10

MONADS AND ALGEBRAS

In Chapter 9, the adjoint functor theorem was seen to imply that the category of algebras for an equational theory T always has a "free T-algebra" functor, left adjoint to the forgetful functor into **Sets**. This adjunction describes the notion of a T-algebra in a way that is independent of the specific syntactic description given by the theory T, the operations and equations of which are rather like a particular *presentation* of that notion. In a certain sense that we are about to make precise, it turns out that *every* adjunction describes, in a "syntax invariant" way, a notion of an "algebra" for an abstract "equational theory."

Toward this end, we begin with yet a third characterization of adjunctions. This one has the virtue of being entirely equational.

10.1 The triangle identities

Suppose we are given an adjunction,

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : U.$$

with unit and counit,

$$\eta : 1_{\mathbf{C}} \to UF$$
$$\epsilon : FU \to 1_{\mathbf{D}}.$$

We can take any $f : FC \to D$ to

$$\phi(f) = U(f) \circ \eta_C : C \to UD,$$

and for any $g : C \to UD$, we have

$$\phi^{-1}(g) = \epsilon_D \circ F(g) : FC \to D.$$

This we know gives the isomorphism

$$\mathrm{Hom}_{\mathbf{D}}(FC, D) \cong_\phi \mathrm{Hom}_{\mathbf{C}}(C, UD).$$

Now put $1_{UD} : UD \to UD$ in place of $g : C \to UD$ in the foregoing consideration. We know that $\phi^{-1}(1_{UD}) = \epsilon_D$, and so

$$\begin{aligned} 1_{UD} &= \phi(\epsilon_D) \\ &= U(\epsilon_D) \circ \eta_{UD}. \end{aligned}$$

And similarly, $\phi(1_{FC}) = \eta_C$, so

$$\begin{aligned} 1_{FC} &= \phi^{-1}(\eta_C) \\ &= \epsilon_{FC} \circ F(\eta_C). \end{aligned}$$

Thus, we have shown that the following two diagrams commute:

$$\begin{tikzcd} UD \arrow[rr, "1_{UD}"] \arrow[dr, "\eta_{UD}"'] & & UD \\ & UFUD \arrow[ur, "U\epsilon_D"'] & \end{tikzcd}$$

$$\begin{tikzcd} FC \arrow[rr, "1_{FC}"] \arrow[dr, "F\eta_C"'] & & FC \\ & FUFC \arrow[ur, "\epsilon_{FC}"'] & \end{tikzcd}$$

Indeed, one has the following equations of natural transformations:

$$U\epsilon \circ \eta_U = 1_U \tag{10.1}$$

$$\epsilon_F \circ F\eta = 1_F \tag{10.2}$$

These are called the "triangle identities."

Proposition 10.1. *Given categories, functors, and natural transformations*

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : \mathbf{U}$$

$$\begin{aligned} \eta &: 1_{\mathbf{C}} \to U \circ F \\ \epsilon &: F \circ U \to 1_{\mathbf{D}} \end{aligned}$$

one has $F \dashv U$ with unit η and counit ϵ iff the triangle identities (10.1) *and* (10.2) *hold.*

Proof. We have already shown one direction. For the other, we just need a natural isomorphism,

$$\phi : \mathrm{Hom}_{\mathbf{D}}(FC, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, UD).$$

As earlier, we put

$$\begin{aligned} \phi(f : FC \to D) &= U(f) \circ \eta_C \\ \vartheta(g : C \to UD) &= \epsilon_D \circ F(g). \end{aligned}$$

Then we check that these are mutually inverse:

$$\begin{aligned}
\phi(\vartheta(g)) &= \phi(\epsilon_D \circ F(g)) \\
&= U(\epsilon_D) \circ UF(g) \circ \eta_C \\
&= U(\epsilon_D) \circ \eta_{UD} \circ g && \eta \text{ natural} \\
&= g && (10.1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\vartheta(\phi(f)) &= \vartheta(U(f) \circ \eta_C) \\
&= \epsilon_D \circ FU(f) \circ F\eta_C \\
&= f \circ \epsilon_{FC} \circ F\eta_C && \epsilon \text{ natural} \\
&= f && (10.2)
\end{aligned}$$

Moreover, this isomorphism is easily seen to be natural. □

The triangle identities have the virtue of being entirely "algebraic"—no quantifiers, limits, Hom-sets, infinite conditions, etc. Thus, anything defined by adjoints such as free groups, product spaces, quantifiers, ... can be defined *equationally*. This is not only a matter of conceptual simplification; it also has important consequences for the existence and properties of the structures that are so determined.

10.2 Monads and adjoints

Next consider an adjunction $F \dashv U$ and the composite functor

$$U \circ F : \mathbf{C} \to \mathbf{D} \to \mathbf{C}.$$

Given *any* category $\mathbf{C}$ and endofunctor

$$T : \mathbf{C} \to \mathbf{C}$$

one can ask the following:

Question: When is $T = U \circ F$ for some adjoint functors $F \dashv U$ to and from another category $\mathbf{D}$?

Thus, we seek necessary and sufficient conditions on the given endofunctor $T : \mathbf{C} \to \mathbf{C}$ for recovering a category $\mathbf{D}$ and adjunction $F \dashv U$. Of course, not every T arises so, and we see that even if $T = U \circ F$ for *some* $\mathbf{D}$ and $F \dashv U$, we cannot always recover *that* adjunction. Thus, a better way to ask the question would be, given an adjunction what sort of "trace" does it leave on a category and can we recover the adjunction from this?

First, suppose we have $\mathbf{D}$ and $F \dashv U$ and T is the composite functor $T = U \circ F$. We then have a natural transformation,

$$\eta : 1 \to T.$$

And from the counit ϵ at FC,

$$\epsilon_{FC} : FUFC \to FC$$

we have $U\epsilon_{FC} : UFUFC \to UFC$, which we call,

$$\mu : T^2 \to T.$$

In general, then, as a first step toward answering our question, if T arises from an adjunction, then it should have such a structure $\eta : 1 \to T$ and $\mu : T^2 \to T$.

