Chapter 2:
PROBABILITY

LEARNING OBJECTIVES

1. Sample spaces and events
2. Interpretations of Probability
3. Addition rules
4. Conditional probability
5. Multiplication and total probability rules
6. Independence
7. Bayes’ theorem
8. Random variables

Sample spaces and events

Example

When a ***six-sided die is rolled***,

- the ***sample space :***
  $S = \{1, 2, 3, 4, 5, 6\}$.
- The ***event*** *A* that an even number is obtained = $\{2, 4, 6\}$.
- The ***event*** *B* that a number greater than 2 is obtained = $\{3, 4, 5, 6\}$.

Sample spaces and events

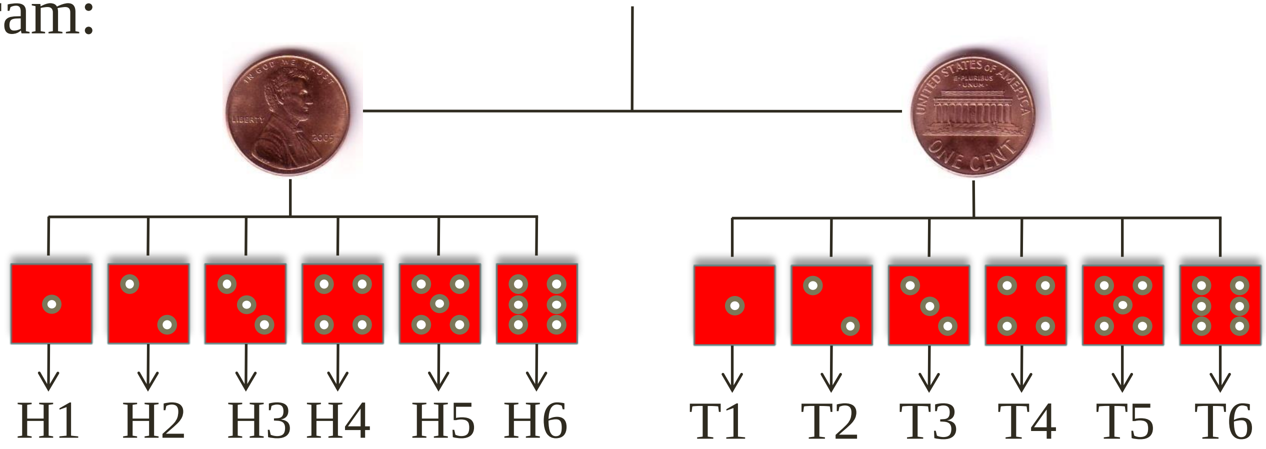
Definition

- An experiment that can result in different outcomes, even though it is repeated in the same manner everytime, is called a ***random experiment.***
- The set of all possible outcomes of a random experiment is called the ***sample space*** (denoted as *S)*
- An ***event*** is a subset of the sample space of a random experiment.

Sample spaces and events

Example 1: A probability experiment consists of tossing a coin and then rolling a six-sided die. Describe the sample space.

Tree diagram:

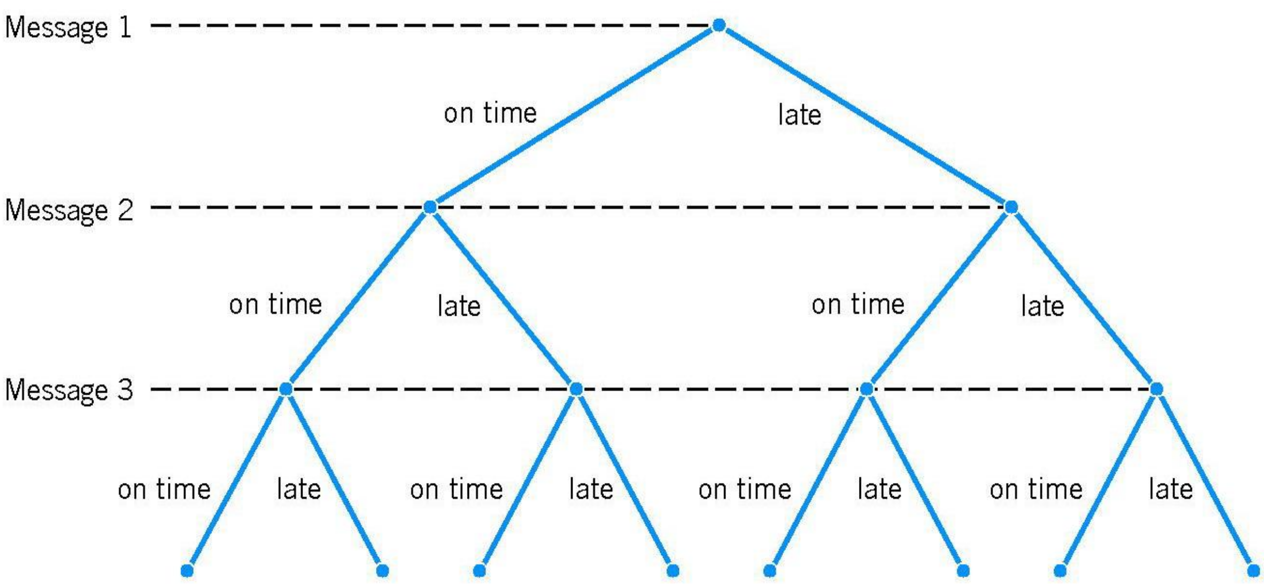


The sample space has 12 outcomes:

S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

Sample spaces and events

Example 2: Each message in a digital communication system is classified as to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.



Sample spaces and events

Basic Set Operations

The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.

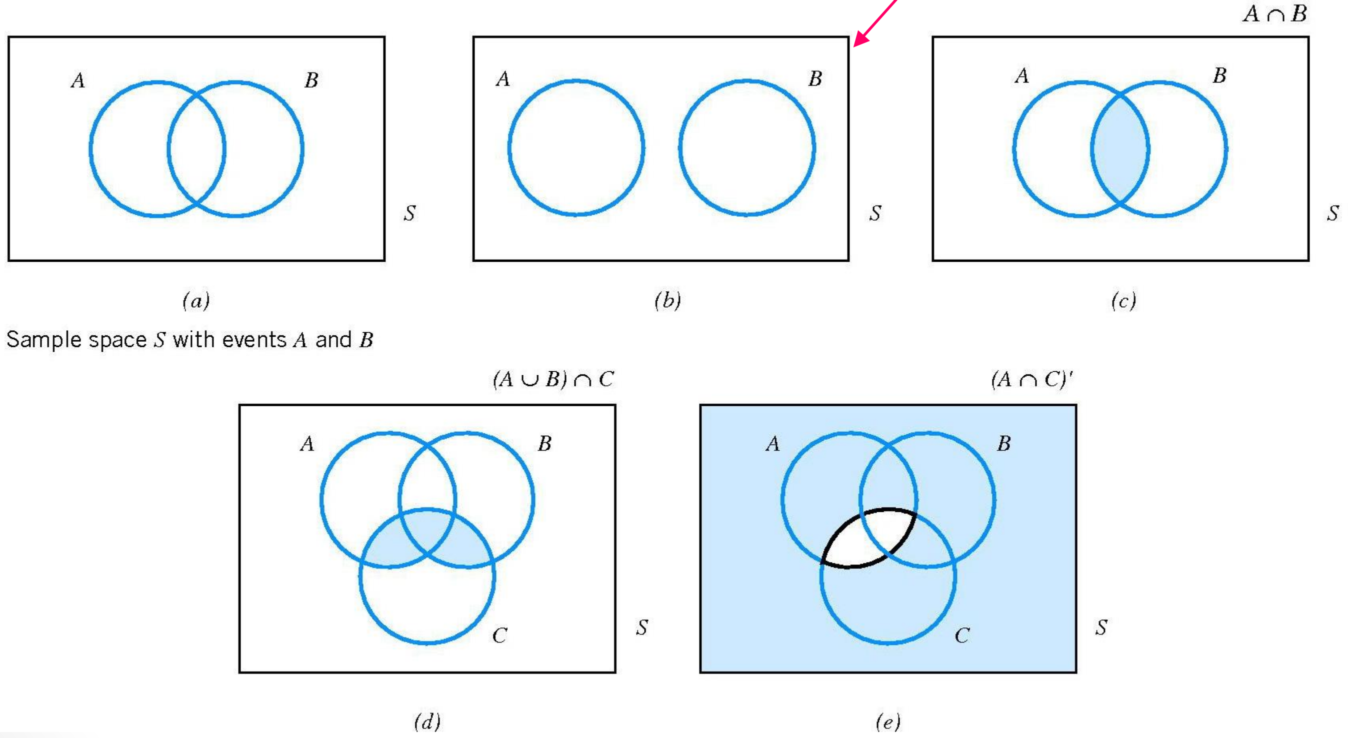
The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.

