Calculus I

A textbook for Pitzer's Ma030 Course

Jane Panangaden

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Introduction

This textbook is under construction! Please check back often to access the latest version at the Github repository.

This course has twin goals. The first is to develop the theory of calculus and learn how to apply its tools to analyze rates of change in continuous systems. The second is to use calculus of an example of how to build up a mathematical theory, and in doing so to get a feeling for what it means to "think mathematically". In service of the second goal, the course begins with a substantial section of foundations of math and logic.

1 Mathematical Language

1.1 Propositional logic

Doing mathematics consists of making rigorous arguments that certain statements are true. Consequently, the pieces of language that we will be most interested in are full sentences that are statements which can be clearly evaluated as either true or false, but not both. Such statements are called propositions.

Definition 1: Proposition

A **proposition** is a statement that is either true or false, but not both.

Let's consider the following chunks of words in English and evaluate whether or not they are propositions.

"Penguins can swim."	This is a full sentence, it is a statement, and it is true. This is a proposition.
"Penguins can fly."	This is a full sentence, it is a statement, and it is false. This is a proposition.
"Look at that penguin!"	This is a full sentence, but it is not a statement. It is a command (imperative). This is not a proposition.
"What is your favorite species of penguin?"	This is a full sentence, but it is not a statement. It is a question. This is not a proposition.
"That penguin's foot"	This is not a full sentence, so it is not a proposition.

We have seen three examples of chunks of words that fail to be propositions because they are not statements. What about the other requirement that the statement be true or false but not both; is it really possible for a statement to be true and false? Can a statement be neither true nor false?

Let's think about this example:

"This sentence is false."

Let's call the sentence above by the name S. If S is true, then S must also be false. However, if S is false, then it must be false that S is false, and hence S must be true. We have just argued that if S can be assigned any value of true or false, it must actually be both true and false. The one thing we have figured out is definitely true is that S is not a proposition.

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Consider the following two sentences.

The next sentence is true. The previous sentence is false.

Are either of the above sentences propositions? Would they be propositions if they were individually written on different pages of this book?

If mathematicians find a particular proposition to be very interesting or cool, and someone has successfully proven that it is true, they will call it a **theorem**.

A proposition can be either true or false, but not both. We can assign a symbol to a proposition that represents whether it is true or false. This symbol is called a truth value.

Definition 2: Truth value

A proposition is assigned a **truth value** of T if the proposition is true and F is the proposition is false.

For example, *"Penguins can swim."* is a proposition with a truth value of T and *"Penguins can fly."* is a proposition with a truth value of F.

1.1.1 Logical operators

Propositions can be combined together to form more complex propositions. One way of doing this is by using logical operators.

Definition 3: Logical operator

A **logical operator** is a machine that takes some propositions as input and returns a new proposition as output whose truth value is determined by the truth values of the inputs.

Four important logical operators that we will define are AND, OR, NOT, and IMPLIES. These logical operators emulate the function of the words “and”, “or”, “not”, and the “if - then” construction in English language. We will define these operators using a **truth table**, which is a table that describes the truth value of the more complex proposition formed by applying the logical operator to simpler propositions, based on the truth values of the simpler input propositions.

Definition 4: Truth table

A **truth table** for a proposition formed using logical operators applied to simple propositions is a table with one column for each simple proposition, a final column for the overall proposition, and one row for each combination of possible truth values of the simple propositions. The final entry of each row shows the corresponding truth value of the overall proposition.

Suppose P is a proposition formed by connecting simple propositions P_1,...,P_n using logical operators. Its truth table is of the following form.

P_1	$\dots$	P_n	P
T	$\dots$	T	$\star$
T	$\dots$	F	$\star$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	$\dots$	T	$\star$
T	$\dots$	F	$\star$

The $\star$ symbols represent spots which can have a value of either T or F and will change depending on which logical operators are used.

AND

Suppose P and Q are two propositions. The truth value of the proposition P AND Q is defined by the following truth table.

P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

This table says that the proposition P AND Q is true only when both P and Q are true. If any one of P or Q is false, then P AND Q is false. Hence, in order to argue that P AND Q is true, we must first argue that P is true and then argue that Q is true.

For example, the proposition *"Penguins can swim AND penguins are birds"* is true, because both *"Penguins can swim.* and *"Penguins are birds."* are true.

However, the proposition *"Penguins can swim AND penguins can fly."* is false, because one of the simpler statements (*"Penguins can fly."*) is false.

OR

Suppose P and Q are two propositions. The truth value of the proposition P OR Q is defined by the following truth table.

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

This table says that the proposition P OR Q is true as long as one of P or Q (or possibly both) are true. In order to argue that P OR Q is true we can either argue that P is true, or we can argue that Q is true. It might be the case that both are true, but showing that one is true is sufficient.

For example, the proposition *"Penguins can swim OR penguins are birds"* is true, because at least one of *"Penguins can swim.* and *"Penguins are birds."* is true. In fact, they are both true.

The proposition *"Penguins can swim OR penguins can fly."* is also true, because one of the simpler statements (*"Penguins can swim"*) is true.

NOT

Suppose P is a proposition. The truth value of NOT P is defined by the following table.

P	NOT P
T	F
F	T

NOT P is true exactly when P is false and vice versa. In order to argue that NOT P is true, we must argue that P is false.

For example, *"NOT penguins can fly."* is true because *"Penguins can fly."* is false. When using English as a natural language we often rewrite the statement *"NOT penguins can fly."* as *"Penguins cannot fly."*.

People sometimes abbreviate NOT P by using the symbol $\sim$ P or the symbol $\neg$ P.

IMPLIES

The final logical operator we'll consider is a little more subtle. We want to have a formal way of finding out the truth values of propositions that take the form of sentences like

"If a cat has spots then it is a leopard."

The above proposition is false because there are some cats that have spots but are not leopards. For example, my pet cat John Brown has spots, but he is not a leopard. There is a particular cat for whom the first part of the sentence is true (it has spots) but the second part is false (it is not a leopard). This is the phenomenon that causes the overall sentence to be false.

So, what we want our truth table to capture is that the whole proposition should be true if whenever the first part of the sentence following "if" is true, then the second part of the sentence following"then" is also true.

Let P and Q be two propositions. Then the truth value of P IMPLIES Q, which is equivalent to *"if P then Q"* in natural language, is given by the following table.

P	Q	P IMPLIES Q
T	T	T
T	F	F
F	T	T
F	F	T

The only way that P IMPLIES Q can be false is if P is true and yet Q is false.

We will abbreviate IMPLIES by the symbol $\implies$.

Let's look at examples of all four rows in this truth table.

"If penguins can swim then the sky is blue."	This is true, because both parts are true.

"If penguins can swim then the sky is green."	This is false, because the first part is true but the second part is false.
"If penguins can fly then the sky is blue."	This is true, because the first part is false and the second part is true.
"If penguins can fly then the sky is green."	This is true, because although the second part is false, the first part is also false.

When we have a proposition of the form P IMPLIES Q (or $P \implies Q$), then the proposition P is called the **hypothesis** and the proposition Q is called the **conclusion**. In order to argue that $P \implies Q$ is true, we must assume that P is true and show that in this case Q is also true. We don't need to worry about the case that P is false, because if P is false, then $P \implies Q$ is always true regardless of the truth value of Q.

Sometimes, two propositions imply each other. In other words, we may have $P \implies Q$ AND $Q \implies P$. We often abbreviate this by writing "$P \iff Q$" or "P if and only if Q". In order to prove that $P \iff Q$ is true, we must prove that $P \implies Q$ is true, and then prove that $Q \implies P$ is true.

Combining logical operators

We can combine multiple logical operators together to create ever more complicated propositions. For example, we could suppose that P, Q, and R are three propositions and think about the truth value of P AND (Q OR R). Let's write down a truth table to understand this complex proposition. First we'll calculate the truth values of Q OR R, and then we'll tackle the whole thing.

Since each of the three propositions P, Q, and R can take on two different truth values, there are $2 \times 2 \times 2 = 8$ rows in our truth table.

P	Q	R	Q OR R	P AND (Q OR R)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

?

If a proposition is formed by combining n different simple propositions using logical operators, how many rows will it have in its truth table?

Sometimes we'll find that two different ways of combining propositions yields exactly the same truth table. For example, let's work out the truth table of (NOT P) AND (NOT Q).

P	Q	NOT P	NOT Q	(NOT P) AND (NOT Q)
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Now, let's write down the truth table of NOT (P OR Q), which is a completely different construction.

P	Q	P OR Q	NOT (P OR Q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Looking at the first two and the last columns we see that these two truth tables are exactly the same! When this happens, we say the two propositions are logically equivalent because one is true exactly when the other is true.

Definition 5: Logical equivalence

If two propositions formed out of simple propositions and logical operators have the same truth table, we say they are **logically equivalent**.

It is useful to identify logically equivalent propositions because if we can argue that one of them must be true, we will also automatically know that the other one is true as well.

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EXOR (read "exclusive or") is another logical operator defined by the following truth table.

P	Q	P EXOR Q
T	T	F
T	F	T
F	T	T
F	F	F

Can you find a proposition constructed using P, Q, AND, OR, and NOT that is logically equivalent to P EXOR Q?

The contrapositive

There is one example of logical equivalence that turns out to be particularly useful called the contrapositive. It gives us a proposition that is equivalent to $P \implies Q$.

Proposition 1.1.1: Contrapositive

$P \implies Q$ is logically equivalent to $(NOTQ) \implies (NOTP)$.

Proof.

We will calculate the truth table of $(NOTQ) \implies (NOTP)$ and compare it to the truth table of $P \implies Q$.

P	Q	NOT Q	NOT P	(NOT Q) $\implies$ (NOT P)
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Looking at the first two and the last column, we see that this is identical to the truth table for $P \implies Q$. $\square$

Consider the proposition,

If penguins can swim, the sky is blue.

We can write this using symbols as: penguins can swim $\implies$ the sky is blue. To find the contrapositive of this propositions, we reverse the order of the hypothesis and the conclusion and negate both of them. In this example, the contrapositive is: NOT(the sky is blue) $\implies$ NOT(penguins can swim), or in natural language,

If the sky isn't blue, then penguins cannot swim.

This contrapositive is logically equivalent to the original proposition. One or the other may be easier to establish, but once we've established one the other follows automatically.

1.2 Sets

A set is the most basic mathematical concept that we will use to start building up our theory.

Definition 6: Set

A **set** is a container with mathematical objects inside of it. These objects can be numbers, shapes, words, symbols, other sets, or anything we want.

We denote a set by writing curly brackets $\{\}$ with a list of the objects the set contains inside the brackets, separated by commas. For example,

$$\{3, \triangle, cat\}$$

is the set containing the number 3, the symbol $\triangle$, and the word *cat*. Similarly,

$$\{4.2, a, \{0, 1\}\}$$

is the set containing the number 4.2, the letter a, and the set $\{0, 1\}$, which itself is a set containing the numbers 0 and 1. We may want to express the idea that a certain object is inside a set.

