PDEs and numerical methods - Introduction to finite differences

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Exercise 1 – *1D schemes* See also NoteBook `SchemeBuilding`

1.1 Derive a symmetric scheme for $f''(x)$ with highest possible order using the points $x-2h$, $x-h$, $x+h$ and $x+2h$.

1.2 We want to derive finite difference schemes for $f^{(3)}(x)$ and $f^{(4)}(x)$ using the values of f at points $x-3h$, $x-2h$, $x-h$, x, $x+h$, $x+2h$, $x+3h$. Which approximation order can we expect? (do not derive the approximation schemes).

1.3 Same question with $x, x+h, x+2h, x+3h$.

1.4 Derive left-sided schemes of order 2 and 3 for $f''(x_0)$, assuming that one already knows that $f'(x_0) = 0$.

Exercise 2 – *Transfer functions* See also NoteBook `TransferFunctions`

2.1 Compare the transfer function of the second-order derivative with the transfer function of its usual second-order centered approximation scheme $\dfrac{u(x-h) - 2u(x) + u(x+h)}{h^2}$.

2.2 Same question for the interpolation operator $\dfrac{u(x-h) + u(x+h)}{2}$. Comment on the quality of the approximation for high frequencies.

Exercise 3 – *Equivalent differential equation* See also NoteBook `Sheet2-Exercise3`

Let the ODE : $u''(x) + \mu^2\, u(x) = 0$ for $x \in (a,b)$ discretized on a regular finite difference grid.

3.1 What is the equivalent differential equation if u'' is discretized using a standard two-sided second-order scheme ? What is the effect of the dominant error term?

3.2 We now suppose that u'' is discretized with a fourth-order scheme, but that $u(x)$ is approximated by $\dfrac{1}{2}\left(u(x+h) + u(x-h)\right)$. Same questions.

Exercise 4 – *Compact finite differences*

A drawback of usual finite difference schemes is that it is necessary to use a wide stencil to get a high order of accuracy. In this exercise, we introduce the so-called *compact schemes*, that address this point.

4.1 Let consider the formal relation

$$\alpha u'_{i-1} + u'_i + \alpha u'_{i+1} \simeq a\,\frac{u_{i+1} - u_{i-1}}{2h}$$

4.1.a Find α and a in order for this formula to be as accurate as possible. What is the order of approximation ?

4.1.b How can such a scheme be used in practice ? What about its computation cost ?

4.2 Generalize this approach starting with

$$\beta u'_{i-2} + \alpha u'_{i-1} + u'_i + \alpha u'_{i+1} + \beta u'_{i+2} \simeq a\,\frac{u_{i+1} - u_{i-1}}{2h} + b\,\frac{u_{i+2} - u_{i-2}}{4h}$$

4.3 What are the transfer functions of the preceding schemes ?

4.4 Propose a similar approach for the second-order derivative.

Exercise 5 – *Shooting method*

Let the ODE

$$(P) \quad \left\{ \begin{array}{lcll} y''(x) & = & a(x)y'(x) + b(x)y(x) + c(x) & \quad x \in (\alpha, \beta) \\ y(\alpha) & = & A, \quad y(\beta) = B & \end{array} \right.$$

where a, b, c are given regular functions and A, B are given real numbers. Due to the two Dirichlet boundary conditions, a direct one-step method (Euler, Runge-Kutta...) cannot be used. That is why a finite difference scheme is usually implemented. However an alternative method, called *shooting method* can be used, which reduces this boundary value problem to two auxiliary initial value problems.

5.1 Prove that the solution of (P) is given by $\quad y(x) = Z(x) + \dfrac{B - Z(\beta)}{Y(\beta)} Y(x), \quad x \in (\alpha, \beta)$
where $Z(x)$ and $Y(x)$ are the solutions of

$$(P_c) \quad \left\{ \begin{array}{lcl} Z''(x) & = & a(x)Z'(x) + b(x)Z(x) + c(x) \\ Z(\alpha) & = & A, \quad Z'(\alpha) = 0 \end{array} \right.$$

and

$$(P_0) \quad \left\{ \begin{array}{lcl} Y''(x) & = & a(x)Y'(x) + b(x)Y(x) \\ Y(\alpha) & = & 0, \quad Y'(\alpha) = 1 \end{array} \right.$$

5.2 Give a numerical method to solve (P), using the Euler method.

Exercise 6 – *2D Laplacian*

6.1 Let a regular 2-D grid, with space steps h and k in directions x and y.

6.1.a Derive a second-order finite difference scheme for Δf using the five grid points $(x, y), (x \pm h, y), (x, y \pm k)$.

6.1.b Let suppose $h = k$. Simplify the preceding scheme. Derive another five-point scheme using the other set of five grid points $(x, y), (x \pm h, y \pm k)$.

6.1.c Is it possible to get a higher order scheme by combining those two schemes ?

6.2 Derive a scheme for Δu at point (x, y) as a function of the values of u at points $(x + a, y)$, $(x - b, y)$, $(x, y + c)$, $(x, y - d)$.

Exercise 7 – *2D Taylor formula*

7.1 The values of a real function f are given on the four vertices of a rectangle. Build a finite difference interpolation formulation for $f(M)$, for any M within the rectangle.

7.2 The values of a real function f are given on the three vertices A, B and C of a triangle. We seek an approximation of $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}$ at point M within the triangle, under the form $\alpha f(A) + \beta f(B) + \gamma f(C)$. Derive the corresponding linear system for α, β and γ (do not solve it). What is the approximation order ?