Note Title 12-06-2019

Set Cover:

ILP $\to$ LP $\to$ round.

For $i = 1..t$

Pick each set w/ prob. x_S

& add to T_i

Return $\cup T_i$

① $\mathbb{P}r[T = \cup T_i$ is a set cover$] \geq 1 - \frac{1}{n}$. $(n = |X|)$

② $\mathbb{E}|T| \leq \alpha \cdot OPT$.

Thm: $\mathbb{E}|T| \leq O(\log n)\, OPT$.

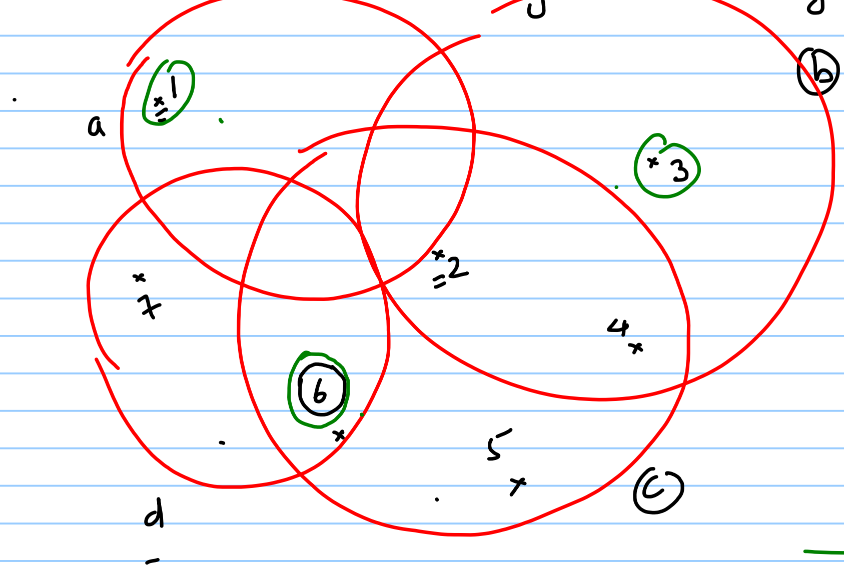
Proof: $\mathbb{E}[|T_i|] = \sum x_S$ [linearity of expectation]

$= OPT_{LP} \leq OPT$

$$\mathbb{E}[|T|] \leq 2\ln n \cdot OPT_{LP} \leq 2\ln n \cdot OPT \quad \square$$

Thm [Feige '02, Lund & Yannakakis '98]: Better than a $c\ln n$-approx. for Set Cover would imply $P \neq NP$.

. Look at Set Cover in a ~~geometric~~ Setting.



$$\mathcal{H} = (X, \mathcal{S})$$

$$X = \{1, \dots, 7\}, \quad \mathcal{S} = \{a, b, c, d\}.$$

Given a set System

$$\mathcal{H} = (X, \mathcal{S})$$

$$\downarrow$$

Dual set System

$$\mathcal{H}^* = (\mathcal{S}, \{S_x\}_{x \in X})$$

$$\mathcal{H}^* = \{ \{a, b, c, d\}, \{a\}_1, \{b, c\}_2, \{d, c\}, \dots \}$$

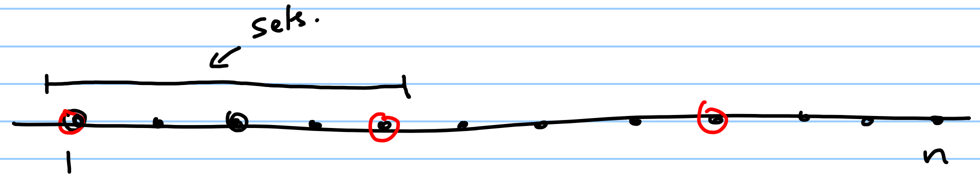
Hitting set: Pick $Y \subseteq X$, of smallest size:

$$\forall S \in \mathcal{S}, \quad Y \cap S \neq \emptyset$$

ε-nets:

- Given $(X, \mathcal{S})$, $|X| = n$, a set $S \in \mathcal{S}$ is large if $|S| \geq \varepsilon n$; for some $\varepsilon > 0$.
- A set $Y \subseteq X$ is an ε-net if $Y \cap S \neq \emptyset$ $\forall$ S large.

