Week 7

Interiors

Let $\Pi = (\mathcal{P}, \mathcal{L}, d)$ be a Pasch geometry. Given distinct points A and B in Π, we can easily definte the interior of the line segment $\overline{AB}$ and the interior of the ray $\overrightarrow{AB}$ as

$$\operatorname{int}\left(\overline{AB}\right) = \overline{AB} - \{A, B\} \qquad \text{and} \qquad \operatorname{int}(\overrightarrow{AB}) = \overrightarrow{AB} - \{A\}.$$

Proposition 1. $\operatorname{int}(\overrightarrow{AB})$ *and* $\operatorname{int}(\overline{AB})$ *are convex.*

Proof. Let $C, D \in \operatorname{int}(\overrightarrow{AB})$ and let $P \in \overline{CD}$. Choose a rule $f : \overleftrightarrow{AB} \to \mathbf{R}$ with

$$f(A) = 0 \qquad \text{and} \qquad f(B) > 0,$$

so $\operatorname{int}(\overrightarrow{AB}) = \{Q \in \overleftrightarrow{AB} : f(Q) > 0\}$. Since $f(P)$ is between $f(C) > 0$ and $f(D) > 0$, we see that $f(P) > 0$, hence $f(P) \in \operatorname{int}(\overrightarrow{AB})$. This being true for every $P \in \overline{CD}$ implies that $\overline{CD} \subset \operatorname{int}(\overrightarrow{AB})$, showing that $\operatorname{int}(\overrightarrow{AB})$ is convex.

To see that $\operatorname{int}(\overline{AB})$ is convex, first note that

$$\operatorname{int}(\overline{AB}) = \operatorname{int}(\overrightarrow{AB}) \cap \operatorname{int}(\overrightarrow{BA})$$

and apply proposition 2 from week 6.

□

Proposition 1 allows us to prove an analogue of (PSA) for lines:

Theorem 2. *Let l be a line in Π and let P be a point on l. There exist disjoint convex sets H_1 and H_2 such that:*

1. *$l - \{P\} = H_1 \cup H_2$;*
2. *If $P_1 \in H_1$ and $P_2 \in H_2$, then $P \in \overline{P_1 P_2}$.*

Proof. Given l and $P \in l$, choose a rule $f : l \to \mathbf{R}$ with $f(P) = 0$. Let A be a point with $f(A) < 0$ and let B be a point with $f(B) > 0$. Define

$$H_1 = \operatorname{int}(\overrightarrow{PA}) \qquad \text{and} \qquad H_2 = \operatorname{int}(\overrightarrow{PB}).$$

By proposition 1, H_1 and H_2 are convex. Note that

$$H_1 = \{Q \in l : f(Q) < 0\} \qquad \text{and} \qquad H_2 = \{Q \in l : f(Q) > 0\},$$

so $H_1 \cap H_2 = \emptyset$ and $l - \{P\} = H_1 \cup H_2$. Finally, let $P_1 \in H_1$ and $P_2 \in H_2$. Since $f(P) = 0$ is between $f(P_1) < 0$ and $f(P_2) > 0$, we see that P is between P_1 and P_2, so $P \in \overline{P_1P_2}$.

□

We would like to prove analogues of proposition 1 for more general objects. First, consider non-collinear points A, B and C in Π. Recall that $\angle ABC$ is the union of two rays:

$$\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}.$$

By (PSA), there exists a unique half-plane determined by the line $\overleftrightarrow{BA}$ containing the point C. This half-plane will be denoted by $H_{AB,C}$. Similarly, we can consider $H_{BC,A}$, the unique half-plane determined by $\overleftrightarrow{BC}$ containing the point A.

Definition 3. The *interior* of the triangle $\angle ABC$ is the set

$$\operatorname{int}(\angle ABC) = H_{BC,A} \cap H_{AB,C}.$$

Since, by (PSA), half-planes are convex, we can use proposition 2 from week 6 to see that:

Proposition 4. $\operatorname{int}(\angle ABC)$ *is convex.*

Using our intuition from $(\mathbf{R}^2, d_2)$, we expect a point P to be in $\operatorname{int}(\angle ABC)$ if and only if the points A and C are on opposite sides of the line $\overleftrightarrow{BP}$. It turns out that this philosophy is true for every Pasch geometry, and it is called the *crossbar theorem*. In order to prove it, we just need a preliminary result about the intersection of rays:

Lemma 5 (Z-theorem). *Let A and B be distinct points on a line l. If C and D are points on opposite sides of l, then*

$$\overrightarrow{AC} \cap \overrightarrow{BD} = \emptyset.$$

Proof. Since $\overrightarrow{AC} \cap l = \{A\}$, $\overrightarrow{BD} \cap l = \{B\}$ and $A \neq B$, we see that

$$\overrightarrow{AC} \cap \overrightarrow{BD} = \operatorname{int}(\overrightarrow{AC}) \cap \operatorname{int}(\overrightarrow{BD}).$$

Let H_1 and H_2 be the half-planes determined by l with $C \in H_1$ and $D \in H_2$. Since $\operatorname{int}(\overrightarrow{AC}) \cap l = \emptyset$ and $C \in H_1$, we have

$$\operatorname{int}(\overrightarrow{AC}) \subset H_1.$$

By a similar argument, $\operatorname{int}(\overrightarrow{BD}) \subset H_2$, so

$$\overrightarrow{AC} \cap \overrightarrow{BD} = \operatorname{int}(\overrightarrow{AC}) \cap \operatorname{int}(\overrightarrow{BD}) \subset H_1 \cap H_2 = \emptyset.$$

□

Theorem 6 (**Crossbar**). *Let A, B and C be non-collinear points. A fourth point P is in* $\mathrm{int}(\angle ABC)$ *if and only if* $\overrightarrow{BP} \cap \overline{AC} = \{D\}$*, for some point D between A and C.*

Proof. First, assume $\overrightarrow{BP} \cap \overline{AC} = \{D\}$, for some point D between A and C. We will show that $P \in \mathrm{int}(\angle ABC)$. There are two cases to consider. First, if $P = D$, then P is between A and C, so $\overline{PC} \cap \overleftrightarrow{AB} = \emptyset$, showing that $P \in H_{AB,C}$. Similarly, $\overline{PA} \cap \overleftrightarrow{BC} = \emptyset$, so $P \in H_{BC,A}$. It follows that

$$P \in H_{AB,C} \cap H_{BC,A} = \mathrm{int}(\angle ABC).$$

Next, we consider the case when $P \neq D$. In this case, P, D and C are non-collinear. Since B is not between D and P, and $\overleftrightarrow{AB} \cap \overleftrightarrow{DP} = \{B\}$, we see that

$$\overleftrightarrow{AB} \cap \overline{DP} = \emptyset.$$

Likewise,

$$\overleftrightarrow{AB} \cap \overline{DC} = \emptyset.$$

By (PP), $\overleftrightarrow{AB} \cap \overline{PC} = \emptyset$, implying that $P \in H_{AB,C}$. A similar argument shows that $P \in H_{BC,A}$, hence

$$P \in H_{AB,C} \cap H_{BC,A} = \mathrm{int}(\angle ABC).$$

Conversely, assume $P \in \mathrm{int}(\angle ABC)$. We will first show that $\overrightarrow{BP} \cap \overline{AC} \neq \emptyset$. In order to apply (PP), we introduce a new point C' on $\overleftrightarrow{BC}$ with B between C' and C. Since A, B and C are non-collinear, the points C', A and C are also non-collinear. Note that the line $\overleftrightarrow{BP}$ does not pass through C', C or A, and it intersects the side $\overline{C'C}$ of $\triangle C'AC$.

Claim. $\overrightarrow{BP} \cap \overline{C'A} = \emptyset$*.*

Proof. Since C and C' are on opposite sides of $\overleftrightarrow{AB}$, and P and C are on the same side of $\overleftrightarrow{AB}$, we see that C' and P are on opposite sides of $\overleftrightarrow{AB}$. By the Z-theorem, $\overrightarrow{BP} \cap \overrightarrow{AC'} = \emptyset$. In particular, $\overrightarrow{BP} \cap \overline{C'A} = \emptyset$ $\square$

Claim. *Let P' be a point on $\overleftrightarrow{BP}$ with B between P' and P. Then,* $\overrightarrow{BP'} \cap \overline{C'A} = \emptyset$*.*

Proof. Since P and P' are on opposite sides of $\overleftrightarrow{BC}$, and P and A are on the same side of $\overleftrightarrow{BC}$, we see that A and P' are on opposite sides of $\overleftrightarrow{BC}$. By the Z-theorem, $\overrightarrow{BP'} \cap \overrightarrow{C'A} = \emptyset$. In particular, $\overrightarrow{BP} \cap \overline{C'A} = \emptyset$. $\square$

Combining the two statements above, we see that $\overleftrightarrow{BP} \cap \overline{C'A} = \emptyset$. Since $\overleftrightarrow{BP} \cap \overline{C'C} = \{B\} \neq \emptyset$, we can apply (PP) to conclude that $\overleftrightarrow{BP} \cap \overline{AC} = \{D\}$, for some point $D \in \overline{AC}$.