Definition 7: Element

If S is a set containing the object x, we say that x is an **element** of S. We denote this using the symbol $\in$ by writing $x \in S$.

For example, we say that 3 is an element of $\{3, \triangle, cat\}$ or $3 \in \{3, \triangle, cat\}$. If a certain object is not in a set, we say that the object is not an element of a set and use the symbol $\notin$. For example, $4 \notin \{3, \triangle, cat\}$.

A set can even have nothing inside it.

Definition 8: Empty set

A set with nothing inside it is called the **empty set** and it is written either as $\{\}$ or ϕ.

A set may also have infinitely many things inside it. However, we can't list out infinitely many elements when writing such a set down, because it would take forever. Sometimes we use ellipses to suggest a pattern. For example

$$\{1, 2, 3, 4, \ldots\}$$

denotes the set containing every positive whole number. This set has infinitely many elements. We will see another way to write infinite sets later in this section.

1.2.1 Subsets and equality

A natural question to ask is how we can compare two sets to each other. We'll start with a notion of one set being "smaller" than another.

Definition 9: Subset

A set S_1 is a **subset** of a set S_2 if every element of S_1 is also an element of S_2. We denote this by $S_1 \subset S_2$.

To check if a set S_1 is a subset of S_2, we have to go through each element of S_1 and verify that it is an element of S_2 as well.

For example, $\{1, 3\}$ is a subset of $\{1, 2, 3, 4\}$. We can check this by going through each element of the first set and seeing that it is in the second set. First we check that 1 is in $\{1, 2, 3, 4\}$, which it is, and then we check that 3 is in $\{1, 2, 3, 4\}$, which it is.

On the other hand $\{1, 5\}$ is not a subset of $\{1, 2, 3\}$ because there is an element of the first set, namely 5, that is not in the second set.

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What are all the subsets of the empty set?

Next we'll define a notion of equality for subsets.

Definition 10: Equality of sets

Two sets are equal if they have exactly the same elements.

Notice that the order that elements are listed in the sets does not matter. For example, $\{0, 1\}$ and $\{1, 0\}$ are equal as sets because they contain exactly the same two elements.

When dealing with numbers x and y, we know that $x = y$ exactly when x is smaller than or equal to y ($x \leq y$) and y is smaller than or equal to x ($y \leq x$). A similar statement is true for sets, subsets, and equality of sets.

Proposition 1.2.1: Subsets and equality

Let S_1 and S_2 be sets.

$S_1 = S_2 \iff S_1 \subset S_2$ and $S_2 \subset S_1$.

Proof.

We have two propositions to prove, and both involve the IMPLIES operator. Let's begin with the first one: "$S_1 = S_2 \implies S_1 \subset S_2$ and $S_2 \subset S_1$. "

The hypothesis is $S_1 = S_2$. We will assume that the hypothesis is true. Now we will try to argue that the conclusion is also true.

In order for ($S_1 \subset S_2$) AND ($S_2 \subset S_1$) to be true, both ($S_1 \subset S_2$) and ($S_2 \subset S_1$) must be true. Let's first show that ($S_1 \subset S_2$) is true.

Let x be any element of S_1. Since $S_1 = S_2$, x must also be an element of S_2. Any element of S_1 is also in S_2. Hence, $S_1 \subset S_2$.

Similarly, let x be any any element of S_2. Since $S_1 = S_2$, x must also be an element of S_1. Hence, $S_2 \subset S_1$.

We have shown that if $S_1 = S_2$, then ($S_1 \subset S_2$) and ($S_2 \subset S_1$).

Now let's show the second statement, which is that ($S_1 \subset S_2$) and ($S_2 \subset S_1$) $\implies$ $S_1 = S_2$. Now the hypothesis is that ($S_1 \subset S_2$) and ($S_2 \subset S_1$). We'll assume this hypothesis is true. We need to show, then, that $S_1 = S_2$.

Let x be any element of S_1. Since $S_1 \subset S_2$, x is also an element of S_2. Let y be any element of S_2. Since $S_2 \subset S_1$, y is also an element of S_1. Hence, S_1 and S_2 have exactly the same elements and $S_1 = S_2$. $\square$

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Suppose you have three sets S_1, S_2, and S_3 satisfying $S_1 \subset S_2$, $S_2 \subset S_3$ and $S_3 \subset S_1$. Are all three of the sets equal to each other?

1.2.2 Set builder notation

Set-builder notation is a convenient way to define subsets of an already established set. The idea is that we will select out certain elements of the set which satisfy some criteria.

Definition 11: Set-builder subset

Let S be a set. We denote by

$$\{x \in S | \text{criteria on } x\}$$

the set which contains all the elements of S that satisfy the criteria written to the right of the vertical line.

For example, let $S = \{1, 2, 3, 4, 5, 6\}$. We define a subset of S by

$$S' = \{x \in S | x \text{ is an even number}\}.$$

Now, in order to write down a list of the elements in S' explicitly, we will go through each element of S and check if it is an even number. If it is, we will include it in S'. Following this procedure we find that 1 is not even, so it is not included in S', 2 is even and is included in S' and so on until we get

$$S' = \{2, 4, 6\}.$$

We can write more complicated criteria using logical operators. For example

$$S'' = \{x \in S | x \text{ is even and } x \text{ is prime}\}.$$

Only one element of S is both an even number and a prime number, namely the number 2. Writing out a list of its elements we get $S'' = \{2\}$.

1.2.3 Set operations

We will now study ways of combining and modifying sets to form new, more complicated sets.

Definition 12: Set operation

A **set operation** is a machine which takes as input some number of sets and gives a new set as output.

A **binary** set operation takes two sets as input.

We will introduce five set operations: UNION, INTERSECTION, SET MINUS, COMPLEMENT, and DIRECT PRODUCT. We shall see that these set operations have some connection to the logical operators we studied earlier.

UNION

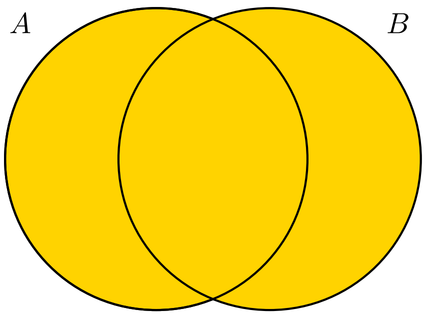
The union is a binary set operation. It takes all the elements in two sets and bundles them together in a new set.

Definition 13: Union of sets

Let A and B be two sets. The **union** of A and B is

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

Notice that the definition of the union relies on the logical operator OR. We can represent the union of sets visually on a Venn diagram.



The entirety of the circles representing A and B are colored in, because any element that is in either set will also be in the union.

For example, let $A = \{1, \triangle, \square\}$ and $B = \{0, 1, 2\}$. The set $A \cup B$ consists of all the elements that appear in either A or B, or possibly both. If an element does appear in both A and B, it is counted only once in the union. So, we have

$$A \cup B = \{0, 1, 2, \triangle, \square\}.$$

INTERSECTION

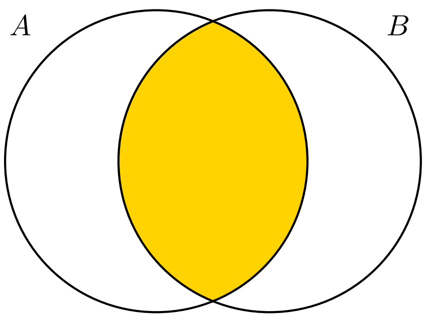
The intersection is also a binary set operation which takes two sets as input and gives one new set as output.

Definition 14: Intersection of sets

Let A and B be two sets. The **intersection** of A and B is

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

The definition of the intersection relies on the logical operator AND. Representing the intersection of sets A and B on a Venn diagram, we color in only the section where A and B overlap, as an element must be in both A and B in order to be in $A \cap B$.



Taking the same example as before, let $A = \{1, \triangle, \square\}$ and $B = \{0, 1, 2\}$. There is only one element that appears in both A and B, namely 1. Hence

$$A \cap B = \{1\}.$$

It could be the case that there are no elements in the intersection. For example, if $C = \{1, 3, 5\}$ and $D = \{2, 4, 6\}$ then no element appears in both C and D. In this case we say the intersection is empty, or write $C \cap D = \phi$.

SET MINUS

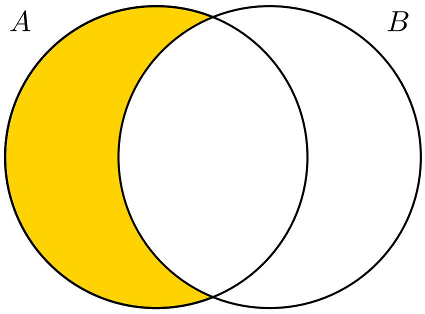
Set minus is a binary operation which takes two sets as input and gives one new set as output. However, contrary to the previous two operations, the order of the input sets matters.

Definition 15: Set minus

Let A and B be two sets. Then A set minus B is

$$A \setminus B = \{x | x \in A \text{ and } x \notin B\}.$$

The definition of set minus uses both the AND and the NOT logical operators. Representing $A \setminus B$ on a Venn diagram, only the part of A which lies outside of B is colored.



Let's consider our trusty example of $A = \{1, \triangle, \square\}$ and $B = \{0, 1, 2\}$. We must identify those elements of A which do not appear in B. Noticing that 1 appears in B, but $\triangle$ and $\square$ do not, we find that

$$A \setminus B = \{\triangle, \square\}.$$

?

What would $B \setminus A$ be for the above example? Is it the same as $A \setminus B$?

COMPLEMENT

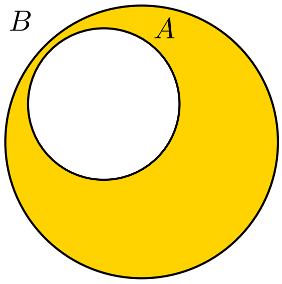
Our next set operation is slightly different. It still takes two sets as input, but one of these sets must be a subset of the other.

Definition 16: Complement

Let A and B be two sets such that $A \subset B$. Then the complement of A (in B) is

$$A^C = \{x \in B | x \notin A\}.$$

Think of B as a background set in which A is embedded. A^C takes all of the elements in the background that are outside of A. Representing this on a Venn diagram, we draw A inside B and color in everything outside of A.



For example, let $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{2, 3\}$. Then, with respect to the background B, the complement of A is

$$A^C = \{1, 4, 5, 6, 7, 8, 9, 10\}.$$

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In the special case that $A \subset B$, rewrite A^C (with respect to B) using the SET MINUS operation.

Direct Products

The final set operation is the direct product. It is a binary operation which can be thought of as the set version of multiplication for numbers.

Definition 17: Direct product

Let A and B be two sets. The **direct product** of A and B is the set

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}.$$

Here (x, y) is an *ordered pair*, so for example $(1, 2)$ is not the same as $(2, 1)$. Notice that if A and B are both finite with n and m elements respectively, the direct product $A \times B$ has $n \times m$ elements.

