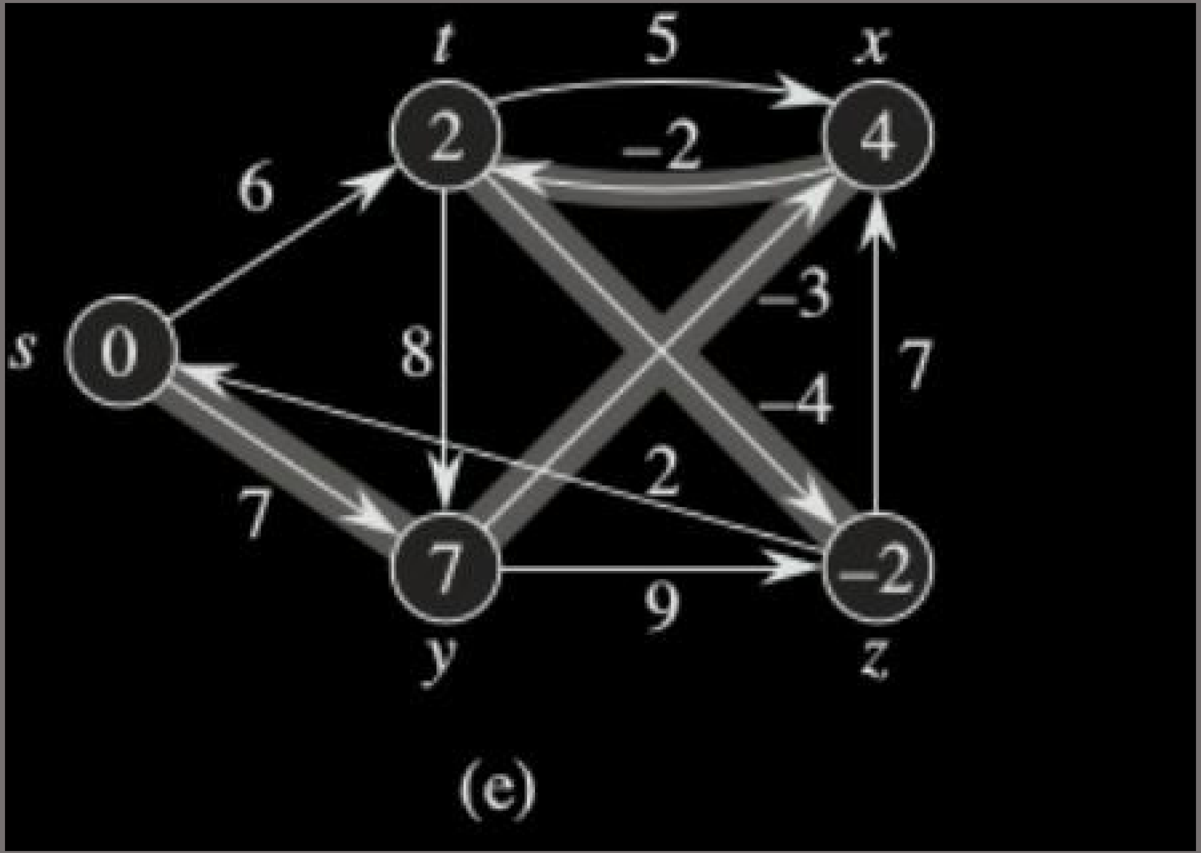
Bellman–Ford algorithm – Iteration 5



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	–2

(t, x)	(t, y)	(t, z)	(x, t)						

