

Calculus I

A textbook for Pitzer's Ma030 Course

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Introduction

This textbook is under construction! Please check back often to access the latest version at the Github repository.

This course has twin goals. The first is to develop the theory of calculus and learn how to apply its tools to analyze rates of change in continuous systems. The second is to use calculus of an example of how to build up a mathematical theory, and in doing so to get a feeling for what it means to “think mathematically”. In service of the second goal, the course begins with a substantial section of foundations of math and logic.

1 Mathematical Language

1.1 Propositional Logic

Doing mathematics consists of making rigorous arguments that certain statements are true. Consequently, the pieces of language that we will be most interested in are full sentences that are statements which can be clearly evaluated as either true or false, but not both. Such statements are called propositions.

Definition 1: Proposition

A **proposition** is a statement that is either true or false, but not both.

Let's consider the following chunks of words in English and evaluate whether or not they are propositions.

“Penguins can swim.”

This is a full sentence, it is a statement, and it is true. This is a proposition.

“Penguins can fly.”

This is a full sentence, it is a statement, and it is false. This is a proposition.

“Look at that penguin!”

This is a full sentence, but it is not a statement. It is a command (imperative). This is not a proposition.

“What is your favorite species of penguin?”

This is a full sentence, but it is not a statement. It is a question. This is not a proposition.

“That penguin’s foot”

This is not a full sentence, so it is not a proposition.

We have seen three examples of chunks of words that fail to be propositions because they are not statements. What about the other requirement that the statement be true or false but not both; is it really possible for a statement to be true and false? Can a statement be neither true nor false?

Let's think about this example:

"This sentence is false."

Let's call the sentence above by the name S . If S is true, then S must also be false. However, if S is false, then it must be false that S is false, and hence S must be true. We have just argued that if S can be assigned any value of true or false, it must actually be both true and false. The one thing we have figured out is definitely true is that S is not a proposition.

?

Consider the following two sentences.

The next sentence is true. The previous sentence is false.

Are either of the above sentences propositions? Would they be propositions if they were individually written on different pages of this book?

If mathematicians find a particular proposition to be very interesting or cool, and someone has successfully proven that it is true, they will call it a **theorem**.

A proposition can be either true or false, but not both. We can assign a symbol to a proposition that represents whether it is true or false. This symbol is called a truth value.

Definition 2: Truth value

A proposition is assigned a **truth value** of T if the proposition is true and F if the proposition is false.

For example, "*Penguins can swim.*" is a proposition with a truth value of T and "*Penguins can fly.*" is a proposition with a truth value of F .

1.1.1 Logical operators

Propositions can be combined together to form more complex propositions. One way of doing this is by using logical operators.

Definition 3: Logical operator

A **logical operator** is a machine that takes some propositions as input and returns a new proposition as output whose truth value is determined by the truth values of the inputs.

Four important logical operators that we will define are AND, OR, NOT, and IMPLIES. These logical operators emulate the function of the words “and”, “or”, “not”, and the “if - then” construction in English language. We will define these operators using a **truth table**, which is a table that describes the truth value of the more complex proposition formed by applying the logical operator to simpler propositions, based on the truth values of the simpler input propositions.

Definition 4: Truth table

A **truth table** for a proposition formed using logical operators applied to simple propositions is a table with one column for each simple proposition, a final column for the overall proposition, and one row for each combination of possible truth values of the simple propositions. The final entry of each row shows the corresponding truth value of the overall proposition.

Suppose P is a proposition formed by connecting simple propositions P_1, \dots, P_n using logical operators. Its truth table is of the following form.

| P_1 | \dots | P_n | P |
|----------|----------|----------|----------|
| T | \dots | T | \star |
| T | \dots | F | \star |
| \vdots | \vdots | \vdots | \vdots |
| F | \dots | T | \star |
| T | \dots | F | \star |

The \star symbols represent spots which can have a value of either T or F and will change depending on which logical operators are used.

AND

Suppose P and Q are two propositions. The truth value of the proposition P AND Q is defined by the following truth table.

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| P | Q | $P \text{ AND } Q$ |
|-----|-----|--------------------|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

This table says that the proposition $P \text{ AND } Q$ is true only when both P and Q are true. If any one of P or Q is false, then $P \text{ AND } Q$ is false. Hence, in order to argue that $P \text{ AND } Q$ is true, we must first argue that P is true and then argue that Q is true.

For example, the proposition “*Penguins can swim AND penguins are birds*” is true, because both “*Penguins can swim.*” and “*Penguins are birds.*” are true.

However, the proposition “*Penguins can swim AND penguins can fly.*” is false, because one of the simpler statements (“*Penguins can fly.*”) is false.

OR

Suppose P and Q are two propositions. The truth value of the proposition $P \text{ OR } Q$ is defined by the following truth table.

| P | Q | $P \text{ OR } Q$ |
|-----|-----|-------------------|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

This table says that the proposition $P \text{ OR } Q$ is true as long as one of P or Q (or possibly both) are true. In order to argue that $P \text{ OR } Q$ is true we can either argue that P is true, or we can argue that Q is true. It might be the case that both are true, but showing that one is true is sufficient.

For example, the proposition “*Penguins can swim OR penguins are birds*” is true, because at least one of “*Penguins can swim.*” and “*Penguins are birds.*” is true. In fact, they are both true.

The proposition “*Penguins can swim OR penguins can fly.*” is also true, because one of the simpler statements (“*Penguins can swim*”) is true.

NOT

Suppose P is a proposition. The truth value of NOT P is defined by the following table.

| P | NOT P |
|-----|---------|
| T | F |
| F | T |

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NOT P is true exactly when P is false and vice versa. In order to argue that NOT P is true, we must argue that P is false.

For example, “NOT penguins can fly.” is true because “Penguins can fly.” is false. When using English as a natural language we often rewrite the statement “NOT penguins can fly.” as “Penguins cannot fly.”.

People sometimes abbreviate NOT P by using the symbol $\sim P$ or the symbol $\neg P$.