Going back to our example $A = \{1, \triangle, \square\}$ and $B = \{0, 1, 2\}$, we can calculate that the direct product is

$$A \times B = \{(1, 0), (1, 1), (1, 2), (\triangle, 0), (\triangle, 1), (\triangle, 2), (\square, 0), (\square, 1), (\square, 2)\}.$$

?

Is $A \times B$ always the same as $B \times A$? In the case that A and B are finite, do $A \times B$ and $B \times A$ have the same number of elements?

1.3 Number systems

Number systems are very special sets that have additional structure on them which allows us to perform arithmetical operations like addition, multiplication, subtraction, and division.

The simplest number system is the **natural numbers** . We think of these as the positive whole numbers such as 0, 1, 2, etc. The natural numbers can actually be constructed using sets. This definition is recursive, which means we will define the number 0 and then give a recipe for getting the number $n + 1$ from the number n. By applying this recipe many times, we can get from 0 to whatever natural number we would like.

We begin by defining 0 to be the empty set ϕ. We then define the successor of n to be the set $S(n) = n \cup \{n\}$. We define $n + 1$ to be the set $S(n)$. Let's use this recipe to write out the first few numbers.

$$\begin{array}{c|l} 0 & \phi \\ 1 & 0 \cup \{0\} = \phi \cup \{\phi\} = \{\phi\} \\ 2 & 1 \cup \{1\} = \{\phi\} \cup \{\{\phi\}\} = \{\phi, \{\phi\}\} \\ 3 & 2 \cup \{2\} = \{\phi, \{\phi\}\} \cup \{\{\phi, \{\phi\}\}\} = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\} \end{array}$$

We see that 3 is really $S(S(S(0))) = \{0, 1, 2\}$. Addition is defined by saying $m + n$ is $S...S(n)$ where the successor is taken m times.

For our purposes, we can just think of the natural numbers as the regular old whole numbers that we count on our fingers, but it's nice to know that we can construct them using only sets! We denote the set of natural numbers by

$$\mathbb{N} = \{0, 1, 2, 3, ...\}.$$

After establishing a number system that has addition, it makes sense to ask if we can reverse the addition. This is a process known as subtraction. In the natural numbers, we cannot always perform subtraction. For example $2 - 3$ is not defined in the natural numbers, because there is no natural number that we can add 3 to in order to get a result of 2. Negative numbers were invented to deal with this situation.

We say that $-n$ is the number such that when we add it to n, we get back to 0, i.e. $(-n) + n = 0$. The set of all the negative and positive whole numbers (including 0) is called the **integers** and is denoted by

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}.$$

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> Suppose addition is defined for any positive numbers and subtraction is defined when you are subtracting a smaller number from a larger number. Can you use this to define subtraction for any two integers?

From addition, we define another arithmetical operation called multiplication. We say that $m \times n$ is the number n added to itself m times. Can we reverse multiplication in the same way that we can reverse addition using subtraction? In the integers, we cannot. For example, there is no integer that you can multiply by 2 to arrive at 3.

To accommodate reversing multiplication, we must expand our number system again. The rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} | p, q \in \mathbb{Z} \right\}$$

is the set of all the fractions with integer numerators and denominators. There is a slight problem, because sometimes two fractions really represent the same number, such as $\frac{1}{3}$ and $\frac{2}{6}$. For now, we are just going to pretend that those are two names for the same element of the set $\mathbb{Q}$.

We define multiplication in $\mathbb{Q}$ by

$$\frac{p_1}{q_1} \times \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2}$$

and we define division by

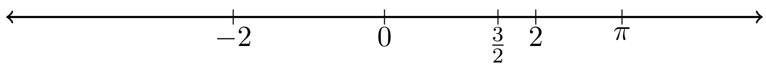
$$\frac{p_1}{q_1} \div \frac{p_2}{q_2} = \frac{p_1 q_2}{q_1 p_2}.$$

Even the rational numbers aren't enough when we want to start talking about areas of math like geometry. For instance, we know there is a number π which is the ratio between the circumference and diameter of a circle, that cannot be written as a fraction. We need an even more expansive number system that contains numbers like this. This system is called the **real numbers**. Defining it carefully is a little tricky. For our purposes we will think of the real numbers as the set of every decimal expansion, including infinite decimal expansion. When two decimal expansions actually represent the same number, like 0.999999.... and 1.0, we will pretend those are two different names for one single element of our set. The set of real numbers is denoted by the symbol $\mathbb{R}$. Real numbers that are not rational numbers are called irrational number.

?

Think about the four number systems as sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. Are any of these subsets of the others?

We often visualize the real numbers as number line, which we imagine extends to infinity at the left and right. We can plot numbers on this line, as in the following example.



?

What is the set $\mathbb{R} \times \mathbb{R}$? How can we visualize it?

There are some special subsets of $\mathbb{R}$ that have their own notation so that we can write them down quickly. They are called **intervals** and they are the sets of all numbers that are between two specific values. Intervals can be open: excluding the endpoints or close: including the endpoints.

Let a and b be real numbers such that $a < b$. The closed interval between points a and b is

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}.$$

The open interval between points a and b is

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}.$$

We may also consider the half open intervals

$$[a, b) = \{x \in \mathbb{R} | a \leq x < b\} \qquad \text{and} \qquad (a, b] = \{x \in \mathbb{R} | a < x \leq b\}.$$

We'll often draw these intervals as line segments with filled or open dots at the end to indicate whether the endpoint is included or excluded.

We can also consider half-lines which are all the real numbers greater than or less than some value, possibly including or excluding the endpoint.

$$
\begin{aligned}
[a, \infty) &= \{x \in \mathbb{R} | x \geq a\} \\
(a, \infty) &= \{x \in \mathbb{R} | x > a\} \\
(-\infty, a] &= \{x \in \mathbb{R} | x \leq a\} \\
(-\infty, a) &= \{x \in \mathbb{R} | x < a\}
\end{aligned}
$$

1.4 Quantifiers

A quantifier is a useful way to combine many (sometimes infinitely many) propositions using AND and OR. We use sets to index and count all the propositions that we combine.

To use quantifiers, we will generate a list of many propositions as follows. We start by taking a proposition "template" $P(x)$, which is a sentence with a variable written in it. For example, the statement

$$P(x) = x \text{ can swim.}$$

is not a proposition because we can't assign it a truth value, but it could be a proposition if we substituted certain animals for x.

Next, we will specify a set that represents all the possible values of x that we are interested in studying. For example, let

$$S = \{\text{penguins}, \text{fish}, \text{whales}\}.$$

We will generate a set of propositions by taking the template $P(x)$ and plugging each value of S into x. We denote this list of propositions by

$$P(x), x \in S.$$

In the example above, the list of propositions would be:

Penguins can swim.
Fish can swim.
Whales can swim.

Universal Quantifier

The universal quantifier allows us to combine many propositions using AND.

Definition 18: Universal quantifier

Consider the proposition template $P(x)$ and the set S. The proposition

$$\forall x \in S, P(x)$$

is the proposition

$$P(x_1) \text{ AND } P(x_2) \text{ AND } ...P(x_i) \text{ AND } ...$$

where x_i are all the elements in S.

The symbol $\forall$ is read aloud as "for all" and is called the universal quantifier. The whole proposition should be read aloud as "For all x in S, $P(x)$".

In our example above with $P(x)$ = "x can swim." and $S = \{\text{penguins}, \text{fish}, \text{whales}\}$, the proposition

$$\forall x \in S, P(x)$$

is the proposition "Penguins can swim AND fish can swim AND whales can swim." It happens to be true.

In general, to show that $\forall x \in S, P(x)$ is true, we must show that $P(x)$ is true no matter which element of S we plug in for x.

Existential Quantifier

The existential quantifier allows us to combine many propositions using OR.

Definition 19: Existential quantifier

> Consider the proposition template $P(x)$ and the set S. The proposition
>
> $$\exists x \in S \; s.t. \; P(x)$$
>
> is the proposition
>
> $$P(x_1) \text{ OR } P(x_2) \text{ OR } ...P(x_i) \text{ OR } ...$$
>
> where x_i are all the elements in S.

The symbol $\exists$ is read aloud as "there exists" and is called the existential quantifier. The whole proposition should be read aloud as "There exists x in S such that $P(x)$".

For example if $P(x)$ = "x can swim." and $S = \{\text{penguins}, \text{chickens}, \text{pigeons}\}$, the proposition

$$\exists x \in S \; s.t. \; P(x)$$

is the proposition "Penguins can swim OR chickens can swim OR pigeons can swim." It happens to be true.

In general, to show that $\exists x \in S \; s.t. \; P(x)$ is true, we must show that $P(x)$ is true for at least one choice of $x \in S$.

Adversary Method

We may want to consider a proposition that contains multiple quantifiers. For example, let S = { birds , mammals , reptiles }. Also, let S_x be the set of all species of animals of type x. So, for example, S_{birds} is the set of all species of birds. Let's think about this proposition:

$$\forall x \in S,\ \exists y \in S_x \text{ s.t. } y \text{ can swim.}$$

In English, this would read: "For all elements x in the set S, there exists an element y in the set S_x such that y can swim. Written out using AND and OR operators this would be:

⋮		⋮		⋮
ostriches can swim		wombats can swim		chameleons can swim
OR		OR		OR
penguins can swim	AND	bats can swim	AND	crocodiles can swim
OR		OR		OR
ravens can swim		whales can swim		snakes can swim
⋮		⋮		⋮

Each column represents one of the "$\exists y \in S_x$ such that y can swim" statements. Each of the three columns is true, because at least one of the components is true. The overall statement is also true, since each of the columns are true.

You can see that this method of evaluating propositions with multiple quantifiers will get unwieldy when we have very large sets or a large number of quantifiers.

?

Suppose S_1 and S_2 are finite sets with n_1 and n_2 elements respectively. Suppose also that $P(x, y)$ is a proposition template which becomes a proposition after plugging an element of S_1 into x and an element of S_2 into y. Can you come up with a diagram similar to the one above that represents the following proposition?

$$\forall x \in S_1, \exists y \in S_2 \ s.t. \ P(x, y)$$

What about this proposition?

$$\exists x \in S_1 \ s.t. \ \forall y \in S_2, P(x, y)$$

Are the diagrams the same?

Here's a useful trick called the Adversary Method. Let's pretend that we are playing a game against an opponent who we call our Adversary. The game is played using a proposition with quantifiers that looks something like this:

$$\forall x_1 \in S_1, \exists x_2 \in S_2, ..., \forall x_n \in S_n, P(x_1, ..., x_n)$$

with a sequence of quantifiers at the beginning and a proposition template at the end that becomes true or false once we plug elements of the appropriate sets into it.

We proceed from left to right. Whenever there is a $\forall$ symbol, our Adversary gets to pick an element of the corresponding set. Whenever there is a $\exists$ symbol, we get to pick an element of the corresponding set. Our goal is to make $P(x_1, ..., x_n)$ true, while our Adversary's goal is to make it false.

The overall proposition is true exactly when we have a winning strategy, i.e. if no matter what choices our Adversary makes we can react and make a choice that still results in us winning.

