Phys Enph 479/879 Assignment 1
Non-Linear Optics: Solving the Optical Bloch Equations

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The optical Bloch equations are used in the field of non-linear optics to model the interaction of a field with a two-level quantum system. Solving these equations gives rise to important "observables" such as the population density of the excited state and the complex coherence of the system. The rotating wave approximation can be used to simplify the equations and demonstrate the Rabi oscillations that occur in the system under the influence of a Gaussian pulse. Moving beyond the rotating wave approximation, the Bloch equations cannot be solved analytically. To solve the full form of the equations, the Runge-Kutta method is implemented to analyse the behaviour of the system. The solutions to these equations are a more accurate description of the field interactions with a two-level quantum system and are used to model the behaviour of the system under the influence of a full Rabi field. Finally, using fast Fourier transforms, the output of the two-level system in the frequency domain can be studied and used to explain the behaviour of the system when the applied field is changed.

I. INTRODUCTION

The interaction between light and a two-level system (TLS) can be modelled using the Optical Bloch Equations (OBEs), also known as the Maxwell-Bloch equations. Solving these equations demonstrates the resonance behaviour of a TLS under the influence of an external field over a given time period [1]. More specifically, the solutions to the OBEs are time dependent functions for the population density of the excited state ($n_e(t)$) and the complex coherence of the system ($u(t)$).
Multiple forms of the OBEs are used to demonstrate the resonant behavior of TLSs. The two variations discussed in this paper will be for an on-resonance system with no dephasing using a rotating wave approximation (RWA) and the full wave equations with no RWA. For the RWA, the fast oscillations of the system are ignored. This approximation is valid when the amplitude of the field (or pulse) applied to the system is small compared to its frequency. When coupled with a time-independent field, this approximation can be solved analytically and using numerical approximations. Different methods for solving OBEs will therefore be tested and compared to the exact solution found when the field is a continuous, monochromatic harmonic wave. Once an efficient numerical approximation method is determined, the TLSs behaviour will modelled by solving the OBEs with and without the rotating wave approximation.
Finally, using fast Fourier transforms, the polarization power spectrum of the system will be studied. Similar to techniques used to measure the frequency of the pulse and system using the power spectrum, $|E(\omega|$, the polarization power spectrum $|u(\omega)|$ can be found for TLSs. For this case, polarization dephasing will be introduced and the resulting system output spectrum shown.

II. METHOD

The Bloch equations without the rotating wave approximation cannot be solved analytically, meaning a robust ODE solver is required to accurately describe TLS field interactions. As mentioned in Sec. I, the OBEs in the RWA can be solved analytically for a continuous wave (CW) field. Under these conditions, the solutions to the OBEs are given by Eq. 1, when $ne(0) = 0$.

$$n_e(t) = \cos^2(\Omega_0 t) \tag{1}$$

Both the Forward Euler and Runge-Kutta methods are used to solve the OBEs numerically and their solutions are compared to the analytical solution. As the Runge-Kutta method's solution agrees excellently with the exact solution, it is used to solve the more complex versions of the OBEs in this paper.

III. THEORY

The first form of the OBEs that will be studied is in the rotating wave approximation. Given by Eq. 2 and Eq. 3, this set of equations can be used when the amplitude of the field interacting with the system is much smaller than the frequency of the system and neglects the fast oscillations of the system [2]. In these equations, n_e represents the population density of the excited state and u is the complex coherence of the system.

$$\frac{du}{dt} = -\gamma_d u - i\Delta_{0L} u + i\frac{\tilde{\Omega}(t)}{2}(2n_e - 1) \tag{2}$$

$$\frac{dn_e}{dt} = -\tilde{\Omega}(t) Im[u] \tag{3}$$

To simplify the equations further, on-resonance excitation is assumed, meaning Δ_{0L} is zero. In addition, no

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dephasing occurs, meaning γ_d is also set to zero. Using this simplified version of the standard RWA equations, the validity of the Area Theorem is demonstrated. This theorem states that a TLS system will undergo exactly n Rabi oscillations (or flops) for an incident pulse of area n$2pi$. The pulse area is found by integrating over the pulse function as shown in Eq. 25 in Ref. [3].

