

# ME685 - APPLIED NUMERICAL METHODS

## PROJECT REPORT

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TOPIC : 2D Incompressible Viscous Flow Analysis: Investigating  
Flow Regime Transition at Varied Reynolds Numbers

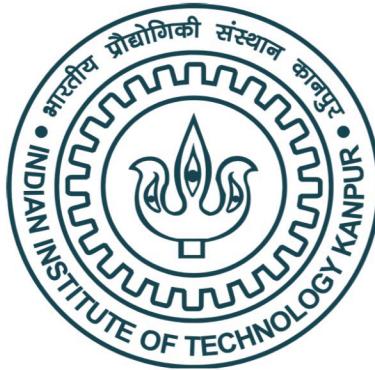
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# Contents

<b>1 Abstract</b>	<b>3</b>
<b>2 The physical problem</b>	<b>3</b>
<b>3 The mathematical model</b>	<b>4</b>
3.1 Initial and boundary conditions . . . . .	5
3.2 Inlet and outlet formulation . . . . .	7
<b>4 The numerical method</b>	<b>9</b>
4.1 Stream function . . . . .	9
4.2 Vorticity function . . . . .	10
<b>5 Influence of the numerical methods</b>	<b>11</b>
5.1 Convergence criterion . . . . .	11
5.2 Optimal parameters . . . . .	13
<b>6 Results</b>	<b>16</b>
6.1 Comparison between $Re$ numbers . . . . .	23
<b>7 Conclusion and References</b>	<b>33</b>

## List of Figures

1	Full geometry shape of the problem . . . . .	5
2	Half geometry of the problem around the line of symmetry (IDC) . . . . .	8
3	Computational time vs. $(\Delta l, \epsilon)$ . . . . .	12
4	Insufficient number of cells at the orifice . . . . .	12
5	Optimal region obtained via parameter space . . . . .	13
6	Too small $\Delta t$ results in insufficient progression . . . . .	14
7	Slightly big $\Delta t$ results in unstable fluctuations . . . . .	14
8	Too big $\Delta t$ results in rapid divergence . . . . .	14
9	Example of succesfull set of parameters within permissible range . . . . .	15

# 1 Abstract

In this project I would like to examine the flow regime that is generated by a fluid profile that is entering a channel at one side, and leaves it with a fully developed flow by the exit. To simplify the model, we will analyze it as a 2D problem in space, incompressible but viscous varying two values of  $Re$  numbers.

## 2 The physical problem

Consider a 2D profile of an  $L$  long channel that is opened at **AH**, narrows at **F** and exits at **BC**. The viscous fluid enters horizontally ( $u_0(y)$ ,  $v_0$ ) and leaves fully developed such that its velocity vector will exhibit parabolic distribution.

Provided that the  $Re$  numbers are relatively small, we should expect getting a laminar ( $Re \leq 2,040$ ) and non-turbulent flow in most of the cavity, apart from the line of symmetry, where counter direction flow might collide.

I shall look for suspicious areas across the problem's shape in sought of influential factors that increase sensitivity (numerical / physical).

### 3 The mathematical model

Using Navier-Stokes equations, we'll write the governing equations for the steady-state ( $\frac{\partial f}{\partial t} = 0$ ), incompressible ( $\frac{\partial \rho}{\partial t} = 0$ ) fluid flow problem, where  $\nabla \bar{p} \rightarrow 0$  and no external force :

$$\text{Continuity : } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3.1)$$

The momentum balance equation in a conservation form :

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \times \mathbf{u}) = -\nabla \bar{p} + \mu \nabla^2 \mathbf{u} + \frac{1}{3} \mu \nabla(\nabla \cdot \mathbf{u}) + \rho g \quad (3.2)$$

and momentum of each flow component where  $u \parallel \hat{x}$  and  $v \parallel \hat{y}$  :

$$\hat{x} : \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v = \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (3.3)$$

$$\hat{y} : \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v = \frac{\mu}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (3.4)$$

Using stream function - vorticity formulation, we can present the following terms :

$$\text{Velocity components : } u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (3.5)$$

$$\text{Define vorticity : } \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \quad (3.6)$$

$$\text{plug } u(\psi), v(\psi) : \quad \omega = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} \quad (3.7)$$

After fair amount of arithmetic manipulations, we get another elliptic equation :

$$\text{Stream function : } \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial x}u + \frac{\partial \omega}{\partial y}v = \frac{\mu}{\rho} \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \quad (3.8)$$

Eq. 3.7 is elliptic w.r.t  $\psi$ , and Eq. 3.8 is parabolic w.r.t time (in the unsteady-state case), such that both are coupled, and can be solved as displayed in the flowchart :

- (i)  $\hat{\omega}(\Delta t) = f(u, v)$
- (ii)  $\hat{\psi} = f(\hat{\omega})$
- (iii)  $\hat{u}, \hat{v} = f(\hat{\psi}) \Rightarrow \text{Return to (i)}$

For the sake of convenience we'll express the equations in a non-dimensional form :

$$\tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{L}, \quad \tilde{u} = \frac{u}{L}, \quad \tilde{v} = \frac{v}{L}, \quad \tilde{\omega} = \frac{\omega}{U_0/L}, \quad \tilde{\psi} = \frac{\psi}{U_0 L}, \quad Re = \frac{UL}{\nu} \quad (3.9)$$

And divide the functions by  $L$ , such that get the following normalized equations :

Vorticity function :  $\frac{\partial \tilde{\omega}}{\partial t} + \frac{\partial \tilde{\omega}}{\partial \tilde{x}} \tilde{u} + \frac{\partial \tilde{\omega}}{\partial \tilde{y}} \tilde{v} = \frac{1}{Re} \left( \frac{\partial^2 \tilde{\omega}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\omega}}{\partial \tilde{y}^2} \right) \quad (3.10)$

Stream function :  $\frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\psi}}{\partial \tilde{y}^2} = \tilde{\omega} \quad (3.11)$

Velocity components :  $\tilde{u} = \frac{\partial \tilde{\psi}}{\partial \tilde{y}}, \quad \tilde{v} = -\frac{\partial \tilde{\psi}}{\partial \tilde{x}} \quad (3.12)$

For presentation convenience, the notation (  $\tilde{\cdot}$  ) will be dismissed from here on.

### 3.1 Initial and boundary conditions

Given the problem's geometry we can portray the full symmetry as such :

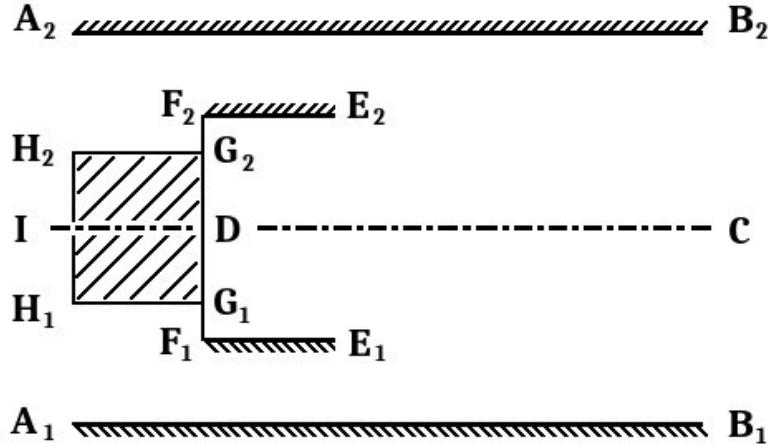


Figure 1: Full geometry shape of the problem

Let  $A_1$  coincide with the origin  $(0, 0)$ , and flow enter horizontally from  $A_1A_2$  entrance, and advances towards  $B_1B_2$  exit. The problem's nature allows assuming symmetry around  $IDC$  line, such that it's suffice to analyze only one chosen half.

All the solid walls satisfy two important conditions ( $u = v = 0$ ) :

$$(i) \text{ No-penetration} \quad (ii) \text{ No-slip} \quad (3.13)$$

The solid walls satisfy conditions **3.13**, such that the boundary layer diffuses vorticity into the free fluid, and can be expressed using *Taylor expansion* :

$$\psi_{i,w \pm \Delta x} = \psi_w \pm \Delta x \frac{\partial \psi}{\partial y} + \frac{\Delta x^2}{2} \left( \frac{\partial^2 \psi}{\partial y^2} \right)_{=\omega} + O(\Delta x^3) \quad (3.14)$$

$$\omega_{i,w} = 2 \cdot \frac{\psi_{w \pm \Delta y} - \psi_{i,w}}{\Delta x^2} + O(\Delta x) \quad (3.15)$$

Conveniently set  $\psi_{A_1} = 0$ , we get the following set of boundary conditions :

Segment	Indices	<b>u</b>	<b>v</b>	$\psi$	$\omega$
$A_1 H_1^{in}$	$i = 1, 1 \leq j \leq \frac{L/20}{\Delta y}$	$u_{in}(y)$	0	$\int_{y_{A_1}}^{y_{A_1} + j \Delta y} u_{in}(y) dy$	$\frac{\partial u_{in}}{\partial y} - \frac{\partial v_{in}}{\partial x} \overset{0}{\nearrow}$
$H_1 G_1$	$1 \leq i \leq \frac{2L/20}{\Delta x}, j = \frac{L/20}{\Delta y}$	0	0	$\int_{y_{A_1}}^{y_{H_1}} u_{in}(y) dy$	$\frac{2}{\Delta_y^2} (\psi_{\frac{L}{20} - \Delta y} - \psi_{\frac{L}{20}})$
$G_1^- F_1$	$i = \left\lfloor \frac{2L/20}{\Delta x} \right\rfloor, \frac{L/40}{\Delta y} \leq j \leq \frac{2L/40}{\Delta y}$	0	0	$\psi_{H_1 G_1}$	$\frac{2}{\Delta_x^2} (\psi_{\frac{L}{10} - \Delta x} - \psi_{\frac{L}{10}})$
$F_1 E_1^-$	$\frac{2L/20}{\Delta x} \leq i \leq \frac{3L/20}{\Delta x}, j = \left\lfloor \frac{L/40}{\Delta y} \right\rfloor$	0	0	$\psi_{G_1^- F_1}$	$\frac{2}{\Delta_y^2} (\psi_{\frac{L}{40} - \Delta y} - \psi_{\frac{L}{40}})$
$F_1 E_1^+$	$\frac{2L/20}{\Delta x} \leq i \leq \frac{3L/20}{\Delta x}, j = \left\lceil \frac{L/40}{\Delta y} \right\rceil$	0	0	$\psi_{F_1 E_1^-}$	$\frac{2}{\Delta_y^2} (\psi_{\frac{L}{40} + \Delta y} - \psi_{\frac{L}{40}})$
$F_1^+ D$	$i = \left\lceil \frac{2L/20}{\Delta x} \right\rceil, \frac{L/40}{\Delta y} \leq j \leq \frac{6L/20}{\Delta y}$	0	0	$\psi_{F_1 E_1^+}$	$\frac{2}{\Delta_x^2} (\psi_{\frac{L}{10} + \Delta x} - \psi_{\frac{L}{10}})$
$DC^{Sym}$	$\left\lceil \frac{2L/20}{\Delta x} \right\rceil \leq i \leq \frac{L}{\Delta x}, j = \frac{6L/20}{\Delta y}$	$f_{extr.}$	0	$\psi_{F_1^+ D} = \text{const.}$	0
$B_1 C^{Out}$	$i = \frac{L}{\Delta x}, 1 \leq j \leq \frac{6L/20}{\Delta y}$	$u_{out}(y)$	0	$\psi_{B_1} \overset{0}{\nearrow} + \int_{y_{B_1}}^{y_C} u_{out}(y) dy$	$\frac{\partial u_{out}}{\partial y} - \frac{\partial v_{out}}{\partial x} \overset{0}{\nearrow}$
$A_1 B_1^{Base}$	$1 \leq i \leq L, j = 1$	0	0	$\psi_{A_1} = \psi_{B_1} = 0$	$\frac{2}{\Delta_y^2} (\psi_{i,2} - \psi_{i,1})$

### 3.2 Inlet and outlet formulation

The last challenge is defining the velocity profiles developed in the inlet, outlet and the symmetry line. Starting from continuity equation for incompressible flow :

$$\nabla \bar{U} = 0 \Rightarrow \frac{\partial u}{\cancel{\partial x}}^0 + \frac{\partial v}{\partial y} = 0 \Rightarrow v = v(y=0)_{\text{const.}} \quad (3.16)$$

