Math, Problem Set #4, Optimization Introduction

OSM Lab, Dr. Barro

Due Friday, July 14 at 8:00am

6.1 Put the following optimization problem in standard form; that is, write an optimization problem in standard form that is equivalent to the following. Given $x, y \in \mathbb{R}^n$, $a, b \in \mathbb{R}$, and $A \in M_n(\mathbb{R})$, choose $w \in \mathbb{R}^n$ in order to

$$\begin{aligned} \text{maximize} \quad & e^{-w^T x} \\ \text{subject to} \quad & w^T x \geq w^T A w - w^T A y + a \\ & y^T w = w^T x + b \end{aligned}$$

Solution In the standard form (see Definition 6.1.8), this is written as

$$\begin{aligned} \text{minimize} \quad & -e^{-w^T x} \\ \text{subject to} \quad & w^T A w - w^T A y - w^T x \leq -a \\ & y^T w - w^T x = b \end{aligned}$$

6.5 A plastics company makes two products: knobs for electronic products and milk cartons. The primary production expenses for each are labor and the raw plastic. Each milk bottle requires 4 grams of plastic and 2 minutes of labor. Each knob takes 3 grams of plastic and 1 minute of labor. During the current production period, the company has 240kg of plastic and 100 hours of labor. Each milk bottle yields a profit of \$0.07 and each knob \$ 0.05. Write an optimization problem in standard form that is equivalent to finding the amount the company should produce of each product in order to maximize its profits.

Solution We use shorthand m for milk bottles, k for knobs. We have a plastic constraint (in grams) $4m+3k \leq 240000$, and time constraint (in hours) $2m+k \leq 6000$.

The profit for the firm is given by $0.07m + 0.05k$.

Writing these constraints in standard form, we get the maximization problem in standard form:

$$\begin{aligned} \underset{m,k}{max} \quad & 0.07m + 0.05k \\ \text{such that} \quad & \begin{bmatrix} 4 & 3 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \leq \begin{bmatrix} 240000 \\ 6000 \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

where the inequalities are element-wise.

6.6 Find and identify all the critical points of the function

$$f(x, y) = 3x^2y + 4xy^2 + xy$$

Determine whether they are the locations of local maxima, minima, or saddle points.

Solution We have the derivatives:

$$\begin{aligned} f_x(x, y) &= 6xy + 4y^2 + y = (6x + 4y + 1)y \\ f_y(x, y) &= 3x^2 + 8xy + x = (3x + 8y + 1)x \end{aligned}$$

So

$$Df(x, y) = \begin{bmatrix} (6x + 4y + 1)y & x(3x + 8y + 1) \end{bmatrix},$$

from which we can easily see that $(0, 0), (0, -\frac{1}{4}), (\frac{1}{3}, 0)$ are solutions. A fourth solution can be found from the linear system of equations:

$$\begin{bmatrix} 6x + 4y + 1 \\ 3x + 8y + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

from which we can deduce that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & 8 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{12} \end{bmatrix},$$

and thus the fourth root $(-\frac{1}{9}, -\frac{1}{12})$.

The Hessian is given by:

$$D^2 f(x, y) = \begin{bmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix},$$

from which we can observe

$$D^2 f(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D^2 f(0, -\frac{1}{4}) = \begin{bmatrix} -\frac{3}{2} & -1 \\ -1 & 0 \end{bmatrix},$$

$$D^2 f(-\frac{1}{3}, 0) = \begin{bmatrix} 0 & -1 \\ -1 & -\frac{8}{3} \end{bmatrix}, \quad D^2 f(-\frac{1}{9}, -\frac{1}{12}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

With mathematica (or tedious algebra), we can observe that $D^2 f(-\frac{1}{9}, -\frac{1}{12})$ is negative definite, so the associated root corresponds to a local maximum. The other three points are saddle points since the Hessian matrix evaluated at them is neither positive nor negative definite.

6.11 Consider a quadratic function $f(x) = ax^2 + bx + c$, where $a > 0$, and $b, c \in \mathbb{R}$. Show that for any initial guess $x_0 \in \mathbb{R}$, one iteration of Newton's method lands at the unique minimizer of f.

Solution Note that $f(x) = a(x+\frac{b}{2a})^2 - a(\frac{b}{2a})^2 + c$ so the minimum is clearly reached at $x = -\frac{b}{2a}$.

Now regardless of what we choose as x_0, iterating one time with Newton's method

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 - \frac{2ax_0 + b}{2a} = -\frac{b}{2a}$$

6.14 Code up the secant method for finding a minimizer of a function.

Solution See the notebook in the same folder.