



Solve multiband ELIASHBERG equations

Outline

This software provides three programs:

1. `ebmb` itself solves the multiband ELIASHBERG equations (Eqs. 1 or 4) on a cut-off imaginary axis and optionally continues the results to the real axis via PADÉ approximants. The normal-state equations (Eq. 7) can also be solved on the real axis.
A material is defined by nothing but an ELIASHBERG spectral function or, as fallback, an EINSTEIN phonon frequency and intra- and interband electron-phonon couplings, COULOMB pseudo-potentials and, if desired, the band densities of BLOCH states, otherwise assumed to be constant.
2. `critical` finds the critical point via the bisection method varying a parameter of choice. Superconductivity is defined by the kernel of the linearized gap equation (Eq. 5 or 6) having an eigenvalue greater than or equal to unity.
3. `tc` finds the critical temperature for each band separately via the bisection method. Superconductivity is defined by the order parameter exceeding a certain threshold. Usually, it is preferable to use `critical`.

Installation

The makefile is designed for the *GNU* or *Intel* Fortran compiler:

```
$ make FC=gfortran FFLAGS='-O3 -fopenmp'
$ python3 -m pip install -e .
```

Reference

`ebmb` is stored on *Zenodo*: <https://doi.org/10.5281/zenodo.13341224>.

The theory is described here: <https://scipost.org/theses/132/>.

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Let $\hbar = k_B = 1$. Fermionic and bosonic MATSUBARA frequencies are defined as $\omega_n = (2n+1)\pi T$ and $\nu_n = 2n\pi T$, respectively. The quantity of interest is the NAMBU self-energy matrix¹

$$\Sigma_i(n) = \underbrace{i\omega_n[1 - Z_i(n)]}_{(\delta\omega_n)_i} \mathbf{1} + \underbrace{Z_i(n) \Delta_i(n)}_{\phi_i(n)} \sigma_1 + \chi_i(n) \sigma_3,$$

where the PAULI matrices are defined as usual and i is a band index. Renormalization $Z_i(n)$, order parameter $\phi_i(n)$ and energy shift $\chi_i(n)$ are determined by the ELIASHBERG equations²

$$\begin{aligned} Z_i(n) &= 1 + \frac{T}{\omega_n} \sum_j \sum_{m=0}^{N-1} \int_{-\infty}^{\infty} d\varepsilon \frac{n_j(\varepsilon)}{n_j(\mu_0)} \frac{\omega_m Z_j(m)}{\Theta_j(\varepsilon, m)} \Lambda_{ij}^-(n, m), \\ \phi_i(n) &= T \sum_j \sum_{m=0}^{N-1} \int_{-\infty}^{\infty} d\varepsilon \frac{n_j(\varepsilon)}{n_j(\mu_0)} \frac{\phi_j(m)}{\Theta_j(\varepsilon, m)} [\Lambda_{ij}^+(n, m) - U_{ij}^*(m)], \\ \chi_i(n) &= \chi_{Ci} - T \sum_j \sum_{m=0}^{N-1} \int_{-\infty}^{\infty} d\varepsilon \frac{n_j(\varepsilon)}{n_j(\mu_0)} \frac{\varepsilon - \mu + \chi_j(m)}{\Theta_j(\varepsilon, m)} \Lambda_{ij}^+(n, m), \\ \Theta_i(\varepsilon, n) &= [\omega_n Z_i(n)]^2 + \phi_i^2(n) + [\varepsilon - \mu + \chi_i(n)]^2, \end{aligned} \quad (1)$$

and may then be analytically continued to the real-axis ($i\omega_n \rightarrow \omega + i\eta$) by means of PADÉ approximants.³ The electron-phonon coupling matrices and the rescaled COULOMB pseudo-potential are connected to the corresponding input parameters via

