Divide and Conquer

Matrix Multiplication

Matrix Multiplication

$\begin{bmatrix} a_{11} \end{bmatrix}$ $\begin{bmatrix} b_{11} \end{bmatrix}$ $\begin{bmatrix} c_{11} \end{bmatrix}$

$c_{11} = a_{11} * b_{11}$

Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$c_{11} = a_{11}*b_{11} + a_{12}*b_{21}$

$c_{12} = a_{11}*b_{12} + a_{12}*b_{22}$

$c_{21} = a_{21}*b_{11} + a_{22}*b_{21}$

$c_{22} = a_{21}*b_{12} + a_{22}*b_{22}$

Matrix Multiplication

```
SQUARE-MATRIX-MULTIPLY(A, B)
1  n = A.rows
2  let C be a new n × n matrix
3  for i = 1 to n
4      for j = 1 to n
5          c_ij = 0
6          for k = 1 to n
7              c_ij = c_ij + a_ik · b_kj
8  return C
```

- Each of the triply–nested for loops runs exactly n iterations, and each execution of line 7 takes constant time
- Hence SQUARE–MATRIX – MULTIPLY procedure takes $\theta(n^3)$ time

Bigger Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \quad \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

$c_{11} = a_{11}*b_{11} + a_{12}*b_{21} + a_{13}*b_{31} + a_{14}*b_{41}$

$c_{12} = a_{11}*b_{12} + a_{12}*b_{22} + a_{13}*b_{32} + a_{14}*b_{42}$

$c_{21} = a_{21}*b_{11} + a_{22}*b_{21} + a_{23}*b_{31} + a_{24}*b_{41}$

$c_{22} = a_{21}*b_{12} + a_{22}*b_{22} + a_{23}*b_{32} + a_{24}*b_{42}$

Matrix Multiplication

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad \left[\begin{array}{cc|cc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{array}\right] \qquad \left[\begin{array}{cc|cc} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ \hline c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{array}\right]$$

