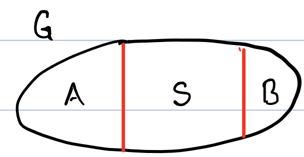
Separators

- Let $G=(V,E)$ be a graph.
- A set $S \subseteq V$ is a separator if $V \setminus S$ separates into two components A, B, with no edges between $u \in A$ & $v \in B$. [Note: either A, or B may be empty]. ie: all paths between a vertex in A, and a vertex in B go through S.



- A balanced separator is a separator $S \subseteq V$, such that $|A|, |B| \leq c.|V|$, for some constant $c < 1$.

- We will be interested in classes of graphs that admit a balanced separator of sublinear size, ie: of size $O(|V|^{1-\delta})$, for some constant $\delta > 0$.

- There are several interesting graph classes that admit such balanced separators of sub-linear size.

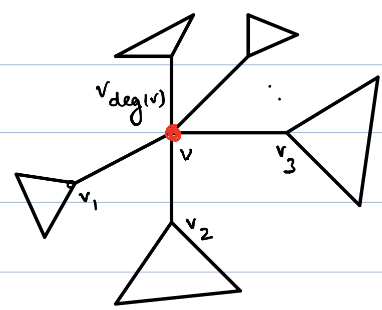
- Since the only separators we will be interested in are ones that are balanced, we will drop the adjective balanced in the rest of these notes.

Example: Let $T=(V,E)$ be a tree.

$\exists S \subseteq V$, $|S| = 1$, such that $|A|, |B| \leq \frac{2}{3}|V|$.

i.e., every tree has a balanced separator of size 1.

- Start with an arbitrary vertex $v \in V$.
- Removing v disconnects T into $\deg(v)$ connected components.



- Let $T_1, \ldots, T_{\deg(v)}$ be the components, ordered such that: $|T_1| \geq |T_2| \geq \cdots \geq |T_{\deg(v)}|$, where T_i is rooted at v_i; neighbor of v.

- If $|T_1| \leq \frac{1}{3}|V|$

  Then, let k be the first index at which $\sum_{i=1}^{k} |T_i| > \frac{2}{3}|V|$.

- Let $A = \bigcup_{i=1}^{k-1} T_i$. Hence, $|A| < \frac{2}{3}|V|$.

- Since $|T_i| < \frac{|V|}{3}$, $\forall i$, $|A| \geq \frac{1}{3}|V| \Rightarrow |B| \leq \frac{2}{3}|V|$.

- Suppose $\frac{1}{3}|V| < |T_1| \leq \frac{2}{3}|V|$

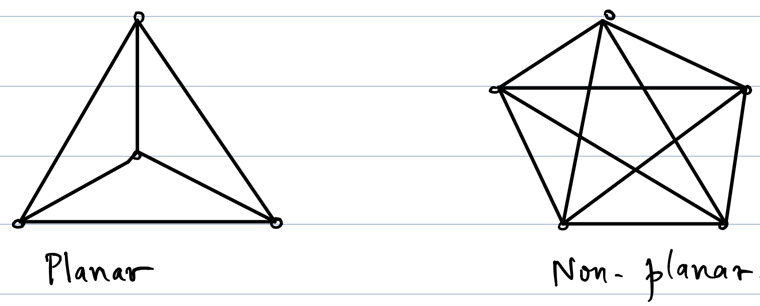
- So, suppose $|T_1| > \frac{2|V|}{3}$.

- We apply the argument above vilth v_1 in place of v.

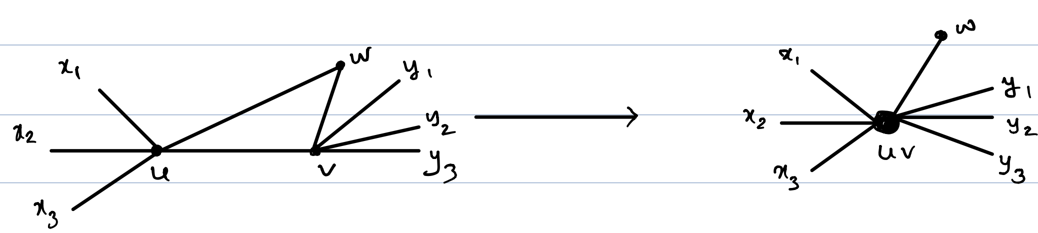
- The process must halt since we are decreasing the size of the largest component created at each step.

