Linear Algebra I for Mathematical Sciences
Week 12 Tutorial Worksheet Winter 2024

Name: ______________________________ Student Number: ____________________

Q1. Determine whether the following are true or false. Give justifications for your answers. If a statement is true, then prove it. If a statement is false, then either explain why it is false or give an example demonstrating why it is false.

1. If V is finite dimensional and $T : V \to V$ is a linear transformation, then T has n distinct eigenvalues.
2. If $T : V \to V$ has one non-zero eigenvector $\mathbf{v}$, then it has infinitely many non-zero eigenvectors.
3. Eigenvalues must be non-zero scalars.
4. If V is finite dimensional, and $T : V \to V$ has fewer than $\dim(V)$ eigenvalues, then it is not diagonalizable.

Q2. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Is A diagonalizable?

Q3. Recall that the λ-eigenspace of $T : V \to V$ is $E_\lambda = \{\mathbf{v} : T(\mathbf{v}) = \lambda\mathbf{v}\}$. Prove that $E_0 = \ker(T)$.

Q4. Prove that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

> The Cayley-Hamilton theorem states that a matrix is a zero of its own characteristic polynomial. In this last question, you will examine a special case of the Cayley-Hamilton Theorem by proving it for diagonalizable matrices.

Q5. Given a polynomial $p \in \mathbf{P}_n(\mathbb{R})$ and a linear transformation $T : V \to V$ we can define a transformation $p(T) : V \to V$. We treat the constant term of the polynomial as a multiple of the identity. For example, consider the following.

$$p(x) = x^2 - 3x + 2 \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

This gives the linear transformation:

$$p(A) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -6 & 7 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ -6 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2).$$

Prove the following:

(a) If λ is an eigenvalue of $T : V \to V$ then $p(\lambda)$ is an eigenvalue of $p(T) : V \to V$.

(b) If T is diagonalizable and $\chi_T(\lambda)$ is its chacteristic polynomial, then $\chi_T(T) = \mathbf{0} \in \mathcal{L}(V, V)$.