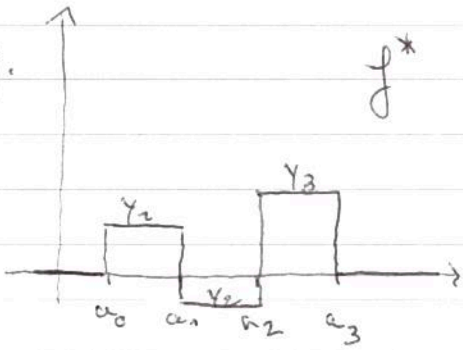
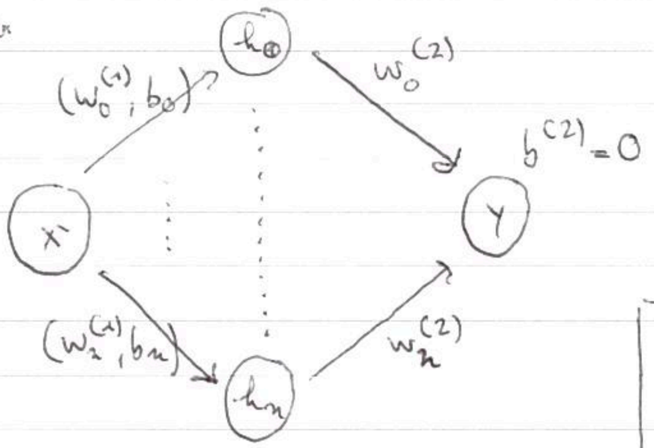
Let $a_0, \dots, a_n \in \mathbb{R}$; $y_1, \dots, y_n \in \mathbb{R}$

$$f^*: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} y_i & x \in [a_{i-1}, a_i[\,, \quad i = 1, \dots, n \\ 0 & \text{else} \end{cases}$$

so, e.g.



Consider



$\longleftrightarrow \quad f: \mathbb{R} \to \mathbb{R}$

$$\boxed{Y := \sum_{j=0}^{n} w_j^{(2)} \cdot h_j}$$
[linear function; activation = identity]

where
$$h_i = \begin{cases} 1 & x \geq a_i \\ 0 & \text{else} \end{cases}$$
[perceptron: $w_i^{(1)} = 1$, $b_i = a_i$]

further choose $w_0^{(2)} := y_1$

and recursively for $i = 1, \dots, n-1$ $\quad w_i^{(2)} := y_{i+1} - \sum_{j=0}^{i-1} w_j^{(2)}$

and $\quad w_n^{(2)} := -\sum_{j=0}^{n-1} w_j^{(2)}$

Then $f = f^*$.

Proof:

(i) $x < a_0 \Rightarrow h_0, \dots, h_n = 0 \Rightarrow y = 0$

(ii) $x \in [a_{i-1}, a_i[\Rightarrow h_0, \dots, h_{i-1} = 1$

$h_i, \dots, h_n = 0$

so $$y = \sum_{j=0}^{n} w_j^{(2)} \cdot h_j = \sum_{j=0}^{i-1} w_j^{(2)}$$

$$= w_{i-1}^{(2)} + \sum_{j=0}^{i-2} w_j^{(2)}$$

$$= y_i - \sum_{j=0}^{i-2} w_j^{(2)} + \sum_{j=0}^{i-2} w_j^{(2)} = y_i$$

(iii) $x \geq a_n \Rightarrow h_0, \dots, h_n = 1$

$$\Rightarrow y = \sum_{j=0}^{n} w_j^{(2)} \cdot 1 = w_n^{(2)} + \sum_{j=0}^{n-1} w_j^{(2)}$$

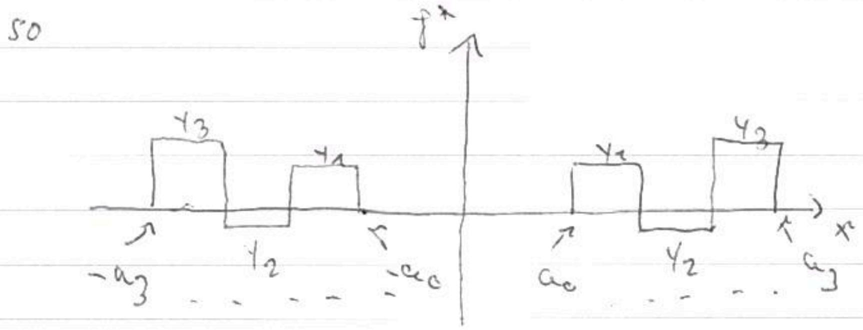
$$= 0$$

$\square$

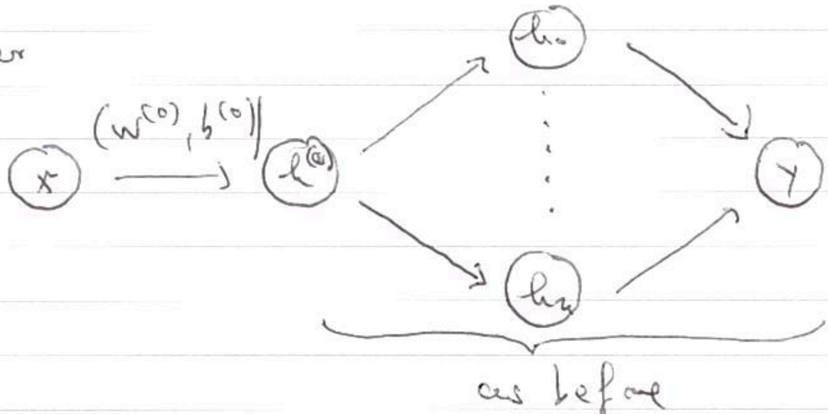
Let $a_0, \dots, a_n \in \mathbb{R}_+$, $y_1, \dots, y_n \in \mathbb{R}$

$$f^*: \mathbb{R} \to \mathbb{R}$$
$$x \longmapsto \begin{cases} y_i & x \in [a_{i-1}, a_i[\\ y_i & x \in]-a_i, -a_{i-1}] \\ 0 & \text{else} \end{cases}$$

so



Consider



and $w^{(0)} = 1$, $b^{(0)} = 0$, activation $a(.) = |.|$

so $h^{(0)} = |w^{(0)} x + b^{(0)}| = |x|$