IMPLIES

The final logical operator we’ll consider is a little more subtle. We want to have a formal way of finding out the truth values of propositions that take the form of sentences like

“If a cat has spots then it is a leopard.”

The above proposition is false because there are some cats that have spots but are not leopards. For example, my pet cat John Brown has spots, but he is not a leopard. There is a particular cat for whom the first part of the sentence is true (it has spots) but the second part is false (it is not a leopard). This is the phenomenon that causes the overall sentence to be false.

So, what we want our truth table to capture is that the whole proposition should be true if whenever the first part of the sentence following “if” is true, then the second part of the sentence following “then” is also true.

Let P and Q be two propositions. Then the truth value of P IMPLIES Q , which is equivalent to “if P then Q ” in natural language, is given by the following table.

| P | Q | P IMPLIES Q |
|-----|-----|-----------------|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The only way that P IMPLIES Q can be false is if P is true and yet Q is false.

We will abbreviate IMPLIES by the symbol \implies .

Let’s look at examples of all four rows in this truth table.

*“If penguins can swim then
the sky is blue.”*

This is true, because both parts are true.

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“If penguins can swim then the sky is green.”

This is false, because the first part is true but the second part is false.

“If penguins can fly then the sky is blue.”

This is true, because the first part is false and the second part is true.

“If penguins can fly then the sky is green.”

This is true, because although the second part is false, the first part is also false.

When we have a proposition of the form P IMPLIES Q (or $P \implies Q$), then the proposition P is called the **hypothesis** and the proposition Q is called the **conclusion**. In order to argue that $P \implies Q$ is true, we must assume that P is true and show that in this case Q is also true. We don't need to worry about the case that P is false, because if P is false, then $P \implies Q$ is always true regardless of the truth value of Q .

Sometimes, two propositions imply each other. In other words, we may have $P \implies Q$ AND $Q \implies P$. We often abbreviate this by writing “ $P \iff Q$ ” or “ P if and only if Q ”. In order to prove that $P \iff Q$ is true, we must prove that $P \implies Q$ is true, and then prove that $Q \implies P$ is true.

Combining logical operators

We can combine multiple logical operators together to create ever more complicated propositions. For example, we could suppose that P , Q , and R are three propositions and think about the truth value of P AND (Q OR R). Let's write down a truth table to understand this complex proposition. First we'll calculate the truth values of Q OR R , and then we'll tackle the whole thing.

Since each of the three propositions P , Q , and R can take on two different truth values, there are $2 \times 2 \times 2 = 8$ rows in our truth table.

| P | Q | R | Q OR R | P AND (Q OR R) |
|-----|-----|-----|------------|------------------------|
| T | T | T | T | T |
| T | T | F | T | T |
| T | F | T | T | T |
| T | F | F | F | F |
| F | T | T | T | F |
| F | T | F | T | F |
| F | F | T | T | F |
| F | F | F | F | F |

?

If a proposition is formed by combining n different simple propositions using logical operators, how many rows will it have in its truth table?

Sometimes we'll find that two different ways of combining propositions yields exactly the same truth table. For example, let's work out the truth table of $(\text{NOT } P) \text{ AND } (\text{NOT } Q)$.

| P | Q | $\text{NOT } P$ | $\text{NOT } Q$ | $(\text{NOT } P) \text{ AND } (\text{NOT } Q)$ |
|-----|-----|-----------------|-----------------|--|
| T | T | F | F | F |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

Now, let's write down the truth table of $\text{NOT } (P \text{ OR } Q)$, which is a completely different construction.

| P | Q | $P \text{ OR } Q$ | $\text{NOT } (P \text{ OR } Q)$ |
|-----|-----|-------------------|---------------------------------|
| T | T | T | F |
| T | F | T | F |
| F | T | T | F |
| F | F | F | T |

Looking at the first two and the last columns we see that these two truth tables are exactly the same! When this happens, we say the two propositions are logically equivalent because one is true exactly when the other is true.

Definition 5: Logical equivalence

If two propositions formed out of simple propositions and logical operators have the same truth table, we say they are **logically equivalent**.

It is useful to identify logically equivalent propositions because if we can argue that one of them must be true, we will also automatically know that the other one is true as well.

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EXOR (read “exclusive or”) is another logical operator defined by the following truth table.

| P | Q | $P \text{ EXOR } Q$ |
|-----|-----|---------------------|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

Can you find a proposition constructed using P , Q , AND, OR, and NOT that is logically equivalent to $P \text{ EXOR } Q$?

The contrapositive

There is one example of logical equivalence that turns out to be particularly useful called the contrapositive. It gives us a proposition that is equivalent to $P \implies Q$.

Proposition 1: Contrapositive

$P \implies Q$ is logically equivalent to $(\text{NOT } Q) \implies (\text{NOT } P)$.

Proof.

We will calculate the truth table of $(\text{NOT } Q) \implies (\text{NOT } P)$ and compare it to the truth table of $P \implies Q$.

| P | Q | $\text{NOT } Q$ | $\text{NOT } P$ | $(\text{NOT } Q) \implies (\text{NOT } P)$ |
|-----|-----|-----------------|-----------------|--|
| T | T | F | F | T |
| T | F | T | F | F |
| F | T | F | T | T |
| F | F | T | T | T |

Looking at the first two and the last column, we see that this is identical to the truth table for $P \implies Q$. \square

Consider the proposition,

If penguins can swim, the sky is blue.

We can write this using symbols as: penguins can swim \implies the sky is blue. To find the contrapositive of this propositions, we reverse the order of the hypothesis and the

conclusion and negate both of them. In this example, the contrapositive is: NOT(the sky is blue) \implies NOT(penguins can swim), or in natural language,

If the sky isn't blue, then penguins cannot swim.

This contrapositive is logically equivalent to the original proposition. One or the other may be easier to establish, but once we've established one the other follows automatically.

1.2 Sets

A set is the most basic mathematical concept that we will use to start building up our theory.