Example.



Qn: How large should Y be?

- Pick every $(1/\varepsilon)^{th}$ point.
- Then $|Y| = O\left(\frac{1}{\varepsilon}\right)$

The size of the ε-net is independent of n!

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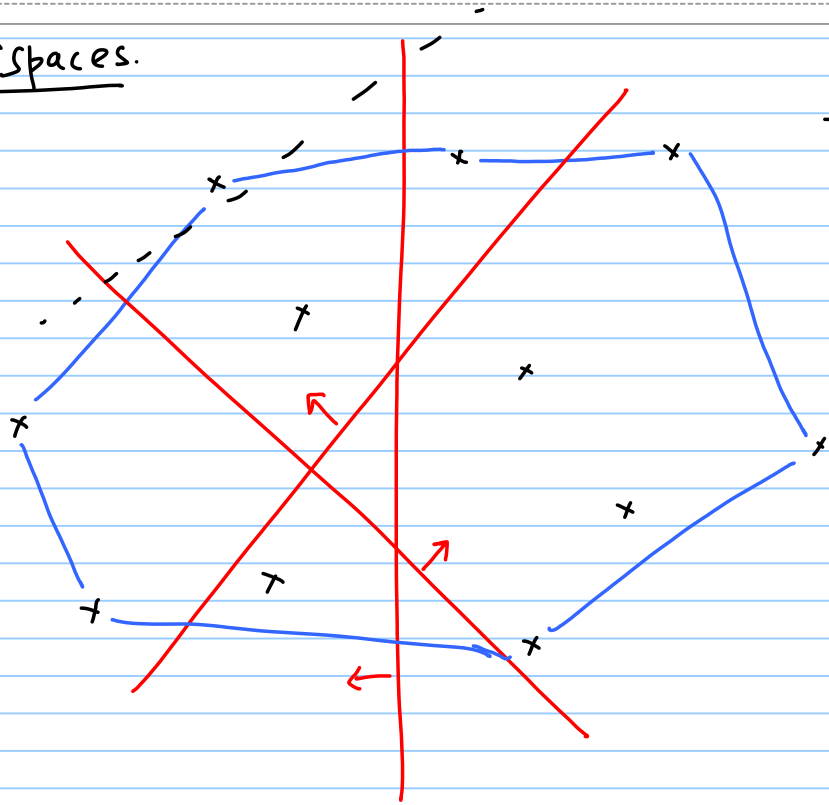
Halfspaces.

