Computational Finance Assignment Report

*COMP6212 Computational Finance 2017/2018 Assignment Part 1 Report

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***Abstract*—In this paper, we solve a series of problems in portfolio optimisation and option pricing. Firstly we try different methods to derive the portfolio efficient frontier and then discuss index tracking and the transaction cost in portfolio optimisation. Secondly, we use Black-Scholes model and Radial Basis Function Network to do the option pricing and approximate some underlying variables.**

***Index Terms*—Portfolio optimisation, Efficient Frontier, FTSE100, Sparse Portfolio, Greedy Forward Selection, Option Pricing, Black-Scholes Model, Implied Volatility, Volatility Smile, RBF Network**

I. Introduction

In the Part A Portfolio Optimisation, we do some calculation on Markowitz Efficient Portfolio and practice with the MATLAB financial toolbox, CVX toolbox to plot the efficient frontier. We use the FTSE100 data and pick 30 stocks to compare the sample-based mean-variance model with 1/N Naive model. And compare the Greedy Forward Selection with Sparse Portfolio Selection method in selecting a part of the 30 FTSE stocks to track the FTSE100 index. Finally, we discuss the transaction cost to optimise the portfolio.

In the Part B Option Pricing, we firstly do some calculation on Black-Scholes model use the model to predict both call and put options with different strike prices. We plot the predictions, the corresponding implied volatilities and try to find the relation between the strike price and volatility. Eventually, we prove that the non-parametric model such as RBF network can be used in option pricing.

II. Part A: Portfolio Optimisation

A. Markowitz Efficient Frontier

The mean and covariance matrix are

$$\mathbf{m} = (0.10 \quad 0.10)^t, \tag{1}$$

$$C = \begin{pmatrix} 0.005 & 0 \\ 0 & 0.005 \end{pmatrix}. \tag{2}$$

Assuming the weight vector is $\pi = (p, 1-p)^t, p \in [0,1]$, since there are only two assets and the weights of all asset should add up to one.

Therefore, the expected return and variance are

$$E = \pi^t \mathbf{m} = (p, 1-p) \begin{pmatrix} 0.10 \\ 0.10 \end{pmatrix} = 0.10, \tag{3}$$

$$\begin{aligned} V &= \pi^t C \pi \\ &= (p, 1-p) \begin{pmatrix} 0.005 & 0 \\ 0 & 0.005 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} \\ &= 0.005(2p^2 - 2p + 1) \end{aligned} \tag{4}$$

The variance varies in [0.0025,0.005] and the expected return is always 0.10(Fig.1).

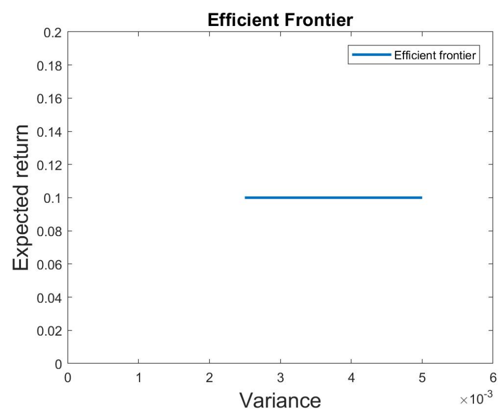

Fig. 1. The Efficient Frontier of the simple portfolio with return and covariance matrix specified in Eq.1 and Eq.2. The frontier is a piece of horizontal line

B. Portfolio Optimisation

1) Three Assets
Firstly, we generate 100 random portfolios π_i and the entries of each vector add up to one. With the portfolios, the specified mean

$$\mathbf{m} = (0.10 \quad 0.20 \quad 0.15)^t \tag{5}$$

and covariance matrix

$$C = \begin{pmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{pmatrix} \tag{6}$$

of the three assets, we can calculate the expected portfolio return $\pi^t \mathbf{m}$ and standard deviation $\pi^t C \pi$. Finally we plot the return and standard deviation combinations and efficient frontiers.

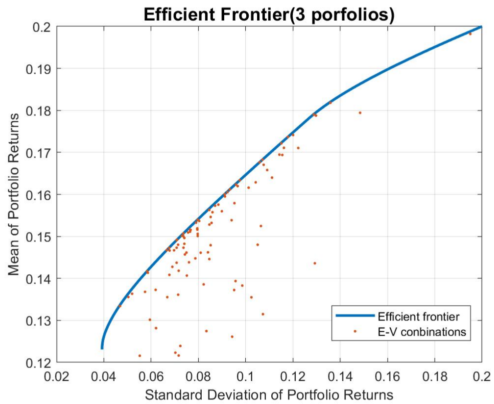

Fig. 2. Efficient Frontier of 3 Portfolios

2) Two Assets
Using the same method, we plot the E-SD combinations and efficient frontiers with only two of the three asset.