Now, what can be said about the structure (T, η, μ)? Actually, quite a bit! Indeed, the triangle equalities give us the following commutative diagrams:

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

$$\mu \circ \mu_T = \mu \circ T\mu \tag{10.3}$$

$$\begin{tikzcd}
T \arrow[r, "\eta_T"] \arrow[dr, "="'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "="] \\
& T &
\end{tikzcd}$$

$$\mu \circ \eta_T = 1_T = \mu \circ T\eta \tag{10.4}$$

To prove the first one, for any $f : X \to Y$ in $\mathbf{D}$, the following square in $\mathbf{C}$ commutes, just since ϵ is natural:

$$\begin{tikzcd}
FUX \arrow[r, "FUf"] \arrow[d, "\epsilon_X"'] & FUY \arrow[d, "\epsilon_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Now take $X = FUY$ and $f = \epsilon_Y$ to get the following:

$$\begin{tikzcd}
FUFUY \arrow[r, "FU\epsilon_Y"] \arrow[d, "\epsilon FUY"'] & FUY \arrow[d, "\epsilon_Y"] \\
FUY \arrow[r, "\epsilon_Y"'] & Y
\end{tikzcd}$$

Putting FC for Y and applying U, therefore, gives this

$$\begin{tikzcd}
UFUFUFC \arrow[r, "UFU\epsilon_{FC}"] \arrow[d, "U\epsilon FUFC"'] & UFUFC \arrow[d, "U\epsilon_{FC}"] \\
UFUFC \arrow[r, "U\epsilon_{FC}"'] & UFC
\end{tikzcd}$$

which has the required form (10.3). The equations (10.4) in the form

$$\begin{tikzcd}
UFC \arrow[r, "\eta_{UFC}"] \arrow[dr, "="'] & UFUFC \arrow[d, "U\epsilon_{FC}"] & UFC \arrow[l, "UF\eta_C"'] \arrow[dl, "="] \\
& UFC &
\end{tikzcd}$$

are simply the triangle identities, once taken at FC, and once under U. We record this data in the following definition.

Definition 10.2. A *monad* on a category $\mathbf{C}$ consists of an endofunctor $T : \mathbf{C} \to \mathbf{C}$, and natural transformations $\eta : 1_{\mathbf{C}} \to T$, and $\mu : T^2 \to T$ satisfying the two commutative diagrams above, that is,

$$\mu \circ \mu_T = \mu \circ T\mu \tag{10.5}$$

$$\mu \circ \eta_T = 1 = \mu \circ T\eta. \tag{10.6}$$

Note the formal analogy to the definition of a monoid. In fact, a monad is exactly the same thing a *monoidal* monoid in the monoidal category $\mathbf{C}^{\mathbf{C}}$ with composition as the monoidal product, $G \otimes F = G \circ F$ (cf. section 7.8). For this reason, the equations (10.5) and (10.6) above are called the *associativity* and *unit* laws, respectively.

We have now shown the following proposition.

Proposition 10.3. *Every adjoint pair $F \dashv U$ with $U : \mathbf{D} \to \mathbf{C}$, unit $\eta : UF \to 1_{\mathbf{C}}$ and counit $\epsilon : 1_{\mathbf{D}} \to FU$ gives rise to a monad (T, η, μ) on $\mathbf{C}$ with*

$$T = U \circ F : \mathbf{C} \to \mathbf{C}$$

$$\eta : 1 \to T \quad \textit{the unit}$$

$$\mu = U\epsilon_F : T^2 \to T.$$

Example 10.4. Let P be a poset. A monad on P is a monotone function $T : P \to P$ with $x \leq Tx$ and $T^2x \leq Tx$. But then $T^2 = T$, that is, T is *idempotent.* Such a T, that is both inflationary and idempotent, is sometimes called a *closure operation* and written $Tp = \bar{p}$, since it acts like the closure operation on the subsets of a topological space. The "possibility operator" $\diamond p$ in modal logic is another example.

In the poset case, we can easily recover an adjunction from the monad. First, let $K = \mathrm{im}(T)(P)$ (the fixed points of T), and let $i : K \to P$ be the inclusion. Then let t be the factorization of T through K, as indicated in

$$\begin{tikzcd} P \arrow[rr, "T"] \arrow[dr, "t"'] & & P \\ & K \arrow[ur, "i"'] & \end{tikzcd}$$

Observe that since $TTp = Tp$, for any element $k \in K$ we then have, for some $p \in P$, the equation $itik = ititp = itp = ik$, whence $tik = k$ since i is monic. We therefore have

$$p \leq ik \quad \text{implies} \quad tp \leq tik = k$$

$$tp \leq k \quad \text{implies} \quad p \leq itp \leq ik$$

So indeed $t \dashv i$.

Example 10.5. Consider the covariant powerset functor

$$\mathcal{P} : \mathbf{Sets} \to \mathbf{Sets}$$

which takes each function $f : X \to Y$ to the image mapping $\mathrm{im}(f) : P(X) \to P(Y)$. Let $\eta_X : X \to \mathcal{P}(X)$ be the singleton operation

$$\eta_X(x) = \{x\}$$

and let $\mu_X : \mathcal{P}\mathcal{P}(X) \to \mathcal{P}(X)$ be the union operation

$$\mu_X(\alpha) = \bigcup \alpha.$$

The reader should verify as an exercise that these operations are in fact natural in X and that this defines a monad $(\mathcal{P}, \{-\}, \bigcup)$ on $\mathbf{Sets}$.

As we see in these examples, monads can, and often do, arise without coming from evident adjunctions. In fact, the notion of a monad originally did occur independently of adjunctions! Monads were originally also known by the names "triples" and sometimes "standard constructions." Despite their independent origin, however, our question "when does an endofunctor T arise from an adjunction?" has the simple answer: just if it is the functor part of a monad.

10.3 Algebras for a monad

Proposition 10.6. *Every monad arises from an adjunction. More precisely, given a monad (T, η, μ) on the category $\mathbf{C}$, there exists a category $\mathbf{D}$ and an adjunction $F \dashv U$, $\eta : 1 \to UF$, $\epsilon : FU \to 1$ with $U : \mathbf{D} \to \mathbf{C}$ such that*

$$\begin{aligned} T &= U \circ F \\ \eta &= \eta \quad \textit{(the unit)} \\ \mu &= U\epsilon_F. \end{aligned}$$

Proof. We first define the important category $\mathbf{C}^T$ called the *Eilenberg–Moore category of T*. This will be our "$\mathbf{D}$." Then we need suitable functors

$$F : \mathbf{C} \rightleftarrows \mathbf{C}^T : U.$$

And, finally, we need natural transformations $\eta : 1 \to UF$ and $\epsilon : FU \to 1$ satisfying the triangle identities.

