^{2} and $\mathscr{L}^1$ Spaces2.9.218$\mathscr{L}^2$ and $\mathscr{L}^1$ Spacessubsection.2.9.2 ^{2} Space2.9.718$\mathscr{L}^2$ Spacetheo.2.9.7 ^{2} Convergence2.9.1321Uniform Converge Implies $\mathscr{L}^2$ Convergencetheo.2.9.13 ^{2}3.3.432Theorem of Convergence in $\mathscr{L}^2$theo.3.3.4 5.1.155$Non-rigorousdevelopmentofthe$ FT $subsection$.5.1.1^1 ($\mathbb{R}$)5.257Fourier Transform in $\mathscr{L}^1$ ($\mathbb{R}$)section.5.2 5.359$LinkBetweenDFTand$ FT $section$.5.3 on $\mathscr{L}(\mathbb{R})$ 5.467Properties of $\mathscr{F}\mathscr{T}$ on $\mathscr{L}(\mathbb{R})$section.5.4 ($\mathbb{R}$) 5.570FT on $\mathscr{S}$ ($\mathbb{R}$)section.5.5 FT on $\mathrm{L}^2(\mathbb{R})$ on $\mathrm{L}^2(\mathbb{R})$6.173Properties of $\mathscr{F}\mathscr{T}$ on $\mathscr{L}^2(\mathbb{R})$section.6.1

Fourier Analysis and Signal Processing

Joel Andrepont, Lecturer : Cecil Amblard

Fall 2019

Contents

Chapter 1

Sources / References

1.1 Bremaud, Mathematical Principles of Sig- nal Processing : Fourier and Wavelet Anal- ysis

- Introduction to Fourier Analysis in L 1
- appendix on Lebesgue integral
- Signal Processing impulse response, frequency response, band-pass Sig- nals, sections on sampling, DSP, Subband Coding.
- Fourier analysis in L 2
- Wavelet Analysis with section on Windowed Fourier transform cover- ing the uncertainty principle with WFT and Gabor's inversion formula. Wavelet transform, orthonormal expansions, construction of an MRA and smooth MRA.

1.2 Boggess and Narcowich, A First Course in Wavelets with Fourier Analysis

- Exercises with solutions!!!
- Inner product spaces : L 1 , L 2 , orthogonality, triangle inequalities, linear operators , Least squares and predicitve coding.
- Fourier Series : Intro, computation convergence theorems
- Fourier Transform : informal development, properties and convolution, linear filters, sampling theorem, uncertainty principle.
- discrete fourier analysis : discrete fourier transform, discrete signals
- Haar wavelet Analysis : why, Haar wavelets and properties, Decomposition and Reconstruction Algorithms.

- MRA framework and what not. Decomposition and Reconstruction and Wavelet spaces.

1.3 Kaiser, A Friendly Guide to Wavelets

- Background and notation for Dual Bases, innder product spaces, recipro- cal bases, function spaces, fourier Series, integrals signal processing.
- Windowed FT : motivation and definition. time-frequency localization. The reconstruction formula. Singals processing in time-frequency domain.
- Continuous wavelet transforms
- Generalized Frames
- Discrete time-frequency : Analysis and Sampling
- Discrete time-scake analysis
- MRA
- other stuff (physics)

1.4 Sidney Burrus and Guo, Introduction to Wavelets and Wavelet Transforms

Covers only wavelets. Goes quite quickly into MRA (p. 10) Should be used just for MRA I think.

1.5 Mallat, A Wavelet Tour of Signal Processing

This one is a monster (630 pages) this will definitely be referenced often. Shouldn't be used until after fourier series are finished. Good for Fourier transform and time-frequency properties.

- Introduction to time-frequency (Time-Frequency Atoms, Bases)
- Fourier Kingdom(Impulse response, transfer function, L 1 , L 2 , Properties and uncertainty principle.
- Discrete Revolution and finite signals.
- Time meets frequency (Time-Frequency Atoms, Windowed Fourier Trans- form, Wavelet Transforms)
- Frames

- Wavelet Zoom
- Wavelet Bases (orthogonal filter banks, classes)

1.6 Stein and Shakarchi, Fourier Analysis : An Introduction

1.7 James, A Student's Guide to Fourier Transforms

1.8 Gasquet and Witomski, Analyse De Fourier et Applications : Filtrage, calcul numerique et ondelettes

Chapter 2

Introduction

2.1 Riemann Integral

In this section we detail some problems that occur with the Riemann integral and help motivate the use of another integral. Much of this follows from Axler, *Measure, Integration Real analysis*.

2.1.1 Basic Definitions

Definition 2.1.1 (Partition)**.** *Supppose $a, b \in \mathbb{R}$ with $a < b$. A partition $[a, b]$ is a finite list of the form $x_0, x_1, \cdots, x_n$ where*

$$a = x_0 < x_1 < \cdots < x_n = b$$

Where

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cdots \cup [x_{n-1}, x_n]$$

Definition 2.1.2 (Notation for infimum and supremum)**.** *If f is a real-valued function and A is a subset of the domain of f, then*

$$\inf_A f = \inf\{f(x) : x \in A\} \quad \textit{and} \quad \sup_A f = \sup\{f(x) : x \in A\}$$

Where the inf is just the greatest lower bound of the set and sup is the least upper bound of the set. The following definition allows us to approximate the area under the graph of a nonnegative function :

Definition 2.1.3 (Lower and upper Riemann sums)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P is a partition $x_0, \cdots, x_n$ of [a,b]. The lower Riemann sum $L\big(f, P, [a, b]\big)$ and the upper Riemann sum $U\big(f, P, [a, b]\big)$ are defined by :*

$$L\big(f, P, [a, b]\big) = \sum_{j=1}^{n} \big(x_j - x_{j-1}\big) \inf_{[x_{j-1}, x_j]} f$$

and

$$U\left(f, P, [a,b]\right) = \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) \sup_{[x_{j-1},x_j]} f$$

Consider x^2 with a partition of 16 on $[0,1]$. Then each partition is of length $\frac{1}{16}$ and the formula gives us

$$L\left(x^2, P_{16}, [0,1]\right) = \frac{1}{16}\sum_{j=1}^{16} \frac{(j-1)^2}{n^2} = \frac{2n^2 - 3n + 1}{6n^2}$$

which are found using the formula

$$1 + 4 + 9 + \cdots + n^2 = \frac{n\left(2n^2 + 3n + 1\right)}{6}$$

2.1.2 Inequalities with Riemann sums

Adding more points to a partition increases or decreases or lower and upper sums, respectively.

Theorem 2.1.4 (Inequalities with Riemann Sums)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a,b]$ such that the list defining P is a subset of the list defining P'. Then*

$$L\left(f, P, [a,b]\right) \leq L'\left(f, P', [a,b]\right) \leq U'\left(f, P', [a,b]\right) \leq U\left(f, P', [a,b]\right)$$

Proof.
Suppose that we have two partitions consisting of

$$P = x_0, \cdots, x_n \quad \text{and} \quad P' = x'_0, \cdots, x'_N$$

of [a,b]. For each j in P we can find the sublist that covers the length $x_{j-1} \to x_j$ from some $k \in \{0, \cdots, N-1\}$ to some m :

$$x_{j-1} = x\prime_{k < x\prime_{k-1} < \cdots < x\prime_{k+m} = x_j}$$

Thus, we now have two sums to compare

$$\begin{aligned}\left(x_j - x_{j-1}\right) \inf_{\left[x_{j-1},x_j\right]} f &= \sum_{i=1}^{m} \left(x'_{k+i} - x'_{k+i-1}\right) \inf_{\left[x_{j-1},x_j\right]} f \\ &\leq \sum_{i=1}^{m} \left(x'_{k+i} - x'_{k+i-1}\right) \inf_{\left[x_{k+i-1},x_{k+i}\right]} f\end{aligned}$$

this is due to the fact that the infimum of each set of real numbers is less than or equal to the supremum of that set, and, as the partition increases and m grows larger, the sub partition of P' approaches the supremum of the original partition P.
For the supremum we use an analogous argument :
Using the same partitions P, P' and the values k, m we compare two sums Thus, we now have two sums to compare

$$\begin{aligned}\left(x_j - x_{j-1}\right) \sup_{\left[x_{j-1},x_j\right]} f &= \sum_{i=1}^{m} \left(x'_{k+i} - x'_{k+i-1}\right) \sup_{\left[x_{j-1},x_j\right]} f \\ &\geq \sum_{i=1}^{m} \left(x'_{k+i} - x'_{k+i-1}\right) \sup_{\left[x_{k+i-1},x_{k+i}\right]} f\end{aligned}$$

this is due to the fact that the sup of each set of real numbers is greater than or equal to the infimum of that set, and, as the partition increases and m grows larger, the sub partition of P' approaches the infimum of the original partition P. □

Theorem 2.1.5 (Lower Riemann sums $\leq$ Upper Riemann Sums)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a, b]$. Then*

$$L\left(f, P, [a, b]\right) \leq U\left(f, P', [a, b]\right)$$

Proof. Using 2.1.4 and P'' as the merging of P, P' we have

$$\begin{aligned}L\left(f, P, [a, b]\right) &\leq L\left(f, P'', [a, b]\right) \\ &\leq U\left(f, P'', [a, b]\right) \\ &\leq L\left(f, U', [a, b]\right)\end{aligned}$$

□

2.1.3 Riemann Integral

Definition 2.1.6 (Lower and upper Riemann integrals)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. The lower Riemann integral $L\left(f, [a, b]\right)$ and the upper Riemann integral $U\left(f, [a, b]\right)$ are defined as*

$$L\left(f, [a, b]\right) = \sup_{P} L\left(f, P, [a, b]\right)$$

and

$$U\left(f, [a, b]\right) = \inf_{P} U\left(f, P, [a, b]\right)$$

Where P denotes all possible partitions of $[a, b]$.

Theorem 2.1.7 (Lower Riemann Integral $\leq$ Upper Riemann Integral). *Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Then*

$$L\left(f, [a, b]\right) \leq U\left(f, [a, b]\right)$$

Proof. This is due to an extension of 2.1.5. Let P^1 be the partition where $x_0 = a, x_1 = b$ and let each number in P^i denote an additional partition, such that P^2 has 2 partitions and

$$P = \cup_{i=1}^{n} P^i$$

Then

$$\sup L\left(f, P, [a, b]\right) = L\left(f, P^n, [a, b]\right)$$

and

$$\inf U\left(f, P, [a, b]\right) = U\left(f, P^n, [a, b]\right)$$

using 2.1.5 we have

$$L\left(f, P^n, [a, b]\right) \leq U\left(f, P^n, [a, b]\right)$$

which by def(2.1.3) completes the proof. □

Definition 2.1.8 (Riemann integrable). *A bounded function on a closed bounded integral is called Riemann integrable if its lower Riemann integral equals its upper Riemann Integral*

Definition 2.1.9 (Riemann integral). *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then the Riemann integral $\int_a^b$ is defined by*

$$\int_a^b f = L\left(f, [a, b]\right) = U\left(f, [a, b]\right)$$

Example 2.1.10 (Computing a Riemann integral). *Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x^2$. Then*

$$U\left(f, [0, 1]\right) \leq \inf_{n \in \mathbb{Z}^+} \frac{2n^2 + 3n + 1}{6n^2} = \frac{1}{3} = \sup_{n \in \mathbb{Z}^+} \frac{2n^2 - 3n + 1}{6n^2} \leq L\left(f, [0, 1]\right)$$

Thus,

$$\int_0^1 f = \frac{1}{3}$$

2.1.4 Continuity and Riemann Integration

Theorem 2.1.11 (Continuous Functions are Riemann Integrable)**.** *Every continuous real-valued function on each closed bounded interval is Riemann integrable.*

Proof. Suppose that $a, b \in \mathbb{R}$ and $a < b$ with $f : [a, b] \to \mathbb{R}$ is continous. Therefore f is bounded and uniformly continuous[1]. Let $\epsilon > 0$. Because f is uniformly continous there exists a $\delta > 0$ such that

$$\left|f(s) - f(t)\right| < \epsilon \quad \text{for all } s, t \in [a, b] \text{ with } |s - t| < \delta$$

Let $n \in \mathbb{Z}$ be such that $\frac{b-a}{n} < \delta$. Let P be the equally spaced partition $a = x_0, x_1, \cdots, x_n = b$ of [a,b] with

$$x_j - x_{j-1} = \frac{b-a}{n}$$

for each $j = 1, \cdots, n$. Also, Since

$$U\left(f, [a, b]\right) = \inf_P U\left(f, P, [a, b]\right) \implies U\left(f, [a, b]\right) \le U\left(f, P, [a, b]\right)$$

The same follows in the opposite direction for the lower sum. Then,

$$\begin{aligned} U\left(f, [a, b]\right) - L\left(f, [a, b]\right) &\le U\left(f, P, [a, b]\right) - L\left(f, P, [a, b]\right) \\ &= \frac{b-a}{n} \sum_{j=1}^{n} \left(\sup_{[x_j - x_{j-1}]} f - \inf_{[x_j - x_{j-1}]} f \right) \\ &= (b-a)\,\epsilon \end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$ the proof is done. □

2.1.5 Exercises

2.2 Riemann Integral Is Not Good Enough

Example 2.2.1 (Function that is not Riemann Integrable)**.**
Let $f : [0, 1] \to \mathbb{R}$ *with*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

if $[a, b] \subset [0, 1]$ *with* $a < b$, *then*

$$\inf_{[a,b]} f = 0 \quad \text{and} \quad \sup_{[a,b]} = 1$$

Then we have $L\left(f, P, [0, 1]\right) = 0$ *and* $U\left(f, P, [a, b]\right) = 1$ *for any partition of* $[0, 1]$ *and the function is not integrable.*

[1] See Rudin, *Principles of Mathematical Analysis* Theorem 4.19

Example 2.2.2 (Riemann integration does not work with unbounded functions)**.** *Define* $f : [a,b] \to \mathbb{R}$ *by*

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \textit{if } 0 < x \leq 1 \\ 0 & \textit{if } x = 0 \end{cases}$$

if the partition is on $[0,1]$ *then we have* $supf = \infty$ *for any partition of* $[0,1]$. *However, we know, from standard calculus that*

$$\lim_{a \to 0} \int_a^1 = \lim_{a \to 0} \left(2 - 2\sqrt{a}\right) = 2$$

2.3 Measure Theory

2.4 Lebesgue Measure

2.5 Convergence of Measurable Functions

2.6 Integration with respect to measure

2.7 Limits of Integrals and Integrals of Limits

2.8 LP theory

2.9 Inner Product Spaces

The material here is summarized from Boggess and Narcowich, *A First Course in Wavelets with Fourier Analysis*.

2.9.1 Definition of Inner Product

Definition 2.9.1 (Inner Product)**.** *for $X, Y \in \mathbb{R}^n$ we define the inner product as*

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_j$$

When studying Fourier series and Fourier transform we will be using the complex exponential. Thus, we can use the conjugate the second factor to get the same result.

Definition 2.9.2 (Complex Inner Product)**.** *for $Z, W \in \mathbb{C}^n$ we define the inner product as*

$$\langle Z, W \rangle = \sum_{i=1}^{n} z_i \overline{w_j}$$

This ensures that the length of a vector in $\mathbb{C}^n$ is real.

Definition 2.9.3 (Length of Complex Vector)**.** *asdfasdf*

$$\begin{aligned} Length(Z) &= \sqrt{\langle Z, Z \rangle} \\ &= \sqrt{\sum_{i=1}^{n} z_i \overline{z_i}} \\ &= \sqrt{\sum_{i=1}^{n} |z_i|^2} \end{aligned}$$

Definition 2.9.4 (Inner Product Space).
A vector space with an inner product.

Definition 2.9.5 (Norm of a vector).

$$\|v\| = \sqrt{\langle v, v\rangle}$$

Definition 2.9.6 (Distance between two Vectors).

$$dist(v, w) = \|v - w\|$$

2.9.2 $\mathscr{L}^2$ and $\mathscr{L}^1$ Spaces

Definition 2.9.7 ($\mathscr{L}^2$ Space).
Consider the interval $[a, b]$ where $t \in [a, b]$, the space $\mathscr{L}^2([a, b])$ is the set of all square integrable functions defined for t.

$$\mathscr{L}^2([a, b]) = \left\{ f : [a, b] \to \mathbb{C}; \int_a^b \left|f(t)\right|^2 \; dt < \infty \right\}$$

This allows for both continuous and discontinuous functions. The condition of a finite square integral physically means tha the total energy of the signal is finite.

Example 2.9.8 (Function membership in $\mathscr{L}^2$).
The set of functions $\left\{1, t, t^2, \cdots\right\}$ is linearly independent and belongs to $\mathscr{L}^2[0, 1]$. The function $f(t) = 1/t$ is not a member of this space since

$$\int_0^1 \left(1/t\right)^2 \; dt = \infty$$

$\mathscr{L}^2$ Inner Product

We first simply consider a discretized space and set $a = 0, b = 1$. Let N be a large integer and let $t_i = i/N$ for $1 \le i \le N$. If a function f is continous, the the values of it can be approximated on the interval $[t_i, t_{i+1})$ by $f(t_i)$. This gives us

$$f_N = \big(f(t_1), f(t_2), \cdots, f(t_N)\big) \in \mathbb{R}^n$$

. Where the approximation is better as N grows larger. Now, consider two approximations f_N, g_N. How do we define $\langle f_N, g_N\rangle$? Using 2.9.2 we have

$$\begin{aligned}\langle f_N, g_N\rangle &= \sum_{i=1}^{N} f(t_i)\overline{g(t_i)} \\ &= \sum_{i=1}^{N} f(i/N)\overline{g(i/N)}\end{aligned}$$

This, however, can result in a large sum as N grows large. Thus, we consider the averaged inner product :

$$\frac{1}{N}\langle f_N, g_N\rangle = \sum_{i=1}^{N} f(t_i)\overline{g(t_i)}\frac{1}{N}$$

which we can take the limit of to get

$$\frac{1}{N}\langle f_N, g_N\rangle = \sum_{i=1}^{N} f(t_i)\overline{g(t_i)}\triangle t \qquad \text{with } \triangle t = 1/N$$

This is a Riemann sum approximation of $\int_0^1 f(t)\overline{g(t)}\,dt$ on the partition $[0, \cdots, t_N = 1]$.

Definition 2.9.9 ($\mathscr{L}^2$ inner product)**.**

$$\langle f, g\rangle_{\mathscr{L}^2} = \int_a^b f(t)\overline{g(t)}\,dt \qquad \textit{for } f, g \in \mathscr{L}^2[a,b]$$

Convergence in $\mathscr{L}^2$

Definition 2.9.10 (Convergence in $\mathscr{L}^2$)**.**
A sequence f_n converges to f in $\mathscr{L}^2[a,b]$ if, for any $\epsilon > 0$ we have $\exists N \in \mathbb{N}$ if $n \geq N$ then

$$\|f_n - f\|_{L^2} < \epsilon$$

This is sometimes called convergence in the mean. We also have two other types of convergence.

