Note Title 11-06-2019

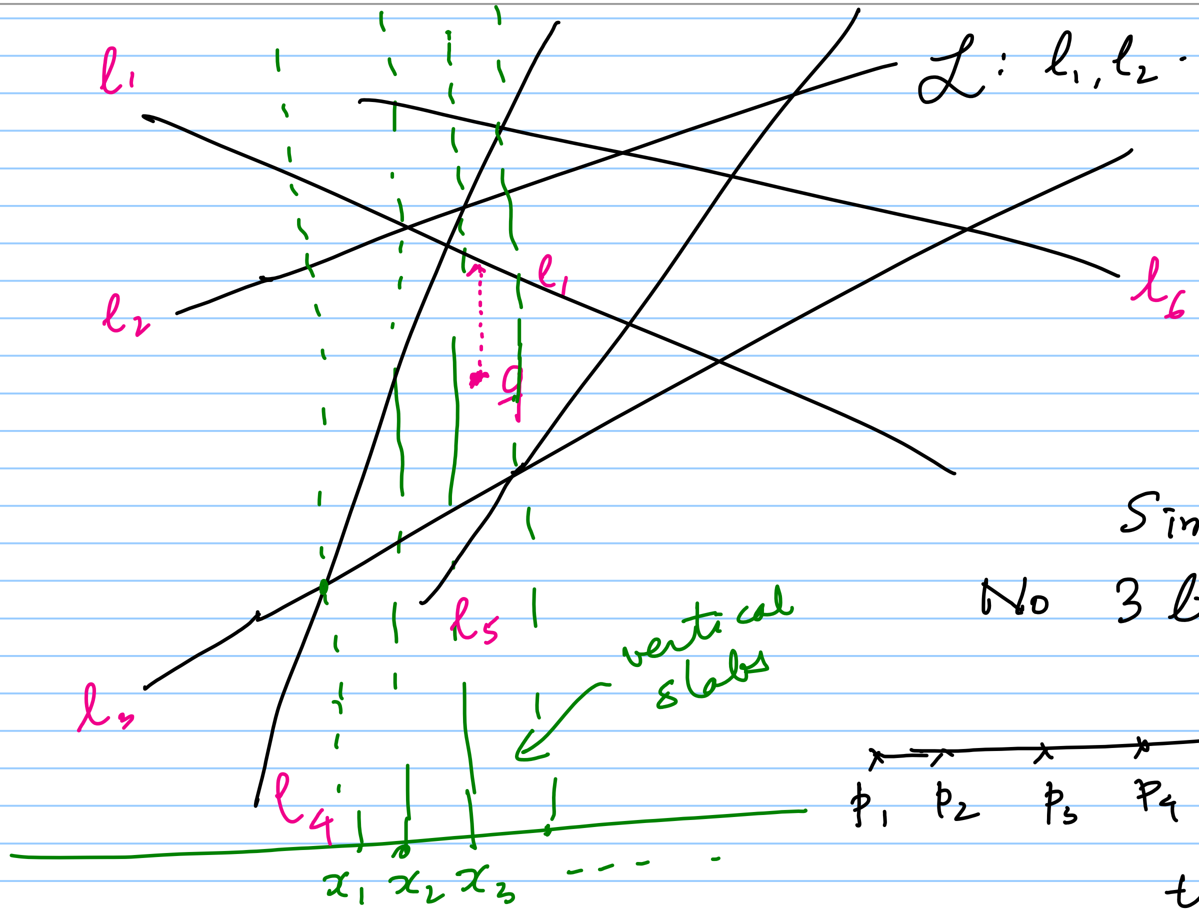
Arrangements

Topics

- Duality
- Lifting transforms, minimization diagrams
- Levels of arrangements
- Inside-outside circle queries
- Zone theorem

Tutorials :

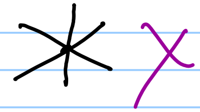
- Sum of squares of faces
- Largest / Smallest Δ
- Collinearity



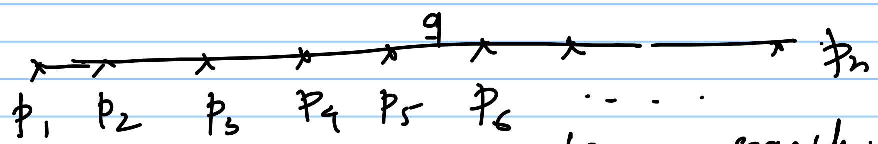
$\mathcal{L}: l_1, l_2 \cdots l_n$

Build a data structure to do point location

Obs: n line intersect in $\binom{n}{2}$ distinct points



Simple arrangement: No 3 lines are concurrent



binary search tells us that q is between p_5 and p_6

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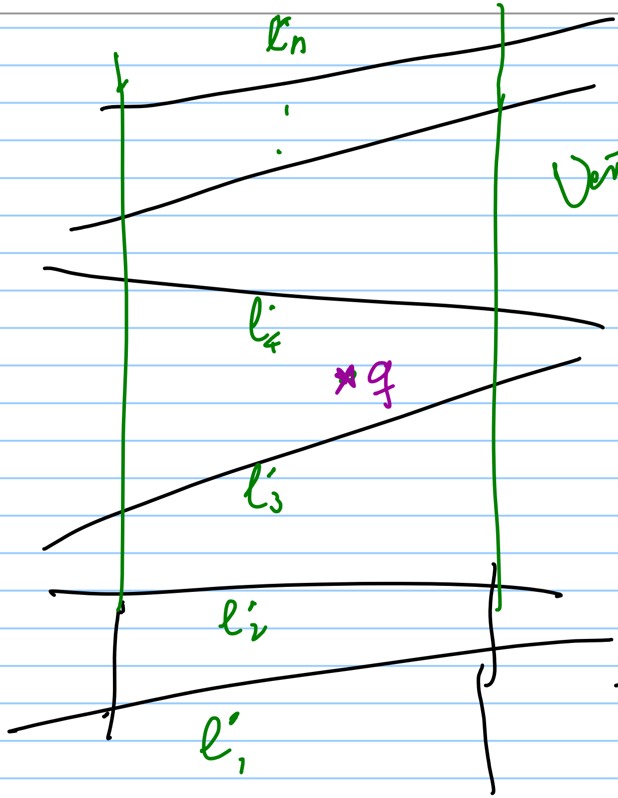