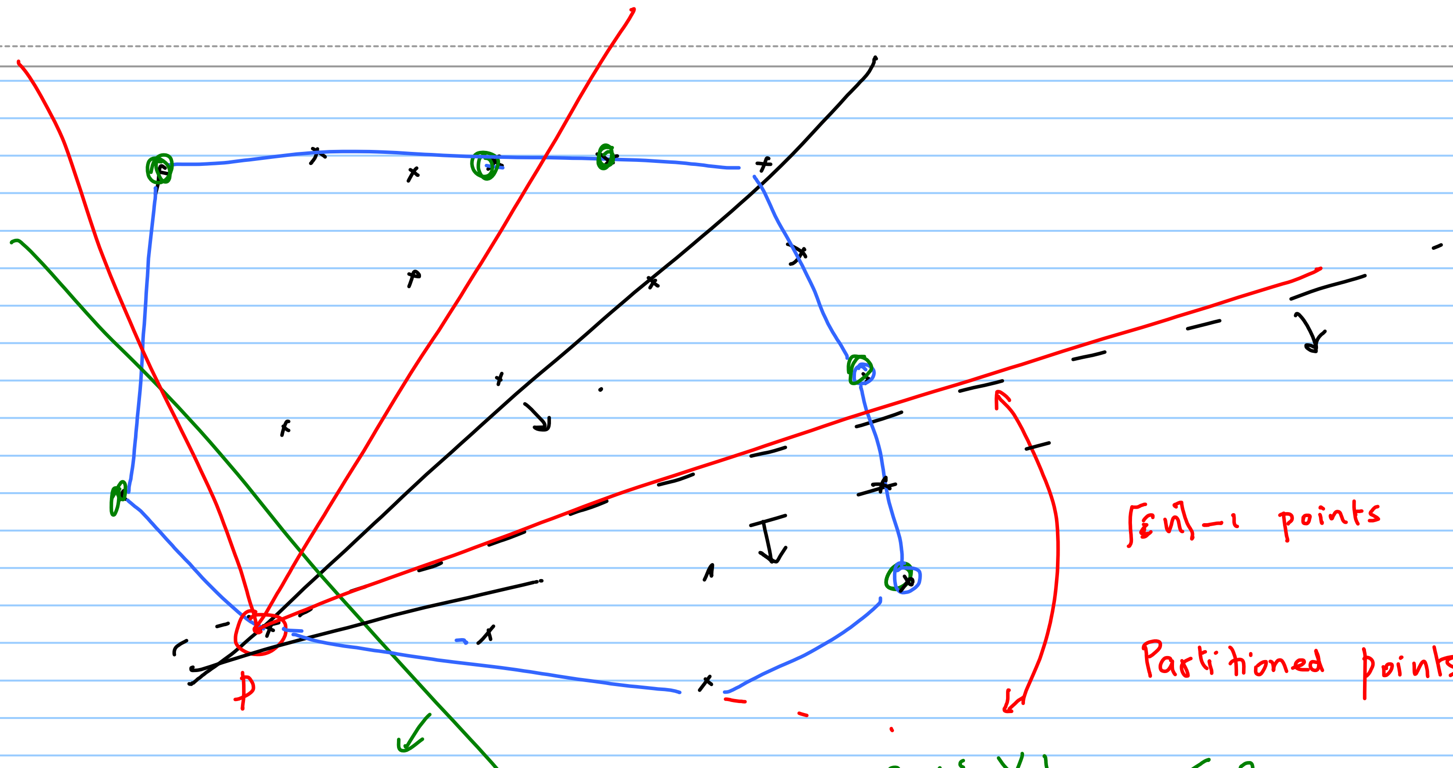
- $X \subset \mathbb{R}^2$, $|X| = n$.
- Sets: Halfspaces in $\mathbb{R}^2$

Qn: How large should Y be?

$Y \subset X$: $Y \cap H \neq \phi$ $\forall$ H. $|H| \geq \varepsilon n$.



Thm: There exist ε-nets of size $O(1/\varepsilon)$.



Claim: Any halfspace containing $\geq \varepsilon n$ points must

$P \cup \{ Y \} = \bullet \leq \frac{2}{\varepsilon}$

Claim: Y is an ε-net.