Planar Graphs

- A graph that can be drawn in the plane, where the vertices are points, & edges are continuous curves between the end-points, such that no pair of edges share a point in their interior is called a planar graph.



Edge contraction: A fundamental operation on planar graphs (that preserves planarity) is edge-contraction:



More formally, given $G=(V,E)$,

$$G/\{u,v\} = \left(\left(V \setminus \{u,v\}\right) \cup \{uv\},\ E \cup F_{uv}\right),$$

where

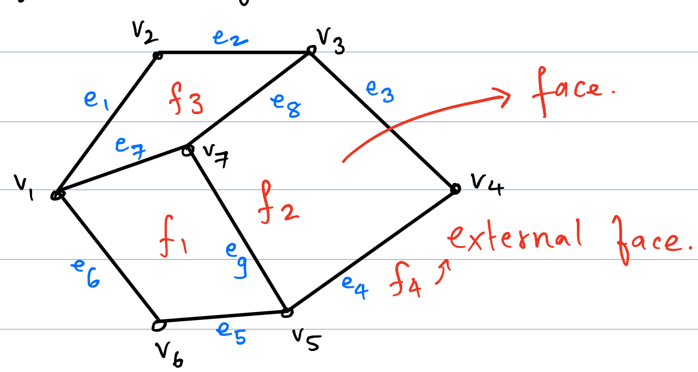
$$F_{uv} = \{ \{uv, x\} : \{x,u\} \in E \text{ or } \{x,v\} \in E \}.$$

Facts about planar graphs.

Theorem [Kuratowski] A graph is planar if and only if we can not obtain K_5, or $K_{3,3}$ via a sequence of edge-contractions & vertex deletions.

Plane drawing: Drawing of a planar graph, such that no pair of edges intersect in their interior.

Theorem [Fáry '40's]: Every planar graph admits a plane drawing with straight-line edges.



Euler's theorem: For any connected planar graph $G=(V,E)$,

$$|V| - |E| + |F| = 2.$$

In the example above: $\underbrace{|V|}_{7} - \underbrace{|E|}_{9} + \underbrace{|F|}_{4} = 2$

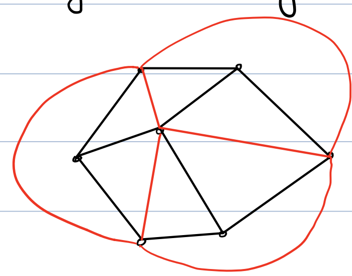
Theorem [4-color theorem]: The vertices of a planar graph can be colored with at most 4 colors so that adjacent vertices receive distinct colors.

- This is a famous theorem, proved by Appel & Haken '71, which extensively used computers in the proof.

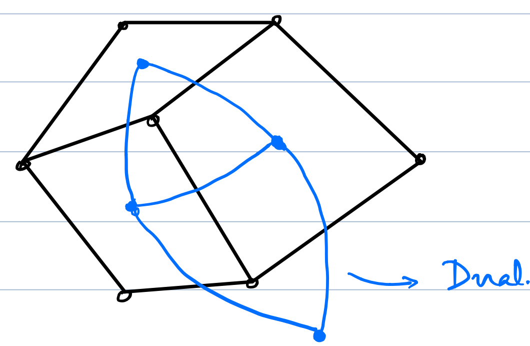
Triangulated planar graphs:

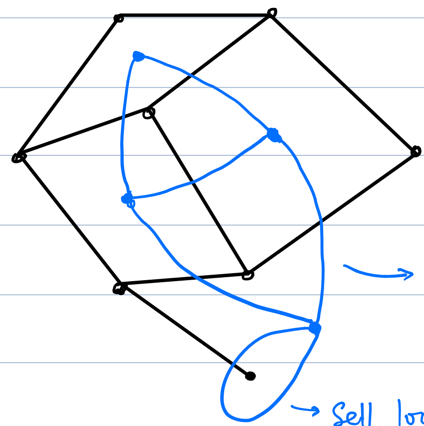
Every face, including the external face is a $\triangle$.