Definition 6: Set

A **set** is a container with mathematical objects inside of it. These objects can be numbers, shapes, words, symbols, other sets, or anything we want.

We denote a set by writing curly brackets $\{\}$ with a list of the objects the set contains inside the brackets, separated by commas. For example,

$$\{3, \triangle, cat\}$$

is the set containing the number 3, the symbol \triangle , and the word *cat*. Similarly,

$$\{4.2, a, \{0, 1\}\}$$

is the set containing the number 4.2, the letter *a*, and the set $\{0, 1\}$, which itself is a set containing the numbers 0 and 1. We may want to express the idea that a certain object is inside a set.

Definition 7: Element

If S is a set containing the object x , we say that x is an **element** of S . We denote this using the symbol \in by writing $x \in S$.

For example, we say that 3 is an element of $\{3, \triangle, cat\}$ or $3 \in \{3, \triangle, cat\}$. If a certain object is not in a set, we say that the object is not an element of a set and use the symbol \notin . For example, $4 \notin \{3, \triangle, cat\}$.

A set can even have nothing inside it.

Definition 8: Empty set

A set with nothing inside it is called the **empty set** and it is written either as $\{\}$ or ϕ .

A set may also have infinitely many things inside it. However, we can't list out infinitely many elements when writing such a set down, because it would take forever. Sometimes we use ellipses to suggest a pattern. For example

$$\{1, 2, 3, 4, \dots\}$$

denotes the set containing every positive whole number. This set has infinitely many elements. We will see another way to write infinite sets later in this section.

1.2.1 Subsets and equality

A natural question to ask is how we can compare two sets to each other. We'll start with a notion of one set being "smaller" than another.

Definition 9: Subset

A set S_1 is a **subset** of a set S_2 if every element of S_1 is also an element of S_2 . We denote this by $S_1 \subset S_2$.

To check if a set S_1 is a subset of S_2 , we have to go through each element of S_1 and verify that it is an element of S_2 as well.

For example, $\{1, 3\}$ is a subset of $\{1, 2, 3, 4\}$. We can check this by going through each element of the first set and seeing that it is in the second set. First we check that 1 is in $\{1, 2, 3, 4\}$, which it is, and then we check that 3 is in $\{1, 2, 3, 4\}$, which it is.

On the other hand $\{1, 5\}$ is not a subset of $\{1, 2, 3\}$ because there is an element of the first set, namely 5, that is not in the second set.

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What are all the subsets of the empty set?

Next we'll define a notion of equality for subsets.

Definition 10: Equality of sets

Two sets are equal if they have exactly the same elements.

Notice that the order that elements are listed in the sets does not matter. For example, $\{0, 1\}$ and $\{1, 0\}$ are equal as sets because they contain exactly the same two elements.

When dealing with numbers x and y , we know that $x = y$ exactly when x is smaller than or equal to y ($x \leq y$) and y is smaller than or equal to x ($y \leq x$). A similar statement is true for sets, subsets, and equality of sets.

Proposition 2: Subsets and equality

Let S_1 and S_2 be sets.

$$S_1 = S_2 \iff S_1 \subset S_2 \text{ and } S_2 \subset S_1.$$

Proof.

We have two propositions to prove, and both involve the IMPLIES operator. Let's begin with the first one: " $S_1 = S_2 \implies S_1 \subset S_2$ and $S_2 \subset S_1$."

The hypothesis is $S_1 = S_2$. We will assume that the hypothesis is true. Now we will try to argue that the conclusion is also true.

In order for $(S_1 \subset S_2)$ AND $(S_2 \subset S_1)$ to be true, both $(S_1 \subset S_2)$ and $(S_2 \subset S_1)$ must be true. Let's first show that $(S_1 \subset S_2)$ is true.

Let x be any element of S_1 . Since $S_1 = S_2$, x must also be an element of S_2 . Any element of S_1 is also in S_2 . Hence, $S_1 \subset S_2$.

Similarly, let x be any any element of S_2 . Since $S_1 = S_2$, x must also be an element of S_1 . Hence, $S_2 \subset S_1$.

We have shown that if $S_1 = S_2$, then $(S_1 \subset S_2)$ and $(S_2 \subset S_1)$.

Now let's show the second statement, which is that $(S_1 \subset S_2)$ and $(S_2 \subset S_1) \implies S_1 = S_2$. Now the hypothesis is that $(S_1 \subset S_2)$ and $(S_2 \subset S_1)$. We'll assume this hypothesis is true. We need to show, then, that $S_1 = S_2$.

Let x be any element of S_1 . Since $S_1 \subset S_2$, x is also an element of S_2 .
 Let y be any element of S_2 . Since $S_2 \subset S_1$, y is also an element of S_1 . Hence, S_1 and S_2 have exactly the same elements and $S_1 = S_2$. \square

?

Suppose you have three sets S_1 , S_2 , and S_3 satisfying $S_1 \subset S_2$, $S_2 \subset S_3$ and $S_3 \subset S_1$. Are all three of the sets equal to each other?

1.2.2 Set builder notation

Set-builder notation is a convenient way to define subsets of an already established set. The idea is that we will select out certain elements of the set which satisfy some criteria.

Definition 11: Set-builder subset

Let S be a set. We denote by

$$\{x \in S \mid \text{criteria on } x\}$$

the set which contains all the elements of S that satisfy the criteria written to the right of the vertical line.

For example, let $S = \{1, 2, 3, 4, 5, 6\}$. We define a subset of S by

$$S' = \{x \in S \mid x \text{ is an even number}\}.$$

Now, in order to write down a list of the elements in S' explicitly, we will go through each element of S and check if it is an even number. If it is, we will include it in S' . Following this procedure we find that 1 is not even, so it is not included in S' , 2 is even and is included in S' and so on until we get

$$S' = \{2, 4, 6\}.$$

We can write more complicated criteria using logical operators. For example

$$S'' = \{x \in S \mid x \text{ is even and } x \text{ is prime}\}.$$

Only one element of S is both an even number and a prime number, namely the number 2. Writing out a list of its elements we get $S'' = \{2\}$.