Let's apply this to our animal example

$$\forall x \in S, \exists y \in S_x \text{ such that } y \text{ can swim.}$$

Where $S = \{$ birds , mammals , reptiles $\}$ and S_x is the set of all species of animals of type x. Our adversary gets to go first and they get to pick either birds, mammals, or reptiles from the set S. Once they have made that choice, it's our job to pick an animal from the set of all animals of that type. What is our strategy? If the Adversary picks birds, we'll pick penguins. If they pick, mammals, we'll pick whales, and if they pick reptiles, we will pick crocodiles. At the end we much check if our chosen animal can swim. We have developed a strategy that guarantees it can. We have a winning strategy, so the proposition is true.

Negating propositions with quantifiers

Let's begin by thinking about what the negation of a proposition constructed with OR is. We'll write out the truth table for NOT (P OR Q).

P	Q	P OR Q	NOT (P OR Q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Notice that this truth table looks like an "upside down" version of the truth table for P AND Q. In fact, it is the same as the truth table for (NOT P) AND (NOT Q). Let's double check that.

P	Q	NOT P	NOT Q	(NOT P) AND (NOT Q)
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Comparing the two truth tables, we see they are the same. Bringing the "NOT" inside the parentheses requires us to flip the OR into an AND.

?

Following the same line of thinking, show that NOT(P AND Q) is logically equivalent to (NOT P) OR (NOT Q).

Thinking of a universal quantifier as a big AND and the existential quantifier as a big OR, we find the following.

Proposition 1.4.1

NOT ($\forall x \in S\ P(x)$) is logically equivalent to $\exists x \in S\ s.t.$ NOT($P(x)$)
NOT ($\exists x \in S\ s.t. P(x)$) is logically equivalent to $\forall x \in S$ NOT($P(x)$)

When we bring a negation inside a quantifier, we must flip the type of the quantifier.

1.5 Functions

Aside from sets, functions are another fundamental object that we use to build up many different fields of math. Functions are a much more general concept than how they may have been portrayed in a high school curriculum.

Definition 20: Function

Let S_1 and S_2 be two sets. A **function** from S_1 to S_2

$$f : S_1 \to S_2$$

is a machine that takes as input an element of S_1 and gives as output an element of S_2.

When we describe a function we always have to specify what its input and output sets are, *and* the method by which it assigns outputs to inputs. We use the notation

$$f : x \mapsto y$$

or alternatively

$$f(x) = y$$

to mean that when we put the element $x \in S_1$ into the function f, we will get out the element $y \in S_2$. When speaking, we can also say "f maps x to y".

For example, let $S_1 = \{1, 2, 3\}$ and $S_2 = \{\triangle, \square\}$. We can define a function from S_1 to S_2 by specifying what element of S_2 it maps each element of S_1 to. Let f be the function that acts as follows.

$$\begin{aligned} f : S_2 &\to S_2 \\ 1 &\mapsto \triangle \\ 2 &\mapsto \square \\ 3 &\mapsto \triangle \end{aligned}$$

In the alternate notation we would write $f(1) = f(3) = \triangle$ and $f(2) = \square$.

?

Can we think of the logical operator AND as a function? What would its input and output sets be?

Sometimes our input set is too large to describe what every single element maps to. We might define a function by writing down a general rule for how outputs are assigned.

For example, let S be the set of all subsets of $\mathbb{N}$ except the empty set. A few sample elements of S are: $\{1\}$, $\{2, 506, 1000000\}$, $\{2, 5, 8, 9\}$. We can define a function

$$\begin{aligned} f : S &\to \mathbb{N} \\ x &\mapsto \text{ the smallest element of } x \end{aligned}$$

Looking at our sample elements, we have $f(\{1\}) = 1$, $f(\{2, 506, 1000000\}) = 2$, and $f(\{2, 5, 8, 9\}) = 2$. We can work out all these specific instances from the general rule.

?

Why did we have to exclude the empty set from S in the above example?

Sometimes we might want to define a function using a few different rules for different parts of the set S_1. Such a function is called a **piece-wise function**. To write one down, we first separate S_1 into a finite number of subsets $T_1, ..., T_n$ that don't overlap (i.e. $T_i \cap T_j = \phi$ whenever $i \neq j$) and that cover the whole set S_1 so that $S_1 = T_1 \cup ... \cup T_n$. Then, we give a different rule for each T_i like so:

$$f(x) = \begin{cases} \text{rule 1} & \text{if } x \in T_1 \\ \dots & \\ \text{rule n} & \text{if } x \in T_n \end{cases}$$

For example, if we let S be the set of all subsets of $\mathbb{N}$ including the empty set, we could define a function $f : S \to \mathbb{N}$ by looking at the empty set and all the other subsets separately like so:

$$f(x) = \begin{cases} \text{the smallest element of } x & \text{if } x \in S \setminus \{\phi\} \\ 0 & \text{if } x \in \{\phi\} \end{cases}$$

1.5.1 Graphs

The graph of a function is a special set which contains all the information about how the function maps elements.

Definition 21: Graph of a function

Let $f : S_1 \to S_2$ be a function. The **graph** of f is the set

$$graph f = \{(x, y) \in S_1 \times S_2 | y = f(x)\}.$$

In other words, we can think of the graph of a function as the set of ordered pairs where the first object in the pair is an element of S_1 and the second object in the pair is the element in S_2 that f maps the first element to.

In our earlier example with $S_1 = \{1, 2, 3\}$ and $S_2 = \{\triangle, \square\}$ and $f : S_1 \to S_2$ defined by $f(1) = f(3) = \triangle$ and $f(2) = \square$, the graph would be the set

$$graph(f) = \{(1, \triangle), (2, \square), (3, \triangle)\}.$$

1.5.2 Real-valued functions

In this class, we will be particularly interested in functions which take real numbers as inputs and give real numbers as outputs, i.e. functions of the form

$$f : \mathbb{R} \to \mathbb{R}.$$

Such functions are called **real-valued functions**. Since $\mathbb{R}$ is a large (infinite!) set, we'll have to describe real-valued functions with a general rule. For example

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x + 1 \end{aligned}$$

is the function that takes any real number as input and gives as output that number plus one. We might abbreviate these functions by just writing down the rule and leave it to the reader to infer from context that the input and output sets are $\mathbb{R}$. In the example we just discussed, we would write simply $f(x) = x + 1$.

The graph of a real-valued function is a set of the form

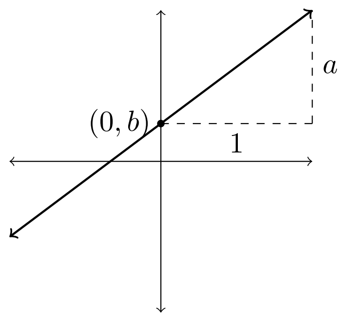
$$graph(f) = \{(x, f(x)) | x \in \mathbb{R}\}.$$

We can visualize this set as a bunch of points in the 2-D plane where the first number in the ordered pair gives a coordinate on the horizontal axis and the second number gives a coordinate on the vertical axis.

Notice that any real number only appears in the first coordinate once, so there cannot be two points in the graph lying along the same vertical line in the plane. This fact is known colloquially as the “vertical line test”.

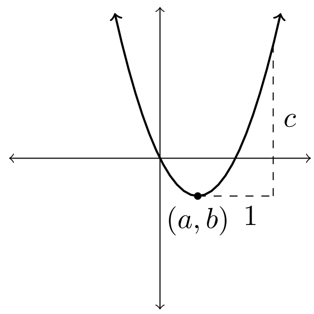
Linear functions

Linear functions are real-valued functions of the form $f(x) = ax + b$. Their graphs are straight lines in the plane which have slope a and cross the vertical axis at b.



Quadratic functions

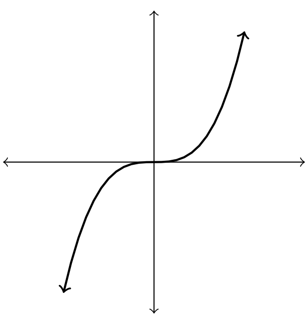
Quadratic functions are real-valued functions of the form $f(x) = c(x - a)^2 + b$. They have graphs which are shaped like parabolas. The base of the parabola is located at the point (a, b) and c determines how stretched the parabola is in the vertical direction.



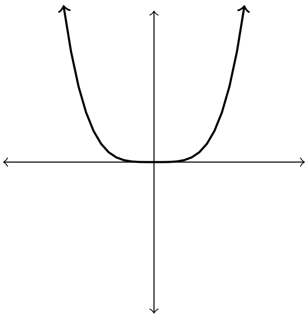
Cubic and quartic functions

Cubic and quartic functions are real-valued functions involving taking third and fourth powers respectively. Graphs of basic cubic and quartic functions are pictured below.

$f(x) = x^3$



$f(x) = x^4$

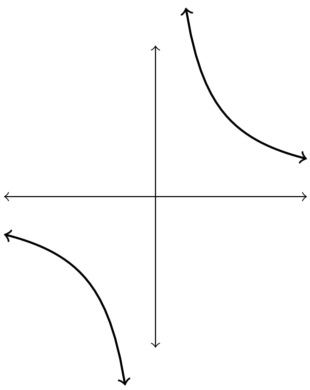


?

What does the graph of the function $f(x) = -x^3$ look like? What about the graph of $f(x) = -x^4$?

Reciprocal function

The reciprocal function is defined on $\mathbb{R} \setminus \{0\}$ by $f(x) = \frac{1}{x}$. It is a decreasing function which has a vertical asymptote at $x = 0$. The graph has a break there.



Translations of this function defined on $\mathbb{R} \setminus a$ by $f(x) = \frac{1}{x-a} + b$ have a vertical asymptote at $x = a$ and a horizontal asymptote at $y = b$.

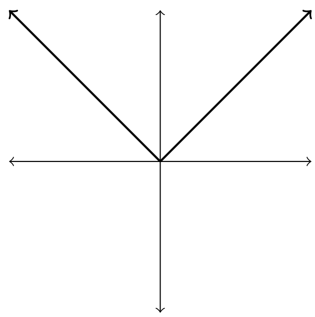
Absolute-value function

The absolute value function is defined piece-wise. When it takes a positive number as input, it outputs the same number. However, when it takes a negative number as input, it returns the negative of that number, which is a positive number. The absolute value function is denoted by $|x|$ and is defined formally as follows.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We can think of the absolute value function as measuring the (positive) distance between a real number and 0 on the number line. We can extend this and think of $|x_1 - x_2|$ as the distance between two numbers x_1 and x_2 on the number line.

The graph of the absolute value function looks like two linear functions stuck together with a sharp corner.



?

What is the relationship between the functions $|x|$ and $|-x|$?

1.6 Proof strategies

When building up a mathematical theory, we repeatedly apply this basic process:

define an object $\longrightarrow$ formulate a proposition $\longrightarrow$ argue that proposition is true

For example, suppose we want to understand how even and odd numbers relate to each other. First, we have to define what even and odd mean. So, we'll give a formal definition.

