Week 6

Pasch geometries

Let $\Pi = (\mathcal{P}, \mathcal{L}, d)$ be a metric geometry.

Definition 1. A subset S of $\mathcal{P}$ is *convex* (in Π) if, for every $A, B \in S$, we have

$$\overline{AB} \subset S.$$

Note that the entire set of points, $\mathcal{P}$, is convex. On the other end of the spectrum, for every $P \in \mathcal{P}$, the singleton, $\{P\}$ is convex. It's also easy to build convex sets from other convex sets:

Proposition 2. *Let $\{S_i\}$ be an arbitrary collection of convex sets in Π. The intersection*

$$S = \bigcap_i S_i$$

is convex.

Proof. Let A, $B \in S$. Then, $A, B \in S_i$, for every i. Since S_i is convex, we have

$$\overline{AB} \subset S_i.$$

This being true for every index i implies that

$$\overline{AB} \subset \bigcap_i S_i = S,$$

showing that S is convex.

□

Let's look at some basic examples of convex sets:

Proposition 3. *Let A and B be distinct points in Π.*

1. *The line $\overleftrightarrow{AB}$ is convex.*
2. *The ray $\overrightarrow{AB}$ is convex.*
3. *The line segment $\overline{AB}$ is convex.*

Proof.

1. Let $C, D \in \overleftrightarrow{AB}$. Note that
$$\overline{CD} \subset \overleftrightarrow{CD} = \overleftrightarrow{AB}.$$

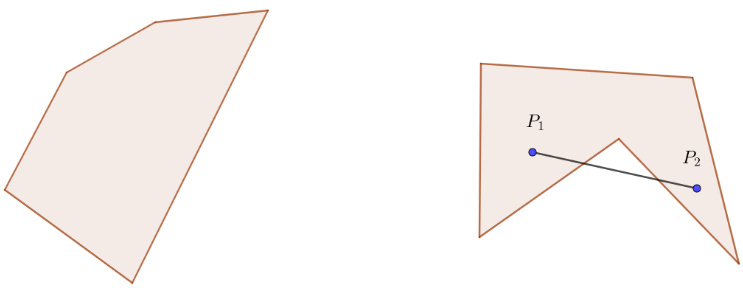
2. Let $C, D \in \overrightarrow{AB}$ and let $P \in \overline{CD}$. Choose a ruler $f : \overleftrightarrow{AB} \to \mathbf{R}$ such that
$$f(A) = 0 \qquad \text{and} \qquad f(B) > 0,$$
in which case
$$\overrightarrow{AB} = \{Q \in \overleftrightarrow{AB} : f(Q) \geq 0\}.$$
Since $C, D \in \overrightarrow{AB}$, we have
$$f(C) \geq 0 \qquad \text{and} \qquad f(D) \geq 0.$$
It follows that $f(P) \geq 0$, so $P \in \overrightarrow{AB}$. This being true for every $P \in \overline{CD}$ implies that $\overline{CD} \subset \overrightarrow{AB}$, showing that $\overrightarrow{AB}$ is convex.

3. By what we just proved, the rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are convex. By proposition 2,
$$\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$$
is convex.

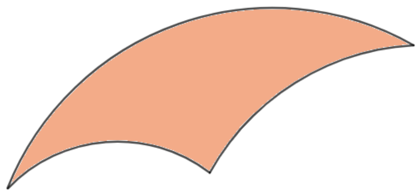
□

Let's now consider some more complicated examples:

Example 4. In the Euclidean plane, $(\mathbf{R}^2, d_2)$, we are used to describing convex sets as sets that do not "curve inward". For example, the figure on the left below depicts a convex set in $\mathbf{R}^2$. However, the region on the right is not convex, since there exist points P_1 and P_2 such that the line segment $\overline{P_1P_2}$ is not contained in the region.



Example 5. The philosophy that convex sets are depicted by regions with no inward indention is useful when Π is the Euclidean plane. Such philosophy might not hold in other metric geometries. For instance, in the next section we will see that the interior of triangles in the Poincaré plane are convex but those regions always have inward indentations:



Given a set S of points in Π, we would like to determine the smallest convex set containing S. If such a set exists, then we call it the *convex hull* of S in Π.