Definition 2.9.11 (Pointwise Convergence)**.** *A sequence f_n converges to f pointwise on the interval $a \leq t \leq b$ if for each $t \in [a,b]$ and each $\epsilon > 0$, there exists a positive integer N such that if $n \geq N$ then*

$$\left|f_n(t) - f(t)\right| < \epsilon$$

Note that N depends on both t and ϵ.

Definition 2.9.12 (Uniform Convergence)**.** *A sequence f_n converges to f uniformly on the interval $a \leq t \leq b$ if for each small tolerance $\epsilon > 0$, there exists a positive integer N such that if $n \geq N$, then*

$$\left| f_n(t) - f(t) \right| < \epsilon \qquad \forall t \in [a, b]$$

Uniform converge only depends on ϵ

If f_n uniformly converges on $[a, b]$ then the values of f_n are close to f over the entire interval. If f_n converges to f pointwise, then for each fixed t, $f_n(t)$ is close to $f(t)$ for large n. However, the rate of convergence is dependent on the point t. Thus,

$$\text{uniform cv} \implies \text{pointwise cv} \quad \text{but} \quad \text{pointwise cv} \not\Rightarrow \text{uniform cv}$$

Finally, if f_n converges to f in $\mathscr{L}^2[a, b]$, then, $f_n \to f$ on average. But there may be values where $f_n(t)$ is far from $f(t)$. This is why this is sometimes called convergence in the mean. If the set of these values is either finite or countably infinite then it has measure zero. Since intervals of length zero have no effect on integration, we say that convergence occurs if the difference is less that $\epsilon > 0$ except for a set a set of measure zero.

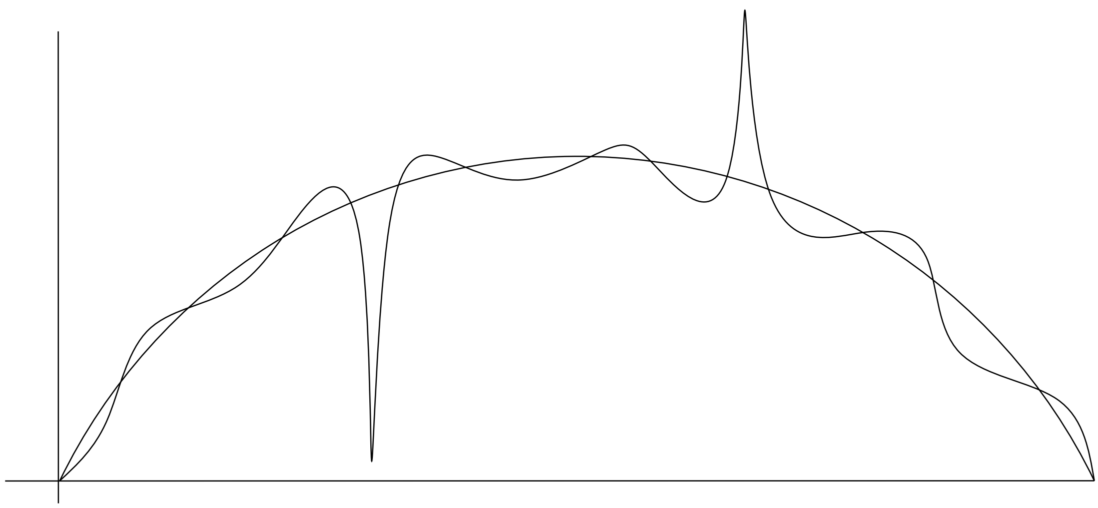


Figure 2.1: convergence in $\mathscr{L}^2$

Theorem 2.9.13 (Uniform Converge Implies $\mathscr{L}^2$ Convergence).
If $f_n \to f$ uniformly to f as $n \to \infty$ on a finite interval $a \le t \le b$, then this sequence also converges in $\mathscr{L}^2[a,b]$. The converse of this is not true.

Proof. Using 2.9.12 we have

$$\left|f_n(t) - f(t)\right| < \epsilon \quad \text{for } n \ge N \text{ and } a \le t \le b$$

Using 2.9.10 we get

$$\begin{aligned}\|f_n - f\|_{L^2} &= \int_a^b \left|f_n(t) - f(t)\right|^2 \, dt \\ &\le \int_a^b \epsilon^2 \, dt \\ &= \epsilon^2 (b-a)\end{aligned}$$

Thus, for $n \ge N$ we have

$$\|f_n - f\|_{L^2} \le \epsilon\sqrt{b-a}$$

For the converse, consider

$$f_n(t) = \begin{cases} 1 & 0 < t < 1/n \\ 0 & \text{otherwise} \end{cases}$$

which has

$$\|f_n\|_{L^2} = \int_0^1 \left|f_n(t)\right|^2 \, dt = \int_0^{1/n} 1 \, dt = 1/n$$

which is dependent on how close t is to the origin. □

2.9.3 Orthogonality

Definition 2.9.14 (Orthogonality).

- *The vectors X and Y are orthogonal if $\langle X, Y\rangle = 0$.*
- *The collection of vectors e_i are said to be orthonormal if each e_i has length 1 and $\langle e_i, e_j\rangle = 0$ for $i \ne j$.*
- *Two subspaces V_1 and V_2 of V are said to be orthogonal if each vector in V_1 is orthogonal to every vector in V_2.*

Example 2.9.15. *For the space $\mathscr{L}^2[0,1]$ any two functions where the first functions is zero, on the set where the second is nonzero will be orthogonal.*

Example 2.9.16. *Let* $f(t) = \sin t$ *and* $g(t) = \cos t$. *These functions are orthogonal in* $\mathscr{L}^2[-\pi, \pi]$

$$
\begin{aligned}
\langle f, g \rangle &= \int_{-\pi}^{\pi} \sin(t)\cos(t)\ dt \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2t)\ dt \\
&= -\frac{1}{4}\cos(2t)\Big|_{-\pi}^{\pi} \\
&= 0
\end{aligned}
$$

Furthermore,

$$\int_{-\pi}^{\pi} \sin^2(t)\ dt = \|\sin(t)\|_{L^2} = \int_{-\pi}^{\pi} \cos^2(t)\ dt = \|\cos(t)\|_{L^2} = \pi$$

The orthonormal functions can be found simply :

$$
\begin{aligned}
\|\cos(t)\|_{L^2} &= \pi \\
\cos(t) &= \sqrt{\pi} \\
\frac{\cos(t)}{\sqrt{\pi}} &= 1
\end{aligned}
$$

Taking the $\mathscr{L}^2[-\pi, \pi]$ *norm we get*

$$
\begin{aligned}
\int_{-pi}^{\pi} \frac{\cos^2(t)}{\pi}\ dt =& \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(t)\ dt \\
&= \int_{-\pi}^{\pi} \frac{1}{2} + \frac{\cos(2t)}{2}\ dt \\
&= \frac{1}{2}t + \frac{1}{4}\sin(2t)\Big|_{-\pi}^{\pi} = 1
\end{aligned}
$$

The same is true for sin with the orthonormal function $\frac{\sin(t)}{\sqrt{\pi}}$

Theorem 2.9.17. *Suppose* V_0 *is a subspace of an inner product space* V. *Suppose*

$e = \{e_1, \cdots, e_N\}$ is an orthonormal basis for V_0. If $v \in V_0$, then

$$v = \sum_{i=1}^{N} \langle v, e_i \rangle e_i$$

Proof. Since e is a basis for we can express any vector as a linear combination

$$v = \sum_{i=1}^{N} \alpha_i e_i$$

For fixed α_k, take the inner product of both sides.

$$\langle v, e_k \rangle = \sum_{i=1}^{N} \langle \alpha_j e_j, e_k \rangle$$

Since these are orthonormal we get one nonzero term at $j = k$.

$$\langle v, e_k \rangle = \alpha_k \langle e_k, e_k \rangle = \alpha_k$$

Thus, $\alpha_k = \langle v, e_k \rangle$ □

2.9.4 Orthogonal Projections

Definition 2.9.18 (Orthogonal Projection)**.** *Suppose V_0 is a finite-dimensional subspace of an inner product space V. For any vector $v \in V$, the orthogonal projecion of v onto V_0 is the unique vector $v_0 \in V_0$ that is closes to v*

$$\|v - v_0\| = \min_{u \in V_0} \|v - u\|$$

Theorem 2.9.19. *Suppose V_0 is a finite-dimensional subspace of an inner product space V. Let b be any element in V. Then its orthogonal projection ,v_0, has the property*

$$\langle v - v_0, u \rangle = 0 \forall u \in V_0$$

Theorem 2.9.20 (Orthogonal Projection)**.** *Suppose V is an inner product space and V_0 is a $N-$dimensional subspace with orthonormal basis e_i. The orthogonal projection of a vector $v \in V$ onto V_0 is given by*

$$v_0 = \sum_{i=1}^{N} \alpha_i e_i \qquad \text{with } \alpha_i = \langle v, e_i \rangle$$

Example 2.9.21 (0.22)**.**

Example 2.9.22 (0.23)**.**

Definition 2.9.23 (Orthogonal Complement)**.** *Suppose V_0 is a subspace of an inner product space V. The orthogonal complement of V_0, denoted $V_0^{\perp}$ is the set of all vectors in V orthogonal to V_0*

Chapter 3

Fourier Series

3.1 Trigonometric Polynomials

We begin by creating a space of functions, to this end we will need to define the functions that constitute this space and their properties. The basis of this space is given by monocromatic functions.

Definition 3.1.1 (Monocromatic Function)**.**

$$D \circ f : e_n(t) = e^{\frac{2i\pi nt}{T}}$$

$e_n(t)$ is the monochromatic function.

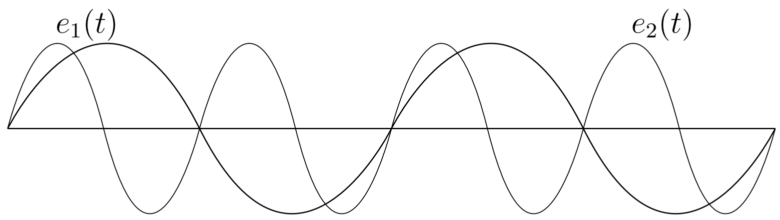


Figure 3.1: monocromatic

We say that there is one frequency, thus one chroma. Since n is fixed the frequency is constant. The n subscript references the frequency, thus for greater n there is a greater frequency.

Definition 3.1.2 (Periodic Function)**.** *A funcion f is called periodic with period a if*

$$\forall t \in \mathbb{R} \quad f(t+a) = f(t)$$

The function $e_n(t)$ has period T for all $n \in \mathbb{N}$, thus, we can write any function p using the following :

Definition 3.1.3 (Trigonometric Polynom)**.** *We define a trigonometric polynomial as :*

$$p(t) = \sum_{n=-N}^{N} c_n e_n(t), \qquad c_n \in \mathbb{R}$$

which has degree less that or equal to N. Expanding this representation we get :

$$p(t) = c_0 + \sum_{n=1}^{N} \left(C_n e^{\frac{2i\pi nt}{T}} + C_{-n} e^{\frac{2i\pi nt}{T}} \right)$$

using the fact that cos *is even and* sin *is odd we get*

$$p(t) = C_0 + \sum_{n=1}^{N} \left((C_n + C_{-n}) \cos\left(\frac{2\pi nt}{T}\right) + i\,(C_n - C_{-n}) \sin\left(\frac{2\pi nt}{T}\right) \right)$$

Which we abbreviate to

$$p(t) = C_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right) \tag{3.1}$$

where

$$a_n = C_n + C_{-n} \quad b_n = i\,(C_n - C_{-n})$$

which gives us the following inverse formulas :

$$c_n = \big(a_n - ib_n)\big)/2 \quad c_{-n} = (a_n + ib_n)/2$$

3.1.1 Orthogonality

We can calculate

$$\begin{aligned}\langle e_n, e_m \rangle &= \int_0^T e_n(t)\bar{e_m}(t) \\ &= \int_0^T e^{2i\pi(n-m)/T} \\ &= \begin{cases} 0 & \text{if } n \neq m \\ T & \text{if } n = m \end{cases}\end{aligned}$$

We use $\mathscr{T}_N$ to denote the set of trigonometric polynomials of order N. It is a vectorial space generated by $\left\{e_n(t)\right\}_{-N \leq n \leq N}$
The functions e_n and e_m are orthogonal in $\mathscr{L}^2\,(T)$, using the scalar product on $\mathscr{L}^2(T)$. Where T denotes the period and thus the interval where the functions are defined.

$$\langle f, g \rangle_{L^2} = \int_0^T f(t)\overline{g(t)}\ dt$$

and

$$\|e_n\|^2 = \langle e_n, e_n \rangle = T$$

and

$$\|e_n\| = \sqrt{T}$$

The orthonormal basis of $\mathscr{T}_N$ is given by $\left\{\frac{e_n(t)}{\sqrt{T}}\right\}$ For $p(t) \in \mathscr{T}_N$.

3.1.2 Calculating Fourier Coefficients

3.2 Need to check these!!!

For Period $= 2\pi$

Using Asmar, *Partial Differential Equations : with Fourier Series and Boundary Value Problems* Section 2.2 we derive the formulas for coefficients that are in 3.1, however, for simplicity we derive this for functions with period 2π. Integrate both sides of 3.1 except we do for x instead of t. Also, the period is 2π so the equation becomes

$$\int_{-\pi}^{\pi} f(x)\ dx = \int_{-\pi}^{\pi} a_0\ dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx)\ dx$$

Since

$$\int_{-\pi}^{\pi} \cos nx\ dx = \int_{-\pi}^{\pi} \sin nx\ dx = 0 \qquad \text{for } n = 1, 2, 3, \cdots$$

we have

$$\int_{-\pi}^{\pi} f(x)\ dx = \int_{-\pi}^{\pi} a_0\ dx = 2\pi a_0$$

or

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

We use orthogonality to derive the other terms.

$$\int_{-\pi}^{\pi} f(x)\cos mx\ dx = \overbrace{\int_{-\pi}^{\pi} a_0 \cos mx\ dx}^{=0} + \sum_{n=1}^{\infty}\overbrace{\int_{-\pi}^{\pi} a_n \cos nx \cos mx\ dx}^{=0 \text{ for } m\neq n} + \sum_{n=1}^{\infty}\overbrace{\int_{-\pi}^{\pi} b_n \sin nx \cos mx\ dx}^{=0}$$

$$= a_m \overbrace{\int_{-\pi}^{\pi} \cos^2 mx\ dx}^{=\pi} = \pi a_m$$

Thus,

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\ dx \quad (m = 1, 2, \cdots)$$

Similarly, we reach

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\ dx \quad (m = 1, 2, \cdots)$$

Period = T

We now use 3.1 as it is, with period T.

$$\int_{0}^{T} p(t) = \int_{0}^{T} C_0 + \sum_{n=1}^{N}\int_{0}^{T}\left(a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)\right)$$

The rightmost expression is always zero for $n = (1, 2, \cdots)$. Thus,

$$\int_{0}^{T} p(t)\ dt = \int_{0}^{T} C_0\ dt = TC_0$$

Gives us

$$\frac{1}{T}\int_{0}^{T} p(t)\ dt = C_0$$

Using orthogonality we can compute cosine representation :

$$
\begin{aligned}
\int_0^T p(t)\cos\left(\frac{2\pi mt}{T}\right)\,dx &= \overbrace{\int_0^T C_0\cos\left(\frac{2\pi mt}{T}\right)\,dt}^{=0} \\
&+ \sum_{n=1}^{\infty}\overbrace{\int_0^T a_n\cos\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi mt}{T}\right)\,dt}^{=0\text{ for }m\neq n} \\
&+ \sum_{n=1}^{\infty}\overbrace{\int_0^T C_0\sin\left(\frac{2\pi nt}{T}\cos\left(\frac{2\pi mt}{T}\right)\right)\,dt}^{=0} \\
&= \int_0^T a_n\cos^2\left(\frac{2\pi mt}{T}\right)\,dt \\
&= \left[\frac{t}{2}+\frac{T}{8\pi m}\sin\left(\frac{4\pi mt}{T}\right)\right]_0^T \left(b/c\int\cos^2 ax\,dx = \frac{x}{2}+\frac{1}{4a}\sin 2ax + C\right) \\
&= \frac{Ta_n}{2}
\end{aligned}
$$

which gives us

$$a_n = \frac{2}{T}\int_0^T p(t)\cos\left(\frac{2\pi mt}{T}\right)\,dt$$

Similarly,

$$b_n = \frac{2}{T}\int_0^T p(t)\sin\left(\frac{2\pi nt}{T}\right)\,dt$$

Using

$$C_n = \int_{-T/2}^{T/2} p(t)e_n(t)\,dt$$

we observe that if $p(t)$ is even then $C_n = C_{-n} \implies b_n = 0$ and if $p(t)$ is odd then $C_n = -C_{-n} \implies a_n = 0$.

Theorem 3.2.1 (Perseval equality).

$$\|p(t)\|^2_{L^2(T)} = \langle \sum_{n=-N}^{N} C_n e_n(t), \sum_{m=-N}^{N} C_m b_m(t) \rangle$$

this gives us

$$T \sum_{n=-N}^{N} |C_n|^2 \iff \sum |C_n|^2 = \frac{1}{T} \int_0^T \left|p(t)\right|^2 \ dt$$

Theorem 3.2.2 (Parseval's Equation (alternate form)).
Let

$$f(x) = \sum_{k=-\infty}^{\infty} \alpha_k e_n(t) \in \mathscr{L}_{(T)}$$

Then

$$\frac{1}{T}\|f\|^2 = \frac{1}{T} \int_0^T \left|f(x)\right|^2 \ dx = \sum_{k=-\infty}^{\infty} |\alpha_k|^2$$

Moreover, for f and g in $\mathscr{L}^2[T]$ we obtain

$$\frac{1}{T}\langle f, g \rangle = \frac{1}{T} \int_0^T f(t)\overline{g(t)} \ dt = \sum_{n=\infty}^{\infty} \alpha_n \overline{\beta_n}$$

A physical interpretation of this equation is that the energy of a signal is simply the sum of the energies from each of its frequency components.

3.3 Fourier Series

Let's consider $\mathscr{L}^2(T) = \left\{ f, \text{T-periodic, and } \int_0^T \left|f(t)\right|^2 dt < \infty \right\}$
On $\mathscr{L}^2(T)$ we use the scalar product

$$\langle f, g \rangle = \int_0^T f(t)\bar{g}(t) \ dt \text{ and } \|f\|^2 = \langle f, f \rangle$$

$\mathscr{L}^2(T)$ is a Hilbert Space.
Remark : What does it mean to say that $\|f\| = 0$? this means that the integral of

the function is zero almost everywhere.

$$\int_0^T |f(t)|^2 \ dt = 0$$

all equalities that we deal with are not necessarily true for each point of the two functions. Thus f and g can be said to be equal almost everywhere even if f is continuous everywhere and g is irregular.

We have $\mathscr{T}_N < \mathscr{L}^2(T)$ thus, $\mathscr{T}_N$ is a vectorial space included in $\mathscr{L}^2(T)$. In fact, we will project elements of $\mathscr{L}^2(T)$ on $\mathscr{T}_N$! View this projection as an approximation of any function of $\mathscr{L}^2(T)$ by a trigonometric polynom.

