Computational Geometry

Computational Geometry – Introduction

- Branch of computer science that studies algorithms for solving geometric problems
- Modern engineering and mathematics, computational geometry has applications in such diverse fields as computer graphics, robotics, VLSI design, computer-aided design, molecular modeling, metallurgy, manufacturing, textile layout, forestry, and statistics

Computational Geometry – Introduction

- Input – Description of a set of geometric objects, such as a set of points, a set of line segments, or the vertices of a polygon in counterclock–wise order
- Output – a response to a query about the objects, such as whether any of the lines intersect, or perhaps a new geometric object, such as the convex hull (smallest enclosing convex polygon) of the set of points.

Computational Geometry – Introduction

- We look at a few computational-geometry algorithms in two dimensions, that is, plane
- We represent each input object by a set of points $\{p_1, p_2, p_3, \ldots\}$, where each $p_i = (x_i , y_i)$ and $x_i, y_i \in R$
- For example, we represent an n-vertex polygon P by a sequence $<p_0, p_1, p_2, \ldots, p_{n-1}>$ of its vertices in order of their appearance on the boundary of P

Computational Geometry – Introduction

- Computational geometry can also apply to three dimensions, and even higher–dimensional spaces, but such problems and their solutions can be very difficult to visualize

Line–segment properties

- A convex combination of two distinct points $p_1 = (x_1 , y_1)$ and $p_2 = (x_2, y_2)$ is any point $p_3 = (x_3 , y_3)$ such that for some α in the range $0 \leq \alpha \leq 1$, we have $x_3 = \alpha x_1 + (1 - \alpha) x_2$ and $y_3 = \alpha y_1 + (1 - \alpha) y_2$
- We also write that $p_3 = \alpha p_1 + (1 - \alpha) p_2$
- Intuitively, p_3 is any point that is on the line passing through p_1 and p_2 and is on or between p_1 and p_2 on the line

Line–segment properties

- Given two distinct points p_1 and p_2 , the line segment $\overline{p_1 p_2}$ is th e set of convex combinations of p_1 and p_2
- We call p_1 and p_2 the endpoints of segment $\overline{p_1 p_2}$
- Sometimes the ordering of p_1 and p_2 matters, and we speak of the directed segment $\overrightarrow{p_1 p_2}$.
- If p_1 is the origin (0, 0), then we can treat the directed segment $\overrightarrow{p_1 p_2}$ as the vector p 2

Line–segment Questions

- Given two directed segments, $\overrightarrow{p_0 p_1}$ and $\overrightarrow{p_0 p_2}$, is $\overrightarrow{p_0 p_1}$ clockwise from $\overrightarrow{p_0 p_2}$ with respect to their common endpoint p_0 ?
- Given two line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$, if we traverse $\overline{p_0 p_1}$ and then $\overline{p_1 p_2}$, do we make a left turn at point p_1 ?
- Do line segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ intersect?

Line–segment Questions

- Can answer each question in O(1) time
- No surprise since the input size of each question is O(1)
- Our methods use only additions, subtractions, multiplications, and comparisons
- Need neither division nor trigonometric functions, both of which can be computationally expensive and prone to problems with round–off error

Line–segment Questions

- For example, the “straightforward” method of determining whether two segments intersect—compute the line equation of the form $y = mx + b$ for each segment (m is the slope and b is the y–intercept),
- Find point of intersection of lines, and check whether this point is on both segments—uses division to find the point of intersection

Line–segment Questions

- When segments are nearly parallel, this method is very sensitive to precision of division operation on real computers
- The method in this section, which avoids division, is much more accurate

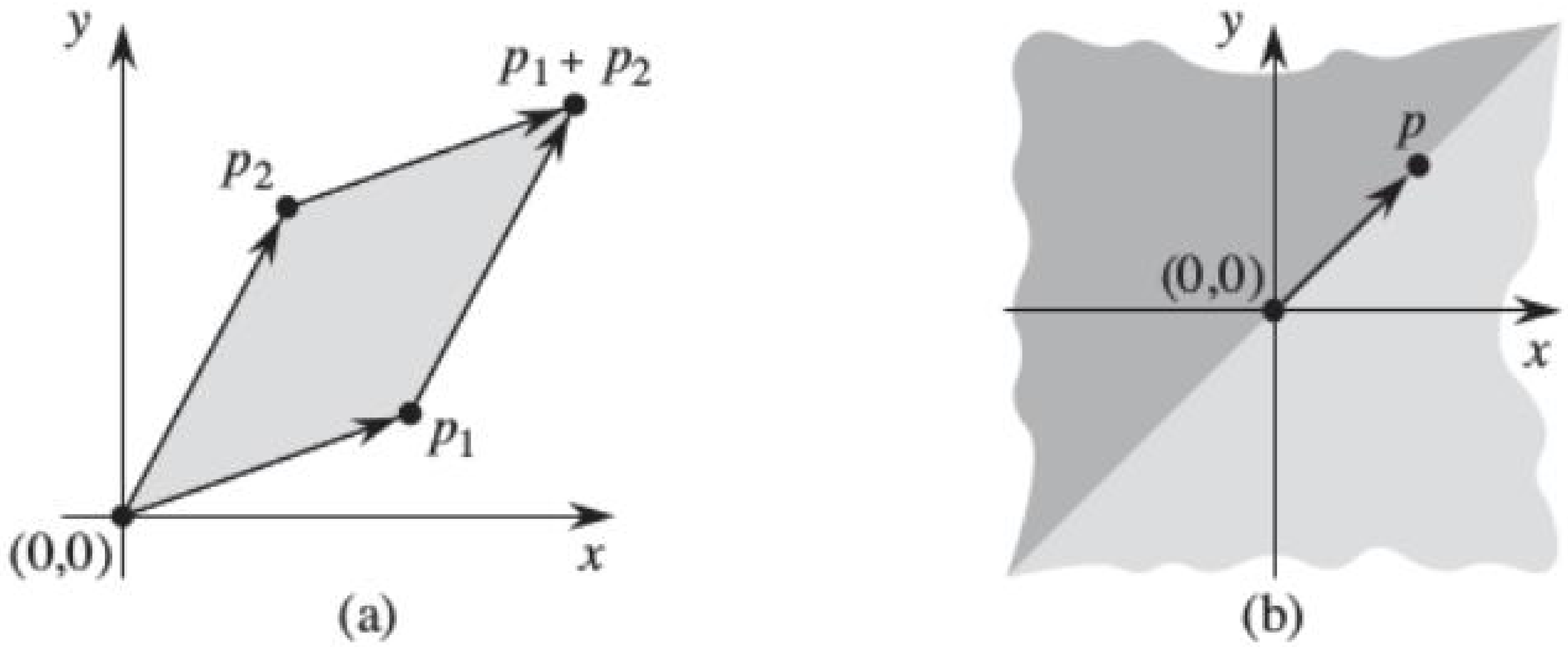
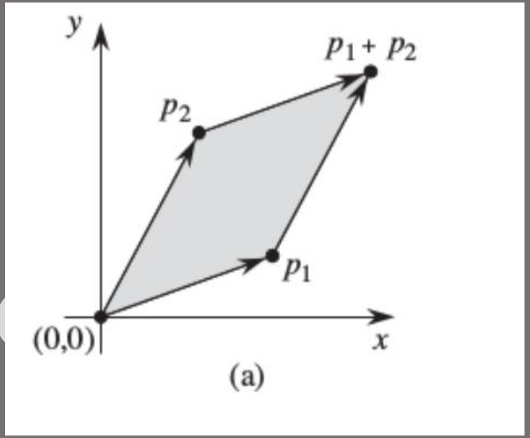


Figure 33.1 **(a)** The cross product of vectors p_1 and p_2 is the signed area of the parallelogram. **(b)** The lightly shaded region contains vectors that are clockwise from p. The darkly shaded region contains vectors that are counterclockwise from p.

