PDEs and numerical methods - Some introductory exercises

The goal of these exercises is to get used to some usual vocabulary, notions and tools frequently encountered when dealing with PDEs.

***Reminder:** A **differential equation** is a relationship involving a function u and its derivatives. It is an **ordinary differential equation (ODE)** if u depends on only one variable, or a **partial differential equation (PDE)** if u depends on several variables.*

Exercise 1 Solve the PDE: $\dfrac{\partial^2 u}{\partial y^2}(x,y) = 1$

Exercise 2 *Some tools for solving PDEs: change of variables*

2.1 Look for solutions of $\quad x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = u \quad$ as $u(x,y) = f(x^2+y^2)$.

2.2 Same question for the PDE $x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = \lambda\, u$, where λ is a real constant.

2.3 Let $u : \mathbb{R}^n \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. Solve the PDE $\displaystyle\sum_{i=1}^{n} x_i\,\frac{\partial u}{\partial x_i} = \lambda\, u$

Exercise 3 *Some tools for solving PDEs : change of variables*

Look for solutions for the PDE $x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = 0$ under the form $u(x,y) = f\left(\dfrac{x}{y}\right)$.

Exercise 4 *Some tools for solving PDEs : change of variables*

Look for solutions of $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$ using the change of variables $(X,Y) = (x+y, x-y)$.

Exercise 5 *Some tools for solving PDEs : Fourier transform*

Let consider the PDE $\dfrac{\partial u}{\partial t}(x,t) - \nu\dfrac{\partial^2 u}{\partial x^2}(x,t) = 0$ $(x \in \mathbb{R}, t > 0, \nu > 0)$ with the initial condition $u(x,0) = u_0(x)$. Solve this equation using a Fourier transform.

Reminder: the Fourier transform of f is $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2i\pi x\xi}\, dx$, and $\widehat{e^{-x^2/a^2}} = \sqrt{\pi} a\, e^{-\pi^2\, a^2\, \xi^2}$.

Exercise 6 *Partial differential operators*

6.1 Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$. Compute $\mathrm{curl}(\nabla\varphi)$.

6.2 Let $\varphi : \mathbb{R}^n \to \mathbb{R}$. Compute $\mathrm{div}(\nabla\varphi)$.

6.3 Let $\psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, $(u,v) = (\partial\psi/\partial y, -\partial\psi/\partial x)$ is the *velocity field derived from the stream-function* ψ. Prove that the velocity field is everywhere tangent to the isolines of ψ. Compute the divergence of the velocity field.

Exercise 7 *Proof of the basic Green formula in the 2D case*

The aim of this exercise is to prove the basic Green formula

$$\int_\Omega \frac{\partial u}{\partial x_k}\, v\, dx = -\int_\Omega u\, \frac{\partial v}{\partial x_k}\, dx + \int_{\partial\Omega} u\, v\, (e_k.n)\, ds$$

where e_k is the unit vector in direction x_k, in the 2D-case ($\Omega \subset \mathbb{R}^2$).
7.1 Prove this formula for Ω being the reference triangle (i.e. with nodes (0,0), (1,0) and (0,1)).
7.2 Discuss the extension of this result to any domain Ω with sufficiently smooth boundary

Exercise 8 *Green formulas*

Perform an integration by parts for the following expressions:

8.1 $\displaystyle\int_\Omega \operatorname{div}(\mathbf{E}(\mathbf{x}))\, d\mathbf{x}$ with $\mathbf{E} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$

8.2 $\displaystyle\int_\Omega \operatorname{div}(k(\mathbf{x})\, \nabla u(\mathbf{x}))\, d\mathbf{x}$ with $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$

8.3 $\displaystyle\int_\Omega \sum_{i=1}^{n} \alpha_i \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}$ with $u, v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $\alpha_1, \ldots, \alpha_n$ given real numbers.

Exercise 9 *Boundary conditions and well-posedness*

9.1 Let the ODE $u''(x) = 0$ in $]a, b[$. Find its solution(s), considering the following different boundary conditions:

9.1.a Dirichlet conditions: $u(a) = \alpha, u(b) = \beta$
9.1.b Neumann conditions: $u'(a) = \alpha, u'(b) = \beta$
9.1.c Robin conditions: $u'(a) + \lambda u(a) = \alpha, u'(b) + \lambda u(b) = \beta$ $(\lambda \neq 0)$
9.1.d Mixed conditions: $u'(a) = \alpha, u(b) = \beta$

9.2 Let consider the Poisson equation $\Delta u = f$ on $\Omega \subset \mathbb{R}^n$. Study the unicity of its solution(s), considering the following different boundary conditions:

9.2.a Dirichlet conditions: $u = g$ on $\partial\Omega$
9.2.b Neumann conditions: $\dfrac{\partial u}{\partial n} = h$ on $\partial\Omega$
9.2.c Mixed conditions: $u = g$ on Γ_0, $\dfrac{\partial u}{\partial n} = h$ on Γ_1, where $\Gamma_0 \cup \Gamma_1 = \partial\Omega$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$

Exercise 10 *Characterization of a PDE*
For each of the following PDEs, give as many indications as possible on its nature: order, linear or not, with or without right hand side, stationary or not, elliptic/parabolic/hyperbolic...

10.a $\displaystyle\frac{\partial u}{\partial t}(x,t) + u^2(x,t)\,\frac{\partial u}{\partial x}(x,t) = 0 \qquad (x \in \mathbb{R}, t > 0)$

10.b $\displaystyle\sum_{i=1}^{n} \alpha_i^2 \,\frac{\partial^2 u}{\partial x_i^2}(\mathbf{x}) = f(\mathbf{x}) \qquad (\mathbf{x} \in \mathbb{R}^n)$

10.c $\displaystyle\frac{\partial^2 u}{\partial t^2}(\mathbf{x},t) - c^2 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(\mathbf{x},t) = f(\mathbf{x},t) \qquad (\mathbf{x} \in \mathbb{R}^n, t > 0, c \neq 0 \text{ given})$

10.d $\displaystyle\frac{\partial u}{\partial t}(\mathbf{x},t) - \nu \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(\mathbf{x},t) = f(\mathbf{x},t) \qquad (\mathbf{x} \in \mathbb{R}^n, t > 0, \nu > 0 \text{ given})$

10.e $\displaystyle\frac{\partial^2 u}{\partial y^2}(x,y) + y\,\frac{\partial^2 u}{\partial x^2}(x,y) = 0$

10.f $\displaystyle x^2\,\frac{\partial^2 u}{\partial x^2}(x,y) + 2xy\,\frac{\partial^2 u}{\partial x\,\partial y}(x,y) + y^2\,\frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y)$