1.2.3 Set operations

We will now study ways of combining and modifying sets to form new, more complicated sets.

Definition 12: Set operation

A **set operation** is a machine which takes as input some number of sets and gives a new set as output.

A **binary** set operation takes two sets as input.

We will introduce five set operations: UNION, INTERSECTION, SET MINUS, COMPLEMENT, and DIRECT PRODUCT. We shall see that these set operations have some connection to the logical operators we studied earlier.

UNION

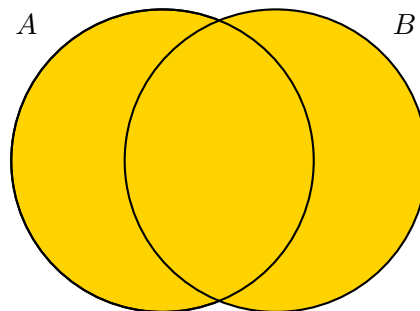
The union is a binary set operation. It takes all the elements in two sets and bundles them together in a new set.

Definition 13: Union of sets

Let A and B be two sets. The **union** of A and B is

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

Notice that the definition of the union relies on the logical operator OR. We can represent the union of sets visually on a Venn diagram.



The entirety of the circles representing A and B are colored in, because any element that is in either set will also be in the union.

For example, let $A = \{1, \triangle, \square\}$ and $B = \{0, 1, 2\}$. The set $A \cup B$ consists of all the elements that appear in either A or B , or possibly both. If an element does appear in both A and B , it is counted only once in the union. So, we have

$$A \cup B = \{0, 1, 2, \triangle, \square\}.$$

INTERSECTION

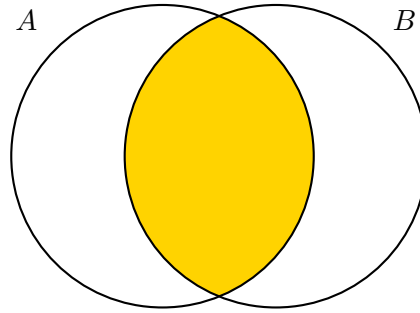
The intersection is also a binary set operation which takes two sets as input and gives one new set as output.

Definition 14: Intersection of sets

Let A and B be two sets. The **intersection** of A and B is

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

The definition of the intersection relies on the logical operator AND. Representing the intersection of sets A and B on a Venn diagram, we color in only the section where A and B overlap, as an element must be in both A and B in order to be in $A \cap B$.



Taking the same example as before, let $A = \{1, \triangle, \square\}$ and $B = \{0, 1, 2\}$. There is only one element that appears in both A and B , namely 1. Hence

$$A \cap B = \{1\}.$$

It could be the case that there are no elements in the intersection. For example, if $C = \{1, 3, 5\}$ and $D = \{2, 4, 6\}$ then no element appears in both C and D . In this case we say the intersection is empty, or write $C \cap D = \phi$.

SET MINUS

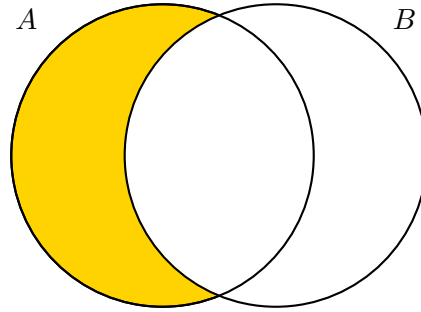
Set minus is a binary operation which takes two sets as input and gives one new set as output. However, contrary to the previous two operations, the order of the input sets matters.

Definition 15: Set minus

Let A and B be two sets. Then A set minus B is

$$A \setminus B = \{x | x \in A \text{ and } x \notin B\}.$$

The definition of set minus uses both the AND and the NOT logical operators. Representing $A \setminus B$ on a Venn diagram, only the part of A which lies outside of B is colored.



Let's consider our trusty example of $A = \{1, \triangle, \square\}$ and $B = \{0, 1, 2\}$. We must identify those elements of A which do not appear in B . Noticing that 1 appears in B , but \triangle and \square do not, we find that

$$A \setminus B = \{\triangle, \square\}.$$

?

What would $B \setminus A$ be for the above example? Is it the same as $A \setminus B$?

COMPLEMENT

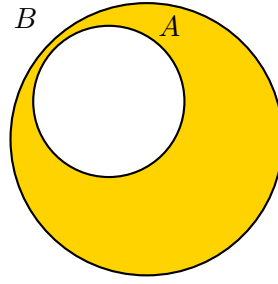
Our next set operation is slightly different. It still takes two sets as input, but one of these sets must be a subset of the other.

Definition 16: Complement

Let A and B be two sets such that $A \subset B$. Then the complement of A (in B) is

$$A^C = \{x \in B | x \notin A\}.$$

Think of B as a background set in which A is embedded. A^C takes all of the elements in the background that are outside of A . Representing this on a Venn diagram, we draw A inside B and color in everything outside of A .



For example, let $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{2, 3\}$. Then, with respect to the background B , the complement of A is

$$A^C = \{1, 4, 5, 6, 7, 8, 9, 10\}.$$

?

In the special case that $A \subset B$, rewrite A^C (with respect to B) using the SET MINUS operation.

Direct Products

The final set operation is the direct product. It is a binary operation which can be thought of as the set version of multiplication for numbers.

Definition 17: Direct product

Let A and B be two sets. The **direct product** of A and B is the set

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}.$$

Here (x, y) is an *ordered pair*, so for example $(1, 2)$ is not the same as $(2, 1)$. Notice that if A and B are both finite with n and m elements respectively, the direct product $A \times B$ has $n \times m$ elements.

Going back to our example $A = \{1, \triangle, \square\}$ and $B = \{0, 1, 2\}$, we can calculate that the direct product is

$$A \times B = \{(1, 0), (1, 1), (1, 2), (\triangle, 0), (\triangle, 1), (\triangle, 2), (\square, 0), (\square, 1), (\square, 2)\}.$$

?