Cross products

- Computing cross products lies at the heart of our line–segment methods
- Consider vectors p_1 and p_2, shown in Figur
- We can interpret the cross product p_1 x p_2 as the signed area of the parallelogram formed by the points (0, 0), p_1 , p_2, and $p_1 + p_2 = (x_1 + x_2 , y_1 + y_2)$



(a)

Cross products

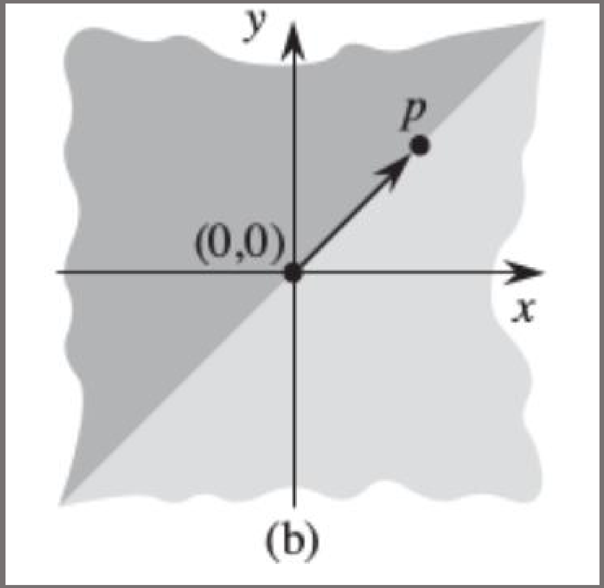
- An equivalent, but more useful, definition gives the cross product as the determinant of a matrix:

$$\begin{aligned} p_1 \times p_2 &= \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \\ &= x_1 y_2 - x_2 y_1 \\ &= -p_2 \times p_1 \,. \end{aligned}$$

- If p_1 X p_2 is positive, then p_1 is clockwise from p_2 with respect to the origin (0, 0), if this cross product is negative, then p_1 is counterclockwise from p_2

Cross products

- Figure 33.1(b) shows the clockwise and counterclockwise regions relative to a vector p
- A boundary condition arises if the cross product is 0; in this case, the vectors are colinear, pointing in either the same or opposite directions



Cross products

- To determine whether a directed segment $\overrightarrow{p_0 p_1}$ is closer to a directed segment $\overrightarrow{p_0 p_2}$ in a clockwise direction or in a counterclockwise direction with respect to their common endpoint p_0, we simply translate to use p_0 as the origin.

Cross products

- That is, we let $p_1 - p_0$ denote the vector $p'_1 = (x'_1, y'_1)$, where $x'_1 = x_1 - x_0$ and $y'_1 = y_1 - y_0$, and we define $p_2 - p_0$ similarly
- We then compute the cross product
- $(p_1 - p_0) \times (p_2 - p_0) = (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)$
- If this cross product is positive, then $\overrightarrow{p_0 p_1}$ is clockwise from $\overrightarrow{p_0 p_2}$; if negative, it is counterclockwise

Determining whether consecutive segments turn left or right

- Whether two consecutive line segments p_0p_1 and p_1p_2 turn left or right at point p_1
- Equivalently, we want a method to determine which way a given angle $\angle p_0 p_1 p_2$ turns
- Cross products allow us to answer this question without computing the angle.

Determining whether consecutive segments turn left or right

- As Figure 33.2 shows, we simply check whether directed segment $\overrightarrow{p_0 p_2}$ is clockwise or counterclockwise relative to directed segment $\overrightarrow{p_0 p_1}$
- To do so, we compute the cross product $(p_2 - p_0) \times (p_1 - p_0)$
- If the sign of this cross product is negative, then $\overrightarrow{p_0 p_2}$ is counterclockwise with respect to $\overrightarrow{p_0 p_1}$, and thus we make a left turn at p_1

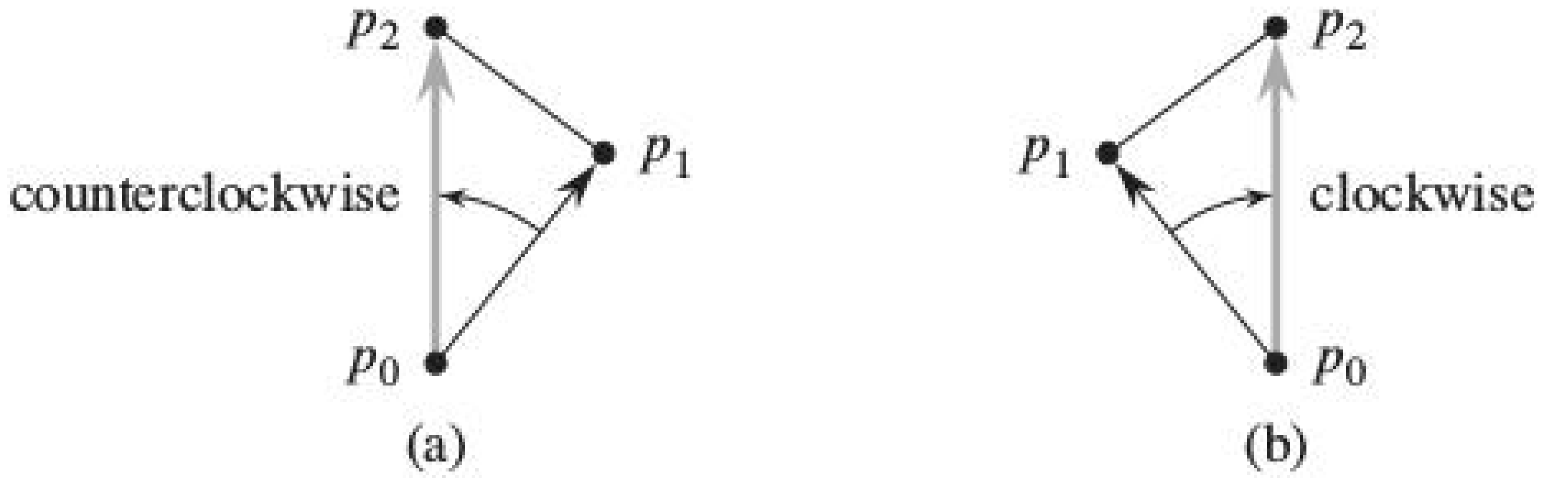


Figure 33.2 Using the cross product to determine how consecutive line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$ turn at point p_1. We check whether the directed segment $\overrightarrow{p_0 p_2}$ is clockwise or counterclockwise relative to the directed segment $\overrightarrow{p_0 p_1}$. **(a)** If counterclockwise, the points make a left turn. **(b)** If clockwise, they make a right turn.

