8

CATEGORIES OF DIAGRAMS

In this chapter, we prove a very useful technical result called the Yoneda lemma, and then employ it in the study of the important categories of set-valued functors or "diagrams." The Yoneda lemma is perhaps the single most used result in category theory. It can be seen as a straightforward generalization of some simple facts about monoids and posets, yet it has much more far-reaching applications.

8.1 Set-valued functor categories

We are going to focus on special functor categories of the form

$$\mathbf{Sets}^{\mathbf{C}}$$

where the category $\mathbf{C}$ is locally small. Thus, the objects are set-valued functors,

$$F, G : \mathbf{C} \to \mathbf{Sets}$$

(sometimes called "diagrams on $\mathbf{C}$"), and the arrows are natural transformations

$$\alpha, \beta : F \to G.$$

Where $\mathbf{C} = \mathrm{P}$, a poset, we have already considered such functors as "variable sets," that is, sets F_i depending on a parameter $i \in \mathrm{P}$. The general case of a non-poset $\mathbf{C}$ similarly admits an interpretation as "variable sets": such a functor F gives a family of sets FC and transitions $FC \to FC'$ showing how the sets change according to every $C \to C'$. For instance, $\mathbf{C}$ might be the category $\mathbf{Sets}_{\mathrm{fin}}$ of all finite sets (of finite sets, ...) and all functions between them. Then in $\mathbf{Sets}^{\mathbf{Sets}_{\mathrm{fin}}}$ there is for example the inclusion functor $U : \mathbf{Sets}_{\mathrm{fin}} \to \mathbf{Sets}$, which can be regarded as a "generic" or variable finite set, along with the functors $U \times U$, $U + U$, etc., which are "variable" structures of these kinds.

Given any such category $\mathbf{Sets}^{\mathbf{C}}$, remember that we can evaluate any commutative diagram,

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[dr, "\beta\alpha"'] & Q \arrow[d, "\beta"] \\ & R \end{tikzcd}$$

at any object C to get a commutative diagram in **Sets**,

$$
\begin{tikzcd}
PC \arrow[r, "\alpha_C"] \arrow[dr, "(\beta\alpha)_C"'] & QC \arrow[d, "\beta_C"] \\
& RC
\end{tikzcd}
\tag{8.1}
$$

Thus, for each object C, there is an evaluation functor

$$\mathrm{ev}_C : \mathbf{Sets}^{\mathbf{C}} \to \mathbf{Sets}.$$

Moreover, naturality means that if we have any arrow $f : D \to C$, we get a "cylinder" over the diagram (8.1) in **Sets**.

Another way of thinking about such functor categories that was already considered in Section 7.7 is suggested by considering the case where $\mathbf{C}$ is the category Γ pictured as

$$
\begin{tikzcd}
1 \arrow[r, shift left] \arrow[r, shift right] & 0
\end{tikzcd}
$$

Then a set-valued functor $G : \Gamma \to \mathbf{Sets}$ is just a graph, and a natural transformation $\alpha : G \to H$ is a graph homomorphism. Thus, for this case,

$$\mathbf{Sets}^{\Gamma} = \mathbf{Graphs}.$$

This suggests regarding an arbitrary category of the form $\mathbf{Sets}^{\mathbf{C}}$ as a generalized "category of structured sets" and their "homomorphisms"; indeed, this is a very useful way of thinking of such functors and their natural transformations.

Another basic example is the category $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$, where the index category Δ is the category of finite ordinals that we already met in Chapter 7. The objects of $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$ are called *simplicial sets*, and are used in topology to compute the homology, cohomology, and homotopy of spaces. Since Δ looks like

$$
\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r, shift left=2] \arrow[r, shift right=2] & 2 \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & 3 \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots
\end{tikzcd}
$$

(satisfying the simplicial identities), a simplicial set $S : \Delta^{\mathrm{op}} \to \mathbf{Sets}$ looks like this:

$$
\begin{tikzcd}
S_0 & S_1 \arrow[l] \arrow[r] & S_2 \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] & S_3 \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] & \cdots
\end{tikzcd}
$$

(satisfying the corresponding identities). For example, one can take $S_n = S^n = S \times \ldots \times S$ (n times) for a fixed set S to get a (rather trivial) simplicial set, with the maps being the evident product projections and generalized diagonals. More interestingly, for a fixed poset P, one takes

$$S(\mathrm{P})_n = \{(p_1, \ldots, p_n) \in \mathrm{P}^n \mid \ p_1 \leq \ldots \leq p_n\},$$

with the evident projections and inclusions; this is called the "simplicial nerve" of the poset P.

8.2 The Yoneda embedding

Among the objects of $\mathbf{Sets}^{\mathbf{C}}$ are certain very special ones, namely the (covariant) representable functors,

$$\mathrm{Hom}_{\mathbf{C}}(C, -) : \mathbf{C} \to \mathbf{Sets}.$$

Observe that for each $h : C \to D$ in $\mathbf{C}$, we have a natural transformation

$$\mathrm{Hom}_{\mathbf{C}}(h, -) : \mathrm{Hom}_{\mathbf{C}}(D, -) \to \mathrm{Hom}_{\mathbf{C}}(C, -)$$

(note the direction!) where the component at X is defined by precomposition:

$$(f : D \to X) \mapsto (f \circ h : C \to X).$$

Thus, we have a *contravariant* functor

$$k : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}^{\mathbf{C}}$$

defined by $k(C) = \mathrm{Hom}_{\mathbf{C}}(C, -)$. Of course, this functor k is just the exponential transpose of the bifunctor

$$\mathrm{Hom}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sets}$$

which was shown as an exercise to be functorial.