The second set of OBEs that will be solved are without the RWA, shown in Eq. 4 and Eq. 5, and are known as the full-wave Bloch equations. For this form of the equations, no analytical solutions exist, so the Runge-Kutta method is used as discussed above.

$$\frac{du}{dt} = -\gamma_d u - i\omega_0 u + i\Omega(t)(2n_e - 1) \tag{4}$$

$$\frac{dn_e}{dt} = -2\Omega(t)Im[u] \tag{5}$$

In both sets of OBEs, $\tilde{\Omega(t)}$ is the field that interacts with the system and is often a laser pulse. For a time-independent pulse (such as a monochromatic laser), $\tilde{\Omega}(t)$=Ω_0 is used. To represent a time-dependent pulse, a Gaussian pulse and full-wave Rabi field given by Eq. 6 and Eq. 7 respectively can be used. In both equations, t_p is the pulse width.

$$\tilde{\Omega} = \Omega_0 \exp(-t^2/t_p^2) \tag{6}$$

$$\tilde{\Omega} = \Omega_0 \exp(-t^2/t_p^2) sin(\omega_l t + \phi) \tag{7}$$

The full-wave equations solution's are compared to those of the rotating wave approximation to explore the varying behaviour of a TLS when both a Gaussian and Rabi pulse are applied.
In addition, using a non-zero polarization decay rate, the full-wave equations are used to study the polarization of the power spectrum $|u(w)|$. To do so, the discrete Fourier transform of the coherence and input pulse are found using the fast Fourier transform (FFT) algorithm. The resulting sets of data $|u(w)|$ and $|\Omega(w)|$ allow us to compare the input and output signals in the frequency domain. These results demonstrate the behaviour of the system's frequency over time when polarizing dephasing occurs.

IV. RESULTS

A. Numerical Approximations of the Bloch Equations' Solutions

The Forward Euler and Runge-Kutta methods were implemented to solve the simplified OBEs in the RWA and their solutions were compared to the exact solution given by Eq. 1. When a small time step was used, both methods

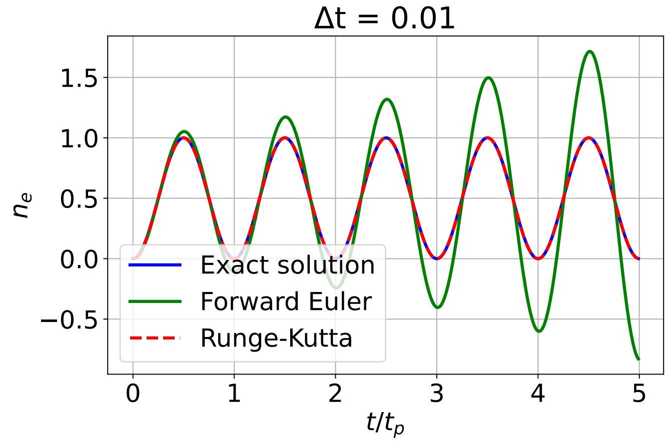

Figure 1. The population density of the excited state, n_e for a two-level quantum system. The density is the solution to the optical Bloch equations in the RWA found using the forward Euler method, the Runge-Kutta method and an analytical solution. The solution found using the Euler method diverges from the analytical solution drastically. The solution found using the Runge-Kutta method is identical to the exact solution.

appeared to produce the same results as the analytical solution. When the step size was reduced, as seen in Fig. 1, it becomes apparent that the Forward Euler solution diverges significantly from the analytical solution. Therefore, the Runge-Kutta method is the preferred method for solving differential equations and is used to solve all the OBEs.
While the Runge-Kutta (RK) method was proven to be effective for solving the OBEs, when compared to a built in Python differential equation solver, such as *odeint* from the SciPy "integrate" package, the Runge-Kutta method is relatively slow. To solve one set of the OBEs, our original code using the RK algorithm took 0.0529 seconds, while SciPy *odeint* found the solutions in 0.0079 seconds. Despite the speed, the RK method produced the exact same solution as the SciPy solver, which provides additional support for the capabilities of the Runge-Kutta method.

B. Rotating Wave Approximation for a Time Dependent Pulse

With an effective differential equation solver found, the solutions of the Bloch equations in the rotating wave approximation were determined. By choosing a Gaussian pulse as seen in Eq. 6, the behaviour of a TLS during an interaction with a time-dependent field was modelled. More specifically, using a pulse area of 2π, the population density of the excited state, n_e, for the duration of a pulse was found. As seen in Fig. 2 the system's population density rose from zero to one, before returning back to zero, also known as a Rabi flop [4]. This behaviour was expected as the Area theorem discussed in Sec. III predicts that the system would undergo one Rabi oscillation due to the pulse area of 2π used.

