Project Ideas in with Robert and Nidham

Jane Du and Alex Townsend

September 15, 2019

1 Randomized ADI method (algorithmic project)

The alternating direction implicit (ADI) method is an iterative solver for linear systems that are *ADI model problems* [4]. These linear systems take the form

$$(H - V)\,x = b, \tag{1}$$

where H and V are normal matrices with distinct eigenvalues (it can be generalized to non-symmetric matrices).

The ADI method involves a two-step iteration: Starting with $x_0 = 0$, one performs the following steps:

$$\begin{aligned} (H + p_j I)x_{j+1/2} &= b + (q_j I - V)x_j, \\ (V + q_j I)x_{j+1} &= b + (p_j I - H)x_{j+1/2}, \end{aligned} \tag{2}$$

where $p_0, p_1, \ldots,$ and $q_0, q_1, \ldots,$ are carefully selected algorithmic parameters. The ADI method is a particularly efficient solver for (1) if the linear systems involving $H - pI$ and $V - qI$ can be rapidly solved and the spectrum of H and V are well-separated.

A special case of (1) is when solving a Sylvester matrix equation of the form $AX - XB = F$. Here, $H = I \otimes A$ and $V = B \otimes I$ and the algorithm in (2) can be rewritten in terms of just A and B, avoiding Kronecker products in the computation. Jane has been thinking about using sketch and project on the linear systems that appear in (2), i.e., having a randomized iterative method to solve the shifted linear systems required by the ADI method.

We suspect that a randomized ADI method could be useful for extremely large-scale Sylvester equations, which we were imagining coming from control theory or the discretization of partial differential equations. Jane has produced some code here: Existing code (Julia). There are remaining questions related to the convergence of this algorithm, the accuracy requirements on the inner iterative solvers, and the potential applications.

2 K-spectral set theory (theoretical project)

A spectral set of a matrix A, denoted by $\Omega(A) \subset \mathbb{C}$, offers an approach to grow the norm of functions of matrices in terms of the supremum norm of the function on $\Omega(A)$. For example, given any analytic function $f : \Omega(A) \to \mathbb{C}$, $\Omega(A)$ is a K-spectral set if

$$\|f(A)\|_2 \leq K \sup_{z\in\Omega(A)} |f(z)|. \tag{3}$$

where K is a constant. If A is restricted to being a symmetric matrix, then the set of eigenvalues of A can serve as a spectral set with $K = 1$. However, for non-symmetric matrices, the spectrum of A is typically not a spectral set. For general matrices, one example of a spectral set is the so-called field of values, i.e., $\Omega(A) = \{x^T Ax : \|x\|_2 = 1\}$, which is a spectral set with $2 \leq K \leq 1 + \sqrt{2}$ (the exact value for K is unknown).

The bound in (3) is useful for understanding the convergence of iterative methods applied to non-symmetric matrices. For example, if x_k is the kth GMRES iterate for solving $Ax = b$, we have [3]

$$\|Ax_k - b\|_2 \leq \inf_{p\in\mathcal{P}_k, p(0)=1} \|p(A)\|_2\|b\|_2 \leq K \left(\inf_{p\in\mathcal{P}_k, p(0)=1} \sup_{z\in\Omega(A)} |p(z)| \right) \|b\|_2,$$

where $\mathcal{P}_k$ is the space of polynomials of degree $\leq k$. Therefore, understanding the convergence of GMRES is converted to finding the smallest monic polynomial on $\Omega(A)$. One might now go and seek near-optimal solutions to this polynomial approximation problem [2].

Similar ideas can be applied to understand the convergence of iterative solvers on non-symmetric linear systems provided that they are based on matrix-vector products.

3 Rational Krylov methods (theoretical project)

Most of the iterative methods for computing eigenvalues of structured or sparse matrices are based on Krylov subspaces. Given a matrix A whose eigenvalues one would like to compute and an initial vector $\mathbf{v}$ we have the following Krylov subspace:

$$\mathcal{K}_k(A, v) = \mathrm{Span}\left\{v, Av, \ldots, A^{k-1}v\right\}.$$

Rational Krylov methods use rational functions of A instead of the standard powers of A. That is, they compute eigenvalues of A using the following subspace:

$$\mathcal{K}_k(A, v) = \mathrm{Span}\left\{\varphi_1(A)v, \varphi_2(A)v, \ldots, \varphi_k(A)v\right\}.$$

where the $\varphi_i(\lambda)$ are rational functions in λ (usually of increasing degree). Rational Krylov methods are particularly efficient when apriori knowledge of the eigenvalues of A is known (i.e., contained in a region of the complex plane) and/or linear systems involving $A - pI$ are fast to solve.

References

[1] C. Badea and B. Beckermann, *Spectral sets*, arXiv preprint arXiv:1302.0546 (2013).

[2] B. Beckermann and M. Crouzeix, *Faber polynomials of matrices for non-convex sets*, arXiv preprint arXiv:1310.1356 (2013).

[3] N. M. Nachtigal, R. C. Satish, and L. N. Trefethen, *How fast are nonsymmetric matrix iterations?*, SIAM J. Mat. Anal. Appl., 13 (1992), pp. 778–795.

[4] E. Wachspress, *The ADI model problem*, New York, Springer, 2013.