Definition: Even and odd numbers

An integer n is called **even** if it can be written as $n = 2k$ where $k \in \mathbb{Z}$.
And integer n is called **odd** if it can be written as $n = 2k + 1$ where $k \in \mathbb{Z}$.

A natural question to ask now is whether a number can be odd and even at the same time. Let's experiment with a few numbers to get a feeling for it.

Consider the number 1. Is it even? If it is even, we have to be able to write it as $1 = 2k$ where k is an integer. But, solving this equation for k we find that the only solution is $k = \frac{1}{2}$, which is not an integer. So 1 is not even. Is 1 odd? Yes, it is, because we can write $1 = 2 \times 0 + 1$, and 0 is an integer.

Consider the number 2. Is it even? Yes it is, because we can write $2 = 2 \times 1$ and 1 is an integer. Is 2 odd? If it were odd, we would be able to write $2 = 2k + 1$ with k an integer. However, solving for k we find that $k = \frac{1}{2}$, which is not an integer. So 2 is not odd.

We're starting to get the sense that a number can't be odd and even at the same time. But, we don't know for sure because we have only checked for a few numbers. Now is a good time to formulate a proposition that we think might be true.

Proposition

An integer cannot be simultaneously odd and even.

?

I've written this proposition in natural language. Can you rewrite this proposition using quantifiers and logical operators?

After stating the proposition, we will attempt to argue that it is true. This is called proving the proposition. By convention, we write "Proof." at the beginning of our argument and this symbol: $\square$ at the end of our argument to help the reader see where it starts and stops.

Proof.

Suppose n is an integer that is both even and odd.

Since n is even, we can write it as $n = 2k_1$, where k_1 is an integer.

Since n is odd, we can write it as $n = 2k_2 + 1$ where k_2 is an integer.

Since n is equal to both of them, we can conclude that $2k_1 = 2k_2 + 1$.

Diving the equation by 2, we get that $k_1 = k_2 + \frac{1}{2}$.

However, k_1 and k_2 are both integers, which means their difference should be an integer. But, their difference is $\frac{1}{2}$. We have arrived at a statement that is false. We have two integers that are separated by a distance of $\frac{1}{2}$. This means that the assumption we made at the beginning must be false.

It is not possible for an integer to be both even and odd. $\square$

Instead of having to check that something is true for every single integer (an impossible task since there are infinitely many of them) we have used reasoning and appealed to our rigorous definitions to find out that a particular statement holds true for every single integer.

We will now describe a few strategies that we can use to approach proofs.

Direct proof

The simplest approach to a proof is to begin with the definitions of objects involved and apply them along with other propositions that you've already established are true until you have convinced the reader that the proposition in question is true.

Doing a proof in math is fundamentally a social exercise. You are speaking to an imagined reader, and you have to provide enough detail to be able to convince your intended audience. Most proofs in math are not written in enough detail to, for example, be verified by a computer. You will need to develop an intuition for how much detail to add, but to start, err on the side of giving lots of explanation.

Let's look at an example of a direct proof, which will rely on the definition of even and odd numbers that we gave earlier.

Proposition

2 is an even number.

Proof.

We can write

$$2 = 2 \times 1$$

where 1 is an integer. An even number is defined as a number that can be written as $2 \times k$ where k is an integer. We have shown that 2 can be written in this form, hence it is an even number. $\square$

We have argued directly why 2 is an even number by showing that is satisfies the definition of an even number. Let's do one more example that requires a bit more work.

Proposition

The square of an even number is even.

Proof.

Let n be any even number. Since n is even, we can write it as

$$n = 2k$$

where k is an integer. Then,

$$n^2 = (2k)^2 = 4k^2 = 2 \times 2k^2.$$

If k is an integer, then so is k^2 and so is $2k^2$. We have written n^2 as an integer times 2. Hence, n^2 is even. $\square$

?

Is the square of an odd number always odd? Is it always even? Experiment with a few numbers, then formulate a proposition that you think is true and prove it.

Proof by contradiction

In a proof by contradiction, we begin by assuming the negation of what we are trying to prove. Then, we argue from there until we obtain a statement that is well known to be false. Finally, we can conclude that our initial assumption was false and hence the proposition we actually want to prove is true.

Let's prove the same proposition as above, that 2 is an even number, but using contradiction this time.

Proof.

Suppose for contradiction that 2 is not an even number. Then there is no integer k such $2 = 2 \times k$.

However, we know that $2 = 2 \times 1$. It follows that 1 cannot be an integer. However, we know that 1 is an integer.

This is a contradiction. Our assumption must be false and hence 2 must be an even number. $\square$

This looks like a more complicated way of proving something that we could do more efficiently with a direct proof. However, some propositions are easier or more convenient to prove using contradiction.

Here's a classic example of something that is convenient to prove using contradiction.

Proposition

$\sqrt{2}$ is not a rational number.

Proof.

Suppose for contradiction that $\sqrt{2}$ is a rational number. Then, it can be written as

$$\sqrt{2} = \frac{a}{b}$$

where a and b are integers with no common factors. (If they have common factors, divide by the factors until there are no common factors remaining.) Squaring both sides of this equation, we get

$$2 = \frac{a^2}{b^2}$$

and multiplying by b^2 we get

$$a^2 = 2b^2.$$

Since a^2 is two times an integer, it is even. Hence, a is also even.

This is because we know that if a is odd then a^2 must be odd. Taking the contrapositive, we get that if a^2 is not odd then a is not odd, or in other words if a^2 is even, then a is even.

Since a is even we can write
$$a = 2c$$
for an integer c. Plugging this into the equation above, we get
$$(2c)^2 = 2b^2 \implies 4c^2 = 2b^2 \implies 2c^2 = b.$$

Hence, b is also even. However, we established that a and b have no common factors! If they are both even they have a common factor of 2. This is a contradiction.

Our initial assumption that $\sqrt{2}$ is a rational number must be false. In fact, $\sqrt{2}$ is not a rational number. $\square$

There is a lot going on in this proof. We have used the definition of an even and odd number, and some facts that we previously proved about even/odd numbers and squares. We have used the definition of a rational number. We have structured the proof as a contradiction argument. We have even used the contrapositive in one step of our logical argument. You should read this proof over several times and chew over each step.

If/then statements and the contrapositive

We will often be interested in proving propositions that take the form $P \implies Q$, or equivalently "If P then Q". In order to do this, we first assume that P is true, and then under that assumption we try to argue that Q must be true as well. Let's look at an example.

Proposition

If n is an odd number, then $n+1$ is an even number.

Here, the proposition playing the role of P, also called the hypothesis, is "n is an odd number". The proposition playing the role of Q, also called the conclusion, is "$n+1$ is an even number." So, we will begin by assuming that n is odd, and then arguing that in that case, $n+1$ must be even.

Proof.

Suppose that n is an odd number. Then, we can write it as

$$n = 2k + 1$$

for an integer k. $n + 1$ can be written as

$$n + 1 = 2k + 1 + 1 = 2k + 2 = 2(k + 1).$$

Since k is an integer, $k + 1$ is also an integer. We have written $n + 1$ as 2 times an integer, which means that $n + 1$ is even. □

Inside our proof we have relied on the definitions that we gave of an even and odd number, and we used some basic arithmetic.

Earlier we proved that the proposition $P \implies Q$ is logically equivalent to its contrapositive NOT $Q \implies$ NOT P. We can take advantage of this. If the original proposition is tricky to prove but the contrapositive is easier, then we can prove the latter and we'll get the former for free. For example, consider the following proposition.

Proposition

If $3n + 1$ is even then n is odd.

We can prove this directly, and we can also prove it using the contrapositive.

Proof.: (Direct)

Suppose that $3n + 1$ is even. Then

$$3n + 1 = 2k$$

for some integer k. We can subtract $2n+1$ from both sides of the equation to get

$$n = 2k - (2n + 1) = 2(k - n - 1) + 1.$$

Since n and k are both integers, so is $k - n - 1$. Hence n is odd. □

This proof was not too long, but we had to think of an idea that isn't totally obvious: subtracting $2n + 1$ from both sides of the equation. Let's see what happens if we try to prove the contrapositive instead.

?

Take a moment to write down the contrapositive of the above proposition.

Proof.: (Contrapositive)

Suppose that n is not odd. Then it is even and we can write it as

$$n = 2k$$

where k is an integer. Then

$$3n + 1 = 3(2k) + 1 = 2(3k) + 1.$$

$3k$ is an integer, so this shows that $3n + 1$ is odd. Hence, $3n + 1$ is not even. $\square$

This proof might feel more "straightforward" as we didn't have to think of subtracting a certain amount from the equation we were working with, we just rearranged the multiplication.

2 Limits

2.1 Definition of the limit

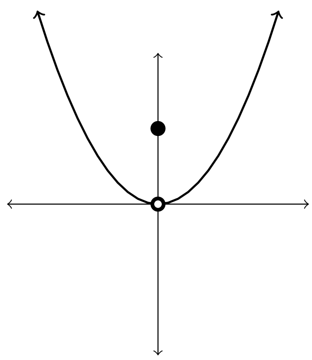
We begin with a real-valued function $f(x)$. We would like to come up with a mathematical definition that captures the following concept:

When we get very close to a point x_0, what value does $f(x)$ get very close to?

For example, consider the piecewise function

$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

which has the graph:



As we get very close to the 0 on the x-axis, $f(x)$ gets very close to 0 as well.

x	$f(x)$
0.1	0.01
0.01	0.0001
0.001	0.000001
...	...

However, the actual value of the function at $x = 0$ is $f(0) = 1$. We'd like to come up with some way of detecting that $f(x)$ is getting close to 0 near $x = 0$ even though $f(0)$ itself is not equal to 0.

In this section, we will define a quantity called the **limit of the function $f(x)$ as x approaches x_0**. This is the number that $f(x)$ gets close to when x is close to x_0. In our example above, the limit of $f(x)$ as x approaches 0 is 0, even though $f(0) = 1$.

2.1.1 Distances in the real numbers

The first issue we must address is that we don't yet have a mathematical notion of what "close to" means in the real numbers. We will use the absolute value function to define the distance between two points.

Definition 22: Distance in the real numbers

The **distance between two real numbers** x and y is

$$d(x, y) = |y - x|.$$

This quantity is the positive length of the line segment with endpoints at x and y.

?

Think of d as a function. What is its input set and its output set?

This distance has some properties that we would expect from a good definition of distance.

Proposition 2.1.1: Properties of the distance function

- $d(x, y) \geq 0$ (The distance is positive.)
- $d(x, x) = 0$ (The distance from a point to itself is 0.)
- $d(x, y) = d(y, x)$ (The distance from x to y is the same as the distance from y to x.

Proof.

These three properties follow from the definition of the absolute value function.

$d(x, y) = |y - x| \geq 0$ because the absolute value of any number is positive.

$d(x, x) = |x - x| = |0| = 0.$

$d(x, y) = |y - x| = |-(x - y)| = |x - y| = d(y, x).$

The absolute value function has a very nice property called the triangle inequality.