The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event E as E'.

Sample spaces and events

Venn Diagrams

mutually exclusive



Sample space S with events A and B

Sample spaces and events

Important properties:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

$$A = (A \cap B) \cup (A \cap B')$$

Interpretations of Probability

Introduction

There are different approaches to assessing the probability of an uncertain event:

1. **a priori classical probability**: the probability of an event is based on prior knowledge of the process involved.

2. **empirical classical probability**: the probability of an event is based on observed data.

Interpretations of Probability

<u>Equally Likely Outcomes</u>

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

1. **a priori classical probability**

$$\text{Probability of Occurrence} = \frac{X}{T} = \frac{\text{number of ways the event can occur}}{\text{total number of possible outcomes}}$$

2. **empirical classical probability**

$$\text{Probability of Occurrence} = \frac{\text{number of favorable outcomes observed}}{\text{total number of outcomes observed}}$$

Interpretations of Probability

Example: Find the probability of selecting a face card (Jack, Queen, or King) from a standard deck of 52 cards.

$$\text{Probability of Face Card} = \frac{X}{T} = \frac{\text{number of face cards}}{\text{total number of cards}}$$

$$\frac{X}{T} = \frac{12 \text{ face cards}}{52 \text{ total cards}} = \frac{3}{13}$$

Interpretations of Probability

Example: Find the probability of selecting a male taking statistics from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

$$\text{Probability of Male Taking Stats} = \frac{\text{number of males taking stats}}{\text{total number of people}}$$

$$= \frac{84}{439} = 0.191$$

Interpretations of Probability

Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

(S is the sample space and E is any event)

1. $P(S) = 1$
2. $0 \le P(E) \le 1$
3. For two events E_1 and E_2 with $E_1 \cap E_2 = \varnothing$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Interpretations of Probability

For a discrete sample space, the probability of an event E, denoted as P(E), equals the sum of the probabilities of the outcomes in E.

Example 1: A random experiment can result in one of the outcomes S = {a, b, c, d} with probabilities 0.1, 0.3, 0.5, and 0.1, respectively.
Let A = {a, b}, B = {b, c, d}, A'= S \ A, B' = S \ B.
Find P(A); P(B); P(A'); P(B'); P(A∩ B); P(A∪ B).

Example 2: A visual inspection of a location on wafers from a semiconductor manufacturing process resulted in the following table.
What is the probability that a wafer contains three or more particles in the inspected location?

Number of contamination particles	Proportion of wafers
0	0.40
1	0.15
2	0.20
3	0.10
4 or more	0.15

Complement rule

If the complement of A, denoted by A', consists of all the outcomes in which the event A does not occur, then we have:

$$P(A) + P(A') = 1$$

Remark: Depending on the problem, it may be easier to find $P(A')$ and then use the above equation to find P(A).

Example: A number is chosen at random from a set of whole numbers from 1 to 50. Calculate the probability that the chosen number is not a perfect square.

Addition rules

1. If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

2. A collection of events, E_1, E_2, …, E_k is said to be **mutually exclusive** if for all pairs,

$$E_i \cap E_j = \varnothing$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \cdots \cup E_k) = P(E_1) + P(E_2) + \cdots + P(E_k)$$

3. General: If A and B are any events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Addition rules

Example: Find the probability of selecting a male or a statistics student from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

P(Male or Stat) = P(M) + P(S) – P(M ∩ S)

= 229/439 + 160/439 – 84/439 = 305/439

Conditional Probability

Definition

The conditional probability of an event B given an event A, denoted as $P(B|A)$, is computed as

$$P(B|A) = \frac{P(B \cap A)}{P(A)}.$$

Special case: All outcomes are equally likely

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

Conditional Probability

Example: Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

What is the probability that a car has a CD player, given that it has AC ?