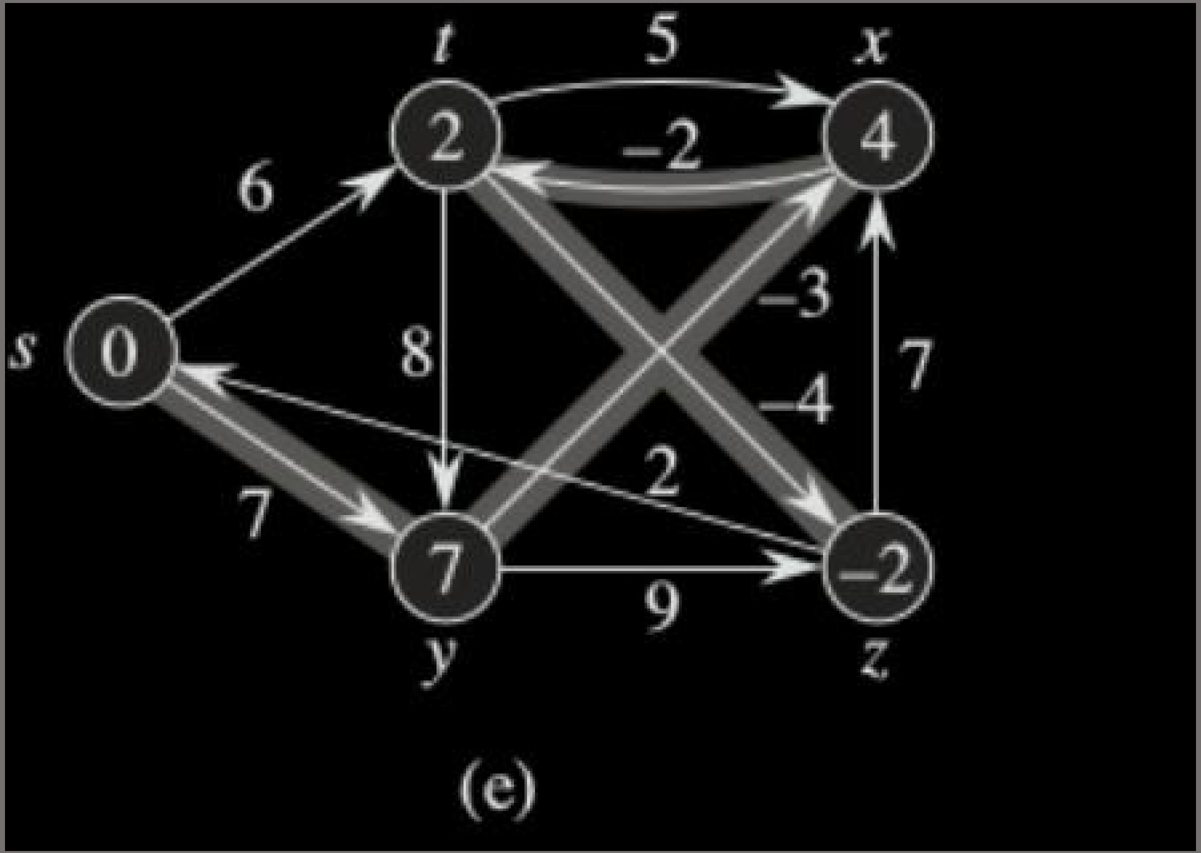
Bellman–Ford algorithm – Iteration 5



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	–2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)					