- We can add edges to triangulate a planar graph.

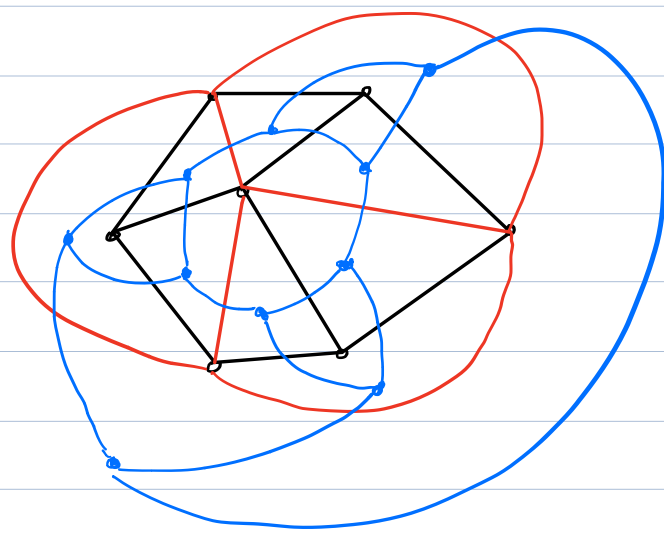


Dual: Every planar graph admits a dual: the vertices of the dual are the faces, and two faces are adjacent if they share an edge.





Lemma: Let $G=(V,E)$ be a triangulated planar graph.
Then, $G^*=(V^*,E^*)$, the planar dual of G is
a ?



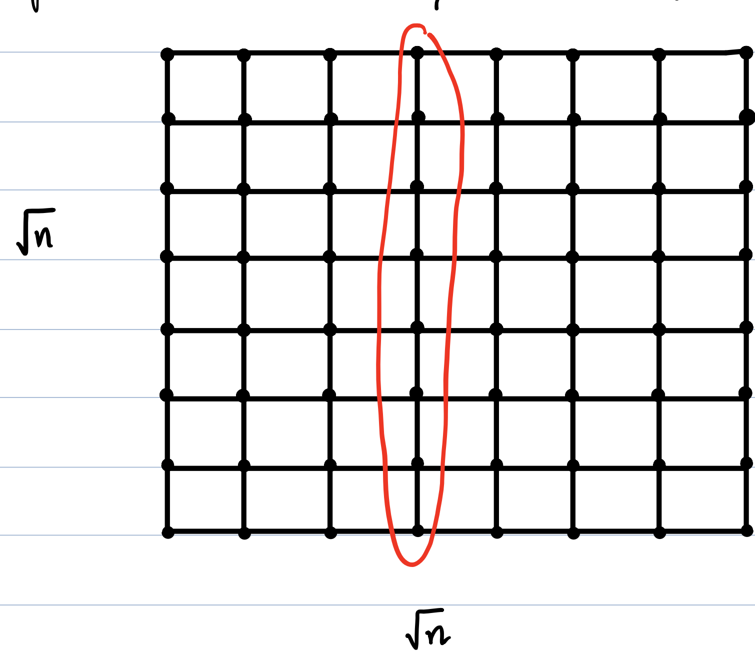
Separators in planar graphs

$G=(V,E)$

Theorem [Lipton, Tarjan '79] Every planar graph, admits a separator of size $O(\sqrt{n})$, such that $|A|, |B| \le \frac{2}{3}|V|$.

Intuitively:

Separator of size $\sqrt{n}$.