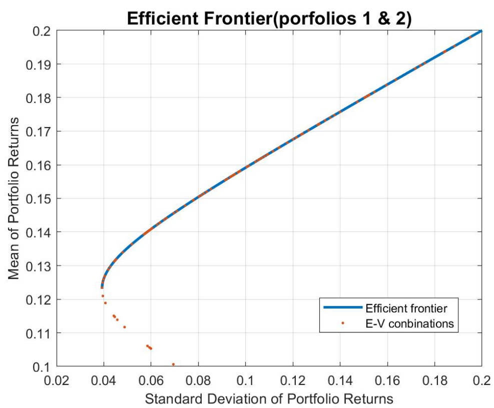

Fig. 3. Efficient Frontier of Portfolios 1 and 2

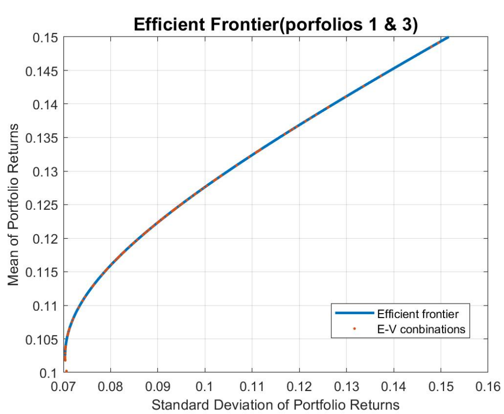

Fig. 4. Efficient Frontier of Portfolios 1 and 3

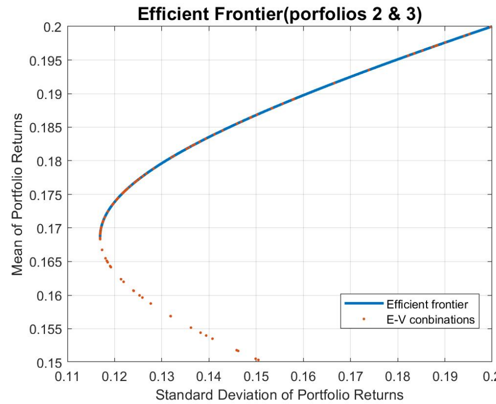

Fig. 5. Efficient Frontier of Portfolios 2 and 3

For portfolio made up by two assets, the expected returns and risks are fixed on the efficient frontier.

C. NaiveMV with CVX toolbox

In this part, we generate the efficient frontier by using the NaiveMV function with CVX toolbox and compare to which with linear programming and quadratic programming functions.

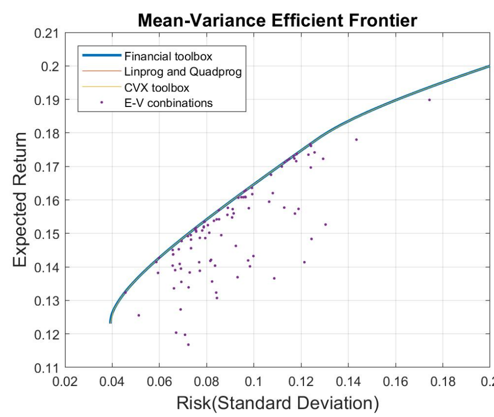

Fig. 6. Efficient Frontier by CVX toolbox vs Lingprog & Quadprog

As is shown on the Figure 6, the curves are quite overlapping and the mean-squared error between the two methods is 2.8273e-14. Therefore, the result can be regarded as identical.

D. FTSE100 data

From the 100 stocks of FTSE100 index, we choose 30 of them to do the analysis, and we download the data from the last three years(26-Feb-2015 to 23-Feb-2018).

We evaluate the 1/N Naive model and M-V model in terms of the expected return and risk. There are totally 4060 assets combinations if randomly picking three from the 30 assets. Therefore, we use the data from the first half of the time period to train the weights for every portfolios with the equation(2) that DeMiguel has discussed in [2]:

$$w = \frac{C^{-1}\mu}{\mathbf{1}_N C^{-1}\mu}. \tag{7}$$

And we derive the expected return and risk for the test set and finally plot the E-V graph for both method.

As is shown in Figure 7, generally speaking, the 1/N Naive method gives better expected return with less risk than the Sample-based Mean-Variance Model.

Therefore, the efficient portfolio constructed from M-V model does not consistently perform better than the 1/N Naive model.

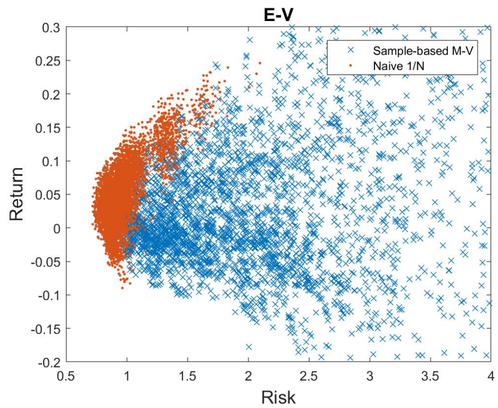

Fig. 7. The expected return is plot against the Risk(standard deviation) to compare the sample-based MV model(blue 'x' points) with 1/N Naive model(red dots).