Vertical slab

<u>Obs</u> : No intersections

Therefore we can order the lines within the slab above/below ordering

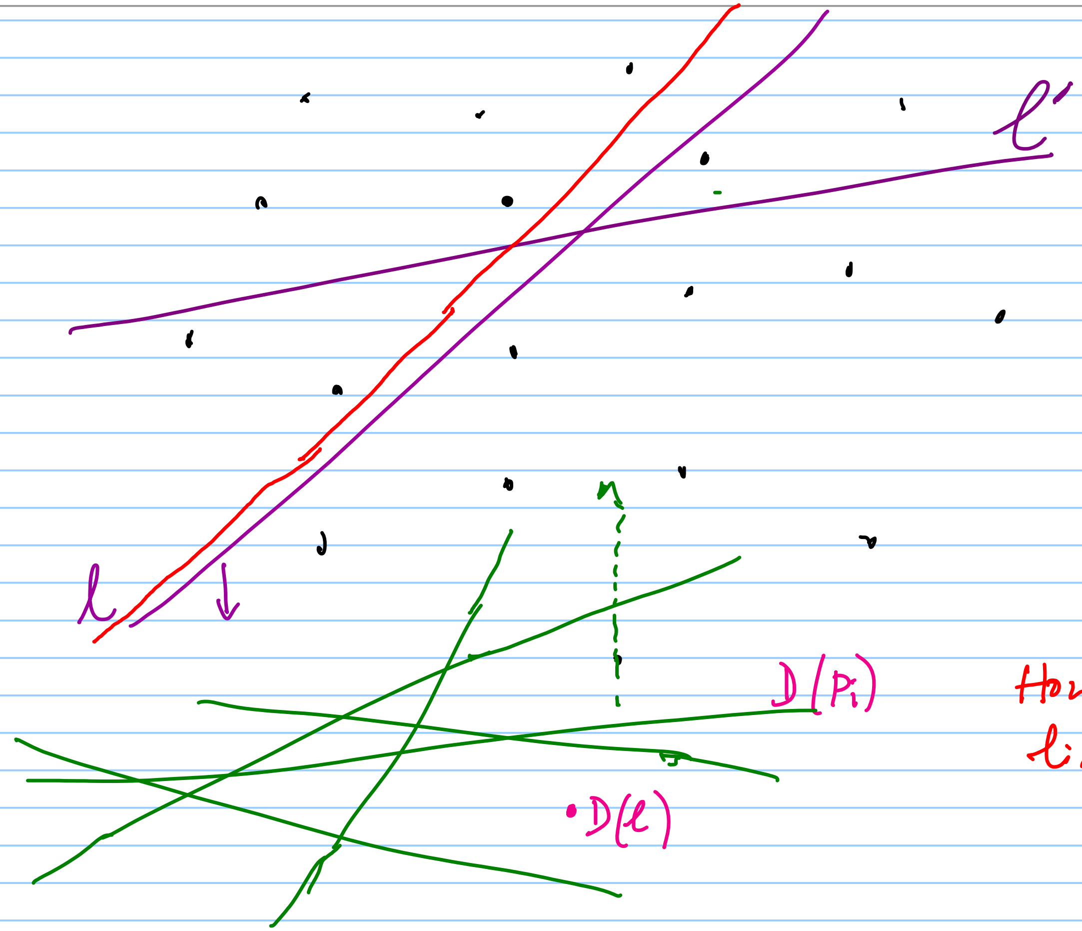
point above/below a line

Two binary searches yields the required result

Query time $2 \log n$: $O(\log n)$

In 2 dimension search data structure has size $\binom{n}{2} \times n \sim n^3$

In d dim this scheme require $O\left(n^{2^{d-1}}\right)$ space



$P : \{p_1, p_2 \ldots p_n\}$

Build a data structure that supports half-plane queries

How many / which ones lie below a query line ℓ

How many distinct answers are possible

How many partition can be induced by lines?

$O(n^2)$ possible

and not 2^n

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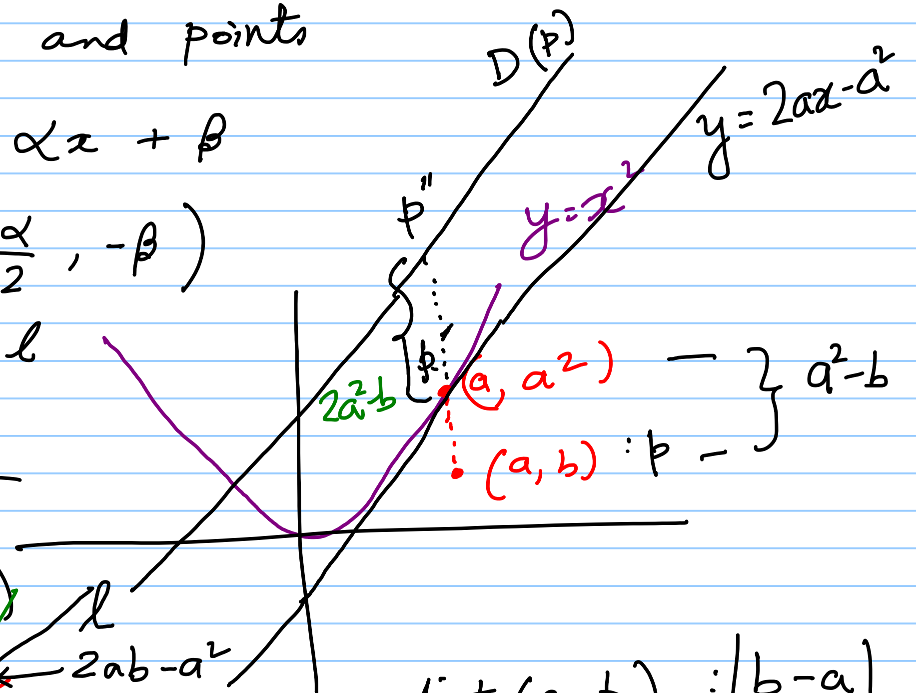
Duality between lines and points

$p: (a, b)$ $\quad$ $l: y = \alpha x + \beta$

$D(p): y = 2ax - b$ $\quad$ $D'(l): \left(\frac{\alpha}{2}, -\beta\right)$

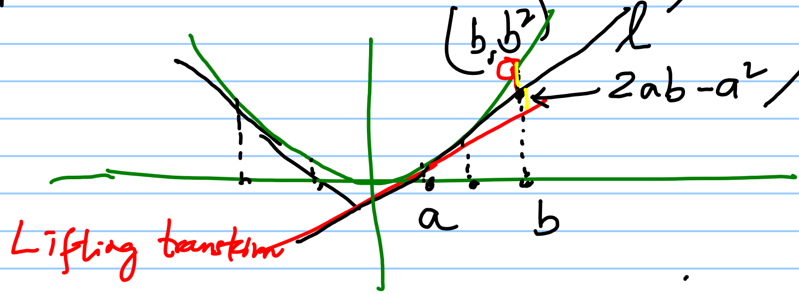
$D'(D(p)) = p$ $\quad$ $D'(D'(l)) = l$

$D: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$



Vertical distance between p and $D(p)$

$\text{dist}(p, p') = \text{dist}(p', p'')$



$\text{dist}(a, b) : |b - a|$

$\text{Vert}((b, b^2), l) = b^2 - (2ab - a^2)$

$= (a-b)^2$

properties of dual transform

1. Vertical distance between p and ℓ is preserved between $D(p)$ and $D(\ell)$

$p:(a,b) \rightarrow 2ax - b$ ✓ $\quad \ell: \ y = \alpha x + \beta \rightarrow \left(\frac{\alpha}{2}, -\beta\right)$ ✓