Theorem [Chan '03], based on [Smith, Wormald '98]

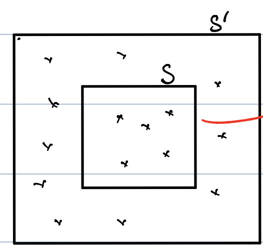
- Let D be a set of n disjoint disks in the plane. There exists a square S such that:

(i) the number of disks lying entirely inside/outside is at most $4n/5$

(ii) the number of disks intersecting the boundary ∂S of S is $O(\sqrt{n})$.

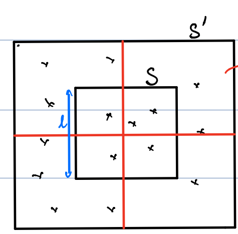
Proof: • Let S be the smallest square containing $1/5$ of the centers of the disks. Let l be the side-length of S.

- Let S' be a square of size twice the size of S. with the same center as S.



smallest square containing $\geq \frac{1}{5}$ of the centers of disks.

Obs 1: S' contains $\leq 4n/5$ centers in its interior, and $\geq n/5$ centers in its exterior.



- Each small square has area $\leq$ area(S).
- By choice of S, each small square contains $< n/5$ centers.

- Since S' can be covered with 4 squares of area $\leq$ area(S), and S is the smallest square containing $n/5$ centers, S' contains $\leq 4n/5$ centers.
- The exterior of S', therefore contains $\geq n - 4n/5 = 1/5$ centers.

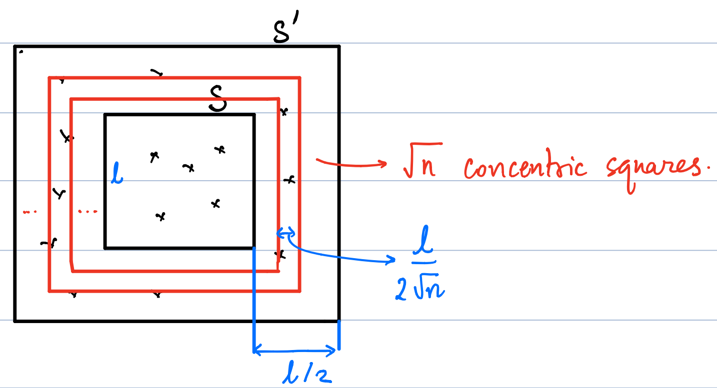
It remains to show that $\exists$ a square between S & S' intersecting $O(\sqrt{n})$ disks.

Let a disk D be called large if $radius(D) \geq \frac{l}{2\sqrt{n}}$

Otherwise, the disk is called small.

Construct $\sqrt{n}$ concentric squares between S & S', equidistant.

Obs 2: The distance between 2 consecutive squares is $l/\sqrt{n}$.



Obs 3: Pick any square S^* between S and S'. The perimeter of S^* is $< 4l$. Hence, the boundary of S^* can be covered with at most $\frac{4l\sqrt{n}}{\sqrt{2}l} < 4l\sqrt{n}$ boxes of side length $\sqrt{2}l/\sqrt{n}$.

the number of large disks intersecting the boundary of $S^* \leq 4\sqrt{n}$.

Obs 4: A small disk intersects at most one concentric square between S & S':

- Each small disk has radius $< \ell/2\sqrt{n}$.
- The distance between two consecutive disks is: $\frac{\ell}{\sqrt{n}}$.

- If each of the $\sqrt{n}$ squares intersect $> \sqrt{n}$ small disks, the total number of small disks $> \sqrt{n}$, a contradiction.

- Therefore, there exists a square intersecting at most $\sqrt{n}$ small disks.

- From Observations 3 & 4, we obtain a square intersecting at most $5\sqrt{n}$ disks.

- Since the interior & exterior each contain $\leq 4n/5$ disks, we obtain a separator.

$\square$

The theorem above can be extended to higher dimensions.

Theorem [Chan'03, Smith & Wormald'98] Given a set of n disjoint balls in $\mathbb{R}^d$, There exists a box B such that:

(i) $int(B)$, $ext(B)$ contain $\leq dn/(d+1)$ balls.

(ii) The boundary B intersects $O(n^{1-d})$ balls.