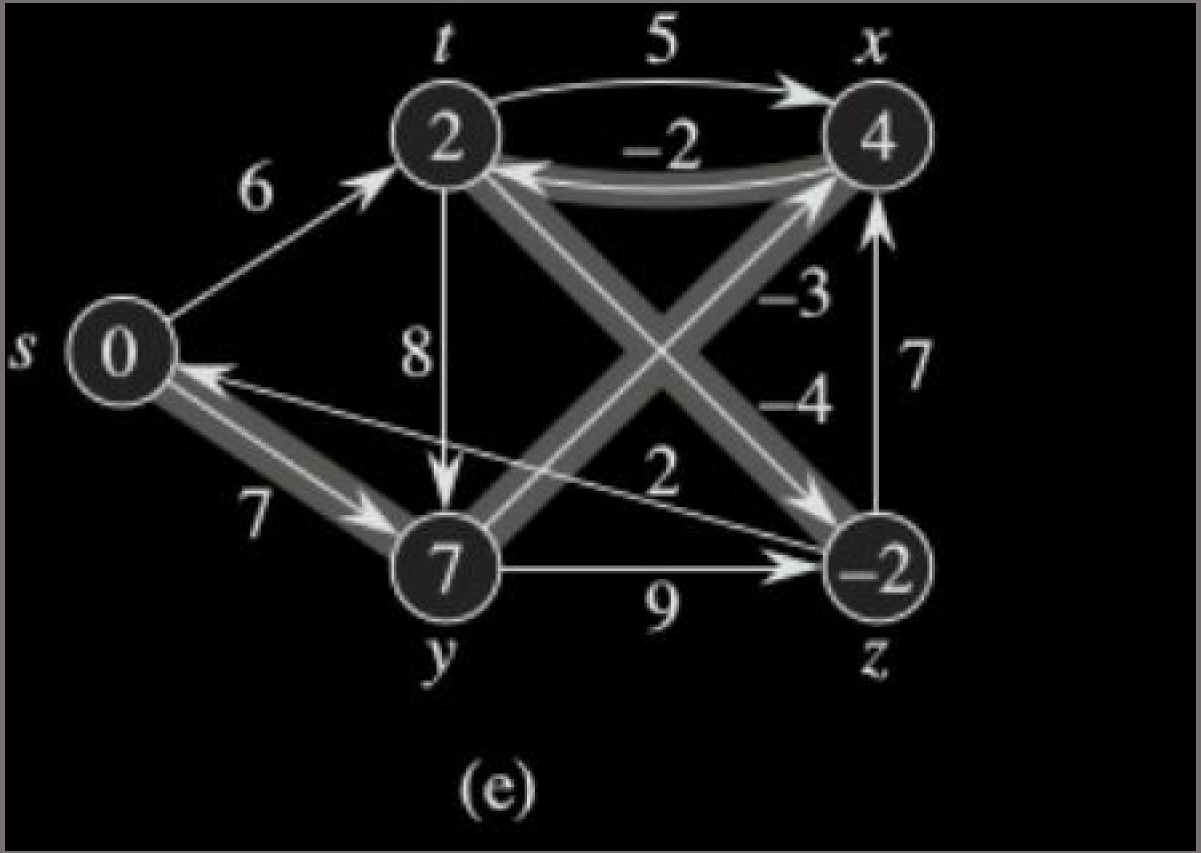
Bellman–Ford algorithm – Iteration 5



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	–2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)				