To begin, $\mathbf{C}^T$ has as *objects* the "T-algebras," which are pairs (A, α) of the form $\alpha : TA \to A$ in $\mathbf{C}$, such that

$$1_A = \alpha \circ \eta_A \quad \text{and} \quad \alpha \circ \mu_A = \alpha \circ T\alpha. \tag{10.7}$$

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

A *morphism* of T-algebras,

$$h : (A, \alpha) \to (B, \beta)$$

is simply an arrow $h : A \to B$ in $\mathbf{C}$, such that,

$$h \circ \alpha = \beta \circ T(h)$$

as indicated in the following diagram:

$$\begin{tikzcd}
TA \arrow[r, "Th"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

It is obvious that $\mathbf{C}^T$ is a category with the expected composites and identities coming from $\mathbf{C}$, and that T is a functor.

Now define the functors,

$$U : \mathbf{C}^T \to \mathbf{C}$$
$$U(A, \alpha) = A$$

and

$$F : \mathbf{C} \to \mathbf{C}^T$$
$$FC = (TC, \mu_C).$$

We need to check that (TC, μ_C) is a T-algebra. The equations (10.7) for T-algebras in this case become

$$\begin{tikzcd}
TC \arrow[r, "\eta_{TC}"] \arrow[dr, "1"'] & T^2C \arrow[d, "\mu_C"] \\
& TC
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3C \arrow[r, "T\mu_C"] \arrow[d, "\mu_{TC}"'] & T^2C \arrow[d, "\mu"] \\
T^2C \arrow[r, "\mu"'] & TC
\end{tikzcd}$$

But these come directly from the definition of a monad.

To see that F is a functor, given any $h : C \to D$ in $\mathbf{C}$, we have

$$\begin{tikzcd}
T^2C \arrow[r, "T^2h"] \arrow[d, "\mu_C"'] & T^2D \arrow[d, "\mu_D"] \\
TC \arrow[r, "Th"'] & TD
\end{tikzcd}$$

since μ is natural. But this is a T-algebra homomorphism $FC \to FD$, so we can put

$$Fh = Th : TC \to TD$$

to get an arrow in $\mathbf{C}^T$.

Now we have defined the category $\mathbf{C}^T$ and the functors

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "F"] & \mathbf{C^T} \arrow[l, shift left, "U"] \end{tikzcd}$$

and we want to show that $F \dashv U$. Next, we need the unit and counit:

$$\begin{aligned} \bar{\eta} &: 1_{\mathbf{C}} \to U \circ F \\ \epsilon &: F \circ U \to 1_{\mathbf{C}^T} \end{aligned}$$

Given $C \in \mathbf{C}$, we have

$$UF(C) = U(TC, \mu_C) = TC.$$

So we can take $\bar{\eta} = \eta : 1_{\mathbf{C}} \to U \circ F$, as required.

Given $(A, \alpha) \in \mathbf{C}^T$,

$$FU(A, \alpha) = (TA, \mu_A)$$

and the definition of a T-algebra makes the following diagram commute:

$$\begin{tikzcd} T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

But this is a morphism $\epsilon_{(A,\alpha)} : (TA, \mu_A) \to (A, \alpha)$ in $\mathbf{C}^T$. Thus we are setting

$$\epsilon_{(A,\alpha)} = \alpha.$$

And ϵ is *natural* by the definition of a morphism of T-algebras, as follows. Given any $h : (A, \alpha) \to (B, \beta)$, we need to show

$$h \circ \epsilon_{(A,\alpha)} = \epsilon_{(B,\beta)} \circ Th.$$

But by the definition of ϵ, that is, $h \circ \alpha = \beta \circ Th$, which holds since h is a T-algebra homomorphism.

Finally, the triangle identities now read as follows:

1. For (A, α), a T-algebra

$$\begin{tikzcd} U(A,\alpha) \arrow[rr] \arrow[dr, "\eta_{U(A,\alpha)}"'] & & U(A,\alpha) \\ & UFU(A,\alpha) \arrow[ur, "U\epsilon_{(A,\alpha)}"'] & \end{tikzcd}$$

which amounts to

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\eta_A"'] & & A \\ & TA \arrow[ur, "\alpha"'] & \end{tikzcd}$$

which holds since (A, α) is T-algebra.

2. For any $C \in \mathbf{C}$

$$\begin{tikzcd} FC \arrow[rr] \arrow[dr, "F\eta_C"'] & & FC \\ & FUFC \arrow[ur, "\epsilon_{FC}"'] & \end{tikzcd}$$

which is

$$\begin{tikzcd} TC \arrow[rr] \arrow[dr, "T\eta_C"'] & & TC \\ & T^2C \arrow[ur, "\mu_C"'] & \end{tikzcd}$$

which holds by one of the unit laws for T.

Finally, note that we indeed have

$$T = U \circ F$$
$$\eta = \text{ unit of } F \dashv U.$$

And for the multiplication,

$$\bar{\mu} = U\epsilon F$$

we have, for any $C \in \mathbf{C}$,

$$\bar{\mu}_C = U\epsilon_{FC} = U\epsilon_{(TC,\mu_C)} = U\mu_C = \mu_C.$$

So $\bar{\mu} = \mu$ and we are done; the adjunction $F \dashv U$ via η and ϵ gives rise to the monad (T, η, μ). $\square$

Example 10.7. Take the free monoid adjunction,

$$F : \mathbf{Sets} \rightleftarrows \mathbf{Mon} : \mathbf{U}.$$

The monad on **Sets** is then $T : \mathbf{Sets} \to \mathbf{Sets}$, where for any set X, $T(X) = UF(X) =$ "strings over X." The unit $\eta : X \to TX$ is the usual "string of length one" operation, but what is the multiplication?

$$\mu : T^2X \to TX$$

Here T^2X is the set of strings of strings,

$$[[x_{11}, \ldots, x_{1n}], [x_{21}, \ldots, x_{2n}], \ldots, [x_{m1}, \ldots, x_{mn}]].$$

And μ of such a string of strings is the string of their elements,

$$\mu([[x_{11}, \ldots, x_{1n}], [x_{21}, \ldots, x_{2n}], \ldots, [x_{m1}, \ldots, x_{mn}]]) = [x_{11}, \ldots, x_{mn}].$$

Now, what is a T-algebra in this case? By the equations for a T-algebra, it is a map,

$$\alpha : TA \to A$$

from strings over A to elements of A, such that

$$\alpha[a] = a$$

and

$$\alpha(\mu([[\ldots], [\ldots], \ldots, [\ldots]])) = \alpha(\alpha[\ldots], \alpha[\ldots], \ldots, \alpha[\ldots]).$$

If we start with a monoid, then we can get a T-algebra $\alpha : TM \to M$ by

$$\alpha[m_1, \ldots, m_n] = m_1 \cdot \ldots \cdot m_n.$$

This clearly satisfies the required conditions. Observe that we can even recover the monoid structure from m by $u = m(-)$ for the unit and $x \cdot y = m(x, y)$ for the multiplication. Indeed, *every* T-algebra is of this form for a *unique* monoid (exercise!).