Is $A \times B$ always the same as $B \times A$? In the case that A and B are finite, do $A \times B$ and $B \times A$ have the same number of elements?

1.3 Number Systems

Number systems are very special sets that have additional structure on them which allows us to perform arithmetical operations like addition, multiplication, subtraction, and division.

The simplest number system is the **natural numbers**. We think of these as the positive whole numbers such as 0, 1, 2, etc. The natural numbers can actually be constructed using sets. This definition is recursive, which means we will define the number 0 and then give a recipe for getting the number $n + 1$ from the number n . By applying this recipe many times, we can get from 0 to whatever natural number we would like.

We begin by defining 0 to be the empty set ϕ . We then define the successor of n to be the set $S(n) = n \cup \{n\}$. We define $n + 1$ to be the set $S(n)$. Let's use this recipe to write out the first few numbers.

$$\begin{array}{l|l} 0 & \phi \\ 1 & 0 \cup \{0\} = \phi \cup \{\phi\} = \{\phi\} \\ 2 & 1 \cup \{1\} = \{\phi\} \cup \{\{\phi\}\} = \{\phi, \{\phi\}\} \\ 3 & 2 \cup \{2\} = \{\phi, \{\phi\}\} \cup \{\{\phi, \{\phi\}\}\} = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\} \end{array}$$

We see that 3 is really $S(S(S(0))) = \{0, 1, 2\}$. Addition is defined by saying $m + n$ is $S...S(n)$ where the successor is taken m times.

For our purposes, we can just think of the natural numbers as the regular old whole numbers that we count on our fingers, but it's nice to know that we can construct them using only sets! We denote the set of natural numbers by

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

After establishing a number system that has addition, it makes sense to ask if we can reverse the addition. This is a process known as subtraction. In the natural numbers, we cannot always perform subtraction. For example $2 - 3$ is not defined in the natural numbers, because there is no natural number that we can add 3 to in order to get a result of 2. Negative numbers were invented to deal with this situation.

We say that $-n$ is the number such that when we add it to n , we get back to 0, i.e. $(-n) + n = 0$. The set of all the negative and positive whole numbers (including 0) is called the **integers** and is denoted by

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

?

Suppose addition is defined for any positive numbers and subtraction is defined when you are subtracting a smaller number from a larger number. Can you use this to define subtraction for any two integers?

1 Mathematical Language

From addition, we define another arithmetical operation called multiplication. We say that $m \times n$ is the number n added to itself m times. Can we reverse multiplication in the same way that we can reverse addition using subtraction? In the integers, we cannot. For example, there is no integer that you can multiply by 2 to arrive at 3.

To accommodate reversing multiplication, we must expand our number system again. The rational numbers is

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z} \right\}$$

the set of all the fractions with integer numerators and denominators. There is a slight problem, because sometimes two fractions really represent the same number, such as $\frac{1}{3}$ and $\frac{2}{6}$. For now, we are just going to pretend that those are two names for the same element of the set \mathbb{Q} .

We define multiplication in \mathbb{Q} by

$$\frac{p_1}{q_1} \times \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2}$$

and we define division by

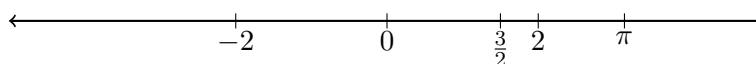
$$\frac{p_1}{q_1} \div \frac{p_2}{q_2} = \frac{p_1 q_2}{q_1 p_2}.$$

Even the rational numbers aren't enough when we want to start talking about areas of math like geometry. For instance, we know there is a number π which is the ratio between the circumference and diameter of a circle, that cannot be written as a fraction. We need an even more expansive number system that contains numbers like this. This system is called the **real numbers**. Defining it carefully is a little tricky. For our purposes we will think of the real numbers as the set of every decimal expansion, including infinite decimal expansion. When two decimal expansions actually represent the same number, like 0.999999.... and 1.0, we will pretend those are two different names for one single element of our set. The set of real numbers is denoted by the symbol \mathbb{R} .

?

Think about the four number systems as sets: \mathbb{N} , \mathbb{Z} , \mathbb{Q} , and \mathbb{R} . Are any of these subsets of the others?

We often visualize the real numbers as number line, which we imagine extends to infinity at the left and right. We can plot numbers on this line, as in the following example.



?

What is the set $\mathbb{R} \times \mathbb{R}$? How can we visualize it?

There are some special subsets of \mathbb{R} that have their own notation so that we can write them down quickly. They are called **intervals** and they are the sets of all numbers that are between two specific values. Intervals can be open: excluding the endpoints or close: including the endpoints.

Let a and b be real numbers such that $a < b$. The closed interval between points a and b is

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}.$$

The open interval between points a and b is

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}.$$

We may also consider the half open intervals

$$[a, b) = \{x \in \mathbb{R} | a \leq x < b\} \quad \text{and} \quad (a, b] = \{x \in \mathbb{R} | a < x \leq b\}.$$

We'll often draw these intervals as line segments with filled or open dots at the end to indicate whether the endpoint is included or excluded.



We can also consider half-lines which are all the real numbers greater than or less than some value, possibly including or excluding the endpoint.

$$[a, \infty) = \{x \in \mathbb{R} | x \geq a\}$$

$$(a, \infty) = \{x \in \mathbb{R} | x > a\}$$

$$(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$$

$$(-\infty, a) = \{x \in \mathbb{R} | x < a\}$$

1.4 Quantifiers

A quantifier is a useful way to combine many (sometimes infinitely many) propositions using AND and OR. We use sets to index and count all the propositions that we combine.

To use quantifiers, we will generate a list of many propositions as follows. We start by taking a proposition “template” $P(x)$, which is a sentence with a variable written in it. For example, the statement

$$P(x) = x \text{ can swim.}$$

is not a proposition because we can’t assign it a truth value, but it could be a proposition if we substituted certain animals for x .