We want, by definition $\forall p \in \mathscr{T}_N,\ p$ orthogonal $(f - f_N) \implies \forall |n| \geq N,\ e_n$ orthogonal $(f - f_N)$ and

$$\langle e_n, f \rangle = \langle e_n, f_N \rangle$$

For $f_N \in \mathscr{T}_N$ we can write f_N as

$$f_N(T) = \sum_{n=-N}^{N} C_n e_n(t)$$

then we have $\langle f_N, e_N \rangle = TC_n$ so

Definition 3.3.1 (Fourier coefficient)**.**

$$C_n = \frac{1}{T} \langle f, e_n \rangle$$

C_n is the docomposition factor of e_n of the projection f on $\mathscr{T}_N$.

Definition 3.3.2 (Using Pythagorous)**.**

$$\|f\|^2 = \|f_N\|^2 + \|f - f_N\|^2$$

$$\|f\|^2 = T \sum_{n=-N}^{N} |C_n|^2 + \|f - f_N\|^2$$

which gives Bessel inequality :

Definition 3.3.3 (Bessel inequality)**.**

$$\sum_{-N}^{N} |C_n|^2 \leq \frac{1}{T} \|f\|^2 \implies \sum_{n=-N}^{N} |C_n|^2 \ \textit{is convergent}$$

Thus, $C_n \to 0$ as $n \to \infty$

Theorem 3.3.4 (Theorem of Convergence in $\mathscr{L}^2$)**.** *Let $f \in \mathscr{L}^2(T)$.*

$$f_N(t) = \sum_{n=-N}^{N} C_n e^{\frac{2i]pint}{T}}$$

$$f_N(t) \to f(t) \quad \text{as } N \to \infty$$

This means that

$$\int_0^T \left|f_N(t) - f(t)\right|^2 \, dt \to 0 \text{ as } N \to \infty$$

it does imply the pointwise convergence.

Proof. The proof is based on a theorem of Hilbert spaces.

Theorem 3.3.5. *Let $(\varphi_n)_n$ an orthonormal basis of H and $(\alpha_n)_n$ scalars. Then*

$$\sum \alpha_n \varphi_n \text{ converges } \iff \sum |\alpha_n|^2 \|\varphi_n\|_H^2 < \infty$$

Assume $\sum \alpha_n \varphi_n < \infty$. By the continuity of the norm

$$\|\sum \alpha_n \varphi_n\|^2 < \infty \implies \sum |\alpha|^2 \|\varphi_n\|^2 < \infty.$$

The other way : We assume $\sum |\alpha_n|^2 \|[hi_n\|^2 < \infty$ we can show that $\sum \alpha_n \varphi_n$ is a Cauchy sequence : .... missed half the proof.

$$p < q, \ \| \sum_{|n|<p} \|$$

□

Theorem 3.3.6 (Perseval equality)**.**

$$T \sum |C_n|^2 = \|f\|^2 - \|f - f_N\|^2 \to 0 \quad N \to \infty$$

$$\sum_{n=-w}^{w} |C_n|^2 = \frac{1}{T} \|f\|^2$$

$$\sum_{n=-w}^{\infty} C_n(g) \bar{C_n}(g) = \frac{1}{T} \int_0^T f(t) \bar{g}(t) \, dt$$

Properties of the Fourier Coefficients.

- Unicity $f = g$ almost surely then $\forall n,\ C_n(f) = C_n(g)$ $f = 0$ almost everywhere then $\forall n$ $C_n(f) = 0$.
- f real function, then $C_n = \bar{C_n}$, $a_n, b_n real$
- f even then C_n even and $b_n = 0$.
- f odd then C_n odd and $(a_n = 0)$
- f even and real $\implies$ (C_n) is even and a real sequence.
- f odd and real $\implies$ (C_n) is a odd and imaginary sequence.

3.4 Pointwise representation of periodic function by its Fourier Series

In practice, we don't know f for $t \in [0, T]$ in fact, in practice, f is represented by a vector if $\mathbb{R}^N$ which is $\big(f(t_0), f(t_1), \cdots, f(t_{N-1})\big)$ using this vector which we denote by $?f(t)$ we approximate the fourier coefficients. So we need to be able to inverse the Fourier Series and obtain the intial signal. So we need to have pointwise convergence for this to be achievable.

To have it, we need to ave some regularity properties of f.

Theorem 3.4.1. *Assume f is T periodic, continuous, and differentiable, almost everywhere. Assume that f' is piecewise continuous, Then :*

1. *The fourier series of f' is obtained from Fourier Series of f differentiating each term.*
2. *Fourier Coefficients of (f) satisfy*
$$\sum_{-\infty}^{\infty} |C_n| < \infty$$
3. *The Fourier Series of f converge uniformly to f.*

Proof. 1. Use integration by parts on

$$C_n(f) = \left[\frac{T}{2i\pi n} f(t) e^{-\frac{2i\pi nt}{T}}\right]_0^T + \frac{1}{T}\frac{T}{2i\pi n}\int_0^T f'(t) e^{-\frac{2i\pi nt}{T}}\ dt$$

$$= \frac{1}{2i\pi n} \int_0^T f'(t) e^{-\frac{2i\pi nt}{T}} \, dt$$

which is $TC_n(f')$ and

$$C_n(f') = \frac{2i\pi n}{t} C_n(f)$$

2.
$$\left|C_n(f)\right| = \frac{T}{2\pi n} \left|C_n\left(f'\right)\right|$$
$$\leq \frac{T}{4\pi} \left(\frac{1}{n^2} + \left|C_n(f')\right|^2\right)$$

3. Normal convergence of $|C_n| \implies \sum |C_n| < \infty \implies f_N$ normal convergence to g.
$$f_N \to f \text{ in } \mathscr{L}^2(t) \implies f_N \text{ uniformly convergent to } g.$$

$$\begin{aligned} &\text{if } f_N \to f \text{ in L2} \\ &\text{if } f_N \to f \text{ in ?} \end{aligned}$$

$\implies f = g$ almost everywhere. f is continuous then $f_N \to f$ uniformly.

□

Theorem 3.4.2 (Regularity Theorem)**.** *f is T periodic. $f \in \mathscr{C}^p \implies \exists k \in \mathbb{R}$ s.t.*

$$\left|C_n(f)\right| \leq \frac{k}{|n|^p}$$

Conclusion : The more regular f is, the more rapidly $C_n(f)$ converges to 0.

1. $f \in \mathscr{L}^1(T) \implies C_n(f) \to 0$
2. $f \in \mathscr{L}^2(T) \implies \sum \left|C_n(f)\right|^2 < \infty$
3. $f \in \mathscr{L}^1(T) \implies \sum \left|C_n(f)\right| < \infty$
4. $f \in \mathscr{C}^2(T) \implies \left|C_n(f)\right| \leq \frac{k}{n^2}$ for some $k \in \mathbb{R}$.
5. $f \in \mathscr{C}^\infty(T) \implies \forall k \in \mathbb{N}$ s.t. $\left|n^k C_n(f)\right| \to 0$

From the exercises with the characteristic function, we see that $\forall n,\ f_n(0) = 0$ Fourier series only capture the continuous or regular parts of a funtion. It does not capture the discontinutities or irregularities of the function. Look to exercise 5 to see that $f = g$ even though g has an impluse of 2 at $\pi/2$.
Irregularities are lost in the Fourier Series representation.

3.5 Exercises 1 : Fourier Series

3.5.1 Speed of convergence of Fourier coefficients and functional regularity

1. Let $f \in \mathscr{C}_p^2(a)$, prove that $\left|C_n(f)\right| \leq \frac{k}{n^2}$
2. Let $f \in \mathscr{C}_P^\infty(a)$, prove that

$$\forall k \in \mathbb{N}, \lim_{|n| \to \infty} \left|n^k c_n(f)\right| = 0$$

3.5.2 Fourier series convergence

Let $f \in \mathscr{L}^2(2\pi)$ defined by :

$$f(t) = \chi_{[0,\pi[}(t) - \chi_{[\pi,2\pi[}(t)$$

where $\chi_{[a,b]}$ is the characteristic function on the interval $[a, b]$. Note $f_N(t)$ the Fourier series of order N.

$$f_N(t) = \sum_{n=-N}^{N} c_n(f) e^{2i\pi nt/a}$$

1. What does it mean to say that f_N converges to f in $\mathscr{L}_P^2(2\pi)$.
2. Calculate $f_1(t), f_2(t), f_3(t), f_5(t)$
3. What are the values of $f(t), f_1(t), f_2(t), f_3(t), f_5(t)$ for $t = \pi$
4. Does $f_N(t)$ pointwise converge to $f(t)$?
5. Let the function $g \in \mathscr{L}_P^2(2\pi)$ be defined as :

$$g(t) = \chi_{[0,\pi/2[}(t) - \chi_{]\pi/2,\pi]}(t) + 2\delta_{\pi/2}(t) - \chi_{[\pi,2\pi[}(t)$$

   Note $g_N(t)$ its Fourier transform of order N. Calculate $g_1(t)$. What can you conclude.

Solutions to 3.5.1
1)

Proof. From the proofs given in 3.4 we know that

$$\left|C_n(f)\right| = \frac{T}{2i\pi n} \left|C_n(f')\right|$$

Thus,

$$\left|C_n(f')\right| = \frac{T}{2i\pi n} \left|C_n(f'')\right|$$

which gives

$$\begin{aligned}\left|C_n(f)\right| &= \frac{T}{2i\pi n}\left(\frac{T}{2\pi n}\left|C_n(f'')\right|\right)\\ &= \frac{T^2}{4\pi^2 n^2}\left|C_n(f'')\right|\end{aligned}$$

We can use this to show that for arbitrary $\epsilon > 0$ we have that there exists an N such that $n \geq N$ satisfies

$$\left|C_n(f)\right| = \frac{T}{2i\pi n}\left|C_n(f')\right| = \frac{T^2}{4\pi^2 n}\left|C_n(f'')\right| < \epsilon$$

Then

$$\left|C_n(f)\right| \leq \epsilon\frac{T^2}{4\pi^2 n^2} = \frac{k}{n^2} \qquad \text{for } k = \epsilon\frac{T^2}{4\pi^2}$$

□

2)

Proof. Since $f \in \mathscr{C}_P^\infty(a)$ and 3.4.1 (2) we have

$$\left|C_n(f)\right| = \frac{T}{2i\pi n}\left|C_n(f')\right| = \cdots = \left(\frac{T}{2\pi n}\right)^q\left|C_n(f^{(q)})\right|$$

for arbitrary q. Thus, for q = 2k we have

$$\left|n^k C_n(f)\right| = \frac{T^{2k}n^k}{2^{2k}\pi^{2k}n^{2k}}\left|C_n(f^{(2k)}\right|$$

since

$$\left|n^k C_n(f)\right| = n^k\left|C_n(f)\right|$$

we have

$$\left|C_n(f)\right| = \frac{T^{2k}}{2^{2k}\pi^{2k}n^k}\left|C_n(f^{(2k)}\right|$$

which goes to zero as $n \to \infty$ Note : (Could have just used $k+1$ rather than $2k$.) □

Solutions to 3.5.2.

1) It means that it converges on average to f. There may be points where this isn't true. This will be show in the 5th exercise.

2) We first consider $f(t)$.

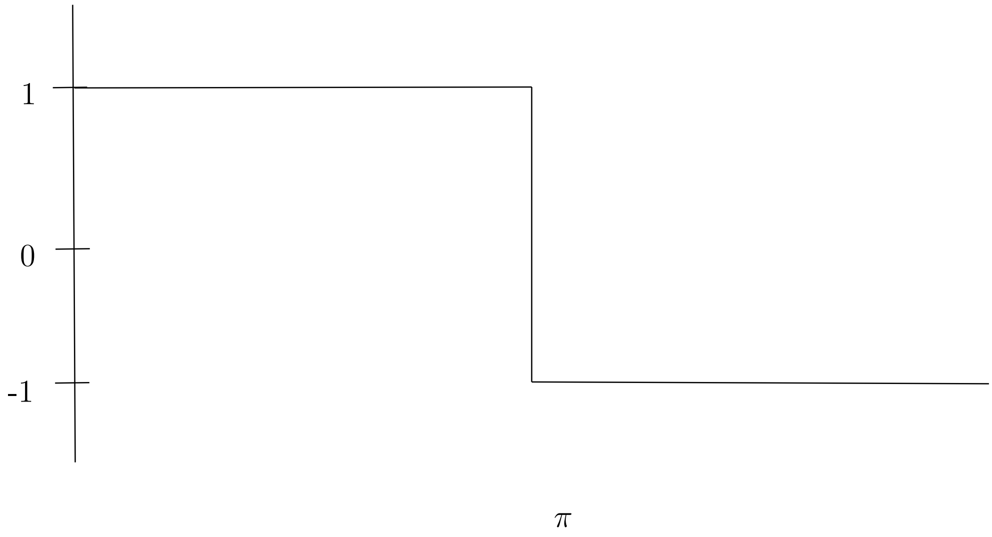


Figure 3.2: f(t)

Which is a real odd function, thus, C_n is odd and imaginary sequence and $a_n = 0$. So we need only calculate the b_n in order to find the coefficients and we need only consider odd numbers.

$$f_n(t) = \frac{2}{\pi}\int_0^{2\pi} \sin\left(\frac{2\pi nt}{2\pi}\right)\ dt$$

$$f_1(t) = \frac{2}{2\pi}\int_0^{2\pi} \sin\left(nt\right)\ dt$$

$$= \int_0^{\pi} \sin(t)\ dt - \int_{\pi}^{2\pi} \sin(t)\ dt$$

$$= -\frac{1}{\pi}\left[\cos(t)\right]_0^{\pi} + \frac{1}{\pi}\left[\cos(t)\right]_{\pi}^{2\pi}$$

$$= -\frac{1}{\pi}\big(1-(-1)\big) + \frac{1}{\pi}(-1-1) = -\frac{4}{\pi}$$

we can generalize this to all n ≥ 1

$$-\frac{1}{\pi}\left[\frac{1}{n}\cos(nt)\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{1}{n}\cos(nt)\right]_{\pi}^{2\pi}$$

Since n is always odd we have

$$= -\frac{1}{\pi n}\Big[\big(1-(-1)\big) - (-1-1)\Big] = -\frac{4}{\pi n}$$

Thus, $b_n = -\frac{4}{\pi n}$ and we can calculate the coefficients C_n, C_{-n} using the identities, where $a_n = 0$, thus

$$C_n = -ib_n/2 \qquad C_{-n} = ib_n/2$$

$$C_1 = -ib_1/2 = \frac{-i\left(-\frac{4}{\pi}\right)}{2} = \frac{2i}{\pi}$$

Furthermore, since f is odd we have $C_n = -C_{-n}$ and

$$C_{-1} = -\frac{2i}{\pi}$$

In general, this gives us

$$C_n = \frac{4i}{2n\pi} \quad \text{and} \quad C_{-n} = -\frac{4i}{2n\pi}$$

We can finally calculate the functions :

$$F_N(t) = \sum_{n=-N}^{N} C_n(f) e^{\frac{2i\pi nt}{2\pi}}$$

which yeilds the simplified form :

$$F_N(t) = \sum_{n=-N}^{N} C_n\left(f\right) e^{int}$$

$$\begin{aligned}
f_1(t) &= \sum_{n=-1}^{1} C_n\left(f\right) e^{int} \\
&= -\frac{2i}{\pi} e^{-it} + \frac{2i}{\pi} e^{it} \\
&= \frac{2i}{\pi}\left(e^{it} - e^{-it}\right) \\
&= \frac{2i}{\pi}\left(\cos t + i\sin t - \cos(-t) - i\sin(-t)\right) \\
&= \frac{2i}{\pi}\left(2i\sin(t)\right) \\
&= -\frac{4}{\pi}\sin t
\end{aligned}$$

$$\begin{aligned}
f_2(t) &= \sum_{n=-2}^{2} C_n(f) e^{int} \\
&= -\frac{i}{\pi} e^{-2it} - \frac{2i}{\pi} e^{-it} + \frac{2i}{\pi} e^{it} + \frac{i}{\pi} e^{2it} \\
&= -\frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t\right)
\end{aligned}$$

$$
\begin{aligned}
f_3(t) &= \sum_{n=-3}^{3} C_n(f)e^{int} \\
&= -\frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t\right)
\end{aligned}
$$

3) For $t = \pi$ we have

$$
\begin{aligned}
f_1(\pi) &= -\frac{4}{\pi}\sin(\pi) = 0 \\
f_3(\pi) &= -\frac{4}{\pi}\left(\sin(\pi) + \frac{1}{3}\sin(3\pi)\right) = 0 \\
f_5(\pi) &= -\frac{4}{\pi}\left(\sin\pi + \frac{1}{3}\sin 3\pi + \frac{1}{5}\sin 5\pi\right) = 0
\end{aligned}
$$

4) It does not converge pointwise to $f(t)$, consider that $f_N(\pi) = 0,\ \forall N \in \mathbb{N}$ and $f(\pi) = -1$.

5)

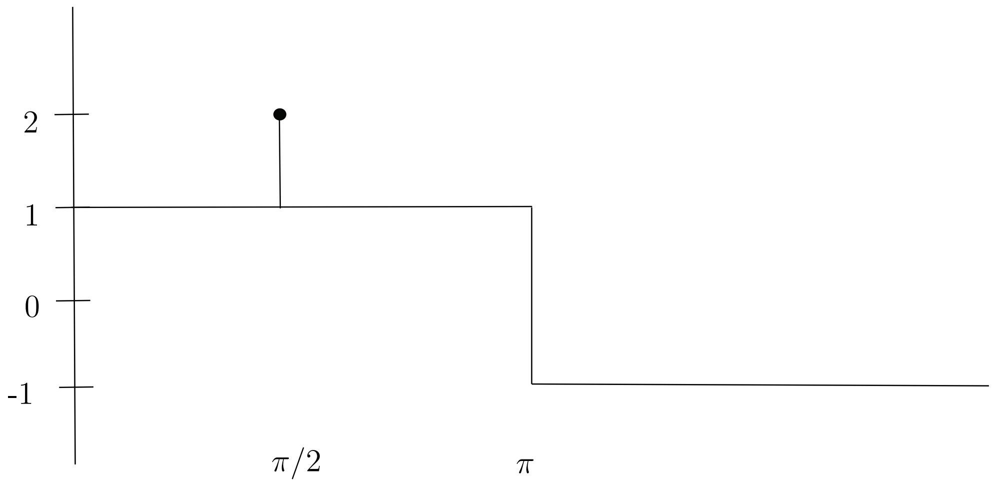


Figure 3.3: Dirac on Characteristic Function

Calculate $g_1(t)$.
First, we consider $C_n(f)$. This has the same Fourier series representation as $f(t)$. We can conlude that Fourier series representations capture the continuous properties or aspects of a function.

