Open Source Macroeconomics Laboratory Boot Camp DSGE Exercises

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DSGE

Exercise 1. For the Brock and Mirman model, find the value of A in the policy function. Show that your value is correct.

For this case find an algebraic solution for the policy function, $k_{t+1} = \Phi(k_t, z_t)$. Couple of good sources for hints are **?**, exercise 2.2, p. 12 and **?**, exercise 1.1, p. 47.

Solution As suggested, we guess that the policy function is $k_{t+1} = \Phi(k_t, z_t) = Ae^{z_t}k_t^\alpha$, for some A.

To verify and find such A, we substitute into both sides of the Euler equation

$$\frac{1}{e^{z_t}k_t^\alpha - k_{t+1}} = \beta E_t\{\frac{\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}}{e^{z_{t+1}}k_{t+1}^\alpha - k_{t+2}}\}$$

We get

$$\frac{1}{e^{z_t}k_t^\alpha(1-A)} = \frac{1}{e^{z_t}k_t^\alpha - Ae^{z_t}k_t^\alpha}$$

$$= \beta E_t\{\frac{\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}}{e^{z_{t+1}}k_{t+1}^\alpha - k_{t+2}}\} = \beta(\frac{\alpha e^{\rho z_t}(Ae^{z_t}k_t^\alpha)^{\alpha-1}}{e^{\rho z_t}(Ae^{z_t}k_t^\alpha)^\alpha(1-A)}) = \frac{\alpha\beta}{Ae^{z_t}k_t^\alpha(1-A)}$$

which simplifies without contradiction to $1 = \frac{\alpha\beta}{A}$, i.e. $A = \alpha\beta$.

Exercise 2. For the model in section 3 of the notes consider the following functional forms:

$$u(c_t, \ell_t) = \ln\, c_t + a\, \ln\, (1-\ell_t)$$
$$F(K_t, L_t, z_t) = e^{z_t}K_t^\alpha L_t^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms.

Can you use the same tricks as in Exercise 1 to solve for the policy function in this case? Why or why not?

Solution Regardless of functional form, we always have the following market clearing conditions

$$\ell_t = L_t, \quad k_t = K_t, \quad w_t = W_t, \quad r_t = R_t,$$

as well as the government budget balance and law of motion of the shock

$$\tau[w_t l_t + (r_t - \delta)k_t] = T_t, \quad z_t = (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2)$$

The remaining characterizing equations with the above-mentioned functional form:

$$c_t = (1-\tau)[w_t \ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1}$$

$$\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}}[(r_{t+1} - \delta)(1-\tau) + 1]$$

$$\frac{a}{1-\ell_t} = \frac{1}{c_t} w_t (1-\tau)$$

$$r_t = \alpha e^{z_t} (\frac{\ell_t}{k_t})^{1-\alpha}$$

$$w_t = (1-\alpha) e^{z_t} (\frac{k_t}{\ell_t})^{\alpha}$$

Exercise 3. For the model in section 3 consider the following functional forms:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \ln(1-\ell_t)$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms.

Solution Regardless of functional form, we always have the following market clearing conditions

$$\ell_t = L_t, \quad k_t = K_t, \quad w_t = W_t, \quad r_t = R_t,$$

as well as the government budget balance and law of motion of the shock

$$\tau[w_t l_t + (r_t - \delta)k_t] = T_t, \quad z_t = (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2)$$

The remaining characterizing equations with the above-mentioned functional form:

$$c_t = (1-\tau)[w_t \ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1}$$

$$\frac{1}{c_t^{\gamma}} = \beta E_t \frac{1}{c_{t+1}^{\gamma}}[(r_{t+1} - \delta)(1-\tau) + 1]$$

$$\frac{a}{1-\ell_t} = \frac{1}{c_t^{\gamma}} w_t (1-\tau)$$

$$r_t = \alpha e^{z_t} (\frac{\ell_t}{k_t})^{1-\alpha}$$

$$w_t = (1-\alpha) e^{z_t} (\frac{k_t}{\ell_t})^{\alpha}$$

Exercise 4. For the model in section 3 consider the following functional forms:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi} - 1}{1-\xi}$$
$$F(K_t, L_t, z_t) = e^{z_t} \left[\alpha K_t^{\eta} + (1-\alpha) L_t^{\eta}\right]^{\frac{1}{\eta}}$$

Write out all the characterizing equations for the model using these functional forms.

Solution Regardless of functional form, we always have the following market clearing conditions

$$\ell_t = L_t, \quad k_t = K_t, \quad w_t = W_t, \quad r_t = R_t,$$

as well as the government budget balance and law of motion of the shock

$$\tau[w_t l_t + (r_t - \delta)k_t] = T_t, \quad z_t = (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2)$$

The remaining characterizing equations with the above-mentioned functional form:

$$c_t = (1-\tau)[w_t \ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1}$$

$$\frac{1}{c_t^{\gamma}} = \beta E_t \frac{1}{c_{t+1}^{\gamma}}[(r_{t+1} - \delta)(1-\tau) + 1]$$

$$\frac{a}{(1-\ell_t)^{\xi}} = \frac{1}{c_t^{\gamma}} w_t (1-\tau)$$

$$r_t = \alpha k_t^{\eta-1} e^{z_t} [\alpha k_t^{\eta} + (1-\alpha)\ell_t^{\eta}]^{\frac{1}{\eta}-1}$$

$$w_t = (1-\alpha) \ell_t^{\eta-1} e^{z_t} [\alpha k_t^{\eta} + (1-\alpha)\ell_t^{\eta}]^{\frac{1}{\eta}-1}$$

Exercise 5. For the model in section 3 abstract from the labor/leisure decision and consider the following functional forms:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$
$$F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms. Assume $\ell_t = 1$.