Claim. *D is between A and C.*

Proof. Assume for a contradiction that $D = A$ or $D = C$. In the first case, we would have $P \in \overleftrightarrow{DB} = \overleftrightarrow{AB}$, contradicting the fact that $P \in \mathrm{int}(\angle ABC) \subset H_{AB,C}$. In the other case, we obtain a similar contradiction.

$\square$

Finally, we show that $D \in \overrightarrow{BP}$. First note that, since D is between A and C, and $\overleftrightarrow{AD} \cap \overleftrightarrow{BC} = \{C\}$, we have $\overline{AD} \cap \overleftrightarrow{BC} = \emptyset$. In other words, A and D are on the same side of $\overleftrightarrow{BC}$. Since, A and P are on the same side of $\overleftrightarrow{BC}$, we conclude that P and D are on the same side of $\overleftrightarrow{BC}$. It follows that B cannot be between D and P, hence $D \in \overrightarrow{BP}$.

$\square$

We finish this section with a discussion about interiors of triangles.

Definition 7. Let A, B and C be non-collinear points in a Pasch geometry. The *interior* of the triangle $\triangle ABC$ is the set

$$\mathrm{int}(\triangle ABC) = H_{BC,A} \cap H_{AC,B} \cap H_{AB,C}.$$

By proposition 2 from week 6, the interior of $\triangle ABC$ is convex. Recall that, in week 6, we wanted to introduce $\mathrm{int}(\triangle ABC)$ so we could describe the convex hull of $S = \{A, B, C\}$ as $\mathrm{int}(\triangle ABC) \cup \triangle ABC$. We need one last ingredient to show that the above description of $\mathrm{Conv}(S)$ is correct.

Let l be a line in a Pasch geometry and let H be a half-plane defined by l. We define the *closure* of H to be

$$\overline{H} = H \cup l.$$

Proposition 8. *$\overline{H}$ is convex.*

Proof. Let $A, B \in \overline{H}$. There are three cases to consider:

- If $A, B \in H$, then the convexity of H implies that

$$\overline{AB} \subset H \subset \overline{H}.$$

- If $A, B \in l$, then

$$\overline{AB} \subset l \subset \overline{H}.$$

- Finally, consider the case when one of those points is in l and the other one is in H, say $A \in l$ and $B \in H$. Given a point C between A and B, we will show that $C \in \overline{H}$. To do so, assume for a contradiction that $C \notin \overline{H}$. Then, by (a) in (PSA), C and B are on opposite sides of l. By (b) in (PSA), $\overline{CB} \cap l \neq \emptyset$. By Bézout's theorem, since $\overleftrightarrow{CB} = \overleftrightarrow{AB}$ and $A \in l$, we must have

$$\overline{CB} \cap l = \{A\},$$

contradicting the fact that C is between A and B.

$\square$

Proposition 9. *Let A, B and C be non-collinear points. The convex hull of $S = \{A, B, C\}$ is*

$$\mathrm{Conv}(S) = \overline{H_{BC,A}} \cap \overline{H_{AC,B}} \cap \overline{H_{AB,C}} = \mathrm{int}(\triangle ABC) \cup \triangle ABC.$$

Proof. Let $T = \mathrm{int}(\triangle ABC) \cup \triangle ABC$. We leave the proof that

$$T = \overline{H_{BC,A}} \cap \overline{H_{AC,B}} \cap \overline{H_{AB,C}}$$

as an exercise for the reader. With the above equality, we can use proposition 2 from week 6 to conclude that T is convex. Let S' be a convex set containing $S = \{A, B, C\}$. By convexity, S' contains the line segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$, so

$$\triangle ABC \subset S'.$$

Let $P \in \mathrm{int}(\triangle ABC)$, and let D be a point between A and B. By (PP), the line $\overleftrightarrow{PD}$ intersects $\triangle ABC$ at some point E with $E \neq D$. Since P and E are on the same side of $\overleftrightarrow{AB}$ and $D \in \overleftrightarrow{AB}$, we see that P is between E and D. Since $E, D \in \triangle ABC \subset S'$, by convexity, $\overline{DE} \subset S'$, implying that $P \in S'$. It follows that $\mathrm{int}(\triangle ABC) \subset S'$, so

$$T = \mathrm{int}(\triangle ABC) \cup \triangle ABC \subset S'.$$

Since T is convex and is contained in any other convex set containing S, we conclude that T is the convex hull of S.

$\square$