E. Index Tracking

In this section, we use two methods to predict the expected return rate of portfolio. We choose six from the 30 stocks of FTSE100 index from last 3 years (26-Feb-2015 to 23-Feb-2018). And the first half time period of daily price change data is used in training the weight and we use the next half of data as the test set. The target value are set to be the FTSE100 data.

- Greedy Forward Selection
  We use the `stepwisefit` function and set the number of stocks we want to select to be six. In the Figure 8, the expected return rate from Greedy forward selection and the true FTSE 100 data is drawn.

Fig. 8. The predicted return by Greedy Forward Selection and compare with true return in test set time period.

  The Mean Squared Error of Greedy Forward Selection is 0.099848, the stocks 15,16,6,7,9,26 are selected. The risk(standard deviation) of this portfolio is 0.6424.
- Sparse Index Tracking
  We use `CVX toolbox` to solve the Sparse portfolio optimisation problem

$$
\begin{aligned}
min_w ||\rho \mathbf{1}_T - Rw||_2^2 + \tau ||w||_1 \\
subjecttow^T \mu = \rho \\
w^T \mathbf{1}_N = 1,
\end{aligned}
\tag{8}
$$

  where the τ is the regularization parameter, the R is the input data, w is the weight that minimize the risk subjected to the average return ρ as the expected return for the empirical implementation [4]. Firstly, we try different regularization parameter to find the value with which the optimisation solution could select out 6 stocks. As is shown on Figure 9, $\tau = 275$. Then we plot Sparse portfolio return with true FTSE 100 data.

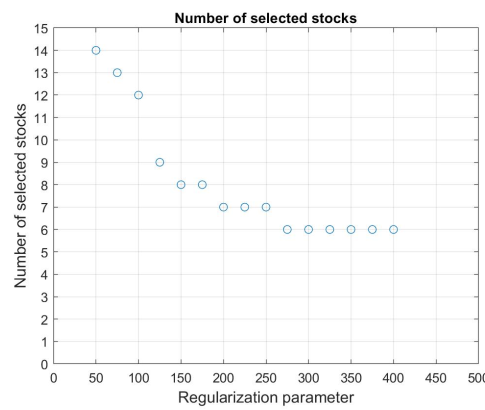

Fig. 9. Number of selected stocks with respect to regularization parameter

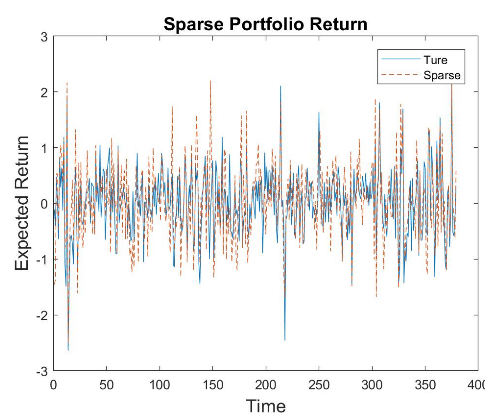

Fig. 10. Sparse Portfolio Return in test time period and compare which with the true value

The Mean Squared Error of Sparse portfolio selection is 0.23529, and the method select the stock 8, 17, 22, 26,27,29. And the risk (standard deviation) of thus portfolio is 0.7129.

In terms of the MSE and risk, the Greedy Forward Selection performs better than Sparse method.

F. Transaction Cost

Lobo *et al.* [5] introduce the portfolio optimisation with transaction cost and constraint on risk.

$$\begin{aligned} Max_x &\quad \bar{a}^T(w+x) \\ s.t. &\quad \mathbf{1}^T x + \Phi(x) \leq 0 \\ &\quad w + x \in S \end{aligned} \tag{9}$$

where w is the portfolio weights, $\bar{a}$ is the return, x is the value of portfolio, $\Phi(x)$ is the transaction cost, S is the feasible set.

The constraints are

- Budget
  $\mathbf{1}^T x + \Phi(x) \leq 0$ is the budget constraint.
- Transaction cost
  The transaction cost are separable for buying and selling assets and assume to be proportional to the portfolio value.

$$\Phi(x) = \sum_{i=1}^{n} \Phi_i(x_i), \tag{10}$$

$$\Phi_i(x_i) = \begin{cases} a_i^+ x_i & x_i \geq 0 \\ -a_i^- x_i & x_i \leq 0 \end{cases} \tag{11}$$

  where a_i^+ and a_i^- are cost rates for buying and selling asset i, and introduce the fixed costs

$$\Phi_i(x_i) = \begin{cases} 0 & x_i = 0 \\ b_i^+ + a_i^+ x_i & x_i \geq 0 \\ b_i^- - a_i^- x_i & x_i \leq 0 \end{cases} \tag{12}$$