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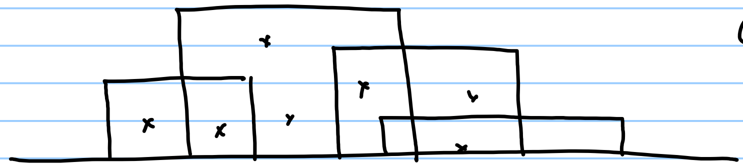
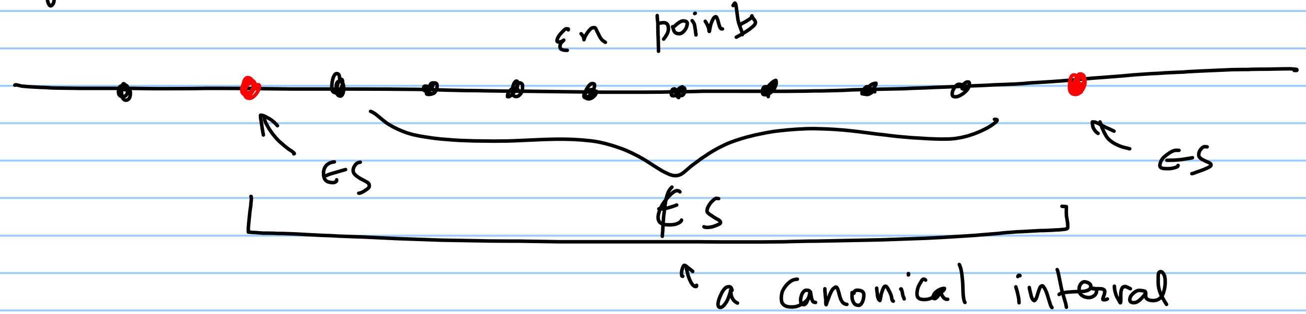


<u>Claim</u>. $\exists$ ε-nets of size $O(1/\varepsilon)$.

Random Sampling.

- Let $S \subseteq X$: constructed by choosing each point of X w/ prob. p.
- We will show that for a suitable p, S is an ε-net, and $|S|$ is independent of n.

When does S fail to be an ε-net?



- The probability of obtaining a canonical interval defined by 2 points.

$$= \underbrace{p^2}_{\text{the 2 end points chosen}} \underbrace{(1-p)^{\varepsilon n}}_{\text{none of the } \varepsilon n \text{ points are chosen.}} \le p^2 e^{-p\varepsilon n} \quad \text{— (1)}$$

- # Canonical intervals $\le n^2$ — (2)

- The expected # of Canonical intervals $\le \underline{n^2 p^2} e^{-p\varepsilon n}$ — (3) from (1), (2)

For: $\underline{p = \frac{c}{\varepsilon n} \ln\left(\frac{1}{\varepsilon}\right)}$, $c > 0$ (3) < 1.

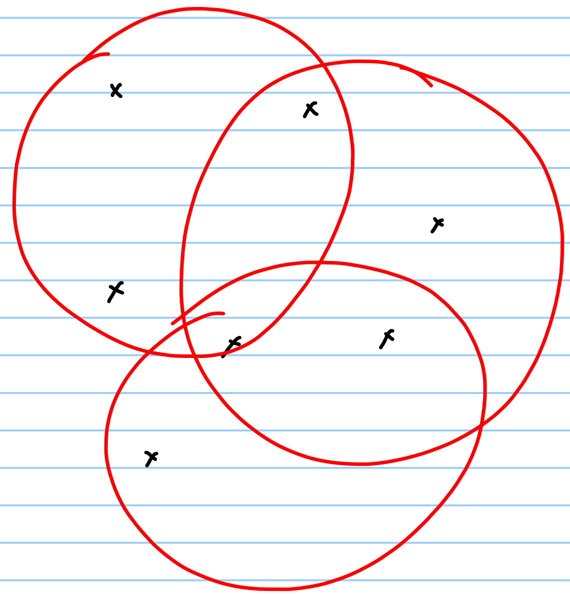
$$\cancel{n^2} \cdot \frac{c^2}{\varepsilon^2 \cancel{n^2}} \ln^2\left(\frac{1}{\varepsilon}\right) \exp\left(-c \frac{\cancel{\varepsilon n}}{\cancel{\varepsilon n}} \cdot \ln\left(\frac{1}{\varepsilon}\right)\right)$$

Since the expectation $< 1 \Rightarrow \exists$ a set S : there are no canonical intervals

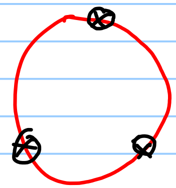
$\Rightarrow S$ is an ε-net.

$$\mathbb{E}[|S|] = np = \cancel{n} \frac{c}{\varepsilon \cancel{n}} \ln\left(\frac{1}{\varepsilon}\right) = O\left(\frac{1}{\varepsilon} \underline{\ln\left(\frac{1}{\varepsilon}\right)}\right)$$

$\boxminus$



Canonical Disk.