	CD	No CD	Total
AC	0.2	0.5	0.7
No AC	0.2	0.1	0.3
Total	0.4	0.6	1.0

$$P(\text{CD} \mid \text{AC}) = \frac{P(\text{CD and AC})}{P(\text{AC})}$$

$$= \frac{0.2}{0.7} = .2857$$

Multiplication rule

Multiplication Rule

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Example: The probability that an automobile battery subject to high engine compartment temperature suffers low charging current is 0.7. The probability that a battery is subject to high engine compartment temperature is 0.05.
The probability that a battery is subject to low charging current and high engine compartment temperature is

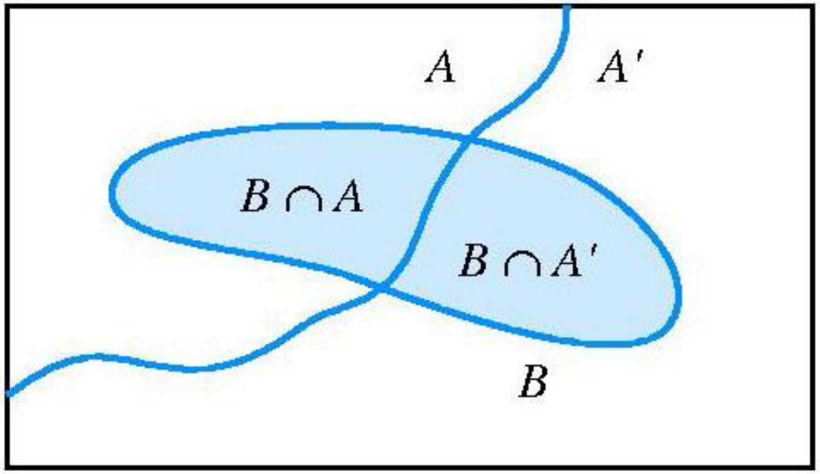
$$P(C \cap T) = P(C|T)P(T) = 0.7 \times 0.05 = 0.035$$

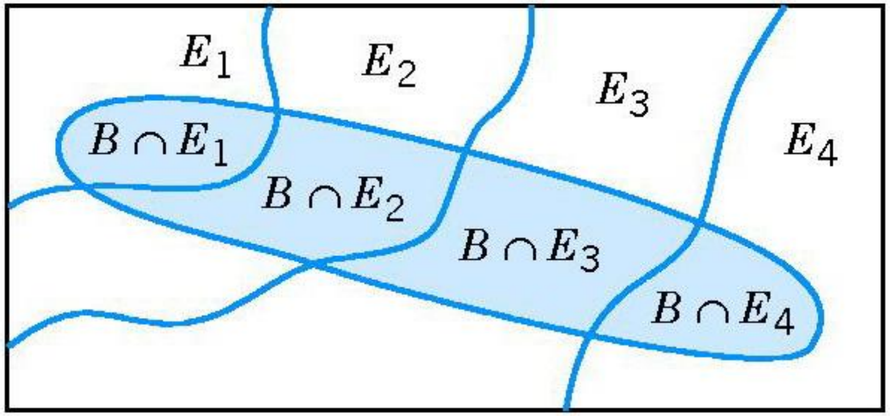
C={a battery suffers low charging current}

T ={a battery is subject to high engine compartment temperature}

Total Probability Rule

Partitioning an event into two mutually exclusive subsets.





Partitioning an event into several mutually exclusive subsets.

$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Total Probability Rule

Total Probability Rule: two events

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A')$$

Total Probability Rule: multiple events

Assume $E_{1,} E_2, \ldots, E_k$ are mutually exclusive and exhaustive events. Then:

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k)$$
$$= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k)$$

Total Probability Rule

Example **Semiconductor Contamination**

The information of the contamination discussion is summarized in the following table:

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.005	No high	0.8

Let F denote the event that the product fail. Find P(F).

Hint:

Let H denote the event that the product is exposed to high level of contamination.

So $P(H) = 0.2$ and $P(H') = 0.8$.

Moreover, $P(F|H) = 0.1$ and $P(F|H') = 0.005$.

Thus, $P(F) = 0.1*0.2 + 0.005*0.8 = 0.024$.

Independence

Definition

Two events are called independent if the occurrence of one event does not change the probability of the other event. Equivalently, any one of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

(2) $P(A \cap B) = P(A)P(B)$

(3) $P(B|A) = P(B)$

Remark: If A and B are independent events, then so are events A and B', events A' and B, and events A' and B'.