- Diversification
  The amount of investment in any asset is limited to certain amount

$$w_i + x_i \leq p_i, i = 1, 2, ..., n, \tag{13}$$

  or proportional to the value of asset

$$w_i + x_i \leq \mathbf{1}^T(w+x), i = 1, 2, ..., n, \tag{14}$$

  or the investment amount is limited by group of assets

$$\sum_{i=1}^{r} (w_i + x_i) \leq \mathbf{1}^T(w+x) \tag{15}$$

- Short selling
  The constraints on short selling can be on individual asset

$$w_i + x_i \geq -s_i, i = 1, 2, ..., n, \tag{16}$$

  or be set on the total short selling assets

$$\sum_{i=1}^{n} (w_i + x_i)_- \leq S \tag{17}$$

- Variance
  The constrain on variance is the limit on risk we wish to take, is written as

$$(w+x)^T C (w+x) \leq \sigma_{max}, \tag{18}$$

  or can be written equally as

$$||C^{\frac{1}{2}}(w+x)|| \leq \sigma_{max}, \tag{19}$$

- Shortfall Risk
  We would like to set a constraint that at the end of period the wealth W should be larger than some value W^{low} with a probability greater than η. And we assume the returns $\bar{a}$ obey the joint Gaussian distribution, therefore the shortfall risk is

$$Prob(W \geq W^{low}) \geq \eta, \tag{20}$$

  which can be rewritten as

$$\Phi^{-1}(\eta)||C^{\frac{1}{2}}(w+x)|| \leq \bar{a}(w+x) - W^{low}, \tag{21}$$

  where $\eta \geq 0.5$. The fig.2 in [5] is the cumulative Gaussian distribution of the return shows that the probability we would like the return to be greater than specified value.

Taking all of the constraints into account, the optimisation problem can be written as

$$\begin{aligned} Max_x &\quad \bar{a}^T(w + x^+ - x^-) \\ s.t. &\quad \mathbf{1}^T(x^+ - x^-) + \sum_{i=1}^{n} \left(a_i^+ x_i^+ + a_i^- x_i^-\right) \leq 0, \\ &\quad x_i^+ \geq 0, x_i^- \geq 0, i = 1, 2, ..., n, \\ &\quad w + x^+ - x^- \geq s_i, i = 1, 2, ..., n, \\ &\quad \Phi^{-1}(\eta_j)||C^{\frac{1}{2}}(w+x)|| \leq \bar{a}(w+x) - W_j^{low}, j = 1, 2. \end{aligned} \tag{22}$$

To use this model, we can set the transaction costs for the FTSE 100 stocks and use the same methods to do the portfolio optimisation.

III. Part B: Option Pricing

A. Black-Scholes Differential Equation

1) Gaussian distribution
   The cumulative Normal distribution is

$$\begin{aligned} N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy \\ &= \frac{1}{2}\left(erf(\frac{x}{\sqrt{2}}) + 1\right). \end{aligned} \tag{23}$$

   The derivative is

$$\begin{aligned} N'(x) &= \frac{\partial N(x)}{\partial x} \\ &= \frac{1}{2}\frac{\partial}{\partial x}\left(erf(\frac{x}{\sqrt{2}}) + 1\right), \end{aligned} \tag{24}$$

   where erf(x) is the error function.
   Let

$$\begin{aligned} u &= \frac{x}{\sqrt{2}}, \\ \frac{d}{du}\left(erf(u)\right) &= \frac{2e^{-u^2}}{\sqrt{\pi}}. \end{aligned} \tag{25}$$

Then we have the derivative of Normal Distribution

$$N'(x) = \frac{\partial\left(N(x)\right)}{\partial x} = \frac{1}{2}\frac{2exp\left\{-\frac{x^2}{2}\cdot\frac{d}{dx}(\frac{x}{\sqrt{2}})\right\}}{\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{26}$$

2) Prove the equation

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2) \tag{27}$$

Firstly, substituting

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} \tag{28}$$

The RHS of Eq(27) can be written as

$$\begin{aligned} &Ke^{-r(T-t)}N'(d_2) \\ =&Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}exp(-\frac{1}{2}d_2^2) \end{aligned} \tag{29}$$

Substituting

$$d_2 = d_1 - \sigma\sqrt{T-t}, \tag{30}$$

we have the RHS of Eq(27)

$$\begin{aligned} &Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}\left(d_1-\sigma\sqrt{T-t}\right)^2\right) \\ =&Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}e^{d_1\sigma\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)} \end{aligned} \tag{31}$$

Substituting the following equations

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}, \tag{32}$$

$$d_1 = \frac{log(S/K) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{33}$$

the RHS of Eq(27) can be written as

$$\begin{aligned} &Ke^{-r(T-t)}N'(d_1)e^{log(S/K)+(r+\frac{\sigma^2}{2})(T-t)-\frac{1}{2}\sigma^2(T-t)} \\ =&Ke^{-r(T-t)}N'(d_1)e^{log(S/K)+r(T-t)} \\ =&SN'(d_1) \end{aligned} \tag{34}$$

The RHS and LHS of Eq(27) are equivalent.