Momentum balance in  $\hat{x}$  direction :

$$\frac{\partial u}{\cancel{\partial t}}^0 + \frac{\partial u}{\cancel{\partial x}}^0 u + \frac{\partial u}{\cancel{\partial y}}^0 v = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\cancel{\partial x}^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (3.17)$$

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} = \text{const.} = c \Rightarrow u(y) = \frac{c}{2} y^2 + dy + e \quad (3.18)$$

We got a general expression for the developed velocity. Now we would like to Taylor it each of the orifices, starting with  $u_{in}(y)$  entering the inlet :

$$\text{Parabolic dist. : } u_{in}(0) = 0 \Rightarrow e = 0; \quad u_{in}\left(\frac{L}{20}\right) = 0 \Rightarrow d = -\frac{cL}{40};$$

$$\text{Maximal flow normalized : } u_{in}\left(\frac{L}{40}\right) = 1 \Rightarrow c = -2\left(\frac{40}{L}\right)^2 = -8$$

And we get the following profiles at the inlet :

$$u_{in}(y) = -4y^2 + 4y \quad (3.19)$$

$$\psi_{in}(y) = \psi_{A_1} + \int_0^y u_{in}(y) dy = \left[ -\frac{4}{3}y^3 + 2y^2 \right]_0^y \quad (3.20)$$

$$\omega_{in}(y) = \frac{\partial u_{in}}{\partial y} - \frac{\partial v_{in}}{\cancel{\partial x}}^0 = -8y + 4 \quad (3.21)$$

And similar analysis for the outlet, starting with its parabolic properties :

$$u_{out}(0) = 0 \Rightarrow e = 0; \quad u'_{out}\left(\frac{6L}{20}\right) = 0 \Rightarrow d = -\frac{6cL}{20} = -6c;$$

Assuming mass flux conservation from both sides (lower symmetric half) :

$$\begin{aligned}\dot{m}_{in} = \dot{m}_{out} &\Leftrightarrow \rho b \int_0^{L/20} u_{in}(y) dy = \rho b \int_0^{6L/20} u_{out}(y) dy \\ \frac{2}{3} \rho b = \rho b \int_0^{6L/20} \left( \frac{c}{2} y^2 - 6cy \right) dy &\Rightarrow \frac{2}{3} = c \left[ \frac{1}{6} y^3 - 3y^2 \right]_0^6 \Rightarrow c = -\frac{1}{108}\end{aligned}$$

And we get the following profiles at the outlet :

$$u_{out}(y) = -\frac{1}{216}y^2 + \frac{1}{18}y \quad (3.22)$$

$$\psi_{out}(y) = \psi_{B_1} + \int_0^y u_{out}(y) dy = \left[ -\frac{1}{648}y^3 + \frac{1}{9}y^2 \right]_0^y \quad (3.23)$$

$$\omega_{out}(y) = \frac{\partial u_{out}}{\partial y} - \frac{\partial v_{out}}{\partial x} = -\frac{1}{108}y + \frac{1}{18} \quad (3.24)$$

At last we can draw a figurative profile of the flow domain as such :

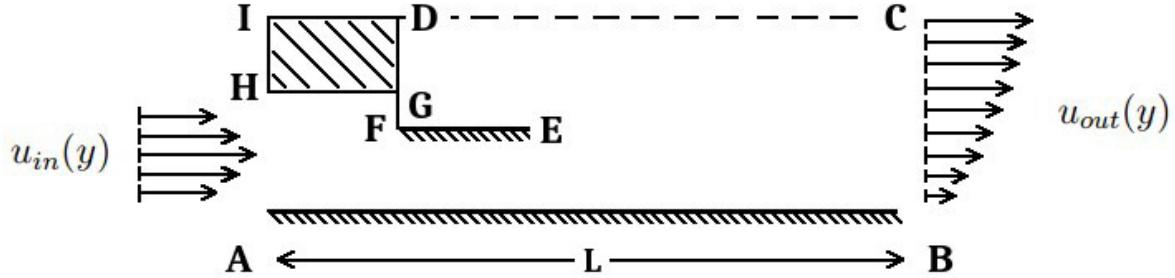


Figure 2: Half geometry of the problem around the line of symmetry (IDC)

Where inlet velocity at **AH** is normalized to '1' in its half height, and exits as a half parabola on **BC**.

## 4 The numerical method

Both functions after being discretized supply their own set of equations, but solved differently. We'll express both using finite differences (=F.D) on the  $x - y$  plane as follows :

$$\begin{aligned} x = i \cdot \Delta x & \quad \forall \quad i = 1, \dots, N + 1 \Leftrightarrow [0, L/\Delta x] \\ y = j \cdot \Delta y & \quad \forall \quad j = 1, \dots, M + 1 \Leftrightarrow [0, \frac{6L}{20}/\Delta y] \end{aligned}$$

### 4.1 Stream function

The stream function (Eq. 3.8) can be solved by a suitable elliptic method, such as implicit *Guass Seidel*, as already seen in previous homeworks :

$$\text{PDE :} \quad \nabla^2 \psi = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = \omega \quad (4.1)$$

$$\text{F.D :} \quad \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} = \omega_{i,j} \quad (4.2)$$

After sorting and extracting the  $\psi_{i,j}$  diagonal, we get the following iterative method :

$$\psi_{i,j}^{(n+1)} = \frac{\Delta y^2 (\psi_{i+1,j}^{(n)} + \psi_{i-1,j}^{(n+1)}) + \Delta x^2 (\psi_{i,j+1}^{(n)} + \psi_{i,j-1}^{(n+1)}) - \Delta x^2 \Delta y^2 \omega_{i,j}^{(n)}}{2(\Delta x^2 + \Delta y^2)} \left| \begin{array}{l} i=1, \dots, N \\ j=1, \dots, M \end{array} \right. \quad (4.3)$$

The local truncation error is  $O(\Delta x^2, \Delta y^2)$ , and to ensure convergence we'll use the S.O.R :

$$\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n)} + \Omega_{opt} (\psi_{i,j}^{(n+1)} - \psi_{i,j}^{(n)}) \quad (4.4)$$

The optimal solution of  $\Omega_{opt}$  is fully developed in the course papers, so it will suffice to bring the final solution of the method's  $\underline{a}_{i,j}$  block matrix and  $\lambda$ 's eigenvalues :

$$\Omega_{opt} = \frac{2(1 - \sqrt{1 - \mu^2})}{\mu^2} \quad , \quad \mu = \frac{1}{2} \left[ \cos\left(\frac{\pi}{M + 1}\right) + \cos\left(\frac{\pi}{N + 1}\right) \right] \quad (4.5)$$

The chosen convergence criterion will be a simple absolute error :  $|\psi_{i,j}^{(n+1)} - \psi_{i,j}^{(n)}| < \epsilon$ .

## 4.2 Vorticity function

The vorticity function (Eq. 3.10) can be solved using *1st order upwind differences* method, where the transport elements are written in accordance with  $u, v$  sign. Divided into 4 possible cases expressed in **F.D**, note that the  $(u, v)$  sign corresponds to  $(\pm, \mp)$  :

$$u_{i,j} > 0 \quad \cap \quad \begin{cases} v_{i,j} > 0 \\ v_{i,j} < 0 \end{cases} : \quad \omega_{i,j}^{(k+1)} = \omega_{i,j}^{(k)} + \Delta t \left[ \Delta y^2 \omega_{i+1,j} + \omega_{i-1,j} (\Delta y^2 + Re \Delta x \Delta y^2 u_{i,j}) + \Delta x^2 \omega_{i,j \pm 1} + \dots \right. \\ \left. \dots + \omega_{i,j \mp 1} (\Delta x^2 \pm Re \Delta y \Delta x^2 v_{i,j}) + \omega_{i,j} (\mp Re \Delta y \Delta x^2 v_{i,j} - Re \Delta x \Delta y^2 u_{i,j} - 2 \Delta y^2 - 2 \Delta x^2) \right]$$

And similarly,

$$u_{i,j} < 0 \quad \cap \quad \begin{cases} v_{i,j} > 0 \\ v_{i,j} < 0 \end{cases} : \quad \omega_{i,j}^{(k+1)} = \omega_{i,j}^{(k)} + \Delta t \left[ \Delta y^2 \omega_{i-1,j} + \omega_{i+1,j} (\Delta y^2 - Re \Delta x \Delta y^2 u_{i,j}) + \Delta x^2 \omega_{i,j \pm 1} + \dots \right. \\ \left. \dots + \omega_{i,j \mp 1} (\Delta x^2 \pm Re \Delta y \Delta x^2 v_{i,j}) + \omega_{i,j} (\mp Re \Delta y \Delta x^2 v_{i,j} + Re \Delta x \Delta y^2 u_{i,j} - 2 \Delta y^2 - 2 \Delta x^2) \right]$$

The  $k$  index stands for the iteration number taken until reaching convergence criterion. The time step however, could be conveniently extracted using *Brauer's Theorem* for a 2D problem, when mesh steps' are equal :

$$\Delta t \leq \min \left\{ \frac{1}{\frac{|u_i|}{\Delta x} + \frac{|v_j|}{\Delta y} + \frac{4}{Re \Delta x \Delta y}} \right\}_{\Delta x = \Delta y} \quad \forall \quad \begin{cases} i = 1, \dots, N \\ j = 1, \dots, M \end{cases} \quad (4.6)$$

Later we'll see that equal step sizes approach is not fully compatible with our specific problem since the geometry imposes  $x - y$  ratio that takes a heavy computational cost. Therefore an empirical optimization approach of  $\Delta t$  vs.  $(\Delta x, \Delta y)$  will be held.

## 5 Influence of the numerical methods

In this chapter I will elaborate on the numerical tuning process, and the optimal values eventually taken. The comprehensive examination of the chosen method's influence is a tough task for the high number of degrees of freedom. The best I could was to take a "frozen" case with some constant parameters, and commence an optimization process in sought of optimality.

### 5.1 Convergence criterion

Given four different functions that are evolving from initial condition into steady state, one must verify that each consecutive difference reaches steady-state :

$$[u^{(n+1)} - u^{(n)}, v^{(n+1)} - v^{(n)}, \psi^{(n+1)} - \psi^{(n)}] < \epsilon \quad (5.1)$$

And since iterative convergence might not be obtained for the big sizes of  $\omega$  at some specific points, it is advised to use the following criterion (and check every few iterations) :

$$\left| \frac{\omega^{(n+1)} - \omega^{(n)}}{\omega_{max}} \right| < \epsilon \quad (5.2)$$

The trade-off between the mesh resolution and  $\Delta t$  was examined along that chapter, and many cases were avoided for their impossible computational time. The first step I did was defining equal step sizes ( $\Delta x = \Delta y \Leftrightarrow \Delta l$ ), and examining the duration w.r.t convergence size ( $10^{-5} \leq \epsilon \leq 10^{-2}$ ). Smaller sizes were simply impractical to be calculated, as they generate matrices with  $10^6$  cells, and higher.

The next figure shows a set of tests in order to examine how the execution responds to different convergence criteria, and to the mesh density.