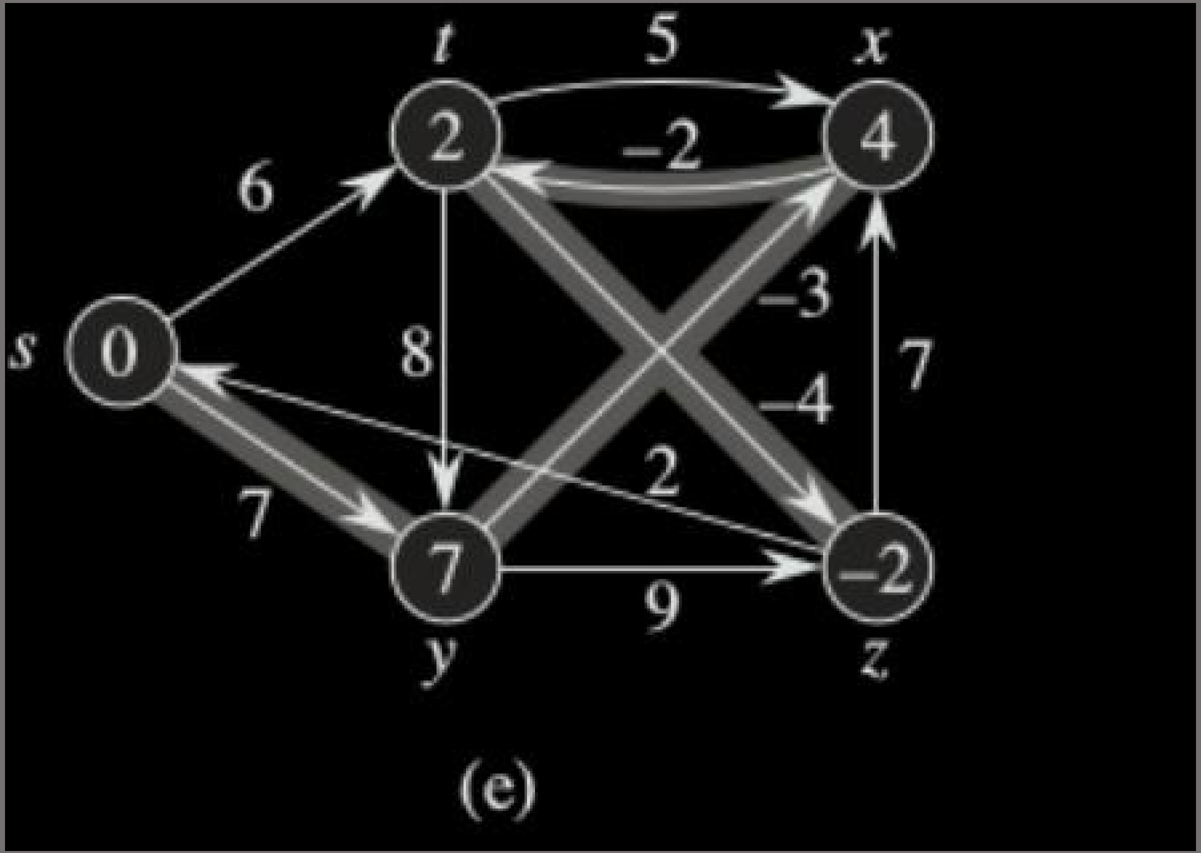
Bellman–Ford algorithm – Iteration 5



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)			

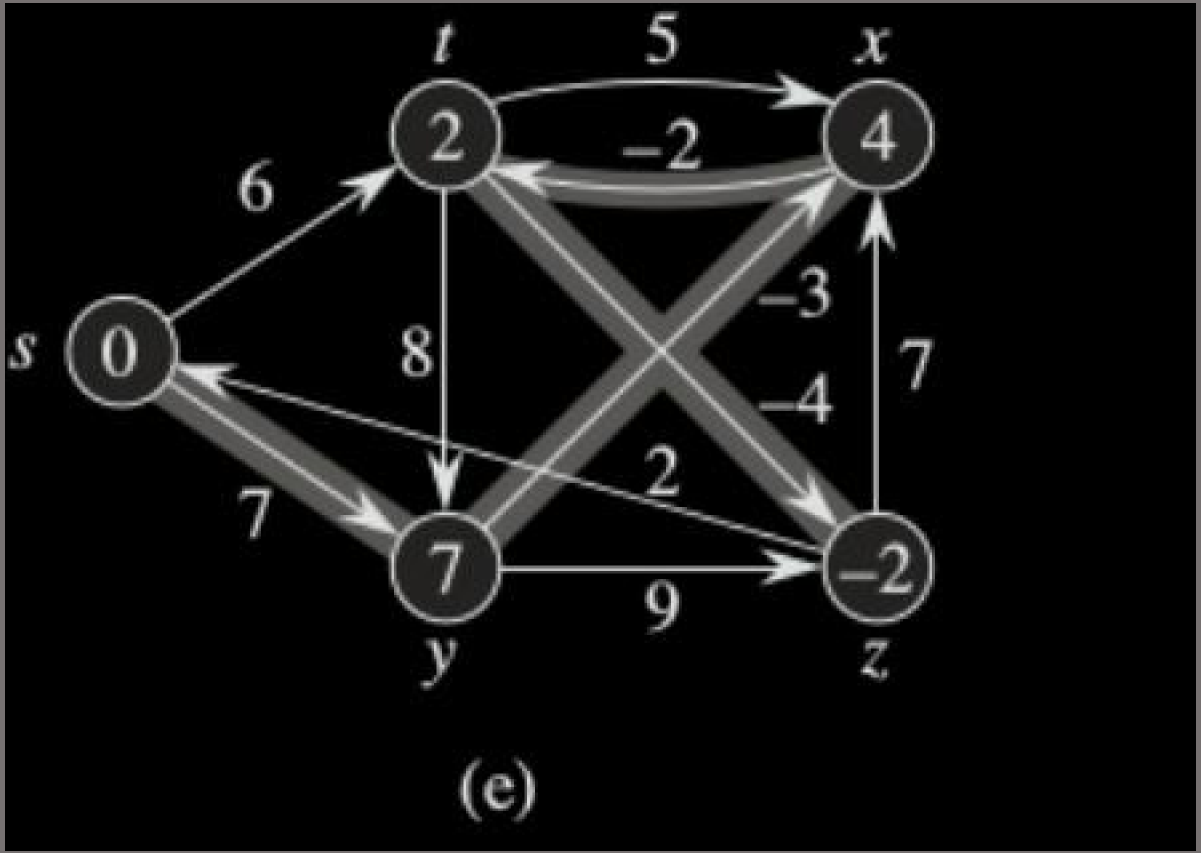
Bellman–Ford algorithm – Iteration 5



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)		