3) Calculation on partial derivative of d_1 and d_2
We have the expression of d_1 and d_2 mentioned in Eq(33) and Eq(30).
The derivatives are

$$\begin{aligned} \frac{\partial d_1}{\partial S} =&\frac{\partial}{\partial S}\left(\frac{log(S/K) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ =&\frac{1}{\sigma\sqrt{T-t}}\frac{\partial}{\partial S}\left(log(S/K)\right) \\ =&\frac{1}{S\sigma\sqrt{T-t}} \end{aligned} \tag{35}$$

and

$$\begin{aligned} \frac{\partial d_2}{\partial S} =&\frac{\partial}{\partial S}\left(d_1 - \sigma\sqrt{T-t}\right) \\ =&-\frac{1}{S^2\sigma\sqrt{T-t}}. \end{aligned} \tag{36}$$

4) Calculate the partial derivative $\frac{\partial c}{\partial t}$
The call option price is given by

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2). \tag{37}$$

The derivative is

$$\begin{aligned} \frac{\partial c}{\partial t} =&\frac{\partial(SN(d_1))}{\partial t} - \frac{\partial}{\partial t}\left(Ke^{-r(T-t)}N(d_2)\right) \\ =&SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) \\ &- Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t} \end{aligned} \tag{38}$$

Substitute Eq(27) that we have proved before. The derivative can be written as

$$\begin{aligned} \frac{\partial c}{\partial t} =&SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) \\ &- SN'(d_1)\frac{\partial d_2}{\partial t} \end{aligned} \tag{39}$$

And substitute Eq(30) to calculate the last term of Eq(39), the derivative is

$$\begin{aligned} \frac{\partial c}{\partial t} =&SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) \\ &- SN'(d_1)\frac{\partial d_1 - \sigma\sqrt{T-t}}{\partial t} \\ =&-rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} \end{aligned} \tag{40}$$

5) Prove the equation

$$\frac{\partial c}{\partial S} = N(d_1) \tag{41}$$

From the Eq(37), the derivative of call price is

$$\begin{aligned} \frac{\partial c}{\partial S} =&\frac{\partial}{\partial S}\left(SN(d_1) - Ke^{-r(T-t)}N(d_2)\right) \\ =&N(d_1) + S\frac{\partial(N(d_1))}{\partial S} - Ke^{-r(T-t)}\frac{\partial(N(d_2))}{\partial S} \end{aligned} \tag{42}$$

Substituting Eq(27), we have the derivative

$$\begin{aligned} \frac{\partial c}{\partial S} =&N(d_1) + S\left(\frac{\partial N(d_1)}{\partial S} - \frac{N'(d_1)}{N'(d_2)}\frac{\partial N(d_2)}{\partial S}\right) \\ =&N(d_1) \end{aligned} \tag{43}$$

6) Calculate $\frac{\partial^2 c}{\partial S^2}$

$$\begin{aligned} \frac{\partial^2 c}{\partial S^2} =&\frac{\partial N(d_1)}{\partial S} \\ =&N'(d_1)\frac{\partial d_1}{\partial S} \end{aligned} \tag{44}$$

Substituting Eq(35) we have calculated previously, we get

$$\frac{\partial^2 c}{\partial S^2} = N'(d_1)\frac{1}{S\sigma\sqrt{T-t}} \tag{45}$$

7) Prove the call option price is the solution of equation

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} - rc = 0, \tag{46}$$

where c is the call option price.
Substituting Eq(40), (41), (45)and (37), it is obvious that the LHS of (46) equals to zero.
Therefore, the call option price is indeed the solution to the Black-Scholes differential equation.

B. Option Price Estimation

- Using the solution of Black-Scholes Eq(37) to predict the price of call and put options with different strike prices. Following graphs are the estimation of option with different strike price. We plot the estimation separately and calculate the mean squared error.