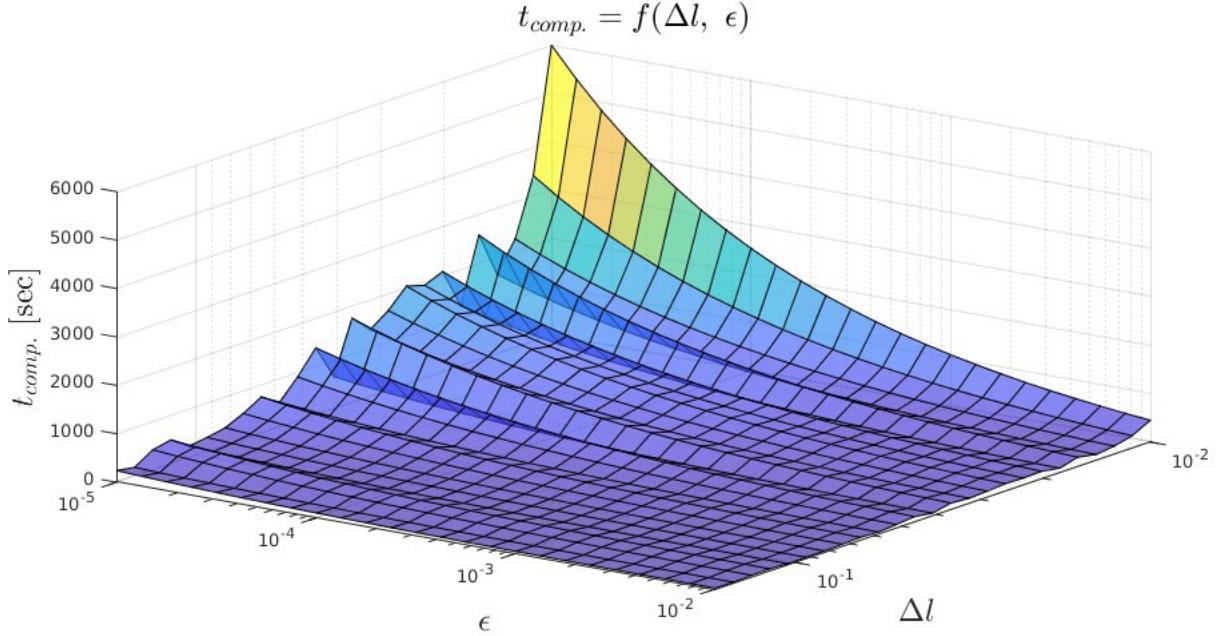


Figure 3: Computational time vs.  $(\Delta l, \epsilon)$

We can see that computation cost rises sharply as  $(\epsilon, \Delta l)$  get smaller, and vice versa. Note that  $\Delta l$  has an upper bound of  $0.5/3$  which is the minimal number of cells required for the orifice width (red). Smaller sizes "freeze" the calculation process :

	1	2	3	4	5	6	7	8	9
1	0	0.0405	0.1440	0.2835	0.4320	0.5625	0.6480	0.6480	NaN
2	0	-72.8345	-127.8782	-153.6858	-147.6457	-114.3476	-62.0813	0.6480	NaN
3	0	-163.4801	-285.2224	-339.6193	-323.1407	-248.3492	-134.5226	0.6480	NaN
4	0	-295.7888	-510.0339	-596.7439	-557.0577	-421.5086	-226.3795	0.6480	NaN
5	0	-509.6845	-862.4807	-979.8614	-887.4716	-654.3954	-346.0529	0.6480	NaN
6	0	-878.7231	-1.4521e...	-1.5747e...	-1.3577e...	-962.5086	-496.5426	0.6480	NaN
7	0	-1.5487e...	-2.4908e...	-2.5071e...	-2.0083e...	-1.3422e...	-666.0899	0.6480	NaN
8	0	-2.7762e...	-4.4589e...	-3.9583e...	-2.8253e...	-1.7311e...	-811.9798	0.6480	NaN
9	0	-4.1944e...	-8.6095e...	-6.0420e...	-3.6042e...	-1.9453e...	-845.3140	0.6480	NaN
10	0	-4.1467e...	-1.9001e...	-8.1751e...	-3.6116e...	-1.6006e...	-624.3866	0.6480	NaN
11	0	0.6480	0.6480	0.6480	0.6480	0.6480	0.6480	0.6480	NaN
12	0	0.6480	NaN	NaN	NaN	NaN	NaN	NaN	NaN
13	0	0.6480	NaN	NaN	NaN	NaN	NaN	NaN	NaN
14	0	0.6480	NaN	NaN	0.6480	0.6480	0.6480	0.6480	0.6480
15	0	0.6480	NaN	NaN	0.6480	2.0418e...	3.7097e...	4.8280e...	5.3988e...
16	0	0.6480	NaN	NaN	0.6480	4.4503e...	7.9691e...	1.0203e...	1.1237e...
17	0	0.6480	NaN	NaN	0.6480	7.7708e...	1.3513e...	1.6780e...	1.7983e...
18	0	0.6480	NaN	NaN	0.6480	1.3073e...	2.1533e...	2.5420e...	2.6130e...
19	0	0.6480	NaN	NaN	0.6480	2.2860e...	3.4125e...	3.7238e...	3.6097e...
20	0	0.6480	NaN	NaN	0.6480	4.3835e...	5.4870e...	5.3309e...	4.7879e...
21	0	0.6480	0.6480	0.6480	0.6480	9.5954e...	8.8211e...	7.3248e...	6.0291e...
22	0	-5.5474e...	5.1770e...	1.0161e...	4.4450e...	2.1105e...	1.2878e...	9.1179e...	6.9957e...
23	0	-4.8088e...	-4.2771e...	2.4427e...	2.4684e...	1.7579e...	1.2468e...	9.2732e...	7.2215e...

Figure 4: Insufficient number of cells at the orifice

For practical runtimes, I chose convergence size of  $\epsilon < 10^{-3}$ , which will be shortly justified.

## 5.2 Optimal parameters

In this subsection I conducted a comparison between different parameters w.r.t computation time until reaching **steady-state**.

After setting  $\epsilon$ 's size, I made a set of tests in sought of convenient  $\Delta x/\Delta y$  ratio, that will suit practically to the geometry of the problem. The long length at x-axis ( $=L$ ) should be handled differently than that of y-axis, which is shorter. Moreover, the orifice is very thin ( $=L/40$ ), and thus generates lots of reactions that we would like to focus on.

The best ratio was found to be  $\Delta x/\Delta y = 5$ , and its size is simply notated as  $\Delta l$ . The next figure exhibits a long lasting empirical tests in sought of optimal **computation time** (z-axis), and the grey-scale colorbar (=right) shows the **number of iterations**.

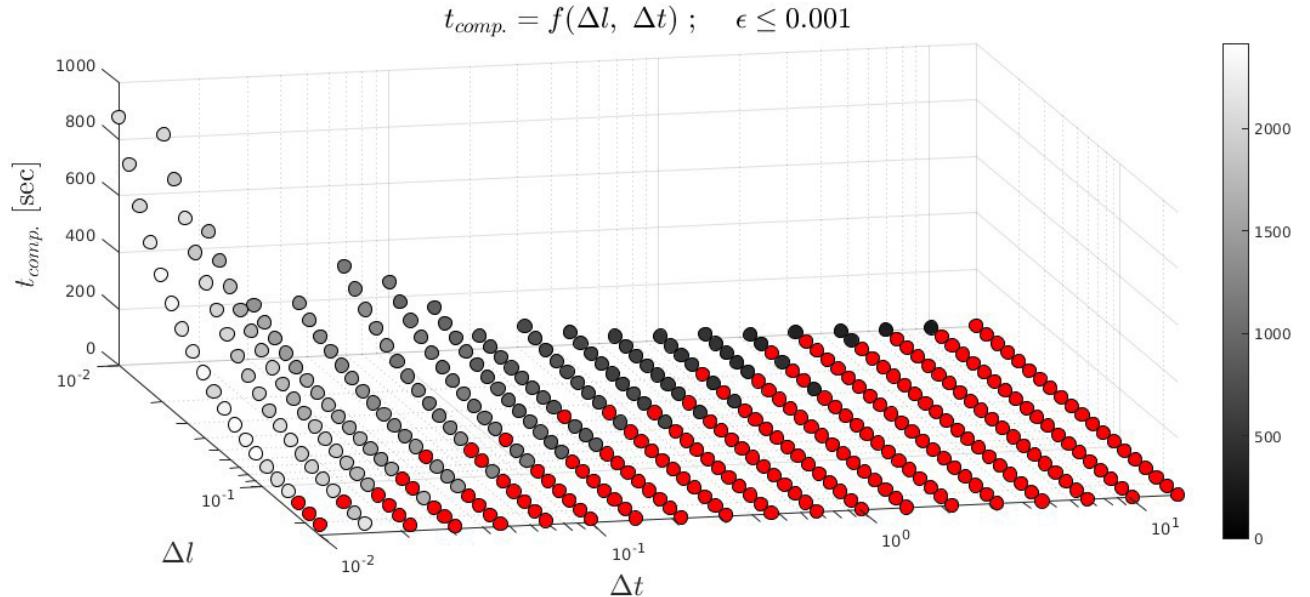


Figure 5: Optimal region obtained via parameter space

Note that only grey-scale points stand for converged executions, while *red* points represent diverged executions that were ignored. Anyway, we can see an interesting relation between  $(\Delta l, \Delta t)$ . As the grid turns denser the computation time rises sharply ( $\Delta l \downarrow : t_{comp.} \uparrow$ ), but more cases manage to converge. But as we relieve the mesh size, the computation time reduces drastically ( $\Delta l \uparrow : t_{comp.} \downarrow$ ), and convergence is less guaranteed unless smaller time steps are used. Additionally, too big time steps ( $\Delta t \uparrow$ ) resulted also in divergence, failing to restrain the vorticity gradient :  $\omega^{k+1} = \omega^k + \nabla \omega^k \Delta t$ .

The grey-scale **face-color** of the points represent the amount of iterations needed to reach steady-state. In this case, the greater the time step, the shorter the computation time ( $\Delta t \uparrow : t_{comp.} \downarrow$ ). The reason is understandable since greater time steps accelerate the progression rate. The following figures demonstrate some of the edge cases :

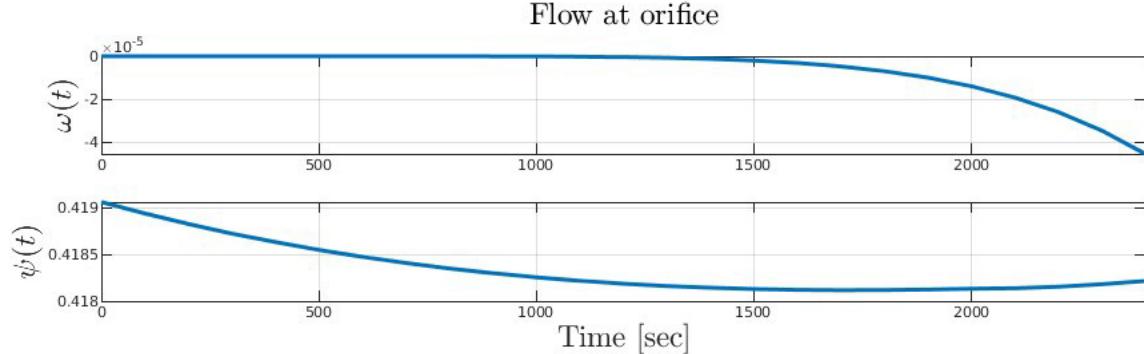


Figure 6: Too small  $\Delta t$  results in insufficient progression

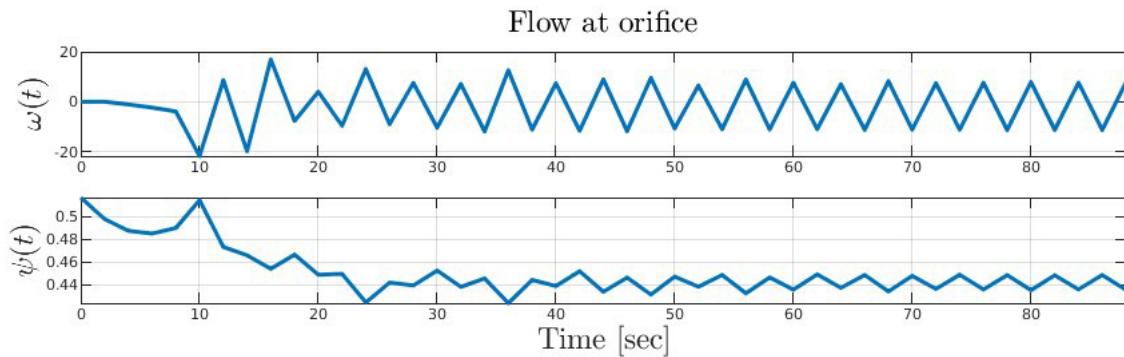


Figure 7: Slightly big  $\Delta t$  results in unstable fluctuations

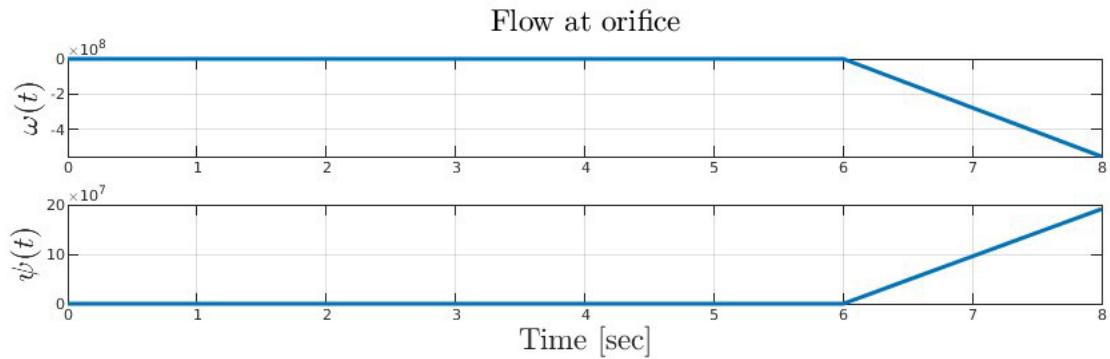


Figure 8: Too big  $\Delta t$  results in rapid divergence

However, values in the permissible region exhibited in fig. 5.2 perform well within a reasonable amount time, either by  $t$  and  $t_{comp}$  :

