9

ADJOINTS

This chapter represents the high point of this book, the goal toward which we have been working steadily. The notion of adjoint functor, first discovered by D. Kan in the 1950s, applies everything that we have learned up to now to unify and subsume all of the different universal mapping properties that we have encountered, from free groups to limits to exponentials. But more importantly, it also captures an important mathematical phenomenon that is invisible without the lens of category theory. Indeed, I will make the admittedly provocative claim that adjointness is a concept of fundamental logical and mathematical importance that is not captured elsewhere in mathematics.

Many of the most striking applications of category theory involve adjoints, and many important and fundamental mathematical notions are instances of adjoint functors. As such, they share the common behavior and formal properties of all adjoints, and in many cases this fact alone accounts for all of their essential features.

9.1 Preliminary definition

We begin by recalling the universal mapping property (UMP) of free monoids: every monoid M has an underlying set $U(M)$, and every set X has a free monoid $F(X)$, and there is a function

$$i_X : X \to UF(X)$$

with the following UMP:

> For every monoid M and every function $f : X \to U(M)$, there is a unique homomorphism $g : F(X) \to M$ such that $f = U(g) \circ i_X$, all as indicated in the following diagram:

$$\begin{tikzcd}
F(X) \arrow[r, dotted, "g"] & M \\
U(F(X)) \arrow[r, "U(g)"] & U(M) \\
X \arrow[u, "i_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Now consider the following map:

$$\phi : \mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \to \mathrm{Hom}_{\mathbf{Sets}}(X, U(M))$$

defined by

$$g \mapsto U(g) \circ i_X.$$

The UMP given above says exactly that ϕ is an isomorphism,

$$\mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \;\cong\; \mathrm{Hom}_{\mathbf{Sets}}(X, U(M)). \tag{9.1}$$

This bijection (9.1) can also be written schematically as a *two-way rule*:

$$\frac{F(X) \longrightarrow M}{X \longrightarrow U(M)}$$

where one gets from an arrow g of the upper form to one $\phi(g)$ of the lower form by the recipe

$$\phi(g) = U(g) \circ i_X.$$

We pattern our *preliminary* definition of adjunction on this situation. It is preliminary because it really only gives half of the picture; in Section 9.2 an equivalent definition emerges as both more convenient and conceptually clearer.

Definition 9.1 (preliminary). An *adjunction* between categories $\mathbf{C}$ and $\mathbf{D}$ consists of functors

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : \mathbf{U}$$

and a natural transformation

$$\eta : 1_{\mathbf{C}} \to U \circ F$$

with the property:

(*) For any $C \in \mathbf{C}$, $D \in \mathbf{D}$, and $f : C \to U(D)$, there exists a unique $g : FC \to D$ such that

$$f = U(g) \circ \eta_C$$

as indicated in

$$\begin{tikzcd} F(C) \arrow[r, dotted, "g"] & D \end{tikzcd}$$

$$\begin{tikzcd} U(F(C)) \arrow[r, "U(g)"] & U(D) \\ C \arrow[u, "\eta_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

Terminology and notation:

- F is called the *left adjoint*, U is called the *right adjoint*, and η is called the *unit* of the adjunction.
- One sometimes writes $F \dashv U$ for "F is left and U right adjoint."
- The statement (*) is the UMP of the unit η.

Note that the situation $F \dashv U$ is a generalization of equivalence of categories, in that a pseudo-inverse is an adjoint. In that case, however, it is the relation between categories that one is interested in. Here, one is concerned with the relation between specific functors. That is to say, it is not the relation on categories "there exists an adjunction," but rather "this functor has an adjoint" that we are concerned with.

Suppose now that we have an adjunction,

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift right, "F"'] & \mathbf{D} \arrow[l, shift right, "U"'] \end{tikzcd}.$$

Then, as in the example of monoids, take $C \in \mathbf{C}$ and $D \in \mathbf{D}$ and consider the operation

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \to \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

given by $\phi(g) = U(g) \circ \eta_C$. Since, by the UMP of η, every $f : C \to UD$ is $\phi(g)$ for a unique g, just as in our example we see that ϕ is an isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(F(C), D) \cong \mathrm{Hom}_{\mathbf{C}}(C, U(D)) \tag{9.2}$$

which, again, can be displayed as the two-way rule:

$$\frac{F(C) \longrightarrow D}{C \longrightarrow U(D)}$$

Example 9.2. Consider the "diagonal" functor,

$$\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$$

defined on objects by

$$\Delta(C) = (C, C)$$

and on arrows by

$$\Delta(f : C \to C') = (f, f) : (C, C) \to (C', C').$$

What would it mean for this functor to have a right adjoint? We would need a functor $R : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ such that for all $C \in \mathbf{C}$ and $(X, Y) \in \mathbf{C} \times \mathbf{C}$, there is a bijection:

$$\frac{\Delta C \longrightarrow (X, Y)}{C \longrightarrow R(X, Y)}$$

That is, we would have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{C}}(C, R(X,Y)) &\cong \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta C, (X,Y)) \\ &\cong \mathrm{Hom}_{\mathbf{C}}(C,X) \times \mathrm{Hom}_{\mathbf{C}}(C,Y).\end{aligned}$$

We therefore must have $R(X,Y) \cong X \times Y$, suggesting that Δ has as a right adjoint the product functor $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$,

$$\Delta \dashv \times.$$

The counit η would have the form $\eta_C : C \to C \times C$, so we propose the "diagonal arrow" $\eta_C = \langle 1_C, 1_C \rangle$, and we need to check the UMP indicated in the following diagram:

$$\begin{tikzcd}
(C,C) \arrow[r, dotted, "{(f_1, f_2)}"] & (X,Y) \\
C \times C \arrow[r, "f_1 \times f_2"] & X \times Y \\
C \arrow[u, "\eta_C"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Indeed, given any $f : C \to X \times Y$, we have unique f_1 and f_2 with $f = \langle f_1, f_2 \rangle$, for which, we then have

$$\begin{aligned}(f_1 \times f_2) \circ \eta_C &= \langle f_1\pi_1, f_2\pi_2 \rangle \eta_C \\ &= \langle f_1\pi_1\eta_C, f_2\pi_2\eta_C \rangle \\ &= \langle f_1, f_2 \rangle \\ &= f.\end{aligned}$$

Thus in sum, the functor Δ has a right adjoint if and only if $\mathbf{C}$ has binary products.

Example 9.3. For an example of a different sort, consider the category **Pos** of posets and monotone maps and $\mathcal{C}$**Pos** of cocomplete posets and cocontinuous maps. A poset $\mathcal{C}$ is cocomplete just if it has a join $\bigvee_i c_i$ for every family of elements $(c_i)_{i \in I}$ indexed by a set I, and a monotone map $f : \mathcal{C} \to \mathcal{D}$ is cocontinuous if it preserves all such joins, $f(\bigvee_i c_i) = \bigvee_i f(c_i)$. There is an obvious forgetful functor

$$U : \mathcal{C}\mathbf{Pos} \to \mathbf{Pos}.$$

What would a left adjoint $F \dashv U$ be? There would have to be a monotone map $\eta : P \to UF(P)$ with the property: given any cocomplete poset $\mathcal{C}$ and monotone

$f : P \to U(\mathcal{C})$, there exists a unique cocontinuous $\bar{f} : F(P) \to \mathcal{C}$ such that $f = U(\bar{f}) \circ \eta_P$, as indicated in

$$\begin{tikzcd} F(P) \arrow[r, dotted, "\bar{f}"] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} UF(P) \arrow[r, "U(\bar{f})"] & U(\mathcal{C}) \\ P \arrow[u, "\eta"] \arrow[ur, "f"'] & \end{tikzcd}$$

In this precise sense, such a poset $F(P)$ would be a "free cocompletion" of P, and $\eta : P \to UF(P)$ a "best approximation" of P by a cocomplete poset.

We leave it to the reader to show that such a "cocompletion" always exists, namely the poset of *lower sets*,

$$\mathrm{Low}(P) = \{U \subseteq P \mid p' \leq p \in U \text{ implies } p' \in U\}.$$

9.2 Hom-set definition

The following proposition shows that the isomorphism (9.2) is in fact natural in both C and D.

Proposition 9.4. *Given categories and functors,*

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift right, "F"'] & \mathbf{D} \arrow[l, shift right, "U"'] \end{tikzcd}$$

the following conditions are equivalent:

1. *F is left adjoint to U; that is, there is a natural transformation*

$$\eta : 1_{\mathbf{C}} \to U \circ F$$

that has the UMP of the unit:

For any $C \in \mathbf{C}$, $D \in \mathbf{D}$ and $f : C \to U(D)$, there exists a unique $g : FC \to D$ such that

$$f = U(g) \circ \eta_C.$$

2. *For any $C \in \mathbf{C}$ and $D \in \mathbf{D}$, there is an isomorphism,*

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

that is natural in both C and D.

Moreover, the two conditions are related by the formulas

$$\phi(g) = U(g) \circ \eta_C$$
$$\eta_C = \phi(1_{FC}).$$

Proof. (1 implies 2) The recipe for ϕ, given η is just the one stated and we have already observed it to be an isomorphism, given the UMP of the unit. For naturality in C, take $h : C' \to C$ and consider the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(FC, D) \arrow[r, "\phi_{C,D}", "\cong"'] \arrow[d, "(Fh)^*"'] & \mathrm{Hom}_{\mathbf{C}}(C, UD) \arrow[d, "h^*"] \\
\mathrm{Hom}_{\mathbf{D}}(FC', D) \arrow[r, "\cong", "\phi_{C',D}"'] & \mathrm{Hom}_{\mathbf{C}}(C', UD)
\end{tikzcd}$$

Then for any $f : FC \to D$, we have

$$\begin{aligned}
h^*(\phi_{C,D}(f)) &= (U(f) \circ \eta_C) \circ h \\
&= U(f) \circ UF(h) \circ \eta_{C'} \\
&= U(f \circ F(h)) \circ \eta_{C'} \\
&= \phi_{C',D}(F(h)^*(f)).
\end{aligned}$$

For naturality in D, take $g : D \to D'$ and consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(FC, D) \arrow[r, "\phi_{C,D}", "\cong"'] \arrow[d, "g_*"'] & \mathrm{Hom}_{\mathbf{C}}(C, UD) \arrow[d, "U(g)_*"] \\
\mathrm{Hom}_{\mathbf{D}}(FC, D') \arrow[r, "\cong", "\phi_{C',D}"'] & \mathrm{Hom}_{\mathbf{C}}(C, UD')
\end{tikzcd}$$

Then for any $f : FC \to D$ we have

$$\begin{aligned}
U(g)_*(\phi_{C,D}(f)) &= U(g) \circ (U(f) \circ \eta_C) \\
&= U(g \circ f) \circ \eta_C \\
&= \phi_{C',D}(g \circ f) \\
&= \phi_{C',D}(g_*(f)).
\end{aligned}$$

So ϕ is indeed natural.