Proposition 2.1.2: Triangle inequality

For any real numbers a and b

$$|a+b| \leq |a| + |b|.$$

The name "triangle inequality" comes from imagining a "squashed" triangle in one dimension.

The third side of the triangle $(a+b)$, which is formed by drawing from the tail of the first side (a) to the tip of the second side (b), can't get longer than the sum of the other

Proof.

First notice that from the definition of the absolute value, $|x| \geq x$ and $|x| \geq -x$. To see why, consider two cases:

Case 1: $x < 0$. Then $|x| = -x > 0 > x$. Also, $|x| = -x$.
Case 2: $x \geq 0$. Then $|x| = x \geq 0 \geq -x$. Also, $|x| = x$.

Now let's consider the quantity $|a+b|$. We will again split into two cases.

Case 1: $|a+b| = a+b$. We know that $a \leq |a|$ and $b \leq |b|$ (see above). So, $|a+b| \leq |a| + |b|$.

Case 2: $|a+b| = -(a+b)$. We know that $-a \leq |a|$ and $-b \leq |b|$ (see above). So, $|a+b| = -(a+b) = -a + (-b) \leq |a| + |b|$.

The triangle inequality has an implication for the distance that we just defined. Imagine we have three points on the number line a, b, and c. Then

$$d(a,c) \leq d(a,b) + d(b,c).$$

The distance from the first point to the last point can't be more than the distance from the first point to an intermediate point added to the distance from the intermediate point to the final point. (If the intermediate point is in between the end points, we have an equality.) We can obtain this result by applying the triangle inequality:

$$d(a,c) = |c-a| = |(c-b) + (b-a)| \leq |c-b| + |b-a| = d(b,c) + d(a,b).$$

2.1.2 Epsilon-Delta definition

Armed with a notion of distance, we can now define the limit of a function rigorously. This is the idea. We will say that the limit of a function $f(x)$ as x approaches a point x_0 has a value of L if for any interval around L, no matter how small, we can find a region around x_0 so that if we keep x in that region, $f(x)$ will stay in the interval around L.

It's helpful to rewrite what we just described using quantifiers.

Definition 23: Limit of a function

We say that

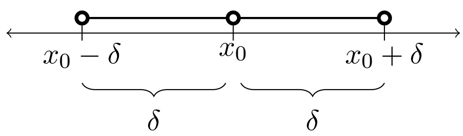
$$\lim_{x \to x_0} f(x) = L$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

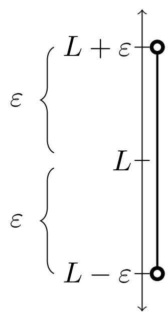
Remember that $|x - x_0|$ is the distance between x and x_0. So,

$$\{x \in \mathbb{R} | 0 < |x - x_0| < \delta\}$$

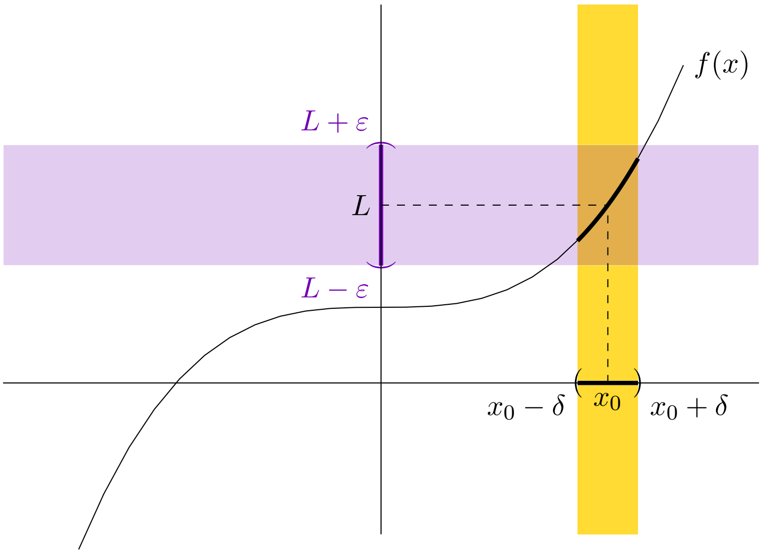
is an open interval centered at x_0 of radius δ and with the center point removed.



As a shorthand, we can think of x being in this set as x being "δ-close" to x_0. Similarly, if $|f(x) - L| < \varepsilon$, then $f(x)$ is in an open interval around L of radius ε. We'll draw this set on the y axis, since that is where we plot $f(x)$ when drawing a graph.



If $f(x)$ is in this interval, we can say that $f(x)$ is "ε-close" to L for short. Our definition of the limit is saying that for any choice of ε, there is some choice of δ such that whenever x is δ-close to x_0, then $f(x)$ stays ε-close to L.



Applying the Adversary Method to this definition, and referring to the above diagram, the definition of the limit says that

$$\lim_{x \to x_0} f(x) = L$$

if for any ε that our Adversary picks (the width of the purple strip), we can pick a δ (the width of the yellow strip), so that the piece of the graph that lies in the yellow strip stays inside the purple strip. As our Adversary shrinks down the purple strip, we are always able to shrink down our yellow trip accordingly to keep the bolded piece of the graph in the purple zone.

2.1.3 Proofs using ϵ-δ

To prove that a limit exists and is equal to a number L, we must follow these steps.

1. Let $\varepsilon > 0$ be some number that we have no control over.
2. Give a formula for how to choose δ depending on ε.
3. Assume that $0 < |x - x_0| < \delta$ and prove that in this case $|f(x) - L| < \varepsilon$.

Let's look at an example. Let $f(x) = x$. Its graph is a straight line with slope 1. Let's calculate $\lim_{x\to 1} x$. First, let's try to intuit what it should be. It seems like as we get close to 1 on the x-axis, because the slope is 1, we are also getting close to 1 on the y-axis. We guess that $\lim_{x\to 1} x = 1$, but we need to prove it.

Proposition

$$\lim_{x \to 1} x = 1$$

Proof.

Fix $\varepsilon > 0$. Let $\delta = \varepsilon$. Suppose that $0 < |x - 1| < \delta$. Then,

$$|f(x) - L| = |x - 1| < \delta = \varepsilon.$$

In other words, when $|x - 1| < \delta$ we have shown that $|f(x) - 1| < \varepsilon$.

In this proof, the main innovation was that we realized that if our Adversary chose some number ε as the tolerance on the y-axis, we could make the choice $\delta = \varepsilon$ for the tolerance on the x-axis. In general, the trickiest part of proving that a limit exists is figuring out the correct recipe for finding δ in terms of ε.

?

How would we have to change this proposition and proof if we instead considered $f(x) = 3x$, the function whose graph is a straight line with slope 3?

Let's try a slightly harder example and return to the function from the beginning of this section:

$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

We suspect, from testing points, that $\lim_{x\to 0} f(x) = 0$. Let's prove it.

Proposition

$$\lim_{x \to 0} f(x) = 0$$

Proof.

Fix $\varepsilon > 0$. Let $\delta = \sqrt{\varepsilon}$. Suppose that $0 < |x - 0| < \delta$. Then,

$$|f(x) - L| = |x^2 - 0| = |x^2| = |x|^2 < \delta^2 = (\sqrt{\varepsilon})^2 = \varepsilon.$$

We have shown that whenever $0 < |x - 0| < \delta$, $|f(x) - 0| < \varepsilon$.

In the examples we've looked at so far, there is a certain amount of symmetry of the slopes of the functions around the point where we are trying to evaluate the limit. If instead our function slopes a lot more steeply on one side compared to the other, we might have to think even harder about how to choose our δ.

Proposition

$$\lim_{x \to 1}(x^2 + 3) = 4$$

Before we begin the proof, we'll do a little scratch work. The thing that we want to show is less than ε is $|f(x) - L|$, so let's investigate this quantity.

$$|f(x) - L| = |x^2 + 3 - 4| = |x^2 - 1| = |x + 1||x - 1|$$

There are two factors here that we need to deal with. Notice that if $|x - 1| < 1$, this is equivalent to saying

$$-1 < x - 1 < 1 \implies -1 + 2 < x + 1 < 1 + 2 \implies 1 < x + 1 < 3$$

which implies that $|x + 1| < 3$. We're going to need to pick our δ in such a way that it keeps $|x - 1| < 1$ and hence keeps the other term $|x + 1| < 3$. We'll also need to include a factor of $1/3$ to counterbalance the factor of 3 coming from the first term.

Proof.

Fix $\varepsilon < 0$. Let $\delta = min(1, \frac{\varepsilon}{3})$. Suppose that $|x - 1| < \delta$. Then

$$|x^2 + 3 - 4| = |x^2 - 1| = |x + 1||x - 1| < 3\delta \leq 3\frac{\varepsilon}{3} = \varepsilon.$$

We may also want to prove that a limit does not exist. One way to do this is to use a proof by contradiction. We first assume that the limit does exist and is equal to some quantity L, and then we argue that this implies something patently false. Let's consider an example.

Proposition

Let

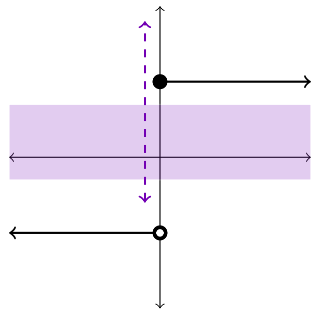
$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}.$$

The limit

$$\lim_{x \to 0} f(x)$$

does not exist.

The intuition behind why this limit does not exist is that if $\varepsilon < 1$, then the purple strip has a width of less than two and there is nowhere we can position the strip so that both the left and right pieces of the function fall inside it.



Proof.

Suppose the limit does exist and is equal to L. Pick $\varepsilon = \frac{1}{2}$. We will consider two cases: $L \geq 0$ and $L < 0$.

Case 1: $L \geq 0$. Pick any $\delta > 0$. Then, consider the point $-\frac{\delta}{2}$. It is δ-close to 0 since $-\delta < -\frac{\delta}{2} < 0$. Also, $f(-\frac{\delta}{2}) = -1$. Since $L \geq 0$, we have that $L - 1 \geq -1 = f(-\frac{\delta}{2})$. This implies that $|L - f(-\frac{\delta}{2})| \geq 1$. Hence, $-\frac{\delta}{2}$ is not ε-close to L.

Case 2: $L < 0$. Pick any $\delta > 0$. Then, consider the point $\frac{\delta}{2}$. It is δ-close to 0 since $0 < \frac{\delta}{2} < \delta$. Also, $f(\frac{\delta}{2}) = 1$. Since $L < 0$, we have that $L+1 \leq 1 = f(\frac{\delta}{2})$. Then $L - f(\frac{\delta}{2}) \leq -1$ and multiplying by -1 we get $f(\frac{\delta}{2}) - L \geq 1$. This implies that $|L - f(\frac{\delta}{2})| \geq 1$. Hence, $\frac{\delta}{2}$ is not ε-close to L.

