Programming and Numerical Analysis

solving method for nonlinear equation

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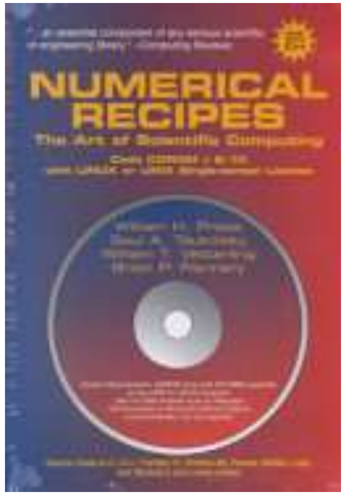
9.0 Introduction



We now consider that most basic of tasks, solving equations numerically.

$$f(x) = 0$$

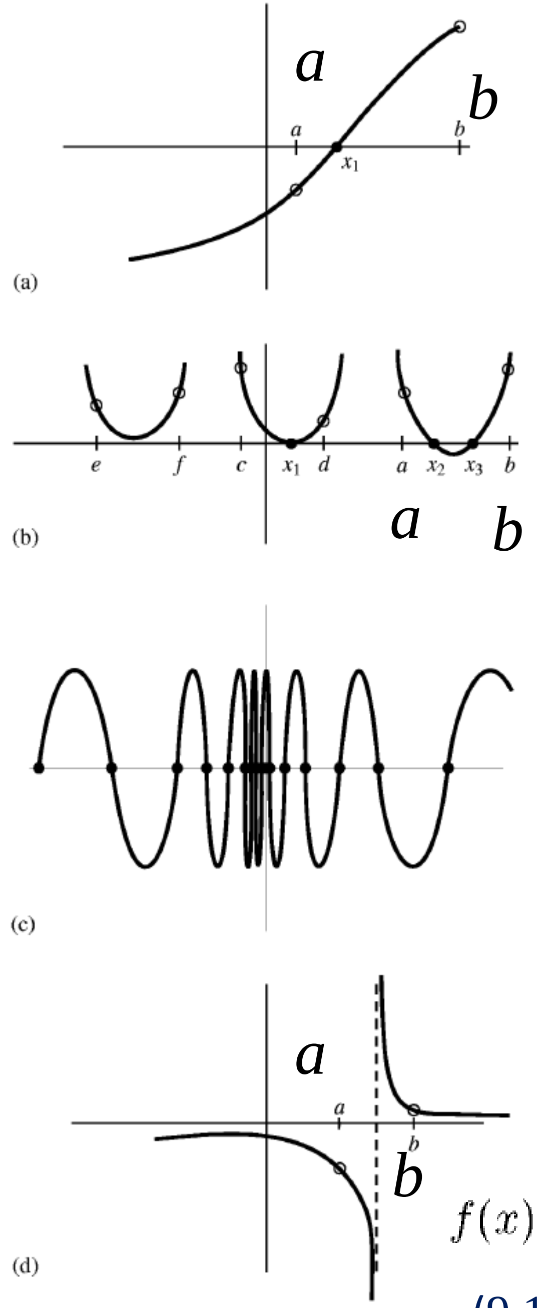
When there is only one independent variable, the problem is *one-dimensional*, namely to find the root or roots of a function. In vector notation, we want to find one or more *N*-dimensional solution vectors **x** such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$

Except in linear problems, root finding invariably proceeds by iteration. **Starting from** some approximate **trial solution**, a useful algorithm will improve the solution until some predetermined convergence criterion is satisfied. For smoothly varying functions, good algorithms will always converge, *provided* that the initial guess is good enough.

9.1 Bracketing and Bisection

- We will say that a root is *bracketed* in the interval (a, b) if $f(a)$ and $f(b)$ have opposite signs. If the function is continuous, then at least one root must lie in that interval (the *intermediate value theorem*).
- If the function is discontinuous, but bounded, then instead of a root there might be a step discontinuity which crosses zero (see Figure d).
- Some root-finding algorithms (e.g., bisection in this section) will readily converge to c in (9.1.1).
- If you are given a function in a black box, there is no sure way of bracketing its roots, or of even determining that it has roots.