If we instead transpose $\mathrm{Hom}_{\mathbf{C}}$ with respect to its other argument, we get a *covariant* functor,

$$y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$$

from $\mathbf{C}$ to a category of *contravariant* set-valued functors, sometimes called "presheaves." (Or, what amounts to the same thing, we can put $\mathbf{D} = \mathbf{C}^{\mathrm{op}}$ and apply the previous considerations to $\mathbf{D}$ in place of $\mathbf{C}$.) More formally:

Definition 8.1. The *Yoneda embedding* is the functor $y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ taking $C \in \mathbf{C}$ to the contravariant representable functor,

$$yC = \mathrm{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$$

and taking $f : C \to D$ to the natural transformation,

$$yf = \mathrm{Hom}_{\mathbf{C}}(-, f) : \mathrm{Hom}_{\mathbf{C}}(-, C) \to \mathrm{Hom}_{\mathbf{C}}(-, D).$$

A functor $F : \mathbf{C} \to \mathbf{D}$ is called an *embedding* if it is full, faithful, and injective on objects. We soon show that y really *is* an embedding; this is a corollary of the Yoneda lemma.

One should thus think of the Yoneda embedding y as a "representation" of $\mathbf{C}$ in a category of set-valued functors and natural transformations on *some* index category. Compared to the Cayley representation considered in Section 1.5, this has the virtue of being *full*: any map $\vartheta : yC \to yD$ in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ comes from a unique map $h : C \to D$ in $\mathbf{C}$ as $yh = \vartheta$. Indeed, recall that the Cayley representation of a group G was an injective group homomorphism

$$G \rightarrowtail \mathrm{Aut}(|G|) \subseteq |G|^{|G|}$$

where each $g \in G$ is represented as an automorphism $\tilde{g}$ of the set $|G|$ of elements (i.e., a "permutation"), by letting it "act on the left,"

$$\widetilde{g}(x) = g \cdot x$$

and the group multiplication is represented by composition of permutations,

$$\widetilde{g \cdot h} = \tilde{g} \circ \tilde{h}.$$

We also showed a generalization of this representation to arbitrary categories. Thus for any monoid M, there is an analogous representation

$$M \rightarrowtail \mathrm{End}(|M|) \subseteq |M|^{|M|}$$

by left action, representing the elements of M as endomorphisms of $|M|$.

Similarly, any poset P can be represented as a poset of subsets and inclusions by considering the poset $\mathrm{Low}(P)$ of "lower sets" $A \subseteq P$, that is, subsets that are "closed down" in the sense that $a' \leq a \in A$ implies $a' \in A$, ordered by inclusion. Taking the "principal lower set"

$$\downarrow(p) = \{q \in P \mid q \leq p\}$$

of each element $p \in P$ determines a monotone injection

$$\downarrow : P \rightarrowtail \mathrm{Low}(P) \subseteq \mathcal{P}(|P|)$$

such that $p \leq q$ iff $\downarrow(p) \subseteq \downarrow(q)$.

The representation given by the Yoneda embedding is closely related to these, but "better" in that it cuts down the arrows in the codomain category to just those in the image of the representation functor $y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ (since y is full). Indeed, there may be many automorphisms $\alpha : G \to G$ of a group G that are not left actions by an element, but if we require α to commute with all *right* actions $\alpha(x \cdot g) = \alpha(x) \cdot g$, then α must itself be a left action. This is what the Yoneda embedding does in general; it adds enough "structure" to the objects yA in the image of the representation that the only "homomorphisms" $\vartheta : yA \to yB$ between those objects are the representable ones $\vartheta = yh$ for some $h : A \to B$. In this sense, the Yoneda embedding y represents the objects and arrows of $\mathbf{C}$ as certain "structured sets" and (*all of*) their "homomorphisms."

8.3 The Yoneda lemma

Lemma 8.2 (Yoneda). *Let* $\mathbf{C}$ *be locally small. For any object* $C \in \mathbf{C}$ *and functor* $F \in \mathbf{Sets},^{\mathbf{C}^{\mathrm{op}}}$ *there is an isomorphism*

$$\mathrm{Hom}(yC, F) \cong FC$$

which, moreover, is natural in both F and C.

Here

(1) the Hom is $\mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}}$,

(2) naturality in F means that, given any $\vartheta : F \to G$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(yC, F) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}(yC,\vartheta)"'] & FC \arrow[d, "\vartheta_C"] \\
\mathrm{Hom}(yC, G) \arrow[r, "\cong"'] & GC
\end{tikzcd}$$

(3) naturality in C means that, given any $h : C \to D$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(yC, F) \arrow[r, "\cong"] & FC \\
\mathrm{Hom}(yD, F) \arrow[r, "\cong"'] \arrow[u, "\mathrm{Hom}(yh,F)"] & FD \arrow[u, "Fh"']
\end{tikzcd}$$

Proof. To define the desired isomorphism,

$$\eta_{C,F} : \mathrm{Hom}(yC, F) \xrightarrow{\cong} FC$$

take $\vartheta : yC \to F$ and let

$$\eta_{C,F}(\vartheta) = \vartheta_C(1_C)$$

which we also write as

$$x_\vartheta = \vartheta_C(1_C) \tag{8.2}$$

where $\vartheta_C : \mathbf{C}(C, C) \to FC$ and so $\vartheta_C(1_C) \in FC$.

Conversely, given any $a \in FC$, we define the natural transformation $\vartheta_a : yC \to F$ as follows. Given any C', we define the component

$$(\vartheta_a)_{C'} : \mathrm{Hom}(C', C) \to FC'$$

by setting

$$(\vartheta_a)_{C'}(h) = F(h)(a) \tag{8.3}$$

for $h : C' \to C$.

