ézier Curves and Bernstein polynomials2.434Tutorial 2: Bézier Curves and Bernstein polynomialssection.2.4

Geometry

Joel Andrepont, Lecturer : Boris Thibert

Fall 2019

Contents

Chapter 1

Introduction

1.1 Needs of CAGD

- Design curves or surfaces
- Bezier curves :
  Introduced in the 70s
  - Pierre Bezier (Renault)
  - De Casteljau (Citreon)
- We want polynomials that are smooth without sharp corners. Want local movement so we can change aspects of the curve rather than the entire curve.
- We stitch together polynomials in order to avoid Runge phenomenon.

1.2 Lagrange Interpolation

Definition 1.2.1 (Problem)**.**
We have $y_0, \cdots, y_n \in \mathbb{R}^d$ and parameters $t_0, \cdots, t_n \in \mathbb{R}$. We want to find a function

$$f : [t_0, \cdots, t_n] \to \mathbb{R}^d, \mathscr{C}^d \textit{ such that } f(t_i) = y_i \qquad 0 \leq i \leq n$$

1.2.1 Lagrange polynomial

We find a solution which is a polynomial.

Theorem 1.2.2 (Lagrange Polynomial).
Let $y_0, \cdots, y_n \in \mathbb{R}^d$ with $a = t_0 < \cdots < t_n = b$ real numbers.
There exists a unique polynomial L_n that satisfies

$$L_n(t_i) = y_i \qquad deg\left(L_n\right) \leq n$$

see drawing for cases of degree $< n$. For instance, in the case where all the points fall on a straight line.

The polynomial function L_n is given by

$$L_n(t) = \sum_{i=0}^{n} y_i P_i(t)$$

where

$$P_i(t) = \frac{\prod_{j\neq i}\left(t - t_j\right)}{\prod_{j\neq i}\left(t_i - t_j\right)} \qquad \begin{cases} P_i(t_j) & = 0 \text{ if } j \neq i \\ & = 1 \text{ if } j = i \end{cases}$$

Proof.
We denote E : the space of polynomial functions of degree $\leq n$. $E \subset \mathbb{R}^{n+1}$. Consider the map

$$\begin{aligned} \varphi :& E \to \mathbb{R}^{n+1} \\ & P \to \left(P(t_0), \cdots, P(t_n)\right) \end{aligned}$$

is a linear map, to show this consider P, Q and $\alpha \in \mathbb{R}$.

$$\begin{aligned} (\alpha P + Q) &= \begin{pmatrix} \alpha P\left(t_0\right) + Q\left(t_0\right) \\ \vdots \\ \alpha P\left(t_n\right) + Q\left(t_n\right) \end{pmatrix} = \left(\begin{pmatrix} \alpha P\left(t_0\right) \\ \vdots \\ \alpha P\left(t_n\right) \end{pmatrix} + \begin{pmatrix} Q\left(t_0\right) \\ \vdots \\ Q\left(t_n\right) \end{pmatrix}\right) \\ &= \alpha \begin{pmatrix} P\left(t_0\right) \\ \vdots \\ P\left(t_n\right) \end{pmatrix} + \begin{pmatrix} Q\left(t_0\right) \\ \vdots \\ Q\left(t_n\right) \end{pmatrix} \\ &= \alpha\varphi\left(P\right) + \varphi\left(Q\right) \end{aligned}$$

For uniqueness we need to show that φ is bijective. We will prove that φ and indeed any linear map is injective if the null space of the map is the set $\{0\}$.

.

Lemma 1.2.3 (Linear maps take 0 to 0).
Suppose T is a linear map that takes $V \to W$. Then $T(0) = 0$.

Proof. Since T is linear, we have :

$$T(0) = T(0+0) = T(0) + T(0)$$

which is only true when $T(0) = 0$ $\square$

Lemma 1.2.4 (Injectivity is equivalent to null space equals $\{0\}$)**.**
Let $T \in (V, W)$. Then T is injective if and only if null $T = \{0\}$

Proof. First, suppose that T is injective. We want to prove that null $T = \{0\}$. By 1.2.3 we know that $\{0\} \subset$ null T. To prove the inclusion in the other direction suppose that $v \in N(T)$(null of T). Then

$$T(v) = 0 = T(0)$$

Since T is injective, this implies that $v = 0$. Therefore we can conclude that $N(T) = \{0\}$ as desired. To prove the other direction we begin with $N(T) = \{0\}$ and want to show that T is injective. Suppose $u, v \in V$. and $Tu = Tv$. Then

$$0 = Tu - Tv = T(u - v)$$

thus $u - v$ is in $N(T)$ which is $\{0\}$. Hence, $u - v = 0$ which implies that $u = v$. Hence T is injective. $\square$

The last theorem we need is the Fundamental Theorem for Linear Maps.

Theorem 1.2.5 (Fundamental Theorem of Linear Maps)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then the range T is finite-dimensional and*

$$dim\ V = \ dim\ \mathcal{N}(T) + \dim \mathcal{R}(T)$$

where $\mathcal{R}(T)$ is the range.

We supply the definition for surjective and injective functions in order to begin the proof.

Definition 1.2.6 (Injective)**.** *A function $T : V \to W$ is called injective if $Tu = Tv$ implies $u = v$.*

Definition 1.2.7 (Surjective)**.** *A function $T : V \to W$ is called surjective if its range equals W.*

We can finally now prove 1.2.2. We have that φ is a linear map. The kernel/nullspace of φ is given by.

$$\mathcal{N}(\varphi) = \big\{P \in E : \varphi(P) = 0\big\}$$

Thus,

$$\exists P \in E, \qquad \varphi = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then $P(t_i) = 0 \ \forall i \in \{0, \cdots, n\}$ where P is of degree $\leq n$. Now, we can write our polynomial as

$$P = a_1 + a_2 x + a_3 x^2 + \cdots + a_n x^{n-1}$$

Where $P(t_0) = a_1 + a_2 t_0 + \cdots + a_n t_0 = 0$. We can proceed by induction using differentiation to show that

$$P' = a_2 + 2a_3(t_1) + \cdots + (n-1)\, a_n t_2^{n-2} \implies a_2 = 0$$

and so on. Giving us that the kernel of φ is $\{0\}$. Using 1.2.5 we have

$$\begin{aligned} \dim E &= \dim \mathcal{N}(\varphi) + \dim \mathcal{R}(\varphi) \\ &= 0 + \dim \mathcal{R}(\varphi) \end{aligned}$$

By 1.2.4 and 1.2.7 we have that φ is bijective and uniqueness of the polynomial used to represent each set of real numbers $(y_0, \cdots, y_n) \in \mathbb{R}^d$ □

1.2.2 Runge Phenomenon

Consider the function

$$f(x) = \frac{1}{1 + 25x^2} \qquad \text{on } [-1, 1]$$

where $t_0, \cdots, t_n$ are uniformly distributed on the interval and $y_i = f(t_i)$

1.2.3 Approximation Result

Theorem 1.2.8.
Let $f : [a, b] \to \mathbb{R}$ of class $\mathscr{C}^{n+1}$ and L_n the lagrance polynomial associated to f and the nodes $a = t_0 < \cdots < t_n = b$
Then

$$\|f - L_n\|_\infty \leq \frac{1}{(n+1)!} \|q_{n+1}\|_\infty \|f^{(n+1)}\|_\infty$$

where $q_{n+1}(t) = \prod_{i=0}^{n} (t - t_i)$

$\|q_{n+1}\|_\infty$ is dependent on the interval. Also, $\|f^{(n+1)}\|_\infty$ is also unknown. These two can go towards infinity depending on the function and the interval.

Proof. We introduce the error function

$$g := f - L_n$$

and we fix $t \in [a, b] \setminus \{t_i\}$ and we also define

$$k(u) := g(u) - \frac{q_{n+1}(u) g(t)}{q_{n+1}(t)}$$

Note that $\forall i$ we have

$$\begin{aligned}
g(t_i) &= f(t_i) - L_n\left(t_i\right) \\
&= f(t_i) - \sum_{k=0}^{n} y_k P_k(t_i) \\
P_k\left(t_i\right) &= 0 \ \forall k \neq i \text{ and } = 1 \text{ if } k = i \\
g\left(t_i\right) &= f\left(t_i\right) - y_k\left(1\right) \\
0 &= y_k - y_k
\end{aligned}$$

Furthermore,

1.2.4 Need to check this

$$\begin{aligned}
k(t_i) &= g(t_i) - \frac{q_{n+1}(t_i)g(t)}{q_{n+1}(t)} \\
&= 0 - \frac{q_{n+1}(t_i)\left(0\right)}{q_{n+1}(t)} = 0
\end{aligned}$$

We now consider when $t \neq t_i$. The function g has n+2 distinct real roots, each at $t = x$ and $t = x_i$ for $i \in \{0, \cdots, n\}$. We also have $k(t) = 0$ then k vanishes at $(n+2)$ points then by Rolle's theorem (Mean Value Theorem) k' vanishes at (n+1)
$\implies k^{(n+1)}$ vanishes at one point denoted by ξ.

$$k^{n+1}(\xi) = 0$$

A calculation gives

$$0 = k^{(n+1)}(\xi) = g^{(n+1)}(\xi) - q_{n+1}^{(n+1)}(\xi)\frac{g(t)}{q_{n+1}(t)}$$

However, Since L_n has dimension $\leq n$ then $g^{(n+1)} = f^{(n+1)}$ and $q_{n+1}(\xi) = (n+1)!$ since it is simply the $(n+1)$ derivative of a polynomial with leading term $t^{(n+1)}$. Then

$$f^{(n+1)}\left(\xi\right) = (n+1)!\frac{g(t)}{q_{n+1}(t)}$$

Then

$$\left|g(t)\right| = \frac{1}{(n+1)!}\|q_{n+1}(t)\|\left|f^{(n+1)}(\xi)\right| \leq \frac{1}{(n+1)!}\|q_{n+1}^{(t)}\|_\infty\|f^{(n+1)}\|_\infty$$

$\square$

1.3 Exercise 1 : Lagrange interpolation

We first recall the Lagrange Polynomial

Definition 1.3.1 (Lagrange Polynomial). *Let $fL[a,b] \to \mathbb{R}$ be a function and $a \leq x_0 \leq \cdots \leq x_n \leq b$ in [a,b]. The Lagrange polynomial is given by*

$$L_n(x) = \sum_{i=0}^{n} y_i P_i(x), \text{ where } P_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

The goal of this exercise is to use another formulation for the Lagrange Polynomial, whose calculation is less costly.

1. Using the above formulation, estimate the number of operations to evaluate $L_n(x)$.
2. Show that the $n+1$ functions

   $$x \to 1, \text{ and } x \to (x - x_0) \cdots (x - x_k), \ 0 \leq k \leq n-1$$

   form a basis for the set of polynomial functions on $[a, b]$ of degree $\leq n$.
3. We now denote by L_k the lagrange polynomial of degree $\leq n$ that satisfies $L_k(x_i) = f(x_i)$ for $0 \leq i \leq k$. We also denote by $f[x_0, \cdots, x_k]$ the dominant coefficient. Show by induction that

   $$L_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, \cdots, x_k] (x - x_0) \cdots (x - x_{k-1})$$

4. Show that for every $k \geq 1$

   $$f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}}{x_k - x_0} \text{ and } f[x_i] = f(x_i)$$

1.3.1 Solutions

1)

We have n+1 points, thus, for

$$\sum_{i=0}^{n} y_i P_i(x)$$

$P_i(x)$ gives us n multiplications and n divisions, which is $\mathcal{O}(n)$. Since we do this n times we have $\mathcal{O}(n^2)$.

2)

Proof. Let ψ_i be functions such that

$$\begin{aligned}
\psi_{-1} &\to 1 \\
\psi_0 &\to (x - x_0) \\
\psi_1 &\to (x - x_0)(x - x_1) \\
&\vdots \\
\psi_n &\to (x - x_0)(x - x_1) \cdots (x - x_{n-1})
\end{aligned}$$

it is sufficient to show that $(\psi_{-1}, \cdots, \psi_{n-1})$ is linearly independent. To do this, consider

$$P(x) = C_0\psi_{-1} + C_1\psi_0 + \cdots + C_n\psi_{n-1} = 0$$

if $x = x_0$ then

$$P(x_0) = C_0 + 0 = 0 \iff C_0 = 0$$

Likewise, if $x = x_1$

$$P(x_1) = 0 + C_1\psi_0 + 0 = 0 \iff C_1 = 0$$

we can continue this for all x_i where $i \in \{-1, \cdots, n-2\}$. Finally giving us

$$P(x_{n-2}) = 0 + 0 + \cdots + 0 + C_n\psi_{n-1} = 0 \iff C_n = 0$$

$\square$

3)

For $k = 0$ we simply have

$$L_0(x_0) = f(x_0)$$

for $k = 1$ we have

$$L_1(x_1) = f(x_0) + f[x_0, x_1](x - x_0)$$

for $k + 1$ we have

$$\begin{aligned}
L_{k+1}(x) &= L_k(x) + f[x_0, \cdots, x_{k-1}]\,(x - x_0) \cdots (x - x_k) \\
\underbrace{L_{k+1}(x) - L_k(x)}_{\text{polynomial of degree } \leq k+1} &= f[x_0, \cdots, x_{k-1}]\,(x - x_0) \cdots (x - x_k)
\end{aligned}$$

this polynomial vanishes at $x_0, \cdots, x_k$ or $k + 1$ values. Then α is the dominant coefficient given by $L_k(x) + f[x_0, \cdots, x_{k-1}]$.

1.4 Interpolation with Splines

The goal is to have piecewise polynomial curve with smooth gluings.