Planar Separator theorem from the separator theorem for balls:

The Koebe-Andreev-Thurston Theorem

Every planar graph $G=(V,E)$ can be represented as the "touching graph" of interior-disjoint disks in the plane.
ie: We can assign a disk D_v corresponding to each $v \in V$, such that the disks are interior-disjoint; and such that two disks D_u, D_v touch if and only if $\{u,v\} \in E$.

· From the Koebe-Andreev-Thurston theorem, we obtain a set of touching disks. Shrinking the disks slightly, we obtain a set of disjoint disks. Now, we apply the theorem above for disjoint disks to obtain a separator theorem for planar graphs.

Weighted Separators:

We can extend the separator theorem to a weighted set of disjoint balls. However, note that the theorem only guarantees that the number of disks intersecting the boundary of a separating box is $O(n^{1-d})$, and not the weight of the disks. The total weight of the disks in the interior, and exterior can be bounded by $2/3\,W$, where W is the total weight of the balls.

Theorem: Given a set of disjoint balls $\mathcal{D}$ in $\mathbb{R}^d$, with a weight fn. $\omega : \mathcal{D} \to \mathbb{R}_+$, $\sum_{D \in \mathcal{D}} w(D) = W$, there exists a box B in $\mathbb{R}^d$, such that

(i) $\sum_{D \in int(B)} w(D), \sum_{D \in ext(B)} w(D) \leq \frac{2}{3} W$.

(ii) The number of balls intersecting the boundary of B is $O(n^{1-d})$.

. The original proof of Lipton & Tarjan for the planar separator theorem also allowed weights on the faces.

Theorem [Lipton-Tarjan '79] Let $G=(V,E)$ be a planar graph set of faces. Let $\omega : V \to \mathbb{R}_+$ be a weight function, and assume that $\sum_{x \in V \cup F} \omega(x) = 1$.

There exists $S \subseteq V$, such that: $G \setminus S$ separates into 2 components A, B:

(i) $\sum_{x \in A} \omega(x) \leq \frac{2}{3}$ $\sum_{x \in B} \omega(x) \leq \frac{2}{3}$

(ii) $|S| \leq 2\sqrt{2}\sqrt{n}$.

Cycle Separators:

- For triangulated planar graphs, we can obtain a seperator that is a simple cycle in G.

Theorem [Miller'80s] For a triangulated planar graph $G=(V,E)$, there with weights $w: V \to \mathbb{R}_+$, $\sum_{v \in V} w(x) = 1$, exists a simple cycle C in G, such that

(i) $\sum_{x \in int(C)} w(x)$, $\sum_{x \in ext(C)} w(x) \leq 2/3$.

(ii) $|C| = O(\sqrt{n})$.

Independent Sets in planar graphs

- Let $G=(V,E)$ be a planar graph. We want to compute a maximum independent set (MIS) K of G.

- We saw earlier that if G is an arbitrary graph, the MIS problem cannot be approximated beyond $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless $NP = ZPP$.

- Computing MIS for planar graphs is NP-hard. However, we can obtain better approximation factors than for general graphs.

- In fact, we can obtain a PTAS (polynomial-time approximation Scheme).

  i.e. $\forall \varepsilon > 0$, we can obtain a $(1-\varepsilon)$-approximation in time $O(n^{O(1/\varepsilon)})$.

  ie: For any constant $\varepsilon > 0$, we obtain an independent set K, such that $|K| \geq (1-\varepsilon)OPT$, and the algorithm runs in polynomial time.

- However, we pay for the accuracy of the solution in the running time.

A PTAS for MIS in planar graphs:

- The algorithm is a divide-and-conquer algorithm:

Alg:

- We repeatedly find a separator in each connected component of the graph, until each connected component has at most $1/\varepsilon^2$ vertices.
- We compute an independent set in each component by brute-force, ignoring the vertices in the separators.
- Since we removed a separator at each step, the union of independent sets from each component is an independent set.

- We will show that the total number of vertices in the separators is at most $O(\varepsilon n)$. Assuming this fact for now, we can prove that the algorithm above is a PTAS.