To show that ϑ_a is natural, take any $f : C'' \to C'$, and consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}(C'', C) \arrow[r, "(\vartheta_a)_{C''}"] & FC'' \\
\mathrm{Hom}(C', C) \arrow[u, "\mathrm{Hom}(f, C)"] \arrow[r, "(\vartheta_a)_{C'}"'] & FC' \arrow[u, "F(f)"']
\end{tikzcd}$$

We then calculate, for any $h \in yC(C')$

$$\begin{aligned}
(\vartheta_a)_{C''} \circ \mathrm{Hom}(f, C)(h) &= (\vartheta_a)_{C''}(h \circ f) \\
&= F(h \circ f)(a) \\
&= F(f) \circ F(h)(a) \\
&= F(f)(\vartheta_a)_{C'}(h).
\end{aligned}$$

So ϑ_a is indeed natural.

Now to show that ϑ_a and x_ϑ are mutually inverse, let us calculate ϑ_{x_ϑ} for a given $\vartheta : yC \to F$. First, just from the definitions (8.2) and (8.3), we have that for any $h : C' \to C$,

$$(\vartheta_{(x_\vartheta)})_{C'}(h) = F(h)(\vartheta_C(1_C)).$$

But since ϑ is natural, the following commutes:

$$\begin{tikzcd}
yC(C) \arrow[r, "\vartheta_C"] \arrow[d, "yC(h)"'] & FC \arrow[d, "Fh"] \\
yC(C') \arrow[r, "\vartheta_{C'}"'] & FC'
\end{tikzcd}$$

So, continuing,

$$\begin{aligned}
(\vartheta_{(x_\vartheta)})_{C'}(h) &= F(h)(\vartheta_C(1_C)) \\
&= \vartheta_{C'} \circ yC(h)(1_C) \\
&= \vartheta_{C'}(h).
\end{aligned}$$

Therefore, $\vartheta_{(x_\vartheta)} = \vartheta$.

Going the other way around, for any $a \in FC$, we have

$$\begin{aligned} x_{\vartheta_a} &= (\vartheta_a)_C(1_C) \\ &= F(1_C)(a) \\ &= 1_{FC}(a) \\ &= a. \end{aligned}$$

Thus, $\mathrm{Hom}(yC, F) \cong FC$, as required.

The naturality claims are also easy: given $\phi : F \to F'$, taking $\vartheta \in \mathrm{Hom}(yC, F)$, and chasing around the diagram

$$\begin{tikzcd} \mathrm{Hom}(yC, F) \arrow[r, "\eta_{C,F}"] \arrow[d, "\mathrm{Hom}(yC,\phi)"'] & FC \arrow[d, "\phi_C"] \\ \mathrm{Hom}(yC, F') \arrow[r, "\eta_{C,F'}"'] & F'C \end{tikzcd}$$

we get

$$\begin{aligned} \phi_C(x_\vartheta) &= \phi_C(\vartheta_C(1_C)) \\ &= (\phi\vartheta)_C(1_C) \\ &= x_{(\phi\vartheta)} \\ &= \eta_{C,F'}(\mathrm{Hom}(yC, \phi)(\vartheta)). \end{aligned}$$

For naturality in C, take some $f : C' \to C$. We then have

$$\begin{aligned} \eta_{C'}(yf)^*(\vartheta) &= \eta_{C'}(\vartheta \circ yf) \\ &= (\vartheta \circ yf)_{C'}(1_{C'}) \\ &= \vartheta_{C'} \circ (yf)_{C'}(1_{C'}) \\ &= \vartheta_{C'}(f \circ 1_{C'}) \\ &= \vartheta_{C'}(f) \\ &= \vartheta_{C'}(1_C \circ f) \\ &= \vartheta_{C'} \circ (yC)(f)(1_C) \\ &= F(f) \circ \vartheta_C(1_C) \\ &= F(f)\eta_C(\vartheta). \end{aligned}$$

The penultimate equation is by the naturality square:

$$\begin{tikzcd} yC(C) \arrow[r, "\vartheta_C"] \arrow[d, "yC(f)"'] & F(C) \arrow[d, "F(f)"] \\ yC(C') \arrow[r, "\vartheta_{C'}"'] & F(C') \end{tikzcd}$$

Therefore, $\eta_{C'} \circ (yf)^* = F(f) \circ \eta_C$.

□

The Yoneda lemma is used to prove our first "theorem."

Theorem 8.3. *The Yoneda embedding* $y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ *is full and faithful.*

Proof. For any objects $C, D \in \mathbf{C}$, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(C, D) = yD(C) \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}}(yC, yD).$$

And this isomorphism is indeed induced by the functor y, since by (8.3) it takes an element $h : C \to D$ of $yD(C)$ to the natural transformation $\vartheta_h : yC \to yD$ given by

$$\begin{aligned} (\vartheta_h)_{C'}(f : C' \to C) &= yD(f)(h) \\ &= \mathrm{Hom}_{\mathbf{C}}(f, D)(h) \\ &= h \circ f \\ &= (yh)_{C'}(f), \end{aligned}$$

where $yh : yC \to yD$ has component at C':

$$\begin{aligned} (yh)_{C'} : \mathrm{Hom}(C', C) &\longrightarrow \mathrm{Hom}(C', D) \\ f &\longmapsto h \circ f \end{aligned}$$

So, $\vartheta_h = y(h)$. □

Remark 8.4. Note the following:

- If $\mathbf{C}$ is small, then $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is locally small, and so $\mathrm{Hom}(yC, P)$ in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is a set.
- If $\mathbf{C}$ is locally small, then $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ need not be locally small. In this case, the Yoneda lemma tells us that $\mathrm{Hom}(yC, P)$ is always a set.
- If $\mathbf{C}$ is not locally small, then $y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ will not even be defined, so the Yoneda lemma does not apply.

Finally, observe that the Yoneda embedding $y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is also injective on objects. For, given objects A, B in $\mathbf{C}$, if $yA = yB$ then $1_C \in \mathrm{Hom}(C, C) = yC(C) = yD(C) = \mathrm{Hom}(C, D)$ implies $C = D$.