$\text{Vert}(p, \ell) = b - (\alpha a + \beta)$

$= \text{Vert}(D(p), D(\ell))$

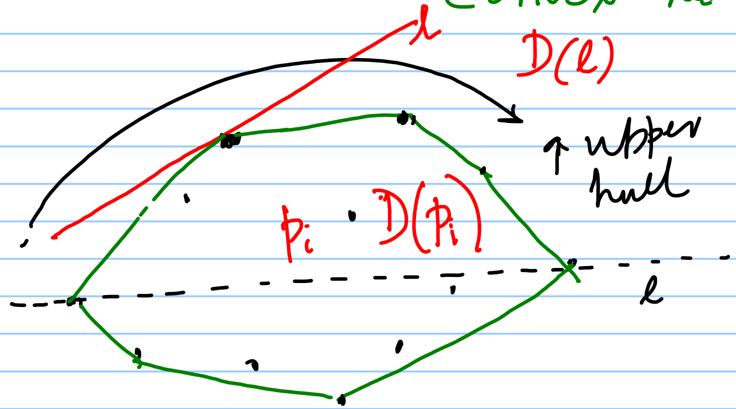
$\text{Vertical}(D(p), D(\ell)) : -\beta - (\alpha a - b)$

$= b - (\alpha a + \beta)$

$\Rightarrow$ If p is incident on ℓ then $D(\ell)$ is incident on $D(p)$
(vertical distance $= 0$)

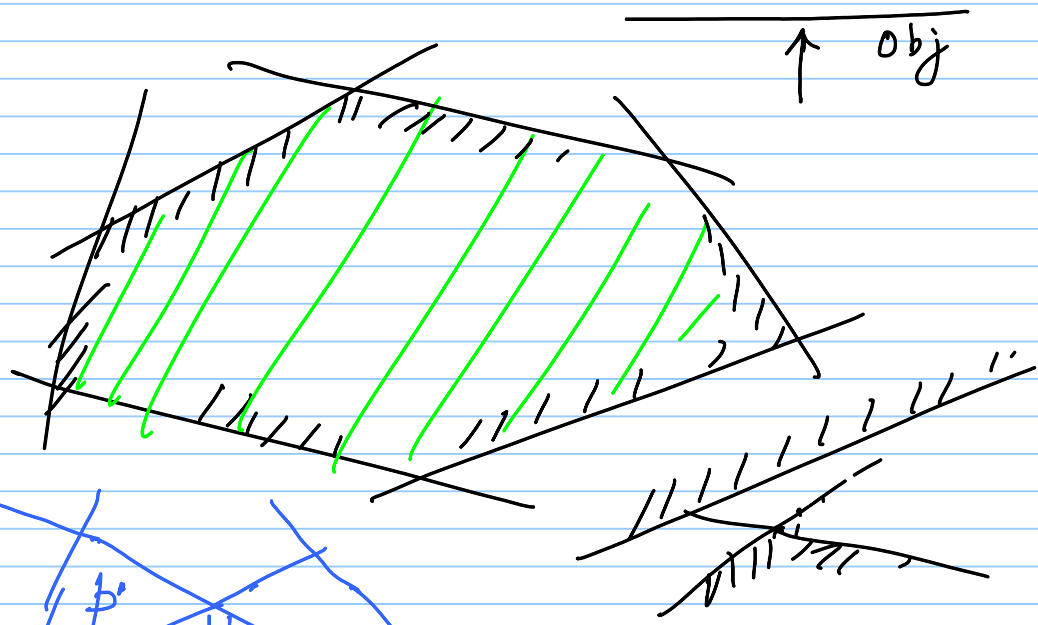
2. Above/Below If p lies above ℓ, then $D(\ell)$ lies above $D(p)$

Convex hulls and Intersection of half planes

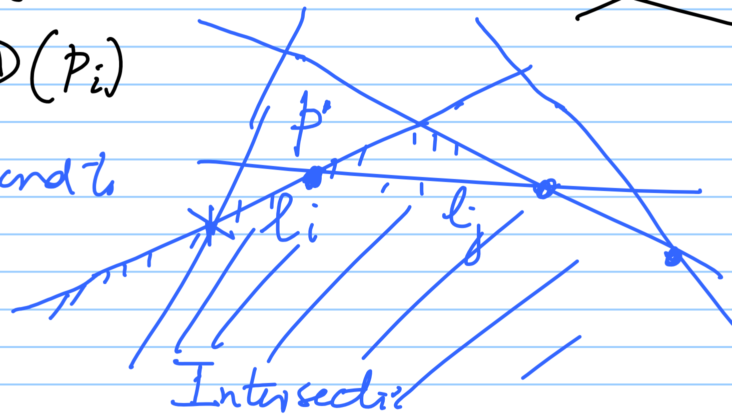


p_i is below l

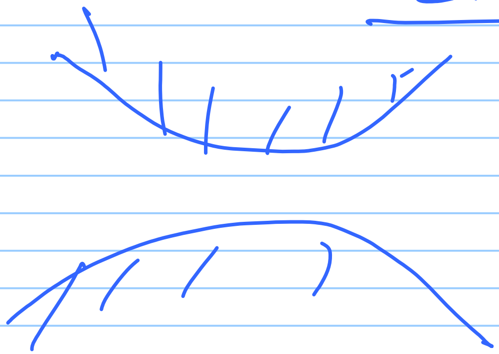
$\Rightarrow$ $D(l)$ is below $D(p_i)$



Obs : Intersection point $p' : (l_i, l_j)$ correspond to line passing through $D(l_i)$ and $D(l_j)$



Emptiness



Dual transform in d dimensions

$$D(a_1, a_2, a_3 \ldots a_d) : x_d = 2a_1x_1 + 2a_2x_2 + \cdots - a_d$$

$\mathcal{L} : \{l_1, l_2, \ldots l_n\}$

$A(\mathcal{L})$: arrangement of lines

What is the structure induced by $\mathcal{L}$

In a simple arrangement no 3 lines are concurrent

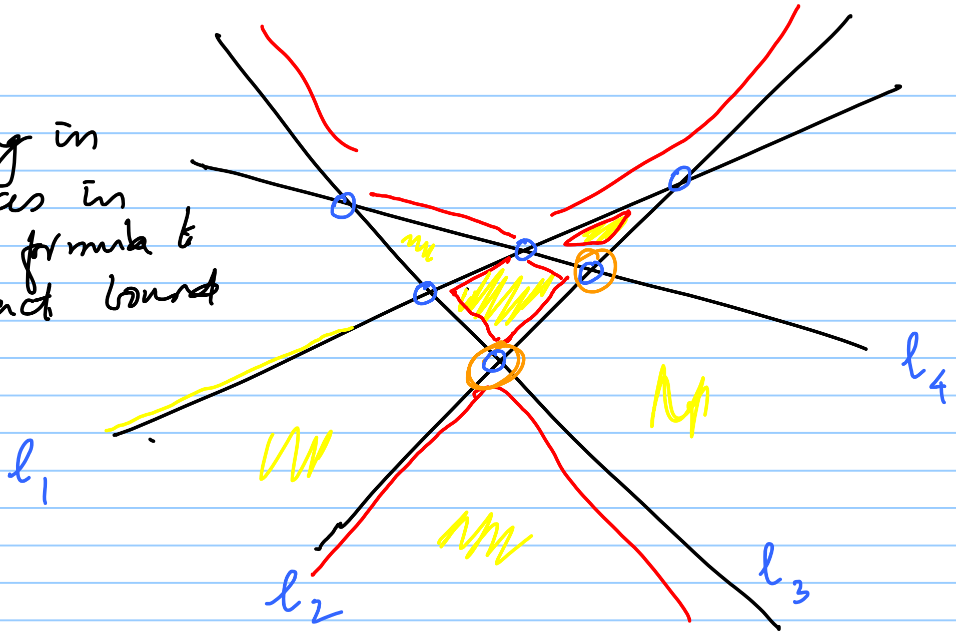
Intersections : $\binom{n}{2} \sim \frac{n^2}{2}$