$$\begin{aligned} \Lambda_{ij}^{\pm}(n, m) &= \lambda_{ij}(n - m) \pm \lambda_{ij}(n + m + 1), \quad \lambda_{ij}(n) = \int_0^{\infty} d\omega \frac{2\omega \alpha^2 F_{ij}(\omega)}{\omega^2 + \nu_n^2} \underset{\text{Einstein}}{=} \frac{\lambda_{ij}}{1 + \left[\frac{\nu_n}{\omega_E}\right]^2}, \\ U_{ij}^*(m) &= \begin{cases} 2\mu_{ij}^*(\omega_{N_C}) & \text{for } m < N_C, \\ 0 & \text{otherwise,} \end{cases} \quad \frac{1}{\mu_{ij}^*(\omega_{N_C})} = \frac{1}{\mu_{ij}^*} + \ln \frac{\omega_E}{\omega_{N_C}} \end{aligned} \quad (2)$$

with the ELIASHBERG spectral function $\alpha^2 F_{ij}(\omega)$ and $\mu_{ij}^* = \mu_{ij}^*(\omega_E)$ per definition. Alternatively, if the density of states $n_i(\varepsilon)$ per spin as a function of energy ε is given,

$$\frac{1}{\mu_{ij}^*(\omega_{N_C})} = \frac{1}{\mu_{ij}} + \frac{1}{\pi} \int_{-\infty}^{\infty} d\varepsilon \frac{n_j(\varepsilon)}{n_j(\mu_0)} \begin{cases} \frac{1}{\varepsilon - \mu_0} \arctan \frac{\varepsilon - \mu_0}{\omega_{N_C}} & \text{for } \varepsilon \neq \mu_0, \\ \frac{1}{\omega_{N_C}} & \text{otherwise,} \end{cases} \quad (3)$$

where D is the electronic bandwidth. μ_0 and μ are the chemical potentials for free and interacting particles, whose number n_0, n (including a factor of 2 for the spin) is usually conserved:

$$\sum_i \int_{-\infty}^{\infty} d\varepsilon \frac{2n_i(\varepsilon)}{e^{(\varepsilon - \mu_0)/T} + 1} = n_0 \stackrel{!}{=} n = \sum_i \int_{-\infty}^{\infty} d\varepsilon n_i(\varepsilon) \left[1 - 4T \sum_{n=0}^{N-1} \frac{\varepsilon - \mu + \chi_i(n)}{\Theta_i(\varepsilon, n)} - \frac{2}{\pi} \arctan \frac{\varepsilon - \mu + \chi_{Ci}}{\omega_N} \right].$$

It is unusual but possible to also consider the COULOMB contribution to the energy shift:

$$\chi_{Ci} = \sum_j \int_{-\infty}^{\infty} d\varepsilon \frac{n_j(\varepsilon)}{n_j(\mu_0)} \left[2T \sum_{m=0}^{N-1} \frac{\varepsilon - \mu + \chi_j(m)}{\Theta_j(\varepsilon, m)} + \frac{1}{\pi} \arctan \frac{\varepsilon - \mu + \chi_{Cj}}{\omega_N} \right] \mu_{ij}.$$

¹Y. NAMBU, Phys. Rev. **117**, 648 (1960)

²G. M. ELIASHBERG, Soviet Phys. JETP **11**, 696 (1960).

A comprehensive review is given by P. B. ALLEN and B. MITROVIĆ in Solid state physics **37** (1982)

³H. J. VIDBERG and J. W. SERENE, J. Low Temp. Phys. **29**, 179 (1977)

For a given scalar $\alpha^2 F(\omega)$, an effective phonon frequency can be calculated in different ways. We follow ALLEN and DYNES,⁴ who define the logarithmic and the second-moment average frequency and use the latter as ω_E in Eqs. 2 and 3 for rescaling μ^* :

$$\omega_{\log} = \exp \left[\frac{2}{\lambda} \int_0^\infty \frac{d\omega}{\omega} \alpha^2 F(\omega) \ln(\omega) \right], \quad \bar{\omega}_2 = \sqrt{\frac{2}{\lambda} \int_0^\infty d\omega \alpha^2 F(\omega) \omega}.$$