Chapter 4

Discrete Fourier Transform

In practice we don't work with the function f, but with the finite vector

$$\big(f(t_0), f(t_1), \cdots, f(t_{N-1})\big) \tag{4.1}$$

where $t_i \in [0,T]$ are the N samples assuming that timestep is dt or h and $h = t_{k+1} - t_k,\ \forall k \in N$. Because we have only N points we can only calculate N coefficients.
We want to calculate $(C_n)\,,\ n \in \mathbb{Z}$ using 4.1. A motivating example for calculating the coefficients can be shown through the trapazoidal rule. This is a numerical approximation technique that follows the idea of the Riemann integral as a sum of areas under the curve of a function. For a function that is sampled N times, we can partition it along [0,T] such that we have partitions p_i where $p_i = [t_i, t_{i+1}]$.

Definition 4.0.1 (Trapazoidal Rule). *Let $\big(f(t_0), f(t_1), \cdots, f(t_{N-1})\big)$ be a function sampled N times over the interval $[a,b]$ and let $\{x_k\}$ be a partition of this interval such that*

$$a = x_0 < x_1 < \cdots < x_{N-1} = b$$

where the lengt is given by $\triangle x_k = x_k - x_{k-1}$ then

$$\int_a^b f(x)\ dx \approx \sum_{k=1}^{N} \frac{f(x_{k-1}) + f(x_k)}{2} \triangle x_k \tag{4.2}$$

Thus, we will be able to calculate only N coefficients, because $C_n \to 0$ we calculate $\{C_n\}_{-\frac{N}{2} \le n \le \frac{N}{2}}$. We will calculate N approximations of $\{C_n\}_{-\frac{N}{2} \le n \le \frac{N}{2}}$ using $(f_0, \ldots, f_{N-1})$.

To calculate C_n^N, we will assume that

$$\forall k \in [0, N-1], \qquad \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} C_n^N e_n(t_k) = f(t_k) = f_k$$

expanding $e_n(t_k)$

$$\forall k \in [0, N-1], \qquad \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} C_n^N e^{\frac{2i\pi nk}{N}} = f_k$$

$$\forall k \in [0, N-1], \qquad \sum_{n=-\frac{N}{2}}^{-1} C_n^N e^{\frac{2i\pi nk}{N}} + \sum_{n=0}^{\frac{N}{2}-1} C_n^N e^{\frac{2i\pi nk}{N}} = f_k$$

Lets denote $\omega_N = e^{2i\pi/N}$ then $e^{2i\pi nk/N} = \omega_N^{nk}$

$$\forall k \in [0, N-1] \qquad \sum_{n=-\frac{N}{2}}^{-1} C_n^N \omega_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} C_N^n \omega_N^{kn} = f_k$$

$$\sum_{p=\frac{N}{2}}^{N-1} C_{p-N}^N \omega_N^{kp} + \sum_{n=0}^{\frac{N}{2}-1} C_n^N \omega_N^{kn} = f_k$$

$$f_k = \sum_{n=0}^{N-1} F_n \omega_N$$

where $F_n = C_n^N$ for $0 \leq n \leq \frac{N}{2}$ and $F_n = C_{n-N}^N$ for $\frac{N}{2} \leq n \leq N$.

The discrete Fourier transform calculates

$$(F_n) \quad \text{using} \quad (f_k)$$

DFT is performed using a fast algorithm called the fast fourier transform.

Remark:

$$\sum_{k=0}^{N-1} f_k \omega_N^{-kp} = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} F_n \omega_n^{nk} \omega_N^{-kp}$$
$$= \sum_{n=0}^{N-1} F_n \sum_{k=0}^{N-1} \omega_n^{k(n-p)}$$

However,

$$\sum_{k=0}^{N-1} \omega_N^{k(n-p)} = \begin{cases} \dfrac{\overbrace{1-\omega_N^{N(n-p)}}^{=0}}{1-\omega_N^{n-p}} & \text{if } n \neq p \\ N & \text{if } n = p \end{cases}$$

so we have

$$\sum_{k=0}^{N-1} f_k \omega_N^{-kp} = NF_?$$

4.0.1 DFT

$$f_0, \cdots f_{N-1} \to F_0, \cdots, F_{N-1} \text{ using } F_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k \omega_N^{-nk}$$

$$F_0, \cdots, F_{N-1} \to f_0, \cdots, f_{N-1} \text{ using } f_k = \sum_{n=0}^{N-1} F_n \omega_N^{kn}$$

and F_N follows the same coefficient labeling as before right before mentioning the DFT

If we note Ω_N the matrix

$$\Omega_n[n,k] = \omega_N^{nk}$$

which is

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{n-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(n-1)} \\ \vdots & & \ddots & & \vdots \\ 1 & \omega_N^{n-1} & \omega_N^{2(n-1)} & \cdots & \omega_N^{(n-1)^2} \end{pmatrix}$$

$f = \Omega_N F$, $F = \frac{1}{N}\overline{\Omega}_N f$, $\left(\Omega_N^{-1} = \frac{1}{N}\overline{\Omega}_N\right)$.

Just a note :

DFT also allows to obtain F from f and f from F. $F = \frac{1}{N}\Omega_N\overline{f}$.

4.1 Properties of DFT

1. (f_k) is N-periodic vector
2. (F_n) is also a periodic vector
3. $\left(C_n^N\right)$ is also a N periodic sequence, while (C_n) is not a periodic sequence.
   - $(C_n \to 0)$ as $n \to \omega$
   - C_n^N is an approximation of C_n only for $-\frac{N}{2} \le n \le \frac{N}{2}$

1.

$$(f_k) \to (F_n)$$

$$(f_{-k}) \to F_{-n}$$

$$\left(\overline{f_k}\right) \to \overline{F_{-n}} \text{ and vice versa}$$

2. (f_k) odd (even) $\implies$ F_n odd(even)
3. (f_k) real $\implies$ $F_{-n} = \overline{F_n}$
4. (f_k) real and even then F_n real and even
5. (f_k) real and odd then F_n is pure imaginary and odd.

(Question) Why are frequencies centered around zero? Why do we shift to -N/2 N/2?

4.2 FFT : Pease algorithm

Let $N = 2^n,\ n \in \mathbb{N}$. $W_N = e^{\frac{2i\pi}{N}},\ m = \frac{N}{2}$.

$$
\begin{aligned}
F_k &= \frac{1}{N}\sum_{j=0}^{N-1} f_j \omega_N^{-kj} \\
&= \frac{1}{N}\sum_{j=0}^{\frac{N}{2}-1} f_j \omega_N^{-kj} + \underbrace{\left(\sum_{j=\frac{N}{2}}^{N-1} f_j \omega_N^{-kj}\right)}_{j=s-\frac{N}{2}} \\
&= \frac{1}{N}\sum_{j=0}^{\frac{N}{2}-1} f_j \omega_N^{-kj} + \frac{1}{N}\sum_{s=0}^{\frac{N}{2}-1} f_{s+\frac{N}{2}} \omega_N^{-\left(s+\frac{N}{2}\right)k}
\end{aligned}
$$

However,

$$
\omega_N^{-k\frac{N}{2}} = e^{\frac{-2i\pi k N}{2N}} = e^{-ik\pi} = (-1)^k
$$

Thus,

$$
F_k = \frac{1}{N}\sum_{j=0}^{\frac{N}{2}-1} \omega_N^{-jk}\left(f_j + (-1)^k f_{j+\frac{N}{2}}\right)
$$

If $k = 2l$ for $l \in [0, \cdots, \frac{N}{2}-1]$ we have

$$
\begin{gathered}
F_{2l} = \frac{1}{N}\sum_{j=0}^{\frac{N}{2}-1} \omega_N^{-2jl}\left(f_j + f_{j+\frac{N}{2}}\right) \\
\text{Using } \omega_N^{-2jl} = e^{\frac{2i\pi j 2l}{N}} = \omega_m^{jl} \\
= \frac{1}{m}\sum_{j=0}^{m-1} \omega_m^{-jl}\left(f_j + f_{j+m}\right)
\end{gathered}
$$

If $k = 2l + 1,\ k \in [0, N/2 - 1]$

$$\begin{aligned}
F_{2l+1} &= \frac{1}{N} \sum_{j=0}^{N/2-1} \omega_N^{-j(2l+1)} \left(f_j - f_{j+N/2}\right) \\
&= \frac{1}{N} \sum_{j=0}^{N/2} \omega_m^{-jl} \omega_N^{-j} \left(f_j - f_{j+N/2}\right) \\
&= \frac{1}{2m} \sum_{j=0}^{m-1} \omega_m^{-jl} \omega_N^{-j} \left(f_j - f_{j+N/2}\right)
\end{aligned}$$

Thus, we have

$$\forall l \in [0, \cdots, m-1]$$

$$\begin{aligned}
F_{2l} &= \frac{1}{2} DFT \left[f_j + f_{j+m}\right] \\
F_{2l+1} &= \frac{1}{2} DFT \left[\omega_N^{-j} \left(f_j - f_{j+m}\right)\right]
\end{aligned}$$

We have replaced the intial DFT of order N by 2 DFT of order N/2.

4.2.1 Visual Representation of the Algorithm and Computation

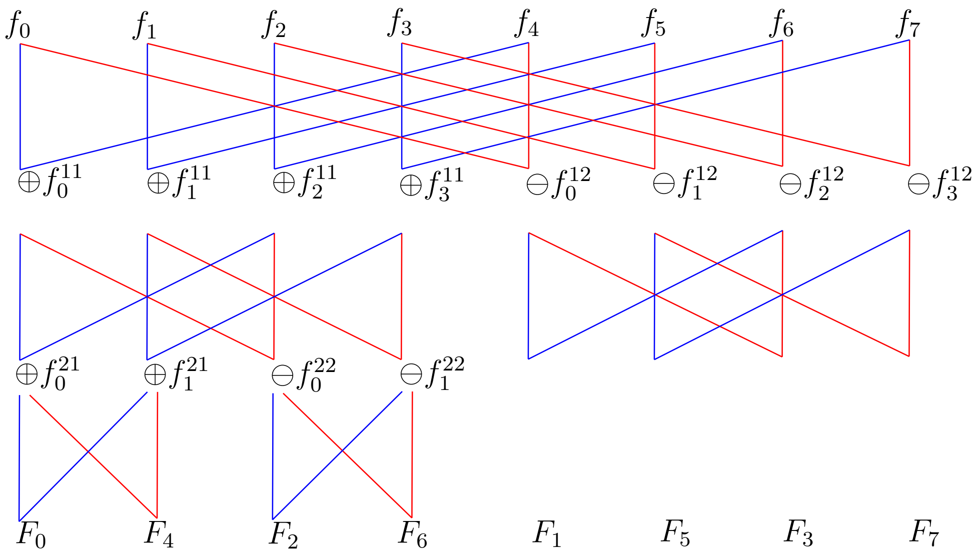


Figure 4.1: peaseAlgorithm

Example 4.2.1 (Computation of F_4)**.**
Consider, $k = 2 * 2$, then we have $l = 2$ and

$$F_4 = \frac{1}{8}\sum_{j=0}^{3} \omega_8^{-2j2}\left(f_j + f_{j+\frac{N}{2}}\right)$$

which can be simplified using

$$\omega_N^{-2jl} = \omega_N^{-4j} = e^{2i\pi j2l/8} = \omega_4^{2l}$$

this gives,

$$\frac{1}{4}\sum_{j=0}^{3} \omega_4^{-2j}\left(f_j + f_{j+m}\right)$$

$$
\begin{aligned}
j = 0 &\Longrightarrow \omega_4^0 \left(f_0 + f_4\right) \\
&= \left(f_0 + f_4\right) \\
j = 1 &\Longrightarrow \omega_4^{-2} \left(f_1 + f_5\right) \\
&= e^{i\pi} \left(f_1 + f_5\right) \\
&= -\left(f_1 + f_5\right) \\
j = 2 &\Longrightarrow \omega_4^{-4} \left(f_2 + f_6\right) \\
&= e^{2i\pi} \left(f_2 + f_6\right) \\
&= \left(f_2 + f_6\right) \\
j = 3 &\Longrightarrow \omega_4^{-6} \left(f_3 + f_7\right) \\
&= e^{3i\pi} \left(f_3 + f_7\right) \\
&= -\left(f_3 + f_7\right)
\end{aligned}
$$

which gives

$$
\begin{aligned}
F_4 &= \frac{1}{4} \left(\left(f_0 + f_4 + +f_2 + f_6\right) - \left(f_1 + f_5 + f_3 + f_7\right)\right) \\
&= \frac{1}{4} \left(\left(f_0^{11} + f_2^{11}\right) - \left(f_1^{11} + f_3^{11}\right)\right) \\
&= \frac{1}{4} \left(f_0^{21} - f_1^{21}\right)
\end{aligned}
$$

which is equivalent to 4.1.

4.3 Exercises FFT: Pease Algorithm

Assume that $N = 2^n,\ n \in \mathbb{N}$. We want to quickly calculate the Discrete Fourier Transform

$$F = \left(F_k\right)\ k = 0, \cdots, N-1$$

of the vector

$$f = \left(f_j\right) j = 0, \cdots N-1$$

Let Ω_N be the matrix of

$$\left(\Omega_n\right)_{k,l} = \omega_N^{kl} = e^{\frac{2i\pi kl}{N}}$$

1. Check that $f = \Omega_N F$
2. Write F_k using Ω_N and f.

Let $A_k(f) = \sum_{j=0}^{N-1} \omega_N^{kj} f_j$, the k^{th} componant of the vector $\Omega_N f$. We can write A_k as the sum of the two summations.

$$A_k = \sum_{j=0}^{N/2-1} \omega_N^{kj} f_j + \sum_{j=N/2}^{N-1} \omega_N^{kj} f_j$$

1. Write A_{2k} and A_{sk+1} for $k = 0, \cdots, N/2 - 1$.

2. Using the prperties $\omega_N^{2kj} = \omega_{N/2}^{kj}$ and $\omega_{N/2}^{k\left(j+lN/2\right)} = \omega_{N/2}^{kj}$ write A_{2lk} as a Discrete Fourier Transform of size $N/2$. Do the same thing for A_{2k+1}

3. If the complexity of a DFT of size N is $O\left(N^2\right)$, what is the interest of the previous approach?

4. The Pease algorithm is an iteration of this process. The two DFT of size $N/2$ will also be performed using the DFT of size $N/4$ etc. Draw for $N = 8$ the data movings and transformations of the 3 steps. Count the number of operations. What is the complexity of the whole process?

5. Write the algorithm for $N = 2^n$, assuming you have an algorithm to reorder the coefficients at the end.

6. What is the complexity of this algorithm?

4.3.1 Solutions

1)

1. We check that $f = \Omega_N F$

$$F_0 = \frac{1}{N} \sum_{k=0}^{N-1} f_k$$
$$F_1 = \frac{1}{N} \sum_{k=0}^{N-1} f_k \omega_N^{-k}$$
$$\vdots$$
$$F_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k \omega_N^{-nk}$$

Thus, $\Omega_N F$ gives us

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{n-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(n-1)} \\ \vdots & & \ddots & & \vdots \\ 1 & \omega_N^{n-1} & \omega_N^{2(n-1)} & \cdots & \omega_N^{(n-1)^2} \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}$$

$$= \begin{pmatrix} F_0 & + & F_1 & + & \cdots & + & F_{n-1} \\ F_0 & + & F_1\omega_N & + & \cdots & + & F_{n-1}\omega_N^{n-1} \\ F_0 & + & F_1\omega_N^2 & + & \cdots & + & F_{n-1}\omega_N^{2(n-1)} \\ & & & & \vdots & & \\ F_0 & + & F_1\omega_N^{n-1} & + & \cdots & + & F_{n-1}\omega_N^{(n-1)^2} \end{pmatrix}$$

$$\begin{pmatrix} \sum_{n=0}^{N-1} F_n \\ \sum_{n=0}^{N-1} F_n\omega_N^n \\ \sum_{n=0}^{N-1} F_n\omega_N^{2n} \\ \vdots \\ \sum_{n=0}^{N-1} F_n\omega_N^{(N-1)n} \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

2.

$$F_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n\omega_N^{-nk}$$

2)

Solutions to all of these exercises are given in 4.2

4.4 Exercises : Circular Convolution and DFT

1. **Convolution of two discrete N - periodic signals :**
   Assume that $N = 2^n$, with $n \in \mathbb{N}$. The numerical sequence $(x_k)_{k\in\mathbb{Z}}$ is N periodic if

   $$\forall k \in \mathbb{Z}, x_{k+N} = x_k$$

   The circular convolution between two N - periodic sequences $(x_k)_{k\in\mathbb{Z}}$ and $(y_k)_{k\in\mathbb{Z}}$ is defined by

   $$\forall k \in \mathbb{Z}, z_k = (x * y)_k = \sum_{q=0}^{N-1} x_q y_{k-q} \tag{4.3}$$

   Let $(X_p)_{p\in\mathbb{Z}}$ the DFT of the N-periodic sequence $(x_k)_{k\in\mathbb{Z}}$

   $$X_p = \frac{1}{N}\sum_{k=0}^{N-1} x_k\omega_N^{-kp}$$

   with $\omega_N = e^{\frac{2i\pi}{N}}$. The sequences $(Y_k)_{k\in\mathbb{Z}}$ and $(Z_k)_{k\in\mathbb{Z}}$ are the DFT of y_k, z_k.

   (a) Show that z_k is a N-periodic sequence. So it is enough to calculate z_k only for $k = 0, \cdots, (N-1)$.

   (b) Prove that $(x * y)_k = (y * x)_k$

(c) How many operations are needed for the calculation of $(z_k)_{k=0,\cdots,N-1}$ using formula 4.3. Give a number for $n = 64$.

(d) Prove that $(X_p)_{p\in\mathbb{Z}}$ is also N-periodic.

(e) Show that $Z_p = NX_pY_p, \forall p \in \mathbb{Z}$.

(f) Propose another method to calculate (z_k). What is the complexity? Give a number for n = 64.

(g) Prove that if $\forall k \in \mathbb{Z}, w_k = x_k y_k$, then

$$W_p = \sum_{q=0}^{N-1} X_q Y_{p-q}$$

2. **Convolution of two discrete non-periodic signals :** Let the sequences $(x_k)_{l\in\{0,\cdots,(M-1)\}}$ and $(h_k)_{l\in\{0,\cdots,(Q-1)\}}$ Assume $Q < M$ and let

$$y_k = (y * h)_k = \sum_{q=0}^{Q-1} h_q x_{k-q}$$

(a) Show that the support of y is $[0, M+Q-1]$

(b) If you directly use the convolution formula, what is the cost of calculation of all non null y_k?

(c) In order to perform this calculation, we can :

- Search $N = 2^p$ such as $N \geq (M+Q-1)$
- extend the sequences x_k, h_k to $xk, h_k, k \in \{0, N-1\}$ adding some zeros and using FFT to calculate y_k. Using this technique, what is the cost of calculation?

(d) Compare the two methods for $Q = 200$ and $M = 500$.

(e) Compare for $Q = 85, M = 1000$.

(f) What can you conclude?