8.4 Applications of the Yoneda lemma

One frequent sort of application of the Yoneda lemma is of the following form: given objects A, B in a category $\mathbf{C}$, to show that $A \cong B$ it suffices to show that $yA \cong yB$ in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$. This "Yoneda principle" results from the foregoing theorem and the fact that, if $F : \mathbf{C} \to \mathbf{D}$ is any full and faithful functor, then $FA \cong FB$ clearly implies $A \cong B$. We record this as the following.

Corollary 8.5 (Yoneda principle). *Given objects A and B in any locally small category $\mathbf{C}$,*

$$yA \cong yB \qquad \textit{implies} \qquad A \cong B.$$

A typical such case is this. In any cartesian closed category $\mathbf{C}$, we know there is always an isomorphism,

$$(A^B)^C \cong A^{(B \times C)},$$

for any objects A, B, C. But recall how involved it was to prove this directly, using the compound universal mapping property (or a lengthy calculation in λ-calculus). Now, however, by the Yoneda principle, we just need to show that

$$y((A^B)^C) \;\cong\; y(A^{(B \times C)}).$$

To that end, take any object $X \in \mathbf{C}$; then we have isomorphisms:

$$\begin{aligned}
\mathrm{Hom}(X, (A^B)^C) &\cong \mathrm{Hom}(X \times C, A^B) \\
&\cong \mathrm{Hom}((X \times C) \times B, A) \\
&\cong \mathrm{Hom}(X \times (B \times C), A) \\
&\cong \mathrm{Hom}(X, A^{(B \times C)}).
\end{aligned}$$

Of course, it must be checked that these isomorphisms are natural in X, but that is straightforward. For instance, for the first one suppose we have $f : X' \to X$. Then, the naturality of the first isomorphism means that for any $g : X \to (A^B)^C$, we have

$$\overline{g \circ f} = \overline{g} \circ (f \times 1),$$

which is clearly true by the uniqueness of transposition (the reader should draw the diagram).

Here is another sample application of the Yoneda principle.

Proposition 8.6. *If the cartesian closed category $\mathbf{C}$ has coproducts, then $\mathbf{C}$ is "distributive," that is, there is always a canonical isomorphism,*

$$(A \times B) + (A \times C) \cong A \times (B + C).$$

Proof. As in the previous proposition, we check that

$$\begin{aligned}\mathrm{Hom}(A\times(B+C),X) &\cong \mathrm{Hom}(B+C,X^A)\\ &\cong \mathrm{Hom}(B,X^A)\times\mathrm{Hom}(C,X^A)\\ &\cong \mathrm{Hom}(A\times B,X)\times\mathrm{Hom}(A\times C,X)\\ &\cong \mathrm{Hom}((A\times B)+(A\times C),X).\end{aligned}$$

Finally, as in the foregoing example, one sees easily that these isos are all natural in X. □

We have already used a simple logical version of the Yoneda lemma several times: to show that in the propositional calculus one has $\varphi \dashv\vdash \psi$ for some formulas φ, ψ, it suffices to show that for any formula ϑ, one has $\vartheta \vdash \varphi$ iff $\vartheta \vdash \psi$.

More generally, given any objects A, B in a locally small category $\mathbf{C}$, to find an arrow $h : A \to B$ it suffices to give one $\vartheta : yA \to yB$ in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, for then there is a unique h with $\vartheta = yh$. Why should it be easier to give an arrow $yA \to yB$ than one $A \to B$? The key difference is that in general $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ has much more structure to work with than does $\mathbf{C}$; as we see, it is complete, cocomplete, cartesian closed, and more. So one can use various "higher-order" tools, from limits to λ-calculus; and if the result is an arrow of the form $yA \to yB$, then it comes from a unique one $A \to B$, despite the fact that $\mathbf{C}$ itself may not admit the "higher-order" constructions. In that sense, the category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is like an extension of $\mathbf{C}$ by "ideal elements" that permit calculations which cannot be done in $\mathbf{C}$. This is something like passing to the complex numbers to solve equations in the reals, or adding higher types to an elementary logical theory.

8.5 Limits in categories of diagrams

Recall that a category $\mathcal{E}$ is said to be *complete* if it has all small limits; that is, for any small category J and functor $F : J \to \mathcal{E}$, there is a limit $L = \varprojlim_{j\in J} Fj$ in $\mathcal{E}$ and a "cone" $\eta : \Delta L \to F$ in $\mathcal{E}^J$, universal among arrows from constant functors ΔE. Here, the constant functor $\Delta : \mathcal{E} \to \mathcal{E}^J$ is the transposed projection $\mathcal{E} \times J \to \mathcal{E}$.

Proposition 8.7. *For any locally small category* $\mathbf{C}$*, the functor category* $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ *is complete. Moreover, for every object* $C \in \mathbf{C}$*, the evaluation functor*

$$\mathrm{ev}_C : \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Sets}$$

preserves all limits.