Approximating $n_i(\varepsilon) \approx n_i(\mu_0)$ yields $\chi_i(n) = 0$ and the constant-DOS ELIASHBERG equations

$$\begin{aligned} Z_i(n) &= 1 + \frac{\pi T}{\omega_n} \sum_j \sum_{m=0}^{N-1} \frac{\omega_m}{\sqrt{\omega_m^2 + \Delta_j^2(m)}} \Lambda_{ij}^-(n, m), \\ \Delta_i(n) &= \frac{\pi T}{Z(n)} \sum_j \sum_{m=0}^{N-1} \frac{\Delta_j(m)}{\sqrt{\omega_m^2 + \Delta_j^2(m)}} [\Lambda_{ij}^+(n, m) - U_{ij}^*(m)]. \end{aligned} \quad (4)$$

At the critical temperature, $\phi_j(m)$ is infinitesimal and negligible relative to ω_m . This yields

$$\begin{aligned} \phi_i(n) &= \sum_j \sum_{m=0}^{N-1} K_{ij}(n, m) \phi_j(m), \\ K_{ij}(n, m) &= T \int_{-\infty}^\infty d\varepsilon \frac{n_j(\varepsilon)}{n_j(\mu_0)} \frac{\Lambda_{ij}^+(n, m) - U_{ij}^*(m)}{\Theta_j(\varepsilon, m)}, \end{aligned} \quad (5)$$

where $\Theta_j(\varepsilon, m)$ is obtained from Eqs. 1 for $\phi_j(m) = 0$. Similarly, in the constant-DOS case,

$$\begin{aligned} \Delta_i(n) &= \sum_j \sum_{m=0}^{N-1} K_{ij}(n, m) \Delta_j(m), \\ K_{ij}(n, m) &= \frac{1}{2m+1} [\Lambda_{ij}^+(n, m) - \delta_{ij} \delta_{nm} D_i^N(n) - U_{ij}^*(m)], \\ D_i^N(n) &= \sum_j \sum_{m=0}^{N-1} \Lambda_{ij}^-(n, m) \stackrel{N=\infty}{=} \sum_j \left[\lambda_{ij} + 2 \sum_{m=1}^n \lambda_{ij}(m) \right]. \end{aligned} \quad (6)$$

$Z_i(n)$ is not biased by the cutoff if $D_i^\infty(n)$ is used in place of $D_i^N(n)$ in the kernel $K_{ij}(n, m)$.

The ELIASHBERG equations can also be solved on the real axis,⁵ which allows for exact analytic continuation without Padé approximants. They are implemented for the normal state:

$$\Sigma_{11i}(\omega) = \underbrace{\sum_j \int_{-\infty}^\infty d\varepsilon \frac{A_j(\varepsilon)}{n_j(\mu_0)} \left[\mu_{ij} \left(\frac{1}{2} - f(\varepsilon) \right) \right]}_{\chi C_i} + \int_0^\infty d\omega' \alpha^2 F_{ij}(\omega') \sum_{\pm} \pm \frac{f(\varepsilon) + n(\pm\omega')}{\omega - \varepsilon \pm \omega'} \quad (7)$$

with the Fermi function $f(\varepsilon) = 1/(e^{\varepsilon/T} + 1)$ and the Bose function $n(\omega) = 1/(e^{\omega/T} - 1)$. The quasiparticle density of states $A_i(\omega) = -\frac{1}{\pi} \text{Im } G_i(\omega + i\eta)$ follows from the Green function

$$G_i(\omega) = - \int_{-\infty}^\infty d\varepsilon n_i(\varepsilon) \frac{\omega Z_i(\omega) + \varepsilon - \mu + \chi_i(\omega)}{\Theta_i(\varepsilon, \omega)} \stackrel{\phi=0}{=} \int_{-\infty}^\infty d\varepsilon \frac{n_i(\varepsilon)}{\omega - \varepsilon + \mu - \Sigma_{11i}(\omega)}.$$

Note that the code can replace $\Sigma_{11i}(\omega + i\eta)$ by $\text{Re } \Sigma_{11i}(\omega + i\eta) + i \text{Im } \Sigma_{11i}(\omega + i0^+)$.

⁴P. B. ALLEN and R. C. DYNES, Phys. Rev. B **12**, 905 (1975)

⁵D. J. SCALAPINO, J. R. SCHRIEFFER and J. W. WILKINS, Phys. Rev. **148**, 263 (1966).