A11 A12 A21 A22

B11 B12 B21 B22

C11 C12 C13 C14

$C_{11} = A_{11}*B_{11} + A_{12}*B_{21}$

$C_{12} = A_{11}*B_{12} + A_{12}*B_{22}$

$C_{21} = A_{21}*B_{11} + A_{22}*B_{21}$

$C_{22} = A_{21}*B_{12} + A_{22}*B_{22}$

Recursive Matrix Multiplication

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B21)
    C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B22)
    C21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B11)
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B21)
    C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
            + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
return C
```

Divide and Conquer

- Divide: Calculating indices to split a square matrix of dimension 'n' into four matrices of dimension 'n/2'
- Conquer: Find product of the sub–matrices by calling itself
- Combine: To get the elements of the original matrix, do matrix addition of the resultant matrices obtained by multiplying submatrices

Recurrence Relation

- Divide: $\theta(1)$
- Conquer: $8 * T(n/2)$
- Combine: $\theta(n^2)$

Recurrence Relation

$$T(n) = \theta(1) \text{ if } n = 1$$

$$= 8 * T(n/2) + \theta(n^2) \text{ if } n > 1$$

Master's Theorem for Divide and Conquer

Consider a recurrence relation of the form:

$T(n) = aT(n/b) + \theta(n^k \log^p n)$

1) If $a > b^k$, then $T(n) = \theta(n^{\log_b a})$

2) If $a = b^k$, then

    a. If $p > -1$, then $T(n) = \theta(n^{\log_b a} \log^{p+1} n)$

    b. If $p = -1$, then $T(n) = \theta(n^{\log_b a} \log \log n)$

    c. If $p < -1$ then $T(n) = \theta(n^{\log_b a})$

Master's Theorem for Divide and Conquer

3) If $a<b^k$,

a. If $p \geqslant 0$, then $T(n) = \theta(n^k \log^p n)$

b. If $p < 0$, then $T(n) = \theta(n^k)$

Strassen’s method

- Make recursion tree slightly less bushy
- Instead of performing eight recursive multiplications of n/2 * n/2 matrices, it performs only seven
- Cost of eliminating one matrix multiplication will be some constant number of additions of n/2*n/2 matrices
- Constant number of matrix additions will be subsumed by θ notation

Strassen's method

- not at all obvious

Step 1: Divide the input matrices A and B and output matrix C into n/2 *n/2 submatrices as in SQUARE–MATRIX–MULTIPLY–RECURSIVE

This step takes θ (1) time by index calculation.

Strassen's method

Step 2: Create 10 matrices S_1, S_2,…,S_{10}, each of which is n/2 x n/2 and is the sum or difference of two matrices created in step 1.

We can create all 10 matrices in $\theta(n^2)$ time

Step 3: Using the submatrices created in step 1 and the 10 matrices created in step 2, recursively compute seven matrix products P_1,P_2 ,…, P_7 . Each matrix P_i is n/2*n/2.

Strassen’s method

Step 4: Compute the desired submatrices C_{11},C_{12},C_{21},C_{22} of the result matrix C by adding and subtracting various combinations of the P_i matrices

Compute all four submatrices in $\theta(n^2)$ time

Recurrence Relation of Strassen's method

- Once the matrix size n gets down to 1, we perform a simple scalar multiplication
- When $n > 1$, steps 1, 2, and 4 take a total of $\theta(n^2)$ time, and step 3 requires us to perform seven multiplications of $n/2 \times n/2$ matrices
- $T(n) = \theta(1)$ if $n = 1$
- $\quad = 7 * T(n/2) + \theta(n^2) \Rightarrow O(n^{\log 7})$

Strassen's method

10 matrices created in step 2 are:

$S_1 = B_{12} - B_{22}$

$S_2 = A_{11} + A_{12}$

$S_3 = A_{21} + A_{22}$

$S_4 = B_{21} - B_{11}$

$S_5 = A_{11} + A_{22}$

$S_6 = B_{11} + B_{22}$

Strassen's method

$S_7 = A_{12} - A_{22}$

$S_8 = B_{21} + B_{22}$

$S_9 = A_{11} - A_{21}$

$S_{10} = B_{11} + B_{12}$

Since we must add or subtract n/2 x n/2 matrices 10 times, this step does indeed take $O(n^2)$ time.

Strassen's method

In step 3, we recursively multiply n/2 x n/2 matrices seven times to compute the following n/2 x n/2 matrices, each of which is the sum or difference of products of A and B submatrices:

Step 3 Strassen's method

$P_1 = A_{11} \, . \, S_1 = A_{11}.B_{12} - A_{11}.B_{22}$

$P_2 = S_2 \, .B_{22} = A_{11}.B_{22} + A_{12}.B_{22}$

$P_3 = S_3.B_{11} = A_{21}.B_{11} + A_{22}.B_{11}$

$P_4 = A_{22}.S_4 = A_{22}.B_{21} - A_{22}.B_{11}$

$P_5 = S_5 \, . \, S_6 = A_{11}.B_{11} + A_{11}.B_{22} + A_{22}.B_{11} + A_{22}.B_{22}$

$P_6 = S_7 \, . \, S_8 = A_{12}.B_{21} + A_{12}.B_{22} - A_{22}.B_{21} - A_{22}.B_{22}$

$P_7 = S_9 \, . \, S_{10} = A_{11}.B_{11} + A_{11}.B_{12} - A_{21}.B_{11} - A_{21}.B_{12}$

Step 4 Strassen's method

adds and subtracts the P_i matrices created in step 3 to construct the four n/2 x n/2 submatrices of the product C

$$C_{11} = P_5 + P_4 - P_2 + P_6$$

Step 4 Strassen's method

$$
\begin{array}{rrrrrrr}
A_{11}\cdot B_{11} & +\,A_{11}\cdot B_{22} & +\,A_{22}\cdot B_{11} & +\,A_{22}\cdot B_{22} & & & \\
 & & -\,A_{22}\cdot B_{11} & & +\,A_{22}\cdot B_{21} & & \\
 & -\,A_{11}\cdot B_{22} & & & & -\,A_{12}\cdot B_{22} & \\
 & & & -\,A_{22}\cdot B_{22} & -\,A_{22}\cdot B_{21} & +\,A_{12}\cdot B_{22} & +\,A_{12}\cdot B_{21} \\
\hline
A_{11}\cdot B_{11} & & & & & & +\,A_{12}\cdot B_{21}\ ,
\end{array}
$$

Step 4 Strassen's method

$C_{12} = P_1 + P_2$

$$
\begin{array}{l}
A_{11} \cdot B_{12} - A_{11} \cdot B_{22} \\
\qquad\qquad + A_{11} \cdot B_{22} + A_{12} \cdot B_{22} \\
\hline
A_{11} \cdot B_{12} \qquad\qquad\qquad + A_{12} \cdot B_{22} ,
\end{array}
$$

Step 4 Strassen's method

$C_{21} = P_3 + P_4$

$$
\begin{array}{l}
A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\
\qquad\qquad - A_{22} \cdot B_{11} + A_{22} \cdot B_{21} \\
\hline
A_{21} \cdot B_{11} \qquad\qquad\qquad\quad + A_{22} \cdot B_{21} ,
\end{array}
$$

Step 4 Strassen's method

$C_{22} = P_5 + P_1 - P_3 - P_7$

$$
\begin{array}{l}
A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\
\qquad\quad - A_{11} \cdot B_{22} \qquad\qquad\qquad\qquad\quad + A_{11} \cdot B_{12} \\
\qquad\qquad\qquad\quad - A_{22} \cdot B_{11} \qquad\qquad\qquad\qquad\quad - A_{21} \cdot B_{11} \\
- A_{11} \cdot B_{11} \qquad\qquad\qquad\qquad\qquad\quad - A_{11} \cdot B_{12} + A_{21} \cdot B_{11} + A_{21} \cdot B_{12} \\
\hline
\qquad\qquad\qquad\qquad\qquad A_{22} \cdot B_{22} \qquad\qquad\qquad\qquad\quad + A_{21} \cdot B_{12} \, ,
\end{array}
$$

Altogether, we add or subtract n/2 x n/2 matrices eight times in step 4, and so this step indeed takes $\theta\ (n^2)$ time