CHSH Violation and the incompatibility of Quantum Mechanics and Local Hidden Variable Theory

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(Dated: 3/24/17)

In this experiment, we use an Entanglement Demonstrator to produce entangled photons by the spontaneous parametric down conversion. From here, we use rotating polarizers to measure the coincidence count of these resulting photons at various angles to show the violation of CHSH. Thus, we are able to confirm the incompatibility of quantum mechanics with local hidden variable thoery.

INTRODUCTION

Quantum mechanics gives rise to the problem of indeterminacy from our interpretation of the wave function, Ψ. This function gives a probability distribution, but collapses when a measurement is taken. As a result, there have been theories proposed involving hidden variables that would lead us to believe that the final state of a particle before measurement is determined. In the Quantum Cakes thought experiment by Kwiat and Hardy, we came across the puzzling result that although we found a lower bound to the probability for getting two good tasting cakes on both sides of the conveyor belt, our calculations gave us zero probability for that case. Even when we attempted to fake this quantum mechanical result with nonlocal theories.
The Bell's inequality describes a limit to the local hidden variable theory, which quantum mechanics defies, and thus we will be strengthening the case that the wave function, Ψ, does in fact contain all the knowable information about a particle when only considering only hidden variable theories that are local. We will start with the full derivation of both the CHSH and Bell's inequalities, then display our measurements taken on entangled photons to show the incompatibility of Quantum mechanics with local hidden variable theory.

Derivation of the CHSH Inequality

First, let us assume that the minus Bell state is used in the following calculations:

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|H_1\rangle|V_2\rangle - |V_1\rangle|H_2\rangle$$

By changing the orientations of our polarizers, we could see two different bases. The following expresses polarization measurements when the polarizers are at an arbitrary angle, α:

$$|V(\alpha)\rangle = \cos\alpha\,|V\rangle - \sin\alpha\,|H\rangle$$

$$|H(\alpha)\rangle = \sin\alpha\,|V\rangle + \cos\alpha\,|H\rangle$$

From here, it's easy to use these expressions as transformation matrices. There are four measurements we could make:

1. $|V(\alpha)\rangle$, $|V(\beta)\rangle$
2. $|V(\alpha)\rangle$, $|H(\beta)\rangle$
3. $|H(\alpha)\rangle$, $|V(\beta)\rangle$
4. $|H(\alpha)\rangle$, $|H(\beta)\rangle$

The probabilities for the four cases above can be computed as follows.

$$P_{VV}(\alpha,\beta) = |\,\langle V(\alpha)|_1\,\langle V(\beta)|_1\,|\Psi_-\rangle\,|^2$$

$$= \frac{1}{2}|cos\ \alpha\ sin\ \beta\ -\ cos\ \beta\ sin\ \alpha|^2$$

$$= \frac{1}{2}\sin^2(\alpha-\beta)$$

Similarly,

$$P_{VH}(\alpha,\beta) = \frac{1}{2}\cos^2(\alpha-\beta)$$

$$P_{HV}(\alpha,\beta) = \frac{1}{2}\cos^2(\alpha-\beta)$$

$$P_{HH}(\alpha,\beta) = \frac{1}{2}\sin^2(\alpha-\beta)$$

Just as in the quantum cakes thought experiment, we have two different bases. In order to find the correlations across the bases, we define the following quantity:

$$S = E(a,b) - E(a,b') + E(a',b) + E(a',b')$$

In the equation above, we used the following angles in place of the values for a, b, a', and b':

$$\{a, b, a', b'\} = \{-45°, -22.5°, 0°, 22.5°\}$$

Plugging in these values, we a value for the correlation: $S = 2\sqrt{2}$. At this point, we can consider the local hidden variable. It is reasonable to assume that this set of hidden variables are carried by a set we'll call λ, with a normalized distribution described by $\rho(\lambda)$. The probability distribution of the hidden variable can be used to calculated the following correlation:

$$P(\hat{x_1}, \hat{x_2}) = -\int d\lambda \rho(\lambda) S_-(\hat{x_1}) S_-(\hat{x_2})$$

We can then include this in our calculation for the distribution of coincidence counts.

$$P_{VV}(\alpha, \beta) = \int d\lambda\ \rho(\lambda) \frac{1 + A(\lambda, \alpha)}{2} \frac{1 + B(\lambda, \beta)}{2}$$

$$P_{VH}(\alpha, \beta) = \int d\lambda\ \rho(\lambda) \frac{1 + A(\lambda, \alpha)}{2} \frac{1 - B(\lambda, \beta)}{2}$$

$$P_{HV}(\alpha, \beta) = \int d\lambda\ \rho(\lambda) \frac{1 - A(\lambda, \alpha)}{2} \frac{1 + B(\lambda, \beta)}{2}$$

$$P_{HH}(\alpha, \beta) = \int d\lambda\ \rho(\lambda) \frac{1 - A(\lambda, \alpha)}{2} \frac{1 - B(\lambda, \beta)}{2}$$