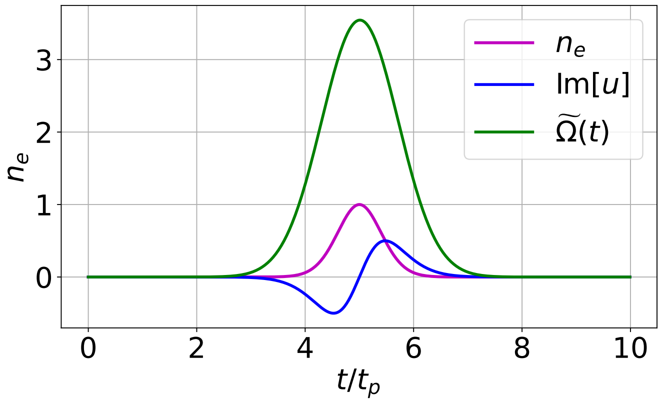

Figure 2. The behaviour of a two-level quantum system for the duration of a single pulse with a pulse area of 2π. The population density, n_e, rises to one and falls back to zero once from the applied pulse.

C. Moving Beyond the Rotating Wave Approximation

Using our knowledge from the rotating wave approximation, the full-wave Bloch equations were solved while using a full-wave Rabi field. Using Eq. 35 from the Area theorem in [3], it is found that Ω_0 will always be equal to the pulse area divided by $\sqrt{\pi}$. Therefore, for a nominal pulse of 2π, $\Omega^0_{2\pi}$ is $\frac{2\pi}{\sqrt{\pi}}$. Using this value of $\Omega^0_{2\pi}$, the frequency of the laser, ω_L was varied to demonstrate the effects of the input frequency on the system. Using values of $\omega_L = \Omega^0_{2\pi}$, $2\Omega^0_{2\pi}$ and $8\Omega^0_{2\pi}$, these results were compared to the RWA solution found earlier. As ω_L increases, the full-wave solution begins to conform to the Area theorem described in I. This behaviour is shown as $u(t)$ begins to fill in the 'envelope' created by the RWA solution as shown in Fig. 3.

Looking at n_e in Fig. 4, the system exhibits a single Rabi flop most clearly at the highest laser frequency of $8\Omega^0_{2\pi}$. Because a pulse area of 2π is being used, this is the expected behaviour of the system under the Area theorem. This trend agrees with the previous observation that the full-wave solutions resemble those from the RWA at higher laser frequencies.

When the phase ϕ in Eq. 7 is changed to $\pi/2$, the coherence and population density curves from the full-wave Bloch equations are affected. The most significant changes were in the lower frequency plots. For ω_L equal to $\Omega^0_{2\pi}$ and $2\Omega^0_{2\pi}$, the peaks of the curve as well as the amplitudes were significantly different than those when ϕ was equal to zero. However, for the case where ω_L is equal to $8\Omega^0_{2\pi}$, the u_t and n_e curves for the system remained observably unchanged. Because it was at this high frequency that the solutions were closest to those found using the RWA, the RWA model likely would not account for the phase of the pulse.

When ω_L was fixed at $2\Omega^0_{2\pi}$, the effects on the system from changing the pulse area were found. Fig. 5 demonstrates the behaviour of the system for pulse areas of $\pi/2$, 4π and 16π. For the smallest pulse area, $\pi/2$, the system does not have enough energy to complete one Rabi flop, and the population density therefore rises but does not follow the oscillatory pattern that arises when the pulse was exactly 2π. When the pulse area is increased, the system exhibits two Rabi flops, as is expected for a pulse area of $2 \times 2\pi$. Lastly, when the system interacts with a pulse of area 16π, the system's oscillations become chaotic and unpredictable, before ending up in an energized state. This behaviour can be explained by observing the signal in the frequency domain as shown in Sec. IV D.