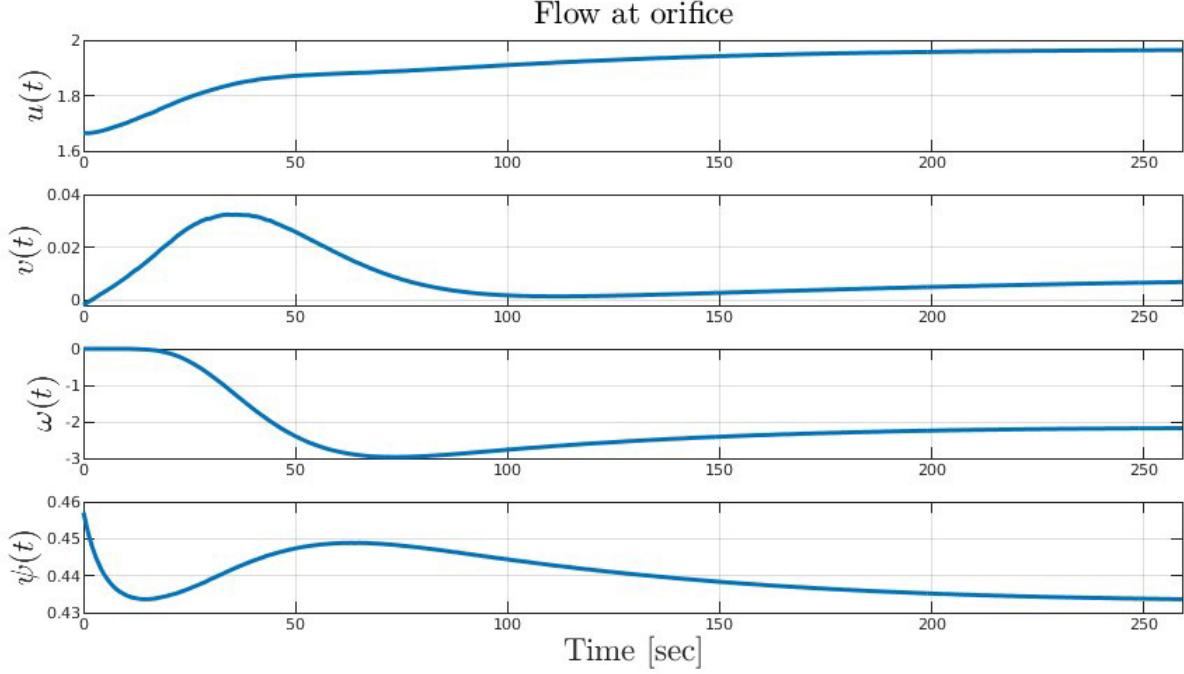


Figure 9: Example of succesfull set of parameters within permissible range

We can see a "healthy" shape of a dynamic response decaying towards a stable steady-state, under a sufficient convergence criterion. Note that values were not taken from "0" initialization, but from 2nd iteration for numerical reasons.

So now we are ready to move on to the results section, presenting a comparison for two case studies :  $Re = [100, 750]$ . The results are comprised of  $u, v, \psi, \omega$  and the dynamic response of each of them, until reaching steady state. The chosen parameters were obtained using the mentioned above analysis :

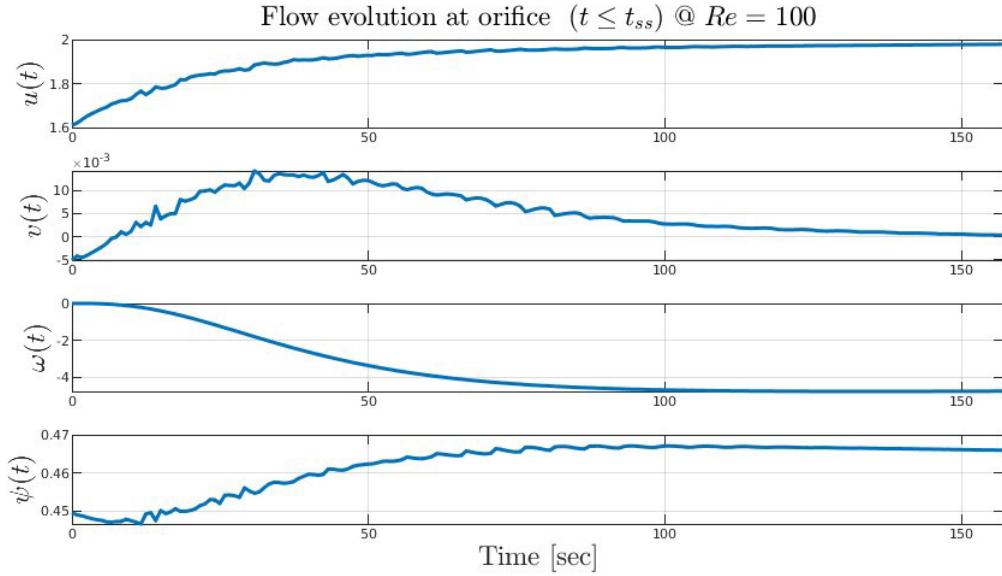
$$\epsilon = 10^{-3} \quad \Delta t = 0.831 \quad \Delta y = \Delta x/5 = 0.04$$

$$[M, N] = [\frac{L}{\Delta x} + 1, \frac{6L/20}{\Delta y} + 1] = [101, 151]$$

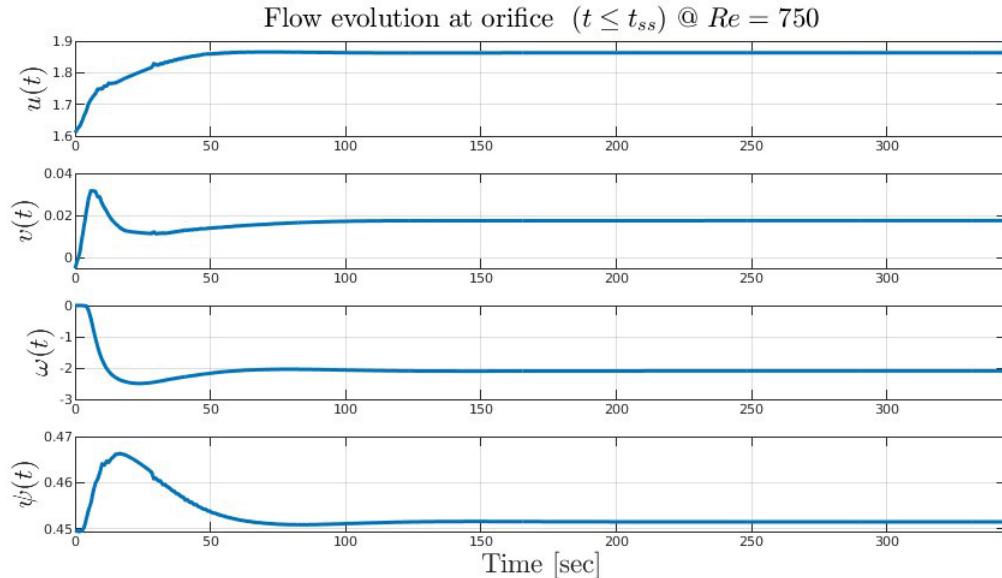
$$\text{S.O.R} : [\Omega_{opt}, \mu] = [1.949, 0.997]$$

## 6 Results

Starting off with a high level sight of the time dependent response of both cases :

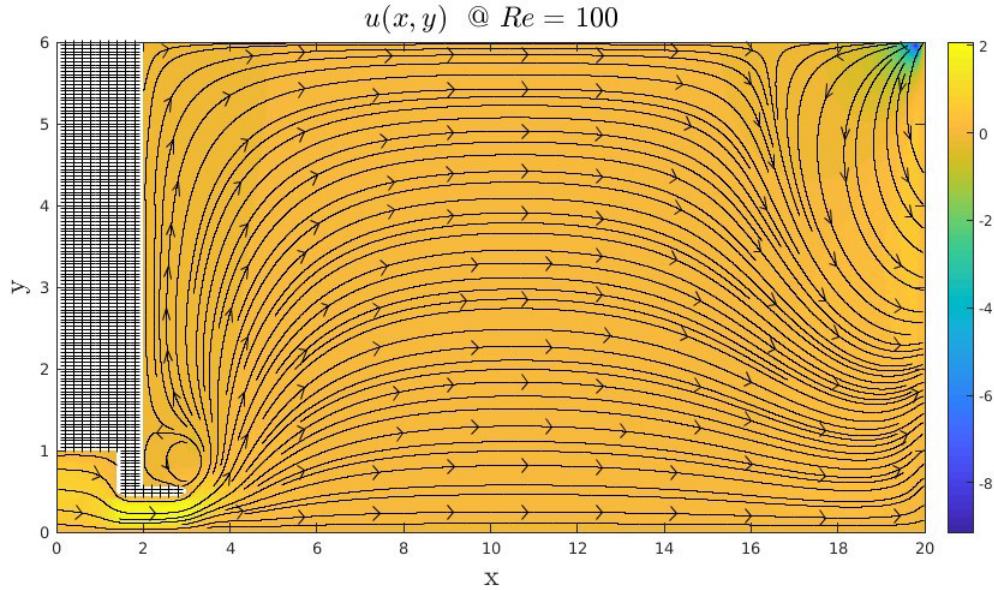


The lower Reynolds reaches steady state faster than higher values ( $t_{ss}^{100} < t_{ss}^{750}$ ), but is also subjected to stronger influence of the time interval, shown by the fluctuated spikes.

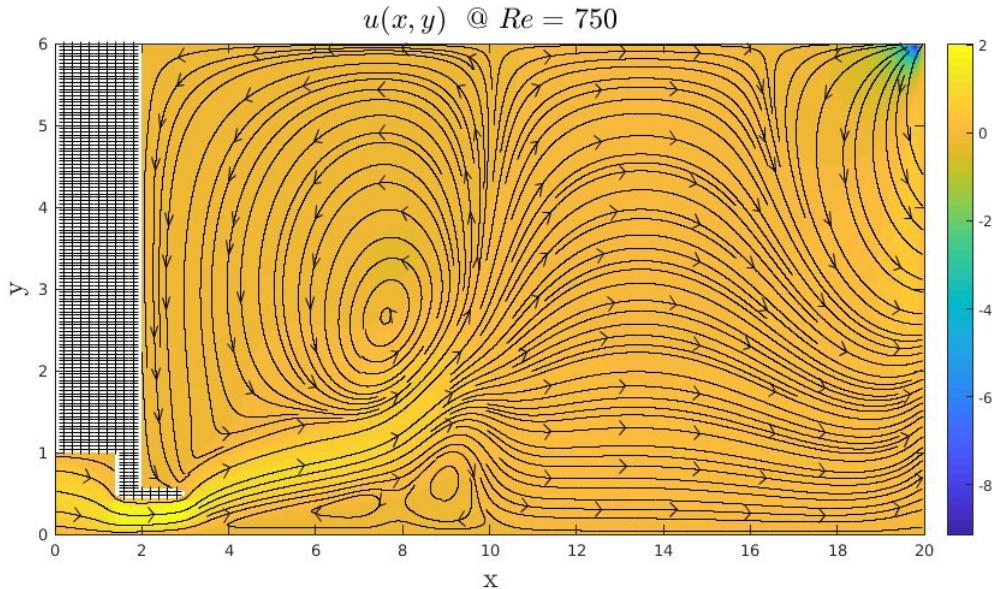


Higher  $Re$  exhibits sort of an "overshoot" that after a while is restrained and stability seems to be obtained quicker, but convergence lasts longer.