Next, we will specify a set that represents all the possible values of x that we are interested in studying. For example, let

$$S = \{\text{penguins, fish, whales}\}.$$

We will generate a set of propositions by taking the template $P(x)$ and plugging each value of S into x . We denote this list of propositions by

$$P(x), x \in S.$$

In the example above, the list of propositions would be:

Penguins can swim.

Fish can swim.

Whales can swim.

Universal Quantifier

The universal quantifier allows us to combine many propositions using AND.

Definition 18: Universal quantifier

Consider the proposition template $P(x)$ and the set S . The proposition

$$\forall x \in S, P(x)$$

is the proposition

$$P(x_1) \text{ AND } P(x_2) \text{ AND } \dots P(x_i) \text{ AND } \dots$$

where x_i are all the elements in S .

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The symbol \forall is read aloud as “for all” and is called the universal quantifier. The whole proposition should be read aloud as “For all x in S , $P(x)$ ”.

In our example above with $P(x) = “x \text{ can swim.}”$ and $S = \{\text{penguins, fish, whales}\}$, the proposition

$$\forall x \in S, P(x)$$

is the proposition “Penguins can swim AND fish can swim AND whales can swim.” It happens to be true.

In general, to show that $\forall x \in S, P(x)$ is true, we must show that $P(x)$ is true no matter which element of S we plug in for x .

Existential Quantifier

The existential quantifier allows us to combine many propositions using OR.

Definition 19: Existential quantifier

Consider the proposition template $P(x)$ and the set S . The proposition

$$\exists x \in S, P(x)$$

is the proposition

$$P(x_1) \text{ OR } P(x_2) \text{ OR } \dots P(x_i) \text{ OR } \dots$$

where x_i are all the elements in S .

The symbol \exists is read aloud as “there exists” and is called the existential quantifier. The whole proposition should be read aloud as “There exists x in S such that $P(x)$ ”.

For example if $P(x) = “x \text{ can swim.}”$ and $S = \{\text{penguins, chickens, pigeons}\}$, the proposition

$$\exists x \in S, P(x)$$

is the proposition “Penguins can swim OR chickens can swim OR pigeons can swim.” It happens to be true.

In general, to show that $\exists x \in S, P(x)$ is true, we must show that $P(x)$ is true for at least one choice of $x \in S$.

Adversary Method

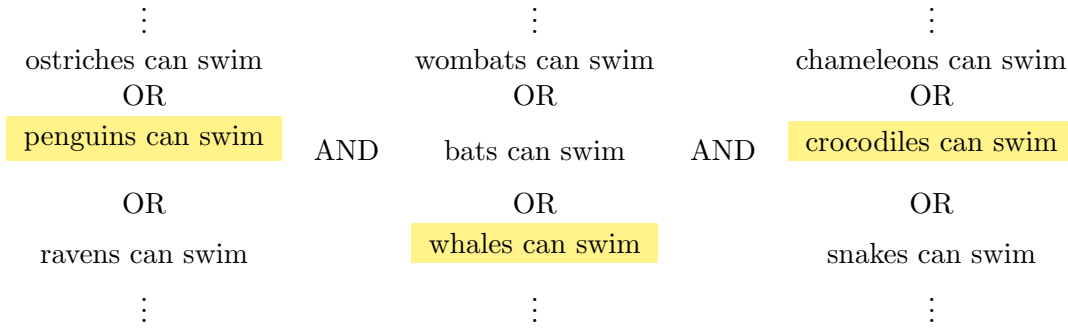
We may want to consider a proposition that contains multiple quantifiers. For example, let $S = \{\text{birds, mammals, reptiles}\}$. Also, let S_x be the set of all species of animals

1 Mathematical Language

of type x . So, for example, S_{birds} is the set of all species of birds. Let's think about this proposition:

$$\forall x \in S, \exists y \in S_x \text{ such that } y \text{ can swim.}$$

In English, this would read: "For all elements x in the set S , there exists an element y in the set S_x such that y can swim. Written out using AND and OR operators this would be:



Each column represents one of the " $\exists y \in S_x$ such that y can swim" statements. Each of the three columns is true, because at least one of the components is true. The overall statement is also true, since each of the columns are true.

You can see that this method of evaluating propositions with multiple quantifiers will get unwieldy when we have very large sets or a large number of quantifiers.

?

Suppose S_1 and S_2 are finite sets with n_1 and n_2 elements respectively. Suppose also that $P(x, y)$ is a proposition template which becomes a proposition after plugging an element of S_1 into x and an element of S_2 into y . Can you come up with a diagram similar to the one above that represents the following proposition?

$$\forall x \in S_1, \exists y \in S_2, P(x, y)$$

What about this proposition?

$$\exists x \in S_1, \forall y \in S_2, P(x, y)$$

Are the diagrams the same?

Here's a useful trick called the Adversary Method. Let's pretend that we are playing a game against an opponent who we call our Adversary. The game is played using a proposition with quantifiers that looks something like this:

$$\forall x_1 \in S_1, \exists x_2 \in S_2, \dots, \forall x_n \in S_n, P(x_1, \dots, x_n)$$

with a sequence of quantifiers at the beginning and a proposition template at the end that becomes true or false once we plug elements of the appropriate sets into it.

We proceed from left to right. Whenever there is a \forall symbol, our Adversary gets to pick an element of the corresponding set. Whenever there is a \exists symbol, we get to pick an element of the corresponding set. Our goal is to make $P(x_1, \dots, x_n)$ true, while our Adversary's goal is to make it false.

The overall proposition is true exactly when we have a winning strategy, i.e. if no matter what choices our Adversary makes we can react and make a choice that still results in us winning.

Let's apply this to our animal example

$$\forall x \in S, \exists y \in S_x \text{ such that } y \text{ can swim.}$$

Where $S = \{ \text{birds, mammals, reptiles} \}$ and S_x is the set of all species of animals of type x . Our adversary gets to go first and they get to pick either birds, mammals, or reptiles from the set S . Once they have made that choice, it's our job to pick an animal from the set of all animals of that type. What is our strategy? If the Adversary picks birds, we'll pick penguins. If they pick, mammals, we'll pick whales, and if they pick reptiles, we will pick crocodiles. At the end we much check if our chosen animal can swim. We have developed a strategy that guarantees it can. We have a winning strategy, so the proposition is true.