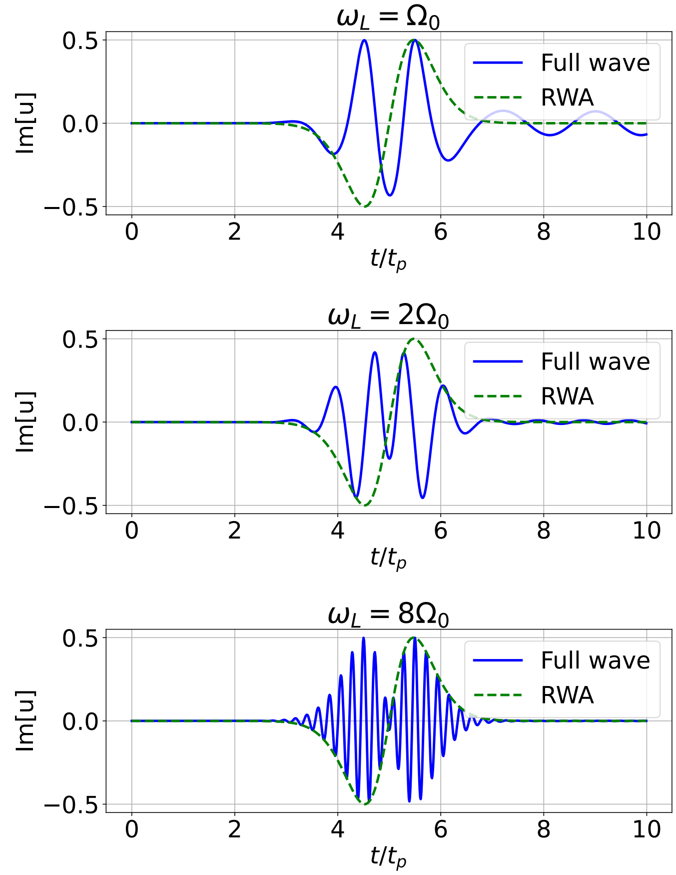

Figure 3. The three plots show the imaginary part of the coherence of a two-level quantum system when a Gaussian pulse interacts with it. The solutions were obtained using the full-wave Bloch equations and are compared to the RWA solutions. In this example, the pulse has a fixed pulse are of 2π and the frequency of the pulse, ω_L is varied. At the highest frequency, the full-wave solution conforms to the envelope created by the RWA solution.

D. The Frequency Domain

To observe the behaviour of the TLS in the frequency domain, the fast Fourier transform algorithm was used to find the discrete Fourier transform of the pulse and the coherence. In this case, the full-wave Bloch equations were used with a dephasing constant, γ_d, of $\pi/2$. As shown in Fig. 6, the polarization of the power spectrum, $|u(w)|$ was then studied for the three pulse areas used in Sec. IV C. For all cases, when a longer vertical axis is used, the solutions decay to zero because of the non-zero γ_d term. For the lowest pulse area, $\pi/2$, the output spectrum follows

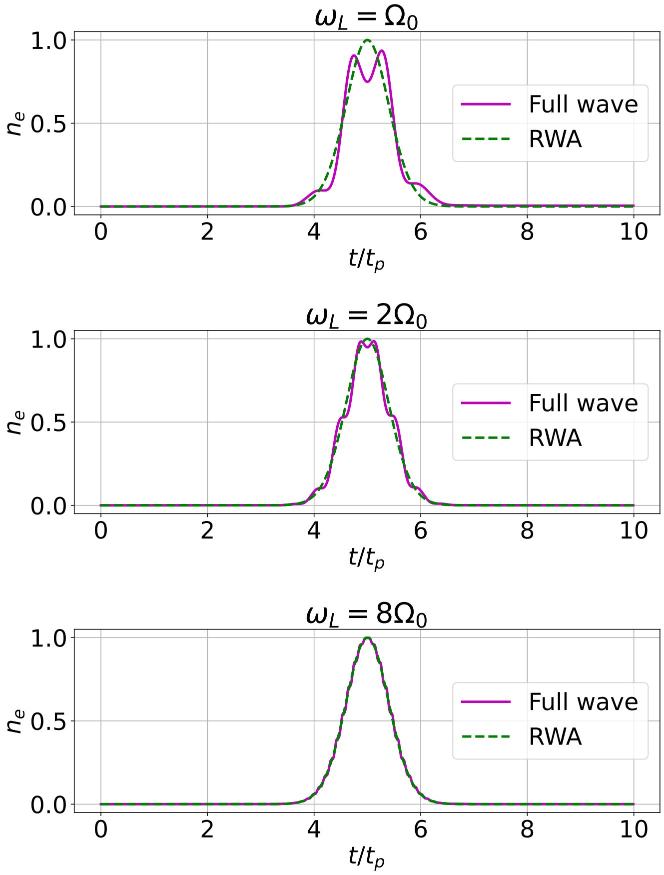

Figure 4. The population density of the excited state, n_e, for a two-level quantum system found by solving the full-wave and rotating wave approximation (RWA) optical Bloch equations. A fixed pulse area of 2pi is used while the frequency of the pulse, ω_L, is varied. The population density from the full-wave solutions increasingly resembles the RWA solution as the frequency of the pulse is increased.