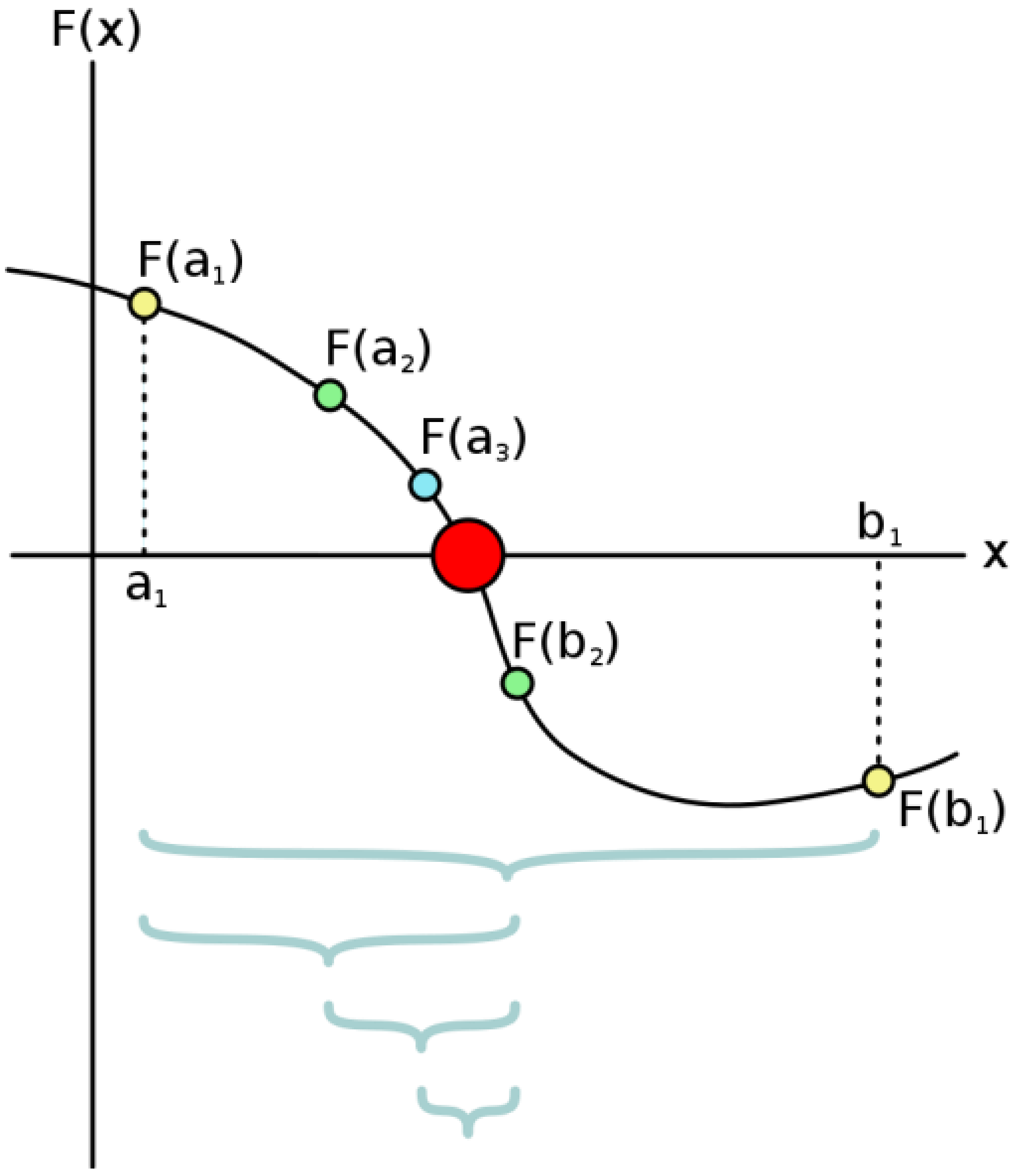


$$f(x) = \frac{1}{x - c}$$

(9.1.1)

Bisection Method

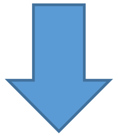
1. Calculate *c*, the midpoint of the interval, $c = (a + b)/2$.
2. Calculate the function value at the midpoint, $f(c)$.
3. If convergence is satisfactory (that is, $c - a$ is sufficiently small, or $|f(c)|$ is sufficiently small), return *c* and stop iterating.
4. Examine the sign of $f(c)$ and replace either $(a, f(a))$ or $(b, f(b))$ with $(c, f(c))$ so that there is a zero crossing within the new interval.