We have now given constructions back and forth between adjunctions and monads. And we know that if we start with a monad $T : \mathbf{C} \to \mathbf{C}$, and then take the adjunction,

$$F^T : \mathbf{C} \rightleftarrows \mathbf{C}^T : U^T$$

then we can get the monad back by $T = U^T \circ F^T$. Thus, in particular, every monoid arises from *some* adjunction. But are $\mathbf{C}^T, U^T, F^T$ unique with this property?

In general, the answer is *no*. There may be many different categories $\mathbf{D}$ and adjunctions $F \dashv U : \mathbf{D} \to \mathbf{C}$, all giving the same monad on $\mathbf{C}$. We have used the Eilenberg–Moore category $\mathbf{C}^T$, but there is also something called the "Kleisli category," which is in general different from $\mathbf{C}^T$, but also has an adjoint pair to $\mathbf{C}$ giving rise to the same monad (see the exercises).

If we start with an adjunction $F \dashv U$ and construct $\mathbf{C}^T$ for $T = U \circ F$, we then get a comparison functor $\Phi : \mathbf{D} \to \mathbf{C}^T$, with

$$U^T \circ \Phi \cong U$$

$$\Phi \circ F = F^T$$

$$\begin{tikzcd}
\mathbf{D} \arrow[rr, "\Phi"] \arrow[dr, shift left, "U"] & & \mathbf{C}^T \arrow[dl, shift left, "U^T"] \\
& \mathbf{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^T"] &
\end{tikzcd}$$

In fact, Φ is *unique* with this property. A functor $U : \mathbf{D} \to \mathbf{C}$ is called *monadic* if it has a left adjoint $F \dashv U$, such that this comparison functor is an equivalence of categories,

$$\mathbf{D} \xrightarrow[\cong]{\Phi} \mathbf{C}^T$$

for $T = UF$.

Typical examples of monadic forgetful functors $U : \mathbf{C} \to \mathbf{Sets}$ are those from the "algebraic" categories arising as models for equational theories, like monoids, groups, rings, etc. Indeed, one can reasonably take monadicity as the *definition* of being "algebraic."

An example of a right adjoint that is *not* monadic is the forgetful functor from posets,

$$U : \mathbf{Pos} \to \mathbf{Sets}.$$

Its left adjoint F is the discrete poset functor. For any set X, therefore, one has as the unit the identity function $X = UF(X)$. The reader can easily show that the Eilenberg–Moore category for $T = 1_{\mathbf{Sets}}$ is then just $\mathbf{Sets}$ itself.

10.4 Comonads and coalgebras

By definition, a *comonad* on a category $\mathbf{C}$ is a monad on $\mathbf{C}^{\mathrm{op}}$. Explicitly, this consists of an endofunctor $G : \mathbf{C} \to \mathbf{C}$ and natural transformations,

$$\begin{array}{ll} \epsilon : G \to 1 & \text{the counit} \\ \delta : G \to G^2 & \text{comultiplication} \end{array}$$

satisfying the duals of the equations for a monad, namely

$$\begin{aligned} \delta_G \circ \delta &= G\delta \circ \delta \\ \epsilon_G \circ \delta &= 1_G = G\epsilon \circ \delta. \end{aligned}$$

We leave it as an exercise in duality to verify that an adjoint pair $F \dashv U$ with $U : \mathbf{D} \to \mathbf{C}$ and $F : \mathbf{C} \to \mathbf{D}$ and $\eta : 1_{\mathbf{C}} \to UF$ and $\epsilon : FU \to 1_{\mathbf{D}}$ gives rise to a comonad (G, ϵ, δ) on $\mathbf{D}$, where

$$\begin{aligned} G &= F \circ U : \mathbf{D} \to \mathbf{D} \\ \epsilon &: G \to 1 \\ \delta &= F\eta_U : G \to G^2. \end{aligned}$$

The notions of coalgebra for a comonad, and of a comonadic functor, are of course also precisely dual to the corresponding ones for monads. Why do we even bother to study these notions separately, rather than just considering their duals? As in other examples of duality, there are actually two distinct reasons:

1. We may be interested in a particular category with special properties not had by its dual. A comonad on $\mathbf{Sets}^{\mathbf{C}}$ is of course a monad on $(\mathbf{Sets}^{\mathbf{C}})^{\mathrm{op}}$, but as we now know, $\mathbf{Sets}^{\mathbf{C}}$ has many special properties that its dual does not have (e.g., it is a topos!). So we can profitably consider the notion of a comonad on such a category.

   A simple example of this kind is the comonad $G = \Delta \circ \varprojlim$ resulting from composing the "constant functor" functor $\Delta : \mathbf{Sets} \to \mathbf{Sets}^{\mathbf{C}}$ with the "limit" functor $\varprojlim : \mathbf{Sets}^{\mathbf{C}} \to \mathbf{Sets}$. It can be shown in general that the coalgebras for this comonad again form a topos. In fact, they are just the constant functors $\Delta(S)$ for sets S, and the category $\mathbf{Sets}$ is thus comonadic over $\mathbf{Sets}^{\mathbf{C}}$.

2. It may happen that both structures—monad and comonad—occur together, and interact. Taking the opposite category will not alter this situation! This happens for instance when a system of *three* adjoint functors are composed:

$$L \dashv U \dashv R \qquad \begin{tikzcd} \mathbf{C} \arrow[r, shift left=8, "R"] \arrow[r, shift right=8, "L"'] & \mathbf{D} \arrow[l, "U"'] \end{tikzcd}$$

   resulting in a monad $T = U \circ L$ and a comonad $G = U \circ R$, both on $\mathbf{C}$. In such a case, T and G are then of course also adjoint $T \dashv G$.