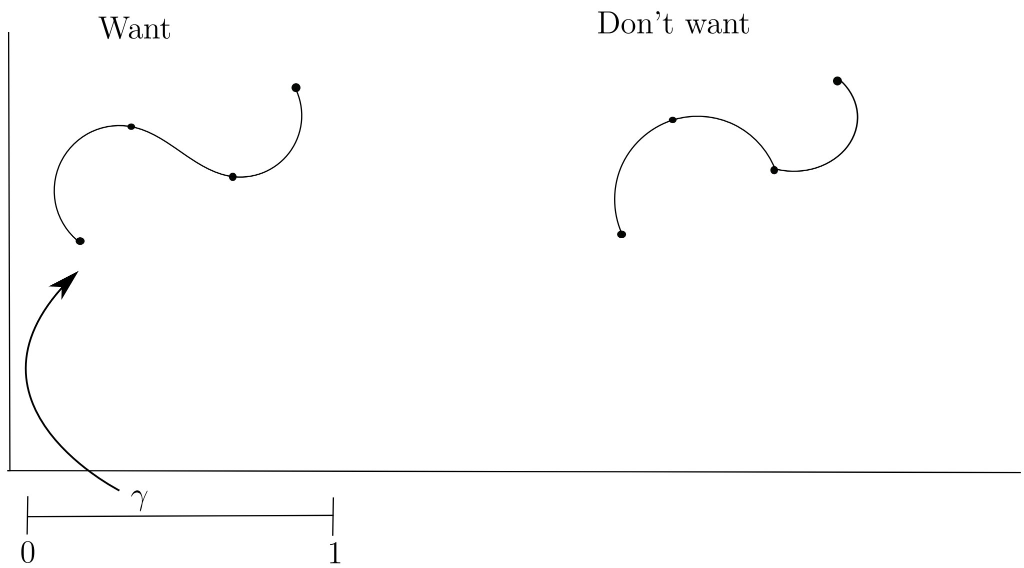

Figure 1.1: piecewise polynomial interpolation

Theorem 1.4.1 (Cubic B-spline)**.** *Let $y_0, \cdots, y_n \in \mathbb{R}^{n+1}$ with $a = t_0 < \cdots < t_n = b$ and $\alpha, \beta \in \mathbb{R}$. There exists a unique*

$$S : [a, b] \to \mathbb{R} \qquad \textit{such that}$$

(i) $S_{[t_i,t_{i+1}]}$ *is polynomial of degree* ≤ 3.

(ii) S *is of class* $\mathscr{C}^2$

(iii) $S(t_i) = y_i$

(iv) $S'(a) = \alpha$ *and* $S'(b) = \beta$

Sketch of proof

Proof.

1. We denote $\mathscr{P}$ the space of functions $f : [a, b] \to \mathbb{R}$ of class $\mathscr{C}^2$ which are polynomials of degree ≤ 3 on each $[t_i, t_{i+1}]$.
   - $\mathscr{P}$ is a vectorial space
   - dim $(\mathscr{P})$ is $4 * n$ where n is the number of intervals and 4 is given by the number of parameters to find each point.
   - We have 3 conditions for each point if we want to uphold $\mathscr{C}^2$ continuity.

- $P_i(t_i) = P_{i+1}(t_i)$
- $P'_i(t_i) = P'_{i+1}(t_i)$
- $P''_i(t_i) = P''_{i+1}(t_i)$

2. A solution $S \in \mathscr{P}$ of the theorem satisfies

   (a) $S'(a) = \alpha$

   (b) $S'(b) = \beta$

   (c) $S(t_i) = y_i, \; \forall i \in \{0, \cdots, n\}$

   Each line is a linear system in the set of paramters i.e. $S'(a) = \alpha$ and we know that $S(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ on $[t_0, t_1]$ and $S'(\alpha) = a_1 + 2x^2 + 3a_3 x^2 = \alpha$ linear in (a_0, a_1, a_2, a_3). These equations are independent (ADMITTED). Thus, there exists a unique solution.

□

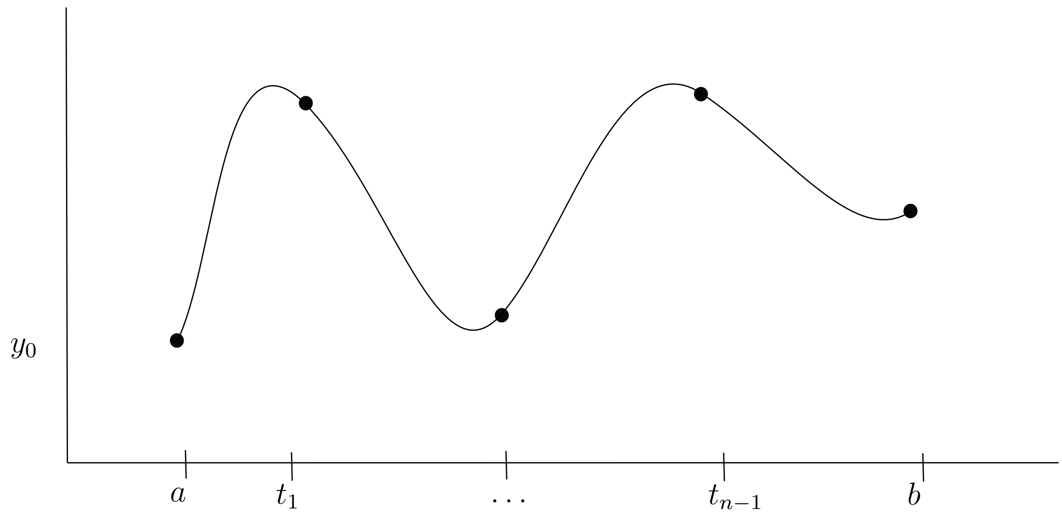

Figure 1.2: cubicBSpline

1.4.1 Minimization Result

Theorem 1.4.2. *Let $y_0, \cdots, y_n \in \mathbb{R}^{n+1}$, $a = t_0, \cdots, t_n = b$. S the spline associated to this interval. Then,*

$$S = argmin \int_a^b f''(t)^2 \, dt \qquad f \in E$$

Where $E = \left\{ f : [a,b] \to \mathbb{R} \text{ of class } \mathscr{C}^2, f'(a) = \alpha, f'(b) = \beta, f(t_i) = y_i \right\}$ *This is to say that the spline is the solution to this problem with the least curvature or minimal energy.*

Proof. Let $f \in E$, $e = f - S$ the error.

1. First, we show that for every function $h : [a,b]$ piecewise linear continuous on each $[t_i, t_{i+1}]$

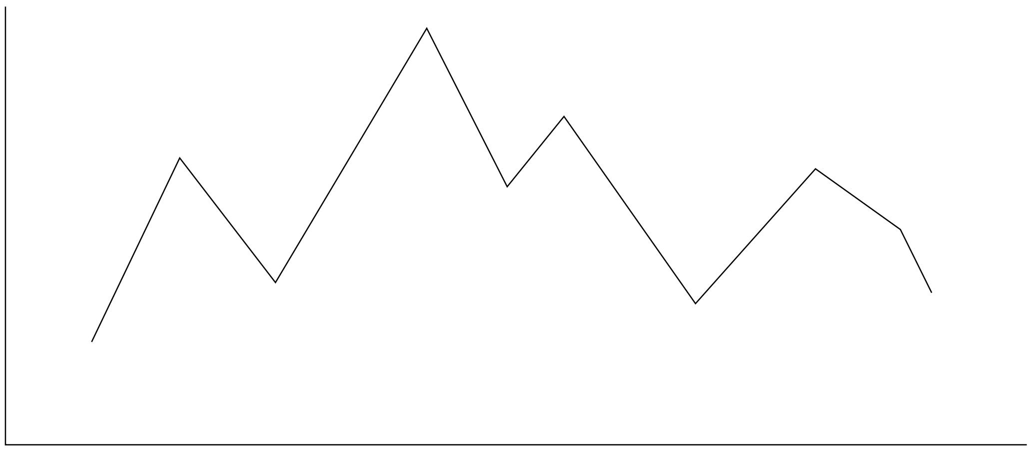

Figure 1.3: h(t)

One has :

$$\int_a^b e''(x)h(x) = 0$$

Indeed,

$$\begin{aligned}\int_a^b e''(x)h(x) &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} e''(x)h(x)\ dx \\ &= \sum_{i=0}^{n-1} \left(\left[e'(x)h(x) \right]_{t_i}^{t_{i+1}} - \int_{t_i}^{t_{i+1}} e'(x)h'(x)\ dx \right)\end{aligned}$$

for the left hand term we have

$$e'(b)h(b) - e'(a)h(a) = 0$$

since

$$e'(b) = f'(b) - S'(b) = 0 \text{ and } e'(a) = f'(a) - S'(a) = 0$$

For the right hand term

$$h'(x) = \lambda_i \text{ on } [t_i, t_{i+1}]$$

Then

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} e'(x)\lambda_i \, dx = -\sum_{i=0}^{n-1} \lambda_i \Big(e\left(t_{i+1}\right) - e(t_i) \Big) \, dx$$

indeed

$$e(t_i) = f(t_i) - S(t_i) = y_i - y_i = 0$$

2.

$$\int_a^b \left(f''(x)\right)^2 \, dx = \int_a^b \left(e''(x) + S''(x)\right)^2 \, dx$$
$$= \int_a^b \left(e''(x)\right)^2 \, dx + \int_a^b \left(S''(x)\right)^2 \, dx + 2\int_a^b e''(x)S''(x) \, dx$$

S is piecewise poly of degree ≤ 3. Then $h = S''$ piecewise linear then

$$\int_a^b e''h \, dx = 0$$

from step (1). Then

$$\int_a^b \left(f''(x)\right)^2 \, dx = \int_a^b \left(e''(x)\right)^2 \, dx + \int_a^b \left(S''(x)\right)^2 \, dx$$

Then

$$\int_a^b \left(f''(x)\right)^2 \, dx \geq \int_a^b \left(S''(x)\right)^2 \, dx$$

(S is a minimizer, by assumption). We have equality if and only if

$$\int_a^b \left(e''(x)\right)^2 \, dx = 0 \iff e''(x) = 0 \text{ because } e'' \text{ is continuous}$$

Using $e(t_i) = 0$ $e'(a) = e'(b) = ?e \equiv 0$.

$\square$

If $f'' \equiv 0 \implies f' = a \implies f(x) = Ax + B$ we don't want zero, but we do want minimization of curvature.

1.4.2 Approximation Result

Theorem 1.4.3. *Let* $f : [a,b] \to \mathbb{R} \in \mathscr{C}^2$. *S the spline associated to* $a = t_0 < \cdots < t_n = b$. *and* $y_i = f(t_i)$. *Then,*

$$\|f - S\|_\infty \leq \frac{h^{3/2}}{2} \|f\|_2$$

with $h = \max |t_i - t_{i+1}|$

$$\|f' - S'\|_\infty \leq h^{1/2} \|f''\|_2$$

Proof. We put $e = f - S$.

$$\|e''\|_2^2 = \int_a^b e''^2 = \int_a^b f''^2 - \int_a^b S''^2$$

$$\leq \int_a^b f''^2 = \|f''\|_2^2 \ (*) \tag{1.1}$$

$\forall i,\ e(t_i) = 0$ By Rolle's theorem, this implies $\forall i,\ \exists \xi i \in [t_i, t_{i+1}]$ s.t. $e'(\xi_i) = 0$ Then

$$\forall i \in [a,b],\ \exists t' \in [a,b] \text{ s.t. } e'(t') = 0 \text{ and } \left|t - t'\right| \leq h$$

Then

$$\begin{aligned}
\left|a'(t)\right| &\leq \left|e'(t) - e'(t')\right| \\
&\leq \left| \int_t^{t'} e''(s)\ ds \right| \\
&\leq \left| \int_t^{t'} ds \right|^{1/2} \left| \int_t^{t'} e''(s)^2 ds \right|^{1/2} \quad \text{C.S.} \\
&\leq h^{1/2} \|e''\|_2 \\
&\leq h^{1/2} \|f''\|_2 \text{ by } (*)
\end{aligned}$$

Second inequality :
Since $e(t_i) = 0 \implies \forall t \in [a,b]\ \exists t''$ s.t.

$$\left|t' - t''\right| < \frac{h}{2} \text{ and } e(t'') = 0$$

$$\begin{aligned}
\left|e(t)\right| &\leq \left|e(t) - e(t'')\right| \\
&\leq \left|\int_{t}^{t''} e'(s)\ ds\right| \\
&\leq \left|t - t''\right| \|e'\|_\infty
\end{aligned}$$

The left side we have $\leq \frac{h}{2}$ and for the right side we have $h^{1/2}\|f''\|_2$ Which gives :

$$\frac{h^{3/2}}{2}\|f''\|_2$$

□

1.5 Exercise 2 : Hermite Interpolation

Let p and q be two point of $\mathbb{R}^d$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$ two vectors. The goal of this exercise is to find a polynomial curve $\gamma : [0,1] \to \mathbb{R}^d$ that interpolates the points and its derivatives, namely that satisfies

$$\gamma(0) = p \quad \gamma'(0) = \boldsymbol{u} \quad \gamma(1) = q \quad \gamma'(1) = \boldsymbol{v}$$

We first assume that $d = 1$. The function $\gamma : [0,1] \to \mathbb{R}$ is thus a polynomial function, whose degree is denoted by k.

1. What is the minimal degree k that we have to take if we want to expect a unique solution for any $p, q, \boldsymbol{u}, \boldsymbol{v}$?
2. Calculate the coefficients of such a polynomial function γ
3. Write γ under the form

$$\gamma(x) = ph_0(x) + qh_1(x) + \boldsymbol{u}h_2(x) + \boldsymbol{v}h_3(x)$$

   where h_i are the polynomial functions to be determined.
   We now suppose that $d \geq 1$.
4. Can we still write γ under the form

$$\gamma(x) = ph_0(x) + qh_1(x) + \boldsymbol{u}h_2(x) + \boldsymbol{v}h_3(x)$$

5. Consider $[a,b]$ with partition $a = x_0 < \cdots < x_n = b$ of $[a,b]$ with a set of points $p_0, \cdots, p_n$ and set of vectors $\boldsymbol{u}_0, \cdots, \boldsymbol{u_n}$ Determine the curve $\gamma : [a,b] \to \mathbb{R}^d$ which is polynomial of degree k on each interval $[x_i, x_{i+1}]$ and that satisfies

$$\gamma(x_i) = p_i \quad \text{ and } \quad \gamma'(x_i) = \boldsymbol{u}_i$$

1.5.1 Solutions

1)

We treat each of these conditions as a linear system. For the solution to be unique we must solve a system with 4 conditions and 4 unknowns, therefore $k = 4$.