In either case, no matter which δ we pick, there is always a point x that is δ-close to 0 but such that $f(x)$ is not ε-close to L. This contradicts the fact that $\lim_{x\to 0} f(x) = L$.

In making this contradiction argument we have also used the fact that we must flip quantifiers when negating statements. In particular,

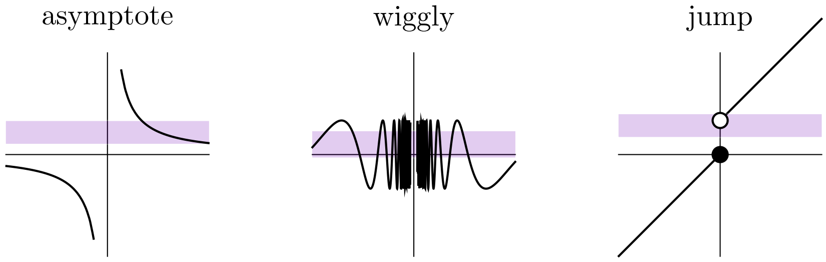
$$\text{NOT } (\forall \varepsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |x - x_0| < \delta \text{ then } |f(x) - L| < \varepsilon)$$

is logically equivalent to

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0 \text{ NOT } (\text{if } 0 < |x - x_0| < \delta \text{ then } |f(x) - L| < \varepsilon)$$

In other words, if $\lim_{x\to x_0} f(x) \neq L$, we must be able to find one value of ε so that no matter how small our Adversary makes δ, there is always a point that is δ-close to x_0 where nevertheless $f(x)$ is not ε-close to L.

There are a few different reasons why a limit might not exist. As in the example we just explored, a function might have a "jump". A function could also have an asymptote at the point we are studying, or be too wiggly.



?

Imagine dragging the purple strip up and down in the three pictures above. Can you position it anywhere so that a chunk of the graph centered at 0 stays inside the strip?

2.2 Right and left-handed limits

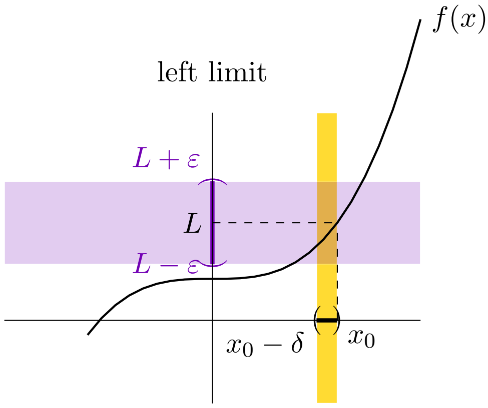
In the definition of a limit of a function, we were interested in what happened whenever we approach, or "get close to" a certain point x_0 on the x-axis. We may also be interested in the same question but when approaching only from the positive (right) direction or the negative (left) direction. This motivates the following two definitions.

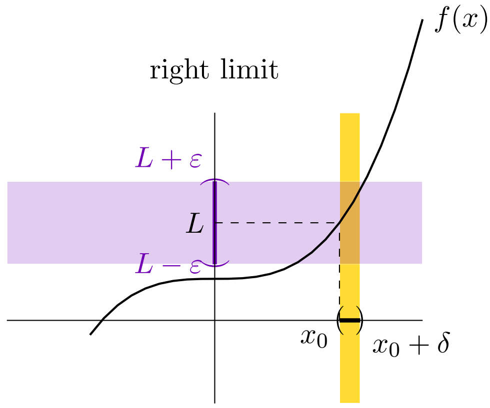
Definition 24: Right and left limits

(Right limit) We say that
$$\lim_{x \to x_0^+} f(x) = L$$
if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $x \in (x_0, x_0 + \delta)$ then $|f(x) = L| < \varepsilon$.

(Left limit) We say that
$$\lim_{x \to x_0^-} f(x) = L$$
if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $x \in (x_0 - \delta, x_0)$ then $|f(x) = L| < \varepsilon$.

In the picture, the difference is that instead of having a small yellow strip centered around x_0, we now have a small yellow strip just to the right or just to the left of x_0.





Earlier we showed that for the function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

which has a jump at $x = 0$, the full limit

$$\lim_{x \to 0} f(x)$$

does not exist. However, our intuition suggests that the right and left limits do exist, for as we approach 0 from the right, $f(x)$ gets close to 1 and when we approach 0 from the left, $f(x)$ gets close to -1. Let's show this using the definition of a left or right limit.

Proposition

Let

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}.$$

Then, $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$

Proof.

Fix $\varepsilon > 0$. Let $\delta = 1$. Then if $x \in (0, 1)$, $f(x) = 1$ and

$$|f(x) - L| = |1 - 1| = 0 < \varepsilon.$$

Fix $\varepsilon > 0$. Let $\delta = 1$. Then if $x \in (-1, 0)$, $f(x) = -1$ and

$$|f(x) - L| = |-1 - (-1)| = 0 < \varepsilon.$$

?

How would we have to modify the above proof if we instead considered the following function?

$$f(x) = \begin{cases} x-1 & x < 0 \\ x+1 & x \geq 0 \end{cases}$$

We have seen that the left and right limits may exist even if the overall limit does not exist. Imagine pinching the two pieces of the function in the previous example together to close the jump. In this case, the left and right limits will become the same number, and the overall limit will once again exist. This idea gives us a useful criterion.

Proposition 2.2.1

Let $f(x)$ be a real-valued function and x_0 a real number. Then

$$\lim_{x \to x_0} f(x)$$

exists if and only if

$$\lim_{x \to x_0^+} f(x) \text{ and } \lim_{x \to x_0^-} f(x)$$

both exist and are equal to each other.

Proof.

First suppose that

$$\lim_{x \to x_0} f(x)$$

exists and is equal to some number L.

Fix $\varepsilon > 0$. By the hypothesis, there is some $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.

Now, if $x \in (x_0, x_0 + \delta)$, then $0 < |x - x_0| < \delta$. Hence, $|f(x) - L| < \varepsilon$. This shows that

$$\lim_{x \to x_0^+} f(x) = L.$$

Similarly, if $x \in (x_0 - \delta, x_0)$, then $0 < |x - x_0| < \delta$. Hence, $|f(x) - L| < \varepsilon$. This shows that

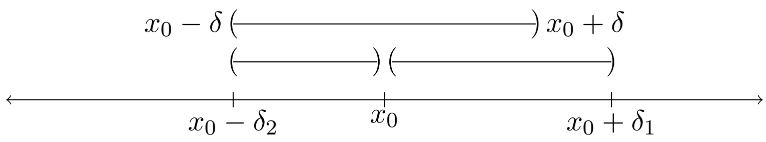
$$\lim_{x \to x_0^-} f(x) = L.$$

Conversely, suppose that

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L.$$

Fix $\varepsilon > 0$. Then there is $\delta_1 > 0$ such that if $x \in (x_0, x_0 + \delta_1)$, then $|f(x) - L| < \varepsilon$. There is also $\delta_2 > 0$ such that if $x \in (x_0 - \delta_2, x_0)$, then $|f(x) - L| < \varepsilon$.

Take $\delta = min(\delta_1, \delta_2)$, the smallest of the two numbers δ_1 and δ_2. Now, if $0 < |x - x_0| < \delta$ then $x \in (x_0, x_0 + \delta_1)$ or $x \in (x_0 - \delta_2, x_0)$.



Either way, $|f(x) - L| < \varepsilon$. This shows that

$$\lim_{x \to x_0} f(x) = L.$$

Now we know that if we calculate the left and right limits and find that they are not the same, we can conclude that the overall limit does not exist.

?

Think about the three pictures labelled "asymptote", "wiggly" and "jump" in the previous section. Do the left and right limits exist? Are they equal to each other?

2.3 Continuity

For some of the functions we have looked at, the limit at a point is consistent with the actual value of the function at that point. Consider for example $f(x) = x^2$. As we plug in values very close to $x = 0$, the output of the function gets very close to 0 as well. At the same time, exactly at $x = 0$, we also have that $f(0) = 0$. When we go to draw the graph of the function, we can draw the region around $x = 0$ with a smooth stroke. We don't have to lift our pen to suddenly move far away from the region where we were initially drawing.

On the other hand, consider the function

$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

The limit as x approached 0 is still 0, but now the output of the function exactly at the point $x = 0$ is 1. This is reflected in the graph. As we try to draw the region near $x = 0$, the curve goes towards $(0, 0)$ but then has to suddenly jump up to $(0, 1)$.

We'll give a definition which allows us to identify functions of the first type, whose limits approach the actual function outputs.

Definition 25: Continuous function

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in \mathbb{R}$ if

1. $f(x_0)$ is defined,
2. $\lim_{x\to x_0} f(x)$ exists, and
3. $f(x_0) = \lim_{x\to x_0} f(x)$.

There are three ways that a function can fail to be continuous at a point. First, the function may not be defined at the point. For example $f(x) = \frac{1}{x}$ is only defined on $\mathbb{R}\backslash\{0\}$, so it is not continuous at 0. This particular example has an asymptote at $x = 0$. Our function could also just have a point missing from its input set, as in the case, $f(x) = x$ for $x \neq 0$. The graph of such a function looks like a single hole in an otherwise nice curve.

Secondly, the limit may not exist at the point. For example, we showed that our function

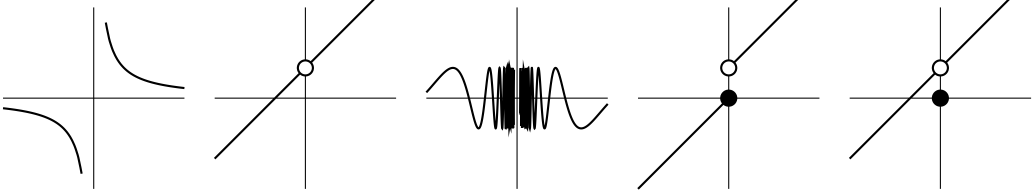
$$f(x) = \begin{cases} 1 & x \geq 1 \\ -2 & x < 0 \end{cases}$$

does not have a limit at 0. So, it is not continuous there. This type of discontinuity, where the limit fails to exist because the left and right limits are not equal, is called a jump discontinuity. The left and right limits themselves also may not exist, as in the wiggly situation, which also leads to a discontinuity.

Finally, the function may be defined at a point and the limit exists there, but the two values are not equal. This is the case for

$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

This type of discontinuity is called a point discontinuity.



To show that functions are continuous, we need to be able to calculate limits. So far, the only way we have to do this is by using the $\varepsilon - \delta$ definition. In the next section, we will use that definition to develop some rules which we can apply to evaluate limits more easily.

2.4 Limit Laws

We'll begin by considering some basic functions. Then, we'll study how limits interact with certain operations like multiplication and addition. This will allow us to build more complicated functions from our basic functions, and evaluate their limits too.

We've already looked at some examples of linear functions in section 2.1.3. Let's now consider a linear function with any non-zero slope.

Proposition 2.4.1

For any real number $a \neq 0$,

$$\lim_{x \to x_0} ax = ax_0.$$

Proof.