Proof. Suppose we have J small and $F : J \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$. The limit of F, if it exists, is an object in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, hence is a functor,

$$(\varprojlim_{j \in J} F_j) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}.$$

By the Yoneda lemma, if we had such a functor, then for each object $C \in \mathbf{C}$ we would have a natural isomorphism,

$$(\varprojlim F_j)(C) \cong \mathrm{Hom}(yC, \varprojlim F_j).$$

But then it would be the case that

$$\begin{aligned} \mathrm{Hom}(yC, \varprojlim F_j) &\cong \varprojlim \mathrm{Hom}(yC, F_j) && \text{in } \mathbf{Sets} \\ &\cong \varprojlim F_j(C) && \text{in } \mathbf{Sets} \end{aligned}$$

where the first isomorphism is because representable functors preserve limits, and the second is Yoneda again. Thus, we are led to *define* the limit $\varprojlim_{j \in J} F_j$ to be

$$(\varprojlim_{j \in J} F_j)(C) = \varprojlim_{j \in J} (F_j C) \tag{8.4}$$

that is, the *pointwise limit* of the functors F_j. The reader can easily work out how $\varprojlim F_j$ acts on $\mathbf{C}$-arrows, and what the universal cone is, and our hypothetical argument then shows that it is indeed a limit in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

Finally, the preservation of limits by evaluation functors is stated by (8.4). □

8.6 Colimits in categories of diagrams

The notion of *cocompleteness* is of course the dual of completeness: a category is cocomplete if it has all (small) colimits. Like the foregoing proposition about the completeness of $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, its cocompleteness actually follows simply from the fact that $\mathbf{Sets}$ is cocomplete. We leave the proof of the following as an exercise.

Proposition 8.8. *Given any categories $\mathbf{C}$ and $\mathbf{D}$, if $\mathbf{D}$ is cocomplete, then so is the functor category $\mathbf{D}^{\mathbf{C}}$, and the colimits in $\mathbf{D}^{\mathbf{C}}$ are "computed pointwise," in the sense that for every $C \in \mathbf{C}$, the evaluation functor*

$$\mathrm{ev}_C : \mathbf{D}^{\mathbf{C}} \to \mathbf{D}$$

preserves colimits. Thus, for any small index category J and functor $A : J \to \mathbf{D}^{\mathbf{C}}$, for each $C \in \mathbf{C}$ there is a canonical isomorphism,

$$(\varinjlim_{j \in J} A_j)(C) \;\cong\; \varinjlim_{j \in J} (A_j C).$$

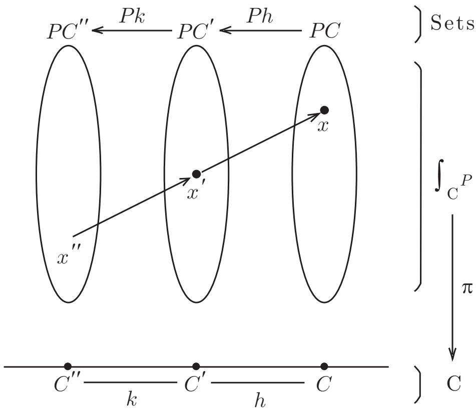

Figure 8.1 Category of elements

Proof. Exercise. □

Corollary 8.9. *For any locally small* $\mathbf{C}$*, the functor category* $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ *is cocomplete, and colimits there are computed pointwise.*

Proposition 8.10. *For any small category* $\mathbf{C}$*, every object* P *in the functor category* $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ *is a colimit of representable functors,*

$$\varinjlim_{j \in J} yC_j \;\cong\; P.$$

More precisely, there is a canonical choice of an index category J *and a functor* $\pi : J \to \mathbf{C}$ *such that there is a natural isomorphism* $\varinjlim_J \, y \circ \pi \;\cong\; P$.

Proof. Given $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$, the index category we need is the so-called *category of elements* of P, written,

$$\int_{\mathbf{C}} P$$

and defined as follows.

Objects: pairs (x, C) where $C \in \mathbf{C}$ and $x \in PC$.

Arrows: an $h : (x', C') \to (x, C)$ is an arrow $h : C' \to C$ in $\mathbf{C}$ such that

$$P(h)(x) = x' \tag{8.5}$$

actually, the arrows are triples of the form $(h, (x', C'), (x, C))$ satisfying (8.5).

The reader can easily work out the obvious identities and composites. See Figure 8.1.

Note that $\int_{\mathbf{C}} P$ is a small category since $\mathbf{C}$ is small. There is a "projection" functor,

$$\pi : \int_{\mathbf{C}} P \to \mathbf{C}$$

defined by $\pi(x, C) = C$ and $\pi(h, (x', C'), (x, C)) = h$.

To define the cocone of the form $y \circ \pi \to P$, take an object $(x, C) \in \int_{\mathbf{C}} P$ and observe that (by the Yoneda lemma) there is a natural, bijective correspondence between

$$\frac{x \in P(C)}{x : yC \to P}$$

which we simply identify notationally. Moreover, given any arrow $h : (x', C') \to (x, C)$ naturality in C implies that there is a commutative triangle

$$\begin{tikzcd} yC \arrow[dr, "x"] & \\ & P \\ yC' \arrow[uu, "yh"] \arrow[ur, "x'"'] & \end{tikzcd}$$

Indeed, the category $\int_{\mathbf{C}} P$ is thus equivalent to the full subcategory of the slice category over P on the objects $yC \to P$ (i.e., arrows in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$) with representable domains.

We can therefore take the component of the desired cocone $y\pi \to P$ at (x, C) to be simply $x : yC \to P$. To see that this is a colimiting cocone, take any cocone $y\pi \to Q$ with components $\vartheta_{(x,C)} : yC \to Q$ and we require a unique natural transformation $\vartheta : P \to Q$ as indicated in the following diagram:

$$\begin{tikzcd} yC \arrow[dr, "x"] \arrow[drr, "\vartheta_{(x,C)}"] & & \\ & P \arrow[r, dotted, "\vartheta"'] & Q \\ yC' \arrow[uu, "yh"] \arrow[ur, "x'"] \arrow[urr, "\vartheta_{(x',C')}"'] & & \end{tikzcd} \tag{8.6}$$

We can define $\vartheta_C : PC \to QC$ by setting

$$\vartheta_C(x) = \vartheta_{(x,C)}$$

where we again identify,

$$\frac{\vartheta_{(x,C)} \in Q(C)}{\vartheta_{(x,C)} : yC \to Q}$$

This assignment is clearly natural in C by the commutativity of the diagram (8.6). For uniqueness, given any $\varphi : P \to Q$ such that $\varphi \circ x = x'$, again by Yoneda we must have $\varphi \circ x = \vartheta(x, c) = \vartheta \circ x$. □

We include the following because it fits naturally here, but defer the proof to Chapter 9, where a neat proof can be given using adjoint functors. As an exercise, the reader may wish to prove it at this point using the materials already at hand, which is also quite doable.