Determining whether consecutive segments turn left or right

- A positive cross product indicates a clockwise orientation and a right turn
- A cross product of 0 means that points p_0, p_1, and p_2 are colinear

Determining whether two line segments intersect

- To determine whether two line segments intersect, we check whether each segment straddles the line containing the other
- A segment $\overline{p_1 p_2}$ straddles a line if point p_1 lies on one side of the line and point p_2 lies on the other side
- Two line segments intersect if and only if either (or both) of the following conditions holds:

Determining whether two line segments intersect

1. Each segment straddles the line containing the other.
2. An endpoint of one segment lies on the other segment. (This condition comes from the boundary case.)

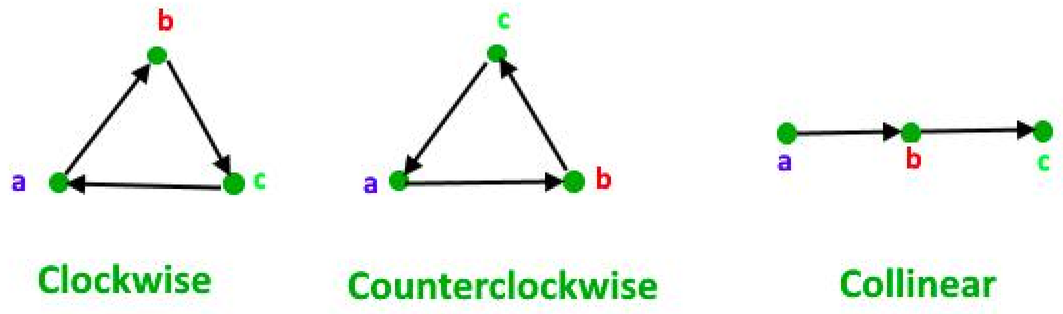
Determining whether two line segments intersect

- The following procedures implement this idea. SEGMENTS – INTERSECT returns TRUE if segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ intersect and FALSE if they do not
- It calls the subroutines DIRECTION, which computes relative orientations using the cross–product method above, and ON–SEGMENT , which determines whether a point known to be colinear with a segment lies on that segment.

```
SEGMENTS-INTERSECT($p_1, p_2, p_3, p_4$)
$d_1$ = DIRECTION($p_3, p_4, p_1$)
$d_2$ = DIRECTION($p_3, p_4, p_2$)
$d_3$ = DIRECTION($p_1, p_2, p_3$)
$d_4$ = DIRECTION($p_1, p_2, p_4$)
if (($d_1 > 0$ and $d_2 < 0$) or ($d_1 < 0$ and $d_2 > 0$)) and
        (($d_3 > 0$ and $d_4 < 0$) or ($d_3 < 0$ and $d_4 > 0$))
    return TRUE
elseif $d_1 == 0$ and ON-SEGMENT($p_3, p_4, p_1$)
    return TRUE
elseif $d_2 == 0$ and ON-SEGMENT($p_3, p_4, p_2$)
    return TRUE
elseif $d_3 == 0$ and ON-SEGMENT($p_1, p_2, p_3$)
    return TRUE
elseif $d_4 == 0$ and ON-SEGMENT($p_1, p_2, p_4$)
    return TRUE
else return FALSE
```

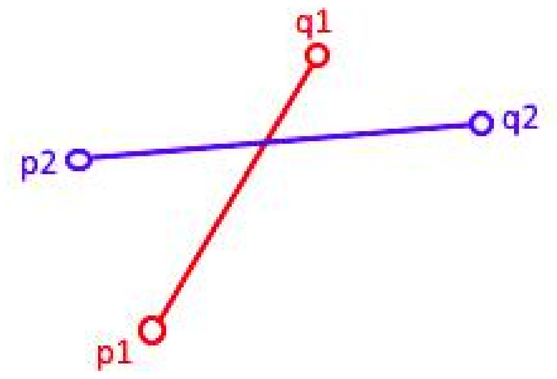
```
DIRECTION($p_i, p_j, p_k$)
1    return $(p_k - p_i) \times (p_j - p_i)$
```

```
ON-SEGMENT($p_i, p_j, p_k$)
1    if $\min(x_i, x_j) \le x_k \le \max(x_i, x_j)$ and $\min(y_i, y_j) \le y_k \le \max(y_i, y_j)$
2        return TRUE
3    else return FALSE
```

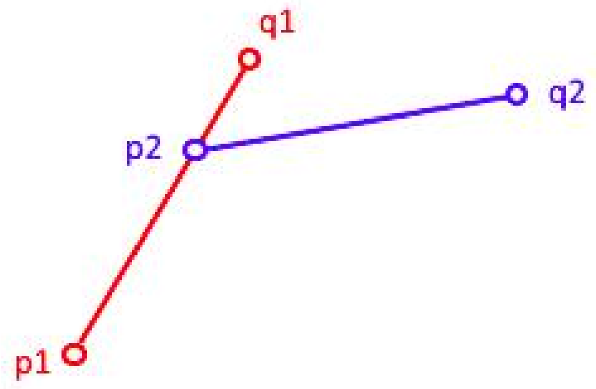

b
a
c
Clockwise
c
a
b
Counterclockwise
a
b
c
Collinear

Determining whether two line segments intersect

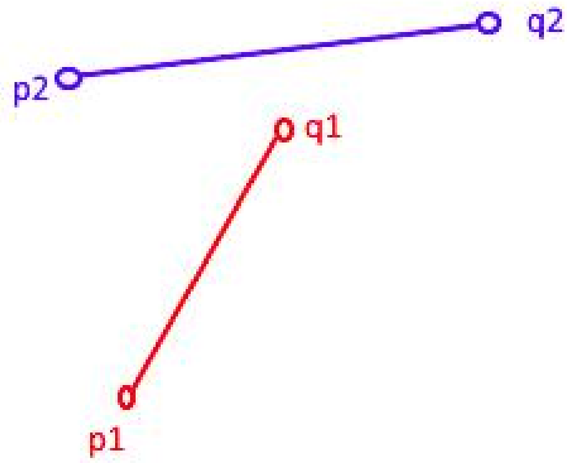
- General Case:
- – (p1, q1, p2) and (p1, q1, q2) have different orientations and
- – (p2, q2, p1) and (p2, q2, q1) have different orientations.



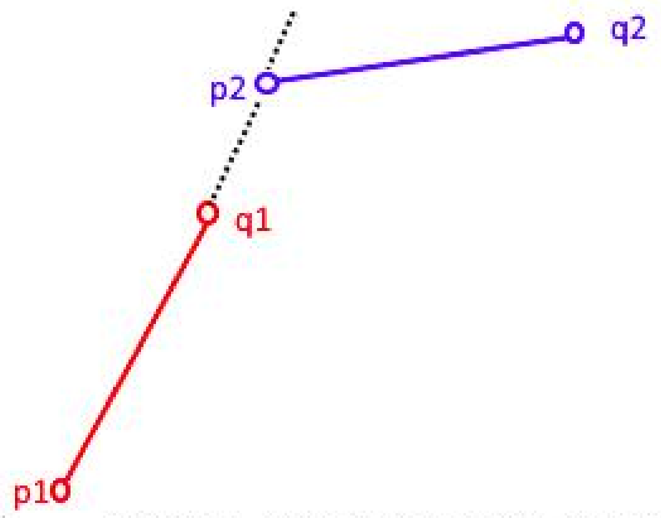
Example : Orientations of (p1,q1,p2) and (p1,q1,q2) are different. Orientations of (p2,q2,p1) and (p2,q2,q1) also differnet.



