Treatment of Special Mathematics

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First Chapter
The Real Numbers

1 Cuts in the set $\mathbb{Q}$ of rational numbers

1.1 Introduction

1° We admit the notion of the natural numbers, which are designated by { 0, 1, 2,...} for the group $\mathbb{N}$.
2° We have constructed (Tome I, 34°) $\mathbb{Z}$ by applying the theorem on the symmetrical property of an associative law, commutative and regular to the addition on $\mathbb{N}$. $\mathbb{Z}$ has been endowed with a ring structure that is commutative, unitary, without division by 0.
3°. We have studied (I, 36) the body $\mathbb{Q}$ of fractions of the ring $\mathbb{Z}$ under the name of rational numbers. // We will define a set $\mathbb{R}$ which we endow a structure, such that $\mathbb{R}$ will be isomorphic to $\mathbb{Q}$, allowing identification in $\mathbb{Q}$. For this we essentially use the relation of strict inequality $<$ defined on $\mathbb{Q}$ in $\deg n$ 36, $\deg 7$ in tome I. We will also use the fact that $\mathbb{Q}$ is archimedian (I, 36, $\deg 8$).

1.2 Cuts in the set $\mathbb{Q}$ of rational numbers(1).

1° Definition : We say that we have effectively made a cut in $\mathbb{Q}$ when formed, by some mathematical procedure, 2 non-empty, disjoint subsets of $\mathbb{Q}$, called the lower (a) and greater class (A), which have the three following properties:

> α) All rational numbers less than an element of (a) is a member of (a),
>
> β) All rational numbers greater than an element of (A) is a member of (A),
>
> γ) The two classes are adjacent, which signifies that, whatever positive rational number ϵ, there exists a couple of elements a of (a) and A of (A) such that $|A - a| < \epsilon$.

We say that the subsets (a) and (A) are disjoints, which is to say that the have no common element.

A cut is not necessarily a partition (I, 5, deg 4) of $\mathbb{Q}$, since all rational numbers are not necessarily a class.

2° **Consequences of the definition -**

a) All elements A_0 of (A) is greater than all elements of (a), otherwise A_0 belongs to (a) according to property α, which would be a contradiction in their being disjoint. b) If $r \in \mathbb{Q}$ and is not in a class, it is greater than all elements in (a) by (α) and lesser than all elements of (A) by (β)

c) There can not be two rational numbers, r and r', which are both not in a class. If we suppose that $r < r'$ we have:

$$\left.\begin{array}{l} a < r < r' < A \\ A - a > r' - r \end{array}\right\} \forall a \in (a) \text{ and } \forall A \in (A)$$

which is a contradiction of property (γ).

3° Theorem. **- If A partition of $\mathbb{Q}$ consists of two non-empty classes (a) and (A) containing elements a and A, then the following properties hold :**

(α) All rational numbers less than an element of (a) belongs to (a),

(β) All rational numbers greater than an element of (A)n belongs to (A)

This partition is a cut.

The word partition implies, we recall (I, 10, 3°) that (a) and (A) are disjoint and that all rational numbers are in a class. It is sufficient to show that the property (γ) is a consequence of this hypothesis.

(α) 1

(β) 2

(γ) 3

(δ) 4

(ε) 5

(ζ) 6

(η) 7