V : Intersection points

E : part of the line joining two consecutive intersection points

(we will deal with the semi infinite edges)

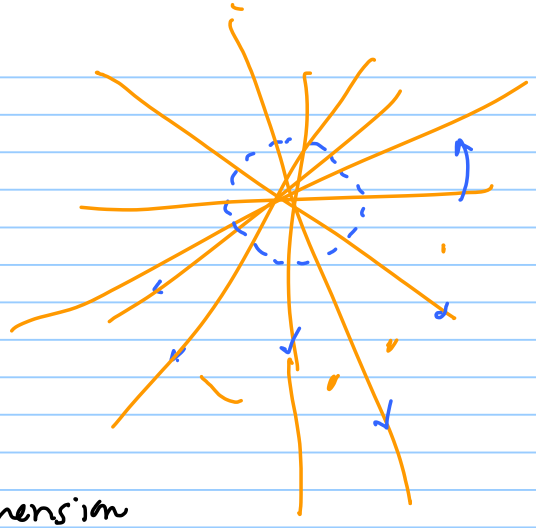
Faces : faces (bounded) + unbounded defined by a cycle



We can plug in the # vertices in the Euler's formula to get an exact bound on # faces

$v - e + f = 2$

Imagine a point Q at ∞ (embed on sphere) such that all semiinfinite edges converge in Q $\Rightarrow$ All faces are bounded



\# unbounded faces $= 2n$

total # faces $= \binom{n}{2} + 2n$

In d dimensions total # vertices (assuming no $d+1$ planes are concurrent)

$$\binom{n}{d} \sim n^d$$

fixed dimension "d" is not scaling

$O(n^d)$

hidden constant $d!$

Total size of $A(\mathcal{L})$ = # faces + # edges + # vertices

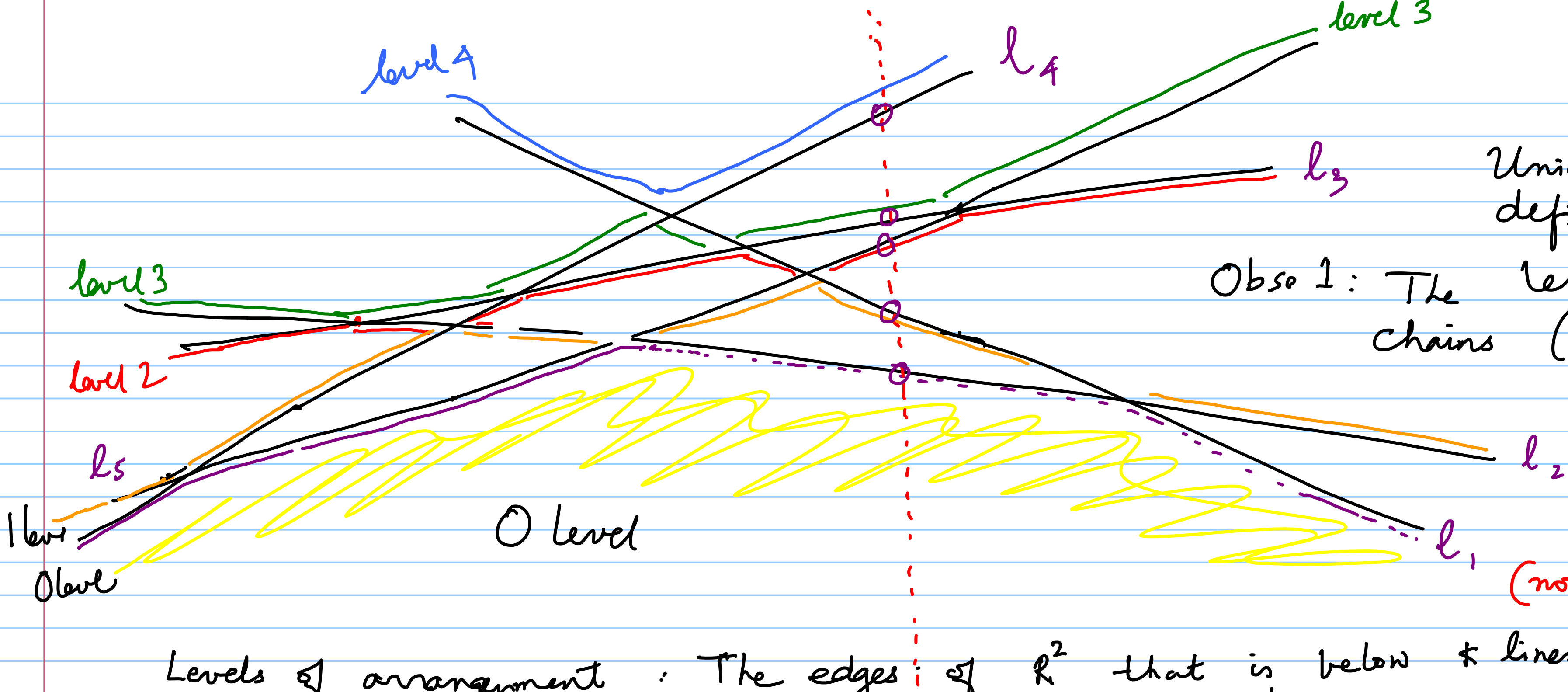
$= O(n^2)$

In d dimensions facets of dimension $0 \le i \le d-1$

\+ d-dimensional faces

Obs : A i dimensional facet is defined by exactly $d-i$ hyperplanes

Total # i dimensional facets $\binom{n}{d-i} \Rightarrow$ Total facets of all dimensions $\sum_i \binom{n}{d-i}$



Union of all levels define $A(\mathcal{L})$

Obs 1: The levels are monotonic chains (w.r.t x direction)

Obs 2: The topmost/bottom most levels are convex (not necessarily true for other levels)