In the above equations, A and B have been defined to be the measurement we make in each polarization arm. Using these distributions, we can rewrite our correlation and our S value:

$$E = \int d\lambda\ \rho(\lambda) A(\lambda, \alpha) B(\lambda, \beta)$$

$$S = \int d\lambda\ \rho(\lambda[(A(\lambda, a)(B(\lambda, b) - B(\lambda, b'))$$

$$+A(\lambda, a')(B(\lambda, b) + B(\lambda, b'))]$$

Since each term in the bracket is bounded by a magnitude of 1, we find that the integral is bounded by 2:

$$|S| \leq 2$$

Therefore, we conclude that CHSH violation occurs when the S value is found to be greater than 2.

Bell's Inequality

Using the correlation calculated with the probability distribution of the set of hidden variables, λ, we can calculate the difference in the two correlations as the following:

$$P(\hat{x_1}, \hat{x_2}) - P(\hat{x_1}, \hat{x_2}') = -\int d\lambda\ \rho(\lambda)(S_-(\lambda, \hat{x_1}) S_-(\lambda, \hat{x_2})$$

$$- S_-(\lambda, \hat{x_a}) S_-(\lambda, \hat{x_2}'))$$

$$= -\int d\lambda\ \rho(\lambda) S_-(\lambda, \hat{x_1}) S_-(\lambda, \hat{x_2})[1 - S_-(\lambda, \hat{x_2}) S_-(\lambda, \hat{x_2}')$$

In the equation above, the term $S_-(\lambda, \hat{x_1}) S_-(\lambda, \hat{x_2})$ is bounded by a magnitude of 1, and the rest of the terms is greater than or equal to 0. As a result, we get the following inequality:

$$|P(\hat{x_1}, \hat{x_2}) - P(\hat{x_1}, \hat{x_2}')| \leq \int d\lambda\ \rho(\lambda)[1 - S_-(\lambda, \hat{x_2}) S_-(\lambda, \hat{x_2}')$$

By using the fact that the spins are related, $S_+(\lambda, \hat{x_2}) = -S_-(\lambda, \hat{x_2})$, the Bell's inequality can be written as follows:

$$|P(\hat{x_1}, \hat{x_2}) - P(\hat{x_1}, \hat{x_2}')| \leq 1 + P(\hat{x_2}, \hat{x_2}')$$

EXPERIMENTAL METHOD

For our experiment, we used the photon Entanglement Demonstrator, consisting of blue laser light shot through a beam splitter, then a Beta-Barium Borate (BBO) crystal. Entangled photons emerged from the crystal through the spontaneous parametric down conversion process. From this birefringent crystal, we got red lasers, which then went through a half wave plate to rotate their polarizations and make the two resulting beams absolutely indistinguishable, yet anticorrelated. From here, there was also installed compensating crystals to bring back the photons into spatial coincidence. Finally, we sent the entangled photons through two rotatable polarizers. This set up was untouched throughout our experiment, with the exception of the rotating polarizer. The main measurement for this experiment was the coincidence count, which was a count of when our detectors saw a photon through both arms. This was proved useful because it blocked out much the error from background light. Using the equation for the correlation from the introduction, we were able to calculate the following values and their errors:

$$E(\alpha, \beta) = P_{VV}(\alpha, \beta) + P_{HH}(\alpha, \beta) - P_{VH}(\alpha, \beta) - P_{HV}(\alpha, \beta)$$

$$S = E(\alpha, \beta) - E(\alpha, \beta') + E(\alpha', \beta) + E(\alpha', \beta')$$

$$\Delta S = \sqrt{\Delta E(a,b)^2 + \Delta E(a',b)^2 + \Delta E(a,b')^2 + \Delta E(a',b')^2}$$

$$\Delta E = 2\sqrt{\frac{(N(a,b) + N(a',b'))(N(a,b') + N(a',b))}{(N(a,b) + N(a',b') + N(a',b) + N(a,b'))^3}}$$

For just the above expression, the primes indicate a 90° rotation of the polarizer, *not* an angle in the different basis.