2)

We have

$$\begin{aligned}\gamma(0) &= p\\ \gamma'(0) &= \boldsymbol{u}\\ \gamma(1) &= q\\ \gamma'(1) &= \boldsymbol{v}\end{aligned}$$

Consier that $k = 4$ we have

$$\left(\begin{array}{cccc|c} a_0 & 0 & 0 & 0 & p\\ 0 & a_1 & 0 & 0 & \boldsymbol{u}\\ a_0 & a_1 & a_2 & a_3 & q\\ 0 & a_1 & 2a_2 & 3a_3 & \boldsymbol{v}\end{array}\right) \implies a_0 = p \text{ and } a_1 = \boldsymbol{u}$$

This gives us

$$p + \boldsymbol{u} + a_2 + a_3 = q \implies a_2 + a_3 = q - p - \boldsymbol{u}$$
$$\boldsymbol{v} = \boldsymbol{u} + 2\,(q - p - \boldsymbol{u}) + a_3 \implies a_3 = \boldsymbol{u} + \boldsymbol{v} - 2q + 2p$$

Simple calculations show that

$$\begin{aligned}a_0 &= p\\ a_1 &= \boldsymbol{u}\\ a_2 &= -3p + 3q - 2\boldsymbol{u} - \boldsymbol{v}\\ a_3 &= 2p - 2q + \boldsymbol{u} + \boldsymbol{v}\end{aligned}$$

3)

The equation can be written as

$$\gamma(x) = ph_0(x) + qh_1(x) + \boldsymbol{u}h_2(x) + \boldsymbol{v}h_3(x)$$

where

$$\begin{aligned}h_0(x) &= 1 - 3x^2 + 2x^3\\ h_1(x) &= 3x^2 - 2x^3\\ h_2(x) &= x - 2x^2 + x^3\\ h_3(x) &= -x^2 + x^3\end{aligned}$$

4)

Yes, we simply denote

$$\gamma_i(x) = p_i h_0(x) + q_i h_1(x) + \boldsymbol{u}_i h_2(x) + \boldsymbol{v}_i h_3(x)$$

we then have

$$\gamma(x) = \begin{pmatrix} p_0 \\ \vdots \\ p_{n-1} \end{pmatrix} h_0(x) + \begin{pmatrix} q_0 \\ \vdots \\ q_{n-1} \end{pmatrix} h_1(x) + \begin{pmatrix} \boldsymbol{u}_0 \\ \vdots \\ \boldsymbol{u}_{n-1} \end{pmatrix} h_2(x) + \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_{n-1} \end{pmatrix} h_0(x)$$

5)

We first want to have t_i, t_{i+1} to be $[0,1]$ Thus, for $t \in [t_i, t_{i+1}]$ we have

$$\frac{t - t_i}{t_{i+1} - t_i}$$

which then gives us

$$\gamma\left(\frac{t - t_i}{t_{i+1} - t_i}\right)$$

However, we see that the derivate of γ gives

$$\gamma'\left(\frac{t - t_i}{t_{i+1} - t_i}\right) = \frac{1}{(t_{i+1} - t_i)}\gamma\left(\frac{t - t_i}{t_{i+1} - t_i}\right)$$

Thus, our equation now becomes

$$\gamma\left(\frac{t - t_i}{t_{i+1} - t_i}\right) = p_i h_0(x) + p_{i+1} h_1(x) + (t_{i+1} - t_i)\,\boldsymbol{u}_i h_2(x) + (t_{i+1} - t_i)\,\boldsymbol{u}_{i+1} h_3(x)$$

Chapter 2

Bezier Curves

Invented by Pierre Bezier (Renault) and Pierre De Casteljau (Citreon). No interpolation points, rather we have "control points".

2.1 Bernstein Polynomials

Definition 2.1.1 (Bernstein Polynomials)**.**
Let $n \in \mathbb{N}$ and $i \in \{0, \ldots, n\}$. Then the Bernstein polynomials are given by

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

where

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$$

2.1.1 Properties

(a) Positivity :

$$\forall i \; B_i^n(t) \geq 0$$

(b) Partition of unity :

$$\sum_{i=0}^{n} B_i^n(t) = 1$$

(c) Linear precision :

$$\sum_{i=0}^{n} \frac{i}{n} B_i^n(t) = t$$

(d) Recursion Formula : for $0 \leq i \leq n$

$$B_i^n(t) = (1-t)\,B_i^{n-1}(t) + B_{i-1}^{n-1}(t)$$

with the convention that $B_j^n = 0$ if $j \notin [0, n]$

(e) Symmetry :

$$B_i^n = B_{n-i}^n\,(1-t)$$

(f) Derivative :

$$B_i^{n\prime}(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$$

(g) Extremum :
B_i^n has an extremum at $t = \frac{i}{n}$

(h) Basis :

$$\{B_i^n, 0 \leq i \leq n\} \text{ is a basis of } \mathbb{R}_n[x]$$

a)

ok

b)

$$\sum_{i=0}^{n} B_i^n = \sum_{i=0}^{n} \binom{n}{i} t^i\,(1-t)^{n-i}$$

Using the binomial theorem, which states that

$$(a+b)^n \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Therefore, we have

$$\left(t + (1-t)\right)^n = 1$$

c)

?

d)

We have, from definition of binomial that

$$\binom{n}{i-1} + \binom{n}{i} = \binom{n+1}{i}$$

Expanding the original equation and simplifying the coefficents.

$$\binom{n}{i} t^i\,(1-t)^{n-i} = \binom{n-1}{i} t^i\,(1-t)^{n-i} + \binom{n-1}{i-1} t^i\,(1-t)^{n-i}$$

We now have

$$\begin{aligned}
&\left(\binom{n-1}{i-1}+\binom{n-1}{i}\right)t^i\left(1-t^{n-i}\right)\\
&=\binom{n}{i}t^i\left(1-t\right)^{n-i}\\
&=B_i^n(t)
\end{aligned}$$

e)

We have from properties of binomials that

$$\binom{n}{k}=\binom{n}{n-k}$$

Then

$$\begin{aligned}
B_{n-i}^n\left(1-t\right)&=\binom{n}{n-i}\left(1-t\right)^{n-i}\left(1-\left(t-1\right)\right)^{n-(n-i)}\\
&=\binom{n}{i}t^i\left(1-t\right)^{n-i}
\end{aligned}$$

f)

$$\begin{aligned}
&n\left(B_{i-1}^{n-1}(t)-B_i^{n-1}(t)\right)\\
&=n\binom{n-1}{i-1}t^{i-1}\left(1-t\right)^{n-i}-n\binom{n-1}{i}t^i\left(1-t\right)^{n-1-i}\\
&=i\binom{n}{i}t^{i-1}\left(1-t\right)^{n-i}-\binom{n}{i}\left(n-i\right)t^i\left(1-t\right)^{n-1-i}\\
&=\binom{n}{i}t^{i-1}\left(1-t\right)^{n-1-i}\left(i\left(1-t\right)-\left(n-i\right)t\right)\\
&=\binom{n}{i}t^{i-1}\left(1-t\right)^{n-1-i}\left(i-tn\right)
\end{aligned}$$

On the other hand

$$B_i'^n(t)=\binom{n}{i}\left(it^{i-1}\left(1-t\right)^{n-i}-t^i\left(1-t\right)^{n-i-1}\left(n-i\right)\right)$$

$$= \binom{n}{i} t^{i-1} (1-t)^{n-1-i} \big(i (1-t) - t (n-i) \big)$$

2.2 Bezier Curves

2.2.1 Refresher on Convex Hull and Barycenter

Proposition 2.2.1. *Let $p_1, \cdots, P_n \in \mathbb{R}^d$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ and $\sum_{i=1}^n \lambda_i \neq 0$*

$$\exists! \in \mathbb{R}^d, \sum_{i=0}^{n} \lambda_i \boldsymbol{GP}_i = \mathbf{0}$$

G is given by

$$\boldsymbol{OG} = \frac{\sum_{i=1}^n \lambda_i \boldsymbol{OP}_i}{\sum_{i=1}^n \lambda_i}$$

G is called the barycenter of P_i with weight λ_i

Proof. Let $o \in \mathbb{R}^d$ be any point

$$\boldsymbol{o} = \sum_{i=1}^{n} \lambda_i \boldsymbol{GP}_i = \sum_{i=1}^{n} \lambda_i \left(\boldsymbol{GO} + \boldsymbol{OP_i} \right)$$
$$= \left(\sum_{i=1}^{n} \lambda_i \right) \boldsymbol{GO} + \sum_{i=1}^{n} \lambda_i \boldsymbol{OP_i}$$

□

Definition 2.2.2 (Convec Set)**.** *$K \subset \mathbb{R}^d$ is convex if*

$$\forall x, y \in K, \quad [x, y] \subset K$$

Definition 2.2.3 (Convex Hull)**.** *Denoted*

$$Conv(K) = \cap_{K \subset K'} K'$$

The smallest convex set that contains the set

Proposition 2.2.4. *$A = \{p_1, \cdots, p_n\}$ then*

$$Conv\ (A) = \left\{ \sum_{i=1}^{n} \lambda_i p_i, \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0 \right\}$$

is the set of barycenter with positive weights

Proof. Let
$$F = \left\{ \sum \lambda_i p_i, \sum \lambda_i = 1, \lambda_i \geq 0 \right\}$$
Conv $(A) \subset F$, F convex.
$$m \in F \implies m = \sum \lambda_i p_i$$
$$n \in F \implies m = \sum \lambda_i' p_i$$

$$q \in [m,n] \implies q = tm + (1-t)n \ t \in [0,1]$$

So
$$q = \sum_{i=1}^{n} \underbrace{\left(t\lambda_i + (1-t)\,\lambda_i'\right)}_{\lambda_i'' \geq 0} p_i \in F$$
Then $[m,n] \subset F$. $F \subset A$ and $\lambda_i'' = 1$

$F \subset \text{Conv}(A)$ By recursion $P(n)$: every barycenter of n points $q_1, \cdots, q_n$ belong to Conv$(q_1, \cdots, q_n)$.
n = 1 :
Let $n \geq 1$ $SqP(n)$
Let $p_1, \cdots, p_{n+1}$ points of $\mathbb{R}^d$ and $\lambda_1, \cdots, \lambda_{n+1} \in \mathbb{R}^d$ with sum = 1. and
$$m = \sum_{i=1}^{n+1} \lambda_i p_i = \sum_{i-1}^{n} \lambda_i p_i + \lambda_{n+1} p_{n+1} + 1$$
Let g denote the barycenter of $p_1, \cdots, p_n$ with associated $_i$ then
$$\sum_{i=1}^{n} \lambda_i p_i = \left(\sum_{i=1}^{n} {}_i \right) g$$
$$= (1 - \lambda_{n+1})\, g$$
then
$$m = (1 - \lambda_{n+1})\, g + \lambda_{n+1} p_{n+1} \in [g, p_{n+1}]$$
by assumption $g \in \text{Conv}\,(A) \implies m \in \text{Conv}\,(A)$.

□

Example 2.2.5.
$$[p_1, p_2] = \left\{ tp_1 + (1-t)\, p_2, t \in [0,1] \right\}$$
Definition 2.2.6. *We see $\mathbb{R}^d$ is both affine and vector space.*

- *$p \in \mathbb{R}^d$ is a point*
- *$p - q = \boldsymbol{pq}$ is a vector*

In particular $\sum_{i=1}^{n} \lambda_i p_i = 0$ gives a vector, barycenter otherwise(a point).

2.2.2 Definition of Bezier Curves

Let $p_0, \cdots, p_n$ points of $\mathbb{R}^d$, $n > 0$. The Bezier curve associated to $p_0, \cdots, p_n$ is the parametrized curve

$$\begin{aligned} P : [0,1] &\to \mathbb{R}^d \\ t &\to \sum_{i=0}^{n} P_i B_i^n(t) \end{aligned}$$

We call $[P_o, \cdots, P_n]$ the Bezier control polygon.

Example 2.2.7 (n=1)**.**

$$\begin{aligned} P(t) &= p_0 B_0^1(t) + B_1^1(t) \\ &= p_0 t + p_1 (1-t) \end{aligned}$$

P is a parametrization of $[p_0, p_1]$

2.2.3 Properties

(a) boundary :

$$P(0) = P_0 \qquad \text{and } P(1) = P_n$$

Proof.

$$P(0) = \sum_{i=0}^{n} B_i^n(0) P_i = P_0$$

□

(b) Convex Hull :

$$P\big([0,1]\big) \subset \mathrm{Conv}\,(\text{control polygon})$$

Proof. $\forall t \in [0,1]$ $P(t) = \sum_{i=0}^{n} B_i^n(t) P_i$ where B_i^n is λ_i which is positive and the sum of which is equal to one. Therefore, P belongs to the convex hull. □

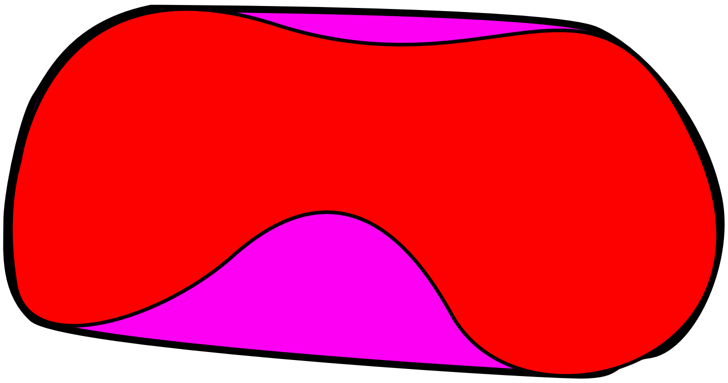

Figure 2.1: A convex set (red) and its Hull (pink)

(c) Affine Invariance : Let $P_1, \cdots, P_n$ and $f : \mathbb{R}^d \to \mathbb{R}^d$ affine transformation. The result is the same if

(a)
$$Q(t) = \sum_{i=0}^{n} B_i^n(t) P_i \rightsquigarrow f \circ Q$$

(b) $\widetilde{Q}(t) = \sum B_i^n(t) f(P_i)$

This just simply says that moving the polygon using an affine transformation gives us the exact same curve.

