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Atoms

The time frequency atoms are functions $\phi_\gamma(t) = \phi(t, \gamma)$ where $t \in \mathbb{R}$ and $\gamma \in \mathbb{R}^d$, $d \in \mathbb{N}$. We assume that $||\phi_\gamma||_{\mathbb{L}^2(\mathbb{R})} = 1$ These atoms allow to analysis a function $f \in \mathbb{L}^2(\mathbb{R})$ via the time frequency transform:

$$Tf(\gamma) = \int_{-\infty}^{+\infty} f(t)\overline{\phi_\gamma(t)}dt$$

1. Show that
$$\int_{-\infty}^{+\infty} f(t)\overline{\phi_\gamma(t)}dt = \int_{-\infty}^{+\infty} \hat{f}(\xi)\overline{\hat{\phi}_\gamma(\xi)}d\xi$$

2. Show that if $\phi_\gamma(t)$ is null outside a neigbourhood $v(u)$ of u and $\hat{\phi}_\gamma(\xi)$ is null outside a neigbourhoud $v(\omega)$ of ω, then $Tf(\gamma)$ uses only the values of f for $t \in v(u)$ and $\lambda \in v(\omega)$.

$|\phi_\gamma(t)|^2$ can be considered as a probability distribution centered in $u_t(\gamma)$ and with variance $\sigma_t^2(\gamma)$ where:

$$\mu_t(\gamma) = \int_{-\infty}^{+\infty} t|\phi_\gamma(t)|^2 dt \text{ and } \sigma_t^2(\gamma) = \int_{-\infty}^{+\infty} (t - u_\gamma)^2|\phi_\gamma(t)|^2 dt$$

$|\hat{\phi}_\gamma(\xi)|^2$ can be considered as a probability distribution centered in $\mu_\xi(\gamma)$ and with variance $\sigma_\xi^2(\gamma)$ where:

$$\mu_\xi(\gamma) = \int_{-\infty}^{+\infty} \xi|\hat{\phi}_\gamma(\xi)|^2 d\xi \text{ and } \sigma_\xi^2(\gamma) = \int_{-\infty}^{+\infty} (\xi - \omega_\gamma)^2|\hat{\phi}_\gamma(\xi)|^2 d\xi.$$

We define in the time frequency space (t, ξ), the Heisenberg's box of ϕ_γ as a rectangle of center $(\mu_t(\gamma), \mu_\xi(\gamma))$ and dimension $\sigma_t(\gamma) * \sigma_\xi(\gamma)$.
Let consider the Short Time Fourier transform:

$$\phi_\gamma(t) = w(t - b)e^{2i\pi\lambda t} \text{ with } \gamma = (b, \lambda)$$

where w is odd.
Let consider the continuous wavelet transform::

$$\phi_\gamma(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t - b}{a}\right) \text{ with } \gamma = (b, a)$$

where $\psi(t)$ is centered at 0 ($|\psi(t)|^2$ is odd) and $\hat{\psi}(\xi) = 0, \forall \xi < 0$ (analytic wavelet).

1. Show that the Heisenberg's box of the Short Time Fourier Transform (STFT) atoms is centered at $\gamma = (b, \lambda)$ and that its dimensions do not depend on γ.

2. Show that the Heisenberg's box of the STFT atoms is centered at $(b, \frac{\eta}{a})$ with

$$\eta = \int_0^{+\infty} \xi |\hat{\psi}(\xi)|^2 d\xi$$

   and that its dimension depend on a.

3. Represent in the time frequency space all these boxes and compare them.

4. What is about the Fourier Transform?