3. **Application to matrix diagonalization :**
Assume that, as in C language, we begin with 0 for an array. A N circulant matrix is a matrix of complex numbers $a_0, \cdots, a_{N-1}$ organized such that $M_{k,l} = a_{(k-l)\text{Modulo}(N)}$. Notice that $a_{(k)}$Modulo(N), $k \in \mathbb{Z}$ is a periodic sequence. Let M be a circulant matrix, y and x are of length N such that $y = Mx$.

(a) Draw a circulant matrix for $N = 4$.

(b) Show that

$$y_k = \sum_{q=0}^{N-1} x_q a_{(k-q)\text{Modulo}(N)}$$

(c) Deduce that M can be diagonalized using DFT

$$\frac{1}{N}\Omega_N M \Omega_N = \operatorname{diag}(\Omega_N a)$$

where diag(v) denotes a diagonal matrix where diagonal elements are compenets of v. $v = (v_0, v_1, \cdots, v_{N-1})$ and $(\Omega_N)_{k,l} = \omega_N^{kl}$

4.4.1 Solutions

1)

(a) ?

(b)

$$\begin{aligned}
(x * y)_k &= \sum_{q=0}^{N-1} x_q y_{k-q}, \text{ let } p = k - q \\
&= \sum_{p=k}^{k-N+1} x_{k-p} y_p \\
&= \sum_{p=k+1-N}^{k} x_{k-p} y_p \\
&= \sum_{p=k+1-N}^{-1} x_{k-p} y_p + \sum_{p=0}^{k} x_{x-p} y_p \qquad \text{let } s = p + N \\
&= \sum_{p=0}^{k} x_{k-p} y_p + \sum_{s=k+1}^{N-1} x_{k+N-s} y_{s-N}
\end{aligned}$$

since $x_k y_k$ are N periodic we have

$$\begin{aligned}
(x * y)_k &= \sum_{p=0}^{k} x_{k-p} y_p + \sum_{s=k+1}^{N-1} x_{k-s} y_s \\
&= \sum_{p=0}^{N-1} x_{k-p} y_p \\
&= (y * x)_k
\end{aligned}$$

(c) $z_k = N\text{add} * N\text{mult}$, $64^2 = 4096$.

(d)

$$X_p = \frac{1}{N}\sum_{k=0}^{N-1} x_k \omega_N^{-kp}$$
$$X_{p+N} = \frac{1}{N}\sum_{k=0}^{N-1} x_k \underbrace{\omega_N^{-k(p+N)}}_{e^{-2i\pi k}=1}$$

(e)

$$\begin{aligned}
Z_p &= NX_pY_p \\
&= N\left(\frac{1}{N}\sum_{k=0}^{N-1} x_k\omega_N^{-pk}\right)\left(\frac{1}{N}\sum_{k=0}^{N-1} y_k\omega_N^{-pk}\right) \\
&= \frac{1}{N}\left(\sum x_k\omega_N^{-pk}\right)\left(\sum y_k\omega_N^{-pk}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{q=0}^{k-1} x_q y_{k-q}\omega_N^{-kp}
\end{aligned}$$

However,

$$\omega_N^{-kp} = \omega_N^{-qp}\omega_N^{-(k-q)p}$$

$$\begin{aligned}
Z_p &= \frac{1}{N}\sum_{k,q=0}^{N-1} x_q\omega_N^{-qp} y_{k-q}\omega_N^{-(k-q)p} \\
&= \frac{1}{N}\sum_{q=0}^{N-1}\sum_{l=q}^{N-q-1} x_q\omega_N^{-qp} y_l\omega_N^{-lp}
\end{aligned}$$

Note :

$$\begin{aligned}
\sum_{l=-q}^{N-q-1}\cdots &= \sum_{-q}^{-1} + \sum_{0}^{n-q-1} \\
&= \sum_{N-q}^{n-1} + \sum_{0}^{N-q-1} \quad \text{since N periodic} \\
&= \frac{1}{N}\sum_{q=0}^{N-1} x_q\omega_N^{-kq}\sum_{l=0}^{N-1} y_l\omega_N^{-pl} \\
&= NX_pY_p
\end{aligned}$$

(f) Calc N fourier coefficients for each series is $2\frac{N}{2}\log_2(N)$. $Z_p = NX_pY_p = N$ multiplications. We also obtain z_k using DFT of $Z_p \to \frac{N}{2}log_2(N)$ the cost is $\frac{3N}{2}\log_2(N) + N$. Direct method $\mathcal{O}(N^2)$. FFT is $\mathcal{O}\left(N\log_2(N)\right)$

(g) use the same proof as (e).

2)

(a)

4.5 Approximation Error

We want to quantify the error of our approximation. We have approximated C_n using only N values, and we calculate only N approximate coefficients. Idea : Using N pts $f_0, \cdots, f_{N-1}$ we calculat C_n^N for $-\frac{N}{2} \leq n \leq \frac{N}{2}$. C_n^N is an appromation of C_n Assume

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{2i\pi nt}{T}} \qquad \left(f \in \mathscr{C}_{\mathscr{P}}(\mathscr{T})\right)$$

So

$$\forall k \in [0, N-1] \quad f_k = f\left(\frac{kT}{N}\right) = \sum_{n=-\infty}^{\omega} C_n e^{\frac{2i\pi nkT}{NT}}$$

$$f_k = \sum_{-\infty}^{\omega} C_n \omega_N^{nk}$$

We know that

$$\sum_{\infty}^{\infty} |C_n| < \infty \quad \text{thus absolutely convergent}$$

meaning we can compute the terms and still retain convergence.

$$f_k = \sum_{-\omega}^{\omega} C_n \omega_N^{nk} = \sum_{n=0}^{N-1} \sum_{q=-\omega}^{\omega} C_{k(qN+n)} \omega_N^{qN+n}$$

However, we have $f_k = \sum_{n=0}^{N-1} F_n \omega_N^{nk}$ and for $0 \leq n \leq \frac{N}{2}$, $C_n^N = F_n$ and for $-\frac{N}{2} \leq n < 0$, $C_{n+N}^N F_n$ By definition we can conlude that

$$f_k = \sum_{n=0}^{N-1} \sum_{q=-\omega}^{\omega} C_{qN+m} \omega_N^{nk}$$

By identity we deduce that

Theorem 4.5.1.

$$C_n^N = \sum_{q=-\omega}^{\omega} C_{qN+n} \implies C_n^N - C_n = \sum_{q\neq 0} C_{qN+n}$$

Conclusion :

$C_n \to 0$ quicker the higher N, better the approximation. The more regular the function the faster the coefficients converge to 0.

.

Example 4.5.2 (Convergence)**.** *Assume*

$$f(t) = \sum_{n=-6}^{6} C_n e^{\frac{2i\pi nt}{T}}$$

1. *How many points do we need to get C_n?*
   To calculate all the coefficients we need $N \geq 2p + 1 = 13$ which gets us all of the coefficients without error.
   . If we use $N = 4$. We have the error

   $$C_0^4 - C_0 = C_4 + C_{-4}$$
   $$C_1^4 - C_1 = C_5 + C_{-3}$$

   If we use $N = 8$, we have

   $$C_0^8 - C_0 = 0$$
   $$C_1^8 - C_0 = C_9 + C_{-7} = 0$$
   $$C_2^8 - C_0 = C_{-6} \neq 0$$

Conclusion again

DFT tool for periodic functions, the more regular f the more regular the error goes to o. Thus, DFT is for regular periodic functions that are regular. The FFT we used demands $N = 2^n$.

Chapter 5

Fourier Transform

5.1 Introduction

The Fourier transform is a representation of a temporal signal. Why would we want to represent a signal in another space and why use FT? The FT has wonderful properties that allow us to easily handle convolution or derivation in the frequency space. The $\mathscr{FT}$ can handle non periodic signals.

5.1.1 Non-rigorous development of the $\mathscr{FT}$

We begin by recalling the definiton of a Fourier series for a function f defined on the interval $= l \leq x \leq l$ and we let l go to infinity.

$$f(x) = \sum_{n=\infty}^{\infty} C_n e^{in\pi x/l} \tag{5.1}$$

which gives us

$$C_n = \frac{1}{2l} \int_{-l}^{l} f(t) e^{-in\pi t/l} \, dt$$

If f is defined on the entire real line then we can take l to infinity and see how it affects the formulas. Substituting the expression of C_n into 5.1 we obtain

$$\begin{aligned} f(x) &= \lim_{l\to\infty} \left[\sum_{n=\infty}^{\infty} \left(\frac{1}{2l} \int_{-l}^{l} f(t) e^{-in\pi t/l} \, dt \right) e^{in\pi x/l} \right] \\ &= \lim_{l\to\infty} \left[\sum_{n=-\infty}^{\infty} \frac{1}{2l} \int_{-l}^{l} f(t) e^{in\pi(x-t)/l} \, dt \right] \end{aligned}$$

We want to transform this into a Riemann sum formulation of an integral. Thus, we partition l using our infinite sum by letting

$$\lambda_n = \frac{n\pi}{l} \quad \text{and} \quad \Delta\lambda = \lambda_{n+1} - \lambda_n = \frac{\pi}{l}$$

This gives us, $\frac{1}{2l} \cdot \frac{\pi}{\pi} = \frac{1}{2\pi} \cdot \Delta\lambda$ and

$$f(x) = \lim_{l\to\infty} \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-l}^{l} f(t) e^{\lambda_n i(x-t)}\ dt \right] \Delta\lambda \tag{5.2}$$

we denote the inner expression as

$$F_l(\lambda) = \frac{1}{2\pi} \int_{-l}^{l} f(t) e^{\lambda i(x-t)}\ dt \tag{5.3}$$

and this with 5.2 we have

$$\sum_{-\infty}^{\infty} F_l(\lambda_n) \Delta\lambda$$

which is similar to the definition of a riemann sum for the integral

$$\int_{-\infty}^{\infty} F_l(\lambda)\ d\lambda$$

As $l \to \infty,\ \Delta\lambda \to 0$ so $\Delta\lambda$ becomes $d\lambda$ and 5.2 becomes

$$f(x) = \lim_{l\to\infty} \int_{-\infty}^{\infty} F_l(\lambda)\ d\lambda$$

As $l \to \infty,\ $ 5.3 becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\lambda(x-t)}\ dt \tag{5.4}$$

which gives us

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\lambda(x-t)}\ dt\ d\lambda$$

which is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t}\ dt \right) e^{i\lambda x}\ dx \tag{5.5}$$

We denote

$$\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\lambda t}\, dt$$

and we now have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\lambda)e^{i\lambda x}\, d\lambda$$

This now allows us to define the Fourier transform.

5.2 Fourier Transform in $\mathscr{L}^1(\mathbb{R})$

Assume $f \in \mathscr{L}^1(\mathbb{R})$ where

$$\mathscr{F}f(\lambda) = \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-2i\pi\lambda t}\, dt$$

We also have

$$\overline{\mathscr{F}}f(\lambda) = \int_{-\infty}^{\infty} f(t)e^{2i\pi\lambda t}\, dt$$

$\mathscr{F}f(\lambda)$ is defined for $f \in \mathscr{L}^1(\mathbb{R})$. We introduce the notation

$$h(\lambda, x) = f(t)e^{-2i\pi\lambda t}$$

For λ, we have $\left|h(\lambda,t)\right| \leq \left|f(t)\right|$ and $|f|$ is integrable.

$$\lambda \to h(\lambda, t) \text{ is continuous}$$

Applying the theorem of continuous? we have

$$\lambda \to \int h(\lambda, t)\, dt \;\text{ is continuous}$$

We continue ?

$$\frac{dh(\lambda,t)}{d\lambda} = -2i\pi t h(\lambda,t)$$

$$\left|\frac{dh(\lambda,t)}{d\lambda}\right| \leq 2\pi\left|th(\lambda,t)\right| \leq 2\pi\left|tf(t)\right|$$

If $\left|tf(t)\right|$ is in $\mathscr{L}^1(\mathbb{R})$ we can apply the theorem of derivation and we have

$$\frac{d\mathscr{F}f(\lambda)}{d\lambda} = \frac{d}{d\lambda}\int h(\lambda,t) = \int \frac{d}{d\lambda}h(\lambda,t) = \int -2i\pi t f(t)e^{-2i\pi\lambda t}\, dt = -\widehat{2i\pi t f}(t)(\lambda)$$

Definition 5.2.1.

$$\widehat{f}(\lambda) = \mathscr{F}f(\lambda) = \int e^{-2i\pi\lambda t} f(t)\ dt$$

Theorem 5.2.2 (Riemann-Lebesgue)**.** *Let $f \in \mathscr{L}^1(\mathbb{R})$ and let*

$$\mathscr{F}f(\lambda) = \widehat{f}(\lambda) = \int f(t)e^{-2i\pi\lambda t}\ dt$$

1. *$\widehat{f}(\lambda)$is continuous and bounded*
2. *The operator $\mathscr{F}$ is continuous linear from $L^1(\mathbb{R})$ to $\mathscr{L}^\infty(\mathbb{R})$ and*

$$\|\widehat{f}\|_\infty \leq \|f\|_1$$

3. $$\lim_{|\lambda|\to\infty} \left|\widehat{f}(\lambda)\right| = 0$$

1. Already have (theorem of continuity of integral)
2. $$\begin{aligned} \forall\lambda \quad \left|\widehat{f}(\lambda)\right| &= \left|\int f(t)e_{-2i\pi\lambda t}\ dt\right| \\ &\leq \int \left|f(t)e^{-2i\pi\lambda t}\ dt\right| \\ &\leq \int \left|f(t)\right|\ dt \\ &\leq \|f\|_1 < \infty \end{aligned}$$

   its true for each lambda, then

$$\|\widehat{f}\|_\infty \leq \|f\|_1$$

3. To prove the result, we will use the density of the simple function space in $\mathscr{L}^1(\mathbb{R})$
   - Prove the result for simple function space $\chi_{[a,b]}(t)$

- Using the density.

1.

$$f(t) = \chi_{[a,b]}(t)$$

$$f(\lambda) = \int_a^b e^{-2i\pi\lambda t}\, dt$$

If $\lambda = 0$ we get

$$\widehat{f}(0) = b - a$$

If $\lambda \neq 0$.

$$\begin{aligned}
\widehat{f}(\lambda) &= -\frac{1}{2i\pi\lambda}\left[e^{-2i\pi\lambda t}\right]_a^b \\
&= -\frac{1}{\pi\lambda}\frac{e^{-2i\pi\lambda b} - e^{-2i\pi\lambda a}}{2i} \\
&= \frac{1}{\pi}e^{\frac{-2i\pi\lambda(a+b)}{2}}\frac{e^{\frac{-2i\pi\lambda(a+b)}{2}} - e^{\frac{-2i\pi\lambda(a+b)}{2i}}}{2}
\end{aligned}$$

$$\left|\widehat{f}(\lambda)\right| = \left|\frac{\sin \pi\lambda(b-a)}{\pi\lambda}\right| \leq \left|\frac{1}{\pi\lambda}\right| \to 0 \text{ as } |\lambda| \to \infty$$

The simple function space is dense in $\mathscr{L}^1(\mathbb{R})$ then there exists a sequence of simple function such that

$$\lim_{n\to\infty} \|f - g_n\|_1 = 0$$

we have

$$\left|\widehat{g}_n(\lambda)\right| \to 0 \text{ as } |\lambda| \to \infty$$

Furthermore,

$$\left|\widehat{f}(\lambda)\right| \leq \underbrace{\left|\widehat{f}(\lambda) - \widehat{g}_n(\lambda)\right|}_{\leq \|f-g_n\|_{\mathscr{L}^1 \to 0}} + \underbrace{\left|\widehat{g}_n(\lambda)\right|}_{\to 0}$$

Thus,

$$\left|\widehat{f}(\lambda)\right| \to 0$$

5.3 Link Between DFT and $\mathscr{F}\mathscr{T}$

DFT

f is T periodic,

$$f(t) = \sum C_n e^{\frac{2i\pi nt}{T}}$$

$$\begin{aligned} C_n &= \frac{1}{T}\int f(t)e^{\frac{2i\pi nt}{T}} \\ &= \frac{1}{T}\widehat{f_T}\left(\frac{n}{T}\right) \text{ with } f_T = f\cdot\chi_{[0,\pi[} \end{aligned}$$

We have only N points $f_0, \cdots, f_{N-1}$ and calculated C_n^N approximation of C_n. We have N coefficients C_n^N If we are looking at a periodic signal the DFT creates a spectrum with ticks at $\frac{1}{T}$. More Coefficients are calculated for larger N but the step between them does not change. DFT transform a T periodic signal into a peaks spectrum where the step between two peaks is $\frac{1}{T}$ where T is the period. The approximation error between C_n and C_n^N will decrease but the step between the two peaks does not change.

$\mathscr{F}\mathscr{T}$

f non-periodic, $f \in \mathscr{L}^1(\mathbb{R})$ and

$$\widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{-2i\pi\lambda t}\, dt$$

In DFT we consider that T periodic signal contains sinusoids, $e^{\frac{2i\pi nt}{T}}$. Now we consider that f contains all sinusoid $e^{2i\pi\lambda t}$, $\lambda \in \mathbb{R}$. This signal is observed during a finite time period , where we denote the time observed as T_{obs}. The signal is sampled.

- We have N observed points denoted bu $f_0, f_1, \cdots, f_{N-1}$ where $f_n = f\left(\frac{kT_{\text{obs}}}{N}\right)$. and h = the time step between 2 time observations.

In practice we can use the trapazoidal rule to approximate the integral. $\widehat{f}(\lambda)$ will be approximate by

$$\begin{aligned} \widehat{f}^N_{T_{obs}}() &= h\sum_{k=0}^{N-1} f_k e^{\frac{-2i\pi\lambda k T_{obs}}{N}} \\ &= h\sum_{k=0}^{N-1} f_k e^{-2i\pi\lambda kh} \end{aligned}$$

Note :

Theorem 5.3.1 (Note a theorem). *$\widehat{f}(\lambda)$ is 1/h periodic.*

$$\widehat{f}^N\left(\lambda + 1/h\right) = \widehat{f}^N(\lambda)$$

So it is also 1/h periodic. But,

$$\left|\widehat{f}(\lambda)\right| \to 0 \ as \ |\lambda| \to \infty$$

$\widehat{f}^N(\lambda)$ is an approx of $\widehat{f}(\lambda)$. for λ "around 0". It is an approximation of $\widehat{f}$ for $\lambda \in [-\frac{1}{2h}, \frac{1}{2h}[$ If we apply DFT on $f_0, \cdots, F_{N-1}$ it is like we consider f $= T_{obs}$ periodic and we obtain N coefficients. The step between 2 coeff is $\frac{1}{hN}$. Remember that $h = \frac{T_{obs}}{N}$ and the frequencied step is $\frac{1}{hN} = \frac{1}{T_{obs}}$.
In fact, we have decomposed f on the functions $e^{\frac{2i\pi nt}{T_{obs}}}$ $n \in \mathbb{N}$. As $n \uparrow$ and T_{obs} does not change, we are able to calculate more values of approx of $\widehat{f}(\lambda)$ but the step between frequencies does not change.