Proposition 6. *For every set S of points in Π, its convex hull exists and is unique. Explicitly, the convex hull of S in Π is:*

$$\mathrm{Conv}(S) = \bigcap_{\substack{X \supset S \\ X \text{ convex}}} X.$$

Proof. By proposition 2,

$$\bigcap_{\substack{X \supset S \\ X \text{ convex}}} X$$

is a convex set. By its definition, the above intersection contains S and is contained in every convex subset of $\mathcal{P}$ containing S, so the above intersection is a convex hull of S.

To show that the convex hull of S is unique, let S' and S'' be convex hulls of S. Since S' contains S and is contained in every convex subset of $\mathcal{P}$ containing S, the minimality of S'' implies that $S'' \subset S'$. Similarly, since S'' contains S and is contained in every convex subset of $\mathcal{P}$ containing S, the minimality of S' implies that $S' \subset S''$. It follows that $S' = S''$.

□

Although proposition 6 provides an explicit description of $\mathrm{Conv}(S)$, finding all convex sets containing S and then taking their intersection is not very practical. If S is convex already, then $\mathrm{Conv}(S) = S$, so the natural question is: when S is not convex, is there an algorithm to determine $\mathrm{Conv}(S)$? Here, by an algorithm, we loosely mean a way to describe $\mathrm{Conv}(S)$ using finitely many steps and basic objects such as lines, rays and line segments. Of course, we should restrict ourselves to finite sets. When $|S| = 1$ or 2, the procedure is simple:

Proposition 7.

1. *If $S = \{P_1\}$, then*
$$\mathrm{Conv}(S) = \{P_1\}.$$
2. *If $S = \{P_1, P_2\}$ with $P_1 \neq P_2$, then*
$$\mathrm{Conv}(S) = \overline{P_1 P_2}.$$

Proof.

1. In this case, $S = \{P_1\}$ is convex already, so

$$\mathrm{Conv}(S) = S = \{P_1\}.$$

2. Since $\mathrm{Conv}(S)$ is a convex set and contains $S = \{P_1, P_2\}$, we have

$$\overline{P_1P_2} \subset \mathrm{Conv}(S).$$

By proposition 3, the line segment $\overline{P_1P_2}$ is a convex set containing $S = \{P_1, P_2\}$, so

$$\mathrm{Conv}(S) \subset \overline{P_1P_2}.$$

It follows that $\mathrm{Conv}(S) = \overline{P_1P_2}$.

□

The case $|S| = 3$ is more complicated. In the special case when S consists of 3 collinear points P_1, P_2 and P_3, say all lying on a line l, then we choose a ruler

$$f : l \to \mathbf{R}$$

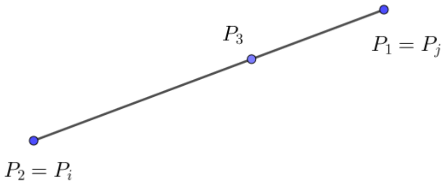
and select indices i and j with

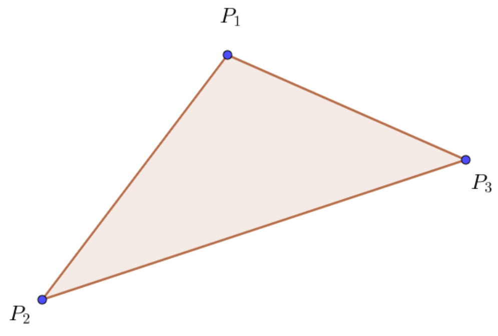
$$f(P_i) = \min\{f(P_1), f(P_2), f(P_3)\} \qquad \text{and} \qquad f(P_j) = \max\{f(P_1), f(P_2), f(P_3)\}.$$

In this case,

$$\mathrm{Conv}(S) = \overline{P_iP_j}.$$

When the points P_1, P_2 and P_3 are not collinear, our intuition from $(\mathbf{R}^2, d_2)$ says that the convex hull of S should be the triangle $\triangle P_1P_2P_3$ together with its interior:





The issue with the above intuition is that, in a general metric geometry, it is not clear what the interior of a triangle is. Perhaps, in order to preserve our intuition, we should define the interior of $\triangle P_1P_2P_3$ to be the convex hull of $\{P_1, P_2, P_3\}$ minus $\triangle P_1P_2P_3$. However, we will opt for a more traditional approach and introduce a new axiom that will allow us to define the interior of general polygons:

Definition 8. A *Pasch geometry* is a metric geometry $(\mathcal{P}, \mathcal{L}, d)$ satisfying the *plane separation axiom*:

(PSA) For every $l \in \mathcal{L}$, there exist disjoint convex sets of points H_1 and H_2 such that:

(a) $\mathcal{P} - l = H_1 \cup H_2$;

(b) If $P_1 \in H_1$ and $P_2 \in H_2$, then $\overline{P_1 P_2} \cap l \neq \emptyset$.

The convex sets H_1 and H_2 are referred to as the *half-planes* determined by l. For each line l, we can introduce an equivalence relation between points as follows: we say $A \sim B$ if there exists a half-plane H determined by l such that $A, B \in H$. In this case, we say that A and B are *on the same side* (of the line l). Otherwise, we say that A and B are *on opposite sides* (of l). The next proposition summarizes the main properties of this sidedness relation:

Proposition 9. *Let l be a line in a Pasch geometry. Let A, B and C be distinct points not on l.*

1. *If A and B are on the same side of l, and B and C are on the same side of l, then A and C are on the same side of l. In other words, the relation $\sim$ introduced above is transitive.*
2. *If A and B are on opposite sides of l, and B and C are on opposite sides of l, then A and C are on the same side of l.*
3. *A and B are on the same side of l if and only if $\overline{AB} \cap l = \emptyset$.*
4. *A and B are on opposite sides of l if and only if $\overline{AB} \cap l \neq \emptyset$.*

Proof.

1. Let H and H' be half-planes determined by l with $A, B \in H$ and $B, C \in H'$. Since the half-planes determined by l are disjoint, $B \in H \cap H'$ implies that $H = H'$. Therefore, $A, C \in H = H'$, showing that A and C are on the same side of l.

2. Let H and H' be the two disjoint half-planes determined by l such that $A \in H$ and $B \in H'$. Since B and C are on opposite sides of l, and $B \in H'$, we must have $C \in H$. Therefore, $A, C \in H$, showing that A and C are on the same side of l.

3. First, suppose that A and B are on the same side of l. Then, there exists a half-plane H determined by l such that $A, B \in H$. Since H is convex, $\overline{AB} \subset H$. By (a) in (PSA),

$$\overline{AB} \cap l \subset H \cap l = \emptyset.$$

Conversely, suppose that A and B are on opposite sides of l. Then, by (b) in (PSA), we have

$$\overline{AB} \cap l \neq \emptyset.$$

4. This statement is equivalent to (3) via negation.

□

To illustrate the usefulness of the above proposition, let's use proposition 9 to show that the half-planes determined by l cannot be empty.

Corollary 10. *Let l be a line in a Pasch geometry. Then, the half-planes H_1 and H_2 determined by l are not empty sets.*

Proof. By the incidence axiom (I3), $\mathcal{P} - l \neq \emptyset$, so $H_1 \neq \emptyset$ or $H_2 \neq \emptyset$. Without loss of generality, let's say that $H_1 \neq \emptyset$. Let $P_1 \in H_1$. Let $P \in l$ and consider the unique line l' joining P_1 and P. Choose a ruler

$$f : l' \to \mathbf{R}$$

and a point $P_2 \in l'$ such that $f(P)$ is between $f(P_1)$ and $f(P_2)$. Then, P is between P_1 and P_2 on the line l', so $P \in \overline{P_1P_2}$. It follows that

$$\overline{P_1P_2} \cap l = \{P\} \neq \emptyset.$$

By proposition 9, P_1 and P_2 are on opposite sides of l, implying that $P_2 \in H_2$, hence $H_2 \neq \emptyset$. □