Secant Method

For functions that are smooth near a root, the methods known respectively as ***false position*** and ***secant** method* generally converge faster than bisection.

$$y = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1) + f(x_1)$$



$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

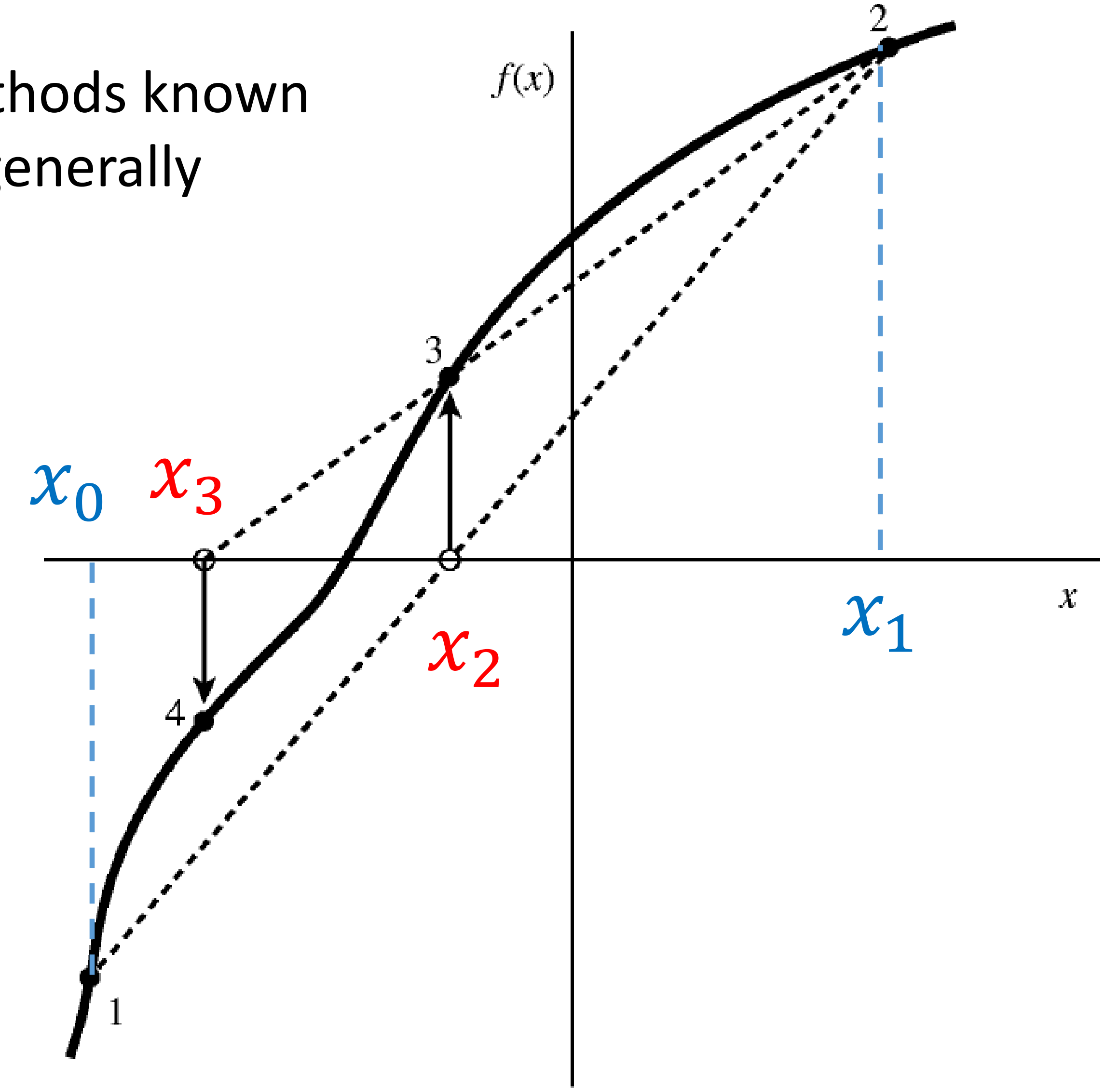
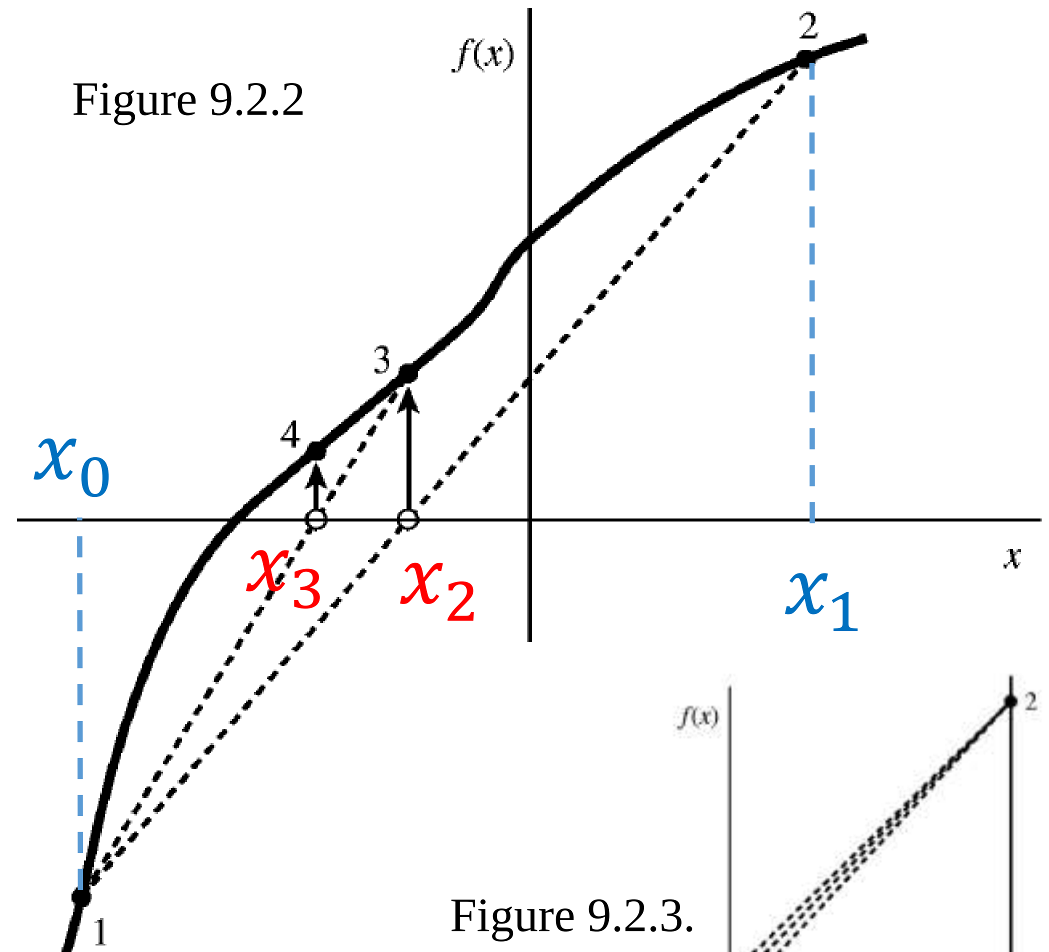