Example: Orientations of (p1,q1,p2) and (p1,q1,q2) are different. Orientations of (p2,q2,p1) and (p2,q2,q1) also different



Example: Orientations of (p1,q1,p2) and (p1,q1,q2) are different. Orientations of (p2,q2,p1) and (p2,q2,q1) are same



Example: Orientations of (p1,q1,p2) and (p1,q1,q2) are different. Orientations of (p2,q2,p1) and (p2,q2,q1) are same.

Determining whether two line segments intersect

- S EGMENTS –I NTERSECT works as follows. Lines 1 – 4 compute the relative orientation di of each endpoint pi with respect to the other segment.
- If all the relative orientations are nonzero, then we can easily determine whether segments $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ intersect, as follows.

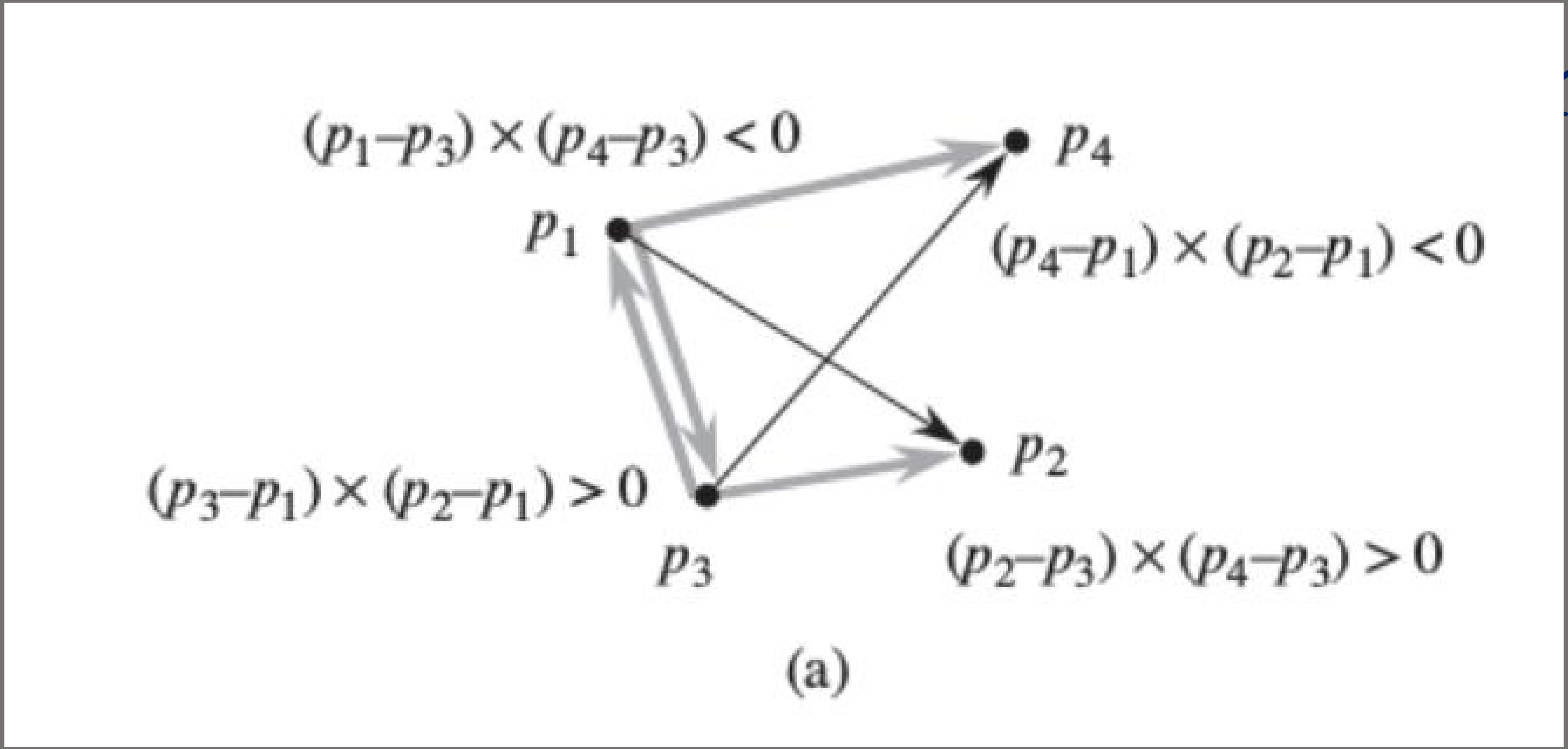
Determining whether two line segments intersect

- Segment $\overline{p_1 p_2}$ straddles the line containing segment $\overline{p_3 p_4}$ if if directed segments $\overrightarrow{p_3 p_1}$ and $\overrightarrow{p_3 p_2}$ have opposite orientations relative to $\overrightarrow{p_3 p_4}$
- In this case, the signs of d_1 and d_2 differ
- Similarly, $\overline{p_3 p_4}$ straddles the line containing $\overline{p_1 p_2}$ if the signs of d_3 and d_4 differ

Determining whether two line segments intersect

- If the test of line 5 is true, then the segments straddle each other, and S EGMENTS –I NTERSECT returns TRUE .
- Figure 33.3(a) shows this case