Proposition 8.11. *For any small category* $\mathbf{C}$*, the Yoneda embedding*

$$y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$$

is the "free cocompletion" of $\mathbf{C}$*, in the following sense. Given any cocomplete category* $\mathcal{E}$ *and functor* $F : \mathbf{C} \to \mathcal{E}$*, there is a colimit preserving functor* $F_! : \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \to \mathcal{E}$*, unique up to natural isomorphism with the property*

$$F_! \circ y \cong A$$

as indicated in the following diagram:

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \arrow[r, dotted, "F_!"] & \mathcal{E} \\
\mathbf{C} \arrow[u, "y"] \arrow[ur, "F"'] &
\end{tikzcd}$$

Proof. (Sketch, see proposition 9.16.) Given $F : \mathbf{C} \to \mathcal{E}$, define $F_!$ as follows. For any $P \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, let

$$\varinjlim_{j \in J} yA_j \;\cong\; P$$

be the canonical presentation of P as a colimit of representables with $J = \int_{\mathbf{C}} P$, the category of elements of P. Then set,

$$F_!(P) = \varinjlim_{j \in J} F(A_j)$$

which exists since $\mathcal{E}$ is cocomplete. □

8.7 Exponentials in categories of diagrams

As an application, let us consider exponentials in categories of the form $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ for small $\mathbf{C}$. We need the following lemma.

Lemma 8.12. *For any small index category J, functor $A : J \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and diagram $B \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, there is a natural isomorphism*

$$\varinjlim_{j} (A_j \times B) \;\cong\; (\varinjlim_{j} A_j) \times B. \tag{8.7}$$

Briefly, the functor $- \times B : \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ preserves colimits.

Proof. To specify the canonical natural transformation mentioned in (8.7), start with the cocone,

$$\vartheta_j : A_j \to \varinjlim_{j} A_j, \qquad j \in J$$

apply the functor $- \times B$ to get a cocone,

$$\vartheta_j \times B : A_j \times B \to (\varinjlim_{j} A_j) \times B, \qquad j \in J$$

and so there is a unique "comparison arrow" from the colimit,

$$\vartheta : \varinjlim_{j}(A_j \times B) \to (\varinjlim_{j} A_j) \times B,$$

which we claim is a natural isomorphism.

By exercise 7 of Chapter 7, it suffices to show that each component,

$$\vartheta_C : (\varinjlim_{j}(A_j \times B))(C) \to ((\varinjlim_{j} A_j) \times B)(C)$$

is iso. But since the limits and colimits involved are all computed pointwise, it therefore suffices to show (8.7) under the assumption that the A_j and B are just sets. To that end, take any set X and consider the following isomorphisms in **Sets**,

$$\begin{aligned}
\mathrm{Hom}(\varinjlim_{j}(A_j \times B), X) &\cong \varprojlim_{j} \mathrm{Hom}(A_j \times B, X) \\
&\cong \varprojlim_{j} \mathrm{Hom}(A_j, X^B) && (\mathbf{Sets} \text{ is CCC}) \\
&\cong \mathrm{Hom}(\varinjlim_{j} A_j, X^B) \\
&\cong \mathrm{Hom}((\varinjlim_{j} A_j) \times B, X).
\end{aligned}$$

Since these are natural in X, the claim follows by Yoneda. □

Now suppose we have functors P and Q and we want Q^P. The reader should try to construct the exponential "pointwise,"

$$Q^P(C) \stackrel{?}{=} Q(C)^{P(C)}$$

to see that it *does not* work (it is not functorial in C, as the exponent is contravariant in C).

Let us instead reason as follows: if we had such an exponential Q^P, we could compute its value at any object $C \in \mathbf{C}$ by Yoneda:

$$Q^P(C) \cong \mathrm{Hom}(yC, Q^P)$$

And if it is to be an exponential, then we must also have

$$\mathrm{Hom}(yC, Q^P) \cong \mathrm{Hom}(yC \times P, Q).$$

But this latter set *does* exist, and it *is* functorial in C. Thus, we are led to *define*

$$Q^P(C) = \mathrm{Hom}(yC \times P, Q) \tag{8.8}$$

with the action on $h : C' \to C$ being

$$Q^P(h) = \mathrm{Hom}(yh \times 1_P, Q).$$

This is clearly a contravariant, set-valued functor on $\mathbf{C}$. Let us now check that it indeed gives an exponential of P and Q.

Proposition 8.13. *For any objects X, P, Q in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, there is an isomorphism, natural in X,*

$$\mathrm{Hom}(X, Q^P) \cong \mathrm{Hom}(X \times P, Q).$$

Proof. By proposition 8.10, for a suitable index category J, we can write X as a colimit of representables,

$$X \cong \varinjlim_{j \in J} yC_j.$$

Thus we have isomorphisms,

$$\begin{aligned}
\mathrm{Hom}(X, Q^P) &\cong \mathrm{Hom}(\varinjlim_j yC_j, Q^P) \\
&\cong \varprojlim_j \mathrm{Hom}(yC_j, Q^P) \\
&\cong \varprojlim_j Q^P(C_j) && \text{(by Yoneda)} \\
&\cong \varprojlim \mathrm{Hom}(yC_j \times P, Q) && \text{(by 8.8)} \\
&\cong \mathrm{Hom}(\varinjlim_j (yC_j \times P), Q) \\
&\cong \mathrm{Hom}(\varinjlim_j (yC_j) \times P, Q) && \text{(Lemma 8.12)} \\
&\cong \mathrm{Hom}(X \times P, Q).
\end{aligned}$$

And as usual these isos are clearly natural in X. □

Theorem 8.14. *For any small category* $\mathbf{C}$*, the category of diagrams* $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ *is cartesian closed. Moreover, the Yoneda embedding*

$$y : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$$

preserves all products and exponentials that exist in $\mathbf{C}$.