(2 implies 1) We are given a bijection ϕ,

$$\frac{F(C) \longrightarrow D}{C \longrightarrow U(D)} \tag{9.3}$$

for each C, D, that is natural in C and D. In detail, this means that given a commutative triangle

$$\begin{tikzcd} F(C) \arrow[r, "f"] \arrow[dr, "g \circ f"'] & D \arrow[d, "g"] \\ & D' \end{tikzcd}$$

there are two ways to get an arrow of the form $C \to UD'$, namely

$$\begin{tikzcd} C \arrow[r, "\phi(f)"] \arrow[dr, "\phi(g \circ f)"'] & UD \arrow[d, "Ug"] \\ & UD' \end{tikzcd}$$

Naturality in D means that this diagram commutes,

$$\phi(g \circ f) = Ug \circ \phi(f). \tag{9.4}$$

Dually, naturality in C means that given

$$\begin{tikzcd} C' \arrow[d, "h"'] \arrow[dr, "f \circ h"] & \\ C \arrow[r, "f"'] & UD \end{tikzcd}$$

and writing $\psi = \phi^{-1}$, the following commutes:

$$\begin{tikzcd} FC' \arrow[d, "Fh"'] \arrow[dr, "\psi(f \circ h)"] & \\ FC \arrow[r, "\psi(f)"'] & D \end{tikzcd}$$

That is,

$$\psi(f \circ h) = \psi(f) \circ Fh.$$

Now, given such a natural bijection ϕ, we want a natural transformation

$$\eta : 1_{\mathbf{C}} \to U \circ F$$

with the UMP of the unit. To find

$$\eta_C : C \to UFC$$

put FC for D and $1_{FC} : FC \to FC$ in the adjoint schema (9.3) to get

$$\phi\,\frac{1_{FC} : \quad FC \longrightarrow FC}{\eta_C : \quad C \longrightarrow UFC}$$

That is, define

$$\eta_C = \phi(1_{FC}).$$

We leave it as an exercise to show that η so defined really is natural in C. Finally, to see that η has the required UMP of the unit, it clearly suffices to show that for all $g : FC \to D$, we have

$$\phi(g) = Ug \circ \eta_C$$

since we are assuming that ϕ is iso. But, using (9.4),

$$\begin{aligned} Ug \circ \eta_C &= Ug \circ \phi(1_{FC}) \\ &= \phi(g \circ 1_{FC}) \\ &= \phi(g). \end{aligned}$$

□

Note that the second condition in the foregoing proposition is symmetric, but the first condition is not. This implies that we also have the following dual proposition.

Corollary 9.5. *Given categories and functors*

$$\mathbf{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{D}$$

the following conditions are equivalent:

1. *For any $C \in \mathbf{C}$, $D \in \mathbf{D}$, there is an isomorphism*

   $$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD)$$

   that is natural in C and D.
2. *There is a natural transformation*

   $$\epsilon : F \circ U \to 1_{\mathbf{D}}$$

   with the following UMP:

   > *For any $C \in \mathbf{C}$, $D \in \mathbf{D}$ and $g : F(C) \to D$, there exists a unique $f : C \to UD$ such that*
   >
   > $$g = \epsilon_D \circ F(f)$$

as indicated in the following diagram:

$$
\begin{tikzcd}
C \arrow[r, dotted, "f"] & U(D) \\
F(C) \arrow[r, "F(f)"] \arrow[dr, "g"'] & FU(D) \arrow[d, "\epsilon_D"] \\
& D
\end{tikzcd}
$$

Moreover, the two conditions are related by the equations

$$\psi(f) = \epsilon_D \circ F(f)$$

$$\epsilon_D = \psi(1_{UD})$$

where $\psi = \phi^{-1}$*.*

Proof. Duality. □

We take the symmetric "Hom-set" formulation as our "official" definition of an adjunction.

Definition 9.6 "official." An *adjunction* consists of functors

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : \mathbf{U}$$

and a natural isomorphism

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD) : \psi.$$

This definition has the virtue of being symmetric in F and U. The unit $\eta : 1_{\mathbf{C}} \to U \circ F$ and the *counit* $\epsilon : F \circ U \to 1_{\mathbf{D}}$ of the adjunction are then determined as

$$\eta_C = \phi(1_{FC})$$

$$\epsilon_D = \psi(1_{UD}).$$

9.3 Examples of adjoints

Example 9.7. Suppose $\mathbf{C}$ has binary products. Take a fixed object $A \in \mathbf{C}$, and consider the product functor

$$- \times A : \mathbf{C} \to \mathbf{C}$$

defined on objects by

$$X \mapsto X \times A$$

and on arrows by

$$(h : X \to Y) \mapsto (h \times 1_A : X \times A \longrightarrow Y \times A).$$

When does $- \times A$ have a right adjoint?

We would need a functor

$$U : \mathbf{C} \to \mathbf{C}$$

such that for all $X, Y \in \mathbf{C}$, there is a natural bijection

$$\frac{X \times A \longrightarrow Y}{X \longrightarrow U(Y)}$$

So let us try defining U by

$$U(Y) = Y^A$$

on objects, and on arrows by

$$U(g : Y \to Z) = g^A : Y^A \longrightarrow Z^A.$$

Putting $U(Y)$ for X in the adjunction schema given above then gives the counit:

$$\frac{Y^A \times A \xrightarrow{\epsilon} Y}{Y^A \xrightarrow[1]{} Y^A}$$

This is, therefore, an adjunction if there is always such a map ϵ with the following UMP:

> For any $f : X \times A \to Y$, there is a unique $\bar{f} : X \to Y^A$ such that $f = \epsilon \circ (\bar{f} \times 1_A)$.

But this is exactly the UMP of the exponential! Thus, we do indeed have an adjunction:

$$(-) \times A \dashv (-)^A$$

Example 9.8. Here is a much more simple example. For any category $\mathbf{C}$, consider the unique functor to the terminal category $\mathbf{1}$,

$$! : \mathbf{C} \to \mathbf{1}.$$

Now we ask, when does ! have a right adjoint? This would be an object $U : \mathbf{1} \to \mathbf{C}$ such that for any $C \in \mathbf{C}$, there is a bijective correspondence,

$$\frac{!C \longrightarrow *}{C \longrightarrow U(*)}$$

Such a U would have to be a terminal object in $\mathbf{C}$. So ! has a right adjoint iff $\mathbf{C}$ has a terminal object. What would a left adjoint be?

This last example is a clear case of the following general fact.

Proposition 9.9. *Adjoints are unique up to isomorphism. Specifically, given a functor* $F : \mathbf{C} \to \mathbf{D}$ *and right adjoints* $U, V : \mathbf{D} \to \mathbf{C}$,

$$F \dashv U \quad \textit{and} \quad F \dashv V$$

we then have $U \cong V$.

Proof. Here is the easy way. For any $D \in \mathbf{D}$, and $C \in \mathbf{C}$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(C, UD) &\cong \mathrm{Hom}_{\mathbf{D}}(FC, D) \quad \text{naturally, since } F \dashv U \\ &\cong \mathrm{Hom}_{\mathbf{C}}(C, VD) \quad \text{naturally, since } F \dashv V. \end{aligned}$$

Thus, by Yoneda, $UD \cong VD$. But this isomorphism is natural in D, again by adjointness. □

This proposition implies that one can use the condition of being right or left adjoint to a given functor to define (uniquely characterize up to isomorphism) a new functor. This sort of characterization, like a UMP, determines an object or construction "structurally" or "intrinsically," in terms of its relation to some other given construction. Many important constructions turn out to be adjoints to particularly simple ones.

For example, what do you suppose would be a *left* adjoint to the diagonal functor

$$\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$$

in the earlier example 9.2, where $\Delta(C) = (C, C)$ and we had $\Delta \dashv \times$? It would have to be functor $L(X, Y)$ standing in the correspondence

$$\frac{L(X,Y) \longrightarrow C}{(X,Y) \longrightarrow (C,C)}$$

Thus, it could only be the coproduct $L(X, Y) = X + Y$. Therefore, Δ has a left adjoint if and only if $\mathbf{C}$ has binary coproducts,

$$+ \dashv \Delta.$$

Next, note that $\mathbf{C} \times \mathbf{C} \cong \mathbf{C}^2$ where 2 is the discrete two-object category (i.e., any two-element set). Then $\Delta(C)$ is the constant C-valued functor, for each $C \in \mathbf{C}$. Let us now replace 2 by any small index category $\mathbf{J}$ and consider possible adjoints to the corresponding diagonal functor

$$\Delta_{\mathbf{J}} : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$$

with $\Delta_{\mathbf{J}}(C)(j) = C$ for all $C \in \mathbf{C}$ and $j \in \mathbf{J}$. In this case, one has left and right adjoints

$$\varinjlim_{\mathbf{J}} \dashv \Delta_{\mathbf{J}} \dashv \varprojlim_{\mathbf{J}}$$

if and only if **C** has colimits and limits, respectively, of type **J**. Thus, all particular limits and colimits we met earlier, such as pullbacks and coequalizers are instances of adjoints. What are the units and counits of these adjunctions?

Example 9.10. *Polynomial rings:* Let R be a commutative ring ($\mathbb{Z}$ if you like) and consider the ring $R[x]$ of polynomials in one indeterminate x with coefficients in R. The elements of $R[x]$ all look like this:

$$r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n \tag{9.5}$$

with the coefficients $r_i \in R$. Of course, there may be some identifications between such expressions depending on the ring R.

There is an evident homomorphism $\eta : R \to R[x]$, taking elements r to constant polynomials $r = r_0$, and this map has the following UMP:

> Given any ring A, homomorphism $\alpha : R \to A$, and element $a \in A$, there is a unique homomorphism
>
> $$a^* : R[x] \to A$$
>
> such that $a^*(x) = a$ and $a^*\eta = \alpha$.

$$\begin{tikzcd} R[x] \arrow[r, "a^*"] & A \\ R \arrow[u, "\eta"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Namely, for a^*, we take the "formal evaluation at a"

$$a^*(r(x)) = \alpha(r)(a/x)$$

given by applying α to the coefficients r_i, substituting a for x, and evaluating the result in A,

$$a^*(r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n) = \alpha(r_0) + \alpha(r_1)a + \alpha(r_2)a^2 + \cdots + \alpha(r_n)a^n.$$

To describe this in terms of adjoints, define $\mathbf{Rings}_*$ to be the category of "pointed" rings, with objects of the form (A, a), where A is a ring and $a \in A$, and arrows $h : (A, a) \to (B, b)$ are homomorphisms $h : A \to B$ that preserve the distinguished point, $h(a) = b$. (Cf. pointed sets, example 7.27.)

The UMP just given says exactly that the functor

$$U : \mathbf{Rings}_* \to \mathbf{Rings}$$

that "forgets the point" $U(A, a) = A$ has as left adjoint the functor

$$[x] : \mathbf{Rings} \to \mathbf{Rings}_*$$

that "adjoins an indeterminate"

$$[x](R) = (R[x], x)$$

and $\eta : R \to R[x]$ is the unit of the adjunction. The reader should have no difficulty working out the details of this example. This provides a characterization of the polynomial ring $R[x]$ by adjointness, one that does not depend on the somewhat vague description in terms of "formal polynomial expressions" like (9.5).

9.4 Order adjoints

Let P be a *preordered set*, that is, a category in which there is at most one arrow $x \to y$ between any two objects. A poset is a preorder that is skeletal. As usual, we define an ordering relation on the objects of P by

$$x \leq y \quad \text{iff there exists an arrow } x \to y.$$

Given another such preorder Q, suppose we have adjoint functors:

$$P \underset{U}{\overset{F}{\rightleftarrows}} Q \qquad\qquad F \dashv U$$

Then the correspondence $Q(Fa, x) \cong P(a, Ux)$ comes down to the simple condition $Fa \leq x$ iff $a \leq Ux$. Thus, an adjunction on preorders consists simply of order-preserving maps F, U satisfying the two-way rule or "bicondition":

$$\frac{Fa \leq x}{a \leq Ux}$$

For each $p \in P$, the unit is therefore an element $p \leq UFp$ that is least among all x with $p \leq Ux$. Dually, for each $q \in Q$ the counit is an element $FUq \leq q$ that is greatest among all y with $Fy \leq q$.

Such a setup on preordered sets is sometimes called a *Galois connection.*

Example 9.11. A basic example is the interior operation on the subsets of a topological space X. Let $\mathcal{O}(X)$ be the set of open subsets of X and consider the operations of inclusion of the opens into the powerset $\mathcal{P}(X)$, and interior:

$$\text{inc} : \mathcal{O}(X) \to \mathcal{P}(X)$$

$$\text{int} : \mathcal{P}(X) \to \mathcal{O}(X)$$

For any subset A and open subset U, the valid bicondition

$$\frac{U \subseteq A}{U \subseteq \text{int}(A)}$$

means that the interior operation is right adjoint to the inclusion of the open subsets among all the subsets:

$$\text{inc} \dashv \text{int}$$

The counit here is the inclusion $\mathrm{int}(A) \subseteq A$, valid for all subsets A. The case of closed subsets and the closure operation is dual.