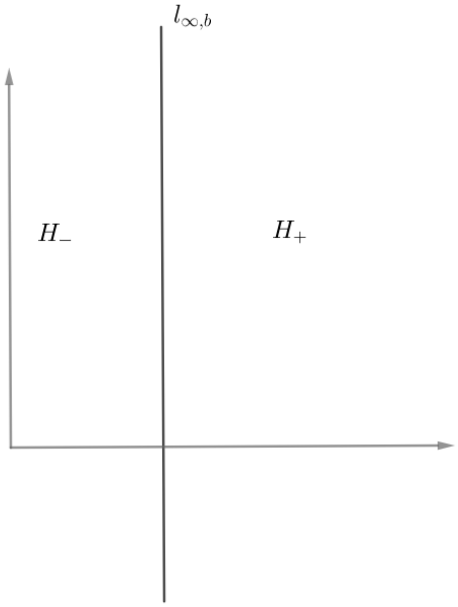
Fortunately, most geometries that we are used to are Pasch. Below we describe the half-planes in the Euclidean plane and in the Poincaré plane, and we leave the verification of (PSA) as an exercise for the reader:

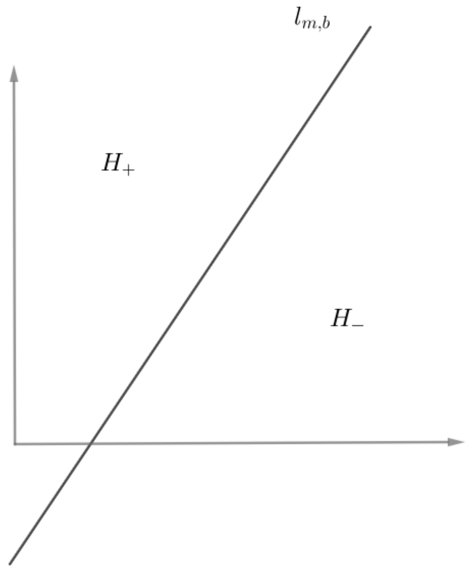
Example 11. The Euclidean plane $(\mathbf{R}^2, d_2)$ is Pasch. The half-planes determined by the line $l_{\infty,b}$ are

$$H_+ = \{(x,y) : x > b\} \qquad \text{and} \qquad H_- = \{(x,y) : x < b\}.$$

The half-planes determined by the line $l_{m,b}$ are

$$H_+ = \{(x,y) : y > mx + b\} \qquad \text{and} \qquad H_- = \{(x,y) : y < mx + b\}.$$



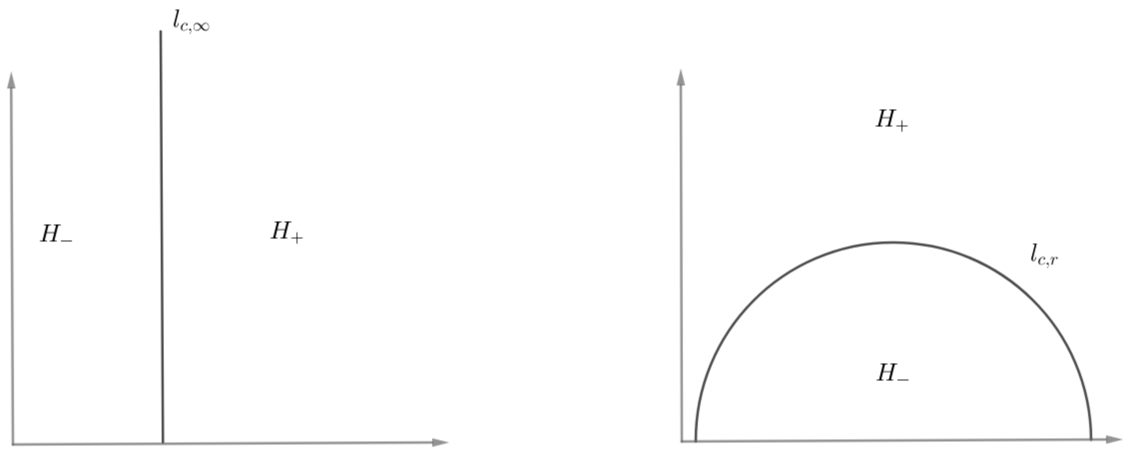


Example 12. The Poincaré plane $(\mathbf{H}, d_h)$ is Pasch. The half-planes determined by the line $l_{c,\infty}$ are

$$H_+ = \{(x,y) : x > c,\ y > 0\} \qquad \text{and} \qquad H_- = \{(x,y) : x < c,\ y > 0\}.$$