I added *Matlab*'s streamlines option such that the contour lines can better demonstrate the concentration gradient. However, **horizontal velocity** varies widely w.r.t the  $Re$  number:

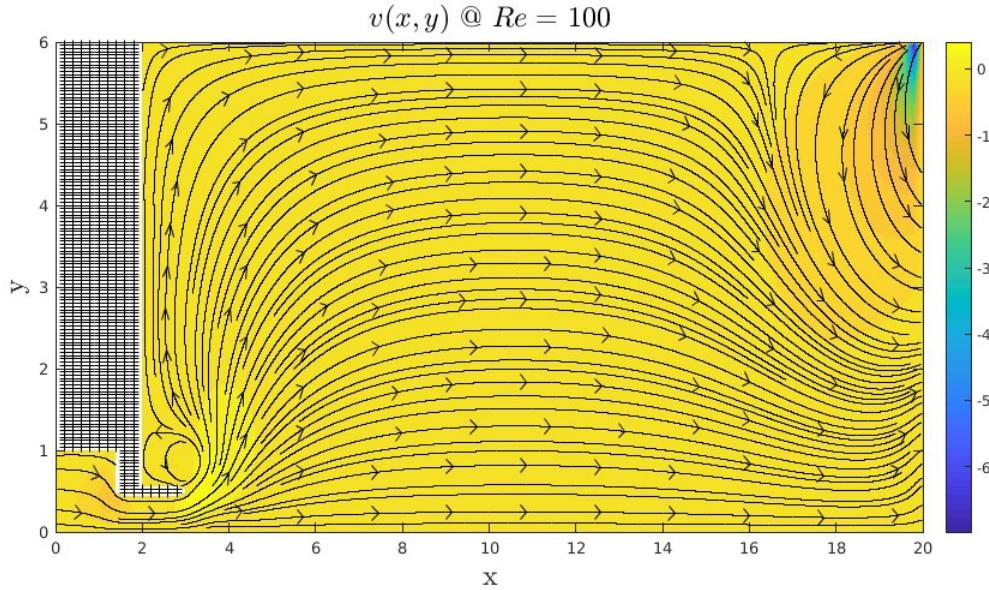


Low  $Re$  exhibits an expected behavior of  $u$  increase as the cross-section narrows (*yellow*), and the streamlines perform a "calm" disperse after leaving the orifice contraction.

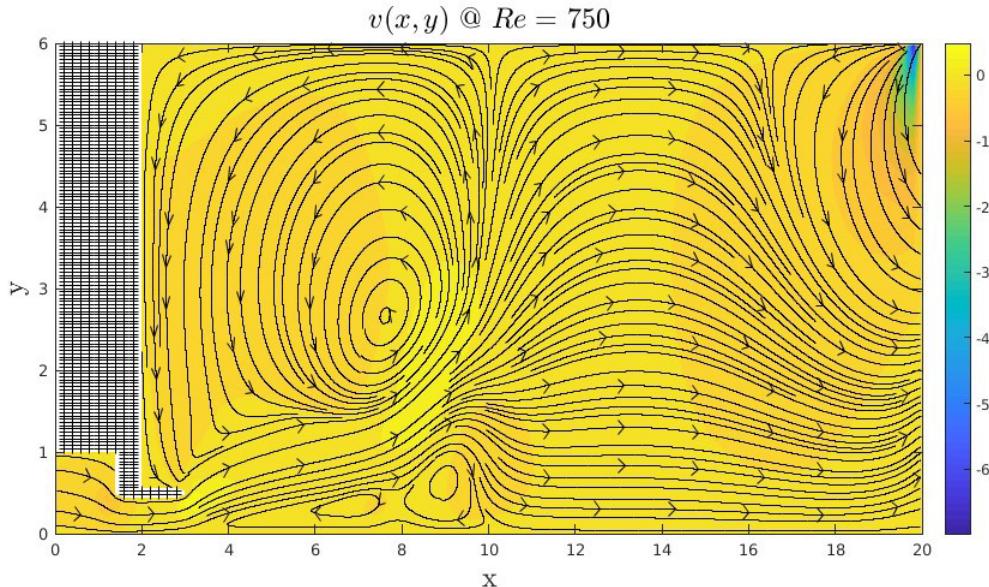


High  $Re$  exhibits a more aggressive policy as the flow remains high, long after leaving the orifice. The streamlines as well perform higher rates of vorticity causing few local turbulences across the plane, and next to the walls.

The **vertical velocity** behaves entirely different since the incoming velocity is completely horizontal and only does the geometrical constraints, exert  $u \rightarrow v$ .

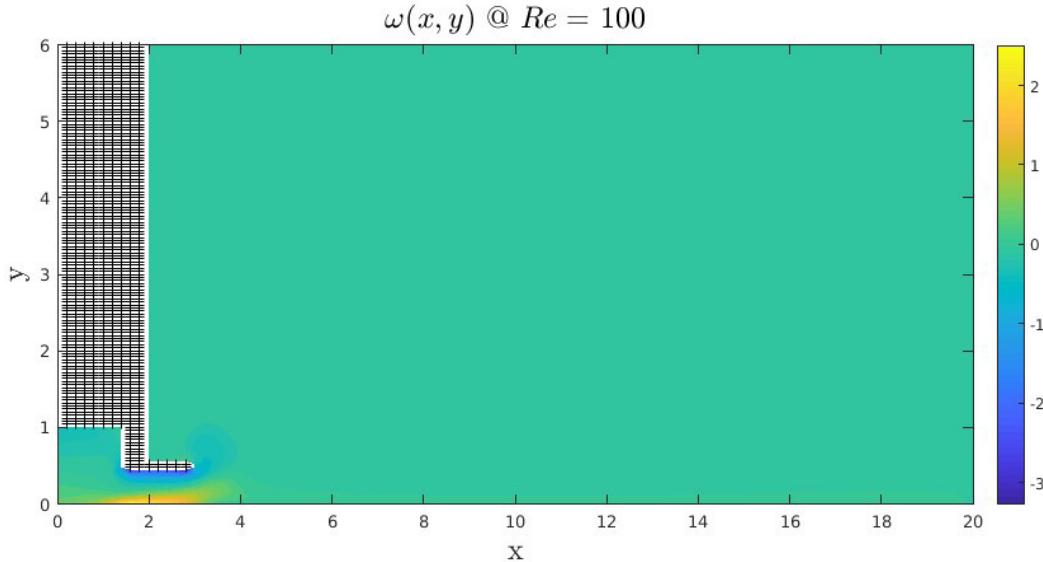


Low  $Re$  number manages to conserve  $u$  component as seen either by the *yellow* color level and by the dominant horizontal streamlines ( $v \approx 0$ ).

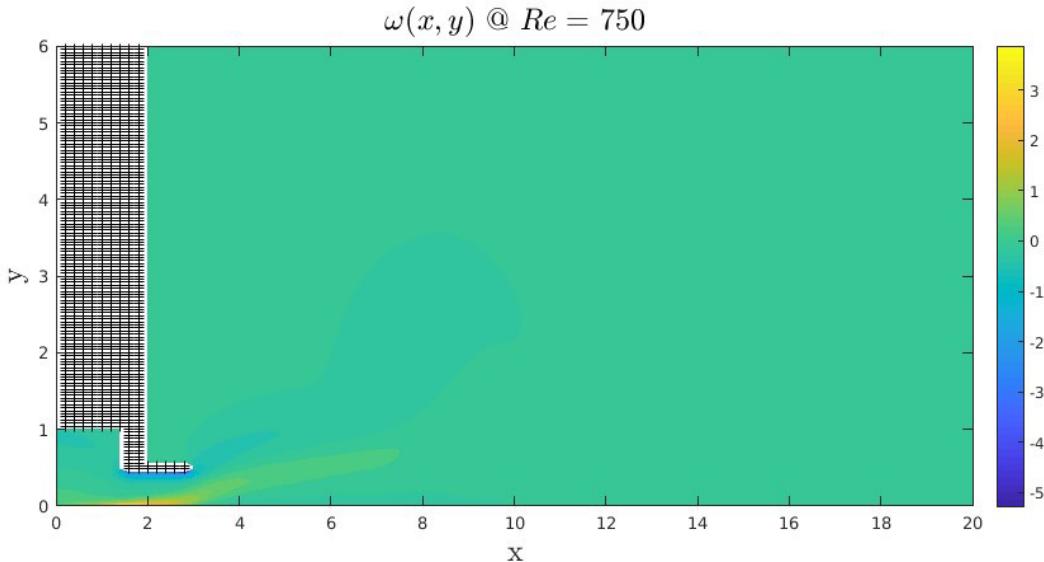


However, when  $Re$  grows, turbulent policy is observed above the orifice and towards the exit. The vertical component becomes lively *orange* wherever turbulences are formed.

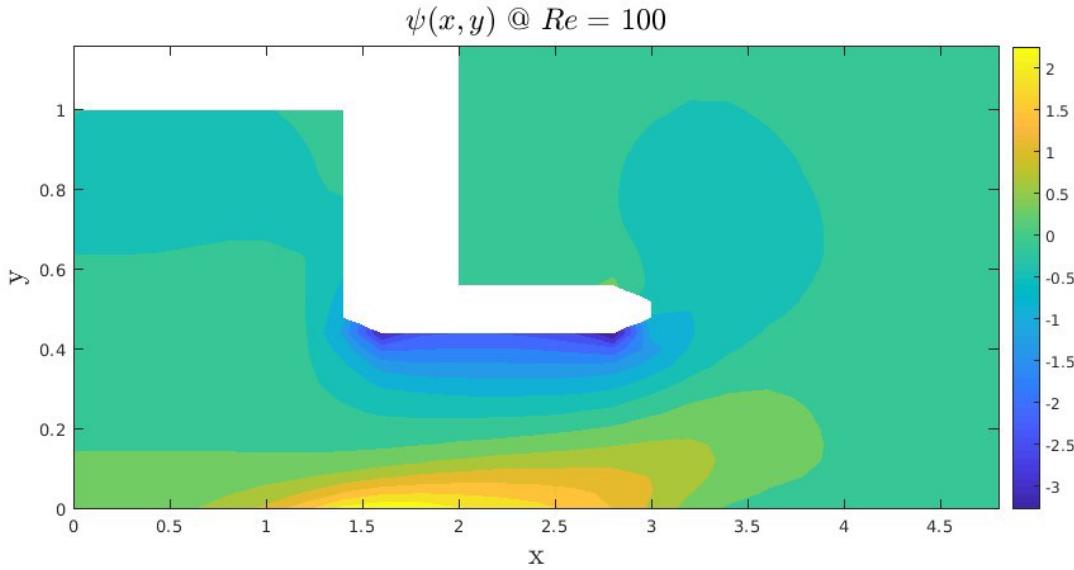
The overall **vorticity** is calculated in two manners : Boundary layers on the walls, and inner flow (further away). Inner flow is calculated by a  $\nabla \omega^k \Delta t$  increment, which remains low relatively to the boundary layers, and high values developed mainly in the orifice exit :



The reason for the dominant *green* color is that all across the shape, low values of vorticity are developed. Contrarily, the contraction effect of the orifice accelerates the stream, which in turn generates turbulences along the walls ( $\neq$  *green*). Note that the parallel walls generate opposite colors, symbolizing the opposite direction of the vorticity.