Example 9.12. A related example is the adjunction on powersets induced by any function $f : A \to B$, between the inverse image operation f^{-1} and the direct image $\mathrm{im}(f)$,

$$\begin{tikzcd} \mathcal{P}(A) \arrow[r, shift right, "\mathrm{im}(f)"'] & \mathcal{P}(B) \arrow[l, shift right, "f^{-1}"'] \end{tikzcd}$$

Here we have an adjunction $\mathrm{im}(f) \dashv f^{-1}$ as indicated by the bicondition

$$\frac{\mathrm{im}(f)(U) \subseteq V}{U \subseteq f^{-1}(V)}$$

which is plainly valid for all subsets $U \subseteq A$ and $V \subseteq B$.

The inverse image operation $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ also has a *right* adjoint, sometimes called the *dual image*, given by

$$f_*(U) = \{b \in B \mid f^{-1}(b) \subseteq U\}$$

which we leave for the reader to verify.

Note that if A and B are topological spaces and $f : A \to B$ is continuous, then f^{-1} restricts to the open sets $f^{-1} : \mathcal{O}(B) \to \mathcal{O}(A)$. Now the left adjoint $\mathrm{im}(f)$ need not exist (on opens), but the right adjoint f_* still does.

$$\begin{tikzcd} \mathcal{O}(A) \arrow[r, shift right, "f_*"'] & \mathcal{O}(B) \arrow[l, shift right, "f^{-1}"'] \end{tikzcd}$$

Example 9.13. Suppose we have a poset P. Then, as we know, P has *meets* iff for all $p, q \in P$, there is an element $p \wedge q \in P$ satisfying the bicondition

$$\frac{r \leq p \wedge q}{r \leq p \text{ and } r \leq q}$$

Dually, P has *joins* if there is always an element $p \vee q \in P$ such that

$$\frac{p \vee q \leq r}{p \leq r \text{ and } q \leq r}$$

The *Heyting implication* $q \Rightarrow r$ is characterized as an exponential by the bicondition

$$\frac{p \wedge q \leq r}{p \leq q \Rightarrow r}$$

Finally, an initial object 0 and a terminal object 1 are determined by the conditions

$$0 \leq p$$

and

$$p \leq 1.$$

In this way, the notion of a Heyting algebra can be formulated entirely in terms of adjoints. Equivalently, the intuitionistic propositional calculus is neatly axiomatized by the "adjoint rules of inference" just given (replace "$\leq$" by "$\vdash$"). Together with the reflexivity and transitivity of entailment $p \vdash q$, these rules are completely sufficient for the propositional logical operations. That is, they can serve as the rules of inference for a logical calculus of "binary sequents" $p \vdash q$, which is equivalent to the usual intuitionistic propositional calculus.

When we furthermore define negation by $\neg p = p \Rightarrow \bot$, we then get the derived rule

$$\frac{q \leq \neg p}{p \wedge q \leq 0}$$

Finally, the classical propositional calculus (resp. the laws of Boolean algebra) result from adding the rule

$$\neg\neg p \leq p.$$

Let us now consider how this adjoint analysis of propositional can be extended to all of first-order logic.

9.5 Quantifiers as adjoints

Traditionally, the main obstacle to the further development of *algebraic logic* has been the treatment of the quantifiers. Categorical logic solves this problem beautifully with the recognition (due to F.W. Lawvere in the 1960s) that they, too, are adjoint functors.

Let $\mathcal{L}$ be a first-order language. For any list $\bar{x} = x_1, \ldots, x_n$ of distinct variables let us denote the set of formulas with at most those variables free by

$$\text{Form}(\bar{x}) = \{\phi(\bar{x}) \mid \phi(\bar{x}) \text{ has at most } \bar{x} \text{ free}\}.$$

Then, $\text{Form}(\bar{x})$ is a preorder under the entailment relation of first-order logic

$$\phi(\bar{x}) \vdash \psi(\bar{x}).$$

Now let y be a variable not in the list $\bar{x}$, and note that we have a trivial operation

$$* : \text{Form}(\bar{x}) \to \text{Form}(\bar{x}, y)$$

taking each $\phi(\bar{x})$ to itself; this is just a matter of observing that if $\phi(\bar{x}) \in \text{Form}(\bar{x})$ then y cannot be free in $\phi(\bar{x})$. Of course, $*$ is trivially a functor since,

$$\phi(\bar{x}) \vdash \psi(\bar{x}) \quad \text{ in Form}(\bar{x})$$

trivially implies

$$*\phi(\bar{x}) \vdash *\psi(\bar{x}) \quad \text{in Form}(\bar{x}, y).$$

Now since for any $\psi(\bar{x}, y) \in \text{Form}(\bar{x}, y)$ there is, of course, no free y in the formula $\forall y.\psi(\bar{x}, y)$, we have a map

$$\forall y : \text{Form}(\bar{x}, y) \to \text{Form}(\bar{x}).$$

We claim that this map is right adjoint to $*$,

$$* \dashv \forall.$$

Indeed, the usual rules of universal introduction and elimination imply that the following two-way rule of inference holds:

$$\frac{*\phi(\bar{x}) \vdash \psi(\bar{x}, y) \qquad \text{Form}(\bar{x}, y)}{\phi(\bar{x}) \vdash \forall y.\psi(\bar{x}, y) \qquad \text{Form}(\bar{x})}$$

The inference downward is just the usual $\forall$-introduction rule, since y cannot occur freely in $\phi(\bar{x})$. And the inference going up follows from the $\forall$-elimination axiom,

$$\forall y.\psi(\bar{x}, y) \vdash \psi(\bar{x}, y). \tag{9.6}$$

Observe that this derived rule saying that the operation $\forall y$, which binds the variable y, is right adjoint to the trivial operation $*$ depends essentially on the usual "bookkeeping" side condition on the quantifier rule.

Conversely, we could instead take this adjoint rule as basic and derive the customary introduction and elimination rules from it. Indeed, the $\forall$-elimination (9.6) is just the counit of the adjunction, and $\forall$-introduction including the usual side condition results directly from the adjunction.

It is now natural to wonder about the other quantifier *exists* of existence; indeed, we have a further adjunction

$$\exists \dashv * \dashv \forall$$

since the following two-way rule also holds:

$$\frac{\exists y.\psi(\bar{x}, y) \vdash \phi(\bar{x})}{\psi(\bar{x}, y) \vdash *\phi(\bar{x})}$$

Here the unit is the existential introduction "axiom"

$$\psi(\bar{x}, y) \vdash \exists y.\psi(\bar{x}, y),$$

and the inference upward is the conventional rule of $\exists$-elimination. It actually follows from these rules that $\exists y$ and $\forall y$ are in particular functors, that is, that $\psi \vdash \phi$ implies $\exists y.\psi \vdash \exists y.\phi$ and similarly for $\forall$.

The adjoint rules just given can thus be used in place of the customary introduction and elimination rules, to give a complete system of deduction for quantificational logic. We emphasize that the somewhat tiresome bookkeeping side conditions typical of the usual logical formulation turn out to be of the essence, since they express the "change of variable context" to which quantifiers are adjoints.

Many typical laws of predicate logic are just simple formal manipulations of adjoints. For example

$$\begin{aligned}
\forall x.\psi(x,y) &\vdash \psi(x,y) && (\text{counit of } * \dashv \forall)\\
\psi(x,y) &\vdash \exists y.\psi(x,y) && (\text{unit of } \exists \dashv *)\\
\forall x.\psi(x,y) &\vdash \exists y.\psi(x,y) && (\text{transitivity of } \vdash)\\
\exists y\forall x.\psi(x,y) &\vdash \exists y.\psi(x,y) && (\exists \dashv *)\\
\exists y\forall x.\psi(x,y) &\vdash \forall x\exists y.\psi(x,y) && (* \dashv \forall)
\end{aligned}$$

The recognition of the quantifiers as adjoints also gives rise to the following *geometric interpretation.* Take any $\mathcal{L}$ structure M and consider a formula $\phi(x)$ in at most one variable x. It determines a subset,

$$[\phi(x)]^M = \{m \in M \mid M \models \phi(m)\} \subseteq M$$

of all elements satisfying the condition expressed by ϕ. Similarly, a formula in several variables determines a subset of the cartesian product

$$[\psi(x_1,\ldots,x_n)]^M = \{(m_1,\ldots,m_n) \mid M \models \psi(m_1,\ldots,m_n)\} \subseteq M^n.$$

For instance, $[x = y]^M$ is the diagonal subset $\{(m,m) \mid m \in M\} \subseteq M \times M$. Let us take two variables x, y and consider the effect of the $*$ operation on these subsets. The assignment $*[\phi(x)] = [*\phi(x)]$ determines a functor

$$* : \mathcal{P}(M) \to \mathcal{P}(M \times M).$$

Explicitly, given $[\phi(x)] \in \mathcal{P}(M)$, we have

$$*[\phi(x)] = \{(m_1,m_2) \in M \times M \mid M \models \phi(m_1)\} = \pi^{-1}([\phi(x)])$$

where $\pi : M \times M \to M$ is the first projection. Thus,

$$* = \pi^{-1},$$

the inverse image under projection. Similarly, the existential quantifier can be regarded as an operation on subsets by $\exists[\psi(x,y)] = [\exists y.\psi(x,y)]$,

$$\exists : \mathcal{P}(M \times M) \to \mathcal{P}(M).$$

Specifically, given $[\psi(x,y)] \subseteq M \times M$, we have

$$\begin{aligned}
\exists[\psi(x,y)] &= [\exists y.\psi(x,y)]\\
&= \{m \mid \text{ for some } y, M \models \psi(m,y)\}\\
&= \mathrm{im}(\pi)[\psi(x,y)].
\end{aligned}$$

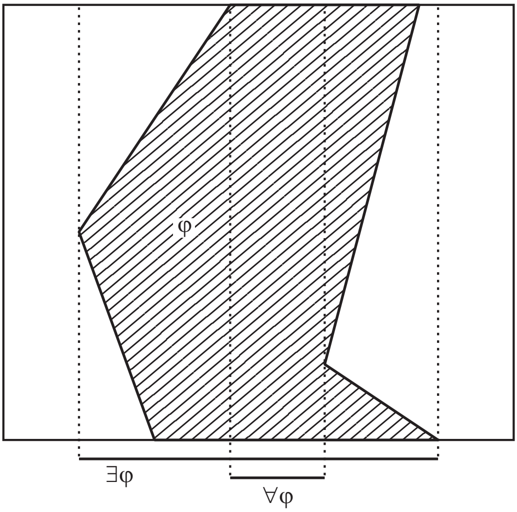

Figure 9.1 Quantifiers as adjoints

Therefore,

$$\exists = \mathrm{im}(\pi),$$

the direct image under projection. In this way, you can actually "see" the logical adjunction:

$$\frac{\exists y.\psi(x,y) \vdash \phi(x)}{\psi(x,y) \vdash \phi(x)}$$

It is essentially the adjunction already considered (example 9.12) between direct and inverse images, applied to the case of a product projection $\pi : M \times M \to M$,

$$\mathrm{im}(\pi) \dashv \pi^{-1}.$$

See Figure 9.1.

Finally, the universal quantifier can also be regarded as an operation of the form

$$\forall : \mathcal{P}(M \times M) \to \mathcal{P}(M)$$

by setting $\forall[\psi(x,y)] = [\forall y.\psi(x,y)]$. Then given $[\psi(x,y)] \subseteq M \times M$, we have

$$\begin{aligned} \forall[\psi(x,y)] &= [\forall y.\psi(x,y)] \\ &= \{m \mid \text{ for all } y, M \models \psi(m,y)\} \\ &= \{m \mid \pi^{-1}\{m\} \subseteq [\psi(x,y)]\} \\ &= \pi_*([\psi(x,y)]). \end{aligned}$$

Therefore,

$$\forall = \pi_*$$

so the universal quantifier is the "dual image," that is, the right adjoint to pullback along the projection π. Again, in Figure 9.1, one can see the adjunction:

$$\frac{\phi(x) \leq \psi(x,y)}{\phi(x) \leq \forall y.\psi(x,y)}$$

by considering the corresponding operations induced on subsets.