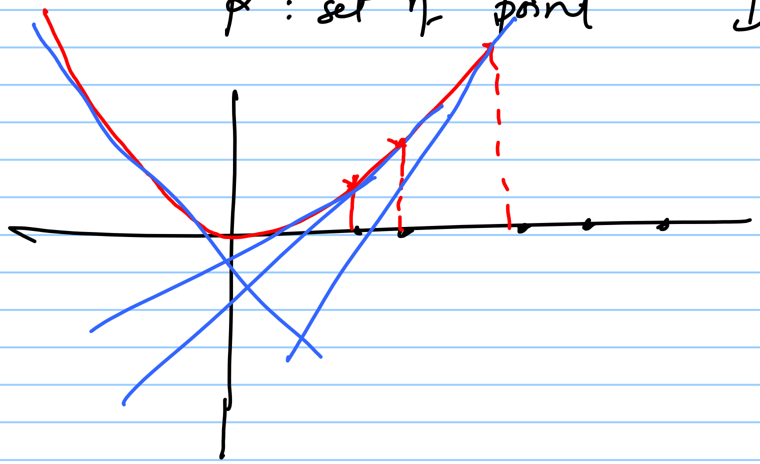
Levels of arrangement: The edges of $\mathbb{R}^2$ that is below k lines is called k-th level of $A(\mathcal{L})$ denote by $A^k(\mathcal{L})$

The subset of $\mathbb{R}^2$ below the k^{th} level $\leq k$ level denoted by $A^{\leq k}(\mathcal{L})$

The arrangement $A(D^*(P))$ corresponds to the k-nearest neighbour Voronoi diagram

dual plane of P

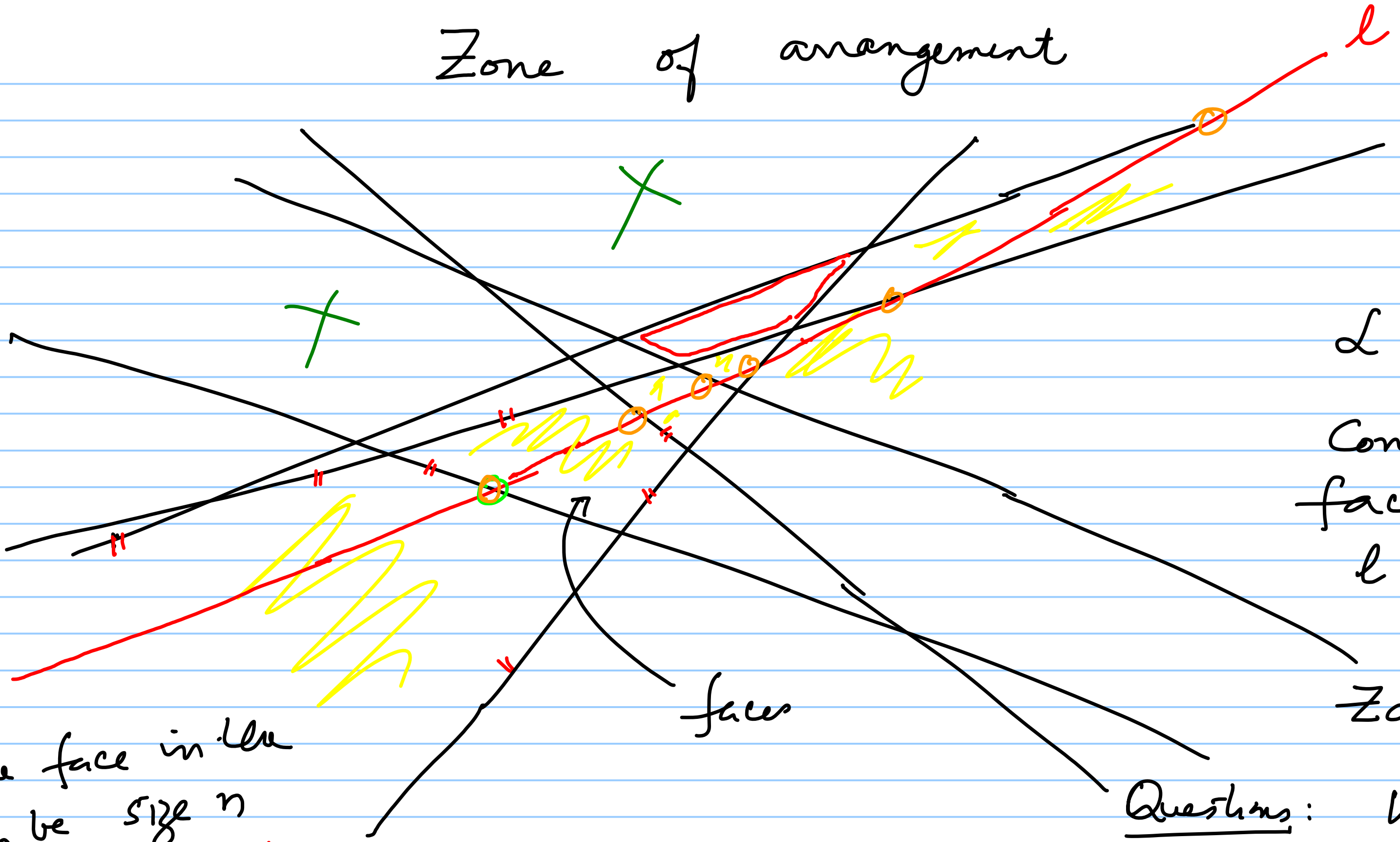
P : set of point



$D^*(P)$ is the lifted transformation where the planes are the tangential planes to the unit paraboloid

The k-th level of $A(D^*(P))$ corresponds to $Vor^k(P)$

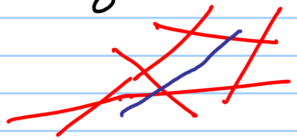
Zone of arrangement



Given lines $\mathcal{L} = \{l_1, l_2 \cdots l_n\}$

Consider all faces of $\mathcal{L}$ that l intersects. This is called the Zone$(l, \mathcal{A}(\mathcal{L}))$ of l

A single face in the zone can be size n



size: $\sum_{e \in \text{Zone}} e \cap \text{Zone} \neq \phi$

Questions: What is the size of the zone (in the worst case)?

Theorem : Zone $(\ell, A(L)) = O(n)$

Use of Zone theorem : How to construct the arrangement

$O(n^2 \log n)$ is easy!

