# Econ, Problem Set #2

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Due Wednesday, July 5 at 8:00am

**Exercise 1, Problem 2** To prove existence, we show that $T = c(1-\beta)+\beta\sum_{k=1}^{K}\max\{w_k,\ x\}\ p_k$ is a contraction mapping. Take any $x, y \in \mathbb{R}$. Then

$$|Tx-Ty| = |\beta\sum_{k=1}^{K}(\max\{w_k, x\}-\max\{w_k, y\})p_k| \leq \beta\sum_{k=1}^{K}|(\max\{w_k, x\}-\max\{w_k, y\})|p_k$$

$$\leq \beta\sum_{k=1}^{K}|x-y|p_k = \beta|x-y|\sum_{k=1}^{K}p_k = \beta|x-y| < |x-y|$$

So T is a contraction mapping on a complete space, $([0, \infty])$ and hence converges to a unique fixed point. $\square$

**Exercise 2, Problem 1** Again, we have an operator U,

$$Uw(y) = u(\sigma(y)) + \beta\int w(f(y-\sigma(y))z)\phi(dz) \qquad (y \in \mathbb{R}_+)$$

and show that U is a contraction mapping by observing:

$$\|Uw(y) - Uw'(y)\| = \|\beta\int(w(f(y-\sigma(y))z) - w'(f(y-\sigma(y))z))\phi(dz)\|$$

$$\leq \beta\int\|w(f(y-\sigma(y))z) - w'(f(y-\sigma(y))z)\|\phi(dz)$$

$$\leq \beta\int sup|w(f(y-\sigma(y))z) - w'(f(y-\sigma(y))z)|\phi(dz)$$

$$\leq \beta\int|w(y) - w'(y)|\phi(dz) = \beta sup|w(y) - w'(y)|\int\phi(dz)$$

$$= \beta sup|w(y) - w'(y)| = \beta\|w(y) - w'(y)\| < \|w(y) - w'(y)\|$$

Once again, observing our space of $C(\mathbb{R}^+)$ is complete under the sup norm, by Banach's fixed point theorem, there exists a unique fixed point. $\square$

**(b)** $v_\sigma(y) = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(\sigma(y_t))\right]$ **is the fixed point.**

$$Uv_\sigma(y) = u(\sigma(y)) + \beta \int v_\sigma(f(y - \sigma(y))z)\phi(dz) \qquad (y \in \mathbb{R}_+)$$

$$= u(\sigma(y)) + \beta\mathbb{E}\left[v_\sigma(y')\right] = u(\sigma(y)) + \beta\mathbb{E}\left[\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(\sigma(y_{t+1}))\right]\right]$$

$$= u(\sigma(y)) + \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^{t+1} u(\sigma(y_{t+1}))\right] = u(\sigma(y)) + \mathbb{E}\left[\sum_{t=1}^{\infty} \beta^t u(\sigma(y_t))\right]$$

$$= \beta^0 u(\sigma(y)) + \mathbb{E}\left[\sum_{t=1}^{\infty} \beta^t u(\sigma(y_t))\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(\sigma(y_t))\right] = v_\sigma(y) \qquad (y \in \mathbb{R}_+)$$

where the time indices on y are necessarily shifted to start in period 1 in the second line since it represents the continuation value.

Thus, we have shown that $v_\sigma(y) = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(\sigma(y_t))\right]$ is the fixed point. □