   This arises, for instance, in the foregoing example with $R = \varprojlim$, and $U = \Delta$, and $L = \varinjlim$ the "colimit" functor. It also occurs in propositional modal logic, with $T = \diamond$ "possibility" and $G = \square$ "necessity," where the adjointness $\diamond \dashv \square$ is equivalent to the law known to modal logicians as "S5."

   A related example is given by the open and closed subsets of a topological space: the topological interior operation on arbitrary subsets is a comonad and closure is a monad. We leave the details as an exercise.

10.5 Algebras for endofunctors

Some very basic kinds of algebraic structures have a more simple description than as algebras for a monad, and this description generalizes to structures that are not algebras for any monad, but still have some algebra-like properties.

As a familiar example, consider first the underlying structure of the notion of a group. We have a set G equipped with operations as indicated in the following:

$$\begin{tikzcd} G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\ & 1 \arrow[u, "u"] & \end{tikzcd}$$

We do not assume, however, that these operations satisfy the group equations of associativity, etc. Observe that this description of what we call a "group structure" can plainly be compressed into a single arrow of the form

$$1 + G + G \times G \xrightarrow{[u,i,m]} G$$

Now let us define the functor $F : \mathbf{Sets} \to \mathbf{Sets}$ by

$$F(X) \; = \; 1 + X + X \times X$$

Then a group structure is simply an arrow,

$$\gamma : F(G) \to G.$$

Moreover, a homomorphism of group structures in the conventional sense

$$h : G \to H,$$

$$h(u_G) = u_H$$
$$h(i(x)) = i(h(x))$$
$$h(m(x,y)) = m(h(x), h(y))$$

is then exactly a function $h : G \to H$ such that the following diagram commutes:

$$\begin{tikzcd} F(G) \arrow[r, "F(h)"] \arrow[d, "\gamma"'] & F(H) \arrow[d, "\vartheta"] \\ G \arrow[r, "h"'] & H \end{tikzcd}$$

where $\vartheta : F(H) \to H$ is the group structure on H. This observation motivates the following definition.

Definition 10.8. Given an endofunctor $P : \mathcal{S} \to \mathcal{S}$ on any category $\mathcal{S}$, a P-*algebra* consists of an object A of $\mathcal{S}$ and an arrow,

$$\alpha : PA \to A.$$

A *homomorphism* $h : (A, \alpha) \to (B, \beta)$ of P-algebras is an arrow $h : A \to B$ in $\mathcal{S}$ such that $h \circ \alpha = \beta \circ P(h)$, as indicated in the following diagram:

$$\begin{tikzcd}
P(A) \arrow[r, "P(h)"] \arrow[d, "\alpha"'] & P(B) \arrow[d, "\beta"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

The category of all such P-algebras and their homomorphisms are denoted as

$$P\text{-Alg}(\mathcal{S})$$

We usually write more simply P-Alg when $\mathcal{S}$ is understood. Also, if there is a monad present, we need to be careful to distinguish between algebras for the monad and algebras for the endofunctor (especially if P is the functor part of the monad!).

Example 10.9. 1. For the functor $P(X) = 1 + X + X \times X$ on **Sets**, we have already seen that the category **GrpStr** of group structures is the same thing as the category of P-algebras,

$$P\text{-Alg} \ = \ \mathbf{GrpStr}.$$

2. Clearly, for any other algebraic structure of finite "signature," that is, consisting of finitely many, finitary operations, there is an analogous description of the structures of that sort as algebras for an associated endofunctor. For instance, a *ring structure*, with two nullary, one unary, and two binary operations is given by the endofunctor

$$R(X) = 2 + X + 2 \times X^2.$$

In general, a functor of the form

$$P(X) = C_0 + C_1 \times X + C_2 \times X^2 + \cdots + C_n \times X^n$$

with natural number coefficients C_k, is called a (finitary) *polynomial functor*, for obvious reasons. These functors present exactly the *finitary structures*. The same thing holds for finitary structures in any category $\mathcal{S}$

with finite products and coproducts; these can always be represented as algebras for a suitable endofunctor.

3. In a category such as **Sets** that is complete and cocomplete, there is an evident generalization to infinitary signatures by using generalized or "infinitary" polynomial functors, that is, ones with infinite sets C_k as coefficients (representing infinitely many operations of a given arity), infinite sets B_k as the exponents X^{B_k} (representing operations of infinite arity), or infinitely many terms (representing infinitely many different arities of operations), or some combination of these. The algebras for such an endofunctor

$$P(X) = \sum_{i \in I} C_i \times X^{B_i}$$

can then be naturally viewed as generalized "algebraic structures." Using locally cartesian closed categories, one can even present this notion without needing (co)completeness.

4. One can of course also consider algebras for an endofunctor $P : \mathcal{S} \to \mathcal{S}$ that is not polynomial at all, such as the covariant powerset functor $\mathcal{P} : \mathbf{Sets} \to \mathbf{Sets}$. This leads to a proper generalization of the notion of an "algebra," which however still shares some of the formal properties of conventional algebras, as seen below.

Let $P : \mathbf{Sets} \to \mathbf{Sets}$ be a polynomial functor, say

$$P(X) = 1 + X^2$$

(what structure is this?). Then the notion of an *initial* P-algebra gives rise to a recursion property analogous to that of the natural numbers. Specifically, let

$$[o, m] : 1 + I^2 \to I$$

be an initial P-algebra, that is, an initial object in the category of P-algebras. Then, explicitly, we have the structure

$$o \in I, \qquad m : I \times I \to I$$

and for any set X with a distinguished element and a binary operation

$$a \in X, \qquad * : X \times X \to X$$

there is a unique function $u : I \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
1 + I^2 \arrow[r, "P(u)"] \arrow[d, "{[o,m]}"'] & 1 + X^2 \arrow[d, "{[a,*]}"] \\
I \arrow[r, "u"'] & X
\end{tikzcd}$$

This of course says that, for all $i, j \in I$,

$$u(o) = a$$
$$u(m(i,j)) = u(i) * u(j)$$

which is exactly a *definition by structural recursion* of the function $u : I \to X$. Indeed, the usual recursion property of the natural numbers $\mathbb{N}$ with zero $0 \in \mathbb{N}$ and successor $s : \mathbb{N} \to \mathbb{N}$ says precisely that $(\mathbb{N}, 0, s)$ is the initial algebra for the endofunctor,

$$P(X) = 1 + X : \mathbf{Sets} \to \mathbf{Sets}$$

as the reader should check.

