PDEs and numerical methods - Laplace and Poisson problems

Exercise 1 *A property of harmonic functions*

Let Ω an open set in $\mathbb{R}^n$, and a function $u : \Omega \to \mathbb{R}$.
u is an harmonic function iff $u \in \mathcal{C}^2(\Omega)$ and $\Delta u = 0$ on Ω.

1.1 Let u a quadratic form: $u(x_1, \dots, x_n) = \displaystyle\sum_{i=1}^{n} a_i\, x_i^2 + \sum_{i<j} b_{ij}\, x_i x_j$. What is the condition for u to be harmonic?
1.2 Prove that, if u is harmonic and if $u \in \mathcal{C}^3(\Omega)$, then $\dfrac{\partial u}{\partial x_i}$ is also harmonic $(i = 1, \dots, n)$.
1.3 Prove that, if u is harmonic and if $u \in \mathcal{C}^{m+2}(\Omega)$, then its partial derivatives up to order m are also harmonic functions.

Exercise 2 *Laplace equation on a rectangle*

Let consider the problem:

$$\begin{cases} \Delta u = 0 \quad \text{in } (0, L_x) \ \times (0, L_y) \\ u(0, y) = h(y), u(L_x, y) = u(x, 0) = u(x, L_y) = 0 \end{cases}$$

Using separation of variables, prove that $u(x, y) = \displaystyle\sum_{k\geq 1} \alpha_k \, \left(e^{\lambda_k x} - e^{\lambda_k(2L_x - x)}\right) \, \sin(\lambda_k y)$

where $\lambda_k = \dfrac{k\pi}{L_y}$ and $\alpha_k = \dfrac{2}{L_y\,(1 - e^{2\lambda_k L_x})} \displaystyle\int_0^{L_y} h(y)\, \sin(\lambda_k y)\, dy$

Exercise 3 *Laplace equation on a disk*

Let Ω the open disk of center $(0, 0)$ and radius R. Let consider the problem:

$$\begin{cases} \Delta u &= 0 & \text{in } \Omega \\ \quad u &= g(\theta) & \text{on } \partial\Omega \end{cases} \tag{1}$$

The Laplacian operator in polar coordinates reads $\Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$.

3.1 Let first solve $\Delta u = 0$ without wondering about boundary conditions, looking for solutions of the form $u(r, \theta) = v(r)w(\theta)$ (separation of variables technique). Prove that this leads to two ODEs, one for v and one for w. Solve first the equation for w, noting that w must be 2π-periodic (since u is $\mathcal{C}^0$ and Ω is a disk). Then solve the equation for v.

The solution at the end should be:

$$u(r,\theta) = \sum_{n\geq 0} (\alpha_n \cos n\theta + \beta_n \sin n\theta)\ r^n \tag{2}$$

3.2 Assuming that $g \in \mathcal{C}^1$, its Fourier series reads $g(\theta) = a_0 + \sum_{n\geq 1} (a_n \cos n\theta + b_n \sin n\theta)$
with $a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\, d\alpha$, $a_n = \frac{1}{\pi}\int_0^{2\pi} g(\alpha)\ \cos n\alpha\ \ d\alpha$, $b_n = \frac{1}{\pi}\int_0^{2\pi} g(\alpha)\ \sin n\alpha\ \ d\alpha.$
Using (2), prove that the solution of (1) is $u(r,\theta) = a_0 + \sum_{n\geq 1} (a_n \cos n\theta + b_n \sin n\theta) \left(\frac{r}{R}\right)^n$

3.3 Prove that the preceding expression can be transformed into

$$u(r,\theta) = \frac{1}{\pi}\left[\frac{1}{2}\int_0^{2\pi} g(\alpha)\ \ d\alpha + \sum_{n\geq 1} \frac{r^n}{R^n}\int_0^{2\pi} g(\alpha)\ \cos n(\theta-\alpha)\ \ d\alpha\right]$$

3.4 Prove that the preceding expression can be transformed into

$$u(r,\theta) = K(r,\theta) * g(\theta) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\theta-\alpha) g(\alpha)\, d\alpha \qquad \text{where } K(r,\theta) = \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos\theta}$$

3.5 Admitting that $\frac{1}{2\pi}\int_0^{2\pi} K(r,\theta-\alpha)\, d\alpha = 1 \quad \forall r < R$ (you can actually prove it if you have 15 minutes left), prove that $u(r,\theta)$ satisfies the maximum principle.

Exercise 4 *1D and 2D Laplacian matrices*

4.1 Let first consider the ODE: $-u''(x) = f(x)$, $x \in (0,L)$, with $u(0) = u(L) = 0$.
What are the eigenvalues and eigenvectors of the second order derivative operator defined on the space $\{u \in \mathcal{C}^2(0,L), u(0) = u(L) = 0\}$?

4.2 Let now a standard second order finite difference discretization of this problem, with mesh step $h = L/(N+1)$. It reads

$$\frac{1}{h^2}\mathbf{A}_N\,\mathbf{U} = \mathbf{F} \qquad \text{with } \mathbf{A}_N = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

By analogy with the continuous case (or by a direct calculation), find the eigenvectors and eigenvalues of $\mathbf{A}_N$ (and make sure that it is consistent with the continuous case).
What is the condition number of $\mathbf{A}_N$?