Proof. In light of the foregoing proposition, it only remains to show that y preserves products and exponentials. We leave this as an easy exercise. □

Remark 8.15. As a corollary, we find that we can sharpen the CCC completeness theorem 6.17 for the simply-typed λ-calculus by restricting to CCCs of the special form $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

8.8 Topoi

Since we are now so close to it, we might as well introduce the important notion of a "topos"—even though this is not the place to develop that theory, as appealing as it is. First we require the following generalization of characteristic functions of subsets.

Definition 8.16. Let $\mathcal{E}$ be a category with all finite limits. A *subobject classifier* in $\mathcal{E}$ consists of an object Ω together with an arrow $t : 1 \to \Omega$ that is a "universal subobject," in the following sense:

> Given any object E and any subobject $U \rightarrowtail E$, there is a unique arrow $u : E \to \Omega$ making the following diagram a pullback:
>
> $$\begin{tikzcd} U \arrow[r] \arrow[d, tail] & 1 \arrow[d, "t"] \\ E \arrow[r, "u"'] & \Omega \end{tikzcd} \tag{8.9}$$

The arrow u is called the *classifying* arrow of the subobject $U \rightarrowtail E$; it can be thought of as taking exactly the part of E that is U to the "point" t of Ω. The most familiar example of a subobject classifier is of course the set $2 = \{0, 1\}$ with a selected element as $t : 1 \to 2$. The fact that every subset $U \subseteq S$ of any set S has a unique characteristic function $u : S \to 2$ is then exactly the subobject classifier condition.

It is easy to show that a subobject classifier is unique up to isomorphism: the pullback condition is clearly equivalent to requiring the contravariant subobject functor,

$$\mathrm{Sub}_{\mathcal{E}}(-) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Sets}$$

(which acts by pullback) to be representable,

$$\mathrm{Sub}_{\mathcal{E}}(-) \;\cong\; \mathrm{Hom}_{\mathcal{E}}(-, \Omega).$$

The required isomorphism is just the pullback condition stated in the definition of a subobject classifier. Now apply the Yoneda principle, corollary 8.5, for two subobject classifiers Ω and Ω'.

Definition 8.17. A *topos* is a category $\mathcal{E}$ such that

1. $\mathcal{E}$ has all finite limits,
2. $\mathcal{E}$ has a subobject classifier,
3. $\mathcal{E}$ has all exponentials.

This compact definition proves to be amazingly rich in consequences: it can be shown for instance that topoi also have all finite colimits, and that every slice category of a topos is again a topos. We refer the reader to the books by Mac Lane and Moerdijk (1992), Johnstone (2002), and McLarty (1995) for information on topoi, and here just give an example (albeit one that covers a very large number of cases).

Proposition 8.18. *For any small category* $\mathbf{C}$*, the category of diagrams* $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ *is a topos.*

Proof. Since we already know that $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ has all limits, and we know that it has exponentials by Section 8.7, we just need to find a subobject classifier. To that end, we define a *sieve* on an object C of $\mathbf{C}$ to be a set S of arrows $f : \cdot \to C$ (with arbitrary domain) that is closed under precomposition; that is, if $f : D \to C$ is in S then so is $f \circ g : E \to D \to C$ for every $g : E \to D$ (think of a sieve as a common generalization of a "lower set" in a poset and an "ideal" in a ring). Then let

$$\Omega(C) \;=\; \{S \subseteq \mathbf{C}_1 \mid S \text{ is a sieve on } C\}$$

and given $h : D \to C$, let

$$h^* : \Omega(C) \to \Omega(D)$$

be defined by

$$h^*(S) = \{g : \cdot \to D \mid h \circ g \in S\}.$$

This clearly defines a presheaf $\Omega : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$, with a distinguished point,

$$t : 1 \to \Omega$$

namely, at each C, the "total sieve"

$$t_C = \{f : \cdot \to C\}.$$

We claim that $t : 1 \to \Omega$ so defined is a subobject classifier for $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$. Indeed, given any object E and subobject $U \rightarrowtail E$, define $u : E \to \Omega$ at any object $C \in \mathbf{C}$ by

$$u_C(e) = \{f : D \to C \mid f^*(e) \in U(D) \rightarrowtail E(D)\}$$

for any $e \in E(C)$. That is, $u_C(e)$ is the sieve of arrows into C that take $e \in E(C)$ back into the subobject U.

□

The notion of a topos first arose in the Grothendieck school of algebraic geometry as a generalization of that of a topological space. But one of the most fascinating aspects of topoi is their relation to logic. In virtue of the association of subobjects $U \rightarrowtail E$ with arrows $u : E \to \Omega$, the subobject classifier Ω can be regarded as an object of "propositions" or "truth-values," with $t = \text{true}$. An arrow $\varphi : E \to \Omega$ is then a "propositional function" of which $U_\varphi \rightarrowtail E$ is the "extension." For, by the pullback condition (8.9), a generalized element $x : X \to E$ is "in" U_φ (i.e., factors through $U_\varphi \rightarrowtail E$) just if $\varphi x = \text{true}$,

$$x \in_E U_\varphi \qquad \text{iff} \qquad \varphi x = \text{true}$$

so that, again in the notation of Section 5.1,

$$U_\varphi = \{x \in E \mid \varphi x = \text{true}\}.$$

This permits an interpretation of first-order logic in any topos, since topoi also have a way of modeling the logical quantifiers $\exists$ and $\forall$ as adjoints to pullbacks (as described in Section 9.5).