We next briefly investigate the question: When does an endofunctor have an initial algebra? The existence is constrained by the fact that initial algebras, when they exist, must have the following noteworthy property.

Lemma 10.10 (Lambek). *Given any endofunctor $P : \mathcal{S} \to \mathcal{S}$ on an arbitrary category $\mathcal{S}$, if $i : P(I) \to I$ is an initial P-algebra, then i is an isomorphism,*

$$P(I) \cong I.$$

We leave the proof as an easy exercise.

In this sense, the initial algebra for an endofunctor $P : \mathcal{S} \to \mathcal{S}$ is a "least fixed point" for P. Such algebras are often used in computer science to model "recursive datatypes" determined by the so-called fixed point equations $X = P(X)$.

Example 10.11. 1. For the polynomial functor,

$$P(X) = 1 + X^2$$

(monoid structure!), let us "unwind" the initial algebra,

$$[*, @] : 1 + I \times I \cong I.$$

Given any element $x \in I$, it is thus either of the form $*$ or of the form $x_1 @ x_2$ for some elements $x_1, x_2 \in I$. Each of these x_i, in turn, is either of the form $*$ or of the form $x_{i1} @ x_{i2}$, and so on. Continuing in this way, we have a representation of x as a finite, binary tree. For instance, an element of the form $x = *@(*@*)$ looks like

$$\begin{tikzcd}
& x \arrow[dl] \arrow[dr] & & \\
* & & *@* \arrow[dl] \arrow[dr] & \\
& * & & *
\end{tikzcd}$$

We can present the monoid structure explicitly by letting

$$I = \{t \mid t \text{ is a finite, binary tree}\}$$

with

$$\begin{aligned}
* &= \text{“the empty tree”} \\
@(t_1, t_2) &= t_1@t_2
\end{aligned}$$

$$= \begin{tikzcd}
& t_1@t_2 \arrow[dl] \arrow[dr] & \\
t_1 & & t_2
\end{tikzcd}$$

The isomorphism,

$$[*, @] : 1 + I \times I \to I$$

here is plain to see.

2. Similarly, for any other polynomial functor,

$$P(X) = C_0 + C_1 \times X + C_2 \times X^2 + \cdots + C_n \times X^n$$

we can describe the initial algebra (in **Sets**),

$$P(I) \cong I$$

as a set of trees with branching types and labels determined by P. For instance, consider the polynomial

$$P(X) = 1 + A \times X$$

for some set A. What is the initial algebra? Since,

$$[*, @] : 1 + A \times I \cong I$$

we can unwind an element x as

$$x = * \text{ or } a_1@x_1$$
$$x_1 = * \text{ or } a_2@x_2$$
$$\dots$$

Thus, we essentially have $x = a_1@a_2@\cdots@a_n$. So I can be represented as the set A-List of (finite) lists of elements $a_1, a_2, \dots$ of A, with the structure

$$* = \text{“the empty list”}$$
$$@(a, \ell) = a@\ell$$

The usual procedure of “recursive definition” follows from initiality. For example, the length function for lists length : A-List $\to \mathbb{N}$ is usually defined by

$$\text{length}(*) = 0 \tag{10.8}$$
$$\text{length}(\text{a}@\ell) = 1 + \text{length}(\ell) \tag{10.9}$$

We can do this by equipping $\mathbb{N}$ with a suitable $P(X) = 1 + A \times X$ structure, namely,

$$[0, m] : 1 + A \times \mathbb{N} \to \mathbb{N}$$

where $m(a, n) = 1+n$ for all $n \in \mathbb{N}$. Then by the universal mapping property of the initial algebra, we get a unique function length : A-List $\to \mathbb{N}$ making a commutative square:

$$\begin{tikzcd}
1 + A \times A\text{-List} \arrow[r, "1 + A \times \text{length}"] \arrow[d, "{[*, @]}"'] & 1 + A \times \mathbb{N} \arrow[d, "{[0, m]}"] \\
A\text{-List} \arrow[r, "\text{length}"'] & \mathbb{N}
\end{tikzcd}$$

But this commutativity is, of course, precisely equivalent to the equations (10.8) and (10.9).

In virtue of Lambek's lemma, we at least know that not all endofunctors can have initial algebras. For, consider the covariant powerset functor $\mathcal{P}$: **Sets** $\to$ **Sets**. An initial algebra for this would give us a set I with the property that $\mathcal{P}(I) \cong I$, which is impossible by the well-known theorem of Cantor!

The following proposition gives a useful sufficient condition for the existence of an initial algebra.

Proposition 10.12. *If the category $\mathcal{S}$ has an initial object* 0 *and colimits of diagrams of type ω (call them “ω-colimits”), and the functor*

$$P : \mathcal{S} \to \mathcal{S}$$

preserves ω-colimits, then P has an initial algebra.

Proof. Note that this generalizes a very similar result for posets already given above as proposition 5.34. And even the proof by "Newton's method" is essentially the same! Take the ω-sequence

$$0 \to P0 \to P^2 0 \to \cdots$$

and let I be the colimit

$$I = \varinjlim_n P^n 0.$$

Then, since P preserves the colimit, there is an isomorphism

$$P(I) = P(\varinjlim_n P^n 0) \cong \varinjlim_n P(P^n 0) = \varinjlim_n P^n 0 = I$$

which is seen to be an initial algebra for P by an easy diagram chase. $\square$

Since (as the reader should verify) every polynomial functor $P : \mathbf{Sets} \to \mathbf{Sets}$ preserves ω-colimits, we have

Corollary 10.13. *Every polynomial functor $P : \mathbf{Sets} \to \mathbf{Sets}$ has an initial algebra.*

Finally, we ask, what is the relationship between algebras for endofunctors and algebras for monads? The following proposition, which is a sort of "folk theorem," gives the answer.