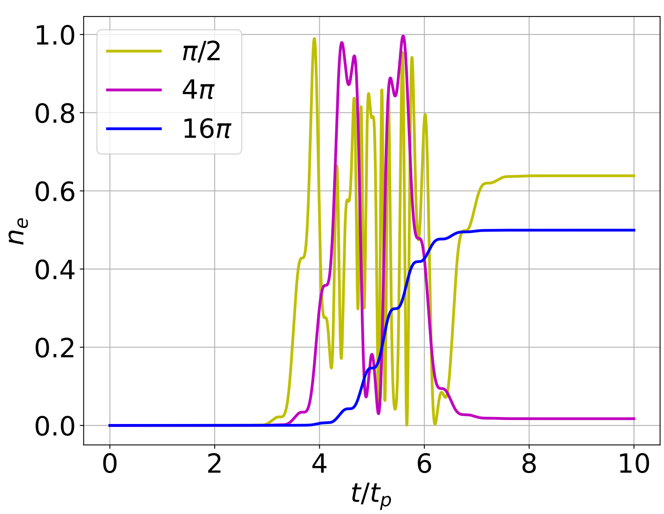

Figure 5. The population density, n_e, of a two-level quantum system from the full-wave optical Bloch equations. The pulse area of the field is varied from pi/2 to 16pi. At 2π, the energy of the system is too low for oscillations to occur. At 4π, the system exhibits two Rabi flops. At 16π, the behaviour of the system becomes chaotic and unpredictable.

the curve of the input pulse. Both curves peak around (w/w_L) equal to one as expected for a pulse area of $\pi/2$. This trend means the system has a similar distribution of power at the same frequencies as the source for a small pulse area. For the higher pulse areas however, the output is less predictable. The polarization power spectrum for 16π, for example, demonstrates the chaotic behaviour of the system that was present in Fig. 5. In the frequency domain, it is apparent that as we increase the area of the pulse, the system responds with unpredictable oscillation of the output power of the system.

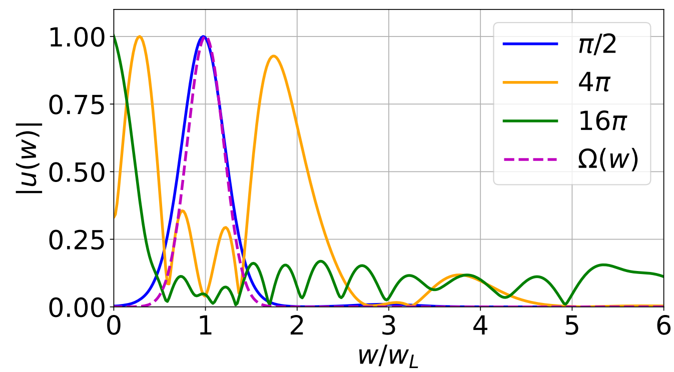

Figure 6. The fast Fourier transform of the coherence, $|u(w)|$, plotted with the input pulse $\Omega(w)$. All pulses have a fixed frequency and the pulse area is varied from $\pi/2$ to 16π. The frequency domain demonstrates visually the relationship between the input and output signals.

V. CONCLUSION

Using the Runge-Kutta method, the behaviour of a two-level quantum system was modelled by solving the optical Bloch equations in two forms. The first form, the rotating wave approximation, allowed for small oscillations in the system to be ignored and satisfied the Area theorem that relates the pulse area of the applied field to the output of the system. A more accurate version of the OBEs, the full-wave Bloch equations, were also solved to demonstrate the behaviour of a TLS without using the RWA. Varying parameters such as the frequency and area of the applied pulse allowed for characteristics of the system (such as the coherence and population density) to be determined. Using this second version of the OBEs, it was found that the solutions conformed to the same patterns as the RWA equations for higher pulse frequencies. However, at these high frequencies, the system lost it's periodic oscillatory motion. This trend was observed by taking the fast Fourier transform of the input and output signal to study their correlation. Overall, the methods used to solve the Bloch equations worked effectively for both forms of equations and resulted in important insights regarding the behaviour of the TLSs being studied.

VI. REFERENCES

[1] Wilhelmus M. Ruyten, "Magnetic and optical resonance of two-level quantum systems in modulated fields. i. bloch equation approach," Phys. Rev. A **42**, 4226–4245 (1990).

[2] Daniel Zeuch, Fabian Hassler, Jesse J. Slim, and David P. DiVincenzo, "Exact rotating wave approximation," Annals of Physics **423**, 168327 (2020).

[3] Dr. Stephen Hughes, "Sh1 - two level atoms, density matrix theory and optical bloch equations," Course Notes (2024), phys 479/879, Winter Term.

[4] Takeshi Suzuki and Rohan Singh, "Chapter eight - coherent control of a semiconductor quantum dot ensemble," in *Semiconductors and Semimetals*, Vol. 105, edited by Steven T. Cundiff and Mackillo Kira (Elsevier, 2020) pp. 347–385.