ntersect



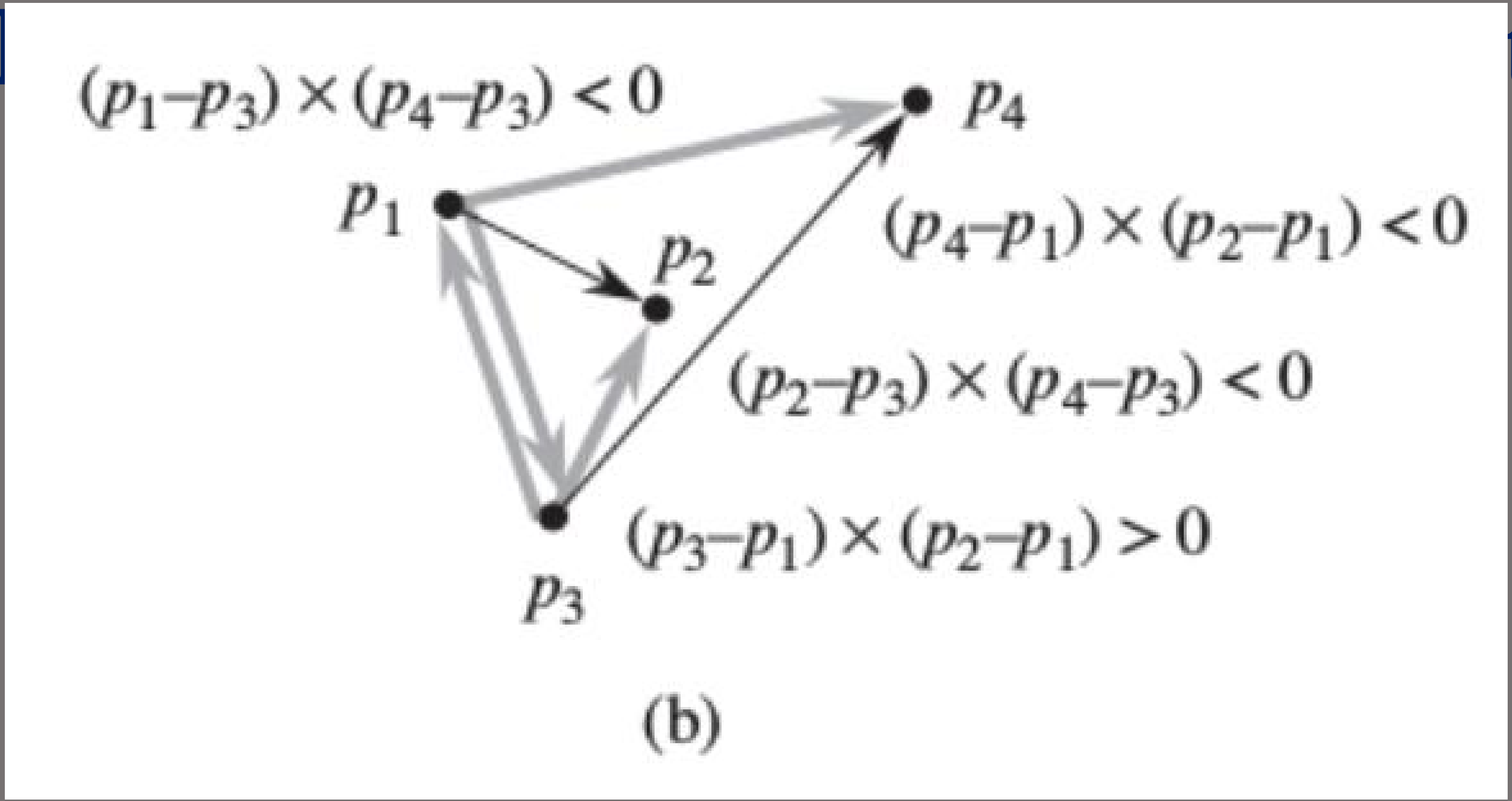
$(p_1-p_3)\times(p_4-p_3)<0$
p_4
p_1
$(p_4-p_1)\times(p_2-p_1)<0$
p_2
$(p_3-p_1)\times(p_2-p_1)>0$
p_3
$(p_2-p_3)\times(p_4-p_3)>0$
(a)

- The segments p_1p_2 and p_3 p_4 straddle each other's lines. Because p_3p_4 straddles the line containing p_1p_2, the signs of the cross products $(p_3 - p_1)$ x $(p_2 - p_1)$ and $(p_4 - p_1)$ x $(p_2 - p_1)$ differ. Because p_1p_2 straddles the line containing p_3 p_4, the signs of the cross products $(p_1 - p_3)$ x $(p_4 - p_3)$ and $(p_2 - p_3)$ x $(p_4 - p_3)$ differ.

Determining whether two line segments intersect

- Otherwise, the segments do not straddle each other’s lines, although a boundary case may apply
- If all the relative orientations are nonzero, no boundary case applies.
- All the tests against 0 in lines 7 – 13 then fail, and SEGMENTS –I NTERSECT returns FALSE in line 15

ts intersect


$(p_1-p_3)\times(p_4-p_3)<0$
p_4
p_1
p_2
$(p_4-p_1)\times(p_2-p_1)<0$
$(p_2-p_3)\times(p_4-p_3)<0$
$(p_3-p_1)\times(p_2-p_1)>0$
p_3
(b)

- Segment p_3p_4 straddles the line containing p_1p_2 , but p_1 p_2 does not straddle the line containing p_3p_4 . The signs of the cross products $(p_1 - p_3) \times (p_4 - p_3)$ and $(p_2 - p_3) \times (p_4 - p_3)$ are the same.

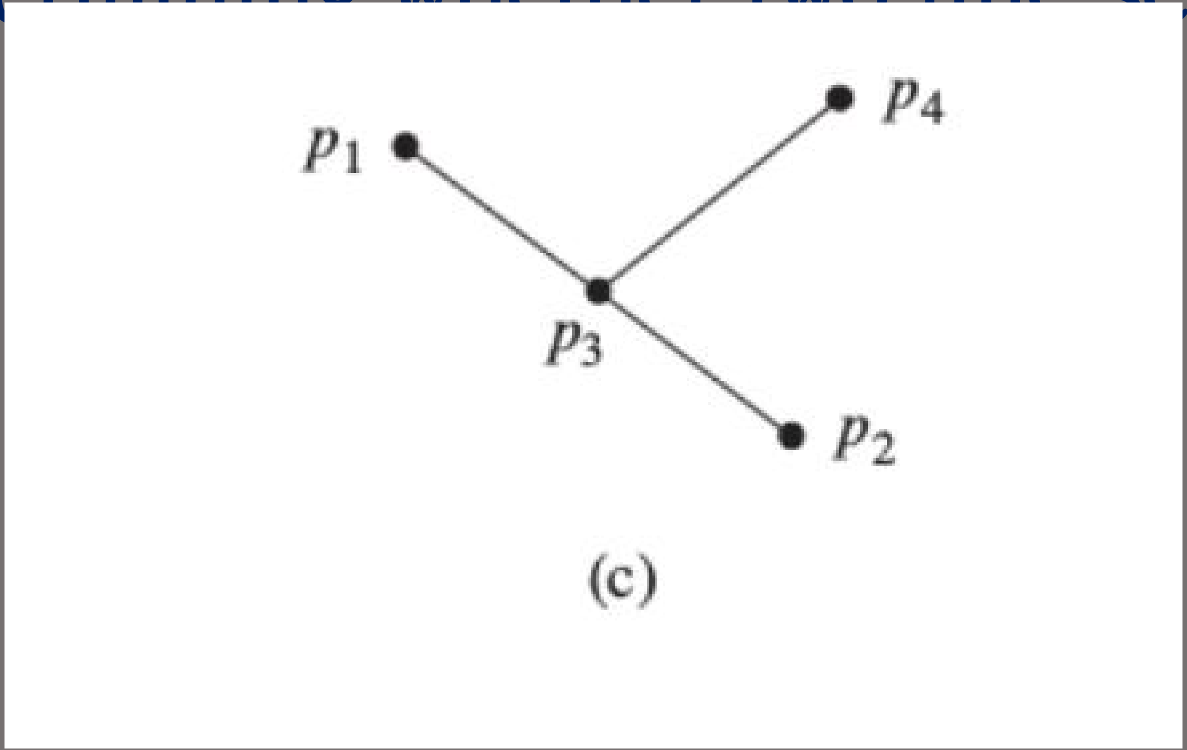
Determining whether two line segments intersect

- A boundary case occurs if any relative orientation d_k is 0.
- Here, we know that pk is colinear with the other segment.
- It is directly on the other segment if and only if it is between the endpoints of the other segment
- The procedure ON –S EGMENT returns whether p_k is between the endpoints of segment $\overline{p_i p_j}$

Determining whether two line segments intersect

- which will be the other segment when called in lines 7 – 13; the procedure assumes that p_k is colinear with segment $\overline{p_i p_j}$
- Figures 33.3(c) and (d) show cases with colinear points
- In Figure 33.3(c), p 3 is on $\overline{p_1 p_2}$,
- and so S EGMENTS –I NTERSECT returns TRUE in line 12.
- No endpoints are on other segments in Figure 33.3(d), and so SEGMENTS –INTERSECT returns FALSE in line 15.