(d) Reduction of Variation

Proposition 2.2.8. *The Bezier curve has less intersection points with any line than its control polygon. For any line, the number of intersections between the curve less that the number of intersections between the polygon and*

Proof. Admitted. □

(e) Matrix Representation :

$$P(t) = \sum_{i=0}^{n} B_i^n(t) P_i \qquad t \in [0,1]$$

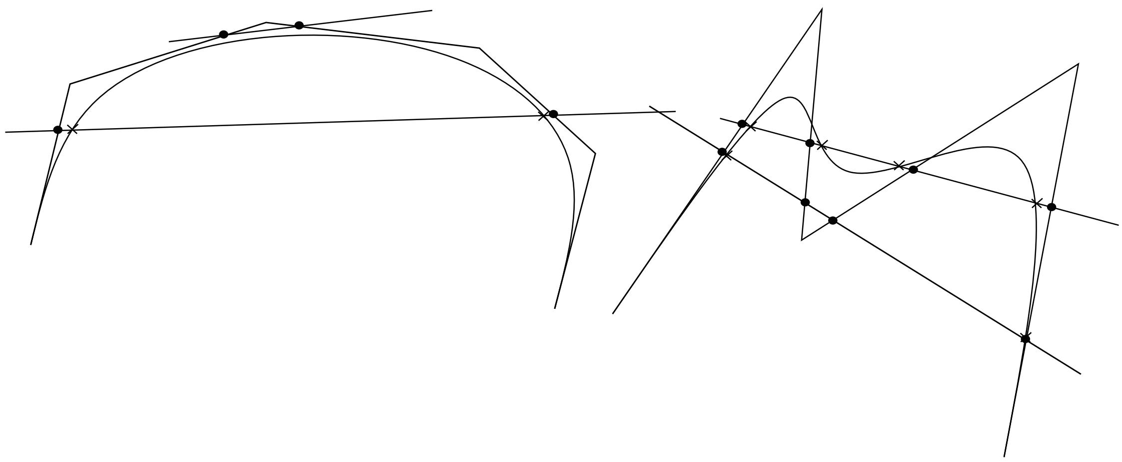

Figure 2.2: Variance Reduction

can be done as a scalar product between vector with Bernstein polynomials where each line is for a point t_i and the vector is the points $P_0, \cdots, P_n$

$$P(t_{=}[P_0, \cdots, P_n] \begin{pmatrix} B_0^n(t) \\ \vdots \\ B_n^n(t) \end{pmatrix}$$

Matrix of change of Basic

$$Q := \begin{pmatrix} B_0^n & \cdots & B_n^n \\ & & \end{pmatrix}$$

$$B_i^n(t) = \sum_{i=0}^{n} q_i 0$$

Then

$$\begin{pmatrix} B_0^n \\ \vdots \\ B_n^n \end{pmatrix} = Q^T \begin{pmatrix} 1 \\ \vdots \\ t^n \end{pmatrix}$$

then

$$P(t) = \underbrace{[P_0, \vdots, P_n]^t Q}_{Q_0, \cdots, Q_n]} \begin{pmatrix} 1 \\ \vdots \\ t^n \end{pmatrix}$$

Proposition 2.2.9.

$$[P_0, \cdots, P_n] = Q^T [Q_0, \cdots, Q_n]$$

Where P... is the control polygon in the bernstein basis, and Q^T is the transpose and expression in the monomial basis

(f) Derivative of Bezier Curves

Proposition 2.2.10.

$$P'(t) = n \sum_{i=0}^{n-1} \underbrace{(P_{i+1} - P_i)}_{\Delta P_i = \frac{P_{i+1} - P_i}{(i+1) - i}} B_i^{n-1}(t)$$

Proof.

$$\begin{aligned} P'(t) &= \sum_{i=0}^{n} P_i B_i^{n\prime}(t) \\ &= \sum_{i=0}^{n} P_i n \left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t) \right) \\ &= n \left(\sum_{i=0}^{n-1} P_{i+1} B_i^{n-1}(t) - \sum_{i=0}^{n-1} P_i B_i^{n-1}(t) \right) \\ &= n \sum_{i=0}^{n-1} (P_{i+1} - P_i) B_i^{n-1}(t) \end{aligned}$$

□

P' is a Bezier curve Associated to $[n\Delta P_0, \cdots, n\Delta P_{n-1}]$ Similarly, one has

Proposition 2.2.11.

$$P''(t) = n\,(n-1)\sum_{i=0}^{n-2} \Delta^2 P_i B_i^{n-2}(t)$$

Where $\Delta^2 P_i = \Delta P_{i+1} - \Delta P_i = P_{i+2} - 2P_{i+1]+P_i}$

Also :

$$P^{(k)}(t) = \frac{n!}{(n-k)!}\sum_{i=0}^{n-k} \Delta^k P_i B_i^{n-k}(t)$$

Where $\Delta^k P_i$ are defined recursively.
In particular :

$$P'(0) = n\Delta P_0 = n\boldsymbol{P_0 P_n}$$
$$P'(1) = n\Delta P_{n-1=n\boldsymbol{P_{n-1}P_n}}$$

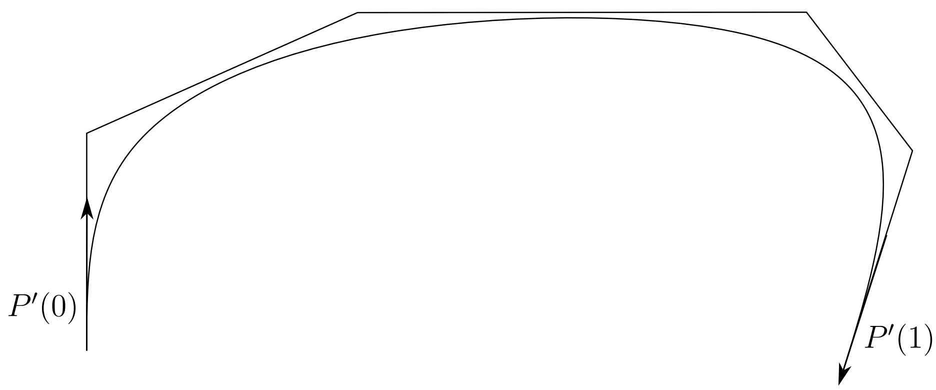

Figure 2.3: Bezier Curve Derivatives

2.3 Algorithms to evaluate Bezier Curves and their derivatives

It is based on the following trick Let $t \in [0,1]$

$$\begin{aligned}
P(t) &= \sum_{i=0}^{n} P_i B_i(t) \\
&= \sum_{i=0}^{n} P_i \left((1-t)\, B_i^{n-1}(t) + t B_{i-1}^{n-1}(t)\right) \\
&= \sum_{i=0}^{n} P_i \,(1-t)\, B_i^{n-1}(t) + \sum_{i=1}^{n} P_i(t) B_{i-1}^{n-1}(t) \\
&= \sum_{i=0}^{n} P_i \,(1-t)\, B_i^{n-1}(t) + \sum_{i=0}^{n-1} P_{i+1}(t) B_i^{n-1}(t) \\
&= \sum_{i=0}^{n-1} \underbrace{(1-t)\, P_i + t P_{i+1}}_{P_i^{(n)}} B_{n-1}^{i}(t) \\
&= \sum_{i=0}^{n-1} P_{(1)}^{i} B_i^{n-1}(t)
\end{aligned}$$

We used the recurrence formula :

$$B_j^n \equiv 0 \text{ if } j \notin [0,1]$$

We iterate the process to give

$$P(t) = \sum_{i=0}^{n-1} P_i^{(1)} B_i^{n-1}(t) = \sum_{i=0}^{n-2} P_i^{(2)} B_{n-2}^{i}(t) = \cdots = P_0^{(n)}$$

$$\begin{pmatrix}
P_0 & & & & & \\
 & P_0^{(1)} & & & & \\
P_1 & & P_0^{(2)} & & & \\
 & P_1^{(1)} & & & P_0^{(1)} & \\
P_2 & & P_1^{(2)} & \dots & & P_0^{(n)} \\
 & P_2^{(1)} & & & P_0^{(1)} & \\
\vdots & & P_{n-2}^{(2)} & & & \\
 & P_{n-1}^{(1)} & & & & \\
P_n & & & & &
\end{pmatrix}$$

This is the triangle scheme of De Casteljaue. To evaluate $P(t)$ when t is fixed.

We denote $\mathscr{P}^{[0]} = [P_0, \cdots, P_n]$ the initial control polygon and $\mathscr{P}^{[1]}$ the concatenation of the 2 diagonals $= [P_0, P_0^{(1)}, \cdots, P_0^{(n)}, P_1^{(n-1)}, P_{n-1}^{(1)}, P_n]$. We use the notation

$$BP[P_0^{(0)}, \cdots, P_0^{(n)}](t) = BP[P_0, \cdots, P_n(\alpha t)$$

as the Bezier curve associated to the control polygon.

Proposition 2.3.1. *(i)*

$$BP[P_0^{(0)}, \cdots, P_0^{(n)}](t) = BP[P_0, \cdots, P_n(\alpha t)$$

(ii)

$$BP[P_0^{(n)}, \cdots, P_n^{(n)}](t) = BP[P_0, \cdots, P_n]\left(\alpha + (1-\alpha)\, t\right)$$

Where α is the parameter used in the De Casteljau

$$\begin{aligned}
&\mathscr{P}^{[0]} \text{ has } \ (n+1) \ \text{ vertices with 2 on the curve.}\\
&\mathscr{P}^{[1]} \text{ has } 2\,(n+1) \ \text{ vertices with 3 on the curve.}\\
&\mathscr{P}^{[2]} \text{ has } 4\,(n+1) \ \text{ vertices with 5 on the curve.}\\
&\quad\vdots\\
&\mathscr{P}^{[n]} \text{ has } 2^k\,(n+1) \ \text{ vertices with } 2^k+1 \text{ on the curve.}
\end{aligned}$$

Then $\mathscr{P}^{[k]}$ converges to the Bézier curve.

2.3.1 Derivative of Bezier curve

1st Method

We use :

$$p^{(k)} = \frac{n!}{(n-k)!} \sum_{k=0}^{n-k} \Delta^k P_i B_i^{n-k}(t)$$

then we calculate $\Delta^k P_i$ given by the De Casteljau algorithm on $\Delta^k P_i$. When k = 0

$$\underbrace{\begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_n \end{pmatrix} \to \begin{pmatrix} \Delta P_0 \\ \Delta P_1 \\ \vdots \\ \Delta P_{n-1} \end{pmatrix} \to \begin{pmatrix} \Delta^2 P_0 \\ \Delta^2 P_1 \\ \vdots \\ \Delta^2 P_{n-2} \end{pmatrix}}_{\text{finite differences}} \underbrace{\to P''(t) \frac{1}{n\,(n-1)}}_{\text{De Casteljau}}$$

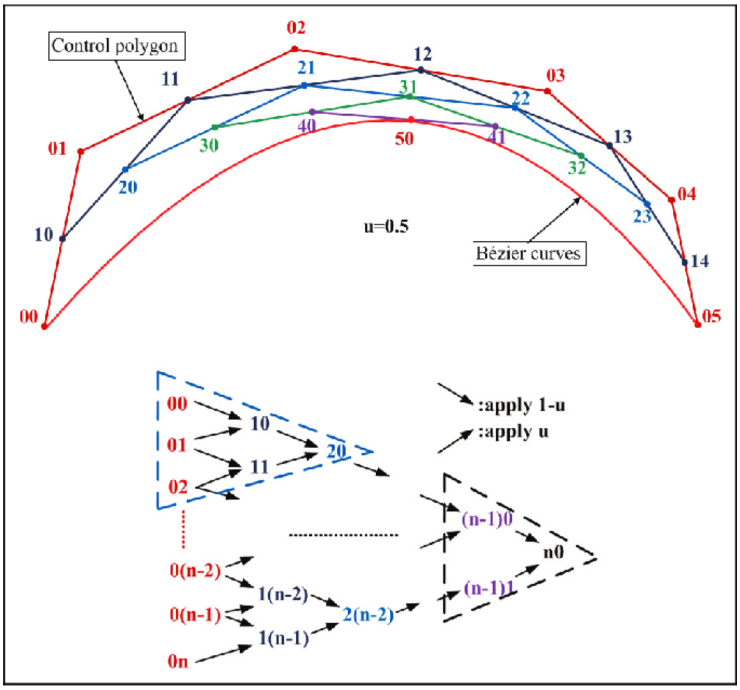

Figure 2.4: De-Casteljau Algorithm, from Ding et al., “Space cutter compensation method for five-axis nonuniform rational basis spline machining”

Second Method

We do not need to calculate $\Delta^k P_i$.