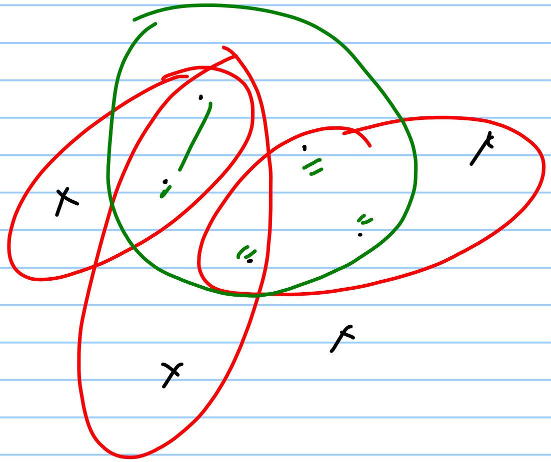


Claim: Every disk containing $\geq \varepsilon n$ points is contained in the union of 2 canonical disks.

Choose points w/ prob. $p = \frac{c}{\varepsilon n} \ln\left(\frac{1}{\varepsilon}\right)$

$\Rightarrow$ If we compute an $\varepsilon/2$- net for canonical disks, we obtain an ε-net for disks

$(X, \mathcal{S})$, $\quad Y \subseteq X$, $\quad \mathcal{S}|_Y = \{ S \cap Y : S \in \mathcal{S} \}$.



The condition we will require:

$\forall\, Y \subseteq X,\ |Y| = n$

$$|\mathcal{S}|_Y| = O(n^d)$$

VC-dimension

Vapnik - Chervonenkis dimension.

$|\mathcal{S}|_Y| = 2^{|Y|}$, $|Y| \leq d$.

For $(X, \mathcal{S})$, If d is the largest subset of X that can be shattered by $\mathcal{S}$, then the VC-dim of $(X, \mathcal{S}) := d$.

Thm. If VC-dim $= d$, $|\mathcal{S}| = O(n^d)$.

Thm: ε-net thm: A set system of VC-dim d have ε-nets of size $O\left(d/\varepsilon \ln (d/\varepsilon)\right)$.

ε-net (weighted): $(X, \mathcal{S})$, $w: X \to \mathbb{R}_+$, $\sum w(x) = 1$, A set is large if $w(S) \geq \varepsilon$.

LP : Hitting Set

$$\min \sum_{v \in X} x_v$$

s.t.

$$\sum_{v \in S} x_S \geq 1 \quad \forall x \in X$$

$$x_v \geq 0.$$

For general Set Systems: $O(\ln n)$-approx.

Thm: Let $(X, \mathcal{S})$ be a set system with VC-dimension d.

Then, $\exists$ a $\log(OPT)$-approx. for Set Cover.

Pf: ① Solve the LP: Let x be the soln.

② Let $W = OPT_{LP}$.

③ For each $v \in X$, set $\omega(v) = \frac{x_v}{W}$.

④ Since x is feasible for the LP:

$$\boxed{\sum_{v \in S} x_S \geq 1 \Rightarrow \sum \frac{x_S}{W} \geq \frac{1}{W}.}$$

⑤ Construct a $\frac{1}{W}$-net for $(X, \mathcal{S})$

By the ε-net theorem, $\exists$ an ε-net of size: $O\left(\frac{d}{\varepsilon} \ln \frac{d}{\varepsilon}\right) = O(W \ln W)$

$$= O(OPT_{LP} \ln OPT_{LP})$$

$$\leq O(OPT \ln OPT)$$

$\Rightarrow$ We have an $O(\ln OPT)$-approx for hitting set.

$\square$