Write out the steady state versions of these equations. Solve algebraically for the steady state value of k as a function of the steady state value of z and the parameters of the model. Numerically solve for the steady state values of all variables using the following parameter values: $\gamma = 2.5$, $\beta = .98$, $\alpha = .40$, $\delta = .10$, $\bar{z} = 0$ and $\tau = .05$. Also solve for the steady state values of output and investment. Compare these values with the ones implied by the algebraic solution.

Solution Regardless of functional form, we always have the following market clearing conditions

$$\ell_t = L_t, \quad k_t = K_t, \quad w_t = W_t, \quad r_t = R_t,$$

as well as the government budget balance and law of motion of the shock

$$\tau[w_t l_t + (r_t - \delta)k_t] = T_t, \quad z_t = (1-\rho_z)\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim i.i.d.(0, \sigma_z^2)$$

The remaining characterizing equations with the above-mentioned functional form:

$$c_t = (1-\tau)[w_t \ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1}$$
$$\frac{1}{c_t^{\gamma}} = \beta E_t \frac{1}{c_{t+1}^{\gamma}}[(r_{t+1} - \delta)(1-\tau) + 1]$$
$$\ell_t = 1, \quad \text{which makes} \quad 0 = u_\ell = \frac{1}{c_t^{\gamma}} w_t(1-\tau) \quad \text{unnecessary}$$
$$r_t = \alpha(\frac{e^{z_t}\ell_t}{k_t})^{1-\alpha}$$

$$w_t = (1-\alpha)e^{z_t}(\frac{k_t}{e^{z_t}})^{\alpha}$$

In the steady state, note that through $e^{\bar{z}} = e^0 = 1$ and budget balance, these become:

$$\bar{c} = \bar{w} + (\bar{r} - \delta)\bar{k}$$

$$\frac{1}{\bar{c}^{\gamma}} = \beta\frac{1}{\bar{c}^{\gamma}}[(\bar{r}-\delta)(1-\tau)+1]$$

$$\bar{r} = \alpha(\frac{1}{\bar{k}})^{1-\alpha}$$

$$\bar{w} = (1-\alpha)(\bar{k})^{\alpha}$$

We can cancel c out of the Euler equation, and immediately solve for $\bar{r}$:

$$\bar{r} = \delta + \frac{1-\beta}{\beta(1-\tau)} = \frac{1-\beta+\delta\beta(1-\tau)}{\beta(1-\tau)}$$

We can then use the third equation to solve for $\bar{k}$:

$$\bar{k} = (\frac{\alpha}{\bar{r}})^{\frac{1}{1-\alpha}} = (\frac{\alpha\beta(1-\tau)}{1-\beta+\delta\beta(1-\tau)})^{\frac{1}{1-\alpha}}$$

and the fourth equation to yield wage:

$$\bar{w} = (1-\alpha)\bar{k}^{\alpha} = (1-\alpha)(\frac{\alpha\beta(1-\tau)}{1-\beta+\delta\beta(1-\tau)})^{\frac{\alpha}{1-\alpha}}$$

Likewise, the remaining variables can be easily computed through substitution:

$$\bar{c} = \bar{w} + (\bar{r}-\delta)\bar{k}, \qquad \bar{Y} = \bar{k}^{\alpha}, \qquad \bar{I} = \delta\bar{k}$$

See the notebook for the calculations.

Exercise 6. For the model in section 3 consider the following functional forms:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi}-1}{1-\xi}$$

$$F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms. Write out the steady state versions of these equations. Numerically solve for the steady state values of all variables using the following parameter values: $\gamma = 2.5$, $\xi = 1.5$, $\beta = .98$, $\alpha = .40$, $a = .5$, $\delta = .10$, $\bar{z} = 0$, and $\tau = .05$. Also solve for the steady state values of output and investment.

Solution Regardless of functional form, we always have the following market clearing conditions

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The remaining characterizing equations with the above-mentioned functional form:

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$$\frac{1}{c_t^\gamma} = \beta E_t \frac{1}{c_{t+1}^\gamma}[(r_{t+1} - \delta)(1-\tau) + 1]$$

$$\frac{a}{(1-\ell_t)^\xi} = \frac{1}{c_t^\gamma} w_t (1-\tau)$$

$$r_t = \alpha e^{z_t}(\frac{e^{z_t}\ell_t}{k_t})^{1-\alpha}$$

$$w_t = (1-\alpha)e^{z_t}(\frac{k_t}{e^{z_t}\ell_t})^\alpha$$

In the steady state:

$$\bar{c} = \bar{w} + (\bar{r}\bar{\ell} - \delta)\bar{k}$$

$$\frac{1}{\bar{c}^\gamma} = \beta\frac{1}{\bar{c}^\gamma}[(\bar{r} - \delta)(1-\tau) + 1]$$

$$\frac{a}{(1-\bar{\ell})^\xi} = \frac{1}{\bar{c}^\gamma}\bar{w}(1-\tau)$$

$$\bar{r} = \alpha e^{\bar{z}} (\frac{\bar{\ell} e^{\bar{z}}}{\bar{k}})^{1-\alpha}$$

$$\bar{w} = (1-\alpha)(\frac{\bar{k}}{\bar{\ell} e^{\bar{z}}})^{\alpha}$$

The remaining exercises can be found in the notebook.