Proposition 2.3.2.

$$P^{(k)}(t) = \frac{n!}{(n-k)!}\Delta^k P_0^{(n-k)}$$

Thus, for $k = 1$ we have

$$P'(t) = n\left(P_1^{(n-1)} - P_0^{(n-1)}\right)$$

k=2

$$P''(t) = n(n-1)\left(P_1^{(n-2)} - 2P_1^{(n-2)} + P_0^{(n-2)}\right)$$

Intuition of the proof :

$$\Delta P_0 = P_1 - P_0$$
$$\Delta P_1 = P_2 - P_1$$

Which Gives

$$\begin{aligned} &= (1-t)\left(P_1 - P_0\right) + t\left(P_2 - P_1\right) \\ &= (1-t)\,P_1 + tP_2 \\ &= -\left((1-t)\,P_0 + P_1\right) \end{aligned}$$

2.4 Tutorial 2: Bézier Curves and Bernstein polynomials

Exercise 1 (Bernstein polynomials)

Show the following :

1. Linear precision :

$$\sum_{i=0}^{n} \frac{i}{n} B_i^n(t) = t \quad \forall t \in [0,1],\ \forall n > 0,\ \forall i \in \{0,\dots,n\}$$

2. Recursive formula

$$B_i^n(t) = (1-t)\, B_i^{n-1}(t) + t B_{i-1}^{n-1}(t) \quad \forall n > 0,\ \forall i \in \{0,\dots,n\}$$

3. The family of Bézier polynomials $(B_9)_{0 \le i \le n}$ is a basis of the space of polynomials of degree $\le n$.

Exercise 2

Let P_0 and P_1 be two points of $\mathbb{R}^d$. Descrie the Bézier curve associated to P_0 and P_1.

Exercise 3

Let $\mathcal{A}$ be the affine space identified to $\mathbb{R}^2$ and γ be the parameterized curve

$$\begin{aligned} \gamma : [0,1] &\to \mathbb{R}^2 \\ t &\to \left(t, t^2\right) \end{aligned}$$

1. Express γ in the monomial basis.
2. Express γ in the Bernstein polyonimals basis.
3. Give the control polygon of γ. Make a drawing with the curve and control polygon.

Exercise 4

Let $[)_0, \cdots, P_6]$ be a control polygon and :

- Express the condition on the P_i to ensure that the Bézier curve

$$P(t) = \sum_{i=0}^{6} P_i B_i^6(t)$$

  is closed of class $\mathscr{C}^2$.

- Draw an example of such a control polygon.

Exercise 5 (Bézier Function)

A Bézier function is a curve of the form

$$\begin{aligned} f : [0,1] &\to \mathbb{R} \\ t &\to \sum_{i=0}^{n} \lambda_i B_i^n(t) \end{aligned}$$

Show that the graph $G(f) := \Big\{ \big(x, f(x)\big), x \in [0,1] \Big\}$ of the function f is a Bézier curve associated to the points $P_i = \big(i/n, \lambda_i\big)$.

Exercise 6

Let $f(t) = \sum_{i=0}^{n} \lambda_i B_i^n(t)$ be a Bézier function.

1. Show that
$$\int_0^1 f(t)\ dt = \frac{\lambda_0 + \cdots + \lambda_n}{n+1}$$
hint : consider the primitive F of f

2. In particular, show that
$$\int_0^1 B_i^n(t) = \frac{1}{n+1}$$

Exercise 7 (Degree elevation)

The idea is to use the observation that a polynomial curve P of degree $\leq n$ can be seen as a polynomial curve of degree $\leq n+1$, namely

$$P(t) = \sum_{i=0}^{n} P_i B_i^n(t) = \sum_{i=0}^{n+1} Q_i B_i^{n+1}(t)$$

1. Show that
$$B_i^n(t) = \frac{n+1-i}{n+1} B_i^{n+1}(t) + \frac{n+1}{i+1}(t)$$

2. Calculate Q_i in terms of the P_j's

2.4.1 Solutions

1)

1. The first index vanishes, so we can rewrite the sum as
$$\sum_{i=1}^{n} \frac{i}{n} B_i^n(t)$$

Furthermore,

$$i\binom{n}{i} = \frac{n\,(n-1)!}{(i-1)!\,(n-i)!} = n\binom{n-1}{i-1}$$

Thus, we have, for the first iteration of the sum

$$\frac{n\,(n-1)!}{n\,(n-1)!}t\,(1-t)^{n-1} = t\,(1-t)^{n-1} = t\left(B_0^{n-1}(t)\right)$$

which allows us to rewrite the sum as

$$t\sum_{i=0}^{n} B_i^n(t) = t\,(1) = t$$

2.

$$\begin{aligned}
B_i^n(t) &= (1-t)\,B_i^{n-1}(t) + tB_{i-1}^{n-1}(t) \\
&= (1-t)\binom{n-1}{i}t^i\,(1-t)^{n-1-i} + t\binom{n-1}{i-1}t^{i-1}\,(1-t)^{n-1-(i-1)} \\
&= \binom{n-1}{i}t^i\,(1-t)^{n-i} + \binom{n-1}{i-1}t^i\,(1-t)^{n-i} \\
&= \left(\binom{n-1}{i} + \binom{n-1}{i-1}\right)t^i\,(1-t)^{n-i} \\
&= \binom{n}{i}t^i\,(1-t)^{n-i} = B_i^n(t)
\end{aligned}$$

3. To show $B_i^n(t)$ as a basis of the space of polynomials of degree $\leq n$ we use derivation. Consider how

$$B_i^n(t) \implies \binom{n}{i}t^i\,(P)$$

where P is some polynomial. Then we can write this as a sum under the form

$$\alpha_0\,(P) + \alpha_1 t\,(P) + \cdots + \alpha_n t^n = 0$$

We check each α_i. Firstly, taking the derivative of this polynomial gives us

$$\alpha_1\,(P) + \cdots + n\alpha_n t^{n-1} = 0$$

Then $\alpha_0 = 0$, this can be done iteratively for each α_i. Then $B_i^n(t)$ is a linear independent set and a basis for polynomials of degree $\leq n$.

2)

The Bézier curve consisting of only 2 points is the straight line :

$$\begin{aligned} P(t) &= \sum_{i=0}^{1} P_i B_i^n(t) \\ &= P_0 (1-t) + P_1 t \\ &= P_0 + (P_1 - P_0) t \end{aligned}$$

3)

1. The monomial basis has representation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^2$$

2. Bernstein basis is given by :

$$B_0^2(t) = (1-t)^2 ; \quad B_1^2(t) = 2t(1-t) ; \quad B_2^2(t) = t^2$$

which results in the following :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} B_0^2(t) + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} B_1^2(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} B_2^2(t)$$

3. The curve is given by :

4)

- In order to ensure that the Bézier curve

$$P(t) = \sum_{i=0}^{6} P_i B_i^6(t)$$

to be closed of class $\mathscr{C}^2$ we must have

1. $P(0) = P(1)$
2. $P'(0) = P'(1)$
3. $P''(0) = P''(1)$

Thus,

1.

$$P_0 = P_n$$

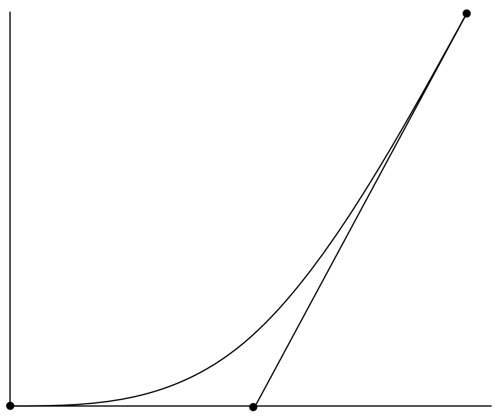

Figure 2.5: e4p3

2.

$$P'(0) = 6\left(P_1 - P_0\right) = P'(1) = 6\left(P_5 - P_6\right)$$
$$\overrightarrow{P_1P_0} = \overrightarrow{P_5P_6}$$

3.

$$P''(0) = 30\left(P_2 - 2P_1 + P_0\right) = P''(1) = 30\left(P_4 - 2P_5 + P_6\right)$$
$$\text{since } P_0 = P_6$$
$$P_2 - 2P_1 = P_4 - 2P_5$$
$$\overrightarrow{P_2P_4} = 2\overrightarrow{P_1P_5}$$

This system of equations results in the figure :

5)

This is found through simple computation. We take

$$P_i = \begin{pmatrix} \frac{i}{n} \\ \lambda_i \end{pmatrix}$$

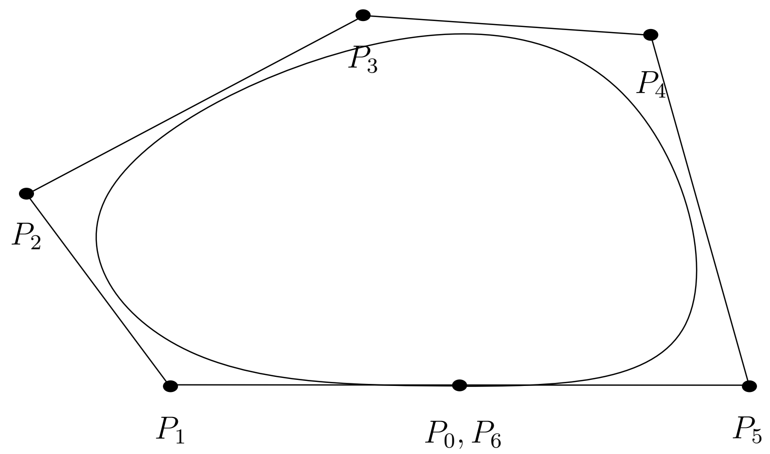

Figure 2.6: Closed of Class $\mathscr{C}^2$

Then

$$\begin{aligned}\sum_{i=0}^{n} \begin{pmatrix} i/n \\ \lambda_i \end{pmatrix} B_i^n(t) &= \begin{pmatrix} \sum_{i=0}^{n} \frac{i}{n} B_i^n(t) \\ \sum_{i=0}^{n} \lambda_i B_i^n(t) \end{pmatrix} \\ &= \begin{pmatrix} t \\ f(t) \end{pmatrix}\end{aligned}$$

6)

1.

$$\int_0^1 f(t)\ dt$$

2.

7)

1.

$$
\begin{aligned}
&\frac{n+1-i}{n+1} B_i^{n+1}(t) + \frac{i+1}{n+1} B_{i+1}^{n+1}(t) \\
&= \frac{n+1-i}{n+1} \binom{n+1}{i} t^i (1-t)^{n+1-i} + \frac{i+1}{n+1} \binom{n+1}{i+1} t^{i+1} (1-t)^{n+1-(i+1)} \\
&= \frac{n!}{i!\,(n-i)!} t^i (1-t)^{n+1-i} + \frac{n!}{i!\,(n-i)!} t^{i+1} (1-t)^{n-i} \\
&= \binom{n}{i} t^i (1-t)^{n-i} \big((1-t)+t\big) \\
&= B_i^n(t)
\end{aligned}
$$

2. We begin by expanding using the form above :

$$
\begin{aligned}
\sum_{i=0}^{n} P_i B_i^n(t) &= \sum_{i=0}^{n} P_i \left(\frac{n+1-i}{n+1} B_i^{n+1}(t) + \frac{i+1}{n+1} B_{i+1}^{n+1}(t) \right) \\
&= \sum_{i=0}^{n} \frac{n+1-i}{n+1} P_i B_i^{n+1}(t) + \sum_{i=1}^{n+1} \frac{i}{n+1} P_{i-1} B_i^{n+1} \\
&= P_0 B_0^{n+1}(t) + \sum_{i=1}^{n} \left(P_i \frac{n+1-i}{n+1} + P_{i-1} \frac{i}{n+1} \right) B_i^{n+1}(t) + P_n B_{n+1}^{n+1}
\end{aligned}
$$

Chapter 3

Curves in the Plane

3.1 Introduction

To represent them, we use

Definition 3.1.1 (Parametrized Curves)**.**

$$\begin{aligned} \gamma : [a, b] &\to \mathbb{R}^d \\ t &\to \gamma(t) \end{aligned}$$

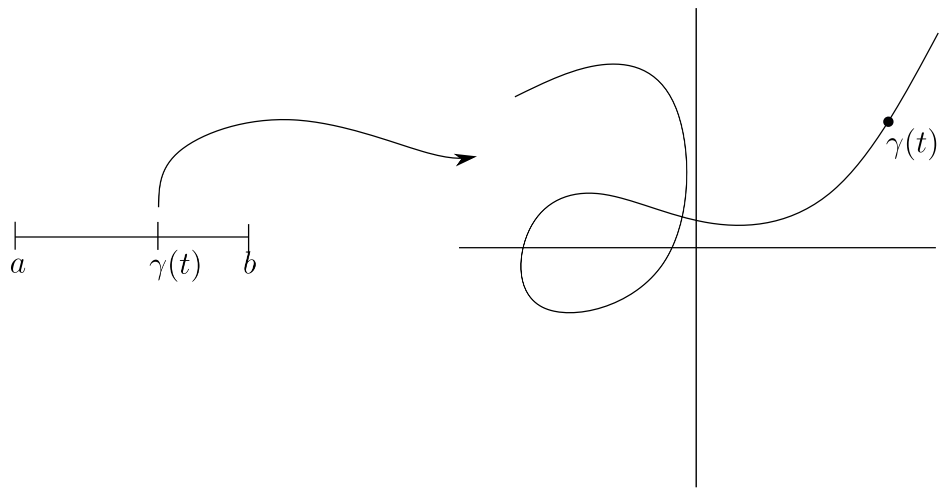

Figure 3.1: Parametrized Planar Curve

Definition 3.1.2 (Implicit Curves)**.** *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *then* $f^{-1}(\{0\}) = \big\{(x, y) \in \mathbb{R}^2, f(x, y) = 0\big\}$ *is a curve.*

Example 3.1.3.