If T_{obs} change, then the step between frequencies changes. 3rd case

Case 1 : N pts, T_{obs}
Case 2 : N' = 2N, T_{obs}
Case 3 : N' = 2N, $T'_{obs} = 2T_{obs}$

Higher N gives us a the ability to analyze higher frequency signals.

5.3.1 Comparison with Fourier series

The complex form of the Fourier transform of f and the corresponding inversion formula are analagous to the complex form of the Fourier series f over the interval $-\frac{T}{2} \leq t \leq \frac{T}{2}$:

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}_n e^{\frac{2i\pi nt}{T}}$$

where

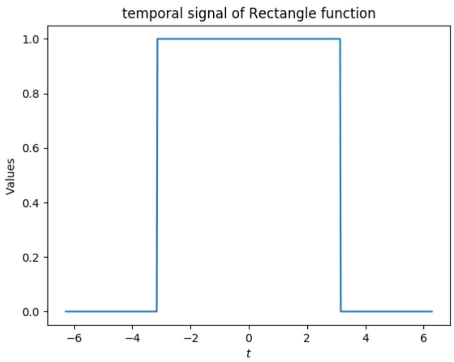
$$\widehat{f}_n = \int_{-T}^{T} f(t) e^{-\frac{2i\pi nt}{T}}$$

we raplce the sum over n from $-\infty$ to ∞ with an intergral with repect to λ from $-\infty$ to ∞. In the case of the fourier series, $\widehat{f}_n$ measures the component of f that oscillates with frequency n. Likewise, $\widehat{f}(\lambda)$ measures the frequency component of f that oscillates with frequency λ.

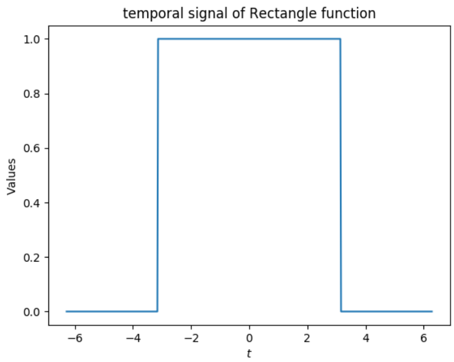
We give some examples of signals and their Fourier transform to give a better idea of what is being given.

Example 5.3.2 (Rectangular wave)**.**

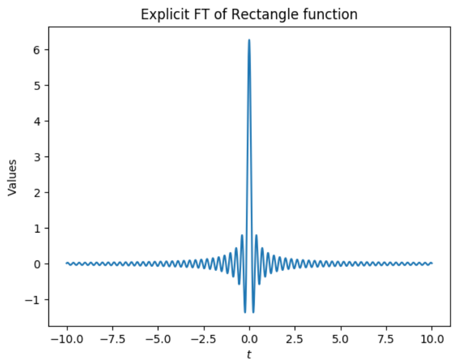
$$f(t) = \begin{cases} 1 & if \ -\pi \leq t \leq \pi \\ 0 & otherwise \end{cases}$$



(a) Rectangle Function $f(t)$



(b) DFFT of $f(t)$



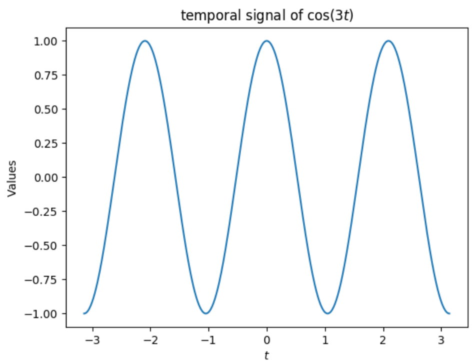
(c) $\widehat{f}(\lambda)$

Since f is an even function $f(t)\sin(\lambda t)$ is an odd fucntion and its integral of the real line is zero. Thus,

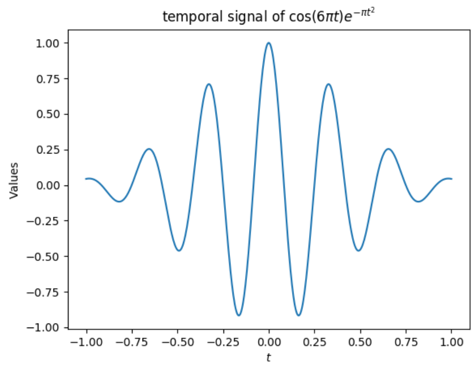
$$\begin{aligned}\widehat{f}(\lambda) &= \int_{-\infty}^{\infty} f(t)\cos(2\pi\lambda t)\ dt \\ &= \int_{-\pi}^{\pi} \cos(2\pi\lambda t)\ dt \\ &= \frac{1}{2\pi\lambda}\sin(2\pi\lambda t)\Big|_{-\pi}^{\pi} \\ &= \frac{1}{\pi\lambda}\sin(2\pi^2\lambda)\end{aligned}$$

Since $\widehat{f}(\lambda)$ measures the frequency component of f that vibrates with frequency λ, we should expect that the largest values of the transform to occur around zero. This is because this constant funtion vibrates wih zero frequency.

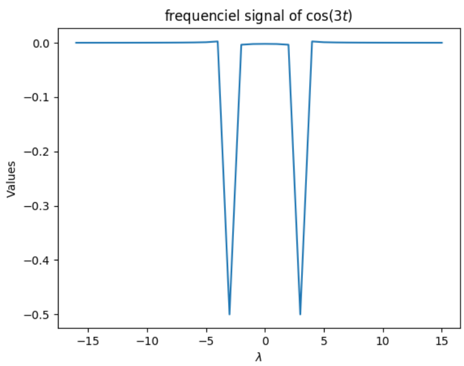
Example 5.3.3 ($\cos(3t)$)**.**



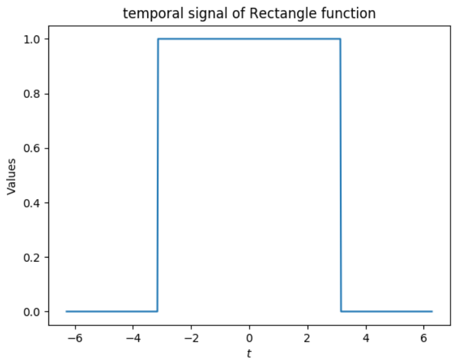
(d) $\cos(3t)$



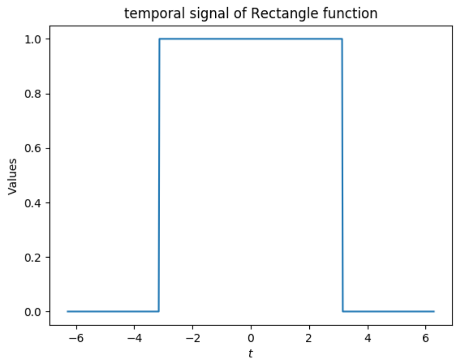
(e) DFFT of $f(t)$



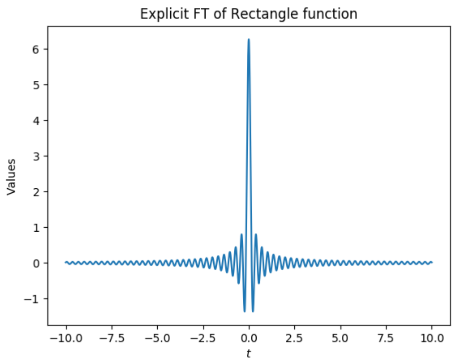
(f) $\widehat{f}(\lambda)$



(g) Rectangle Function $f(t)$



(h) DFFT of $f(t)$



(i) $\widehat{f}(\lambda)$

Example 5.3.4 $(\cos(6\pi t)e^{-\pi t^2})$**.**

What is $f_n = f\left(\frac{kT_{\mathrm{obs}}}{N}\right)$. and what is it to mean that the step is always 1/T for the DFT?

5.4 Properties of $\mathscr{F}\mathscr{T}$ on $\mathscr{L}(\mathbb{R})$

Theorem 5.4.1 (Exchange Property). *Assume* $f, g \in \mathscr{L}^1(\mathbb{R})$ *Then*

$$\int_{-\infty}^{\infty} f(u)\widehat{g}(u)\ du = \int_{-\infty}^{\infty} \widehat{f}(u)g(u)\ du$$

Proof. $-f \in \mathscr{L}^1, \widehat{g} \in \mathscr{C}^\infty \implies f\widehat{g} \in \mathscr{L}^1$ then use fubini's theorem. □

5.4.1 FT and Derivation

1. Assume $x^k f(x)$ in $\mathscr{L}^1(\mathbb{R})$ for $k \in \{0, \cdots, n\}$. Thus $\widehat{f}$ is k times derivable and
$$\widehat{f}^{(k)}(\lambda) = \widehat{(-2i\pi x)^k f(x)}(\lambda)$$
2. Assume $f \in \mathscr{C}^n$ and $f', \cdots, f^{(n)} \in \mathscr{L}^1(\mathbb{R})$. Then
$$\widehat{f^{(k)}}(\lambda) = (2i\pi\lambda)^k \widehat{f}(\lambda)$$
3. $f \in \mathscr{L}^1(\mathbb{R})$, f bounded support then $\widehat{f} \in \mathscr{C}^\infty$

Proof. 1. Direct application of the theorem of derivation

$$\int \frac{\partial}{\partial\lambda} f(\lambda, t) = \frac{\partial}{\partial\lambda} \int f(\lambda, t)\ dt$$

2. $n = 1$. $f' \in \mathscr{L}^1(\mathbb{R})$.

$$\widehat{f'}(\lambda) = \lim_{a\to\infty} \int_a^a e^{-2i\pi\lambda t} f'(t)\ dt$$

integration by parts

$$\widehat{f'}(\lambda) = \lim_{a\to\infty} \left[e^{-2i\pi\lambda t} f(t)\right]_{-a}^{a} + \lim_{a\to\infty} \int_{-a}^{a} 2i\pi\lambda e^{-2i\pi\lambda t} f(t)\ dt$$

$$\lim_{a\to\infty} \left[e^{-2i\pi\lambda t} f(t)\right]_{-a}^{a}$$

$$\lim_{a\to\infty} f(t) = 0 \quad \text{since} \quad f \in \mathscr{L}^1 p(\mathbb{R}).$$

So,

$$\widehat{f'}(\lambda) = 2i\pi\lambda \widehat{f}(\lambda)$$

3. $f \in \mathscr{L}^1(\mathbb{R})$ and f has a bounded support. Then what can we say about $t^k f(t)$. It is also in $\mathscr{L}^1(\mathbb{R})$ and using point 1 we get that $f \in \mathscr{C}^\infty$

□

5.4.2 Notations

symmetry :

$$f_\sigma(x) = f(-x)$$

Translation :

$$\mathscr{T}af(x) = f(x-a)$$

5.4.3 Properties

Assume f in L1 and $\widehat{f}(\lambda) = \mathscr{F}f(\lambda)$.

1. $\overline{\mathscr{F}f} = \overline{\mathscr{F}}\overline{f}$
2. $(\mathscr{F}f)_\sigma = \overline{\mathscr{F}}f = \mathscr{F}f_\sigma$
3. f even $\implies \widehat{f}$ even
4. f odd $\implies \widehat{f}$ odd
5. f real and even then $\widehat{f}$ real and even
6. $\widehat{\mathscr{T}af}(\lambda) = e^{-2i\pi\lambda a}\widehat{f}(\lambda)$ time delay
7. $\mathscr{T}a\widehat{f}(\lambda) = \widehat{e^{2it}f(t)}(\lambda)$ frequency delay

5.4.4 Usual examples

$$\begin{aligned}
e^{-ax}u(x) &= \frac{1}{a+2i\pi\lambda} \\
e^{ax}u(-t) &= \frac{-1}{-a+2i\pi\lambda} \\
\frac{x^k}{k!}e^{-ax}u(x) &= \frac{1}{(a+2i\pi\lambda)^{k+1}} \\
\frac{x^k}{k!}e^{ax} &= \frac{-1}{(-a+2i\pi\lambda)^{k+1}} \\
e^{-a|x|} &= \frac{2a}{a^2+4\pi^2\lambda^2} \\
sign(x)e^{-a|x|} &= \frac{-4i\pi\lambda}{a^2+4\pi^2\lambda^2} \\
e^{-at^2} / & \sqrt{\frac{\pi}{a}}e^{\frac{-\pi^2}{a^2}\lambda^2} \\
\chi_{[-a,a]}(t) &= \frac{\sin(2a\pi\lambda)}{\pi x}
\end{aligned}$$

5.4.5 Inverse of FT

Theorem 5.4.2 (Inverse of FT). *If $f \in \mathscr{L}^1(\mathbb{R})$ and $\widehat{f} \in \mathscr{L}^1(\mathbb{R})$. Then*

$$\overline{\mathscr{F}}\widehat{f}(t) = f(t), \quad \forall t \text{ where } f \text{ cont}$$

But, $f \in \mathscr{L}^1(\mathbb{R})\, notimply \widehat{f}\mathscr{L}^1(\mathbb{R})$

Corollary 5.4.3. *$f \in \mathscr{L}^1(\mathbb{R})$ and $f \in \mathscr{C}^2$ and $f, f', f'' \in \mathscr{L}^1$. Then*

$$\widehat{f} \in \mathscr{L}^1(\mathbb{R}).$$

Proof.

$$\widehat{f''}() = -4\pi^2\lambda^2\widehat{f}(\lambda)$$

and

$$\lim_{|\lambda|\to\omega}\left|\widehat{f''}(\lambda)\right| = 0$$

Then

$$\exists M s.t. |\lambda| > M \implies 4\pi^2\lambda^2\left|\widehat{f}(\lambda)\right| < 1$$

$\left|\widehat{f}\right|$ is continuous (RL theom).

$$\text{At } \omega \quad \left|\widehat{(}\lambda)\right| < \frac{1}{4\pi^2\lambda^2}$$

Then

$$\left|\widehat{f}\right| \in \mathscr{L}^1(\mathbb{R})$$

$\square$

Corollary 5.4.4. *$f, \widehat{f} \in \mathscr{L}^1$, we have*

$$\mathscr{F}\widehat{f}(u) = f_\sigma(u) = f(-u)$$

Example 5.4.5. *Consider the previous examples.*

$$\frac{1}{(a + 2i\pi t)^{k+1}} = \frac{(-\lambda)^k}{k!}e^{a\lambda}u(-\lambda)$$

Also,

$$\frac{-1}{(-a + 2i\pi t)^{k+1}} = \frac{(-\lambda)^k}{k!}e^{-a\lambda}u(\lambda)$$

And

$$\frac{1}{a^2 + t^2} = \frac{\pi}{a}e^{-2i\pi|\lambda|}$$

5.5 FT on $\mathscr{S}(\mathbb{R})$

We need to restrict $\mathscr{L}^1(\mathbb{R})$ in order to

- inverse FT
- Use derivation formulas

In order to do this we use the Schwartz space.

Example 5.5.1 (Schwartz space)**.**

$$\mathscr{S}(\mathbb{R}) \subset \mathscr{L}^1(\mathbb{R})$$

.

Definition 5.5.2 (Rapidly Decreasing Functions)**.**

$$\forall p \in \mathbb{N}, \lim_{p\to\infty} \left|x^p f(x)\right| = 0$$

Theorem 5.5.3 (Property 1)**.** *If $f \in \mathscr{L}^1(\mathbb{R})$ and is a rapidly decreasing function, then*

1. $\forall p, t^p f(t) \in \mathscr{L}^1(\mathbb{R})$
2. $\widehat{f} \in \mathscr{C}^\infty$

Proof. f is rapidly decreasing.

$$\implies \forall p \in \mathbb{N} \quad \lim_{|x|\to\infty} \left|x^{p+2} f(x)\right| = 0$$

$$\exists M s.t. \, |x| > M \quad \left|x^{p+2} f(x)\right| < 1$$

$$\int \left|x^p f(x)\right| \, dx = \int\limits_{|x|<M} \left|x^p f(x)\right| \, dx + \int\limits_{|x|>M} + \int\limits_{|x|>M} \frac{1}{x^2} \left|x^{p+2} f(t)\right| \, dt$$

$$\leq M^p \int\limits_{|x|<M} \left|f(x)\right| \, dx + \int\limits_{|x|>M} \frac{1}{x^2} \, dx$$

$$\leq \infty$$

Then $\forall p, \quad x^p f(x) \in \mathscr{L}^1(\mathbb{R})$. Then

$$\widehat{(-2i\pi t)^k f(t)}$$

exisits and it is $\widehat{f}^{(k)}(\lambda)$

□

Theorem 5.5.4 (Property 2). *Let $f \in \mathscr{C}^\infty$ and $\forall k$ $f^{(k)} \in \mathscr{L}^1(\mathbb{R})$. Then $\widehat{f}$ is rapidly decreasing.*

Proof.

$$\widehat{f^{(k)}}(\lambda) = (2i\pi\lambda)^k \widehat{f}(\lambda)$$

By RL theorem

$$\lim_{|\lambda|\to\infty} \left|\widehat{f^{(k)}}(\lambda)\right| = 0$$

Then

$$\lim_{|\lambda|\to 0} \left|\lambda^k \widehat{f}(\lambda)\right| = 0$$

□

Conclusion

The more regular f is, the more rapidly $\widehat{f}$ converges to zero at infinity. The more rapidly f converges, the more regular $\widehat{f}$ is. Furthermore, we realize that if $f \in \mathscr{C}^\infty$ and f rapidly decreasing, then $\widehat{f}$ rapidly decreasing and $\widehat{f} \in \mathscr{C}^\infty$.

Definition 5.5.5 (Schwartz Space). *The vectorial space $\mathscr{S}(\mathbb{R})$ of function are those function for which*

$$f \in \mathscr{C}^\infty$$

$$f^{(k)} \textit{ are rapidly decreasing for all } k$$

We have for $\mathscr{S}(\mathbb{R}) \subset \mathscr{L}^1(\mathbb{R})$

1. $\mathscr{S}$ stable for derivatives
2. $\mathscr{S}$ stable for $\mathscr{F}\mathscr{T}$
3. $\mathscr{S}$ stable for x by a polynomial

$\mathscr{F}\mathscr{T}$ is a bijection from $\mathscr{S}(\mathbb{R})$ to itself.

Chapter 6

FT on $\mathscr{L}^2(\mathbb{R})$

In signal processing, we often work with

$$\int \left|f(t)\right|^2 \ dt = \text{ the energy of the signal} = \|f\|_2$$

$\mathscr{L}^2(\mathbb{R})$ is the "natural" space for signals. To define $\mathscr{F}\mathscr{T}$ on $\mathscr{L}^2(\mathbb{R})$, we use the density of $\mathscr{S}(\mathbb{R})$ in $\mathscr{L}^2(\mathbb{R})$.

Theorem 6.0.1 (Extension Theorem)**.** *E, F vectorial spaces with norm. F is complete. Let G be a dense subspace of E. Let A be a linear continuous application from G to F. Then $\exists!$ extension of $A : E \to F = \overline{A}$. $\overline{A}$ is lin and cont and $\|\overline{A}\|_{E,F} = \|A\|_{G,F}$. With*

$$\|A\|_{G,F} = Sup\left(\frac{\|Af\|_F}{\|f\|_G}\right)$$

6.1 Properties of $\mathscr{F}\mathscr{T}$ on $\mathscr{L}^2(\mathbb{R})$

1. $\forall f \in \mathscr{L}^2(\mathbb{R})$. $\mathscr{F}\overline{\mathscr{F}}f = \mathscr{F}\mathscr{F}f = f$ almost everywhere.