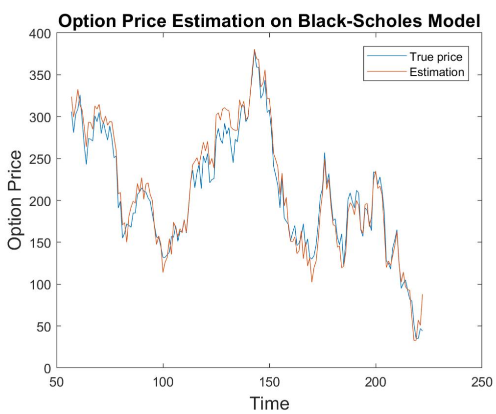

Fig. 11. Call option with strike price K= 2925. MSE =249.61

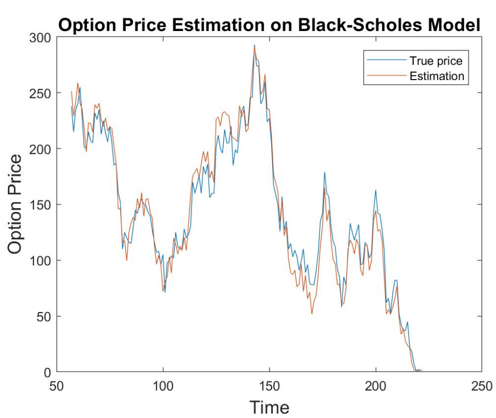

Fig. 12. Call option with strike price K = 3025. MSE =187.32

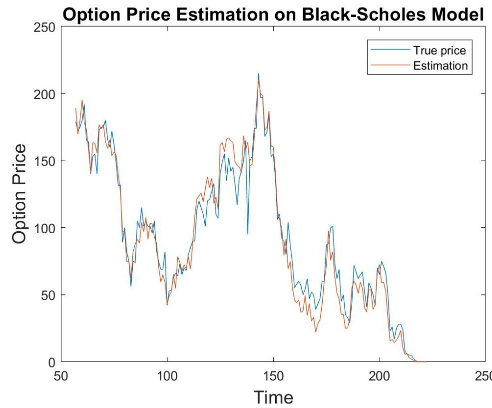

Fig. 13. Call option with strike price K= 3125. MSE = 162.00

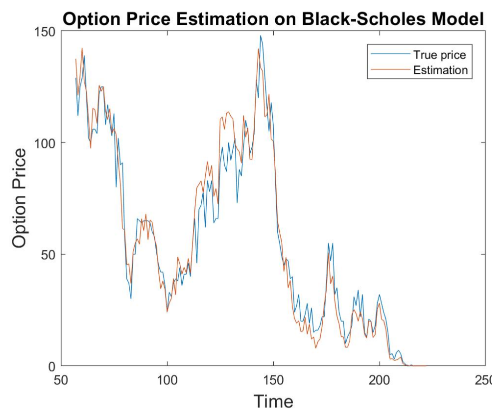

Fig. 14. Call option with strike price K= 3225. MSE =80.97

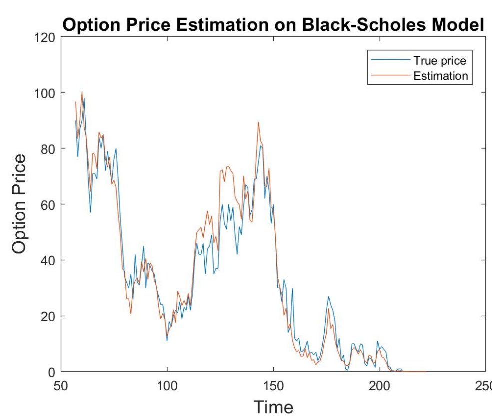

Fig. 15. Call option with strike price K= 3325. MSE =43.60

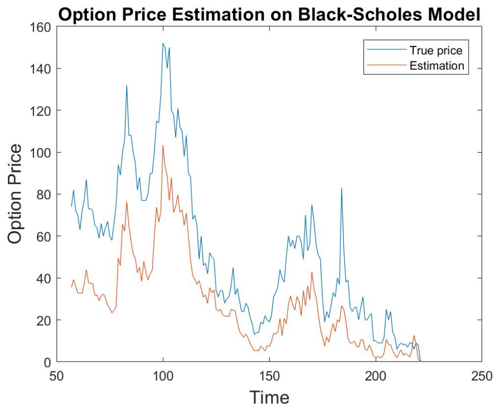

Fig. 16. Put option with strike price K= 2925. MSE =816.65

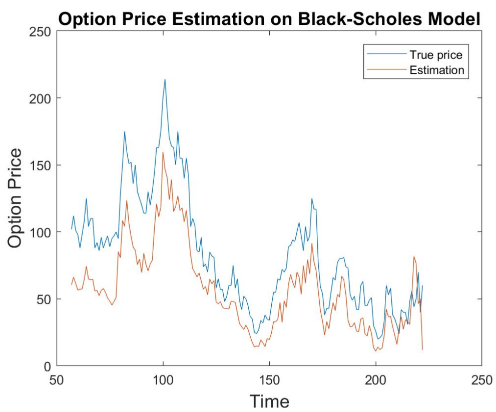

Fig. 17. Put option with strike price K= 3025. MSE =999.65

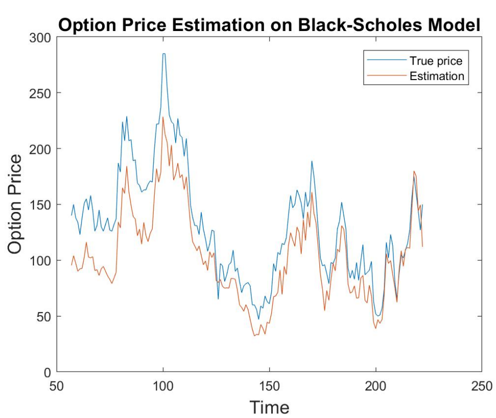

Fig. 18. Put option with strike price K= 3125. MSE = 1003.08

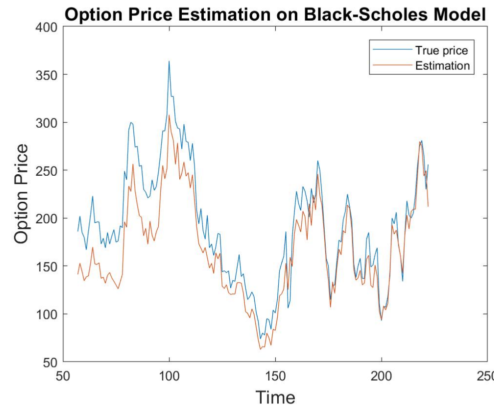

Fig. 19. Put option with strike price K= 3225. MSE = 954.01

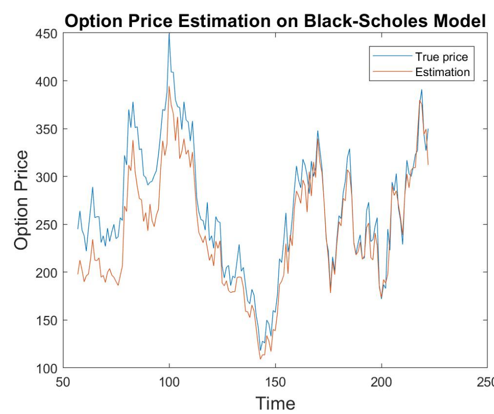

Fig. 20. Put option with strike price K= 3325. MSE = 893.46

As it is shown on the graphs, the Black-Scholes model performs well in option pricing, especially in call option pricing.