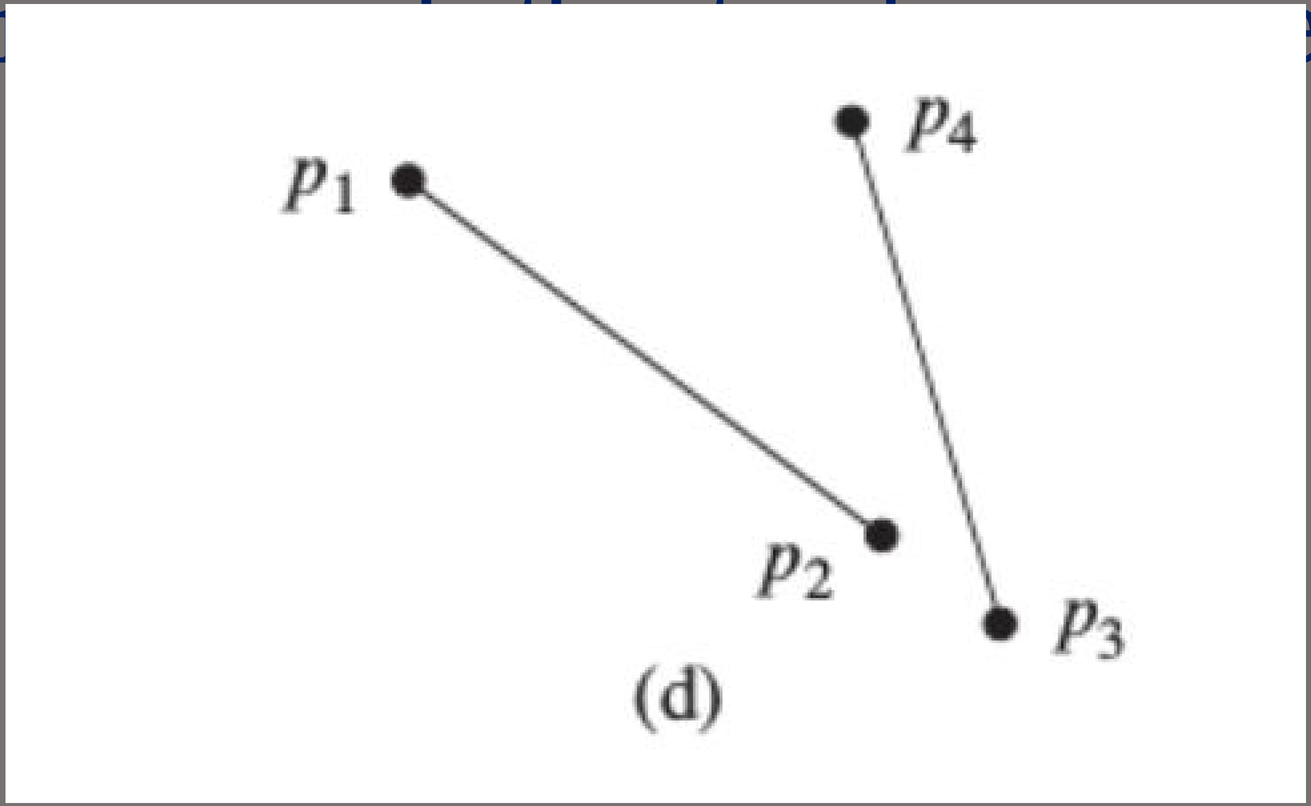
Determining whether two line segments intersect



(c)

- Point p 3 is colinear with p 1 p 2 and is between p 1 and p 2

Dete ents intersect



- Point p3 is colinear with p1p2 , but it is not between 1 and p 2 . The segments do not intersect

Other applications of cross products

- Later sections of this chapter introduce additional uses for cross products.
- In Section 33.3, we shall need to sort a set of points according to their polar angles with respect to a given origin.
- As Exercise 33.1–3 asks you to show, we can use cross products to perform the comparisons in the sorting procedure.

Other applications of cross products

- In Section 33.2, we shall use red–black trees to maintain the vertical ordering of a set of line segments.
- Rather than keeping explicit key values which we compare to each other in the red–black tree code, we shall compute a cross–product to determine which of two segments that intersect a given vertical line is above the other.

Determining whether any pair of segments intersects

- An algorithm to determine whether any two line segments in a set of segments intersect
- uses a technique known as "sweeping," which is common to many computational–geometry algorithms
- algorithm runs in O(n lg n) time, where n is the number of segments given
- check only if any intersection exists

Determining whether any pair of segments intersects

- To find all intersections, it takes $\Omega(n^2)$ time in the worst case

Sweeping Algorithm

- an imaginary vertical sweep line passes through the given set of geometric objects, usually from left to right
- x–dimension is taken as a dimension of time
- Provides a method for ordering geometric objects, usually by placing them into a dynamic data structure, and for taking advantage of relationships among them

Sweeping Algorithm

- Line–segment–intersection algorithm considers all the line–segment endpoints in left–to–right order and checks for an intersection each time it encounters an endpoint

Assumptions

1. No input segment is vertical
2. No three input segments intersect at a single point

Ordering Segments

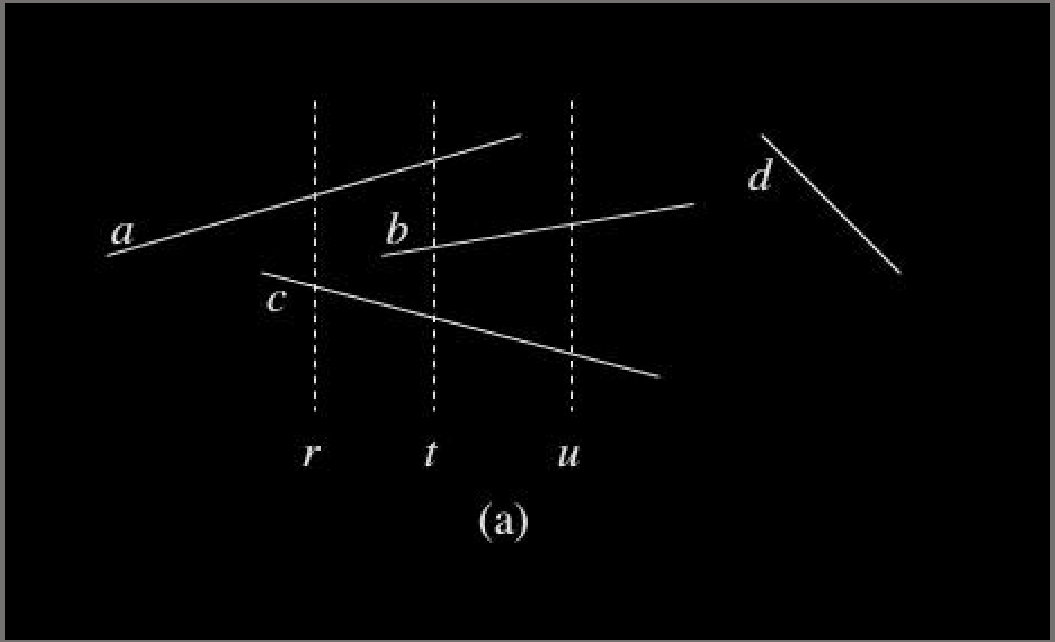
- Any input segment will intersect a given vertical sweep line at a single point
- Thus, we can order the segments that intersect a vertical sweep line according to the y–coordinates of the points of intersection.