$$f(x, y) = x^2 + y^2 - 1$$

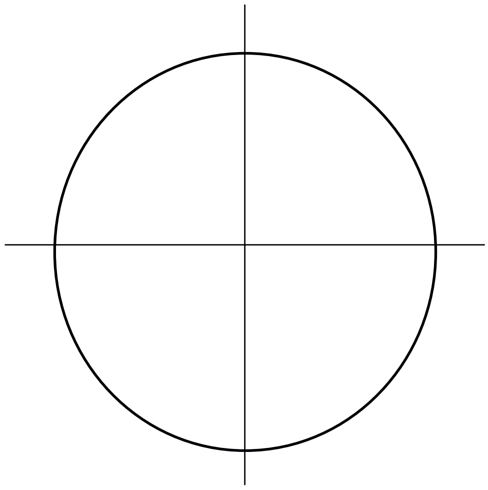

Figure 3.2: Example 3.1.3

Definition 3.1.4 (Graphs of Functions)**.** $\varphi : [a, b] \to \mathbb{R}$.

$$graph(\varphi) = \left\{ \big(t, f(t)\big), t \in [a, b] \right\}$$

Example 3.1.5. $\varphi = \sqrt{1-t^2}$ *and* $t \in [-1, 1]$ *then*

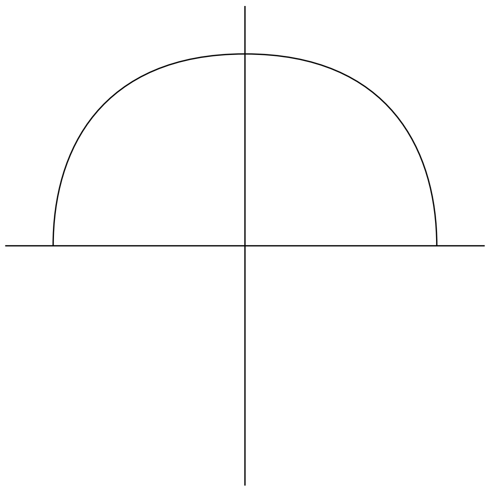

Figure 3.3: Example 3.1.5

3.2 Generalities on Paramaterized Curves

3.2.1 Reminder

Let $f : [a, b] \to \mathbb{R}^2 \in \mathscr{C}^n$ and $t_0 \in [a, b]$ then

$$f(t) = f(t_0) + (t - t_0)\, f'(t_0) + \frac{(t-t_0)^2}{2!} f''(t_0) + \cdots + \frac{(t-t_0)^n}{n!} f^{(n)}(t_0) + \mathcal{O}\left((t-t_0)^n\right)$$

In particular,

$$\frac{f(t) - f(t_0)}{t - t_0} = f'(t_0)$$

let curve be $\mathscr{C} = \{f(t)\}$ and $f(t_0) \in \mathscr{C}$ and $f'(t_0)$ is a vector tangent to $\mathscr{C}$ at $f(t_0)$.

Some notes about 2nd derivative and how it "attracts" the curve.

$$f(t) = f(t_0) + f^{(p)}(t_0) \frac{(t-t_0)^p}{p!} + \cdots + f^{(q)}(t_0) \frac{(t-t_0)^q}{q!} + \mathcal{O}(\dots)$$

p is the smallest k such that $f^{(k)}(t_0) \neq 0$ and q is smallest q such that

$$\left(f^{(q)}(t_0), f^{(k)}(t_0)\right) \text{ independent}$$

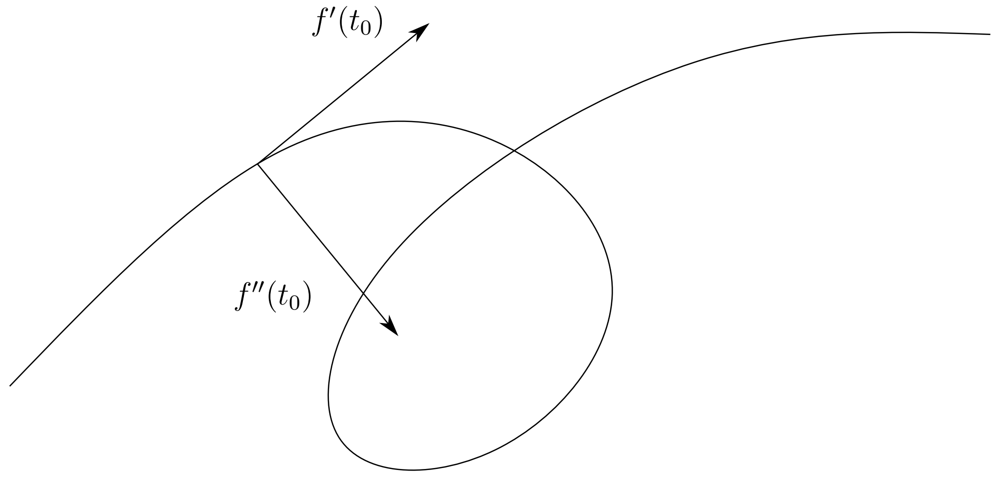

Figure 3.4: linear independence between derivatives

Certain characteristics of the curve can be given by the values of p, q. These values then blah blah blah

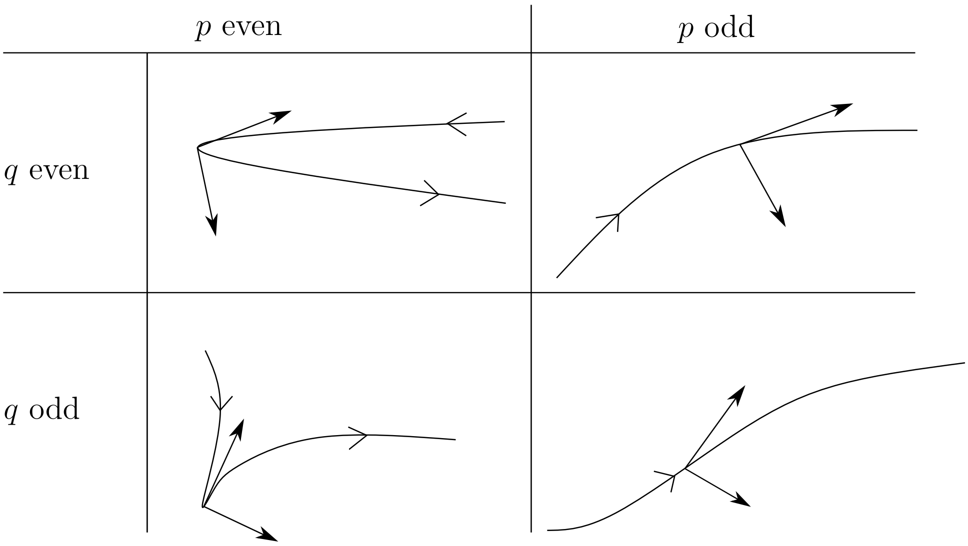

Figure 3.5: pqCurveCharacteristics

3.3 Parametrization and Geometric Curves

Definition 3.3.1 (Parametrized Curve)**.** *A Paramterized curve of class $\mathscr{C}^k$ is a map $f : I \subset \mathbb{R} \to \mathbb{R}^3 \in \mathscr{C}^k$, where I is a union of intervals. We denote (I, f) such a curve.*

Remark :

$$F(I) := \mathscr{C}$$

is the geometric support. Interval connected $\implies$ $\mathscr{C}$ is connected. I compact set $\implies$ $\mathscr{C}$ compact set.
Remark :
Some curve may have 2 paramterization without the same regularity.

Example 3.3.2.

$$t \to \left(t, t^{3/2}\right) t > 0$$
$$t \to \left(|t|, -\sqrt{t^3}\right) t \leq 0$$

another parametrization is

$$t \to \left(t^2, t^3\right) \in \mathscr{C}^\infty$$

3.3.1 ReParametrization

Let $f : I \to \mathbb{R}^3$ param curve in $\mathscr{C}^k$ and $e : J \to I$ is a $\mathscr{C}^k$ diffeomorphism (bijective, $e'(x) \neq 0, \mathscr{C}^k$. Then $f \circ e : J \to \mathbb{R}^3$ has the same "geometric curve" and we say that

- $f \circ e$ is a reparametrization of f
- e is called an admissible change of variable

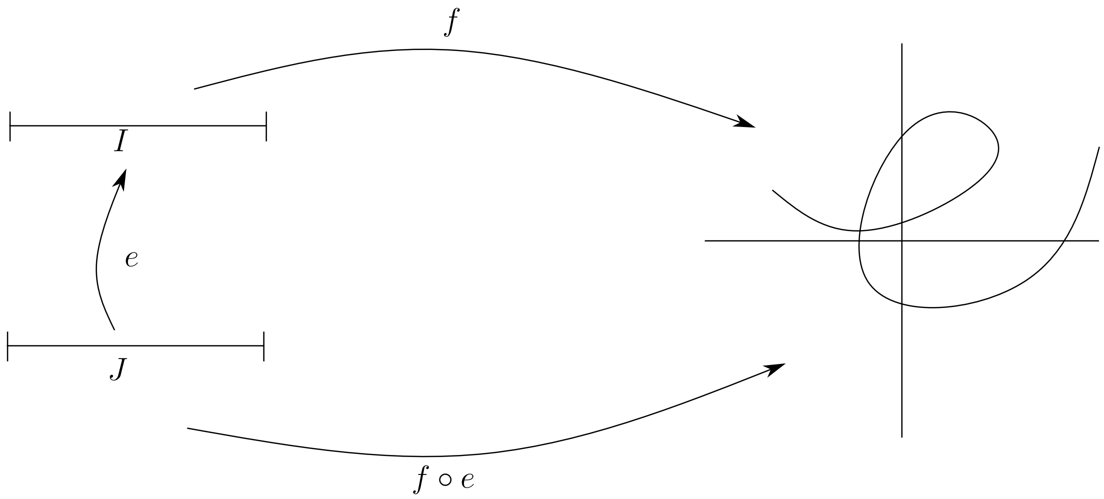

Figure 3.6: Reparamaterized Curve

We consider the following equivalence class

Definition 3.3.3 (Equivalence Class for Curves).
$(I, f) \sim (J, g)$ *if* $\exists e : J \to I, g = f \circ e$ *e admissible change of variable*

Definition 3.3.4 (Geometric Curve). *A geometric curve is an equivalence class of this relation*

3.4 Regular Curve

Definition 3.4.1 (Regular Curve). *Let* $k \geq 1$. *We say that a paramatrized curve* (f, I) *of class* $\mathscr{C}^k$ *is regular if*

$$f'(t) \neq 0 \ \forall t \in I$$

A geometric curve is regular if there exists a paramatrized which is regular.

If $\mathscr{C}$ is of class $\mathscr{C}^1$, then there exists $f : I \to \mathscr{C}$, where $\mathscr{C} = f(I)$

$$f'(t) \neq 0 \quad f'(t) \text{ is tangent to } \mathscr{C}^k \text{ at } f(t)$$

If (I, f) is regular then every reparametrization (J, g) is also regular. Indeed : $\forall t,\ f'(t) \neq 0$ gives

$$g = f \circ e \implies \forall t,\ g'(t) = \underbrace{f'(e(t))}_{\neq 0} \times \underbrace{e'(t)}_{\neq 0} \neq 0$$

Example 3.4.2. *A line segment in $\mathbb{R}^2$ with $t \to (t, at+b)$ regular, can also be reparametrizated by $t \to \left(t^3, at^3+b\right)$ non regular. The reason for this is*

$$f'(t) = \left(3t^2, 3at^2\right) = 0 \ \textit{at } x = 0$$

Remark

This curve does not admit a regular paramatrization.

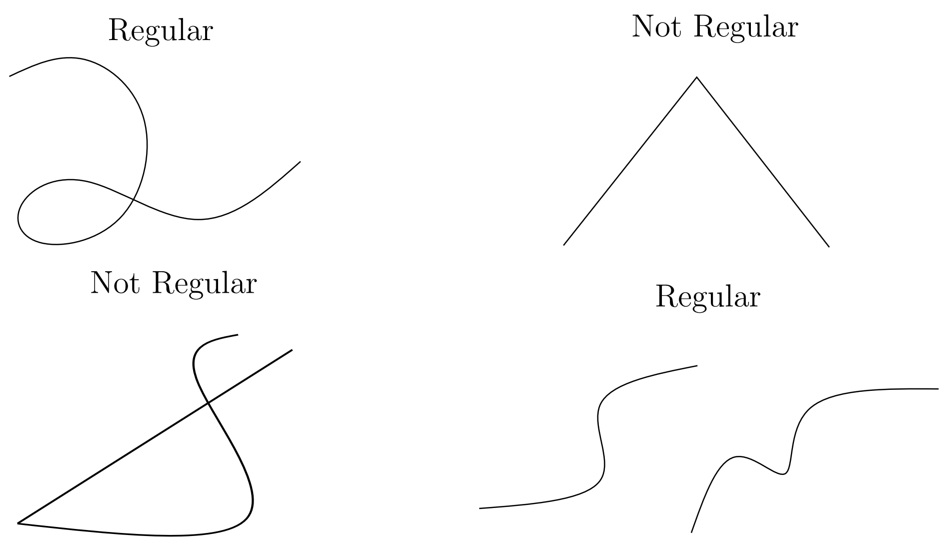

Figure 3.7: Regular and Non-regular curves

However, this is not to say that there does not exist parametrizations of these figures, it is just to say that $f'(a) = 0$ where a is the non-smooth point.

Furthermore, we have

This curve is $\mathscr{C}^1$ since $f''(x) = 0$.

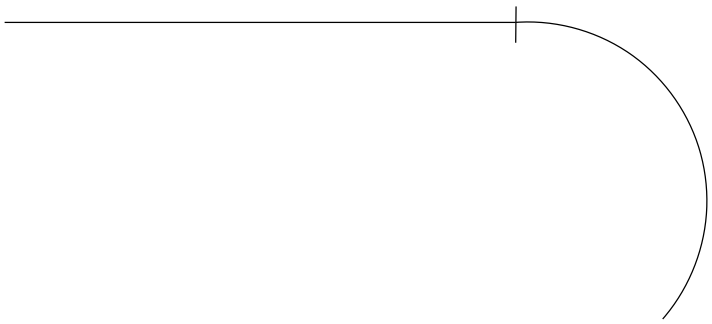

Figure 3.8: $\mathscr{C}^1$ but not $\mathscr{C}^2$

3.5 Metric Properties of Curves

3.5.1 Length of curves

Definition 3.5.1 (Length of a curve)**.** *Let $f : I = [a,b] \to \mathbb{R}^d$. We see that the straight line segments obviously have less length than $\mathscr{C}$.*
Let $\mathscr{S} = \{$ subdivisions $a = t_0 < \cdots < t_n = b\}$.