The half-planes determined by the line $l_{c,r}$ are

$$H_+ = \{(x,y) : (x-c)^2 + y^2 > r^2,\ y > 0\} \qquad \text{and} \qquad H_- = \{(x,y) : (x-c)^2 + y^2 < r^2,\ y > 0\}.$$



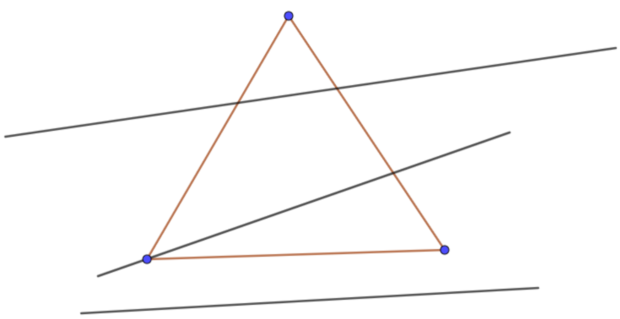
Showing that a metric geometry violates (PSA) directly can be hard. Fortunately, we can show that (PSA) is equivalent to the following more practical axiom:

Definition 13. A metric geometry is said to satisfy *Pasch's postulate* if:

(PP) for every triple of non-collinear points A, B and C, if l is a line with $l \cap \overline{AB} \neq \emptyset$, then

$$l \cap \overline{BC} \neq \emptyset \qquad \text{or} \qquad l \cap \overline{AC} \neq \emptyset.$$

(PP) says that a line never intersects only one side of a triangle. Here's a depiction of the three ways a line can intersect a given triangle:



Before showing that (PSA) and (PP) are equivalent, let's make (PP) more precise:

Lemma 14. *In a metric geometry, (PP) holds if and only, for every triple of non-collinear points A, B, C and, for every l not passing through A, B or C, one of the following holds:*

1. $l \cap \triangle ABC = \emptyset$*; or*
2. l *intersects exactly two sides of* $\triangle ABC$.

Proof. It is clear that this new statement implies (PP). Conversely, suppose (PP) holds. Let A, B and C be non-collinear points, and let l be a line not passing through A, B or C. (PP) guarantees that l does not intersect $\triangle ABC$ or it passes through at least two sides of $\triangle ABC$. To show that the statement in the lemma holds, assume, for a contradiction that l intersects all sides of $\triangle ABC$. Without loss of generality, let's say that

$$l \cap \overline{AB} = \{D\}, \qquad l \cap \overline{BC} = \{E\}, \qquad l \cap \overline{AC} = \{F\},$$

with E between D and F on the line l. Since l passes through D and F but it does not pass through A, we see that the points A, D and F are non-collinear. Let $l' = \overleftrightarrow{BC}$. Note that

$$l' \cap \overline{AF} \subset \overleftrightarrow{BC} \cap \overleftrightarrow{AC} = \{C\}$$

but F is between A and C, so

$$l' \cap \overline{AF} = \emptyset.$$

Similarly, we can see that

$$l' \cap \overline{AD} = \emptyset.$$

However,

$$l' \cap \overline{DF} = \{E\} \neq \emptyset,$$

contradicting (PP).

□

Theorem 15 (Pasch). *A metric geometry satisfies (PSA) if and only if it satisfies (PP).*

Proof. We first assume that (PSA) holds. Let A, B and C be non-collinear points, and let l be a line with $l \cap \overline{AB} \neq \emptyset$. Suppose $l \cap \overline{BC} = \emptyset$. By proposition 9, A and B on opposite sides of l, and B and C are on the same side of l, hence A and C are on opposite sides of l, implying that $l \cap \overline{AC} \neq \emptyset$.