Zone can be constructed incrementally by adding the lines in any sequence and maintain the data structure

$$\sum_{1 \le i \le n} O(i) = O(n^2)$$

Size of level k of arrangement $A^k(\mathcal{L})$

$$\sum_{k=0}^{n-1} A^k(\mathcal{L}) = O(n^2)$$ bounded by the total size of $A(\mathcal{L})$

A single level is not easy except $O(n)$ bound for $A^0(\mathcal{L})$ and $A^{n-1}(\mathcal{L})$

I.e. size of k^{th} level can be bounded by * $O(n k^{1/3})$ [Dey 98]

$$|A^{\le k}(\mathcal{L})| \le O(nk)$$ using Probabilistic Method $\le O(n^{4/3}) \ll O(n^2)$

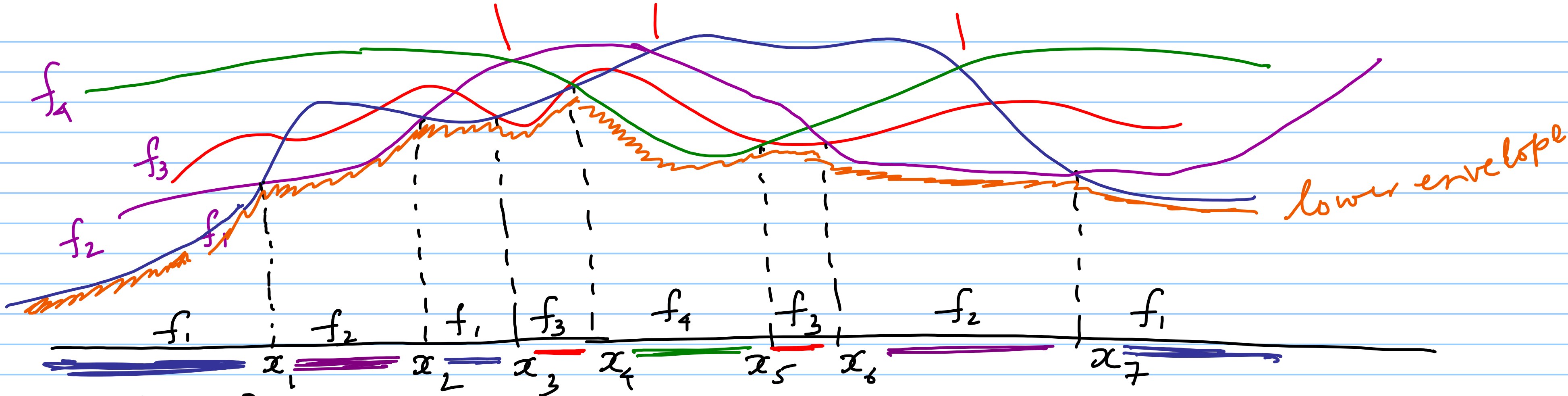
* $n \cdot \min\{k, n-k\}$

The above bounds also hold good for $\mathrm{Vor}^k(P)$

$\Rightarrow$ Nearest and Furthest neighbor Voronoi diagram has $O(n)$ size

since intersection of n half-planes in $\mathbb{R}^3$ has $O(n)$ size

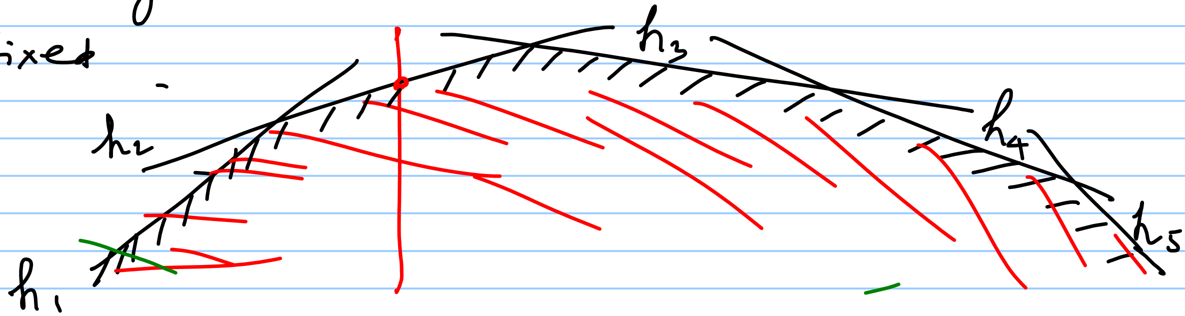
The size of Voronoi diagram in $\mathbb{R}^d$ to convex hulls in $\mathbb{R}^{d+1}$



$F(x) \; \min_i \{f_i(x)\}$

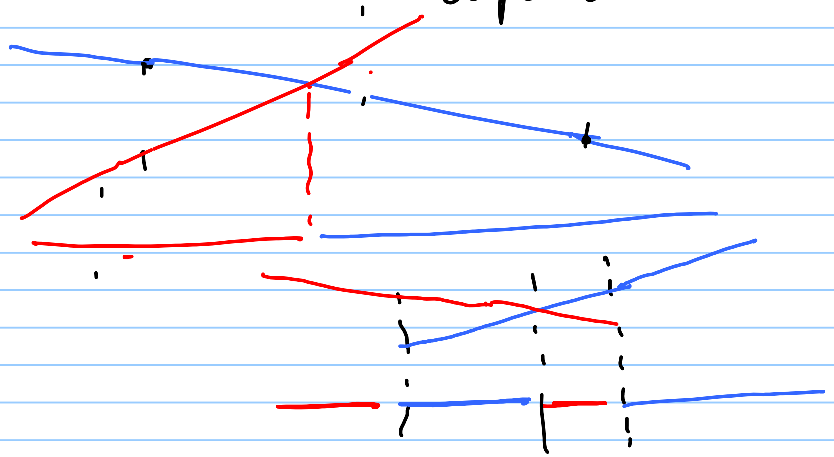
Minimization diagram are maximal intervals over which (argmin) f_i is fixed.

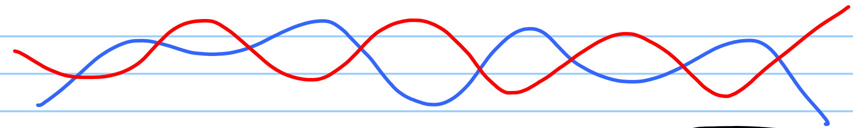
Size of minimization diagram is the # maximal intervals



Complexity of lower envelope

Clearly depends on the nature of f_i : more specifically it depends on how a pair of functions f_i, f_j intersect





Davenport-Schinzel sequences

$\lambda_k(n)$ n functions where a pair has atmost k "intersection"

alterations

What is the size of lower envelop of n line segments?