For $s \in \mathscr{S}$ we denote

$$\gamma(s) = \sum_{i=0}^{N-1} \|f(t_{i+1}) - f(t_i)\|$$

If $\left\{\gamma(s), s \in \mathscr{S}\right\}$ is bounded we say that f is rectifiable. Its lengh is defined by

$$\gamma(f) = \sup_{s\in\mathscr{S}} \gamma(s)$$

Then If $f : [a,b] \to \mathbb{R}^d$ is $\mathscr{C}^1$ then f is rectifiable and

$$\gamma(f) = \int_a^b \|f'(t)\| \; dt$$

Sketch of proof

$$\int_a^b \|f'(t)\| \, dt = \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \|f'(t)\| \, dt$$

The right hand side of this term gives us

$$\begin{aligned} &\sim (t_{i+1} - t_i) \, \|f'(t)\| \\ &\sim (t_{i+1} - t_i) \, \frac{\|f(t_{i+1} - f(t_i))\|}{|t_{i+1} - t_i|} \end{aligned}$$

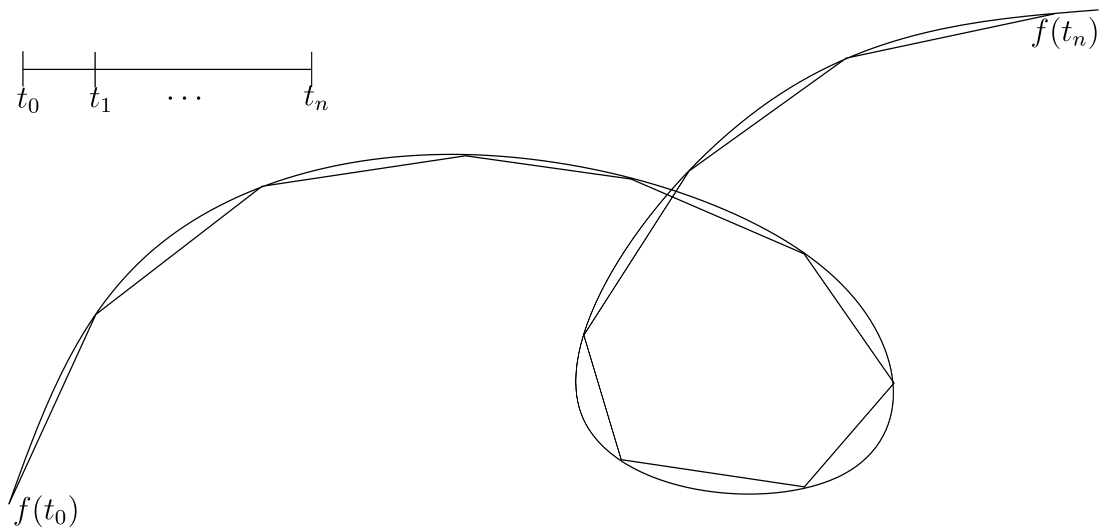

Figure 3.9: Length of a Curve

3.5.2 Arc Length Parametrization

Definition 3.5.2. *Let $(I, f) \in \mathscr{C}^1$ where $I = [a, b]$, $t_0 \in I$. We call arc-length the map*

$$\begin{aligned} &: I \to \mathbb{R} \\ &t \to \int_{t_0}^{t} \|f'(\mu)\| \, d\mu \end{aligned}$$

Remark

$\left|\sigma(t)\right|$ is the length of the curve between $f(t)$ and $f(t_0)$ where $\sigma(t) < 0 \iff t < t_0$

Remark

If f is regular then σ is strictly increasing and $\sigma'(t) = \|f'(t)\| > 0$, therefore, σ is an admissable change of variable of class $\mathscr{C}^1$

Definition 3.5.3. *$f \circ \sigma^{-1}$ is an arc-length paramatrization of the curve.*

So every $\mathscr{C}^1$ regular curve admits an arc-length paramatrization. We use, by convention, S as the parameter of the arc-length parametrization.

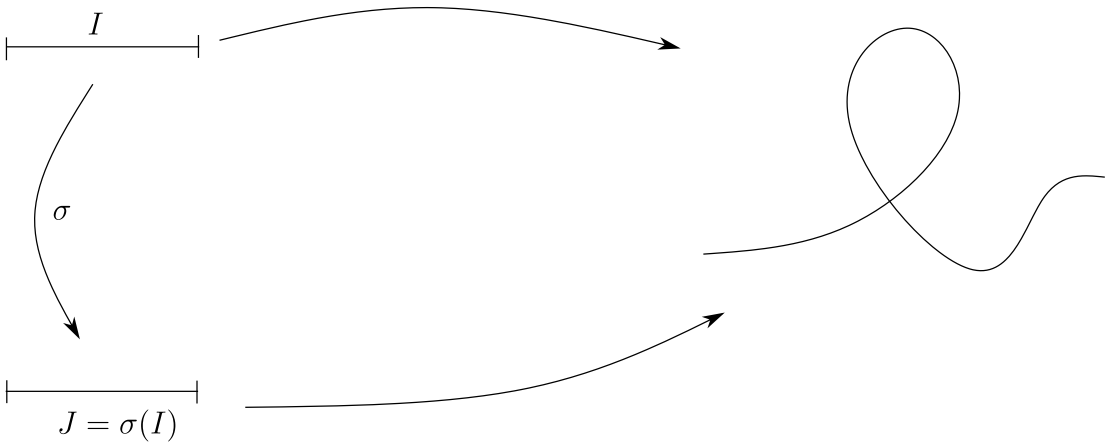

Figure 3.10: Arc-length Paramatrization

Proposition 3.5.4.
- *The arc-length paramatrization is unique up to the parameter t_0 if $I = [a, b],\ t_0 = a$.*
- *$\forall s \in J,\ \|f'(s)\| = 1$ if f arc-length param*
- *$\forall s \in J, f'(s) \perp f''(s)$ for f arc-length*

Proof. Let g be any parametrization and $f = g \circ \sigma^{-1}$. Then

$$\forall s\ f'(s) = g'\left(\sigma^{-1}(s)\right) \times \left(\sigma^{-1}\right)'(s) = \frac{g'\left(\sigma^{-1}(s)\right)}{\|g'\left(\sigma^{-1}(s)\right)\|}$$

where

$$\left(\sigma^{-1}\right)'(s) = \frac{1}{\sigma'\left(\sigma^{-1}(s)\right)} = \frac{1}{\|g'\left(\sigma^{-1}(s)\right)\|}$$

Then $\|f'(s)\| = 1$ and

$$\forall s \in J \ \|f'(s)\|^2 = \langle f'(s), f'(s) \rangle = 1$$

We derive, $\forall s \in J$

$$2\langle f''(s), f'(s) \rangle = 0 \implies f''(s) \perp f'(s)$$

□

3.6 Planar Curves

3.6.1 Serret-Fresnet Frame

Let $f : I \to \mathbb{R}^2, \in \mathscr{C}^1$-regular. Then

$$T(t) = \frac{f'(t)}{\|f'(t)\|} \quad N(t) = \operatorname{rot}_{\frac{\pi}{2}}\big(T(t)\big)$$

So $\big(f(t), \boldsymbol{T}(t), \boldsymbol{N}(t)\big)$ is a frame that is called the Serret-Fresnet Frame.

Remark

If arc-length then we have

$$T(s) = f'(s) \qquad N(s) = \operatorname{rot}_{\frac{\pi}{2}}\big(T(s)\big)$$

3.6.2 Curvature

Definition 3.6.1 (Curvature)**.** *Let $f : I \to \mathbb{R}^2$ arc-length. The curvature at $f(s)$ is defined by*

$$k(s) \coloneqq \langle f''(s), N(s) \rangle = \pm \|f''(s)\|$$

Proposition 3.6.2. *Let $f : I \to \mathbb{R}$ any parametrization. Then*

$$k(u) = \frac{det\big(f'(u), f''(u)\big)}{\|f'(u)\|^3}$$

Proof. We denote $\overline{f} = f \circ \sigma^{-1}$ the arc-length parametrization. We put $u = \sigma^{-1}(s)$. Then

$$\begin{aligned} \overline{f}'(s) &= f'\left(\sigma^{-1}(s)\right) \frac{1}{\|f'\left(\sigma^{-1}(s)\right)\|} \\ &= \frac{f'(u)}{\|f'(u)\|} \end{aligned}$$

We derive again

$$\overline{f}''(s) = \frac{f''(u)}{\|f'(u)\|^2} + f'(u)\frac{d}{ds}$$

where $\frac{d}{ds}$ is the real value result of the rhs above

$$\begin{aligned}\det\left(\overline{f}'(s), \overline{f}''(s)\right) &= \det\left(\frac{f'(u)}{\|f'(u)\|}, \frac{f''(u)}{\|f'(u)\|^2} + \lambda u f'(u)\right)\\ &= \frac{\det\left(f'(u), f''(u)\right)}{\|f'(u)\|^3}\end{aligned}$$

And

$$\begin{aligned}\det\left(\overline{f}'(s), \overline{f}''(s)\right) &= \det\left(T(s), k(s)N(s)\right)\\ &= k(s)\end{aligned}$$

since,

$$f''(s) = k(s)N(s)$$
$$k(s) \coloneqq \langle f''(s), N(s)\rangle = \pm\|f''(s)\|$$

$\square$

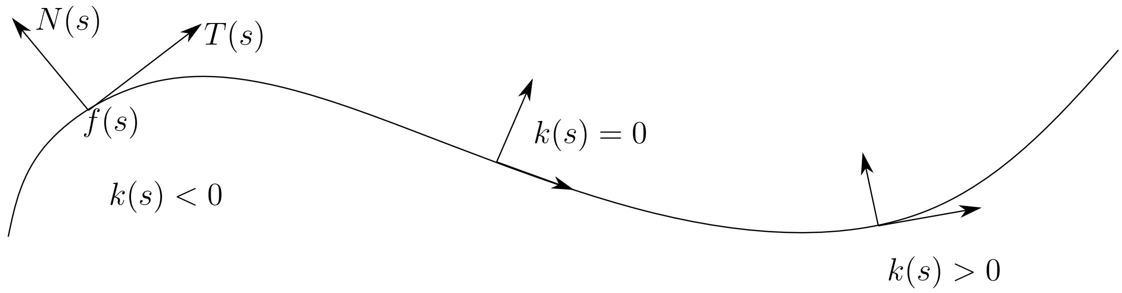

Figure 3.11: curvature

3.6.3 Osculating Circle and Center of Curvature

Definition 3.6.3.

$$c(t) = f(t) + \frac{1}{k(t)} N(t)$$

is called the center of curvature.

$$\frac{1}{|k(t)|}$$

is the radius of curvature at $f(t)$ The circle

$$\mathscr{C}\left(c(t), \frac{1}{|k(t)|}\right)$$

The evolute of f is the set of centers of curvatures.

3.6.4 Serret-Fresnet Formula

Proposition 3.6.4.

$$T'(s) = k(s)N(s) \qquad N'(s) = -k(s)T(s)$$

lhs done is done, and rhs is by definition

3.6.5 Total Curvature

Theorem 3.6.5 (Total Curvature)**.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}^2$ planar curve parametrized by arc-length, then*

$$\int_a^b k(s)\ ds = \theta(a, b)$$

is the angle between the two tangents at a and b.

Proof.

$$f(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \qquad \theta(s) = \big((o, x), T(s)\big)$$

where o is the angle between tangent and the x? Then

$$T(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix} \qquad N(s) = \begin{pmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{pmatrix}$$

However,

$$T'(s) = k(s)N(s) \quad \text{and } T'(s) = \theta'(s)N(s)$$

then $\theta'(s) = k(s)$. then

$$\int_a^b \theta'(s)ds = \theta(b) - \theta(a)$$

which is the difference between the angles. $\square$

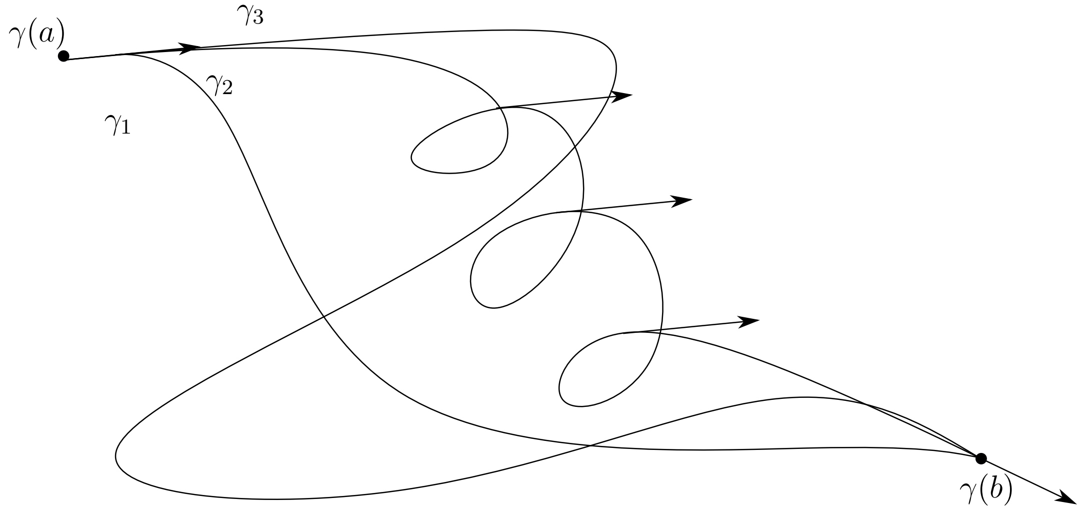

Figure 3.12: Total Curvature

In figure 3.12 we have $\theta_1(a,b) = \theta_2(a,b) = \theta_3(a,b) + 6\pi$. We defined the "winding number" as an index refering to the full revolutions around the a curve that is closed of class $\mathscr{C}^2$.