Fix $\varepsilon > 0$.

Let $\delta = \frac{\varepsilon}{|a|}$.

Now, suppose $0 < |x - x_0| < \delta$. Then,

$$|ax - ax_0| = |a||x - x_0| < |a|\delta = |a|\frac{\varepsilon}{|a|} = \varepsilon.$$

?

Why did we have to exclude the $a = 0$ (zero slope) case we did the above proof?

Let's consider the case for functions whose graphs are horizontal lines separately.

Proposition 2.4.2

$$\lim_{x \to x_0} c = c$$

Proof.

Fix $\varepsilon > 0$.

Let $\delta = 1$.

Now, suppose $0 < |x - x_0| < \delta$. Then,

$$|c - c| = 0 < \varepsilon.$$

?

What else could we have chosen for δ in the above proof?

Let's consider one more basic function. We know that $f(x) = \frac{1}{x}$ does not have a limit at $x = 0$, but what about at the other points, away from the asymptote?

Proposition 2.4.3

For $x_0 \neq 0$,

$$\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}.$$

Proof.

Before we begin, note that so long as $x_0 \neq 0$,

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \left| \frac{x - x_0}{x x_0} \right| = \frac{|x - x_0|}{|x||x_0|}.$$

Fix $\varepsilon > 0$.

Let $\delta = min(\frac{|x_0|}{2}, \frac{|x_0|^2 \varepsilon}{2})$.

Now suppose that $|x - x_0| < \delta$. Then in particular $|x - x_0| < \frac{|x_0|}{2}$,. In the case that $x_0 > 0$, we can rewrite this as

$$\frac{-x_0}{2} < x - x_0 < \frac{x_0}{2} \implies \frac{x_0}{2} < x < \frac{3x_0}{2} \implies |x| > \frac{x_0}{2} \implies \left|\frac{1}{x}\right| < \frac{2}{|x_0|}.$$

Similarly, in the case that $x_0 < 0$, we can rewrite it as

$$\frac{x_0}{2} < x - x_0 < \frac{-x_0}{2} \implies \frac{3x_0}{2} < x < \frac{x_0}{2} \implies |x| > \frac{-x_0}{2} \implies \left|\frac{1}{x}\right| < \frac{2}{|x_0|}.$$

Then,

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x||x_0|} < \frac{\delta}{|x||x_0|} < \frac{2}{|x_0|}\frac{\delta}{|x_0|} \leq \frac{|x_0|^2\varepsilon}{2}\frac{2}{|x_0|^2} = \varepsilon.$$

?

Is $f(x) = \frac{1}{x}$ continuous at points $x_0 \neq 0$?

Now that we've studied some basic functions, let's look at ways that we can use them to build more complicated functions. We can add two functions together according to the rule

$$(f + g)(x) = f(x) + g(x).$$

In other words, we take our input, feed it into the two functions separately, and then add the resulting outputs. It turns out that there is a nice compatibility between this addition and the process of taking limits.

Proposition 2.4.4

$$\lim_{x \to x_0} (f + g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

Proof.

Suppose that

$$\lim_{x \to x_0} f(x) = L_1 \quad \text{and} \quad \lim_{x \to x_0} f(x) = L_1.$$

Fix $\varepsilon > 0$.

We know that there is some $\delta_1 > 0$ such that when $0 < |x - x_0| < \delta_1$, $|f(x) - L_1| < \frac{\varepsilon}{2}$. Similarly, there is some $\delta_2 > 0$ such that when $0 < |x - x_0| < \delta_2$,

$|g(x) - L_2| < \frac{\varepsilon}{2}$.

We will now take $\delta = min(\delta_1, \delta_2)$. Then, we can use the triangle inequality to conclude

$$|f(x)+g(x)-(L_1+L_2)| = |f(x)-L_1+g(x)-L_2| < |f(x)-L_1|+|g(x)-L_2| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{x\to x_0}(f+g)(x) = L_1 + L_2$.

Similarly, we can scale a function by a constant number by taking

$$(cf)(x) = cf(x).$$

This is also compatible with taking limits.

Proposition 2.4.5

$$\lim_{x\to x_0}(cf)(x) = c\lim_{x\to x_0} f(x)$$

Proof.

Suppose that $\lim_{x\to x_0} f(x) = L$.

Fix $\varepsilon > 0$. Then there is a $\delta > 0$ such that when $0 < |x - x_0| < \delta$, $|f(x) - L| < \frac{\varepsilon}{|c|}$.

Suppose that $0 < |x - x_0| < \delta$. Then,

$$|(cf)x - cL| = |c||f(x) - L| < |c|\frac{\varepsilon}{|c|} = \varepsilon.$$

With these two limit laws and our basic functions, we can calculate some more complicated limits. For example,

$$\lim_{x\to 3}\left(\frac{2}{x} + 5x\right) = \lim_{x\to 3}\frac{2}{x} + \lim_{x\to 3} 5x = 2\lim_{x\to 3}\frac{1}{x} + \lim_{x\to 3} 5x = \frac{2}{3} + 15.$$

We can also consider multiplication of functions. We define

$$(fg)(x) = f(x)g(x)$$

the function obtained by feeding the two functions the same input and multiplying the resulting outputs. This process is also compatible with taking limits.

Proposition 2.4.6

$$\lim_{x \to x_0} (fg)(x) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right)$$

Proof.

Suppose that

$$\lim_{x \to x_0} f(x) = L_1 \quad \text{and} \quad \lim_{x \to x_0} f(x) = L_1.$$

Notice that by propositions 2.4.2 and 2.4.4,

$$\lim_{x \to x_0} (f(x) - L_1) = 0 \quad \text{and} \quad \lim_{x \to x_0} (g(x) - L_2) = 0.$$

Fix $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $0 < |x - x_0| < \delta_1$ then $|f(x) - L_1| < \sqrt{\varepsilon}$. Also, there exists $\delta_2 > 0$ such that if $0 < |x - x_0| < \delta_2$ then $|g(x) - L_2| < \sqrt{\varepsilon}$.

Let $\delta = min(\delta_1, \delta_2)$.

Suppose $0 < |x - x_0| < \delta$. Then,

$$|(f(x) - L_1)(g(x) - L_2) - 0| = |f(x) - L_1||g(x) - L_2| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon.$$

This shows that $\lim_{x \to x_0}(f(x) - L_1)(g(x) - L_2) = 0$.

Notice that

$$(f(x) - L_1)(g(x) - L_2) = f(x)g(x) - L_1 g(x) - L_2 f(x) + L_1 L_2$$

and hence that

$$f(x)g(x) = (f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2.$$

Now, using proposition 2.4.4 and 2.4.5 , we can conclude that

$$\begin{aligned} \lim_{x \to 0} f(x)g(x) &= \lim_{x \to 0}(f(x) - L_1)(g(x) - L_2) + L_1 \lim_{x \to 0} g(x) + L_2 \lim_{x \to 0} f(x) - L_1 L_2 \\ &= 0 + L_1 L_2 + L_1 L_2 - L_1 L_2 \\ &= L_1 L_2. \end{aligned}$$

?

A polynomial is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

Explain why polynomial functions are continuous at every point.

We have a final limit law to consider. We define the composition of functions $f(x)$ and $g(x)$ to be

$$(f \circ g)(x) = f(g(x)).$$

We first feed the input x to g and receive the output $g(x)$. Then we feed $g(x)$ as input to f to obtain the output $f(g(x))$.

For example, if $f(x) = x^2$ and $g(x) = x + 3$, then $(f \circ g)(x) = (x + 3)^2$. As long as the limit of g at a point x_0 exists, and f is well-enough behaved near the input value of that limit, this process is also compatible with taking limits in the following sense.

Proposition 2.4.7

Suppose that $\lim_{x \to x_0} g(x)$ exists and that $f(x)$ is continuous at $\lim_{x \to x_0} g(x)$. Then,

$$\lim_{x \to x_0} (f \circ g)(x) = f(\lim_{x \to x_0} g(x)).$$

Proof.

Suppose $\lim_{x \to x_0} g(x) = L'$.

Fix $\varepsilon > 0$.

Since f is continuous at L', $\lim_{y \to L'} f(y) = f(L')$.

Hence, there exists $\delta' > 0$ such that when $0 < |y - L'| < \delta'$, $|f(y) - f(L')| < \varepsilon$.

There also exists $\delta > 0$ such that when $0 < |x - x_0| < \delta$, $|g(x) - L'| < \delta'$.

So, if $0 < |x - x_0| < \delta$, then $|g(x) - L'| < \delta'$, and then $|f(g(x)) - f(L')| < \varepsilon$.

2.5 Limits at infinity and infinite limits

3 Derivatives

3.1 Definition of the derivative and differentiability

3.2 Derivative Rules

3.3 Derivatives of transcendental functions

3.4 Applications

3.5 Fermat's Theorem, Extreme Value Theorem, Mean Value Theorem

3.6 L'Hôpital's Rule

3.7 Higher order derivatives

3.8 Curve sketching

4 Intergals

4.1 Supremum and infimum

4.2 Riemann sums

4.3 Riemann integrals and integrability

4.4 The Fundamental Theorem of Calculus

Symbol Index

Symbol	**Read as**	**Meaning**
P AND Q	"P and Q"	A proposition that is true exactly when P and Q are true
P OR Q	"P or Q"	A proposition that is true when at least one of P and Q is true (possibly both)
NOT P	"NOT P"	A proposition that is true exactly when P is false
$P \implies Q$	"P implies Q" or "if P then Q"	A proposition that is true except when P is true but Q is false
$P \iff Q$	"P if and only if Q"	The proposition $P \implies Q$ AND $Q \implies P$
$x \in A$	"x is in A" or "x is an element of A"	The object x is an element of the set A
$A \subset B$	"A is a subset of B"	Every element of A is also an element of B
$A = B$	"A equals B"	The sets A and B have exactly the same elements
ϕ	"The empty set"	A set with no elements
$A \cup B$	"A union B"	A set consisting of all the elements that are in A or B
$A \cap B$	"A intersection B"	A set consisting of all the elements that are in both A and B
$A \setminus B$	"A set minus B"	A set consisting of all the elements that are in A and are not in B
A^c	"A complement"	A set consisting of all the elements that are not in A. This is usually in reference to some larger set that A lies within.
$\mathbb{N}$	"the natural numbers"	The set of all positive whole numbers (e.g. $0, 1, 2, 3$)
$\mathbb{Z}$	"the integers"	The set of all positive and negative whole numbers (e.g. $-3, -2, -1, 0, 1, 2, 3$)
$\mathbb{Q}$	"the rational numbers"	The set of all fractions
$\mathbb{R}$	"the real numbers"	The set of all finite and infinite decimal expansions
$\forall x \in S, P(x)$	"For all x in S, $P(x)$"	For every value of x in the set S, when I plug x into $P(x)$, $P(x)$ is true
$\exists x \in S$ s.t. $P(x)$	"There exists x in S such that $P(x)$"	There is at least one element x in the set S so that when I plug x into $P(x)$, $P(x)$ is true

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