$$O(n\,\alpha(n))$$

Inverse Ackerman function

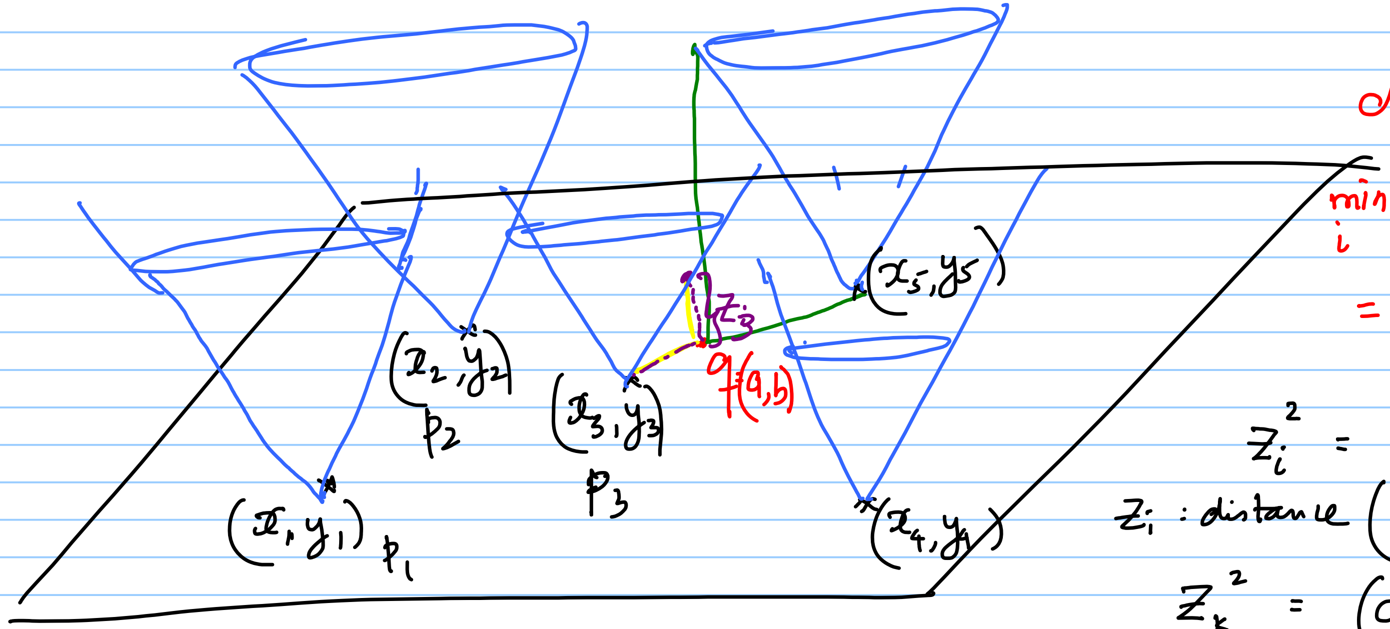
$\log^*(n)$ #times we take log to reduce to 1

$\lambda_4(n)$

base 2

$\log^*(2) = 1$ $\quad \log_*(2^2) = 1+1$ $\quad \log^*(2^{2^2}) = 3$ $\quad \log^*(\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{i}) \sim i$

Voronoi Diagrams as minimization diagram (lower envelopes)



distance (q, p_i) $\quad p_i : (x_i, y_i)$

$$\min_i \sqrt{(a-x_i)^2 + (b-y_i)^2}$$

$$= \min_i \; (a - x_i^2)^2 + (b-y_i)^2$$

$$z_i^2 = (a-x_i)^2 + (b-y_i)^2$$

z_i : distance (q, p_i)

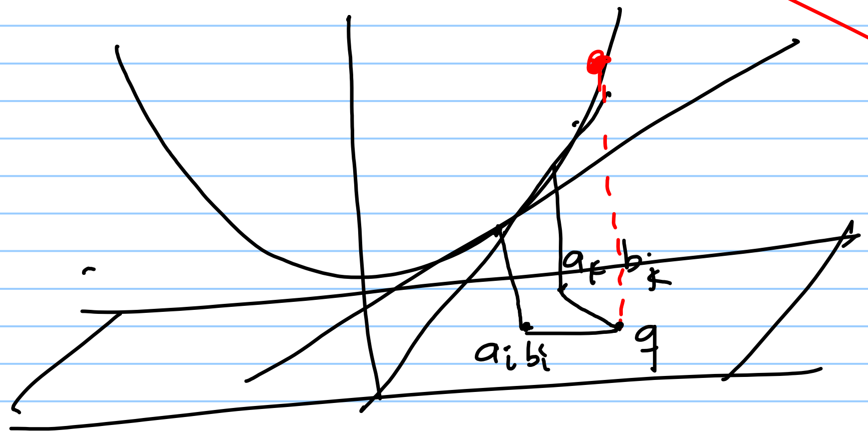
$$z_k^2 = (a-x_k)^2 + (b-y_k)^2$$

$$z_i^2 - z_k^2 = x_i^2 + y_i^2 - x_k^2 - y_k^2 + 2a x_k - 2a x_i + 2b y_k - 2b y_i$$

x_i, y_i are fixed for a given set of points, rename them as a_i, b_i and the query point (a, b) to be (x, y)

Then

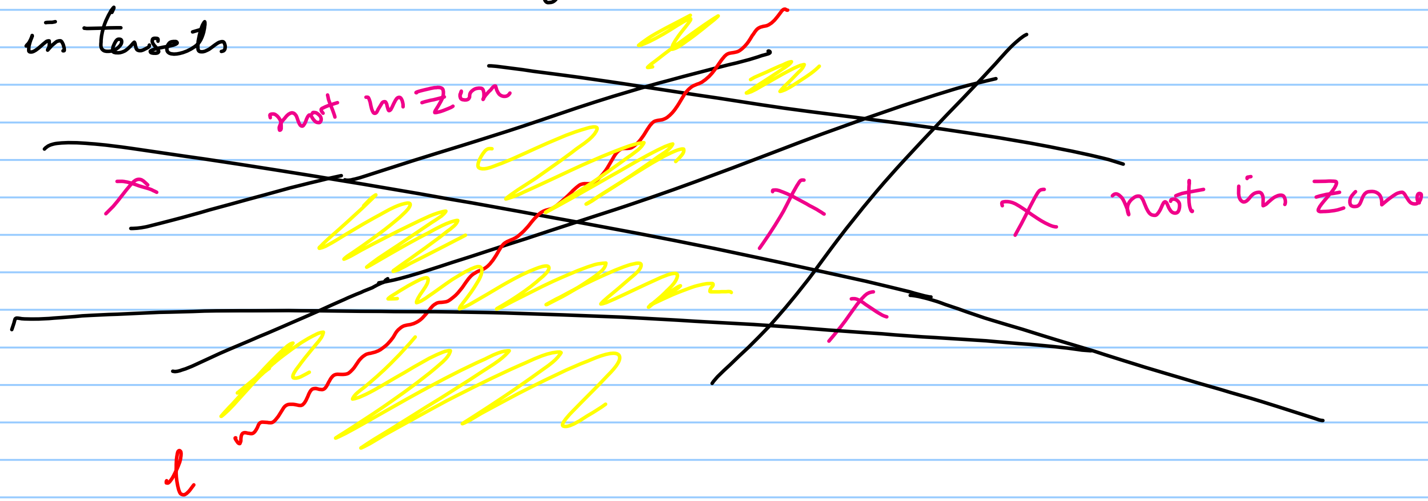
$$z_i^2 - z_k^2 = \boxed{a_i^2 + b_i^2} - a_k^2 - b_k^2 + 2x a_k - \boxed{2x a_i} + 2y b_k - \boxed{2y b_i} \leq 0$$