See also L. X. BENEDICT, C. D. SPATARU and S. G. LOUIE, Phys. Rev. B **66**, 085116 (2002)

I/O

- Parameters are defined on the command line:

\$ `<program> <key 1>=<value 1> <key 2>=<value 2> ...`

The available keys and default values are listed in Table 1.

- The columns `ebmb`, `tc` and `critical` show which keys are used by these programs.
 - The rightmost column indicates which parameters may be chosen as variable for `critical`. The variable is marked with a negative sign; its absolute value is used as initial guess. If no parameter is negative, the critical temperature is searched for.
 - `lambda`, `muStar`, and `muC` expect flattened square matrices of equal size the elements of which are separated by commas. It is impossible to vary more than one element at once. If `diag`, the off-diagonal elements shall be omitted.
 - `dos` has lines ε/eV n_1/eV^{-1} n_2/eV^{-1} ... with ε increasing.
 - `a2F` has lines ω/eV $\alpha^2 F_{1,1}$ $\alpha^2 F_{2,1}$... with ω increasing. If `diag`, the off-diagonal elements shall be omitted.
 - The relative change in the sample spacing of the real-axis frequencies between $\omega = 0$ and $\omega = x$ is $\text{logscale} \cdot |x|$. Thus, `logscale = 0` corresponds to equidistant sampling.
- Unless `tell=false`, the results are printed to standard output.
 - Unless `file=None`, a binary output file is created. For `critical` and `tc` it simply contains one or more double precision floating point numbers, for `ebmb` the format defined in Tables 2 and 3 is used.
 - The provided *Python* wrapper functions load the results into *NumPy* arrays:

```
import ebmb
results = ebmb.get(<program>, <file>, <replace>,
                  <key 1>=<value 1>, <key 2>=<value 2>, ...)
```

`<replace>` decides whether an existing `<file>` is used or overwritten.

Acknowledgment

Parts of the program are inspired by the EPW code⁶ and work of Malte Rösner.

Contact

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⁶See F. GIUSTINO, M. L. COHEN and S. G. LOUIE, Phys. Rev. B **76**, 165108 (2007) for a methodology review. Results related to ELIASHBERG theory are given by E. R. MARGINE and F. GIUSTINO, Phys. Rev. B **87**, 024505 (2013)

key	default	unit	symbol	description	ebmb	T_c	critical	variable
file	none	–	–	output file	+	+	+	–
form	F16.12	–	–	number edit descriptor	+	+	+	–
tell	true	–	–	use standard output?	+	+	+	–
T	10	K	T	temperature	+	+	+	+
omegaE	0.02	eV	ω_E	EINSTEIN frequency	+	+	+	+
cutoff	15	ω_E	ω_N	overall cutoff frequency	+	+	+	–
cutoffC	ω_N	ω_E	ω_{N_c}	COULOMB cutoff frequency	+	+	+	–
lambda, lamda	1	1	λ_{ij}	electron-phonon coupling	+	+	+	+
muStar, mu*	0	1	μ_{ij}^*	rescaled COULOMB potential	+	+	+	+
muC	0	1	μ_{ij}	unscaled COULOMB parameter	+	+	+	+
bands	1	1	–	number of bands	+	+	+	–
diag	false	–	–	only diagonal coupling given?	+	+	+	–
dos, DOS	none	–	–	file with density of states	+	+	+	–
a2f, a2F	none	–	–	file with ELIASHBERG function	+	+	+	–
n	–	1	n_0	initial occupancy number	+	+	+	–
mu	0	eV	μ_0	initial chemical potential	+	+	+	–
conserve	true	–	–	conserve particle number?	+	+	+	–
chi	true	–	–	consider energy shift $\chi_i(n)$?	+	+	+	–
chiC	false	–	–	consider COULOMB part χ_{C_i} ?	+	+	+	–
Sigma	false	–	–	calculate Σ_{11i} ?	+	–	–	–
steps	250000	1	–	maximum number of iterations	+	+	+	–
epsilon	10^{-13}	a.u.	–	negligible float difference	+	+	+	–
toln	10^{-10}	1	–	tolerance for occupancy number	+	+	+	–
error	10^{-5}	a.u.	–	bisection error	–	+	+	–
zero	10^{-10}	eV	–	negligible gap at T_c (threshold)	–	+	–	–
rate	10^{-1}	1	–	growth rate for bound search	–	+	+	–
lower	0	eV	–	minimum real-axis frequency	+	–	–	–
upper	ω_N	eV	–	maximum real-axis frequency	+	–	–	–
points	0	1	–	number of real-axis frequencies	+	–	–	–
logscale	1	1/eV	–	scaling of logarithmic sampling	+	–	–	–
eta, 0+	10^{-3}	eV	η	broadening of retarded objects	+	–	–	–
measurable	false	–	–	find measurable gap?	+	–	–	–
unscale	true	–	–	estimate missing muC from mu*?	+	+	+	–
rescale	true	–	–	use μ_{ij}^* rescaled for cutoff?	+	+	+	–
imitate	false	–	–	use $Z_i(n)$ biased by cutoff?	–	–	+	–
divdos	true	–	–	divide by $n_j(\mu_0)$ in Eqs. 1, 3?	+	+	+	–
stable	false	–	–	calculate $A_i(\omega)$ differently?	+	–	–	–
normal	false	–	–	enforce normal state?	+	–	–	–
realgw	false	–	–	do real-axis GW_0 calculation?	+	–	–	–
etaIm0	true	–	–	send $\eta \rightarrow 0^+$ in $\text{Im } \Sigma_{11i}(\omega + i\eta)$?	+	–	–	–
noZchi	false	–	–	skip Z and χ for realgw?	+	–	–	–
power	true	–	–	power method for single band?	–	–	+	–