1.5 Functions

Aside from sets, functions are another fundamental object that we use to build up many different fields of math. Functions are a much more general concept than how they may have been portrayed in a high school curriculum.

Definition 20: Function

Let S_1 and S_2 be two sets. A **function** from S_1 to S_2

$$f : S_1 \rightarrow S_2$$

is a machine that takes as input an element of S_1 and gives as output an element of S_2 .

When we describe a function we always have to specify what its input and output sets are, *and* the method by which it assigns outputs to inputs. We use the notation

$$f : x \mapsto y$$

or alternatively

$$f(x) = y$$

to mean that when we put the element $x \in S_1$ into the function f , we will get out the element $y \in S_2$. When speaking, we can also say “ f maps x to y ”.

For example, let $S_1 = \{1, 2, 3\}$ and $S_2 = \{\triangle, \square\}$. We can define a function from S_1 to S_2 by specifying what element of S_2 it maps each element of S_1 to. Let f be the function that acts as follows.

$$\begin{aligned} f : S_1 &\rightarrow S_2 \\ 1 &\mapsto \triangle \\ 2 &\mapsto \square \\ 3 &\mapsto \triangle \end{aligned}$$

In the alternate notation we would write $f(1) = f(3) = \triangle$ and $f(2) = \square$.

?

Can we think of the logical operator AND as a function? What would its input and output sets be?

Sometimes our input set is too large to describe what every single element maps to. We might define a function by writing down a general rule for how outputs are assigned.

For example, let S be the set of all subsets of \mathbb{N} except the empty set. A few sample elements of S are: $\{1\}$, $\{2, 506, 1000000\}$, $\{2, 5, 8, 9\}$. We can define a function

$$\begin{aligned} f : S &\rightarrow \mathbb{N} \\ x &\mapsto \text{the smallest element of } x \end{aligned}$$

Looking at our sample elements, we have $f(\{1\}) = 1$, $f(\{2, 506, 1000000\}) = 2$, and $f(\{2, 5, 8, 9\}) = 2$. We can work out all these specific instances from the general rule.

?

Why did we have to exclude the empty set from S in the above example?

Sometimes we might want to define a function using a few different rules for different parts of the set S_1 . Such a function is called a **piece-wise function**. To write one down, we first separate S_1 into a finite number of subsets T_1, \dots, T_n that don't overlap (i.e. $T_i \cap T_j = \emptyset$ whenever $i \neq j$) and that cover the whole set S_1 so that $S_1 = T_1 \cup \dots \cup T_n$. Then, we give a different rule for each T_i like so:

$$f(x) = \begin{cases} \text{rule 1} & \text{if } x \in T_1 \\ \dots & \\ \text{rule n} & \text{if } x \in T_n \end{cases}$$

For example, if we let S be the set of all subsets of \mathbb{N} including the empty set, we could define a function $f : S \rightarrow \mathbb{N}$ by looking at the empty set and all the other subsets separately like so:

$$f(x) = \begin{cases} \text{the smallest element of } x & \text{if } x \in S \setminus \{\emptyset\} \\ 0 & \text{if } x \in \{\emptyset\} \end{cases}$$

1.5.1 Graphs

The graph of a function is a special set which contains all the information about how the function maps elements.

Definition 21: Graph of a function

Let $f : S_1 \rightarrow S_2$ be a function. The **graph** of f is the set

$$\text{graph} f = \{(x, y) \in S_1 \times S_2 \mid y = f(x)\}.$$

In other words, we can think of the graph of a function as the set of ordered pairs where the first object in the pair is an element of S_1 and the second object in the pair is the element in S_2 that f maps the first element to.

In our earlier example with $S_1 = \{1, 2, 3\}$ and $S_2 = \{\triangle, \square\}$ and $f : S_1 \rightarrow S_2$ defined by $f(1) = f(3) = \triangle$ and $f(2) = \square$, the graph would be the set

$$\text{graph}(f) = \{(1, \triangle), (2, \square), (3, \triangle)\}.$$

1.5.2 Real-valued functions

In this class, we will be particularly interested in functions which take real numbers as inputs and give real numbers as outputs, i.e. functions of the form

$$f : \mathbb{R} \rightarrow \mathbb{R}.$$

Such functions are called **real-valued functions**. Since \mathbb{R} is a large (infinite!) set, we'll have to describe real-valued functions with a general rule. For example

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x + 1 \end{aligned}$$

is the function that takes any real number as input and gives as output that number plus one. We might abbreviate these functions by just writing down the rule and leave it to the reader to infer from context that the input and output sets are \mathbb{R} . In the example we just discussed, we would write simply $f(x) = x + 1$.

The graph of a real-valued function is a set of the form

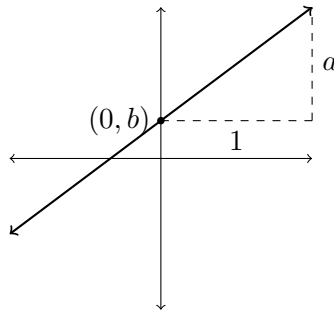
$$\text{graph}(f) = \{(x, f(x)) | x \in \mathbb{R}\}.$$

We can visualize this set as a bunch of points in the 2-D plane where the first number in the ordered pair gives a coordinate on the horizontal axis and the second number gives a coordinate on the vertical axis.

Notice that any real number only appears in the first coordinate once, so there cannot be two points in the graph lying along the same vertical line in the plane. This fact is known colloquially as the “vertical line test”.

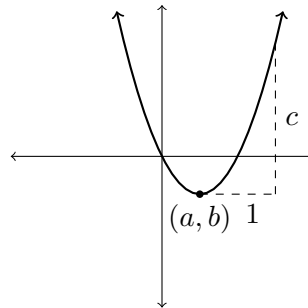
Linear functions

Linear functions are real-valued functions of the form $f(x) = ax + b$. Their graphs are straight lines in the plane which have slope a and cross the vertical axis at b .



