TUTORIAL WEEK 10

- *This week is an introduction to graph theory :-)*

Part 1

Please introduce the complete graph, the star, and the cycle (all on n vertices). For each of them, analyze the number of edges and the degrees of vertices.

Part 2

The assignment teaches the law of total probability (which, at this point, students might have been assuming/using without noticing). The question intends to introduce it, and apply in the context of graphs.

> Law of total probability. A partition of a set Ω is a sequence of non-empty sets $S_1 \ldots S_n$ that are pairwise-disjoint and satisfy $\cup_{i \in [n]} S_i = \Omega$. The law of total probability asserts that for every probability space (Ω,p), and every partition $S_1 \ldots S_n$ of Ω, and every event $E \subseteq \Omega$, it holds that
>
> $$\Pr[E] = \sum_{i \in [n]} \Pr[S_i] \cdot \Pr[E | S_i].$$

Consider a graph G=([n],E). We are choosing a vertex i ∈ [n], but not necessarily uniformly at random. We know that:

- With probability ½, the vertex will have degree n/4.
- With probability ¼, the vertex will have degree n/8.
- With probability ¼, the vertex will be isolated.

1. What is the probability that the vertex has degree at least n/8?

Let E = { i ∈ [n] : deg(i) ≥ n/8 }. We define a partition of [n] as follows:

- S_1 = { i ∈ [n] : deg(i) = n/4 }
- S_2 = { i ∈ [n] : deg(i) = n/8 }
- S_3 = { i ∈ [n] : deg(i) = 0 }

By the law of total probability, we have

$$\Pr[E] = \sum_{i\in[3]} \Pr[S_i] \cdot \Pr[E|S_i] = \Pr[S_1] \cdot 1 + \Pr[S_2] \cdot 1 + \Pr[S_3] \cdot 0 = 3/4$$

2. What is the expected degree of the chosen vertex?

We define a random variable X:[n] → R such that X(i)=deg(i). This random variable has three values, i.e. n/4 and n/8 and 0. We use the property of expected value that we learned:

$$E[X] = \sum_{r\in\{n/4,n/8,0\}} r \cdot \Pr[X = r] = (1/2) \cdot (n/4) + (1/4) \cdot (n/8) + (1/4) \cdot 0$$

$$= n/8 + n/32 = 5n/32$$