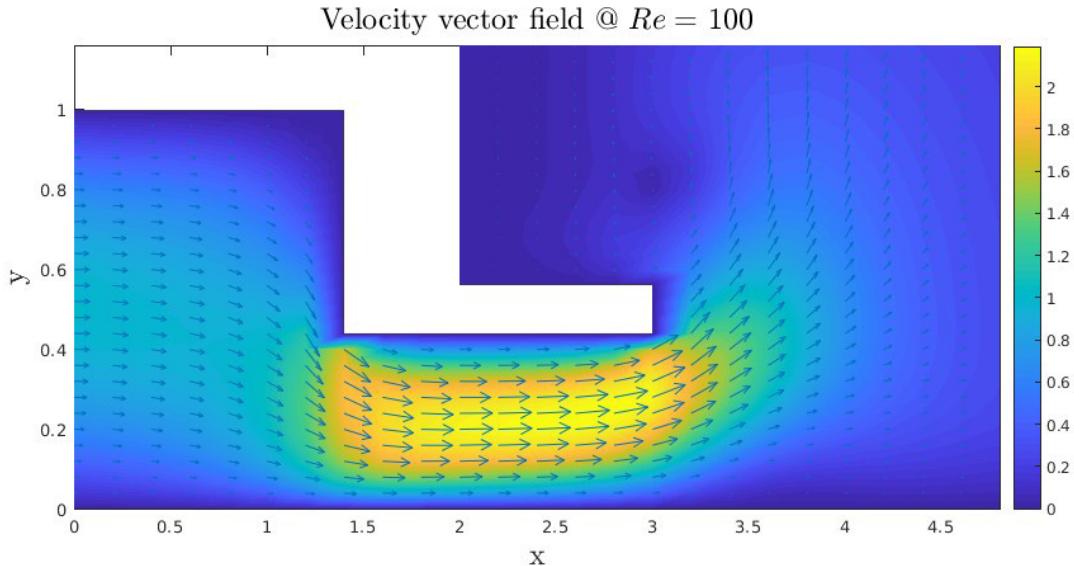
A close up look on the orifice might sharpen that :



The reason for the colorful distribution on almost zero (*green*) background, is that boundary layer is calculated w.r.t the velocity field :

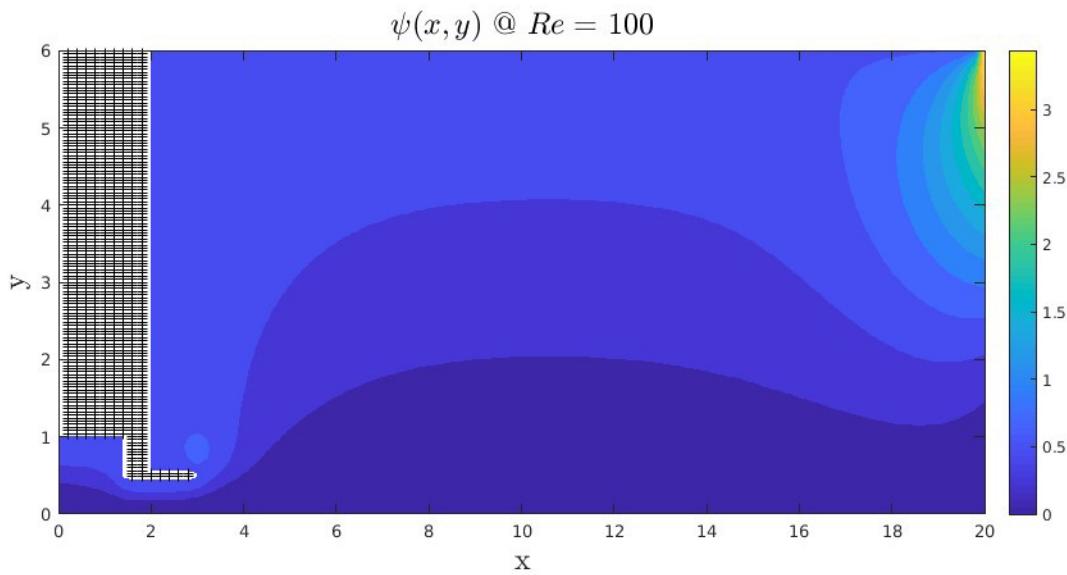
$$\omega_{i,w} = 2 \cdot \frac{\psi_{w\pm\Delta x} - \psi_{i,w}}{\Delta y^2} + O(\Delta x)$$

Such that high values are expected to form wherever the difference (numerator) is big :

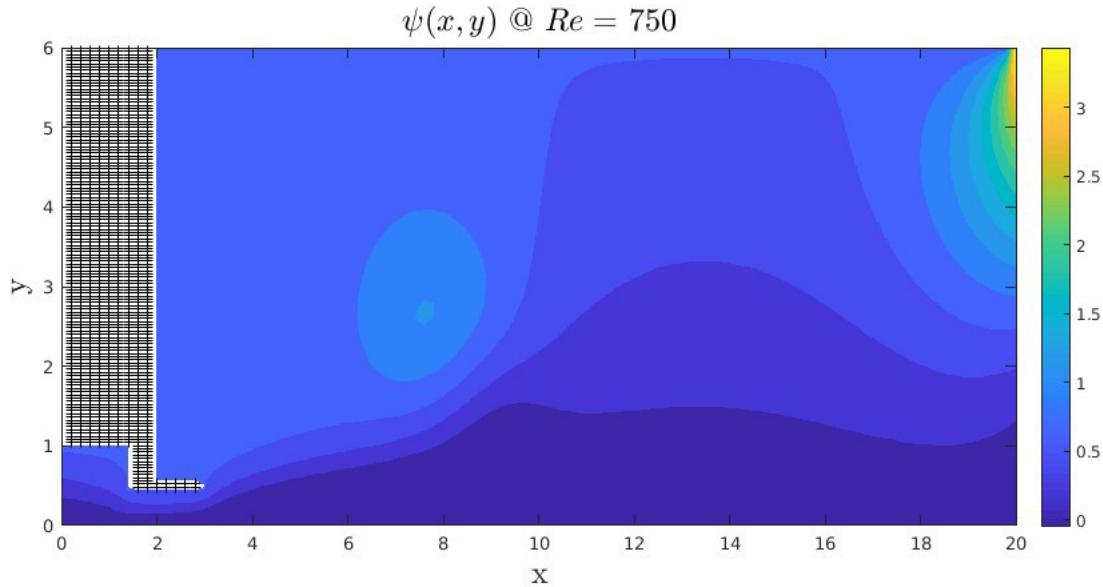


And now we shall examine the most important solution obtained form the execution.

The following figures show the **stream** function obtained from the boundary conditions and the inner field. As seen colorfully, their upmost points satisfy section 3.2:  $[\psi_{in}(1), \psi_{out}(6)] = [\frac{2}{3}, 3\frac{1}{3}]$ . And an integral on both profiles yields equivalence between total input and output, and thus fulfills conservation condition.

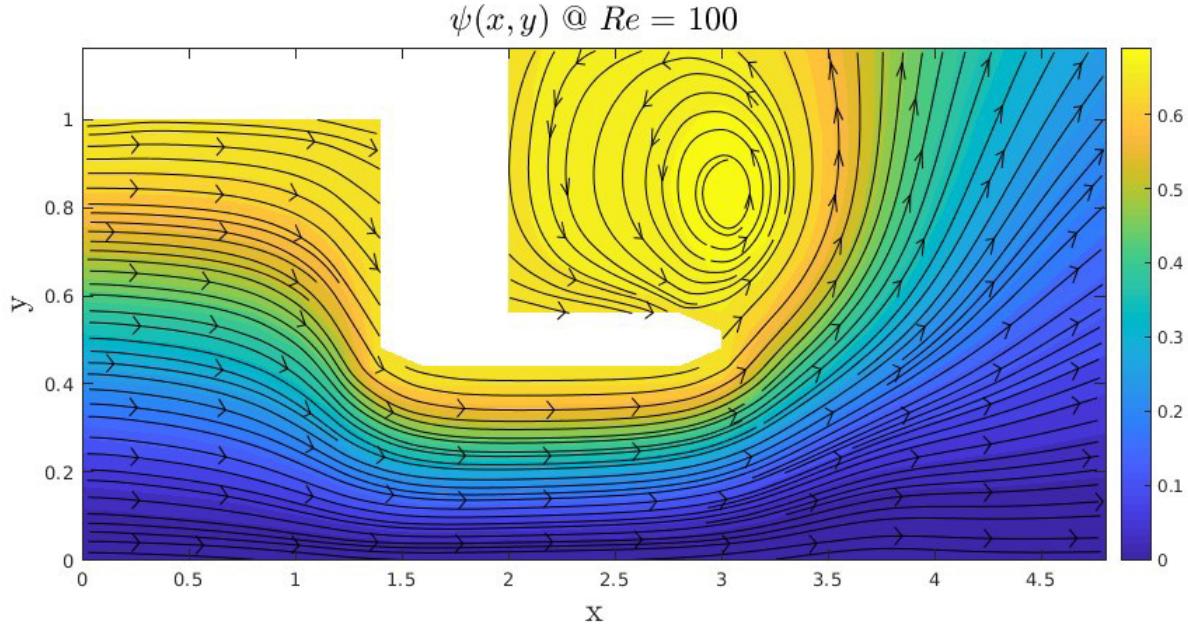


Low  $Re$  shows subtle contour lines that correspond to the stream lines we have just seen. Soft turbulences allow wide dispersion that seems to divide equally to three value levels.

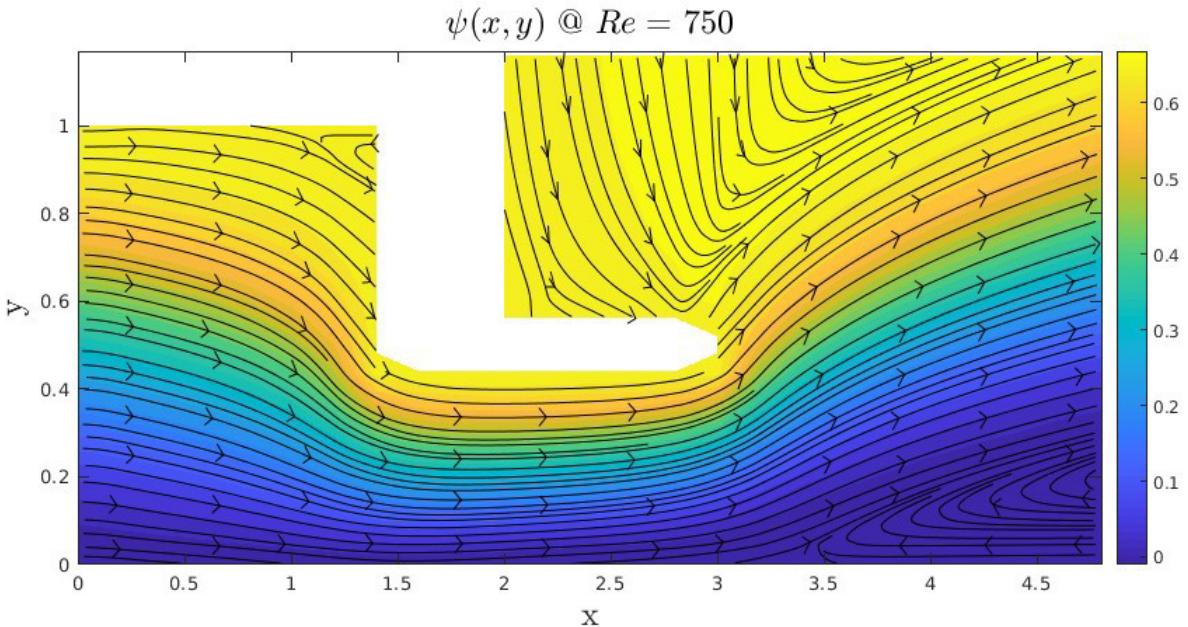


High  $Re$  evokes higher  $\psi$  values alongside turbulences in the shape's center and the outlet.

In order to get a better insight on the **orifice** region, I performed a close up examination :

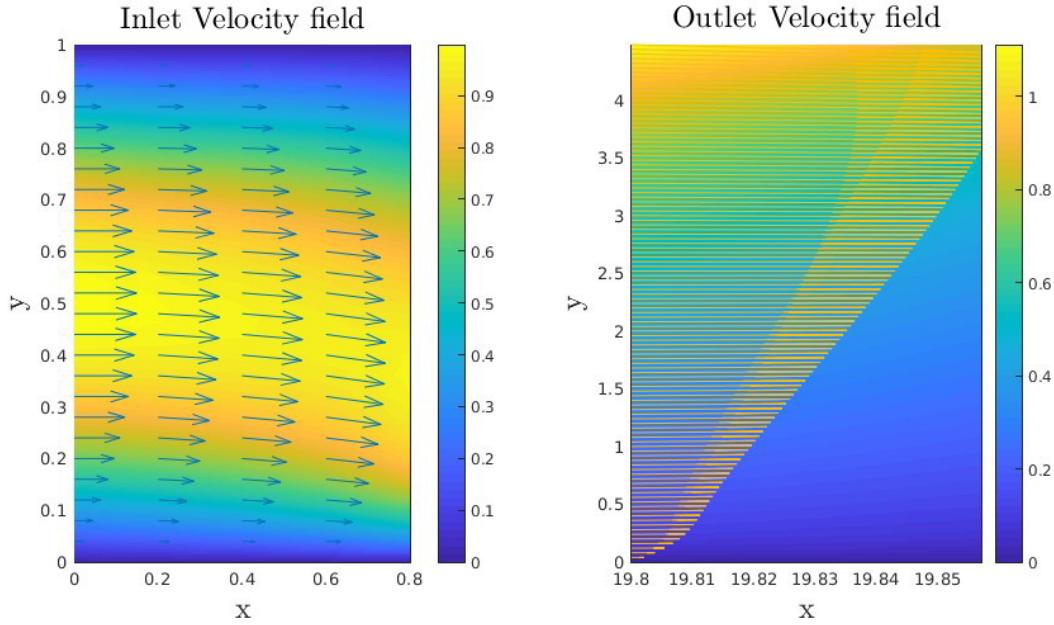


Note that colorbar was re-normalized due to change of interest zone. The low  $Re$  allows flow to follow the walls right after exiting the orifice, and generate circulation.