2. $\forall f, g \in \mathscr{L}^2(\mathbb{R}) \times \mathscr{L}^2(\mathbb{R})$.

$$\int f(t)\overline{g}(t) \ dt = \int \mathscr{F}f(\lambda)\overline{\mathscr{F}g}(\lambda) \ d\lambda$$

$$\langle f, g\rangle_{L^2} = \langle \widehat{f}, \widehat{g}\rangle_{L^2}$$

3.

$$\|f\|_2 = \|\widehat{f}\|_2$$

energy conservation or Parseval equality.

4. Exchange property. $f, g \in \mathscr{L}^2(\mathbb{R}) \times \mathscr{L}^2(\mathbb{R})$

$$\int f(u)\widehat{g}(u)\ du = \int \widehat{f}(u)g(u)\ du$$

Example 6.1.1. *We can do IFT on the functions listed previously to see if we can perform the IFT.*

6.2 FT and Convolution

Part A

$f, g \in \mathscr{L}^2\left(\mathbb{R}\right)$. Then

1. $\widehat{f * g}\,(\lambda) = \widehat{f}(\lambda) \cdot \widehat{g}(\lambda)$
2. $\widehat{f \cdot g}\,(\lambda) = \widehat{f} * \widehat{g}\,(\lambda)$

Part B

$f, g \in \mathscr{S}\left(\mathbb{R}\right)$, then

1. $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$
2. $\widehat{f \cdot g} = \widehat{f} * \widehat{g}$

Part C

$f, g \in \mathscr{L}^2(\mathbb{R})$

1. $f * g(t) = \overline{F}\left(\widehat{f}\widehat{g}\right)(t)$
2. $\widehat{f \cdot g} = \widehat{f} * \widehat{g}(\lambda)$

6.3 Heisenberg Uncertainty Principle

6.3.1 Exercises

6.3.2 Interpretation of Heisenberg's Uncertainty Principle

Let $f(t)$ be our signal and $\widehat{f}(\lambda)$ be its frequential representation. We say that

$$\int \left|f(t)\right|\ dt = \int \left|\widehat{f}(\lambda)\right|^2\ dt$$

is the energy of the signal and $\left|f(t)\right|^2$ is the local density.

- We can view a signal as its density of energy.
- We can summarize $f(t)$ by the box there is between time and frequency accuracy.

$$\mu_N : \sigma_t \sigma_\epsilon > \frac{1}{4\pi}$$

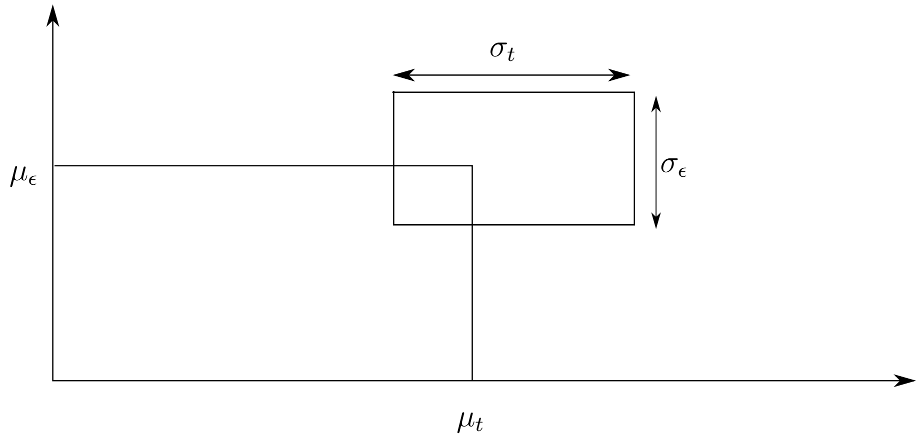


Figure 6.1: heisUnBox

6.4 Conclusion About Fourier Transform

Very useful because :

1. Transform Convolution into product
2. Transforms derivative into polynomial
3. Easy to compute using the FFT algorithm

But, this is a tool adopted for the use of "regular" signals, stationary signals. The tool is not performed for iregular sizes. Example : Show δ_0 and its fourier Transform.

Consider

$f(t) = \big(\sin(2\pi 5t) + \sin(2\pi 20t)\big)\, \mathbb{1}_{[0,1[}$ and $g(t) = \sin(2\pi 5t)\mathbb{1}_{[0,\alpha[} + \sin(2\pi 20t)\mathbb{1}_{[\alpha,1[}$

Use graphs from python to explain!

To correct these drawbacks. We use

- Window Fourier Transform
- Gabor Transform
- Continuous Wavelet Transform and Discrete Wavelet Transform

Chapter 7

Window Fourier Transform

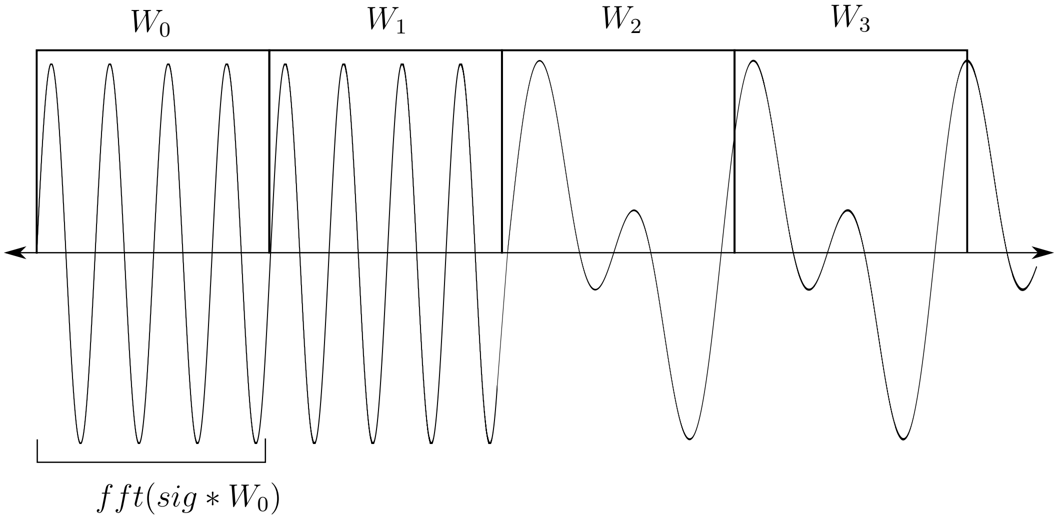


Figure 7.1: WFT of f

Definition 7.0.1 (Window)**.** *Let w a window such that $w \in \mathscr{L}^2 \wedge \mathscr{L}^2$ and $|\widehat{w}|$ is an odd function and $\|w\|_2 = 1$ We define*

$$w_{\lambda,b}(t) = w(t-b)e^{2i\pi\lambda t}$$

then $\forall f \in \mathscr{L}^2(\mathbb{R})$ the WFT is defined as

$$W_f(\lambda, b) = \int_{-\infty}^{\infty} f(t)\overline{w}_{\lambda,b}(t)\ dt$$

Some properties

(a) Conservation of Energy :

$$\int_{-\infty}^{\infty} \left|f(t)\right|^2 \; dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left|W_f \left(\lambda, b\right)\right|^2 \; d\lambda \; db$$

(b) Reconstruction formula :

$$f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\lambda, b) w_{\lambda,b}(t) \; d\lambda \; db$$

Comment:

How to choose w? In practice w is centered and symmetric on 0. We want $w_{\lambda,b}(t)$ centered on (λ, b).

7.1 Exercises : Atoms

Université Grenoble-ALPES
MSIAM
Signal and Image processing
Cecile.Amblard@imag.fr

Atoms

The time frequency atoms are functions $\phi_\gamma(t) = \phi(t,\gamma)$ where $t \in \mathbb{R}$ and $\gamma \in \mathbb{R}^d$, $d \in \mathbb{N}$. We assume that $||\phi_\gamma||_{\mathbb{L}^2(\mathbb{R})} = 1$ These atoms allow to analysis a function $f \in \mathbb{L}^2(\mathbb{R})$ via the time frequency transform:

$$Tf(\gamma) = \int_{-\infty}^{+\infty} f(t)\overline{\phi_\gamma(t)}dt$$

1. Show that
$$\int_{-\infty}^{+\infty} f(t)\overline{\phi_\gamma(t)}dt = \int_{-\infty}^{+\infty} \hat{f}(\xi)\overline{\hat{\phi}_\gamma(\xi)}d\xi$$
2. Show that if $\phi_\gamma(t)$ is null outside a neigbourhood $v(u)$ of u and $\hat{\phi}_\gamma(\xi)$ is null outside a neigbourhoud $v(\omega)$ of ω, then $Tf(\gamma)$ uses only the values of f for $t \in v(u)$ and $\lambda \in v(\omega)$.

$|\phi_\gamma(t)|^2$ can be considered as a probability distribution centered in $u_t(\gamma)$ and with variance $\sigma_t^2(\gamma)$ where:

$$\mu_t(\gamma) = \int_{-\infty}^{+\infty} t|\phi_\gamma(t)|^2 dt \text{ and } \sigma_t^2(\gamma) = \int_{-\infty}^{+\infty} (t-u_\gamma)^2|\phi_\gamma(t)|^2 dt$$

$|\hat{\phi}_\gamma(\xi)|^2$ can be considered as a probability distribution centered in $\mu_\xi(\gamma)$ and with variance $\sigma_\xi^2(\gamma)$ where:

$$\mu_\xi(\gamma) = \int_{-\infty}^{+\infty} \xi|\hat{\phi}_\gamma(\xi)|^2 d\xi \text{ and } \sigma_\xi^2(\gamma) = \int_{-\infty}^{+\infty} (\xi-\omega_\gamma)^2|\hat{\phi}_\gamma(\xi)|^2 d\xi.$$

We define in the time frequency space (t,ξ), the Heisenberg's box of ϕ_γ as a rectangle of center $(\mu_t(\gamma), \mu_\xi(\gamma))$ and dimension $\sigma_t(\gamma) * \sigma_\xi(\gamma)$.
Let consider the Short Time Fourier transform:

$$\phi_\gamma(t) = w(t-b)e^{2i\pi\lambda t} \text{ with } \gamma = (b,\lambda)$$

where w is odd.
Let consider the continuous wavelet transform::

$$\phi_\gamma(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \text{ with } \gamma = (b,a)$$

where $\psi(t)$ is centered at 0 ($|\psi(t)|^2$ is odd) and $\hat{\psi}(\xi) = 0, \forall \xi < 0$ (analytic wavelet).

1. Show that the Heisenberg's box of the Short Time Fourier Transform (STFT) atoms is centered at $\gamma = (b, \lambda)$ and that its dimensions do not depend on γ.

2. Show that the Heisenberg's box of the STFT atoms is centered at $(b, \frac{\eta}{a})$ with

$$\eta = \int_0^{+\infty} \xi |\hat{\psi}(\xi)|^2 d\xi$$

   and that its dimension depend on a.

3. Represent in the time frequency space all these boxes and compare them.

4. What is about the Fourier Transform?

Part 1)

Part 2)

$$\begin{aligned}
\mu_t(\gamma) &= \int_{-\infty}^{\infty} t \left|\phi_\gamma(t)\right|^2 \, dt \\
&= \int_{-\infty}^{\infty} t \left|w(t-b)e^{2i\pi\lambda t}\right|^2 \, dt \\
&= \int_{-\infty}^{\infty} (u+b) \left|w(u)\right|^2 \, du \\
&= b \text{ since } \|w\|_2 = 1; \|w\|^2 \text{ is even}
\end{aligned}$$

$$\begin{aligned}
\sigma_t^2 &= \int_{-\infty}^{\infty} (t-\mu_t)^2 \left|\phi_\gamma(t)\right|^2 \, dt \\
&= \int_{-\infty}^{\infty} (t-b)^2 \left|w\left(t-b\right)\right|^2 \, dt \\
&= \int_{-\infty}^{\infty} u^2 \left|w(u)\right|^2 \, du \\
&= \sigma_w^2
\end{aligned}$$

$$\begin{aligned}
\widehat{\phi}(\xi) &= \int_{-\infty}^{\infty} \phi_\gamma(t) e^{-2i\pi\xi t} \, dt \\
&= \int_{-\infty}^{\infty} w(t-b) e^{-2i\pi t(\xi-\lambda)} \, dt \\
&= \int_{-\infty}^{\infty} w(u) e^{-2i\pi(u+b)(\xi-\lambda)} \, du \\
&= e^{2i(\lambda-\xi)} \int_{-\infty}^{\infty} w(u) e^{-2i\pi u(\xi-\lambda)} \, du \\
&= e^{2i\pi b(\lambda-\xi)} \widehat{w}(\xi-\lambda)
\end{aligned}$$

and we have that

$$\begin{aligned}
\left|\phi_\gamma(\xi)\right| &= \left|\widehat{w}\left(\xi-\lambda\right)\right| \ d\xi \\
\mu_\xi &= \int_{-\infty}^{\infty} \xi \left|\widehat{w}\left(\xi-\lambda\right)\right|^2 \ d\xi \\
&= \int_{-\infty}^{\infty} (f+\lambda) \left|\widehat{w}(f)\right|^2 \ df \\
&= \lambda
\end{aligned}$$

Finally, this all implies that $\sigma_\xi^2 = \int_{-\infty}^{\infty} f^2 \left|\widehat{w}(f)\right|^2 \ df = \sigma_{\widehat{w}}^2$.
w_γ is a signal whose energy can be represented by a box, like 6.1, if γ changes, the center of the box changes but the dimension of the box does not change. Atoms ϕ_γ are represented by temporal spatial boxes of the same size. If the lengths of support of ϕ_γ is not large enough, we can't capture low frequency. If the length of the support is too large, we analyze high frequency component of the signal but lose time accuracy.

Chapter 8

Continuous Wavelet Transform

Motivations

We want to analyze a signal through atoms whose dimensions changes versus frequencies.

Definition 8.0.1 (Analyzing function)**.** *The continuous wavelet transform uses an analyzing function*

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

where b *is the translation parameter and* a *is the scale parameter.*

For each ψ there exists an exact link between scale and frequency.

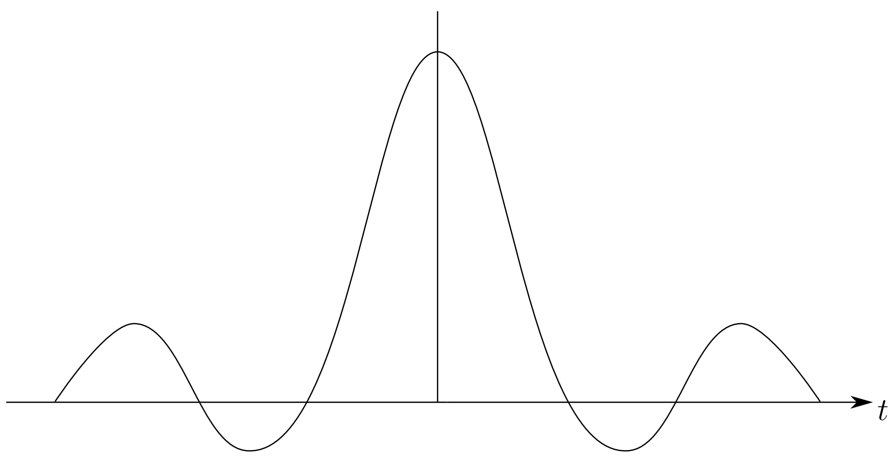


Figure 8.1: $\psi_{[1,0]}$

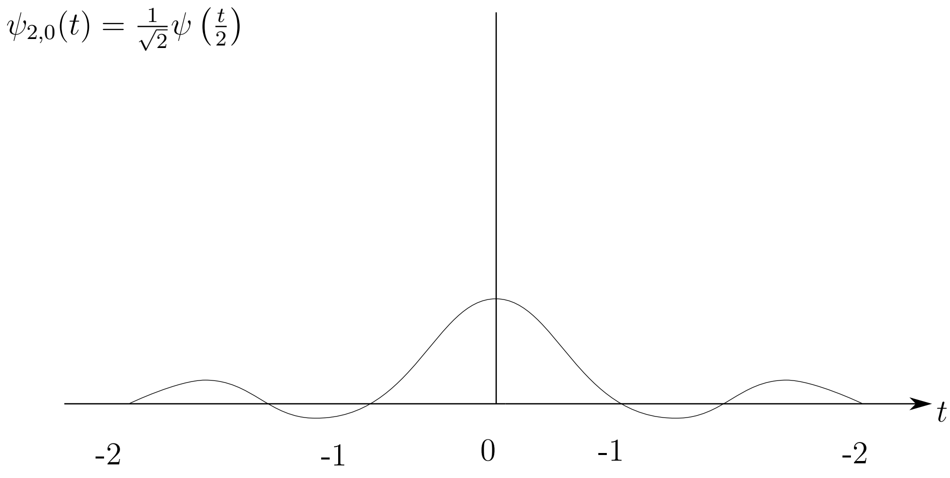


Figure 8.2: Dilation

High scale captures low frequencies, and low scale saptures hight frequency.

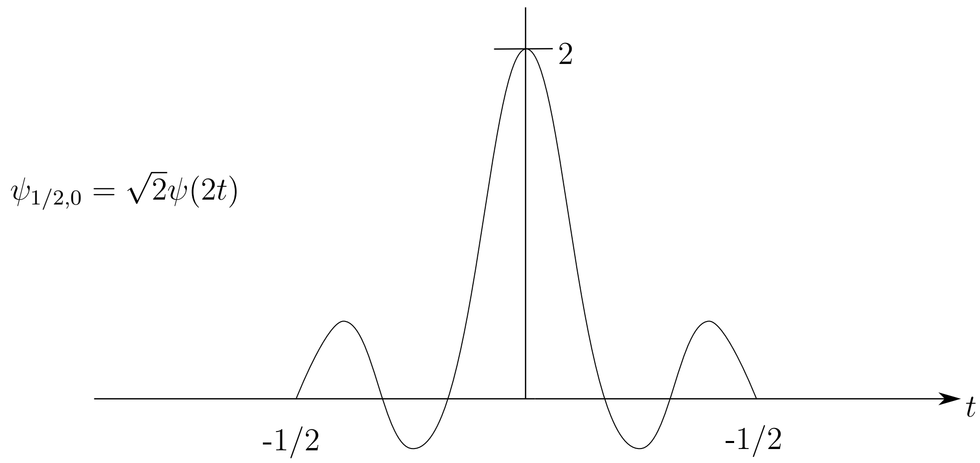


Figure 8.3: contraction

Definition 8.0.2 (Wavelet)**.** *Let* $\psi \in \mathscr{L}^1(\mathbb{R}) \cap \mathscr{L}^2(\mathbb{R})$ *such that*

1.
$$\int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}(\lambda)\right|^2}{|\lambda|}\, d\lambda = K < \infty$$

2. $\|\psi\|_2 = 1$.