- Implied Volatilities

  We plot the Implied Volatilities against the estimated volatilities of both call and put option with strike price $K = 2925$, the time period I choose is $T/4+1$ to $T/4+30$.

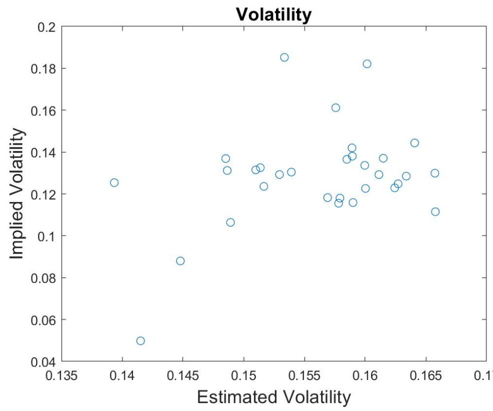

Fig. 21. Volatilities of call option with strike price K=2925 in the day $T/4+1$ to $T/4+30$

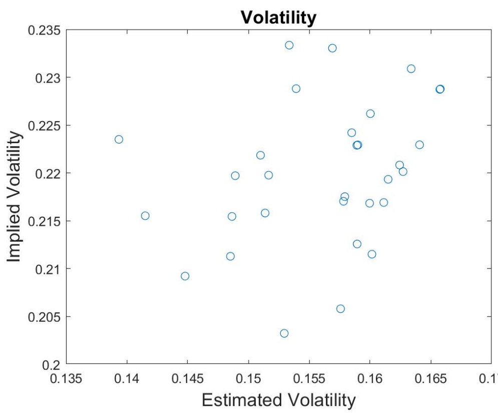

Fig. 22. Volatilities of put option with strike price K=2925 in the day $T/4+1$ to $T/4+30$

And I choose the same time period ($T/4+1$ to $T/4+30$) to plot the Implied Volatilities against different strike prices. But from the volatility surface shown on the figure, the Volatility Smile is not very clear in this case.

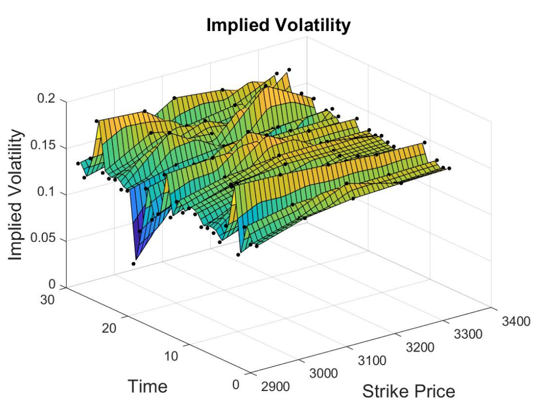

Fig. 23. Implied Volatility against time and strike price

C. Option Price Prediction with RBF Network

In this section, we trained a RBF network to price and delta-hedge the call options.

- Black-Scholes
  Firstly, we normalise and plot all of the estimated call option prices from the previous section(with Black-Scholes Model).