Proposition 10.14. *Let the category $\mathcal{S}$ have finite coproducts. Given an endofunctor $P : \mathcal{S} \to \mathcal{S}$, the following conditions are equivalent:*

1. *The P-algebras are the algebras for a monad. Precisely, there is a monad $(T : \mathcal{S} \to \mathcal{S}, \eta, \mu)$, and an equivalence*

   $$P\text{-Alg}(\mathcal{S}) \simeq \mathcal{S}^T$$

   between the category of P-algebras and the category $\mathcal{S}^T$ of algebras for the monad. Moreover, this equivalence preserves the respective forgetful functors to $\mathcal{S}$.

2. *The forgetful functor $U : P\text{-Alg}(\mathcal{S}) \to \mathcal{S}$ has a left adjoint*

   $$F \vdash U.$$

3. *For each object A of $\mathcal{S}$, the endofunctor*

   $$P_A(X) = A + P(X) : \mathcal{S} \to \mathcal{S}$$

   has an initial algebra.

Proof. That (1) implies (2) is clear.

For (2) implies (3), suppose that U has a left adjoint $F : \mathcal{S} \to P$-Alg and consider the endofunctor $P_A(X) = A + P(X)$. An algebra (X, γ) is a map $\gamma : A + P(X) \to X$. But there is clearly a unique correspondence between the following three types of things:

$$\gamma : A + P(X) \to X$$

$$\begin{tikzcd} & P(X) \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"'] & X \end{tikzcd}$$

$$\alpha : A \to U(X, \beta)$$

Thus, the P_A-algebras can be described equivalently as arrows of the form $\alpha : A \to U(X, \beta)$ for P-algebras (X, β). Moreover, a P_A-homomorphism $h : (\alpha, U(X, \beta)) \to (\alpha', U(X', \beta'))$ is just a P-homomorphism $h : (X, \beta) \to (X', \beta')$ making a commutative triangle with α and $\alpha' : A \to U(X', \beta')$. But an initial object in this category is given by the unit $\eta : A \to UFA$ of the adjunction $F \vdash U$, which shows (3).

Indeed, given just the forgetful functor $U : P$-Alg $\to \mathcal{S}$, the existence of initial objects in the respective categories of arrows $\alpha : A \to U(X, \beta)$, for each A, is exactly what is needed for the existence of a left adjoint F to U. So (3) also implies (2).

Before concluding the proof, it is illuminating to see how the free functor $F : \mathcal{S} \to P$-Alg results from condition (3). For each object A in $\mathcal{S}$, consider the initial P_A-algebra $\alpha : A + P(I_A) \to I_A$. In the notation of recursive type theory,

$$I_A = \mu_X . A + P(X)$$

meaning it is the (least) solution to the “fixed point equation”

$$X = A + P(X).$$

Since α is a map on the coproduct $A + P(I_A)$, we have $\alpha = [\alpha_1, \alpha_2]$, and we can let

$$F(A) = (I_A,\ \alpha_2 : P(I_A) \to I_A)$$

To define the action of F on an arrow $f : A \to B$, let $\beta : B + P(I_B) \to I_B$ be the initial P_B-algebra and consider the diagram

$$\begin{tikzcd}
A + P(I_A) \arrow[r, dotted, "A + P(u)"] \arrow[dd, "\alpha"'] & A + P(I_B) \arrow[d, "f + P(I_B)"] \\
 & B + P(I_B) \arrow[d, "\beta"] \\
I_A \arrow[r, dotted, "u"'] & I_B
\end{tikzcd}$$

The right-hand vertical composite $\beta \circ (f + P(I_B))$ now makes I_B into a P_A-algebra. There is thus a unique P_A-homomorphism u as indicated, and we can set

$$F(f) = u.$$

Finally, to conclude, the fact that (2) implies (1) is an easy application of Beck's Precise Tripleability Theorem, for which we refer the reader to section VI.7 of Mac Lane's *Categories Work* (1971). □

10.6 Exercises

1. Let $\mathbb{T}$ be the equational theory with one constant symbol and one unary function symbol (no axioms). In any category with a terminal object, a natural numbers object (NNO) is just an initial $\mathbb{T}$-model. Show that the natural numbers

   $$(\mathbb{N}, 0 \in \mathbb{N}, n + 1 : \mathbb{N} \to \mathbb{N})$$

   is an NNO in **Sets**, and that any NNO is uniquely isomorphic to it (as a $\mathbb{T}$-model).

   Finally, show that $(\mathbb{N}, 0 \in \mathbb{N}, n + 1 : \mathbb{N} \to \mathbb{N})$ is uniquely characterized (up to isomorphism) as the initial algebra for the endofunctor $F(X) = X + 1$.

2. Let $\mathbf{C}$ be a category and $T : \mathbf{C} \to \mathbf{C}$ an endofunctor. A T-*algebra* consists of an object A and an arrow $a : TA \to A$ in $\mathbf{C}$. A morphism $h : (a, A) \to (b, B)$ of T-algebras is a $\mathbf{C}$-morphism $h : A \to B$ such that $h \circ a = b \circ T(h)$. Let $\mathbf{C}$ be a category with a terminal object 1 and binary coproducts. Let $T : \mathbf{C} \to \mathbf{C}$ be the evident functor with object-part $C \mapsto C + 1$ for all objects C of $\mathbf{C}$. Show (easily) that the categories of T-algebras and $\mathbb{T}$-models ($\mathbb{T}$ as above) (in $\mathbf{C}$) are equivalent:

   $$T\text{-Alg} \simeq \mathbb{T}\text{-Mod}.$$

Conclude that free T-algebras exist in **Sets**, and that an initial T-algebra is the same thing as an NNO.

3. ("Lambek's lemma") Show that for any endofunctor $T : \mathbf{C} \to \mathbf{C}$, if $i : TI \to I$ is an initial T-algebra, then i is an isomorphism. (Hint: consider a diagram of the following form, with suitable arrows.)

$$\begin{tikzcd} TI \arrow[r] \arrow[d, "i"'] & T^2 I \arrow[r] \arrow[d, "Ti"'] & TI \arrow[d] \\ I \arrow[r] & TI \arrow[r] & I \end{tikzcd}$$

Conclude that for any NNO N in any category, there is an isomorphism $N + 1 \cong N$. Also, derive the usual recursion property of the natural numbers from initiality.