9.6 RAPL

In addition to the conceptual unification achieved by recognizing constructions as different as existential quantifiers and free groups as instances of adjoints, there is the practical benefit that one then knows that these operations behave in certain ways that are common to all adjoints. We next consider one of the fundamental properties of adjoints: preservation of limits.

In Section 9.5, we had a string of three adjoints,

$$\exists \dashv * \dashv \forall$$

and it is easy to find other such strings. For example, there is a string of four adjoints between **Cat** and **Sets**,

$$V \dashv F \dashv U \dashv R$$

where $U : \mathbf{Cat} \to \mathbf{Sets}$ is the forgetful functor to the set of objects

$$U(\mathbf{C}) = \mathbf{C}_0.$$

An obvious question in this kind of situation is "are there more?" That is, given a functor does it have an adjoint? A useful necessary condition which shows that, for example, the strings above stop is the following proposition, which is also important in its own right.

Proposition 9.14. *Right adjoints preserve limits (RAPL!), and left adjoints preserve colimits.*

Proof. Here is the easy way: suppose we have an adjunction

$$\mathbf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{D} \qquad\qquad F \dashv U$$

and we are given a diagram $D : J \to \mathbf{D}$ such that the limit $\varprojlim D_j$ exists in $\mathbf{D}$. Then for any $X \in \mathbf{C}$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{C}}(X, U(\varprojlim D_j)) &\cong \mathrm{Hom}_{\mathbf{D}}(FX, \varprojlim D_j) \\
&\cong \varprojlim \mathrm{Hom}_{\mathbf{D}}(FX, D_j) \\
&\cong \varprojlim \mathrm{Hom}_{\mathbf{C}}(X, UD_j) \\
&\cong \mathrm{Hom}_{\mathbf{C}}(X, \varprojlim UD_j)
\end{aligned}$$

whence (by Yoneda), we have the required isomorphism

$$U(\varprojlim D_j) \cong \varprojlim U D_j.$$

It follows by duality that left adjoints preserve colimits. $\square$

It is illuminating to work out what the above argument "really means" in a particular case, say binary products. Given a product $A \times B$ in $\mathbf{D}$, consider the following diagram, in which the part on the left is in $\mathbf{C}$ and that on the right in $\mathbf{D}$:

$$\mathbf{C} \quad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dotted] \arrow[dr, "g"] & \\ UA & U(A \times B) \arrow[l] \arrow[r] & UB \end{tikzcd} \qquad \mathbf{D} \quad \begin{tikzcd} & FX \arrow[dl, "\bar{f}"'] \arrow[d, dotted] \arrow[dr, "\bar{g}"] & \\ A & A \times B \arrow[l] \arrow[r] & B \end{tikzcd}$$

Then given any f and g in $\mathbf{C}$ as indicated, we get the required unique arrow $\langle f, g \rangle$ by adjointness as the transpose

$$\langle f, g \rangle = \overline{\langle \bar{f}, \bar{g} \rangle}$$

where we write $\bar{f}$, etc., for transposition in both directions.

For an example, recall that in the proof that $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ has exponentials we needed the following distributivity law for sets:

$$(\varinjlim_i X_i) \times A \;\cong\; \varinjlim_i (X_i \times A)$$

We now see that this is a consequence of the fact that the functor $(-) \times A$ is a left adjoint (namely to $(-)^A$) and therefore preserves colimits.

It also follows immediately for the propositional calculus (and in any Heyting algebra) that, for example,

$$p \Rightarrow (a \wedge b) \dashv\vdash (p \Rightarrow a) \wedge (p \Rightarrow b)$$

and

$$(a \vee b) \wedge p \dashv\vdash (a \wedge p) \vee (b \wedge p).$$

Similarly, for the quantifiers one has, for example,

$$\forall x(\phi(x) \wedge \psi(x)) \dashv\vdash \forall x \phi(x) \wedge \forall x \psi(x).$$

Note that since this does not hold for $\exists x$, it cannot be a right adjoint to some other "quantifier." Similarly

$$\exists x(\phi(x) \vee \psi(x)) \dashv\vdash \exists x \phi(x) \vee \exists x \psi(x).$$

And, as above, $\forall x$ cannot be a left adjoint, since it does not have this property.

The proposition gives an extremely important and useful property of adjoints. As in the foregoing examples, it can be used to show that a given functor does not have an adjoint by showing that it does not preserve (co)limits. But also, to show that a given functor does preserve all (co)limits, sometimes the easiest way to proceed is to show that it has an adjoint. For example, it is very easy to recognize that the forgetful functor $U : \mathbf{Pos} \to \mathbf{Sets}$ from posets to sets has a left adjoint (what is it?). Thus, we know that limits of posets are limits of the underlying sets (suitably ordered). Dually, you may have shown "by hand" as an exercise that the coproduct of free monoids is the free monoid on the coproduct of their generating sets

$$F(A) + F(B) \cong F(A + B).$$

This now follows simply from the free $\dashv$ forgetful adjunction.

Example 9.15. Our final example of preservation of (co)limits by adjoints involves the UMP of the categories of diagrams $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ studied in Chapter 8. For a small category $\mathbf{C}$, a contravariant functor $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ is often called a *presheaf* on $\mathbf{C}$, and the functor category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ is accordingly called the *category of presheaves* on $\mathbf{C}$, sometimes written as $\hat{\mathbf{C}}$. This cocomplete category is the "free cocompletion" of $\mathbf{C}$ in the following sense.

Proposition 9.16. *For any small category* $\mathbf{C}$*, the Yoneda embedding*

$$\mathrm{y} : \mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$$

has the following UMP: given any cocomplete category $\mathcal{E}$ *and functor* $F : \mathbf{C} \to \mathcal{E}$*, there is a colimit preserving functor* $F_! : \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \to \mathcal{E}$ *such that*

$$F_! \circ \mathrm{y} \cong F \tag{9.7}$$

as indicated in the following diagram:

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \arrow[r, dotted, "F_!"] & \mathcal{E} \\
\mathbf{C} \arrow[u, "\mathrm{y}"] \arrow[ur, "F"'] &
\end{tikzcd}$$

Moreover, up to natural isomorphism, $F_!$ *is the unique cocontinuous functor with this property.*

Proof. We show that there are adjoint functors,

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \arrow[r, shift right, "F_!"'] & \mathcal{E} \arrow[l, shift right, "F^*"'] \\
\mathbf{C} \arrow[u, "\mathrm{y}"] \arrow[ur, "F"'] &
\end{tikzcd} \qquad F_! \dashv F^*$$

with $F_! \circ \mathrm{y} \cong F$. It then follows that $F_!$ preserves all colimits. To define $F_!$, take any presheaf $P \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and write it as a canonical colimit of representables

$$\varinjlim_{j \in J} \mathrm{y}C_j \;\cong\; P$$

with $J = \int_{\mathbf{C}} P$ the category of elements of P, as in proposition 8.10. Then, set

$$F_!(P) = \varinjlim_{j \in J} FC_j$$

with the colimit taken in $\mathcal{E}$, which is cocomplete. (We leave it to the reader to determine how to define $F_!$ on arrows.) Clearly, if $F_!$ is to preserve all colimits and satisfy (9.7), then up to isomorphism this must be its value for P. For F^*, take any $E \in \mathcal{E}$ and $C \in \mathbf{C}$ and observe that by (Yoneda and) the intended adjunction, for $F^*(E)(C)$, we must have

$$\begin{aligned}
F^*(E)(C) &\cong \mathrm{Hom}_{\hat{C}}(\mathrm{y}C, F^*(E)) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(F_!(\mathrm{y}C), E) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(FC, E).
\end{aligned}$$

Thus, we simply set

$$F^*(E)(C) = \mathrm{Hom}_{\mathcal{E}}(FC, E)$$

which is plainly a presheaf on $\mathbf{C}$ (we use here that $\mathcal{E}$ is locally small). Now let us check that indeed $F_! \dashv F^*$. For any $E \in \mathcal{E}$ and $P \in \hat{\mathbf{C}}$, we have natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\hat{\mathbf{C}}}(P, F^*(E)) &\cong \mathrm{Hom}_{\hat{\mathbf{C}}}(\varinjlim_{j\in J} \mathrm{y}C_j\ , F^*(E)) \\
&\cong \varprojlim_{j\in J} \mathrm{Hom}_{\hat{\mathbf{C}}}(\mathrm{y}C_j\ , F^*(E)) \\
&\cong \varprojlim_{j\in J} F^*(E)(C_j) \\
&\cong \varprojlim_{j\in J} \mathrm{Hom}_{\mathcal{E}}(FC_j, E) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(\varinjlim_{j\in J} FC_j, E) \\
&\cong \mathrm{Hom}_{\mathcal{E}}(F_!(P), E).
\end{aligned}$$

Finally, for any object $C \in \mathbf{C}$,

$$F_!(\mathrm{y}C) = \varinjlim_{j\in J} FC_j \cong\ FC$$

since the category of elements J of a representable $\mathrm{y}C$ has a terminal object, namely the element $1_C \in \mathrm{Hom}_{\mathbf{C}}(C, C)$. □

Corollary 9.17. *Let $f : \mathbf{C} \to \mathbf{D}$ be a functor between small categories. The precomposition functor*

$$f^* : \mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$$

given by

$$f^*(Q)(C) = Q(fC)$$

has both left and right adjoints

$$f_! \vdash f^* \vdash f_*$$

Moreover, there is a natural isomorphism

$$f_! \circ \mathrm{y}_{\mathbf{C}} \ \cong\ \mathrm{y}_{\mathbf{D}} \circ f$$

as indicated in the following diagram:

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \arrow[r, shift left=2] \arrow[r, shift right=2, "f_!"'] & \mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}} \arrow[l] \\
\mathbf{C} \arrow[u, "\mathrm{y}_{\mathbf{C}}"] \arrow[r, "f"'] & \mathbf{D} \arrow[u, "\mathrm{y}_{\mathbf{D}}"']
\end{tikzcd}$$

The induced functors $f_!$ and f_* are sometimes referred to in the literature as *(left and right) Kan extensions.*

Proof. First, define

$$F = \mathrm{y}_{\mathbf{D}} \circ f : \mathbf{C} \to \mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}}.$$

Then, by the foregoing proposition, we have adjoints $F_!$ and F^* as indicated in

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \arrow[r, shift right, "F_!"'] & \mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}} \arrow[l, shift right, "F^*"'] \\
\mathbf{C} \arrow[u, "\mathrm{y}_{\mathbf{C}}"] \arrow[r, "f"'] & \mathbf{D} \arrow[u, "\mathrm{y}_{\mathbf{D}}"']
\end{tikzcd}$$

and we know that $F_! \circ \mathrm{y}_{\mathbf{C}} \cong \mathrm{y}_D \circ f$. We claim that $F^* \cong f^*$. Indeed, by the definition of F^*, we have

$$F^*(Q)(C) = \mathrm{Hom}_{\hat{\mathbf{D}}}(FC, Q) \cong \mathrm{Hom}_{\hat{\mathbf{D}}}(\mathrm{y}(fC), Q) \cong Q(fC) = f^*(Q)(C).$$

This, therefore, gives the functors $f_! \dashv f^*$. For f_*, apply the foregoing proposition to the composite

$$f^* \circ \mathrm{y}_{\mathbf{D}} : \mathbf{D} \to \mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}.$$

This gives an adjunction

$$(f^* \circ \mathrm{y}_{\mathbf{D}})_! \dashv (f^* \circ \mathrm{y}_{\mathbf{D}})^*$$

so we just need to show that

$$(f^* \circ \mathrm{y}_{\mathbf{D}})_! \cong f^*$$

in order to get the required right adjoint as $f_* = (f^* \circ \mathrm{y}_{\mathbf{D}})^*$. By the universal property of $\mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}}$, it suffices to show that f^* preserves colimits. But for any colimit $\varinjlim_j Q_j$ in $\mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}}$

$$\begin{aligned}
(f^*(\varinjlim_j Q_j))(C) &\cong (\varinjlim_j Q_j)(fC) \\
&\cong \varinjlim_j (Q_j(fC)) \\
&\cong \varinjlim_j ((f^*Q_j)(C)) \\
&\cong (\varinjlim_j (f^*Q_j))(C).
\end{aligned}$$

□

This corollary says that, in a sense, *every functor has an adjoint*! For, given any $f : \mathbf{C} \to \mathbf{D}$, we indeed have the right adjoint

$$f^* \circ \mathrm{y}_{\mathbf{D}} : \mathbf{D} \to \hat{\mathbf{C}}$$

except that its values are in the "ideal elements" of the cocompletion $\hat{\mathbf{C}} = \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$.