Since topoi are also cartesian closed, they have an internal type theory described by the λ-calculus (see Section 6.6). Combining this with the first-order logic and subobject classifier Ω provides a natural interpretation of *higher-order* logic, employing the exponential Ω^E as a "power object" $P(E)$ of subobjects of E. This logical aspect of topoi is also treated in the books already mentioned.

8.9 Exercises

1. If $F : \mathbf{C} \to \mathbf{D}$ is full and faithful, then $C \cong C'$ iff $FC \cong FC'$.
2. Let $\mathbf{C}$ be a small category. Prove that the representable functors *generate* the diagram category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, in the following sense: given any objects $P, Q \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and natural transformations $\varphi, \psi : P \to Q$, if for every representable functor yC and natural transformation $\vartheta : yC \to P$, one has $\varphi \circ \vartheta = \psi \circ \vartheta$, then $\varphi = \psi$. Thus, the arrows in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ are determined by their effect on generalized elements based at representables.
3. Let $\mathbf{C}$ be a locally small, cartesian closed category. Use the Yoneda embedding to show that for any objects A, B, C in $\mathbf{C}$

$$(A \times B)^C \cong A^C \times B^C$$

(cf. problem 2 Chapter 6).
If $\mathbf{C}$ also has binary coproducts, show that also

$$A^{(B+C)} \cong A^B \times A^C.$$

4. Let Δ be the category of finite ordinal numbers $0, 1, 2, \dots$ and order-preserving maps, and write $[-] : \Delta \to \mathbf{Pos}$ for the evident inclusion. For each poset P, define the simplicial set $S(\mathrm{P})$ by

$$S(\mathrm{P})(n) = \mathrm{Hom}_{\mathbf{Pos}}([n], \mathrm{P}).$$

Show that this specification determines a functor $S : \mathbf{Pos} \to \mathbf{Sets}^{\Delta^{\mathrm{op}}}$ into simplicial sets, and that it coincides with the "simplicial nerve" of P as specified in the text. Is S faithful? Show that S preserves all limits.

5. Generalize the foregoing exercise from posets to (locally small) categories to define the simplicial nerve of a category $\mathbf{C}$.

6. Let $\mathbf{C}$ be any category and $\mathbf{D}$ any complete category. Show that the functor category $\mathbf{D}^{\mathbf{C}}$ is also complete.
Use duality to show that the same is true for cocompleteness in place of completeness.

7. Let $\mathbf{C}$ be a locally small category with binary products, and show that the Yoneda embedding

$$y \colon \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$$

preserves them. (Hint: this involves only a few lines of calculation.)
If $\mathbf{C}$ also has exponentials, show that y also preserves them.

8. Show that if P is a poset and $A : \mathrm{P}^{\mathrm{op}} \to \mathbf{Sets}$ a presheaf on P, then the category of elements $\int_{\mathrm{P}} A$ is also a poset and the projection $\pi : \int_{\mathrm{P}} A \to \mathrm{P}$ is a monotone map.
Show, moreover, that the assignment $A \mapsto (\pi : \int_{\mathrm{P}} A \to \mathrm{P})$ determines a functor,

$$\int_{\mathrm{P}} : \mathbf{Sets}^{\mathrm{P}^{\mathrm{op}}} \longrightarrow \mathbf{Pos}/\mathrm{P}.$$

9. Let $\mathbb{T}$ be a theory in the λ-calculus. For any type symbols σ and τ, let

$$[\sigma \to \tau] = \{M : \sigma \to \tau \mid M \text{ closed}\}$$

be the set of closed terms of type $\sigma \to \tau$. Suppose that for each type symbol ρ, there is a function,

$$f_\rho : [\rho \to \sigma] \to [\rho \to \tau]$$

with the following properties:

- for any closed terms $M, N : \rho \to \sigma$, if $\mathbb{T} \vdash M = N$ (provable equivalence from $\mathbb{T}$), then $f_\rho M = f_\rho N$,

- for any closed terms $M : \mu \to \nu$ and $N : \nu \to \sigma$,

$$\mathbb{T} \vdash f_\mu(\lambda x : \mu.N(Mx)) = \lambda x : \mu.(f_\nu(N))(Mx)$$

Use the Yoneda embedding of the cartesian closed *category of types* $\mathbf{C}_\mathbb{T}$ of $\mathbb{T}$ to show that there is a term $F : \sigma \to \tau$ such that f_ρ is induced by composition with F, in the sense that, for every closed term $R : \rho \to \sigma$,

$$\mathbb{T} \vdash f_\rho(R) = \lambda x : \rho.F(Rx)$$

Show that, moreover, F is unique up to $\mathbb{T}$-provable equivalence.

10. Combine proposition 6.17 with theorem 8.14 to infer that the λ-calculus is deductively complete with respect to categories of diagrams.
11. Show that every slice category $\mathbf{Sets}/X$ is cartesian closed. Calculate the exponential of two objects $A \to X$ and $B \to X$ by first determining the Yoneda embedding $y : X \to \mathbf{Sets}^X$, and then applying the formula for exponentials of presheaves. Finally, observe that $\mathbf{Sets}/X$ is a topos, and determine its subobject classifier.
12. (a) Explicitly determine the subobject classifiers for the topoi $\mathbf{Sets}^\mathbf{2}$ and $\mathbf{Sets}^\omega$, where as always $\mathbf{2}$ is the poset $0 < 1$ and ω is the poset of natural numbers $0 < 1 < 2 < \cdots$.
    (b) Show that $(\mathbf{Sets}_{\mathrm{fin}})^\mathbf{2}$ is a topos.
13. Explicitly determine the graph that is the subobject classifier in the topos of graphs (i.e., what are its edges and vertices?). How many points $1 \to \Omega$ does it have?