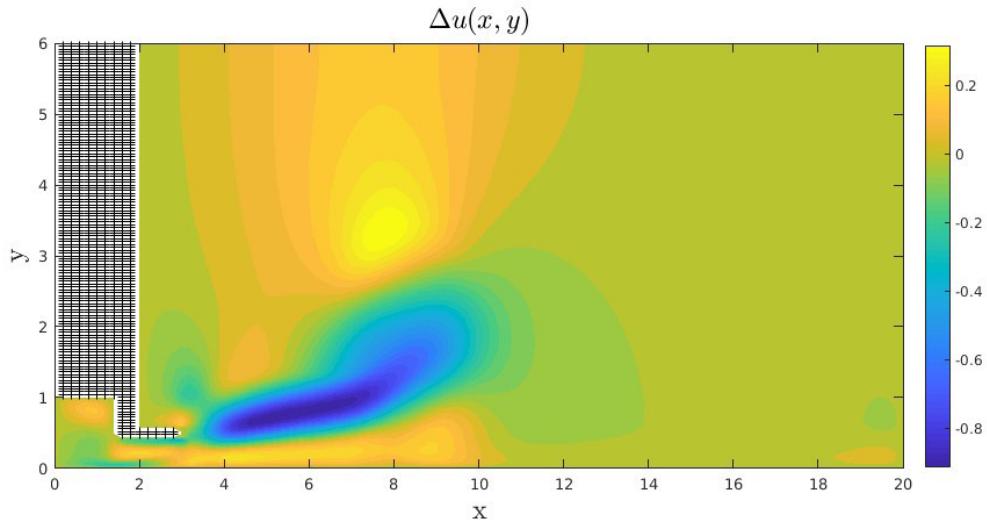
When  $Re$  is higher, the circulation is "postponed" to the center of shape, preventing the smooth tracing of the streamlines along the walls. Eventually, as a sanity check, I wanted

to make sure that the numerical calculations do comply with the fully developed flow formulation on section **3.2**. Expectedly, showing parabolic velocity distribution :



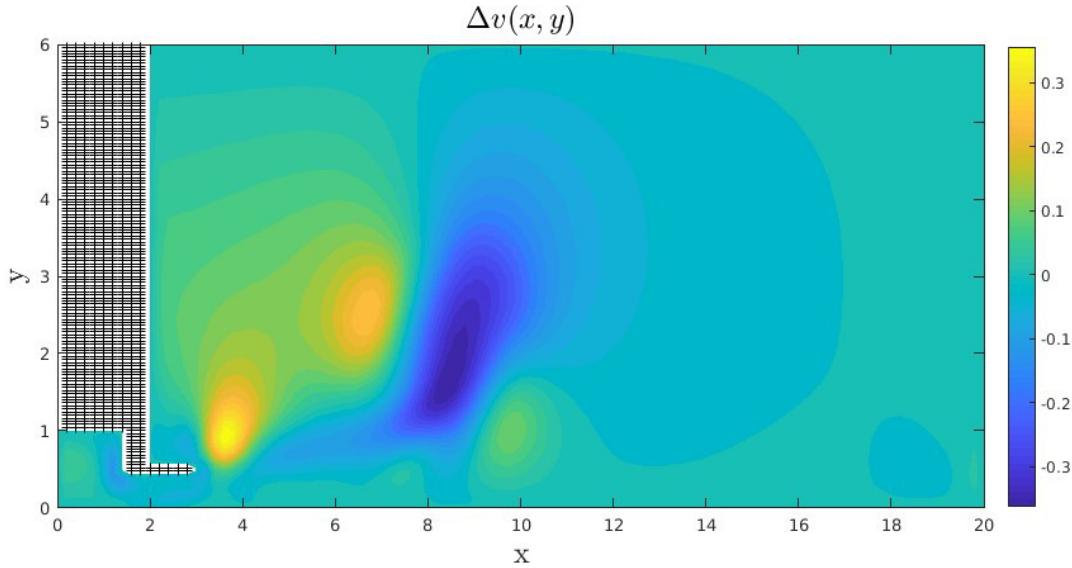
## 6.1 Comparison between $Re$ numbers

The Last results section will show the function difference between  $Re_{(100)} - Re_{(750)}$  :

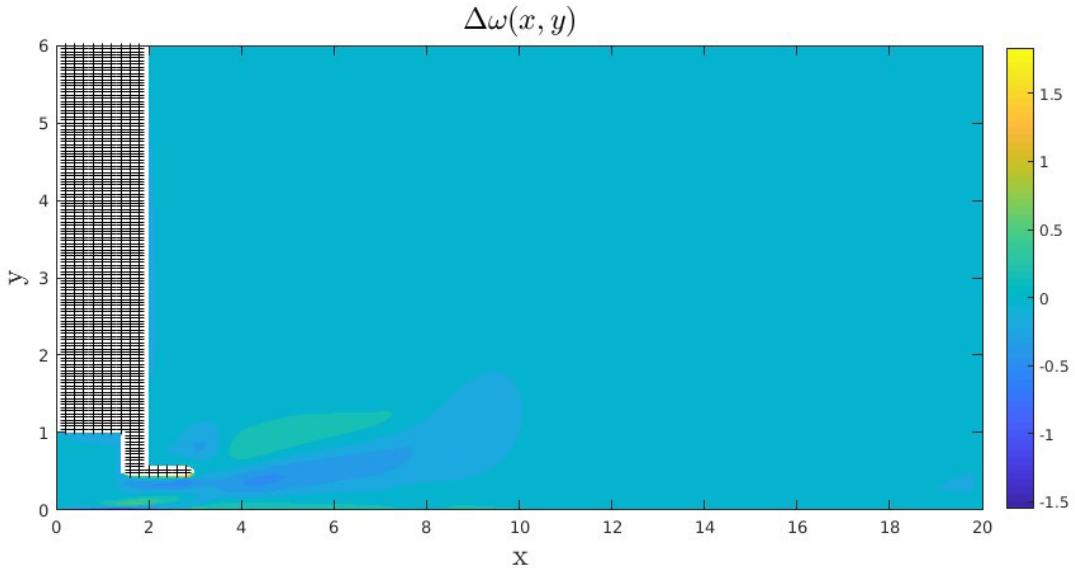


The color difference denotes the change that the  $Re = 750$  generates. That means that negative  $u$  (blue) is due to decrease in horizontal velocity and vice versa (yellow=increase).

Here we can see clearly that higher  $Re$  evoke "singular" circulations across the center. Either here, the color's temperament corresponds to the direction coerced by the proximity to the walls. And as the fluid advances, the circulation happens to grow bigger.

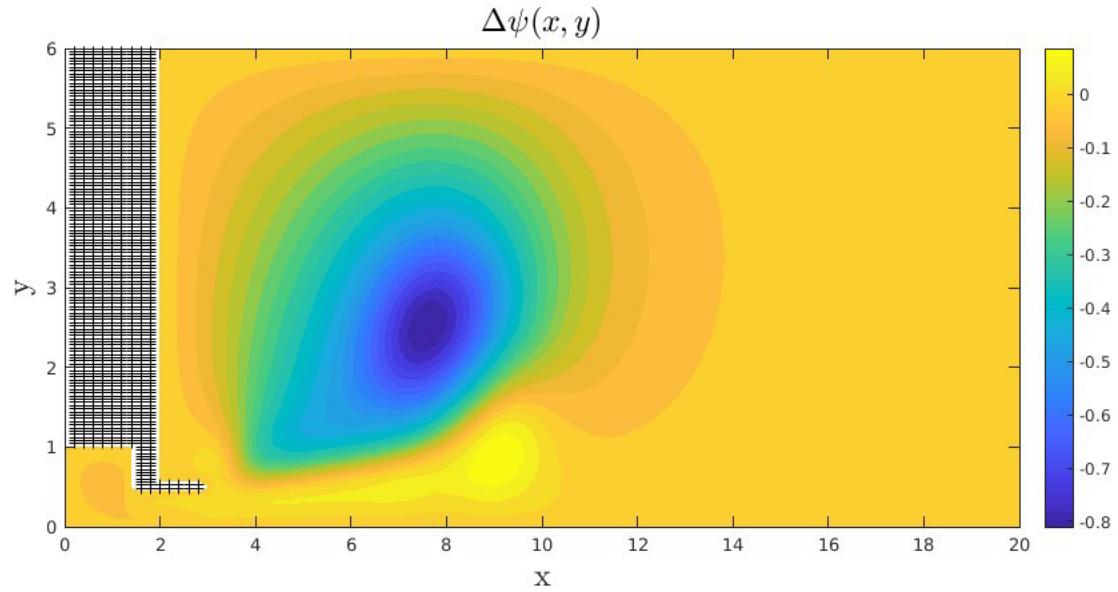


Similarly to the **vorticity** figure we have already seen, the difference goes either here in small numbers, but points on the suspicious area - the orifice exit.



Higher  $Re$  leads to a more aggressive turbulence.

And most important, the stream difference. Beside few areas showing positive stream difference (*yellow*), almost all the areas in the shape show negative values, meaning that the flow regime turns opposite after a certain critical  $Re$  value.



## Algorithms

```
% Final Project - ME685

% Main code to execute all the algorithms

set(0, 'defaultfigurecolor', [1 1 1]);

Re_i = 100; % Re_i=750 for 2nd case
for i = 1:length(Re_i)
    clearvars -except i Re_i % Clear workspace from previous Re number
    Re = Re_i(i); % Index of Reynolds number
    main_Initialization; % Algorithm for initializing values and execution
    main_Boundary_Conditions; % Algorithm for applying boundary conditions
    main_Process_calculation; % Algorithm for the iterative computation process
    main_Visualization; % Script for plotting the graphs
end
```

1) Algorithm for main\_initialization

```
% Initialize run values

global L H dx dy

fig_loc = [400 250 1000 500];
Range = @(i, f) i:f; % Function
L = 20; H = 6*L/20; % Channel's dimensions

% Step size by dx dy
dx = 0.2; dy = dx/5; % Step sizes
M = round(L/dx) + 1; % Defines the points number in y direction
N = round(H/dy) + 1; % Defines the points number in x direction
[evolution_x, evolution_y] = deal(round(M/10)+1, round(N/20)); % Center of shape

% Optimal Omega values
Meu = 0.5*abs(cos(pi/(M+1))+cos(pi/(N+1)));
Omega = 2*((1-sqrt(1-(Meu^2)))/(Meu^2));
[conv_not_achieved, psi_flag] = deal(1, 1); % Convergence condition
[i_step, epsilon] = deal(1, 1e-3); % Convergence condition
[inner_field, u_i, v_i] = deal(1, 1, 0); % velocity values for dt

% Steady state initialization
openfig('plot_epsilon_space');
openfig('plot_mesh_space');
```

```
t = 0; ss_zero = zeros(1e3, 1); % initialize evolution functions
dt = 0.831;
```