Independence

Example: A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected at random, without replacement, from the batch.
Let A ={the first part is defective}, and B ={the second part is defective}.
We suspect that these two events are not independent because knowledge that the first part is defective suggests that it is less likely that the second part selected is defective.

Hint:
P(B|A) = 49/849

$$
\begin{aligned}
P(B) &= P(B|A)P(A) + P(B|A')P(A') \\
&= (49/849)(50/850) + (50/849)(800/850) \\
&= 50/850
\end{aligned}
$$

Conclusion: the two events are not independent, as we suspected.

Independence

Definition

The events $E_1, E_2, \ldots, E_k$ are independent if and only if for any subset of these events

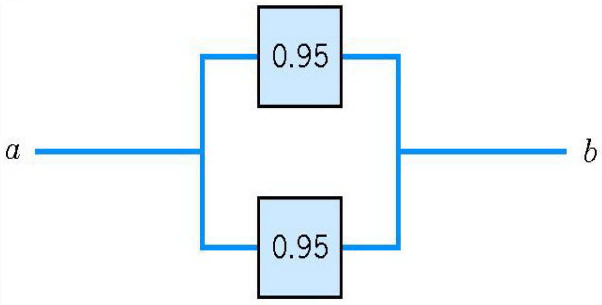
$$P\left(E_{n_1} \cap E_{n_2} \cap \cdots \cap E_{n_m}\right) = P\left(E_{n_1}\right)P\left(E_{n_2}\right)\ldots P\left(E_{n_m}\right)$$

where $1 \leq n_1 < n_2 < \cdots < n_m \leq k$

Question: Two coins are tossed.
Let A denote the event "at most one head on the two tosses" and B denote the event "one head and one tail in both tosses."
Are A and B independent events?

Independence

Example: The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



$$P(T \cup B) = 1 - P[(T \cup B)'] = 1 - P(T' \cap B')$$

Let T and B denote the events that the top and bottom devices operate, respectively.

$$P(T' \cap B') = P(T')*P(B') = (1 - 0.95)^2 = 0.05^2$$

So $P(T \cup B) = 1 - 0.05^2 = 0.9975$

Bayes' Theorem

If E_1, E_2, …, E_k are k mutually exclusive and exhaustive events and B is any event,

$$\boxed{P(E_1 \mid B) = \frac{P(B \mid E_1)P(E_1)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \ldots + P(B \mid E_k)P(E_k)} \quad \text{for } P(B) > 0}$$

A special case:

$$\boxed{P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \qquad \text{for } P(B) > 0}$$

Bayes' Theorem

Example 1: In a state where cars have to be tested for the emission of pollutants, 25% of all cars emit excessive amount of pollutants. When tested, 99% of all cars that emit excessive amount of pollutants will fail, but 17% of all cars that do not emit excessive amount of pollutants will also fail. What is the probability that a car that fails the test actually emits excessive amounts of pollutants?

Hint:

Let A ={a car emits excessive amount of pollutants}, B ={a cars fails the test }, then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A')P(A')} = \frac{0.99*0.25}{0.99*0.25+0.75*0.17} = 0.66$$

Example 2: Two events A and B are such that $P(A \cap B) = 0.15$, $P(A \cup B) = 0.65$, and $P(A|B) = 0.5$. Find $P(B|A)$.

Hint:

Since $\mathrm{P(A|B)} = \dfrac{\mathrm{P(A \cap B)}}{\mathrm{P(B)}}$, we have $\mathrm{P(B)} = \dfrac{\mathrm{P(A \cap B)}}{\mathrm{P(A|B)}} = \dfrac{0.15}{0.5} = 0.3$,

and from $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, then

$P(A) = P(A \cup B) + P(A \cap B) - P(B) = 0.65 + 0.15 - 0.3 = 0.5$, therefore

$P(A|B) = \dfrac{\mathrm{P(A \cap B)}}{\mathrm{P(B)}} = \dfrac{0.15}{0.5} = 0.3$

Random variables

Definition

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

Remark:

1. A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milli-amperes.

2. A discrete random variable is a random variable with a finite (or countable infinite) range.

3. A continuous random variable is a random variable with an interval of real numbers for its range.