Quadratic functions

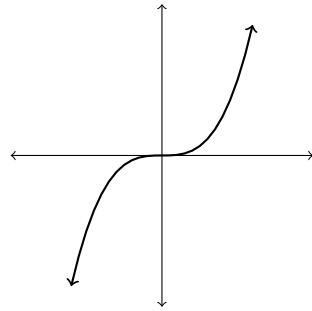
Quadratic functions are real-valued functions of the form $f(x) = c(x - a)^2 + b$. They have graphs which are shaped like parabolas. The base of the parabola is located at the point (a, b) and c determines how stretched the parabola is in the vertical direction.



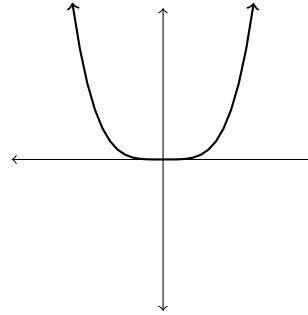
Cubic and quartic functions

Cubic and quartic functions are real-valued functions involving taking third and fourth powers respectively. Graphs of basic cubic and quartic functions are pictured below.

$$f(x) = x^3$$



$$f(x) = x^4$$



?

What does the graph of the function $f(x) = -x^3$ look like? What about the graph of $f(x) = -x^4$?

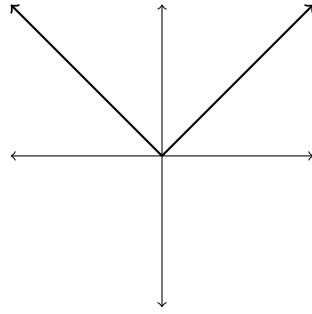
Absolute-value function

The absolute value function is defined piece-wise. When it takes a positive number as input, it outputs the same number. However, when it takes a negative number as input, it returns the negative of that number, which is a positive number. The absolute value function is denoted by $|x|$ and is defined formally as follows.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We can think of the absolute value function as measuring the (positive) distance between a real number and 0 on the number line. We can extend this and think of $|x_1 - x_2|$ as the distance between two numbers x_1 and x_2 on the number line.

The graph of the absolute value function looks like two linear functions stuck together with a sharp corner.



?

What is the relationship between the functions $|x|$ and $|-x|$?

1.6 Proof Strategies

When building up a mathematical theory, we repeatedly apply this basic process:

define an object \longrightarrow formulate a proposition \longrightarrow argue that proposition is true

For example, suppose we want to understand how even and odd numbers relate to each other. First, we have to define what even and odd mean. So, we'll give a formal definition.

Definition 22: Even and odd numbers

An integer n is called **even** if it can be written as $n = 2k$ where $k \in \mathbb{Z}$.

And integer n is called **odd** if it can be written as $n = 2k + 1$ where $k \in \mathbb{Z}$.

A natural question to ask now is whether a number can be odd and even at the same time. Let's experiment with a few numbers to get a feeling for it.

Consider the number 1. Is it even? If it is even, we have to be able to write it as $1 = 2k$ where k is an integer. But, solving this equation for k we find that the only solution is $k = \frac{1}{2}$, which is not an integer. So 1 is not even. Is 1 odd? Yes, it is, because we can write $1 = 2 \times 0 + 1$, and 0 is an integer.

Consider the number 2. Is it even? Yes it is, because we can write $2 = 2 \times 1$ and 1 is an integer. Is 2 odd? If it were odd, we would be able to write $2 = 2k + 1$ with k an integer. However, solving for k we find that $k = \frac{1}{2}$, which is not an integer. So 2 is not

odd.

We're starting to get the sense that a number can't be odd and even at the same time. But, we don't know for sure because we have only checked for a few numbers. Now is a good time to formulate a proposition that we think might be true.

Proposition 3

An integer cannot be simultaneously odd and even.

?

I've written this proposition in natural language. Can you rewrite this proposition using quantifiers and logical operators?

After stating the proposition, we will attempt to argue that it is true. This is called proving the proposition. By convention, we write "Proof." at the beginning of our argument and this symbol: \square at the end of our argument to help the reader see where it starts and stops.

Proof.

Suppose n is an integer that is both even and odd.

Since n is even, we can write it as $n = 2k_1$, where k_1 is an integer.

Since n is odd, we can write it as $n = 2k_2 + 1$ where k_2 is an integer.

Since n is equal to both of them, we can conclude that $2k_1 = 2k_2 + 1$.

Diving the equation by 2, we get that $k_1 = k_2 + \frac{1}{2}$.

However, k_1 and k_2 are both integers, which means their difference should be an integer. But, their difference is $\frac{1}{2}$. We have arrived at a statement that is false. We have two integers that are separated by a distance of $\frac{1}{2}$. This means that the assumption we made at the beginning must be false.

It is not possible for an integer to be both even and odd. \square

Instead of having to check that something is true for every single integer (an impossible task since there are infinitely many of them) we have used reasoning and appealed to our rigorous definitions to find out that a particular statement holds true for every single

integer.

Direct proof

Proof by contradiction

If-then statements and the contrapositive

2 Limits

2.1 Definition of the Limit

2.2 Right and left-handed limits

2.3 Continuity

2.4 Limit Laws

2.5 Limits at infinity and infinite limits

3 Derivatives

3.1 Definition of the derivative and differentiability

3.2 Derivative Rules

3.3 Derivatives of transcendental functions

3.4 Applications

3.5 Fermat's Theorem, Extreme Value Theorem, Mean Value Theorem

3.6 L'Hôpital's Rule

3.7 Higher order derivatives

3.8 Curve sketching

4 Intergals

4.1 Supremum and infimum

4.2 Riemann sums

4.3 Riemann integrals and integrability

4.4 The Fundamental Theorem of Calculus