9.7 Locally cartesian closed categories

A special case of the situation described by corollary 9.17 is the *change of base* for indexed families of sets along a "reindexing" function $\alpha : J \to I$. An arbitrary such function between sets gives rise, by that corollary, to a triple of adjoint functors:

$$\begin{tikzcd} \mathbf{Sets}^J \arrow[r, shift left=4, "\alpha_*"] \arrow[r, shift right=4, "\alpha_!"'] & \mathbf{Sets}^I \arrow[l, "\alpha^*"'] \end{tikzcd}$$

$$\alpha_! \dashv \alpha^* \dashv \alpha_*$$

Let us examine these functors more closely in this special case.

An object A of $\mathbf{Sets}^I$ is an I-indexed family of sets

$$(A_i)_{i \in I}.$$

Then, $\alpha^*(A) = A \circ \alpha$ is the reindexing of A along α to a J-indexed family of sets

$$\alpha^*(A) = (A_{\alpha(j)})_{j \in J}.$$

Given a J-indexed family B, let us calculate $\alpha_!(B)$ and $\alpha_*(B)$.

Consider first the case $I = 1$ and $\alpha = !_J : J \to 1$. Then, $(!_J)^* : \mathbf{Sets} \to \mathbf{Sets}^J$ is the "constant family" or diagonal functor $\Delta(A)(j) = A$, for which we know the adjoints:

$$\begin{tikzcd} \mathbf{Sets}^J \arrow[r, shift left=4, "\Pi"] \arrow[r, shift right=4, "\Sigma"'] & \mathbf{Sets} \arrow[l, "\Delta"'] \end{tikzcd}$$

$$\Sigma \dashv \Delta \dashv \Pi$$

These are, namely, just the (disjoint) sum and cartesian product of the sets in the family

$$\sum_{j \in J} B_j, \qquad \prod_{j \in J} B_j.$$

Recall that we have the adjunctions:

$$\frac{\vartheta_j : B_j \to A}{(\vartheta_j) : \sum_j B_j \to A}, \quad \frac{\vartheta_j : A \to B_j}{\langle \vartheta_j \rangle : A \to \prod_j B_j}$$

By uniqueness of adjoints, it therefore follows that $(!_J)_! \cong \Sigma$ and $(!_J)_* \cong \Pi$.

A general reindexing $\alpha : J \to I$ gives rise to generalized sum and product operations along α

$$\Sigma_\alpha \dashv \alpha^* \dashv \Pi_\alpha$$

defined on J-indexed families (B_j) by

$$(\Sigma_\alpha(B_j))_i = \sum_{\alpha(j)=i} B_j$$

$$(\Pi_\alpha(B_j))_i = \prod_{\alpha(j)=i} B_j.$$

These operations thus assign to an element $i \in I$ the sum, respectively the product, over all the sets indexed by the elements j in the preimage $\alpha^{-1}(i)$ of i under α.

Now let us recall from example 7.29 the equivalence between J-indexed families of sets and the slice category of "sets over J"

$$\mathbf{Sets}^J \simeq \mathbf{Sets}/J.$$

It takes a family $(A_j)_{j\in J}$ to the indexing projection $p : \sum_{j\in J} A_j \to J$ and a map $\pi : A \to J$ to the family $(\pi^{-1}(j))_{j\in J}$. We know, moreover, from an exercise in Chapter 7 that this equivalence respects reindexing, in the sense that for any $\alpha : J \to I$ the following square commutes up to natural isomorphism:

$$\begin{tikzcd}
J \arrow[d, "\alpha"'] & \mathbf{Sets}/J \arrow[r, "\simeq"'] & \mathbf{Sets}^J \\
I & \mathbf{Sets}/I \arrow[u, "\alpha^\sharp"] \arrow[r, "\simeq"] & \mathbf{Sets}^I \arrow[u, "\alpha^*"']
\end{tikzcd}$$

Here we write $\alpha^\sharp$ for the pullback functor along α. Since α^* has both right and left adjoints, we have the diagram of induced adjoints:

$$\begin{tikzcd}
J \arrow[d, "\alpha"'] & \mathbf{Sets}/J \arrow[r, "\simeq"'] \arrow[d, shift right=4, "\alpha_L"'] \arrow[d, shift left=4, "\alpha_\sharp"] & \mathbf{Sets}^J \arrow[d, shift right=4, "\alpha_!"'] \arrow[d, shift left=4, "\alpha_*"] \\
I & \mathbf{Sets}/I \arrow[u, "\alpha^\sharp"] \arrow[r, "\simeq"] & \mathbf{Sets}^I \arrow[u, "\alpha^*"]
\end{tikzcd}$$

Proposition 9.18. *For any function $\alpha : J \to I$, the pullback functor $\alpha^\sharp$: **Sets**$/I \to$ **Sets**$/J$ has both left and right adjoints:*

$$\alpha_L \dashv \alpha^\sharp \dashv \alpha_\sharp$$

In particular, $\alpha^\sharp$ therefore preserves all limits and colimits.

Let us compute the functors explicitly. Given $\pi : A \to J$, let $A_j = \pi^{-1}(j)$ and recall that

$$\alpha_!(A)_i = \sum_{\alpha(j)=i} A_i.$$

But then, we have

$$\begin{aligned}
\alpha_!(A)_i &= \sum_{\alpha(j)=i} A_i \\
&= \sum_{i\in\alpha^{-1}(j)} A_i \\
&= \sum_{i\in\alpha^{-1}(j)} \pi^{-1}(j) \\
&= \pi^{-1}\circ\alpha^{-1}(j) \\
&= (\alpha\circ\pi)^{-1}(i).
\end{aligned}$$

It follows that $\alpha_L(\pi : A \to J)$ is simply the composite $\alpha\circ\pi : A \to J \to I$,

$$\alpha_L(\pi : A \to J) = (\alpha\circ\pi : A \to J \to I).$$

Indeed, the UMP of pullbacks essentially states that composition along any function α is left adjoint to pullback along α.

As for the right adjoint

$$\alpha_\sharp : \mathbf{Sets}/J \longrightarrow \mathbf{Sets}/I$$

given $\pi : A \to J$, the result $\alpha_\sharp(\pi) : \alpha_\sharp(A) \to I$ can be described fiberwise by

$$(\alpha_\sharp(A))_i = \{s : \alpha^{-1}(i) \to A \mid \text{“}s \text{ is a partial section of } \pi\text{”}\}$$

where the condition “s is a partial section of π” means that the following triangle commutes with the canonical inclusion $\alpha^{-1}(i) \subseteq J$ at the base.

$$\begin{tikzcd}
 & A \arrow[d, "\pi"] \\
\alpha^{-1}(i) \arrow[ur, "s"] \arrow[r, hook] & J
\end{tikzcd}$$

Henceforth, we also write these "change of base" adjoints along a map $\alpha : J \to I$ in the form

$$\begin{tikzcd} J \arrow[d, "\alpha"'] & \mathbf{Sets}/J \arrow[d, shift right=6, "\Sigma_\alpha"'] \arrow[d, shift left=6, "\Pi_\alpha"] \\ I & \mathbf{Sets}/I \arrow[u, "\alpha^*"'] \end{tikzcd} \qquad \Sigma_\alpha \dashv \alpha^* \dashv \Pi_\alpha$$

Finally, let us reconsider the case $I = 1$, where these adjoints take the form

$$\begin{tikzcd} J \arrow[d, "!"'] & \mathbf{Sets}/J \arrow[d, shift right=6, "\Sigma_J"'] \arrow[d, shift left=6, "\Pi_J"] \\ 1 & \mathbf{Sets} \arrow[u, "J^*"'] \end{tikzcd} \qquad \Sigma_J \dashv J^* \dashv \Pi_J$$

In this case, we have

$$\begin{aligned} \Sigma_J(\pi : A \to J) &= A \\ J^*(A) &= (p_1 : J \times A \to J) \\ \Pi_J(\pi : A \to J) &= \{s : J \to A \mid \pi \circ s = 1\} \end{aligned}$$

as the reader can easily verify. Moreover, one therefore has

$$\begin{aligned} \Sigma_J J^*(A) &= J \times A \\ \Pi_J J^*(A) &= A^J. \end{aligned}$$

Thus, the product $\dashv$ exponential adjunction can be factored as a composite of adjunctions as follows:

$$\begin{tikzcd} \mathbf{Sets} \arrow[rr, shift left, "J \times (-)"] \arrow[d, equal] & & \mathbf{Sets} \arrow[ll, shift left, "(-)^J"] \arrow[d, equal] \\ \mathbf{Sets} \arrow[r, shift left, "J^*"] & \mathbf{Sets}/J \arrow[l, shift left, "\Pi_J"] \arrow[r, shift left, "\Sigma_J"] & \mathbf{Sets} \arrow[l, shift left, "J^*"] \end{tikzcd}$$

The following definition captures the notion of a category having this sort of adjoint structure. In such a category $\mathcal{E}$, the slice categories can be regarded as categories of abstract-indexed families of objects of $\mathcal{E}$, and the reindexing of such families can be carried out, with associated adjoint operations of sum and product.

Definition 9.19. A category $\mathcal{E}$ is called *locally cartesian closed* if $\mathcal{E}$ has a terminal object and for every arrow $f : A \to B$ in $\mathcal{E}$, the composition functor

$$\Sigma_f : \mathcal{E}/A \to \mathcal{E}/B$$

has a right adjoint f^* which, in turn, has a right adjoint Π_f:

$$\Sigma_f \dashv f^* \dashv \Pi_f$$

The choice of name for such categories is explained by the following important fact.

Proposition 9.20. *For any category $\mathcal{E}$ with a terminal object, the following are equivalent:*

1. *$\mathcal{E}$ is locally cartesian closed.*
2. *Every slice category $\mathcal{E}/A$ of $\mathcal{E}$ is cartesian closed.*

Proof. Let $\mathcal{E}$ be locally cartesian closed. Since $\mathcal{E}$ has a terminal object, products and exponentials in $\mathcal{E}$ can be built as

$$A \times B = \Sigma_B B^* A$$

$$B^A = \Pi_B B^* A.$$

Therefore, $\mathcal{E}$ is cartesian closed. But clearly every slice category $\mathcal{E}/X$ is also locally cartesian closed, since "a slice of a slice is a slice." Thus, every slice of $\mathcal{E}$ is cartesian closed.