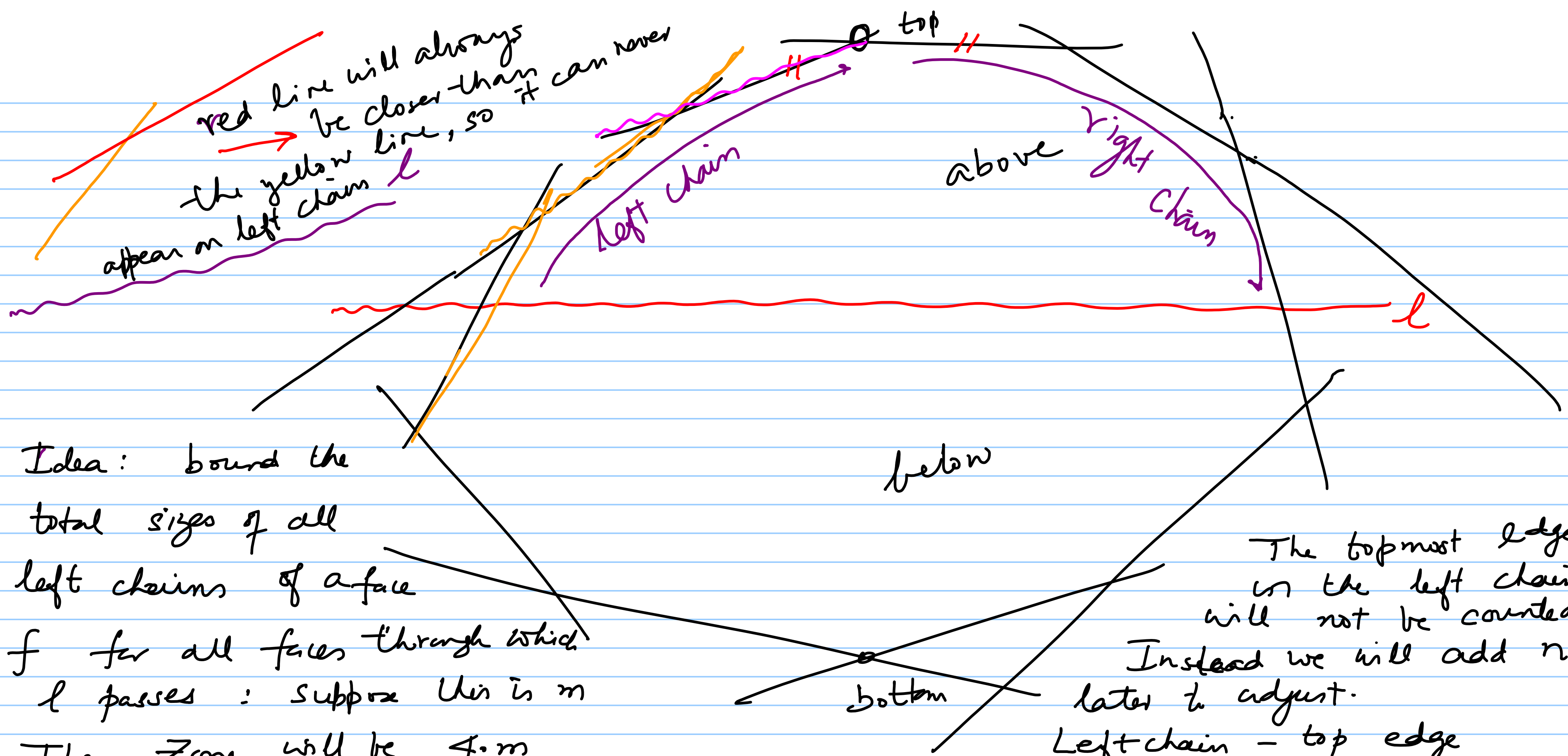
Zone theorem

Input $\mathcal{L} = (\ell_1, \ell_2, \dots \ell_n)$

For a given set of n lines, the zone$(\ell, A(\mathcal{L}))$ is the set of all edges contained in the faces that ℓ intersects



$|Zone(\ell, A(\mathcal{L}))| = O(n)$

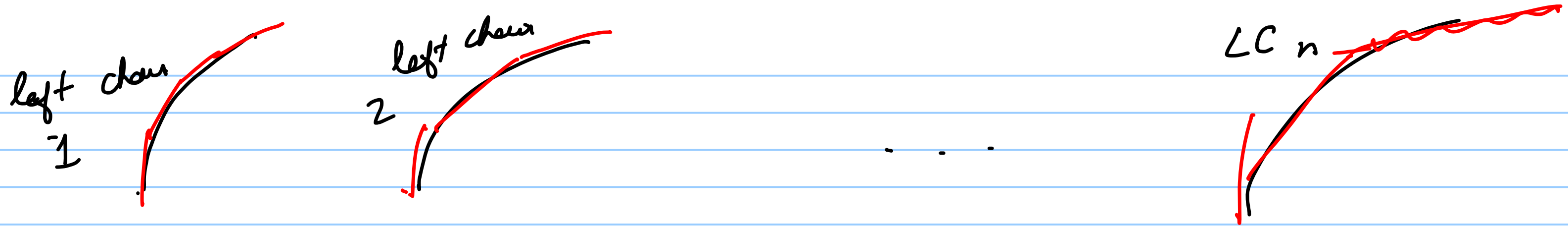


Idea: bound the total sizes of all left chains of a face f for all faces through which l passes : suppose this is m

Then Zone will be 4·m

The topmost edge in the left chain will not be counted. Instead we will add n later to adjust.

Left chain − top edge



<u>Claim</u> : No line contributes to more than 1 edge in the left chains

$\Rightarrow$ Sum of left chains is n Including top edge it is $2n$

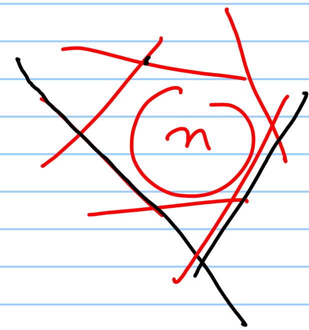
Right chains also bounded by $2n$

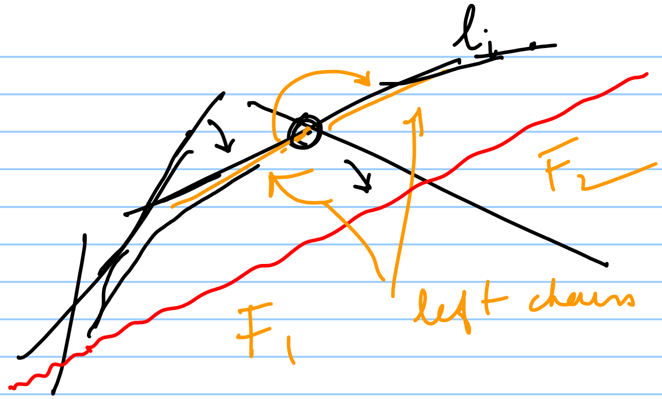
$\Rightarrow$ Total size of zone $\leq (4n) \times 2 = 8n = O(n)$

$$\sum_{f_i \in Zone} |f_i| = O(n)$$

Using this result one can prove

$$\sum_{f_i \in A(L)} |f_i|^2 = O(n^2)$$




l_j
F_1
F_2
left chains