Linear Algebra I for Mathematical Sciences
Week 8 Tutorial Worksheet Winter 2024

Name: ______________________ Student Number: ________________

Q1. Determine whether the following are true or false. Give justifications for your answers. If a statement is true, then prove it. If a statement is false, then either explain why it is false or give an example demonstrating why it is false.

1. If $\dim(V) = n$ we know the values of $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ then we can determine all values of $T(\mathbf{v})$.
2. $\dim(\mathbf{M}_{n\times k}(\mathbb{R}) = n + k$
3. If $\dim(V) = n$ then V has exactly one subspace with $\dim(W_0) = 0$ and exactly one subspace with $\dim(W_n) = n$.

Q2. Prove that V is infinite dimensional if and only if V contains an infinite linearly independent set.

Q3. Suppose that $A \in \mathbf{M}_{n\times n}(\mathbb{R})$. Consider the transformation $T_A : \mathbf{M}_{n\times n}(\mathbb{R}) \to \mathbf{M}_{n\times n}(\mathbb{R})$ given by $T_A(M) = AM$.

- Prove that T_A is linear.
- If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ find a non-zero matrix M such that $T_A(M) = 0$.

Q4. Prove that $R_\theta(R_{\theta'}(\mathbf{v})) = R_{\theta+\theta'}(\mathbf{v})$.

Q5. Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (y, z, x)$. Give a formula for $T^{100}(x, y, z)$. *Note:* The notation T^{100} means that T is composed with itself 100 times.

Q6. Suppose that $S : V \to V$ and $T : V \to V$ are linear transformations. Define their sum by:

$$(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$$

This gives a new linear transformation $S + T : V \to V$.

1. If S and T are injective, is $S + T$ also injective?
2. If S and T are surjective, is $S + T$ also surjective?

Q7. Suppose that $T_1 : V \to \mathbb{R}$ and $T_2 : V \to \mathbb{R}$ are linear transformations with $\ker(T_1) = \ker(T_2)$. Prove that there is a constant $c \in \mathbb{R}$ such that $T_2(\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v} \in V$.

Q8. Find the kernel and image of the following linear transformations given by matrices.

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$
$$T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
2. $T : \mathbb{R}^3 \to \mathbb{R}^2$
$$T = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -4 & 2 \end{bmatrix}$$