Conversely, suppose every slice of $\mathcal{E}$ is cartesian closed. Then $\mathcal{E}$ has pullbacks, since these are just binary products in a slice. Thus, we just need to construct the "relative product" functor $\Pi_f : \mathcal{E}/A \to \mathcal{E}/B$ along a map $f : A \to B$. First, change notation:

$$\mathcal{F} = \mathcal{E}/B$$

$$F = f : A \to B$$

$$\mathcal{F}/F = \mathcal{E}/A$$

Thus, we want to construct $\Pi_F : \mathcal{F}/F \to F$. Given an object $p : X \to F$ in $\mathcal{F}/F$, the object $\Pi_F(p)$ is constructed as the following pullback:

$$\begin{tikzcd}
\Pi_F(p) \arrow[r] \arrow[d] & X^F \arrow[d, "p^F"] \\
1 \arrow[r, "\widetilde{1_F}"'] & F^F
\end{tikzcd} \tag{9.8}$$

where $\widetilde{1_F}$ is the exponential transpose of the composite arrow

$$1 \times F \cong F \xrightarrow{1} F.$$

It is now easy to see from (9.8) that there is a natural bijection of the form

$$\frac{Y \to \Pi_F(p)}{F^*Y \to p}$$

$\square$

Remark 9.21. The reader should be aware that some authors do not require the existence of a terminal object in the definition of a locally cartesian closed category.

Example 9.22 (Presheaves). For any small category $\mathbf{C}$, the category $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves on $\mathbf{C}$ is locally cartesian closed. This is a consequence of the following fact.

Lemma 9.23. *For any object $P \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, there is a small category $\mathbf{D}$ and an equivalence of categories,*

$$\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}/P \;\simeq\; \mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}}.$$

Moreover, there is also a functor $p : \mathbf{D} \to \mathbf{C}$ such that the following diagram commutes (up to natural isomorphism):

$$\begin{tikzcd}
\mathbf{Sets}^{\mathbf{D}^{\mathrm{op}}} \arrow[rr, "\simeq"'] \arrow[dr, "p_!"'] & & \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}/P \arrow[dl, "\Sigma_P"] \\
& \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} &
\end{tikzcd}$$

Proof. One can take

$$\mathbf{D} = \int_{\mathbf{C}} P$$

$$p = \pi : \int_{\mathbf{C}} P \to \mathbf{C}$$

Indeed, recall that by the Yoneda lemma, the category $\int_{\mathbf{C}} P$ of elements of P can be described equivalently (isomorphically, in fact) as the category that we write suggestively as y/P, described as follows:

Objects: pairs (C, x) where $C \in \mathbf{C}$ and $x : \mathrm{y}C \to P$ in $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$

Arrows: all arrows between such objects in the slice category over P

$$\begin{tikzcd}
\mathrm{y}C \arrow[rr, "\vartheta"] \arrow[dr, "x"'] & & \mathrm{y}C' \arrow[dl, "x'"] \\
& P &
\end{tikzcd}$$

Note that by Yoneda, each such arrow is of the form $\vartheta = \mathrm{y}h$ for a unique $h : C \to D$ in $\mathbf{C}$, which, moreover, is such that $P(h)(x') = x$.

Now let $I : \mathrm{y}/P \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}/P$ be the evident (full and faithful) inclusion functor, and define a functor

$$\Phi : \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}/P \to \mathbf{Sets}^{(\mathrm{y}/P)^{\mathrm{op}}}$$

by setting, for any $q : Q \to P$ and $(C, x) \in \mathrm{y}/P$

$$\Phi(q)(C, x) = \mathrm{Hom}_{\hat{\mathbf{C}}/P}(x, q),$$

the elements of which look like

$$\begin{tikzcd} \mathrm{y}C \arrow[rr, dotted] \arrow[dr, "x"'] & & Q \arrow[dl, "q"] \\ & P & \end{tikzcd}$$

In other words, $\Phi(q) = I^*(\mathrm{y}q)$, which is plainly functorial. We leave it to the reader as an exercise to show that this functor establishes an equivalence of categories. □

Combining the foregoing with the fact (theorem 8.14) that categories of presheaves are always cartesian closed now yields the promised:

Corollary 9.24. *For any small category* $\mathbf{C}$*, the category* $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ *of presheaves on* $\mathbf{C}$ *is locally cartesian closed.*

Remark 9.25. Part of the interest in locally cartesian closed categories derives from their use in the semantics of dependent type theory, which has type-indexed families of types

$$x : A \vdash B(x)$$

and type constructors of dependent sum and product

$$\sum_{x:A} B(x) \qquad \prod_{x:A} B(x).$$

Indeed, just as cartesian closed categories provide a categorical interpretation of the simply typed λ-calculus, so locally cartesian closed categories interpret the dependently typed λ-calculus. And since the Yoneda embedding preserves CCC structure, the completeness theorem for λ-calculus with respect to arbitrary CCCs (theorem 6.17) implies completeness with respect to just categories of presheaves $\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$, as was shown in exercise 10 of Chapter 8. Now, just the same sort of completeness theorem holds for dependent type theory as well, by an elementary argument involving the foregoing lemma. More difficult to prove is the fact that one can do even better, retaining completeness while restricting the interpretations to just the "categories of diagrams" on *posets*, $\mathbf{Sets}^{\mathrm{P}}$, which can be regarded as Kripke models (and this of course then also holds for the simply typed λ-calculus as well). In this connection, the following alternate description of such categories is then of particular interest.

Example 9.26 Fibrations of posets. A monotone map of posets $f : X \to P$ is a (discrete) *fibration* if it has the following *lifting property*:

For every $x \in X$ and $p' \leq fx$, there is a unique $x' \leq x$ such that $f(x') = p'$.

One says that x "lies over" $p = f(x)$ and that any $p' \leq p$ "lifts" to a unique $x' \leq x$ lying over it, as indicated in the following diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & & x' \arrow[r, dotted, "\leq"'] & x \\ P & & p' \arrow[r, "\leq"'] & p \end{tikzcd}$$

The identity morphism of a given poset P is clearly a fibration, and the composite of two fibrations is easily seen to be a fibration. Let **Fib** denote the (non-full) subcategory of posets and fibrations between them as arrows.

Lemma 9.27. *For any poset P, the slice category* $\mathbf{Fib}/P$ *is cartesian closed.*

Proof. The category $\mathbf{Fib}/P$ is equivalent to the category of presheaves on P,

$$\mathbf{Fib}/P \;\simeq\; \mathbf{Sets}^{P^{\mathrm{op}}}.$$

To get a functor, $\Phi : \mathbf{Fib}/P \to \mathbf{Sets}^{P^{\mathrm{op}}}$, takes a fibration $q : Q \to P$ to the presheaf defined on objects by

$$\Phi(q)(p) = q^{-1}(p) \qquad \text{for } p \in P.$$

The lifting property then determines the action on arrows $p' \leq p$. For the other direction, $\Psi : \mathbf{Sets}^{P^{\mathrm{op}}} \to \mathbf{Fib}/P$ takes a presheaf $Q : P^{\mathrm{op}} \to \mathbf{Sets}$ to (the indexing projection of) its category of elements,

$$\Psi(Q) = \int_P Q \xrightarrow{\pi} P.$$

These are easily seen to be quasi-inverses. $\square$

The category **Fib** itself is *almost* locally cartesian closed; it only lacks a terminal object (why?). We can "fix" this simply by slicing it.

Corollary 9.28. *For any poset P, the slice category* $\mathbf{Fib}/P$ *is locally cartesian closed.*

This sort of case is not uncommon, which is why the notion "locally cartesian closed" is sometimes formulated without requiring a terminal object.

9.8 Adjoint functor theorem

The question we now want to consider systematically is, when does a functor have an adjoint? Consider first the question, when does a functor of the form $\mathbf{C} \to \mathbf{Sets}$ have a left adjoint? If $U : \mathbf{C} \to \mathbf{Sets}$ has $F \dashv U$, then U is representable $U \cong \mathrm{Hom}(F1, -)$, since $U(C) \cong \mathrm{Hom}(1, UC) \cong \mathrm{Hom}(F1, C)$.

A related condition that makes sense for categories other than **Sets** is preservation of limits. Suppose that $\mathbf{C}$ is complete and $U : \mathbf{C} \to \mathbf{X}$ preserves limits; then we can ask whether U has a left adjoint. The *adjoint functor theorem* (AFT) gives a necessary and sufficient condition for this case.

Theorem 9.29 (Freyd)**.** *Let* $\mathbf{C}$ *be locally small and complete. Given any category* $\mathbf{X}$ *and a limit-preserving functor*

$$U : \mathbf{C} \to \mathbf{X}$$

the following are equivalent:

1. *U has a left adjoint.*
2. *For each object $X \in \mathbf{X}$, the functor U satisfies the following:*
   Solution set condition: There exists a set of objects $(S_i)_{i \in I}$ in $\mathbf{C}$ such that for any object $C \in \mathbf{C}$ and arrow $f : X \to UC$, there exists an $i \in I$ and arrows $\varphi : X \to US_i$ and $\bar{f} : S_i \to C$ such that

$$f = U(\bar{f}) \circ \varphi$$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & US_i \arrow[d, "U\bar{f}"] & S_i \arrow[d, "\bar{f}"] \\ & UC & C \end{tikzcd}$$

   Briefly: "every arrow $X \to UC$ factors through some object S_i in the solution set."

For the proof, we require the following.

Lemma 9.30. *Let* $\mathbf{D}$ *be locally small and complete. Then the following are equivalent:*

1. *$\mathbf{D}$ has an initial object.*
2. *$\mathbf{D}$ satisfies the following:*

Solution set condition: There is a set of objects $(D_i)_{i\in I}$ in $\mathbf{D}$ such that for any object $D \in \mathbf{C}$, there is an arrow $D_i \to D$ for some $i \in I$.

Proof. If $\mathbf{D}$ has an initial object 0, then $\{0\}$ is obviously a solution set.

Conversely, suppose we have a solution set $(D_i)_{i\in I}$ and consider the object

$$W = \prod_{i\in I} D_i,$$

which exists since I is small and $\mathbf{D}$ is complete. Now W is "weakly initial" in the sense that for any object D there is a (not necessarily unique) arrow $W \to D$, namely the composite

$$\prod_{i\in I} D_i \to D_i \to D$$

for a suitable product projection $\prod_{i\in I} D_i \to D_i$. Next, take the joint equalizer of all endomorphisms $d : W \to W$ (which is a set, since $\mathbf{D}$ is locally small), as indicated in the diagram:

$$V \overset{h}{\rightarrowtail} W \underset{\langle d\rangle}{\overset{\Delta}{\rightrightarrows}} \prod_{d:W\to W} W$$

Here, the arrows Δ and $\langle d\rangle$ have the d-projections $1_W : W \to W$ and $d : W \to W$, respectively. This equalizer then has the property that for any endomorphism $d : W \to W$,

$$d \circ h = h. \tag{9.9}$$

Note, moreover, that V is still weakly initial, since for any D there is an arrow $V \rightarrowtail W \to D$. Suppose that for some D there are two arrows $f, g : V \to D$. Take their equalizer $e : U \to V$, and consider the following diagram:

$$\begin{tikzcd}
U \arrow[r, tail, "e"] & V \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, tail, "h"] & D \\
W \arrow[u, "s"] \arrow[r, "hes"'] & W &
\end{tikzcd}$$

in which the arrow s comes from W being weakly initial. So for the endomorphism hes by (9.9), we have

$$hesh = h.$$

Since h is monic, $esh = 1_V$. But then $eshe = e$, and so also $she = 1_U$ since e is monic. Therefore $U \cong V$, and so $f = g$. Thus, V is an initial object. $\square$

Now we can prove the theorem.

Proof. (Theorem) If U has a left adjoint $F \dashv U$, then $\{FX\}$ is itself a solution set for X, since we always have a factorization,

$$
\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "f"'] & UFX \arrow[d, "U(\bar{f})"] & FX \arrow[d, "\bar{f}"] \\
& UC & C
\end{tikzcd}
$$

where $\bar{f} : FX \to C$ is the adjoint transpose of f and $\eta : X \to UFX$ the unit of the adjunction.