Table 1: Input parameters.

$\langle \text{CHARACTERS key} \rangle : \langle n_1 \times \dots \times n_r \text{ NUMBERS value} \rangle$
 associate key with value
 DIM: $\langle \text{INTEGER } r \rangle \langle r \text{ INTEGERS } n_1 \dots n_r \rangle$
 define shape (column-major)
 INT: take NUMBERS as INTEGERS
 REAL: take NUMBERS as DOUBLES

Table 2: Statements allowed in binary output.
 The data types CHARACTER, INTEGER and DOUBLE
 take 1, 4 and 8 bytes of storage, respectively.

imaginary-axis results		
iomega	MATSUBARA frequency (without i)	ω_n
Delta	gap	$\Delta_i(n)$
Z	renormalization	$Z_i(n)$
chi	energy shift (*)	$\chi_i(n) = \text{Re } \Sigma_{11i}(n)$
domega	frequency shift (#) Sigma=true	$(\delta\omega_n)_i = \text{Im } \Sigma_{11i}(n)$
chiC	COULOMB part of energy shift (*)	χ_{C_i}
phiC	COULOMB part of order parameter	ϕ_{C_i}
status	status (steps until convergence or -1)	-
occupancy results		(*) DOS given
states	integral of density of states	$\sum_i \int d\varepsilon n_i(\varepsilon)$
inspect	integral of spectral function (**)	$\sum_i \int d\omega A_i(\omega)$
n0	initial	} occupancy number n_0
n	final	
mu0	initial	} chemical potential μ_0
mu	final	
effective parameters		a2F given
lambda	electron-phonon coupling	λ_{ij}
omegaE	EINSTEIN frequency	ω_E
omegaLog	logarithmic average frequency	ω_{\log}
omega2nd	second-moment average frequency	$\bar{\omega}_2$
real-axis results		(**) points > 0
omega	frequency	ω
Re[Delta]	real	} gap $\Delta_i(\omega)$
Im[Delta]	imaginary	
Re[Z]	real	} renormalization $Z_i(\omega)$
Im[Z]	imaginary	
Re[chi]	real	} energy shift (*) $\chi_i(\omega)$
Im[chi]	imaginary	
Re[Sigma]	real	} normal self-energy (#) $\Sigma_{11i}(\omega)$
Im[Sigma]	imaginary	
DOS	quasiparticle density of states (*)	$A_i(\omega)$
measurable results		measurable=true
Delta0	measurable gap	$\Delta_{0i} = \text{Re}[\Delta_i(\Delta_{0i})]$
status0	status of measurable gap	-

Table 3: Keys used in binary output.