## 2) Algorithm for main\_Boundary\_Conditions

```
% Apply the set of boundary conditions
inner_field = 1;

% Initialization
Mat = zeros(M, N);
for i = 1:M
    for j = 1:N
        if ~check_if_inside_field(i, j, inner_field)
            Mat(i, j) = 0;
        end
    end
end

[u, v, w, psi, dw_dt] = deal(Mat, Mat, Mat, Mat, Mat); % Assign matrices
[x, y] = deal(linspace(0, L, M), linspace(0, H, N)); % x, y axes

% BASE
% A_1 B_1 :: 0 < x(i) < L
j = 1; % y == 0
for i = Range( 1, round(L/dx) + 1 )
    u (i, j) = 0;
    v (i, j) = 0;
    psi(i, j) = 0;
    w (i, j) = (2/dy^2)*( psi(i, 2) - psi(i, 1) );
end

% IN
% A_1 H_1 :: 0 < y(j) < L/20
i = 1; % x == 0
for j = Range( 1, round( (L/20)/dy ) )
    u (i, j) = -4*y(j)^2 + 4*y(j);
    v (i, j) = 0;
    psi(i, j) = -(4/3)*y(j)^3 + 2*y(j)^2;
    w (i, j) = -8*y(j) + 4;
end
psi_in = max(max(psi));

% H_1 G_1 :: 0 < x(i) < L/10
```

```

j = round( (L/20)/dy ) + 1; % y == L/20
for i = Range( 1, round( (L/10)/dx )-2 )
    u (i, j) = 0;
    v (i, j) = 0;
    psi(i, j) = psi_in;
    w (i, j) = (2/dy^2)*( psi(i, j-1) - psi(i, j) );
end

%G_1-F_1 :: L/40 < y(j) < L/20
i = round( (L/10)/dx ) - 2;
for j = Range( round( (L/40)/dy ) , round( (L/20)/dy ) )
    u (i, j) = 0;
    v (i, j) = 0;
    psi(i, j) = psi_in;
    w (i, j) = (2/dx^2)*( psi(i-1, j) - psi(i, j) );
end

% F_1 E_1-:: 2L/20 < x < 3L/20
j = round( (L/40)/dy ) - 1;
for i = Range( round( (2*L/20)/dx ) - 2, round( (3*L/20)/dx ) + 1 )
    u (i, j) = 0;
    v (i, j) = 0;
    psi(i, j) = psi_in;
    w (i, j) = (2/dy^2)*( psi(i, j-1) - psi(i, j) );
end

% Side of compartment handle
i = round( (3*L/20)/dx ) + 1;
for j = Range( round( (L/40)/dy ), round( (L/40)/dy)+1 )
    u (i, j) = 0;
    v (i, j) = 0;
    psi(i, j) = psi_in;
    w (i, j) = (2/dx^2)*( psi(i+1, j) - psi(i, j) );
end

% F_1 E_1+:: 2L/20 < x < 3L/20
j = round( (L/40)/dy ) + 2; j_EF = j;
for i = Range( round( (2*L/20)/dx ) + 1, round( (3*L/20)/dx ) + 1 )
    u (i, j) = 0;
    v (i, j) = 0;
    psi(i, j) = psi_in;
    w (i, j) = (2/dy^2)*( psi(i, j+1) - psi(i, j) );

```

```

end

% F_1+ D :: L/40 < y < L/20
i = round( (L/10)/dx ) + 1; i_FD = i;
for j = Range( j_EF + 1, round( (6*L/20)/dy ) + 1 )
    u (i, j) = 0;
    v (i, j) = 0;
    psi(i, j) = psi_in;
    w (i, j) = (2/dx^2)*( psi(i+1, j) - psi(i, j) );
end

%DC_sym :: 0 < y(j) < L/20
j = round( (6*L/20)/dy ) + 1; % x == L
for i = Range( i_FD, round( L/dx ) + 1 )
    % u (i, j) = 0;
    v (i, j) = 0;
    psi(i, j) = psi_in;
    w (i, j) = 0;
end

% OUT
% B_1 C_o :: 0 < y(j) < 6L/20
i = round( L/dx ) + 1; % x == L
for j = Range( 1, round( (6*L/20)/dy ) + 1 )
    u (i, j) = -(1/216)*y(j)^2 + (1/18)*y(j) ;
    v (i, j) = 0;
    psi(i, j) = -(1/648)*y(j)^3 + (1/9) *y(j)^2 ;
    w (i, j) = -(1/108)*y(j) + (1/18) ;
end

```

### 3) Algorithm for main\_Process\_calculations

```

%Calculates values inside field

while( conv_not_achieved == 1 )
    calculate_dw_dt; % caculates vorticity derivative within field
    w = w + dt*dw_dt; % Euler method :: w_{n+1} = w_n + dt*dw_n
    calculate_stream; % caculates stream within field
    calculate_vorticity; % caculates vorticity on the walls
    calculate_velocity; % caculates velocity within field
    calculate_evolution; % caculates evolution of shape's center
    check_convergence; % Check convergence criteria

```

```

    i_step = i_step + 1;           % updates the iteration index
    t = t + dt;                  % updates time index
end

```

4) Alorithm for main\_Visualization

```

% Plots

inner_field = 0; blank_scatter = zeros(size(psi));

% Nullify zero entries
for i = 1:M
    for j = 1:N
        if ~check_if_inside_field(i, j, inner_field)
            [u(i, j), v(i, j), w(i, j), psi(i, j)] = deal(NaN, NaN, NaN, NaN);
            blank_scatter(i, j) = 1;
        end
    end
end

%% u function
figure('rend', 'painters', 'pos', fig_loc);
contourf(x, y, u', 100, 'edgecolor', 'none'); colorbar;
ind(1) = xlabel('x');
ind(2) = ylabel('y');
ind(3) = title(['$u(x, y)$ \ @ $Re =$', num2str(Re)]);
set(ind, 'Interpreter', 'latex', 'fontsize', 18); clear ind;
hold on; draw_boundaries; ylim([y(1) y(end)]);
h = streamslice(x, y, u', v', 3);
set(h, 'color', 'k', 'linewidth', 0.3); clear ind h;

%% v function
figure('rend', 'painters', 'pos', fig_loc);
contourf(x, y, v', 30, 'edgecolor', 'none');
ind(1) = xlabel('x');
ind(2) = ylabel('y');
ind(3) = title(['$v(x, y)$\ @ $Re =$', num2str(Re)]);
colorbar; set(ind, 'Interpreter', 'latex', 'fontsize', 18); clear ind;
hold on; draw_boundaries; ylim([y(1) y(end)]);
h = streamslice(x, y, u', v', 3);
set(h, 'color', 'k', 'linewidth', 0.3); clear ind h;

```

```

%% Vorticity(x, y)
figure('rend', 'painters', 'pos', fig_loc);
contourf(x, y, w'*1, 40, 'edgecolor', 'none');
ind(1) = xlabel('x');
ind(2) = ylabel('y');
ind(3) = title(['$\omega(x, y)$ @ $Re =$', num2str(Re)]);
set(ind, 'Interpreter', 'latex', 'fontsize', 18); colorbar;
% Add velocity streamline
hold on; draw_boundaries; ylim([y(1) y(end)]);
pbaspect([3 1.5 1]);

%% Vorticity :: Close up on nozzle
figure('rend', 'painters', 'pos', fig_loc);
[x_noz, y_noz] = deal(round(M/4), round(N/5));
contourf(x(1:x_noz), y(1:y_noz), w(1:x_noz, 1:y_noz)'*1, 14, 'edgecolor', 'none');
ind(1) = xlabel('x');
ind(2) = ylabel('y');
ind(3) = title(['$\omega(x, y)$ @ $Re =$', num2str(Re)]);
set(ind, 'Interpreter', 'latex', 'fontsize', 18);
colorbar; pbaspect([3 1.5 1]);

%% psi(x, y) vs. (u, v)
figure('rend', 'painters', 'pos', fig_loc);
contourf(x, y, psi', 15, 'edgecolor', 'none');
ind(1) = xlabel('x');
ind(2) = ylabel('y');
ind(3) = title(['$\psi(x, y)$ @ $Re =$', num2str(Re)]);
set(ind, 'Interpreter', 'latex', 'fontsize', 18); hold on;
% Add velocity streamlines
draw_boundaries; ylim([y(1) y(end)]);
pbaspect([3 1.5 1]); ylim([y(1) y(end)]);

% Stream :: Close up on nozzle
figure('rend', 'painters', 'pos', fig_loc);
[x_noz, y_noz] = deal(round(M/4), round(N/5));
contourf(x(1:x_noz), y(1:y_noz), psi(1:x_noz, 1:y_noz)', 28, 'edgecolor', 'none');
ind(1) = xlabel('x');
ind(2) = ylabel('y');
ind(3) = title(['$\psi(x, y)$ @ $Re =$', num2str(Re)]);
set(ind, 'Interpreter', 'latex', 'fontsize', 18); hold on;
% Add velocity streamlines
h = streamslice(x(1:x_noz), y(1:y_noz), u(1:x_noz, 1:y_noz)', v(1:x_noz, 1:y_noz)', 3);
set(h, 'color', 'k', 'linewidth', 0.4); clear ind h;
colorbar; pbaspect([3 1.5 1]);

%% Nozzle close up - velocity vector field

```

```

figure('rend', 'painters', 'pos', fig_loc); clear X Y
[X,Y] = meshgrid(x(1:x_noz), y(1:y_noz));
pcolor(X, Y, hypot(u(1:x_noz, 1:y_noz)', v(1:x_noz, 1:y_noz)'));
shading interp; hold on;
quiver(imresize(X, 1), imresize(Y, 1), imresize(u(1:x_noz, 1:y_noz)', 1), imresize(v(1:x_noz, 1:y_noz)', 1));
% Fonts
ind(1) = xlabel('x');
ind(2) = ylabel('y');
ind(3) = title(['Velocity vector field @ $Re =$', num2str(Re)]);
set(ind, 'Interpreter', 'latex', 'fontsize', 18);
colorbar; pbaspect([3 1.5 1]);

%% inlet vs. outlet close up - velocity vector field
figure('rend', 'painters', 'pos', fig_loc); clear X Y
subplot(1, 2, 1);
[x_inlet, y_inlet] = deal( round(M/20), round(N/5) );
[X,Y] = meshgrid(x(1:x_inlet), y(1:y_inlet));
pcolor(X, Y, hypot(u(1:x_inlet, 1:y_inlet)', v(1:x_inlet, 1:y_inlet)'));
shading interp; hold on;
quiver(imresize(X, 1), imresize(Y, 1), imresize(u(1:x_inlet, 1:y_inlet)', 1), imresize(v(1:x_inlet, 1:y_inlet)', 1));
% Fonts
ind(1) = xlabel('x'); ind(2) = ylabel('y');
ind(3) = title('Inlet Velocity field');
set(ind, 'Interpreter', 'latex', 'fontsize', 18);
colorbar; ylim([0 1]); %pbaspect([3 1.5 1]);

subplot(1, 2, 2);
[x_outlet, y_outlet] = deal( round(M - 1), round(3*N/4) );
[X,Y] = meshgrid(x(x_outlet:M), y(1:y_outlet));
pcolor(X, Y, hypot(u(x_outlet:M, 1:y_outlet)', v(x_outlet:M, 1:y_outlet)'));
shading interp; hold on;
quiver(imresize(X, 1), imresize(Y, 1), imresize(u(x_outlet:M, 1:y_outlet)', 1), imresize(v(x_outlet:M, 1:y_outlet)', 1));
% color
'color', [1 0.75 0] );
% Fonts
ind(1) = xlabel('x'); ind(2) = ylabel('y');
ind(3) = title('Outlet Velocity field');
set(ind, 'Interpreter', 'latex', 'fontsize', 18);
colorbar; %pbaspect([1 1.5 1]);
xlim([x(x_outlet) x(x_outlet) + dx/3.5]);

%% Flow function vs. time evolution
figure('rend', 'painters', 'pos', fig_loc);
t_lin = (0:i_step-2)*dt;

subplot(4, 1, 1);

```

```

ax(1) = plot(t_lin, u_ss , 'LineWidth', 2.25); grid on;
ind(1) = title(['Flow evolution at orifice \ ($t \leq t_{ss})\ @ $Re =$ ', num2str(Re)]);
ind(2) = ylabel('$u(t)$'); xlim([0 t_lin(end)]);
subplot(4, 1, 2);
ax(2) = plot(t_lin, v_ss , 'LineWidth', 2.25); grid on;
ind(3) = ylabel('$v(t)$'); xlim([0 t_lin(end)]);
subplot(4, 1, 3);
ax(3) = plot(t_lin, w_ss , 'LineWidth', 2.25); grid on;
ind(4) = ylabel('$\omega (t)$'); xlim([0 t_lin(end)]);
subplot(4, 1, 4);
ax(4) = plot(t_lin, psi_ss, 'LineWidth', 2.25); grid on;
ind(5) = ylabel('$\psi (t)$'); ind(6) = xlabel('Time [sec]');
xlim([0 t_lin(end)]);
set(ind, 'Interpreter', 'latex', 'fontsize', 17);

```

## 7 Conclusion

- The problem was solved using 1st order upwind differences for the vorticity and Gauss-Seidel and SOR for the stream function.
- The boundary conditions were not fully defined for sharp corner points, where high vorticity was forced by the acute change of the flow direction.
- The flow further away from the walls, can be said to be laminar as long as Re kept small.
- The optimization stage showed indifference to different Reynolds values.

## 8 References

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