Conversely, consider the following so-called *comma-category* $(X|U)$, with

Objects: are pairs (C, f) with $f : X \to UC$

Arrows: $g : (C, f) \to (C', f')$ are arrows $g : C \to C'$ with $f' = U(g)f$.

$$
\begin{tikzcd}
& UC \arrow[dd, "U(g)"] & C \arrow[dd, "g"] \\
X \arrow[ur, "f"] \arrow[dr, "f'"'] & & \\
& UC' & C'
\end{tikzcd}
$$

Clearly, U has a left adjoint F iff for each object X this category $(X|U)$ has an initial object, $(FX, \eta : X \to UFX)$, which then has the UMP of the unit. Thus, to use the foregoing initial object lemma, we must check

1. $(X|U)$ is locally small.
2. $(X|U)$ satisfies the solution set condition in the lemma.
3. $(X|U)$ is complete.

For (1), we just observe that $\mathbf{C}$ is locally small. For (2), the solution set condition of the theorem implies that there is a set of objects,

$$\{(S_i, \varphi : X \to US_i) \mid i \in I\}$$

such that every object $(C, f : X \to UC)$ has an arrow $\bar{f} : (S_i, \varphi) \to (C, f)$.

$$
\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[dr, "f"'] & US_i \arrow[d, "U\bar{f}"] & S_i \arrow[d, "\bar{f}"] \\
& UC & C
\end{tikzcd}
$$

Finally, to see that $(X|U)$ is complete, one can easily check directly that it has products and equalizers, using the fact that U preserves these. We leave this as an easy exercise for the reader. □

Remark 9.31. 1. The theorem simply does not apply if $\mathbf{C}$ is not complete. In that case, a given functor may have an adjoint, but the AFT will not tell us that.

2. It is essential that the solution set in the theorem be a *set* (and that $\mathbf{C}$ have all set-sized limits).
3. On the other hand, if $\mathbf{C}$ is itself *small* and complete, then we can plainly drop the solution set condition entirely. In that case, we have the following.

Corollary 9.32. *If $\mathbf{C}$ is a small and complete category and $U : \mathbf{C} \to \mathbf{X}$ is a functor that preserves all limits, then U has a left adjoint.*

Example 9.33. For complete posets P, Q, a monotone function $f : P \to Q$ has a right adjoint $g : Q \to P$ iff f is *cocontinuous*, in the sense that $f(\bigvee_i p_i) = \bigvee_i f(p_i)$ for any set-indexed family of elements $(p_i)_{i \in I}$. (Of course, here we are using the dual formulation of the AFT.)

Indeed, we can let

$$g(q) = \bigvee_{f(x) \leq q} x.$$

Then for any $p \in P$ and $q \in Q$, if

$$p \leq g(q)$$

then

$$f(p) \leq fg(q) = f(\bigvee_{f(x) \leq q} x) = \bigvee_{f(x) \leq q} f(x) \leq q.$$

While, conversely, if

$$f(p) \leq q$$

then clearly

$$p \leq \bigvee_{f(x) \leq q} x = g(q).$$

As a further consequence of the AFT, we have the following characterization of representable functors on small complete categories.

Corollary 9.34. *If $\mathbf{C}$ is a small and complete category, then for any functor $U : \mathbf{C} \to \mathbf{Sets}$ the following are equivalent:*

1. *U preserves all limits.*
2. *U has a left adjoint.*
3. *U is representable.*

Proof. Immediate. □

These corollaries are, however, somewhat weaker than it may at first appear, in light of the following fact.

Proposition 9.35. *If* $\mathbf{C}$ *is small and complete, then* $\mathbf{C}$ *is a preorder.*

Proof. Suppose not, and take $C, D \in \mathbf{C}$ with $\mathrm{Hom}(C, D) \geq 2$. Let J be any set, and take the product

$$\prod_J D.$$

There are isomorphisms:

$$\mathrm{Hom}(C, \prod_J D) \cong \prod_J \mathrm{Hom}(C, D) \cong \mathrm{Hom}(C, D)^J$$

So, for the cardinalities of these sets, we have

$$|\,\mathrm{Hom}(C, \prod_J D)| = |\,\mathrm{Hom}(C, D)|^{|J|} \geq 2^{|J|} = |P(J)|.$$

And that is for *any* set J. On the other hand, clearly $|\mathbf{C}_1| \geq |\,\mathrm{Hom}(C, \prod_J D)|$. So taking $J = \mathbf{C}_1$ in the above calculation gives a contradiction. □

Remark 9.36. An important special case of the AFT that often occurs "in nature" is that in which the domain category satisfies certain conditions that eliminate the need for the (rather unpleasant!) solution set condition entirely. Specifically, let $\mathbf{A}$ be a locally small, complete category satisfying the following conditions:

1. $\mathbf{A}$ is *well powered*: each object A has at most a *set* of subobjects $S \rightarrowtail A$.
2. $\mathbf{A}$ has a *cogenerating set*: there is a set of objects $\{A_i \mid i \in I\}$ (I some index *set*), such that for any A, X and $x \neq y : X \rightrightarrows A$ in $\mathbf{A}$, there is some $s : A \to A_i$ (for some i) that "separates" x and y, in the sense that $sx \neq sy$.

Then any functor $U : \mathbf{A} \to \mathbf{X}$ that preserves limits necessarily has a left adjoint. In this form (also originally proved by Freyd), the theorem is usually known as the special adjoint functor theorem ("SAFT"). We refer to Mac Lane, V.8 for the proof, and some sample applications.

Example 9.37. An important application of the AFT is that any *equational theory* T gives rise to a free $\dashv$ forgetful adjunction between **Sets** and the category of models of the theory, or "T-algebras." In somewhat more detail, let T be a (finitary) equational theory, consisting of finitely many operation symbols, each of some finite arity (including nullary operations, i.e., constant symbols), and a set of equations between terms built from these operations and variables. For instance, the theory of groups has a constant u (the group unit), a unary operation g^{-1} (the inverse), and a binary operation $g \cdot h$ (the group product), and a handful of equations such as $g \cdot u = g$. The theory of rings has a further binary operation and some more equations. The theory of fields is not equational, however, because the condition $x \neq 0$ is required for an element x to have a multiplicative inverse. A T-algebra is a set equipped with operations (of the proper arities) corresponding to the operation symbols in T, and satisfying the equations of T. A homomorphism of T-algebras $h : A \to B$ is a function on the underlying sets that preserves all the operations, in the usual sense. Let T-**Alg** be the category of all such algebras and their homomorphisms. There is an evident forgetful functor

$$U : T\text{-}\mathbf{Alg} \to \mathbf{Sets}.$$

The AFT implies that this functor always has a left adjoint F, the "free algebra" functor.

Proposition 9.38. *For any equational theory T, the forgetful functor from T-algebras to* **Sets** *has a left adjoint.*

Rather than proving this general proposition (for which see Mac Lane, chapter V), it is more illuminating to do a simple example.

Example 9.39. Let T be the theory with one constant and one unary operation (no axioms). A T-algebra is a set M with the structure

$$1 \xrightarrow{a} M \xrightarrow{f} M$$

If $1 \xrightarrow{b} N \xrightarrow{g} N$ is another such algebra, a homomorphism of T-algebras $\phi : (M, a, f) \to (N, b, g)$ is a function $\phi : M \to N$ that preserves the element and the operation, in the expected sense that

$$\phi a = b$$

$$\phi f = g\phi.$$

as indicated in the commutative diagram:

$$\begin{tikzcd} & M \arrow[r, "f"] \arrow[d, "\phi"'] & M \arrow[d, "\phi"] \\ 1 \arrow[ur, "a"] \arrow[dr, "b"'] & & \\ & N \arrow[r, "g"'] & N \end{tikzcd}$$

There is an evident forgetful functor (forget the T-algebra structure):

$$U : T\text{-}\mathbf{Alg} \to \mathbf{Sets}.$$

This functor is easily seen to create all limits, as is the case for algebras for any theory T. So in particular, T-**Alg** is complete and U preserves limits. Thus in order to apply the AFT, we just need to check the solution set condition.

To that end, let X be any set and take any function

$$h : X \to M.$$

The image $h(X) \subseteq M$ generates a sub-T-model of (M, a, f) as follows. Define the set "generated by $h(X)$" to be

$$H = \langle h(X) \rangle = \{ f^n(z) \mid n \in \mathbb{N}, z = a \text{ or } z = h(x) \text{ for some } x \in X \}. \qquad (9.10)$$

Then $a \in H$, and f restricts to H to give a function $f' : H \to H$. Moreover, the inclusion $i : H \hookrightarrow M$ is clearly a T-algebra homomorphism

$$\begin{tikzcd} & H \arrow[r, "f'"] \arrow[d, hook, "i"'] & H \arrow[d, hook, "i"] \\ 1 \arrow[ur, "a"] \arrow[r, "a"'] & M \arrow[r, "f"'] & M \end{tikzcd}$$

Furthermore, since $h(X) \subseteq H$ there is a factorization h' of h, as indicated in the following diagram:

$$\begin{tikzcd} X \arrow[r, "h'"] \arrow[dr, "h"'] & H \arrow[d, hook, "i"] \\ & M \end{tikzcd} \qquad (9.11)$$

Now observe that, given X, the cardinality $|H|$ is *bounded*, that is, for a sufficiently large κ independent of h and M, we have

$$|H| \leq \kappa.$$

Indeed, inspecting (9.10), we can take $\kappa = |\mathbb{N}| \times (1 + |X|)$.

To find a solution set for X, let us now take one representative N of each isomorphism class of T-algebras with cardinality at most κ. The set of all such algebras N is then a solution set for X and U. Indeed, as we just showed, any function $h : X \to M$ factors as in (9.11) through an element of this set (namely an isomorphic copy N of H). By the AFT, there thus exists a free functor,

$$F : \mathbf{Sets} \to T\text{-}\mathbf{Alg}.$$

A precisely analogous argument works for any equational theory T.

Finally, let us consider the particular free model $F(\emptyset)$ in T-**Alg**. Since left adjoints preserve colimits, this is an initial object. It follows that $F(\emptyset)$ is a *natural numbers object*, in the following sense.

Definition 9.40. Let **C** be a category with a terminal object 1. A natural numbers object (NNO) in **C** is a structure of the form

$$1 \xrightarrow{0} N \xrightarrow{s} N$$

which is initial among all such structures. Precisely, given any $1 \xrightarrow{a} X \xrightarrow{f} X$ in **C**, there is a unique arrow $\phi : N \to X$ such that the following commutes:

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "\phi"'] & N \arrow[d, "\phi"] \\ 1 \arrow[ur, "0"] \arrow[dr, "a"'] & & \\ & X \arrow[r, "f"'] & X \end{tikzcd}$$

In other words, given any object X, a "starting point" $a \in X$ and an operation $x \mapsto f(x)$ on X, we can build up a unique $\phi : N \to X$ *recursively* by the equations:

$$\phi(0) = a$$
$$\phi(s(n)) = f(\phi(n)) \quad \text{for all } n \in N$$

Thus, the UMP of an NNO says precisely that such an object supports recursive definitions. It is easy to show that the set $\mathbb{N}$ of natural numbers with the canonical structure of 0 and the "successor function" $s(n) = n + 1$ is an NNO, and thus, by the UMP any NNO in **Sets** is isomorphic to it. The characterization of $\mathbb{N}$ in terms of the UMP of recursive definitions is therefore equivalent to the usual logical definition using the Peano axioms in **Sets**. But note that the notion of an NNO (which is due to F.W. Lawvere) also makes sense in many categories where the Peano axioms do not make any sense, since the latter involve logical operations like quantifiers.

Let us consider some simple examples of recursively defined functions using this UMP.