4. Given categories $\mathbf{C}$ and $\mathbf{D}$ and adjoint functors $F : \mathbf{C} \to \mathbf{D}$ and $U : \mathbf{D} \to \mathbf{C}$ with $F \dashv U$, unit $\eta : 1_{\mathbf{C}} \to UF$, and counit $\epsilon : FU \to 1_{\mathbf{D}}$, show that

$$T = U \circ F : \mathbf{C} \to \mathbf{C}$$
$$\eta : 1_{\mathbf{C}} \to T$$
$$\mu = U\epsilon_F : T^2 \to T$$

do indeed determine a monad on $\mathbf{C}$, as stated in the text.

5. Assume given categories $\mathbf{C}$ and $\mathbf{D}$ and adjoint functors

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : U$$

with unit $\eta : 1_{\mathbf{C}} \to UF$ and counit $\epsilon : FU \to 1_{\mathbf{D}}$. Show that every D in $\mathbf{D}$ determines a $T = UF$ algebra $U\epsilon : UFUD \to UD$, and that there is a "comparison functor" $\Phi : \mathbf{D} \to \mathbf{C}^T$ which, moreover, commutes with the "forgetful" functors $U : \mathbf{D} \to \mathbf{C}$ and $U^T : \mathbf{C}^T \to \mathbf{C}$.

$$\begin{tikzcd} \mathbf{D} \arrow[rr, "\Phi"] \arrow[dr, "U"'] & & \mathbf{C}^T \arrow[dl, "U^T"] \\ & \mathbf{C} & \end{tikzcd}$$

6. Show that $(P, s, \cup)$ is a monad on **Sets**, where

- $P : \mathbf{Sets} \to \mathbf{Sets}$ is the covariant powerset functor, which takes each function $f : X \to Y$ to the image mapping

$$P(f) = im(f) : P(X) \to P(Y)$$

- for each set X, the component $s_X : X \to P(X)$ is the singleton mapping, with

$$s_X(x) = \{x\} \subseteq X$$

for each $x \in X$;

- for each set X, the component $\cup_X : PP(X) \to P(X)$ is the union operation, with

$$\cup_X(\alpha) = \{x \in X \mid \exists_{U\in\alpha}.\ x \in U\} \subseteq X$$

for each $\alpha \subseteq P(X)$.

7. Determine the category of (Eilenberg–Moore) algebras for the $(P, s, \cup)$ monad on **Sets** defined in the foregoing problem. (Hint: consider complete lattices.)
8. Consider the free $\dashv$ forgetful adjunction

$$F : \mathbf{Sets} \rightleftarrows \mathbf{Mon} : \mathbf{U}$$

between sets and monoids, and let (T, η^T, μ^T) be the associated monad on **Sets**. Show that any T-algebra $\alpha : TA \to A$ for this monad comes from a monoid structure on A (exhibit the monoid multiplication and unit element).

9. (a) Show that an adjoint pair $F \dashv U$ with $U : \mathbf{D} \to \mathbf{C}$ and $\eta : UF \to 1_{\mathbf{C}}$ and $\epsilon : 1_{\mathbf{D}} \to FU$ also gives rise to a *comonad* (G, ϵ, δ) in $\mathbf{D}$, with

$$G = F \circ U : \mathbf{D} \to \mathbf{D}$$

$$\epsilon : G \to 1 \text{ the counit}$$

$$\delta = F\eta_U : G \to G^2$$

satisfying the duals of the equations for a monad.

(b) Define the notion of a *coalgebra* for a comonad, and show (by duality) that every comonad (G, ϵ, δ) on a category $\mathbf{D}$ "comes from" a (not necessarily unique) adjunction $F \dashv G$ such that $G = FU$ and ϵ is the counit.

(c) Let **End** be the category of sets equipped with an endomorphism, $e : S \to S$. Consider the functor $G : \mathbf{End} \to \mathbf{End}$ defined by

$$G(S, e) = \{x \in S \mid e^{(n+1)}(x) = e^{(n)}(x) \text{ for some } n\}$$

equipped with the restriction of e. Show that this is the functor part of a comonad on **End**.

10. Verify that the open and closed subsets of a topological space give rise to comonad and monad, respectively, on the powerset of the underlying pointset. Moreover, the categories of coalgebras and algebras are isomorphic.

11. (Kleisli category) Given a monad (T, η, μ) on a category $\mathbf{C}$, in addition to the Eilenberg–Moore category, we can construct another category $\mathbf{C}_T$ and an adjunction $F \dashv U$, $\eta : 1 \to UF$, $\epsilon : FU \to 1$ with $U : \mathbf{C}_T \to \mathbf{C}$ such that

$$T = U \circ F$$
$$\eta = \eta \quad \text{(the unit)}$$
$$\mu = U\epsilon_F$$

This category $\mathbf{C}_T$ is called the *Kleisli category* of the adjunction, and is defined as follows:

- the objects are the same as those of $\mathbf{C}$, but written $A_T, B_T, \ldots$,
- an arrow $f_T : A_T \to B_T$ is an arrow $f : A \to TB$ in $\mathbf{C}$,
- the identity arrow $1_{A_T} : A_T \to A_T$ is the arrow $\eta_A : A \to TA$ in $\mathbf{C}$,
- for composition, given $f_T : A_T \to B_T$ and $g_T : B_T \to C_T$, the composite $g_T \circ f_T : A_T \to C_T$ is defined to be

$$\mu_C \circ Tg_T \circ f_T$$

as indicated in the following diagram:

$$\begin{tikzcd} A \arrow[r, "g_T \circ f_T"] \arrow[d, "f_T"'] & TC \\ TB \arrow[r, "Tg_T"'] & TTC \arrow[u, "\mu_C"'] \end{tikzcd}$$

Verify that this indeed defines a category, and that there are adjoint functors $F : \mathbf{C} \to \mathbf{C}_T$ and $U : \mathbf{C}_T \to \mathbf{C}$ giving rise to the monad as $T = UF$, as claimed.

12. Let $P : \mathbf{Sets} \to \mathbf{Sets}$ be a polynomial functor,

$$P(X) = C_0 + C_1 \times X + C_2 \times X^2 + \cdots + C_n \times X^n$$

with natural number coefficients C_k. Show that P preserves ω-colimits.

13. The notion of a *coalgebra* for an endofunctor $P : \mathcal{S} \to \mathcal{S}$ on an arbitrary category $\mathcal{S}$ is exactly dual to that of a P-algebra. Determine the *final* coalgebra for the functor

$$P(X) = 1 + A \times X$$

for a set A. (Hint: Recall that the initial algebra consisted of *finite* lists $a_1, a_2, \ldots$ of elements of A.)