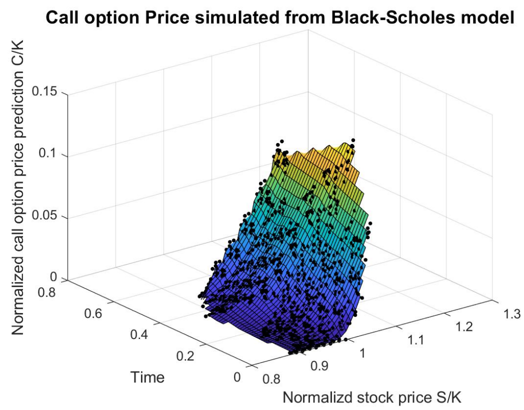

Fig. 24. Call option price simulated from Black-Scholes model

- RBF Network
  We train the radial basis function network by using 60% of the data set as training set and the rest as test set. And we estimate the option price with the RBF model from J. Hutchinson *et al.* [3]

$$C = \sum_{j=1}^{J} \lambda_j \Phi_j(X) + w^T X + w_0, \qquad X = [S/K, (T-t)]^T, \tag{47}$$

where S is the stock price, K is the strike price, $T-t$ is the time to exercise. We adopted four hidden units as the returns diminishing beyond four non-linear term [3]. The Gaussian mixture distribution model is used to fit the data and calculate the center vector m and covariance matrix S. And the activation function $\Phi(X)$ is evaluated by Mahalanobis distance

$$\Phi(X) = \sqrt{(X-m)'S^{-1}(X-m)}. \tag{48}$$

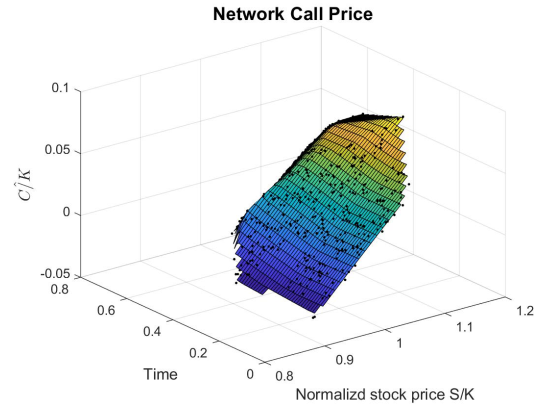

Fig. 25. Network Call Price $\hat{C/K}$

The call price error is evaluated through the difference between the estimation and true value of call option price in test set.

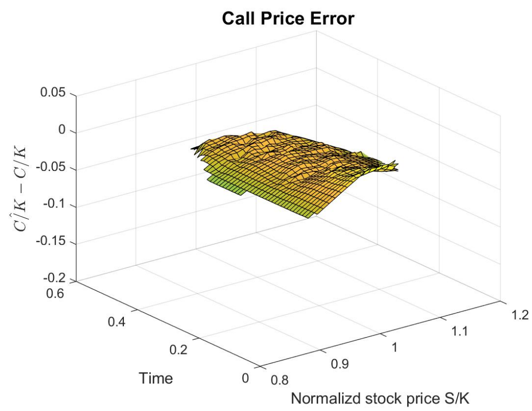

Fig. 26. Call Price Error $\hat{C/K} - C/K$

The delta is calculated by

$$\Delta = \frac{\partial C}{\partial S} = N(d1), \tag{49}$$

with which the Δ can be evaluated by substituting the estimated option price $\hat{C}$ into the solution of Black-Scholes option price equation

$$C = SN(d1) - Ke^{-r(T-t)}N(d2) \tag{50}$$

to get

$$\hat{\Delta} = \frac{(C/K)(e^{-r(T-t)}N(d2))}{S/K}. \tag{51}$$

The σ we used in calculating the $N(d2)$ is the average value of all σ derived from test set option pricing which is $\sigma = 0.1656$.

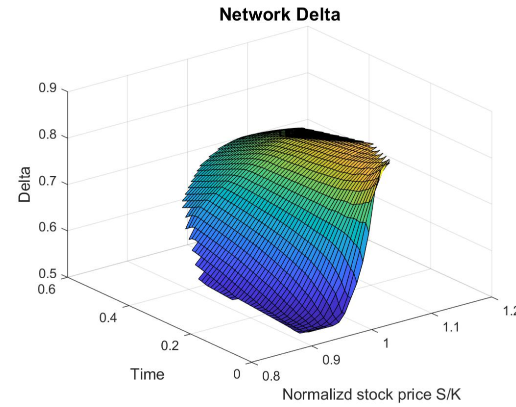

Fig. 27. Network Delta $\frac{\partial \hat{C}}{\partial S}$

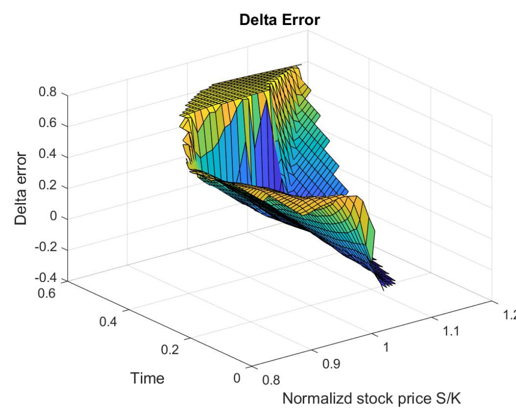

Fig. 28. Delta Error $\frac{\partial \hat{C}}{\partial S} - \frac{\partial C}{\partial S}$

Therefore, the option pricing can be done with non-parametric model such as the Radial Basis Function Network.

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