Example 9.41. 1. Let $(N, 0, s)$ be an NNO in any category **C**. Take any point $a : 1 \to N$, and consider the new structure:

$$1 \xrightarrow{a} N \xrightarrow{s} N$$

Then by the universal property of the NNO, there is a unique morphism $f_a : N \to N$ such that the following commutes:

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "f_a"'] & N \arrow[d, "f_a"] \\ 1 \arrow[ur, "0"] \arrow[dr, "a"'] & & \\ & N \arrow[r, "s"'] & N \end{tikzcd}$$

Thus we have the following "recursion equations":

$$f_a(0) = a$$
$$f_a(s(n)) = s(f_a(n))$$

If we write $f_a(n) = a + n$, then the above equations become the familiar recursive definition of addition:

$$a + 0 = a$$
$$a + (sn) = s(a + n)$$

2. Now take this arrow $a + (-) : N \to N$ together with $0 : 1 \to N$ to get another arrow $g_a : N \to N$, which is the unique one making the following commute:

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "g_a"'] & N \arrow[d, "g_a"] \\ 1 \arrow[ur, "0"] \arrow[dr, "0"'] & & \\ & N \arrow[r, "a+(-)"'] & N \end{tikzcd}$$

We then have the recursion equations:

$$g_a(0) = 0$$
$$g_a(sn) = a + g_a(n)$$

So, writing $g_a(n) = a \cdot n$, the above equations become the familiar recursive definition of multiplication:

$$a \cdot 0 = 0$$
$$a \cdot (sn) = a + a \cdot n$$

3. For an example of a different sort, suppose we have a (small) category $\mathbf{C}$ and an endofunctor $F : \mathbf{C} \to \mathbf{C}$. Then there is a structure

$$1 \xrightarrow{\text{id}} \mathbf{C}^{\mathbf{C}} \xrightarrow{F^{\mathbf{C}}} \mathbf{C}^{\mathbf{C}}$$

where $\text{id} : 1 \to \mathbf{C}^{\mathbf{C}}$ is the transpose of the identity $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ (composed with the iso projection $1 \times \mathbf{C} \cong \mathbf{C}$). We therefore have a unique functor $f : \mathbb{N} \to \mathbf{C}^{\mathbf{C}}$ making the following diagram commute (we use the easy fact, which the reader should check, that the discrete category $\mathbb{N}$ is an NNO in **Cat**):

$$\begin{tikzcd}
 & \mathbb{N} \arrow[r, "s"] \arrow[d, "f"'] & \mathbb{N} \arrow[d, "f"] \\
1 \arrow[ur, "0"] \arrow[dr, "\mathrm{id}"'] & & \\
 & \mathbf{C}^{\mathbf{C}} \arrow[r, "F^{\mathbf{C}}"'] & \mathbf{C}^{\mathbf{C}}
\end{tikzcd}$$

Transposing gives the commutative diagram

$$\begin{tikzcd}
1 \times \mathbf{C} \arrow[r, "0 \times 1_{\mathbf{C}}"] \arrow[d, "\cong"'] & \mathbb{N} \times \mathbf{C} \arrow[r, "s \times 1_{\mathbf{C}}"] \arrow[d, "\bar{f}"'] & \mathbb{N} \times \mathbf{C} \arrow[d, "\bar{f}"] \\
\mathbf{C} \arrow[r, "\mathrm{id}"'] & \mathbf{C} \arrow[r, "F"'] & \mathbf{C}
\end{tikzcd}$$

from which we can read off the recursion equations:

$$\bar{f}(0, C) = C$$
$$\bar{f}(sn, C) = F(\bar{f}(n, C))$$

It follows that $\bar{f}(n, C) = F^{(n)}(C)$, that is, $f(n)$ is the nth iterate of the functor $F : \mathbf{C} \to \mathbf{C}$.

9.9 Exercises

1. Complete the proof that the "Hom-set" definition of adjunction is equivalent to the preliminary one by showing that the specification of the unit $\eta_C : C \to UFC$ as $\eta_C = \phi(1_{FC})$ really is a natural transformation.
2. Show that every monoid M admits a surjection from a free monoid $F(X) \to M$, by considering the counit of the *free* $\dashv$ *forgetful* adjunction.
3. What is the unit of the product $\dashv$ exponential adjunction (say, in **Sets**)?
4. Let 2 be any two-element set and consider the "diagonal functor"

$$\Delta : \mathbf{C} \to \mathbf{C}^2$$

   for any category **C**, that is, the exponential transpose of the first product projection

$$\mathbf{C} \times 2 \to \mathbf{C}.$$

   Show that Δ has a right (resp. left) adjoint if and only if **C** has binary products (resp. coproducts).

Now let $\mathbf{C} = \mathbf{Sets}$ and replace 2 with an arbitrary small category $\mathbf{J}$. Determine both left and right adjoints for $\Delta : \mathbf{Sets} \to \mathbf{Sets}^{\mathbf{J}}$. (Hint: **Sets** is complete and cocomplete.)

5. Let $\mathbf{C}$ be cartesian closed and suppose moreover that $\mathbf{C}$ has all finite colimits. Show that $\mathbf{C}$ is not only distributive,

$$(A + B) \times C \cong (A \times C) + (B \times C)$$

but that also $(-) \times C$ preserves coequalizers. Dually, show that $(-)^C$ preserves products and equalizers.

6. Any category $\mathbf{C}$ determines a preorder $P(\mathbf{C})$ by setting: $A \leq B$ if and only if there is an arrow $A \to B$. Show that the functor P is (left? right?) adjoint to the evident inclusion functor of preorders into categories. Does the inclusion also have an adjoint on the other side?

7. Show that there is a string of four adjoints between **Cat** and **Sets**,

$$V \dashv F \dashv U \dashv R$$

where $U : \mathbf{Cat} \to \mathbf{Sets}$ is the forgetful functor to the set of objects $U(\mathbf{C}) = \mathbf{C}_0$. (Hint: for V, consider the "connected components" of a category.)

8. Given a function $f : A \to B$ between sets, verify that the direct image operation $\mathrm{im}(f) : P(A) \to P(B)$ is left adjoint to the inverse image $f^{-1} : P(B) \to P(A)$. Determine the dual image $f_* : P(A) \to P(B)$ and show that it is right adjoint to f^{-1}.

9. Show that the contravariant powerset functor $\mathcal{P} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ is self-adjoint.

10. Given an object C in a category $\mathbf{C}$ under what conditions does the evident forgetful functor from the slice category $\mathbf{C}/C$

$$U : \mathbf{C}/C \to \mathbf{C}$$

have a right adjoint? What about a left adjoint?

11. (a) A *coHeyting algebra* is a poset P such that P^{op} is a Heyting algebra. Determine the coHeyting implication operation a/b in a lattice L by adjointness (with respect to joins), and show that any Boolean algebra is a coHeyting algebra by explicitly defining this operation a/b in terms of the usual Boolean ones.

    (b) In a coHeyting algebra, there are operations of *coHeyting negation* $\sim p = 1/p$ and *coHeyting boundary* $\partial p = p \wedge \sim p$. State the logical rules of inference for these operations.

    (c) A *biHeyting algebra* is a lattice that is both Heyting and coHeyting. Give an example of a biHeyting algebra that is not Boolean. (Hint: consider the lower sets in a poset.)

12. Let $\mathcal{P}$ be the category of propositions (i.e., the preorder category associated to the propositional calculus, say with countably many propositional

variables $p, q, r, \ldots$, and a unique arrow $p \to q$ if and only if $p \vdash q$). Show that for any fixed object p, there is a functor

$$- \wedge p : \mathcal{P} \to \mathcal{P}$$

and that this functor has a right adjoint. What is the counit of the adjunction? (When) does $- \wedge p$ have a left adjoint?

13. (a) Given any set I, explicitly describe the Yoneda embedding $y : I \to \mathbf{Sets}^I$ of I into the category $\mathbf{Sets}^I$ of I-indexed sets.

(b) Given any function $f : J \to I$ from another set J, prove directly that the following diagram commutes up to natural isomorphism.

$$\begin{tikzcd}
\mathbf{Sets}^I \arrow[r, "f_!"] & \mathbf{Sets}^I \\
J \arrow[u, "\mathrm{y}_J"] \arrow[r, "f"'] & I \arrow[u, "\mathrm{y}_I"']
\end{tikzcd}$$

(c) Describe the result of composing the Yoneda embedding with the equivalence,

$$\mathbf{Sets}^I \simeq \mathbf{Sets}/I.$$

(d) What does the commutativity of the above "change of base" square mean in terms of the categories $\mathbf{Sets}/I$ and $\mathbf{Sets}/J$?

(e) Consider the inclusion functor $i : P(I) \to \mathbf{Sets}/I$ that takes a subset $U \subseteq I$ to its inclusion function $i(U) : U \to I$. Show that this is a functor and that it has a left adjoint

$$\sigma : \mathbf{Sets}/I \longrightarrow P(I).$$

(f) (Lawvere's Hyperdoctrine Diagram) In $\mathbf{Sets}$, given any function $f : I \to J$, consider the following diagram of functors:

$$\begin{tikzcd}
\mathbf{Sets}/I \arrow[r, shift left=2, "\Pi_f"] \arrow[r, shift right=2, "\Sigma_f"'] \arrow[d, shift right, "\sigma_I"'] & \mathbf{Sets}/J \arrow[l, "f^*" description] \arrow[d, shift right, "\sigma_J"'] \\
P(I) \arrow[r, shift left=2, "\forall_f"] \arrow[r, shift right=2, "\exists_f"'] \arrow[u, shift right, "i_I"'] & P(J) \arrow[l, "f^{-1}" description] \arrow[u, shift right, "i_J"']
\end{tikzcd}$$

There are adjunctions $\sigma \dashv i$ (for both I and J), as well as $\Sigma_f \dashv f^* \dashv \Pi_f$ and $\exists_f \dashv f^{-1} \dashv \forall_f$, where $f^* : \mathbf{Sets}/J \to \mathbf{Sets}/I$ is pullback and $f^{-1} : P(J) \to P(I)$ is inverse image.

Consider which of the many possible squares commute.

14. Complete the proof in the text that every slice of a category of presheaves is again a category of presheaves: for any small category $\mathbf{C}$ and presheaf $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$,
$$\mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}/P \;\simeq\; \mathbf{Sets}^{(\int_{\mathbf{C}} P)^{\mathrm{op}}}.$$
15. Let $\mathbf{C}$ be a complete category and $U : \mathbf{C} \to \mathbf{X}$ a continuous functor. Show that for any object $X \in \mathbf{X}$, the comma category $(X|U)$ is also complete.
16. Use the adjoint functor theorem to prove the following facts, which were shown by explicit constructions in Chapter 1:
    (a) Free monoids on sets exist.
    (b) Free categories on graphs exist.
17. Let $1 \xrightarrow{0} N \xrightarrow{s} N$ be an NNO in a cartesian closed category.
    (a) Show how to define the exponentiation operation m^n as an arrow $N \times N \to N$.
    (b) Do the same for the factorial function $n!$.
18. (Freyd's characterization of NNOs) Let $1 \xrightarrow{0} N \xrightarrow{s} N$ be an NNO in $\mathbf{Sets}$ (for your information, however, the following holds in any topos).
    (a) Prove that the following is a coproduct diagram:
    $$1 \xrightarrow{\;0\;} N \xleftarrow{\;s\;} N$$
    So $N \cong 1 + N$.
    (b) Prove that the following is a coequalizer:
    $$\begin{tikzcd} N \arrow[r, shift left, "s"] \arrow[r, shift right, "1_N"'] & N \arrow[r] & 1 \end{tikzcd}$$
    (a) Show that any structure $1 \xrightarrow{0} N \xrightarrow{s} N$ satisfying the foregoing two conditions is an NNO.
19. Recall (from Chapter 1) the category $\mathbf{Rel}$ of relations (between sets), with arrows $R : A \to B$ being the relations $R \subseteq A \times B$ in $\mathbf{Sets}$. Taking the graph of a function $f : A \to B$ gives a relation $\Gamma(f) = \{(a, f(a)) \mid a \in A\} \subseteq A \times B$, and this assignment determines a functor $\Gamma : \mathbf{Sets} \to \mathbf{Rel}$. Show that Γ has a right adjoint. Compute the unit and counit of the adjunction.