Ordering Segments

- To be more precise, consider two segments s_1 and s_2
- We say that these segments are comparable at x if the vertical sweep line with x–coordinate 'x' intersects both of them
- We say that s_1 is above s_2 at x, written $s_1 \geq_x s_2$, if s_1 and s_2 are comparable at x and the intersection of s_1 with the sweep line at x is higher than the intersection of s_2 with the same sweep line, or if s_1 and s_2 intersect at the sweep line

Ordering Segments

- In Figure 33.4(a), for example, we have the relationships a $\geq_r$ c, a $\geq_t$ b, b $\geq_t$ c, a $\geq_t$ c, and b $\geq_u$ c
- Segment d is not comparable with any other segment.



Ordering Segments

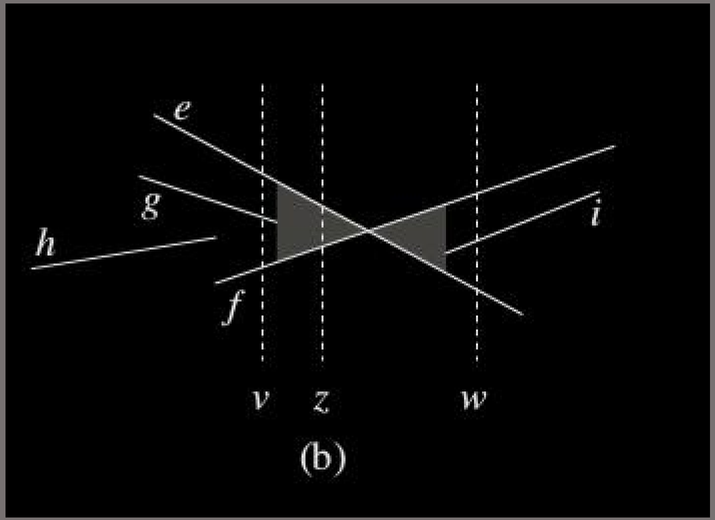
- For any given x, the relation "$\geq_x$" is a total preorder (see Section B.2) for all segments that intersect the sweep line at x
- That is, the relation is transitive, and if segments s_1 and s_2 each intersect the sweep line at x, then either $s_1 \geq_x s_2$ or $s_2 \geq_x s_1$, or both (if s_1 and s_2 intersect at the sweep line).

Ordering Segments

- The relation $\geq_x$ is also reflexive, but neither symmetric nor antisymmetric
- The total preorder may differ for differing values of x, however, as segments enter and leave the ordering.
- A segment enters the ordering when its left endpoint is encountered by the sweep, and it leaves the ordering when its right endpoint is encountered

Ordering Segments

- What happens when the sweep line passes through the intersection of two segments?
- As Figure 33.4(b) shows, the segments reverse their positions in the total preorder.



Ordering Segments

- Sweep lines v and w are to the left and right, respectively, of the point of intersection of segments e and f , and we have e$\geq_v$ f and f $\geq_w$ e.
- Note that because we assume that no three segments intersect at the same point, there must be some vertical sweep line x for which intersecting segments e and f are consecutive in the total preorder $\geq_x$.

Ordering Segments

- Any sweep line that passes through the shaded region of Figure 33.4(b), such as z, t has e and f consecutive in its total preorder.

Moving the sweep line

Sweeping algorithms typically manage two sets of data:

1. The sweep–line status gives the relationships among the objects that the sweep line intersects.
2. The event–point schedule is a sequence of points, called event points, which we order from left to right according to their x–coordinates.

Moving the sweep line

As the sweep progresses from left to right, whenever the sweep line reaches the x–coordinate of an event point, the sweep halts, processes the event point, and then resumes. Changes to the sweep–line status occur only at event points.

Moving the sweep line

For some algorithms (the algorithm asked for in Exercise 33.2–7, for example), the event–point schedule develops dynamically as the algorithm progresses.

The algorithm at hand, however, determines all the event points before the sweep, based solely on simple properties of the input data. In particular, each segment endpoint is an event point.

Moving the sweep line

We sort the segment endpoints by increasing x–coordinate and proceed from left to right.

(If two or more endpoints are covertical, i.e., they have the same x–coordinate, we break the tie by putting all the covertical left endpoints before the covertical right endpoints.

Moving the sweep line

Within a set of covertical left endpoints, we put those with lower y–coordinates first, and we do the same within a set of cover–tical right endpoints.) When we encounter a segment's left endpoint, we insert the segment into the sweep–line status, and we delete the segment from the sweep–line status upon encountering its right endpoint.

Moving the sweep line

Whenever two segments first become consecutive in the total preorder, we check whether they intersect.

Moving the sweep line

The sweep–line status is a total preorder T , for which we require the following operations:

INSERT (T, s): insert segment s into T .

DELETE (T, s): delete segment s from T .

ABOVE (T, s): return segment immediately above segment s in T .

BELOW (T, s): return segment immediately below segment s in T .

Moving the sweep line

- It is possible for segments s 1 and s 2 to be mutually above each other in the total preorder T ; this situation can occur if s 1 and s 2 intersect at the sweep line whose total preorder is given by T
- In this case, the two segments may appear in either order in T .