Figure 9.2.1

False Position Method

The *only* difference between the methods is that secant retains the most recent of the prior estimates (Figure 9.2.1; this requires an arbitrary choice on the first iteration), while false position retains that prior estimate for which the function value has opposite sign from the function value at the current best estimate of the root, so that the two points continue to bracket the root (Figure 9.2.2).

Figure 9.2.2



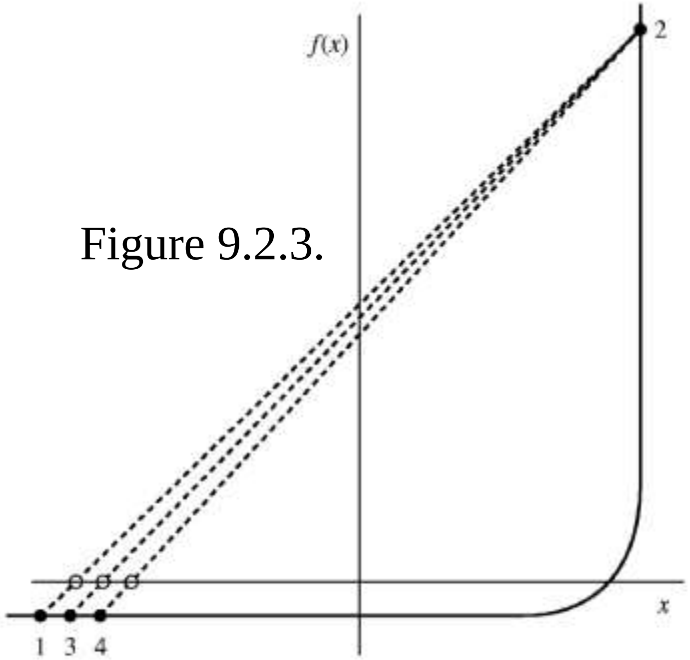
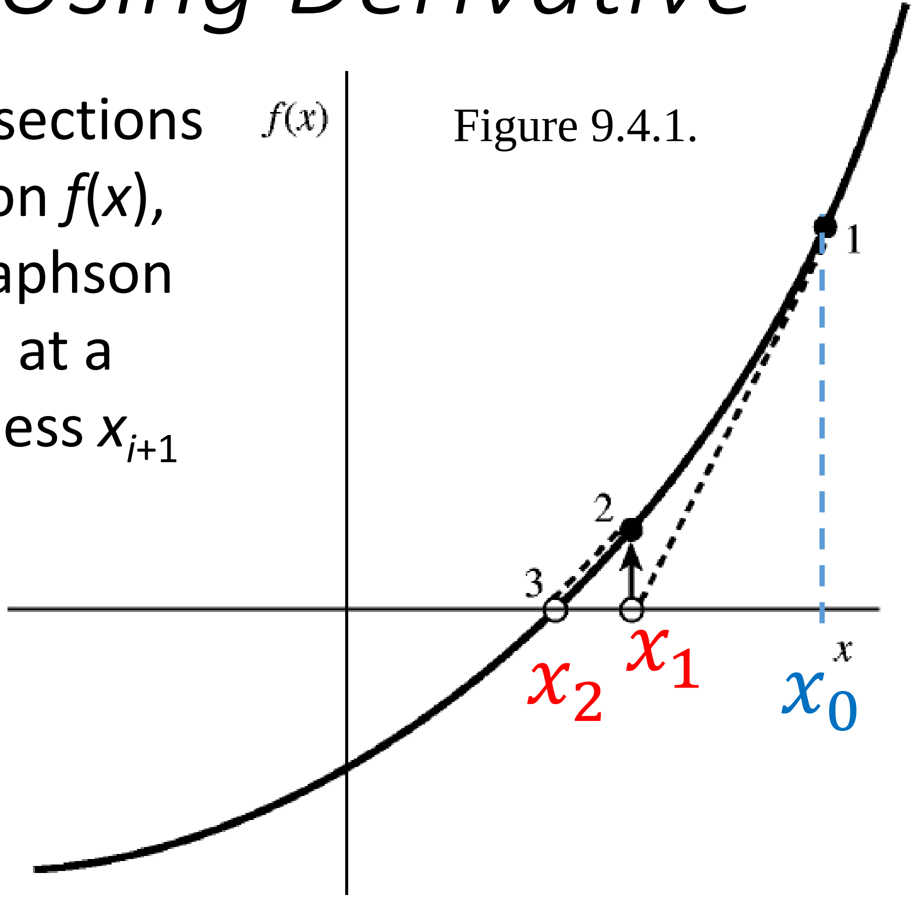


Figure 9.2.3.

Figure 9.2.3 shows the behavior of secant and false-position methods in a difficult situation.

9.4 Newton-Raphson Method Using Derivative

This method is distinguished from the methods of previous sections by the fact that it requires the evaluation of both the function $f(x)$, *and* the derivative $f'(x)$, at arbitrary points x. The Newton-Raphson formula consists geometrically of extending the tangent line at a current point x_i until it crosses zero, then setting the next guess x_{i+1} to the abscissa of that zero-crossing (see Figure 9.4.1).



$$f(x+\delta) \approx f(x) + f'(x)\delta + \frac{f''(x)}{2}\delta^2 + \dots$$

For small enough values of δ, and for well-behaved functions, the terms beyond linear are unimportant, hence $f(x+\delta) = 0$ implies

$$\delta = -\frac{f(x)}{f'(x)} \quad \Rightarrow \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\boxed{\begin{aligned} x_i &= x \\ x_{i+1} &= x + \delta \end{aligned}}$$

Newton-Raphson Method

$$\boxed{f(x+\epsilon)} = \boxed{f(x)} + \epsilon f'(x) + \epsilon^2 \frac{f''(x)}{2} + \cdots$$

$$\boxed{f'(x+\epsilon)} = f'(x) + \epsilon f''(x) + \cdots$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\epsilon_{i+1} = \epsilon_i - \frac{f(x_i)}{f'(x_i)} = \frac{\epsilon_i \boxed{f'(x_i)} - \boxed{f(x_i)}}{f'(x_i)}$$

$x = x_i + \epsilon_i$

$\boxed{f(x) = 0}$

$$\boxed{\epsilon_{i+1} = -\epsilon_i^2 \frac{f''(x)}{2 f'(x)}} \qquad (9.4.6)$$

Equation (9.4.6) says that Newton-Raphson converges *quadratically*. Near a root, the number of significant digits approximately *doubles* with each step.

Newton-Raphson Method

Far from a root, where the higher-order terms in the series are important, the Newton-Raphson formula can give grossly inaccurate, meaningless corrections. For instance, the initial guess for the root might be so far from the true root as to let the search interval include a local maximum or minimum of the function.