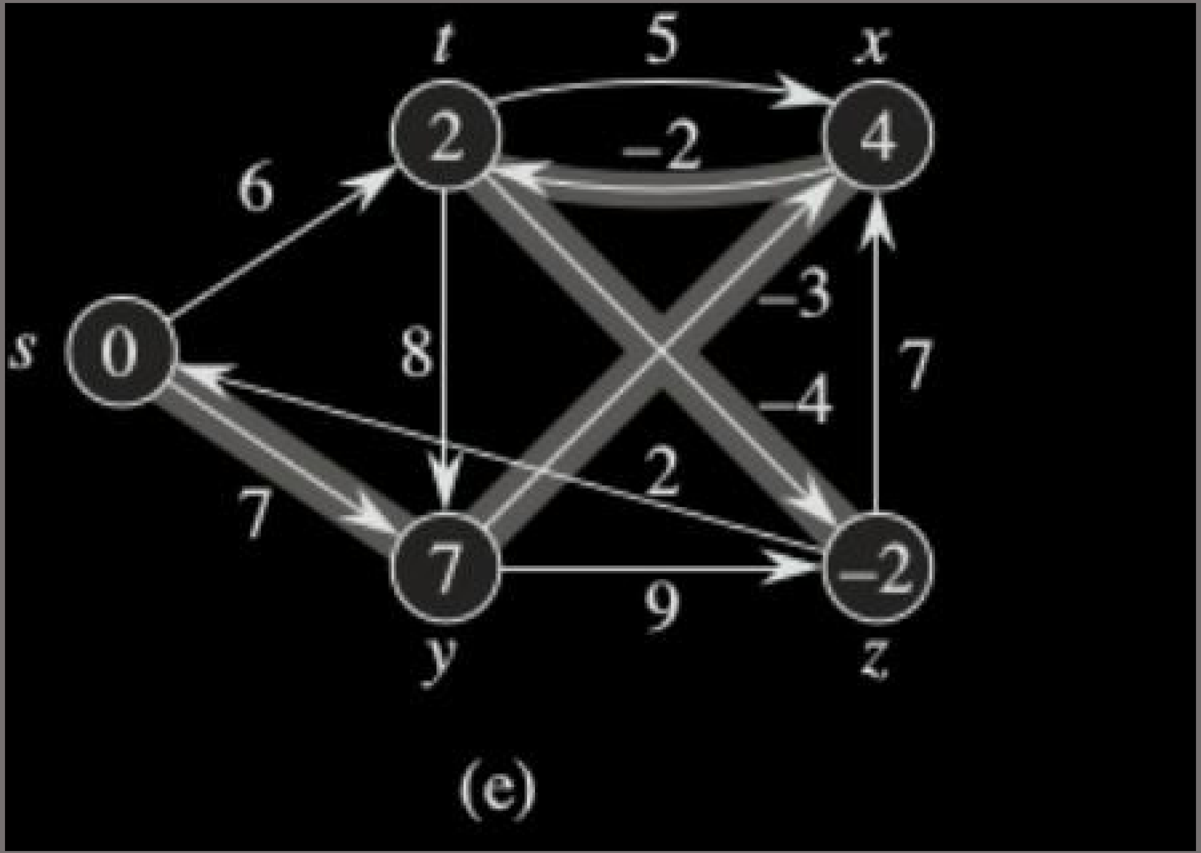
Bellman–Ford algorithm – Iteration 5



Edge Name	Old Cost	Updated Cost
s	0	
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	

Bellman–Ford algorithm – Iteration 5



Edge Name	Old Cost	Updated Cost
s	0	0
t	2	2
x	4	~~7~~ 4
y	7	~~10~~ 7
z	–2	~~16~~ –2

(t, x)	(t, y)	(t, z)	(x, t)	(y, x)	(y,z)	(z, x)	(z,s)	(s,t)	(s,y)

Bellman–Ford algorithm

- time complexity of Bellman–Ford is O(VE), which is more than Dijkstra.