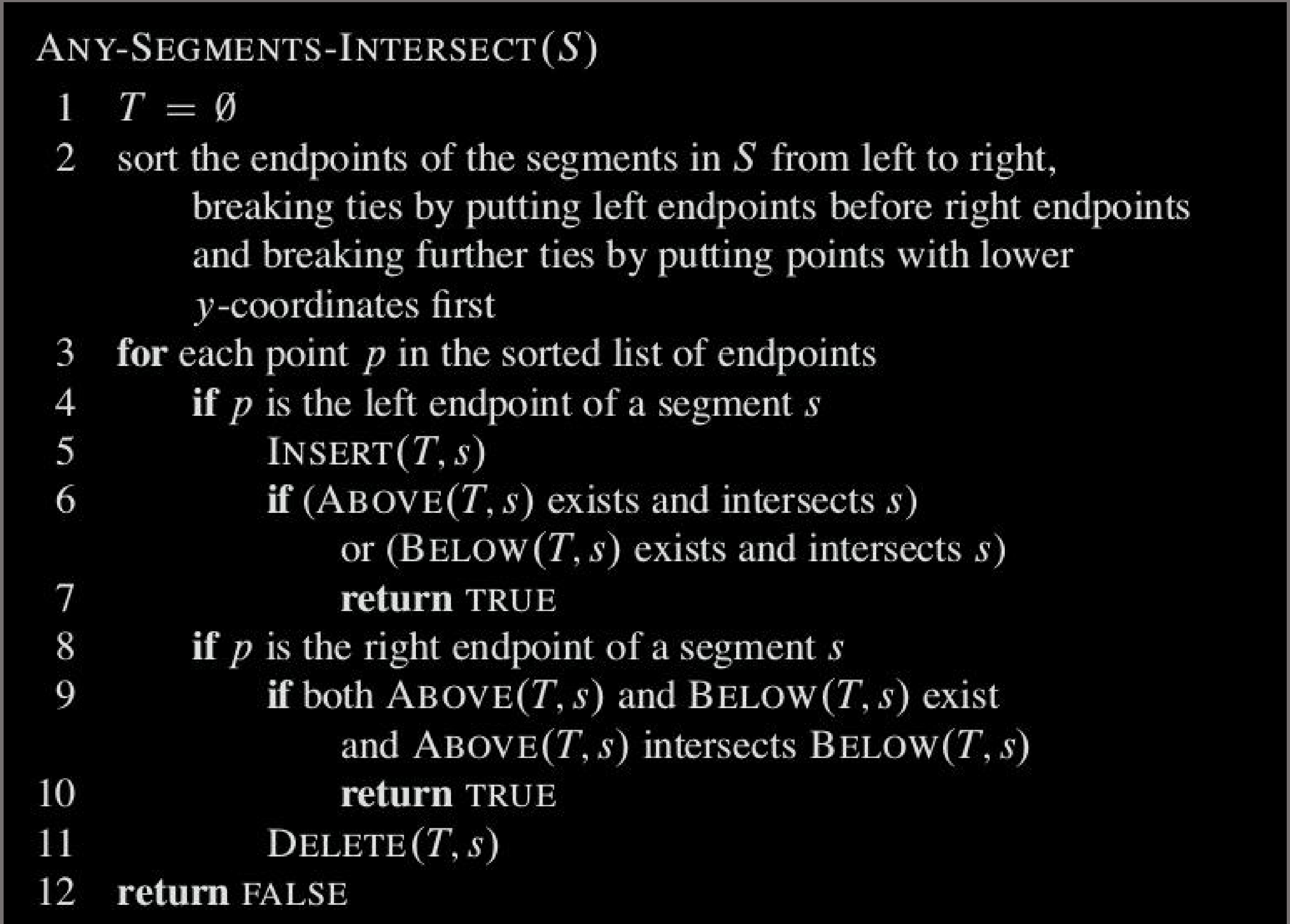
Moving the sweep line

- If the input contains n segments, we can perform each of the operations INSERT , D ELETE , A BOVE , and B ELOW in O(lg n) time using red–black trees.
- Recall that the red–black–tree operations in Chapter 13 involve comparing keys.
- We can replace the key comparisons by comparisons that use cross products to determine the relative ordering of two segments

Segment–intersection pseudocode

- Following algorithm takes as input a set S of n line segments, returning TRUE if any pair of segments in S intersects, and FALSE otherwise.
- A red–black tree maintains the total preorder T .

Segment–intersection pseudocode



```
ANY-SEGMENTS-INTERSECT($S$)
$T = \emptyset$
sort the endpoints of the segments in $S$ from left to right,
        breaking ties by putting left endpoints before right endpoints
        and breaking further ties by putting points with lower
        $y$-coordinates first
for each point $p$ in the sorted list of endpoints
    if $p$ is the left endpoint of a segment $s$
        INSERT($T, s$)
        if (ABOVE($T, s$) exists and intersects $s$)
                or (BELOW($T, s$) exists and intersects $s$)
            return TRUE
    if $p$ is the right endpoint of a segment $s$
        if both ABOVE($T, s$) and BELOW($T, s$) exist
                and ABOVE($T, s$) intersects BELOW($T, s$)
            return TRUE
        DELETE($T, s$)
return FALSE
```

Segment–intersection pseudocode

- Line 1 initializes the total preorder to be empty.
- Line 2 determines the event–point schedule by sorting the 2n segment endpoints from left to right, breaking ties as described above.
- One way to perform line 2 is by lexicographically sorting the endpoints on (x, e, y), where x and y are the usual coordinates, e = 0 for a left endpoint, and e = 1 for a right endpoint.

Segment–intersection pseudocode

- Each iteration of the for loop of lines 3 – 11 processes one event point p.
- If p is the left endpoint of a segment s, line 5 adds s to the total preorder, and lines 6 – 7 return TRUE if s intersects either of the segments it is consecutive with in the total preorder defined by the sweep line passing through p.

Segment–intersection pseudocode

- A boundary condition occurs if p lies on another segment s
- In this case, we require only that s and s' be placed consecutively into T .
- If p is the right endpoint of a segment s, then we need to delete s from the total preorder.

Segment–intersection pseudocode

- But first, lines 9 – 10 return TRUE if there is an intersection between the segments surrounding s in the total preorder defined by the sweep line passing through p.
- If these segments do not intersect, line 11 deletes segment s from the total preorder.
- If the segments surrounding segment s intersect, they would have become consecutive after deleting s had the return statement in line 10 not prevented line 11 from executing

Segment–intersection pseudocode

- Finally, if we never find any intersections after having processed all 2n event points, line 12 returns FALSE .

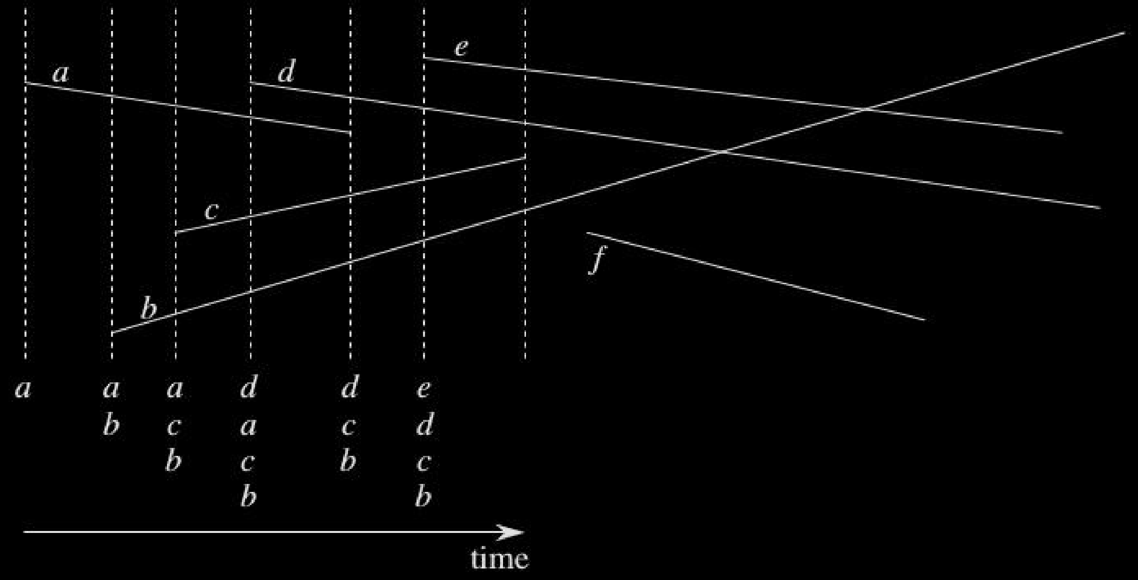


Figure 33.5 The execution of ANY-SEGMENTS-INTERSECT. Each dashed line is the sweep line at an event point. Except for the rightmost sweep line, the ordering of segment names below each sweep line corresponds to the total preorder T at the end of the **for** loop processing the corresponding event point. The rightmost sweep line occurs when processing the right endpoint of segment c; because segments d and b surround c and intersect each other, the procedure returns TRUE.