4.3 Let consider now the 2D Poisson problem $-\Delta u = f$ on $\Omega = (0, L_x) \times (0, L_y)$, with zero Dirichlet boundary condition. We consider a regular discretization grid, with mesh steps $h_x = L_x/(N_x + 1)$ and $h_y = L_y/(N_y + 1)$ in the x and y directions respectively (N_x and N_y are integers).

4.3.1 Write the linear system $\mathbf{A}_{2D}\,\mathbf{U} = \mathbf{F}$ corresponding to the usual 5-point second order finite difference scheme for the Laplacian (with the unknowns ordered as $\mathbf{U}_{i+(j-1)N_x} = u_{ij}$).

The tensor product (also called Kronecker product) of 2 matrices $\mathbf{A}$ and $\mathbf{B}$ being defined, with obvious notations, as the block matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1p}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}\mathbf{B} & a_{n2}\mathbf{B} & \cdots & a_{np}\mathbf{B} \end{bmatrix},$$

prove that $\mathbf{A}_{2D} = \dfrac{1}{h_x^2}\,\mathbf{I}_{N_y} \otimes \mathbf{A}_{N_x} + \dfrac{1}{h_y^2}\,\mathbf{A}_{N_y} \otimes \mathbf{I}_{N_x}$ where $\mathbf{I}_n$ is the identity matrix of size n.

4.3.2 Prove that the eigenvalues and eigenvectors of $\mathbf{A}_{2D}$ are

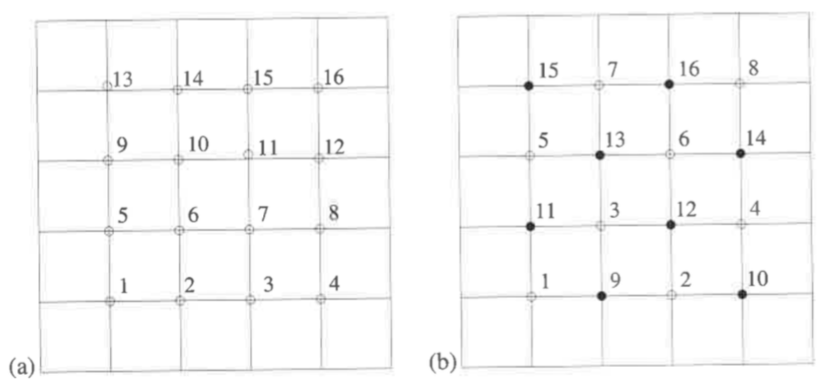
$$\lambda_{k,l} = \frac{4}{h_x^2}\sin^2\left(\frac{k\pi}{2(N_x+1)}\right) + \frac{4}{h_y^2}\sin^2\left(\frac{l\pi}{2(N_y+1)}\right) \qquad (1 \le k \le N_x, 1 \le l \le N_y)$$

and

$$X_{k,l}(i,j) = \sin\frac{k\pi i}{N_x+1}\;\sin\frac{l\pi j}{N_y+1} \qquad (1 \le k \le N_x, 1 \le l \le N_y)$$

4.3.3 What is the condition number of $\mathbf{A}_{2D}$?

4.3.4 Let consider the alternative so called “red-black” order of unknowns, as illustrated below for a 4×4 grid:



Left: natural rowwise ordering; Right: red-black ordering

What about the corresponding new form for $\mathbf{A}_{2D}$?

Exercise 5 *9-point 2D Laplacian - Fourth order scheme*

Let consider the Poisson problem $\Delta u = f$ in $\Omega \subset \mathbb{R}^2$. On a regular 2D grid, let consider the usual 5-point scheme for the Laplacian

$$\Delta_5 u_{ij} = \frac{1}{h^2}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\right]$$

and the alternative 9-point scheme

$$\Delta_9 u_{ij} = \frac{1}{6h^2}\left[4u_{i-1,j} + 4u_{i+1,j} + 4u_{i,j-1} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1} - 20u_{i,j}\right]$$

5.1 What is the dominant error term for each scheme ? Is the 9-point scheme more accurate than the 5-point one ?

Some results from Taylor expansions, to spare time:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} = 4u_{ij} + h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{h^4}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) + \mathcal{O}(h^4)$$

$$u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1} = 4u_{ij} + 2h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\frac{h^4}{12}\left(\frac{\partial^4 u}{\partial x^4} + 6\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) + \mathcal{O}(h^4)$$

5.2 Using the expression of $\Delta(\Delta u)$, prove that the 9-point scheme is actually fourth order accurate if f is an harmonic function (i.e. satisfies $\Delta f = 0$).

5.3 More generally, prove that solving $\Delta_9 u_{ij} = f_{ij}$ and defining

$$f_{ij} = f(x_i, y_j) + \frac{h^2}{12}\Delta f(x_i, y_j)$$

(if f is sufficiently smooth) instead of $f_{ij} = f(x_i, y_j)$ leads to a fourth order accurate method.

This method corresponds to deliberately introducing a $\mathcal{O}(h^2)$ error into the right-hand side of the equation that is chosen to exactly cancel the $\mathcal{O}(h^2)$ part of the local truncation error.

5.4 If f is known only at the grid points (but is known to be sufficiently smooth), prove that we can achieve the same fourth order accuracy by using

$$f_{ij} = f(x_i, y_j) + \frac{h^2}{12}\Delta_5 f(x_i, y_j)$$