Set up and Procedure

Our set up gave us the $|Psi_+\rangle$ state, which we verified by setting both the polarizers at 45° and getting out a minimum coincidence count rate. Once we verified the state, we measured the visibility in two bases. In the horizontal/vertical basis, we calculated a visibility of 0.92.

$$V = (C_{max}C_{min})/(C_{max} + C_{min}) = \frac{390}{424} = 0.919 \approx 92\%$$

From the Entanglement Demonstrator, we calculated the coincidence counts at a time interval of 1000 ms. We also took a total of three measurements of coincidence counts at each angle, and took the average for our calculations. Table 1 shows our 16 measurements. From these measurements we were able to plug them into the calculations in the previous section to compute our value for S, and confirm it indeed violated the CHSH inequality.

EXPERIMENTAL RESULTS AND DISCUSSION

Using the equation for E from before, we calculated the values of E as follows:

$$E(-45°, -22.5°) = \frac{242 + 203 - 1576 - 1475.33}{242 + 203 + 1576 + 1475.33} = -0.745$$

$$\Delta E(-45°, -22.5°) = 2\sqrt{\frac{(242 + 203)(1576 + 1475.33)}{(242 + 203 + 1576 + 1475.33)^3}}$$

$$= 0.00564$$

α (degrees)	β (degrees)	Coincidence count (average)
0	-22.5	1349.67
0	22.5	1727.00
0	67.5	506.67
0	112.5	154.00
-45	-22.5	242.00
-45	22.5	1309.67
-45	67.5	1576.67
-45	112.5	426337
45	-22.5	1475.33
45	22.5	474.00
45	67.5	203.00
45	112.5	1299.67
90	-22.5	456.00
90	22.5	144.67
90	67.5	1428.67
90	112.5	1618.67

TABLE I. Shows the recorded values of coincidence counts for corresponding configurations of our polarizers.

$$E(-45°, 22.5°) = \frac{1309.67 + 1299.67 - 425.33 - 474}{1309.67 + 1299.67 + 426.33 + 474} = 0.487$$

$$\Delta E(-45°, 22.5°) = 2\sqrt{\frac{(1309.67 + 1299.67)(426.33 + 474)}{(1309.67 + 1299.67 + 426.33 + 474)^3}}$$

$$= 0.00737$$

$$E(0°, -22.5°) = \frac{1349.67 + 1428.67 - 506.67 - 456}{1349.67 + 1428.67 + 506.67 + 456} = 0.485$$

$$\Delta E(0°, -22.5°) = 2\sqrt{\frac{(1349.67 + 1428.67)(506.67 + 456)}{(1349.67 + 1428.67 + 506.67 + 456)^3}}$$

$$= 0.00715$$

$$E(0°, 22.5°) = \frac{1727 + 1618.67 - 154 - 144.67}{1727 + 1618.67 + 154 + 144.67} = 0.836$$

$$\Delta E(0°, 22.5°) = 2\sqrt{\frac{(1727 + 1618.67)(154 + 144.67)}{(1727 + 1618.67 + 154 + 144.67)^3}}$$

$$= 0.00454$$

From here, we can compute our S value and its uncertainty:

$$S = 0.745 + 0.487 + 0.485 + 0.836 = 2.553$$

$$\Delta S = \sqrt{0.00564^2 + 0.00737^2 + 0.00715^2 + 0.00454^2} = 0.0126$$

Finally, we calculated the value for S, which is indeed between 2 and $2\sqrt{2}$:

$$S = 2.553 \pm 0.0126$$

We successfully determined the value for S using Bell's inequality, and as expected, computed a value that was between 2 and $2\sqrt{2}$.

CONCLUSION

As you can see from the computation of S, we found our value to indeed violate the CHSH inequality. Even without considering the background noise that could have been counted in the coincidence counts, we got a value that clearly exceeds the upper bound derived in the introduction. Therefore, we did not correct the coincidence counts at all. One main source of error we found difficult to work with was the fluctuation in coincidence counts. At first, we used a time interval of 200 ms, but found that the fluctuations were too large and frequent, making it extremely difficult for us to take measurements. Even after we increased the time interval to 1000 ms, it was difficult to work with the fluctuations and report three random coincidence count rates to average. An improvement could be to increase the time interval further and see if it minimizes the fluctuations.

We also assumed that our set up was producing an optimized Bell state. We were unable to adjust anything in the Entanglement Demonstrator with the exception of the polarizer. Therefore, we were able to take for granted that our set up was aligned correctly. Additionally, we only took measurements for the positive Bell state since we could not change our set up. Nonetheless, we predict the minus Bell state would also violate the CHSH inequality and show that quantum mechanics is indeed incompatible with local hidden variable theory.

SUMMARY

The Entanglement Demonstrator was used to created entangled photons. We studied only the positive Bell's state and took coincidence counts to compute our correlation value. We were able to successfully show the violation of the CHSH inequality, and support our claims regarding the local hidden variable theory.