Let

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

and $\forall f \in \mathscr{L}^2(\mathbb{R})$ *we consider the coefficients of an ondelette to be given by*

$$C_f(a,b) = \int_{-\infty}^{\infty} f(t)\overline{\psi}_{a,b}(t)\ dt$$

We have the following properties.

(a) Conservation of Energy :

$$\frac{1}{K}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|C_f(a,b)\right|^2 \frac{dadb}{a^2}$$

(b) Reconstruction Formula :

$$\frac{1}{K}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_f(a,b)\psi_{a,b}(x)\frac{dadb}{a^2}$$

In practice we choose ψ such that $\widehat{\psi}(0) = 0$ oscilating function.

Give the formulas and graphs for Haar wavelet, Mexican Hat, and Morlet wavelet.

8.0.1 Problem ?

$$\psi_{a,b} = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

$$\begin{aligned}
\mu_t(\gamma) &= \int_{-\infty}^{\infty} t\left|\phi_\gamma(t)\right|^2\ dt \\
&= \int_{-\infty}^{\infty} t\left|\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)\right|^2\ dt \\
u &= \frac{t-b}{a} \implies t = au+b \text{ and } dt = a\ du \\
\mu_t &= \frac{1}{a}\int_{-\infty}^{\infty}(au+b)\left|\psi_{a,b}(u)\right|^2\ adu \\
&= b \text{ since } \psi \text{ even and } \|\psi\|_2^2 = 1
\end{aligned}$$

$$\begin{aligned}
\sigma_t^2 &= \frac{1}{a}\int_{-\infty}^{\infty}(t-b)^2\left|\psi\left(\frac{t-b}{a}\right)\right|^2\ dt \\
&= \int_{-\infty}^{\infty} a^2u^2\left|\psi\left(u\right)\right|^2\ du \\
&= a^2\sigma_4^2
\end{aligned}$$

$$\widehat{\psi}_{a,b}(\lambda) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \psi\left(\frac{t-b}{a}\right) e^{-2i\pi\lambda t}\, dt$$
$$= \sqrt{a} \int_{-\infty}^{\infty} \psi(u)\, e^{-2i\pi\lambda[au+b]}\, dt$$
$$= \sqrt{a} e^{-2i\pi\lambda b} \psi(a\lambda)$$

$$\mu_\xi(\gamma) = \int_{-\infty}^{\infty} \lambda a \left|\widehat{\psi}(a\lambda)\right|^2 d\lambda$$
$$= \int_{-\infty}^{\infty} f \left|\widehat{\psi}(f)\right|^2 \frac{df}{a}$$
$$= \frac{1}{a} \int_{-\infty}^{\infty} f \left|\widehat{\psi}(f)\right|^2 df$$
$$= \frac{?}{a}$$

Where ? is maybe Γ The previous calculation shows that frequency is the inverse of scale.

$$\sigma_\xi^2(\gamma) = \int_{-\infty}^{\infty} \left(\lambda - \frac{\Gamma}{a}\right) a \left|\widehat{\psi}(a\lambda)\right|^2 d\lambda$$
$$= \int_{-\infty}^{\infty} \left(\frac{f-\Gamma}{a}\right)^2 \left|\widehat{\psi}(f)\right|^2 df$$
$$= \frac{1}{a^2} \int_{-\infty}^{\infty} (f-\Gamma)^2 \left|\widehat{\psi}(f)\right|^2 df = \frac{1}{a^2} \sigma_{\widehat{\psi}}^2$$

8.1 Random Discussion

We have that

$$\int \left|f(t)\right|^2 dt$$

gives us the energy density which we can think of as a probability density. Think about the frequency space

In Window Fourier transform, the dimension of boxes are independent of position (μ_t, μ_ξ) of the

In continuous wavelet transform dimension of σ on the scale a.

The aim of CWT is to analyze non-stationary signals that have height variations with less regularity.

8.2 Intro

8.3 Definition and reconstruction theroem

8.4 Properties Or how to choose ψ

Many possible choices for function $\psi \in \mathbb{R}$ or $\mathbb{C}$ and analytic. There is a link between regularity of the signal to analyze f and the wavelet transform $Cf(a,b)$ This link is dependent on the number of null moments of ψ.

Theorem 8.4.1. *Let $f \in \mathscr{C}^n$ and $f^{(n)}$ is bounded and ψ has n null moments defined as*

$$\int t^k \psi(t)\ dt = 0 \quad \forall k \in \{0, \cdots, n\}$$

Then

$$\left|Cf(a,b)\right| = k\left(a^{n+1/2}\right) \quad k \in \mathbb{R}$$

Proof. Let $f \in \mathscr{C}^n$ thus we can expand this using Taylor's formula :

$$f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}}{k!}(t-u)^k + \frac{f^{(n)}(c)}{n!}(t-u)^n \quad u \leq c \leq t$$

$$Cf(a,b) = \frac{1}{\sqrt{a}} \int f(t)\psi(\frac{t-b}{a})\ dt$$

$$Cf(a,b) = \sum_{k=0}^{n-1} \frac{f^k(a)}{k!} \int (t-u)^k \psi\left(\frac{t-b}{a}\right) \frac{dt}{\sqrt{a}}$$

Let $x = \dfrac{t-b}{a}$

$$Cf(a,b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(u)}{k!} \int (ax+b-u)^k \psi(x)\sqrt{a}\ dx + \sqrt{a}\frac{f^{(n)}(c)}{n!} \int (ax+b-u)^n \psi(x)\ dx$$

Using the fact that $\forall 0 \leq k \leq n$ gives

$$\int t^k \psi(t)\ dt = 0$$

we obtain

$$Cf(a,b) = \sqrt{a}\frac{f^{(n)}(c)}{n!} \int (ax+b-u)^n \overline{\psi}(x)\ dx$$

using

$$(\alpha + \beta)^n \leq 2^n \left(|a|^n + |b|^n\right)$$

then if $f \in \mathscr{C}^n$ we can choose $u = b$ then

$$Cf(a,b) \leq ka^{n+1/2}$$

with

$$k = 2^n \frac{\sup(f)^{(n)}}{n!} \int |x^n| \left|\overline{\psi}(x)\right| \, dx$$

□

(a) ψ has p null moments $p < n$

(b) ψ has p null moments $p > n$

$$Cf(a,b) =$$

8.4.1 Properties of ψ

ψ can be either:

- real or complex and either complex analytic or non-analytic.
- Finite support of large size that scales with a
- ψ can be "regular"
- ψ can have n null moments

a) Look at the exercise

$$f(t) = \cos 2\pi\lambda_0 t$$

8.4.2 Include Figures for Analytic vs. Non-analytic Functions

b) $Cf(a,b)$ uses $f(t_0)$ if $t_0 \in \text{supp}(\psi_{a,b})$ the number of values of $Cf(a,b)$ which uses $f(t_0)$ depends on the size of the support of ψ. We don't want all $Cf(a,b)$ to contain t_0 as we want differing information for differening $Cf(a,b)$.

c) Regularity of ψ

Using the reconstruction formula, we write the reconstructed signal as a weighted sum of $\psi_{a,b}$ where

$$f(t) = \int\int Cf(a,b)\psi_{a,b}(t)\frac{da\ db}{a^2}$$

In practice, the reconstructed signal is a finite sum of $\psi_{a,b}$. So if $\psi_{a,b}$ is not continuous f_r wil be non continuous, for example :

8.4.3 Insert image of Haar wavelet reconstruction

$$\psi(t)$$

d) Number of Null Moments

The speed of convergence of $Cf(a,b)$ for $a \to 0$ depends on this number of null moments.

Definition 8.4.2 (Lipshitz function). *f is α Lipshitz at v, $\alpha \geq 0$ means*

$$\exists P_v(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(v)}{k!}(t-v)^k$$

$m = [\alpha]$, then $\forall t \in \mathbb{R}$

$$\left|f(t) - P_v(t)\right| \leq K\,|t-v|^\alpha$$

f is uniformly lipshitz in $[a,b]$ then $\forall v \in [a,b], \forall t \in \mathbb{R}$

$$\left|f(t) - P_v(t)\right| \leq K\,|t-v|^\alpha$$

Also, if $f = g'$ with g being α Lipshitz, then f is $(\alpha - 1)$ Lipshitz.

Example 8.4.3.

$$\delta_0 = u' \textit{ with ...}$$

Theorem 8.4.4. *ψ has n null moments*

$$\forall k \in [0, n-1], \quad \int t^k \psi(t)\ dt = 0$$

f is α Liptshitz , $\alpha < n$ at v. Then

$$\left|Cf(a,b)\right| \leq Aa^{\alpha+1/2}\left(1+\left|\frac{b-v}{a}\right|\alpha\right)$$

Proof. $f(t) = P_v(t) + (t)$ with $\left|(t)\right| \leq k\left|t-v\right|\alpha$

$$Cf(a,b) = \int f(t)\psi_{a,b})(t)\ dt$$

$$= \sum_{k=0}^{m-1}\frac{f^{(k)}(v)}{k!}\int (t-v)^k\,\psi_{a,b}(t)\ dt + \int (t)\psi_{a,b}(t)\ dt$$

Let $x = \frac{t-b}{a}$

$$f(t) = \sum_{k=0}^{m-1}\frac{f^{(k)}(v)}{k!}\int (ax+b-v)^k\,\psi(x)\sqrt{a}\ dx + \sqrt{a}\int (ax+b-v)\psi(x)\ dx$$

$$\left|Cf(a,b)\right| \leq \sqrt{a}\int \left|(ax+b-v)\right|\psi(x)\ dx$$

$$\leq \sqrt{a}K\int \left|ax+b-v\right|^{\alpha}\psi(x)\ dx$$

$|a+b|^{\alpha} \leq 2^{\alpha}\left(|a|^{\alpha}+|b|^{\alpha}\right)$

$$\left|Cf(a,b)\right| \leq k2^{\alpha}\sqrt{a}\left(\int a^{\alpha}\left|x\right|^{\alpha}\psi(x)\ dx + \left|b-v\right|^{\alpha}\left|\psi(x)\right|dx\right)$$

$\square$

If $\alpha = -1$ at v, we have a strong singularity at v, which coefficients $Cf(a,b)$ will be influenced by this. We search $\left\{(a,b)|v \in \text{supp}\psi_{a,b}\right\}$. Consider $\text{supp}\,(\psi) = [-c,c]$ then

$$\left\{(a,b)\ \text{\textbf{,}}\ v \in \text{supp}\left(\psi_{a,b}\right)\right\} = \left\{(a,b)\ \text{\textbf{,}}\ \left|\frac{v-b}{a}\right|\right\}$$

For $\left\{(a,b)\middle/ v-ac < b < v+ac\right\}$ is influenced by the behavoir of f at v. If f is singular at v the $Cf(a,b)$ will have high values in this case.

Theorem 8.4.5. *ψ has n null moments, ψ rapdily decreases to 0 :*

$$\forall m \in \mathbb{N}, \exists C_m s.t. \psi(t) \leq \frac{C_m}{1+|t|^m}$$

then :

$$\psi(t) = (-1)^n \frac{d^n\theta(t)}{dt^n}$$

with θ rapidly decreasing function

$$Cf(a,b) = a^n \frac{d^n}{dt^n}\left(f * \overline{\theta}_a\right)(b)$$

with

$$\theta_a(t) = \frac{1}{\sqrt{a}}\theta\left(-\frac{t}{a}\right)$$

Proof. We prove the first step :
ψ is rapdily decreasing then $\widehat{\psi}$ is $\mathscr{C}^\infty$ and ψ has n null moments then

$$\int t^k \psi(t)\ dt = 0$$

for $k < n$.

$$\widehat{f^{(k)}}(\lambda) = \widehat{(-2i\pi t)^k f(t)}(\lambda)$$

$$\widehat{f}(0) = \int f(t)\ dt$$

$$\int t^k \psi(t)\ dt = \frac{1}{(-2i\pi)^k}\int (-2i\pi t)^k t^k e^{-2i\pi\lambda_0 t}\ dt$$

with $\lambda_0 = 0$

$$0 = \int t^t \psi(t)\ dt0 = (-i)^k\, \widehat{\psi}^{(k)}(0)$$

then $\widehat{\psi}$ is such that $\widehat{\psi}(0) = \widehat{\psi}'(0) = \widehat{\psi}''(0) = \cdots = \widehat{\psi}^{(n-1)} = 0$ then $\widehat{\psi}(\lambda) = (-2i\pi\lambda)^n\,\theta(\lambda)$. Then $\psi(t) = \mathscr{F}^{-1}\left((-2i\pi\lambda)^n\,\widehat{\theta}(\lambda)\right) = (-1)^n\,\theta^{(n)}(t)$ We now show θ is rapidly decreasing.
Assume that $n = 1$ thus, ψ has 1 null moment $\psi(t) = -\theta'(t)$.
We have

$$\forall m, \exists C_m s.t. \left|\psi(t)\right| < \frac{C_m}{1+|t|^m}$$

also,

$$\theta(t) = \int_t^\infty \psi(t)\ dt$$

$$\left|\theta(t)\right| \leq \int_t^\infty \left|\psi(u)\right| du \leq \int_t^\infty \frac{C_{m+1}}{1+\left|u\right|^{m+1}} \leq C_{m+1} \int_t^\infty \frac{1}{\left|u\right|^{m+1}}$$

$$\begin{aligned}\left|\theta(t)\right| &\leq \frac{C_{m+1}}{m}\left[\frac{1}{\left|u\right|^m}\right]_t^\infty \\ &\leq \frac{C_{n+1}}{n}\frac{1}{\left|t\right|^m} \\ &\leq \frac{cst}{\left|t\right| m}\end{aligned}$$

We are supposed to end up with $\leq \frac{dm}{1+|t|^m}$

$$\psi(t) = (-1)^n \frac{\partial^n \theta}{\partial t^n}(t)$$

with θ rapidly decreasing. use $Cf(a,b) = f * \psi_a(b) = f * (-1)^n \theta^{(n)}(b)$ then

$$Cf(a,b) = f * a^{n+1/2}\theta_a^{(n)}(b) = a^{n+1/2}\frac{\partial^n}{\partial b^n}\left(f * \theta_a\right)(b)$$

□

Interpretation

In practice $Cf(a,b)$ is obtained in 2 steps

1. Concolving f by θ_a.
2. Derivative the result

8.4.4 Insert image on convolve with θ_a

θ_a rapidly decreasing then $\frac{\theta_a}{\sqrt{a}} \to_{a\to 0} \delta_0$ and, in the sense of distributions :

$$\forall\varphi \int \frac{\theta_a}{\sqrt{a}}(t)\varphi(t)\, dt = \varphi(0)$$

Assume $f \in \mathscr{C}^n$ then

$$Cf(a,b) = a^n f^{(n)} * \theta_a(b)$$

$$\lim_{a\to 0}\frac{Cf(a,b)}{a^{n+1/2}} = \lim_{a\to 0} f^{(n)} * \frac{\theta_a}{\sqrt{a}}(b) = f^{(n)}(b)$$

Chapter 9

End of FT Lecture, FT of distributions

$\delta =$ u' in the meaning of distributions with $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t \geq 0$

$$\forall \varphi \in \mathscr{D}, \langle \delta, \varphi \rangle = \varphi(0)$$
$$\langle \delta_a, \varphi \rangle = \varphi(a)$$

product $f \in \mathscr{C}^\infty$ and $T \in \mathscr{D}'$. δT defined as

$$\forall \varphi \in \mathscr{D} \ \langle fT, \varphi \rangle = \langle T, f\varphi \rangle$$

Convolution

$s \in \mathscr{C}'$ (distribution of compact support)
$T \in \mathscr{S}$, $S * T \in \mathscr{S}'$

$f\delta_a = f(a)$ and $\Delta_a = \sum \delta_n a$ and $f\Delta_a = \sum f(na)\,\delta_{na}$ which is the sampling of f with step size a.

- $\delta * f = f * \delta = f$
- $\delta_a * f = \mathscr{S}_a f$
- $\Delta_a * f = \sum \mathscr{S}_{na} f =?$
- $\Delta_a * f(t) = \sum f(t - na)$ a periodic signal with period = a built from f

9.0.1 FT of distribution

Definition 9.0.1. *Let $f \in \mathscr{L}^1$ and $\varphi \in \mathscr{D}$*

$$\langle \widehat{f}, \varphi \rangle = \int \int e^{-2i\pi\lambda u} f(u) \ du \ \varphi(\lambda) \ d\lambda$$

$$= \langle f, \widehat{\varphi}(u) \rangle$$

we want to have $\forall T \in \mathscr{S}', \ \langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle$

But it does not work because

$$\forall \varphi \in \mathscr{D}, \widehat{\varphi} \notin \mathscr{D}$$

Then, we must have that T is defined on $\mathscr{S}'$,

$$\forall T \in \mathscr{S}' \ \not\forall \phi \in \mathscr{S}, \ \langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle$$

if $f \in \mathscr{L}^1 \left(\mathscr{L}^2 \right), \ \widehat{T_f} = T_{\widehat{f}}$
if $T \in \mathscr{S}', \mathscr{F} \left(\mathscr{F} T \right) = \widehat{\widehat{T}} = T_\sigma$.

Example 9.0.2. $\widehat{\delta}$ *defined by*

$$\begin{aligned}
\forall \varphi \in \mathscr{L}, \ \langle \widehat{\delta}, \varphi \rangle &= \langle \delta, \widehat{\varphi} \rangle \\
&= \widehat{\phi(0)} \\
&= \int \varphi(t) \ dt \\
&= \langle 1, \varphi \rangle
\end{aligned}$$

$\widehat{\delta} = 1$ *which is the Heisenberg Principle.*

$$\begin{aligned}
\langle \widehat{\delta_a}, \varphi \rangle &= \langle \delta_a, \widehat{\varphi} \rangle \\
&= \widehat{\varphi}(a) \\
&= \int e^{-2i\pi a t} \varphi(t) \ dt \\
&= \langle e^{-2i\pi a \cdot}, \varphi(\cdot) \rangle
\end{aligned}$$

$\widehat{\delta_a} = e^{-2i\pi a \cdot}$ and $\widehat{\delta^{(k)}} = (2i\pi) \delta$

9.0.2 Sawtooth

$$\widehat{\Delta_a} = \widehat{\sum \delta_{na}}(\lambda) = \sum^{n} e^{2i\pi n a \lambda}$$

Assume $f \in \mathscr{L}^1_p(a)$ where $a =$ period Fourier Series of f
Something about calculating fourier series using distributions?

Bibliography

Asmar, Nakhlé H. *Partial Differential Equations : with Fourier Series and Boundary Value Problems*. 2nd edition. Pearson, 2005.

Axler, Shelder. *Measure, Integration Real analysis*. Prelimnary edition, 2019.

Boggess, Albert and Francis J. Narcowich. *A First Course in Wavelets with Fourier Analysis*. Wiley, 2009.

Rudin, Walter. *Principles of Mathematical Analysis*. McGraw-Hill Publishing Co., 1976.