Next, we assume that (PP) holds. Let l be a line. By our incidence axiom (I3), we can fix a point P not on l and define

$$H_1 = \{Q \in \mathcal{P} - l : Q = P \text{ or } l \cap \overline{PQ} = \emptyset\} \qquad \text{and} \qquad H_2 = \{Q \in \mathcal{P} - l : Q \neq P \text{ and } l \cap \overline{PQ} \neq \emptyset\}.$$

By their definition, it is clear that

$$\mathcal{P} - l = H_1 \cup H_2 \qquad \text{and} \qquad H_1 \cap H_2 = \emptyset.$$

Claim. *H_1 is convex.*

Proof. Let A and B be distinct points in $H_1 - \{P\}$, so $l \cap \overline{PA} = \emptyset$ and $l \cap \overline{PB} = \emptyset$. Given $Q \in \overline{AB}$, we will show that $Q \in H_1$. If $Q = P$, then $Q \in H_1$ and we are done. Otherwise, there are two cases to consider. First, suppose A, B and P are collinear. In this case, we have

$$l \cap \overline{PQ} \subset l \cap \big(\overline{PA} \cup \overline{PB}\big) = \big(l \cap \overline{PA}\big) \cup \big(l \cap \overline{PB}\big) = \emptyset,$$

showing that $l \cap \overline{PQ} = \emptyset$, implying that $Q \in H_1$.

Now, we consider the case when A, B and P are not collinear. By (PP), since $l \cap \overline{PA} = \emptyset$ and $l \cap \overline{PB} = \emptyset$, we must have $l \cap \overline{AB} = \emptyset$. In particular, $l \cap \overline{AQ} = \emptyset$. Again, by (PP), since $l \cap \overline{AQ} = \emptyset$ and $l \cap \overline{PA} = \emptyset$, we must have $l \cap \overline{PQ} = \emptyset$, showing that $Q \in H_1$.

□

Claim. *H_2 is convex.*

Proof. Let A and B be distinct points in H_2, so $l \cap \overline{PA} \neq \emptyset$ and $l \cap \overline{PB} \neq \emptyset$. Let $Q \in \overline{AB}$. There are two cases to consider. First, suppose A, B and P are collinear. By Bézout's theorem, there exists $D \in l$ such that

$$l \cap \overline{PA} = l \cap \overline{PB} = \{D\}.$$

Note that P is not between A and B, so $Q \neq P$, and

$$l \cap \overline{PQ} \supset l \cap \big(\overline{PA} \cap \overline{PB}\big) = \{D\} \cap \overline{PB} = \{D\},$$

so $l \cap \overline{PQ} = \{D\} \neq \emptyset$, showing that $Q \in H_2$.

Now, we consider the case when A, B and P are not collinear. By lemma 14, since $l \cap \overline{PA} \neq \emptyset$ and $l \cap \overline{PB} \neq \emptyset$, we must have $l \cap \overline{AB} = \emptyset$. In particular, $l \cap \overline{AQ} = \emptyset$. By (PP), since $l \cap \overline{AQ} = \emptyset$ and $l \cap \overline{PA} \neq \emptyset$, we must have $l \cap \overline{PQ} \neq \emptyset$, showing that $Q \in H_2$.

□

Claim. *If $P_1 \in H_1$ and $P_2 \in H_2$, then $l \cap \overline{P_1P_2} \neq \emptyset$.*

Proof. If $P = P_1$, then the result follows from the definition of H_2. If $P \neq P_1$, then we have two cases to consider. First, suppose P, P_1 and P_2 are collinear. In this case, since $l \cap \overline{PP_1} = \emptyset$ but $l \cap \overline{PP_2} \neq \emptyset$, either we have P between P_1 and P_2 or we have P_1 between P and P_2. If P is between P_1 and P_2, then

$$\overline{P_1P_2} \cap l \supset \overline{PP_2} \cap l \neq \emptyset,$$

so $\overline{P_1P_2} \cap l \neq \emptyset$. If P_1 is between P and P_2, then $\overline{PP_2} = \overline{PP_1} \cup \overline{P_1P_2}$, so

$$l \cap \overline{PP_1} = \emptyset \qquad \text{and} \qquad l \cap \overline{PP_2} \neq \emptyset$$

imply that $l \cap \overline{P_1P_2} \neq \emptyset$.

Now, we consider the case when P_1, P_2 and P are not collinear. By (PP), since $l \cap \overline{PP_1} = \emptyset$ and $l \cap \overline{PB} \neq \emptyset$, we must have $l \cap \overline{P_1P_2} \neq \emptyset$. □

Since the construction of H_1 and H_2 is possible for every line l, we conclude that (PSA) holds.

□