$$k = \frac{l}{2\pi} \int_a^b k(s)\ ds \in \mathbb{Z}$$

Concluding Thoughts

- Metrics of a curve are given by the 1st derivative
- Them shape of a curve is given by the second derivative.
- Arc length parametrization gives constant speed along the curve

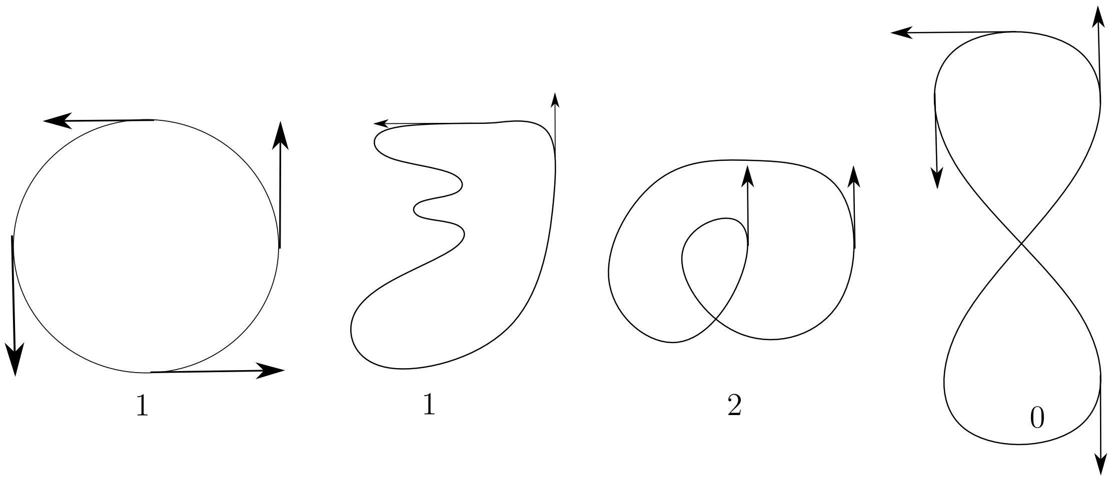

Figure 3.13: Winding Numbers

Chapter 4

Space Curves

4.1 Definition

Definition 4.1.1 (Space Curve)**.** *A space curve is a curve in* $\mathbb{R}^3$ *which is not planar.*

$$f : I \subset \mathbb{R}^2 \to \mathbb{R}^3 \ \mathscr{C}^k \ k \geq 2$$

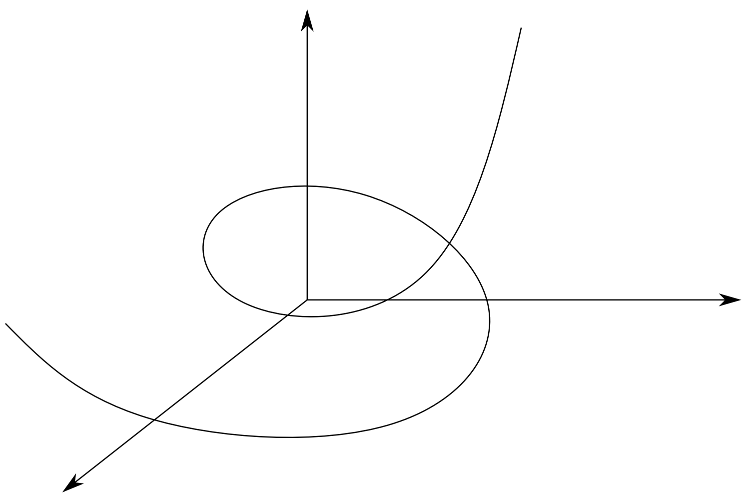

Figure 4.1: Space Curve

Length is calculated the exact same way.

4.2 Curvature and Principle Normal

Definition 4.2.1. *if $f : I \to \mathbb{R}^3$ is arc-length param then the curvature is defined by*

$$k(s) = \|f''(s)\|$$

NOTE:
$k(s) \geq 0$ *for* $\mathbb{R}^3$ *but in* $\mathbb{R}^2, k(s) = \langle f'(s), N(s)\rangle = \pm\|f''(s)\|$

Proposition 4.2.2. *If f is any parametrization*

$$k(t) = \frac{\|f'(t) \wedge f''(t)\|}{\|f'(t)\|^3}$$

Proof. We denote $\bar{f} = f \circ \sigma^{-1}$ the arc-length parametrization.

$$\|\bar{f}'(s)\|^2 = 1 \implies 2\langle \bar{f}''(s), \bar{f}'(s)\rangle = 0$$

$$\implies \bar{f}''(s) \perp \bar{f}'(s)$$

$$k(s) = \|\bar{f}''(s)\| = \|\bar{f}'(s) \wedge \bar{f}''(s)\|$$

and $\bar{f} = f \circ \sigma^{-1}$

$$\implies \bar{f}'(s) = f'\left(\overbrace{\sigma^{-1}(s)}^{=t}\right) \times \left(\sigma^{-1}\right)'(s) = \frac{f'(t)}{\|f'(t)\|}$$

$$\bar{f}''(s) = f''\left(\overbrace{\sigma^{-1}(s)}^{=t}\right) \times \left(\sigma^{-1\prime}(s)\right)^2 + f'\left(\sigma^{-1}(s)\right) \times \lambda \quad \lambda \in \mathbb{R}$$

then

$$\|\bar{f}'(s) \wedge \bar{f}''(s)\| = \|\frac{f'(t)}{\|f'(t)\|} \wedge \frac{f''(t)}{\|f'(t)\|^2}\|$$

□

Definition 4.2.3. *Suppose $f : I \to \mathbb{R}^3$ is arc-length then*

$$\vec{N}(s) = \frac{1}{k(s)}\vec{T}(s) = \frac{f''(s)}{\|f''(s)\|}$$

is called the principle normal and

$$\vec{T}(s) = f'(s) \text{ is tangent at } f(s)$$

a point $f(s)$ is biregular if $f''(s) \neq 0$

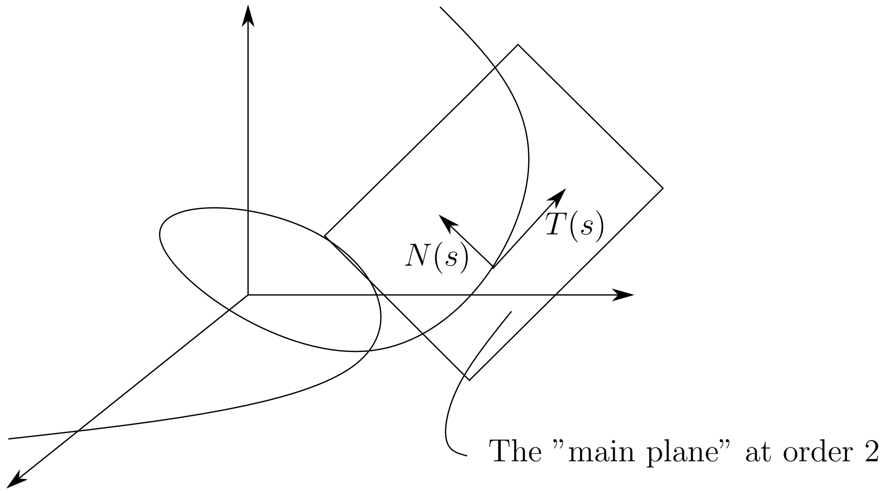

Figure 4.2: NT for R3

Remark

A point is biregular if the curvature is $\neq$ from 0.

Definition 4.2.4 (Osculating Plane). *The osculating plane at a biregular point is spanned by $T(s)$ and $N(s)$*

- $R(s) = \frac{1}{k(s)}$ *is the readius of curvature*
- $C(s) = f(s) + R(s)\vec{N}(s)$ *center of curvature*
- $\mathscr{S}\big(C(s), R(s)\big)$ *osculating sphere*
- $\mathscr{S}\big(C(s), R(s)\big) \cap (\textit{Osculating Plane})$ *gives us the soculating circle*

4.3 Serret-Frenet frame

Definition 4.3.1 (Serret-Fresnet Frame). *if f is defined by arc-length biregular then*

$$B(s) = T(s) \wedge N(s)$$

is called the binormal, and

$$\big(f(s), T(s), N(s), B(s)\big)$$

is called the Serret-Fresnet

Remark

This frame does not exist at non-biregular points

Definition 4.3.2.

- *the plane : $\langle N(s), T(s)\rangle$ osculating plane*
- *the plane : $\langle N(s), B(s)\rangle$ normal plane*
- *the plane : $\langle T(s), B(s)\rangle$ rectifiable plane*

4.4 Tortion

Proposition 4.4.1. *$B'(s)$ is colinear to $N(s)$*

Proof. □

Definition 4.4.2 (Torsion). *At a biregular point, $f \in \mathscr{C}^3$ the torsion $\tau(s)$ is defined by*

$$B'(s) = -\tau(s)N(s)$$

It is a "measure" of how the osculating plane varies and twists

Proposition 4.4.3.

$$\tau(s) = \frac{\det\left(f'(s), f''(s), f'''(s)\right)}{\|f''(s)\|^2}$$

Proof.

$$\tau(s) = -\langle N(s), B'(s)\rangle \quad B(s) = T(s)\wedge N(s)$$

$$\implies B'(s) = T'(s)\wedge N(s) + T(s)\wedge N'(s)$$

$\implies$

$$\tau(s) = \underbrace{-\langle N(s), T'(s)\wedge N(s)\rangle}_{=0} - \langle N(s), T(s)\wedge N'(s)\rangle$$

by definition

$$= \det\left(N(s), T(s), N'(s)\right)$$

However, $N(s) = R(s)f''(s)$ and $T(s) = f'(s)$ which gives

$$N'(s) = R'(s)f''(s) + R(s)f'''(s)$$

$$\tau(s) = \det\Big(R(s)f''(s), f'(s), R'(s)f''(s)_{\text{colinear to}R(s)f''(s)} + R(s)f'''(s)\Big)$$

so we can invert the two vectors and introduce a minus sign to get

$$= \det\left(f'(s), f''(s), f'''(s)\right)R^2(s)$$

and

$$R(s) = \frac{1}{k(s)} = \frac{1}{\|f''(s)\|}$$

□

Proposition 4.4.4. *For any param for biregular points we have*

$$\tau(t) = \frac{\det\big(f'(t), f''(t), f'''(t)\big)}{\|f'(t) \wedge f''(t)\|^2}$$

Proof. Admitted □

Proposition 4.4.5. *$f \in \mathscr{C}^3$ biregular $f : I \to \mathbb{R}^3$ $\tau(s) \equiv 0 \iff f$ planar*

Proof. $\leftarrow$ $T(s)$ and $N(s)$ are in the same plane then $B(s)$ constant and $B'(s) = 0$ then $\tau \equiv 0$.
$\implies$ if $\tau \equiv 0$ and $B'(s) = -\tau(s)N(s) \implies$ B'(s) = 0 $\forall s \implies B(s) = \vec{B}_0$ we show that $\langle B_0, f(s)\rangle =$ constant.
However,

$$\big(\langle B_0, f\rangle\big)' = \langle B_0, \rangle$$

□

4.5 Serret-Fresnet Formla

Definition 4.5.1 (SF Formula).

$$\begin{cases} T'(s) & = k(s)N(s) \\ N'(s) & = k(s)T(s) + \tau(s)B(s) \\ B'(s) & = -\tau(s)N(s) \end{cases}$$

Proof. 1,3 ok.

□

4.6 Fundamental Theorem for Local Theory of Curves

Theorem 4.6.1. *Let $k : [a, b] \to \mathbb{R}^t \in \mathscr{C}^1$ and $\tau : [a, b] \to \mathbb{R} \in \mathscr{C}^1$ then $\exists!$ curve $f : [a, b] \to \mathbb{R}^3 \in \mathscr{C}^3$ parameterized by arc-length with curvature k and torsion τ up to a rigid transformation.*

So a curve is completely determined by its tortion and curvature.

4.7 Tutorial 3: Plane and Space Curves

Exercise 1

Calculate the curvature of the planar curve parametrized by $f(t) = \big(t, \varphi(t)\big)$

Exercise 2

1. Give a parametrization of the circle of center O and radius R.
2. Calculate the curvature, radius of curvature and center of curvature at every point of the circle.

Exercise 3

Show that planar curves with constant curvature are either arc of circles or segments of a line.

Exercise 4 (Circular Helix)

We consider the curve $f : \mathbb{R} \to \mathbb{R}^3$ parameterized by

$$f(t) = (R\cos t, R\sin t, at)$$

1. Determine an arc-length representation
2. Dtermine the Serret-Fresnet frame
3. Calculate the curvature and torsion
4. Calculate the set of curvature centers

Exercise 5

Calculate the evolute (i.e. the set of the centers of curvature) of the ellipse.

Exercise 6

We consider the curve $\gamma(t) = \left(t^2, t^3\right)$ for $t \geq 0$

1. Draw the curve. Is the curve regular?
2. Calculate the arc-length, curvature, and express the curvature with arc-length parametrization for $t > 0$.
3. Provide a parametrixation of the curve of class $\mathscr{C}^1$ and regular.
4. Show that there is no parametrization of class $\mathscr{C}^2$ of the curve at the point $(0, 0)$

4.7.1 Solutions

Bibliography

Ding, Yanyu et al. "Space cutter compensation method for five-axis nonuniform rational basis spline machining". In: *Advances in Mechanical Engineering* 7 (July 2015). DOI: 10.1177/1687814015594125.