Theorem: ALG is a PTAS for MIS in planar graphs.

Proof: • By the 4-color theorem, every planar graph can be colored with 4 colors.

- Each color class is an independent set; the largest independent set has size $\geq n/4$. —(1)
- Fix an optimal independent set OPT.
- For a component C_i on which we apply brute-force, let $OPT_i = C_i \cap OPT$.

- Let K_i be the solution returned by the brute-force algorithm for this component.

Obs: $|K_i| \geq |OPT_i|$.

- Therefore, $\sum_{i \in \mathcal{C}} |K_i| \geq \sum_{i \in \mathcal{C}} |OPT_i|$, where $\mathcal{C}$ is the final set of connected components on which we apply brute-force.
- $|OPT| = \sum_{i \in \mathcal{C}} |OPT_i| + |OPT \cap S|$, where S is the set of vertices in the separator.
- $$\begin{aligned} |OPT| &\leq \sum |OPT_i| + |S| \\ &\leq \sum |OPT_i| + c\varepsilon n \\ &\leq \sum |OPT_i| + 4c\varepsilon(n/4) \\ &\leq \sum |OPT_i| + 4c\varepsilon\, OPT \qquad \text{[from (1)]} \\ &\leq \sum |K_i| + 4c\varepsilon\, OPT \end{aligned}$$

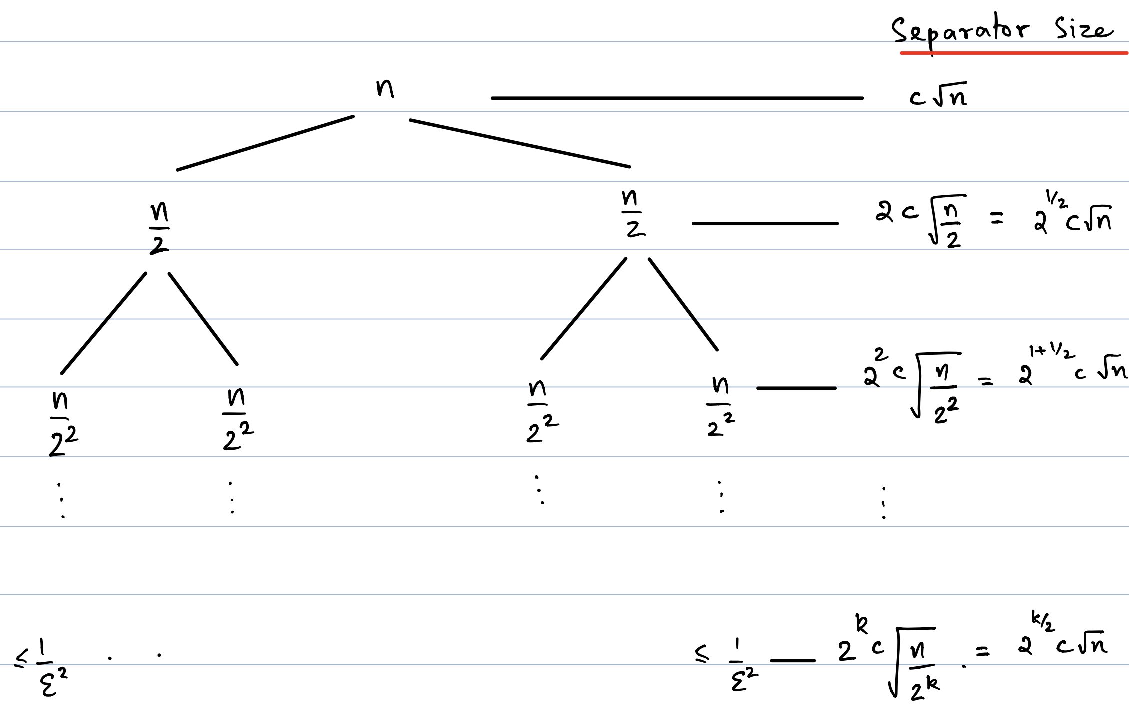
$\Rightarrow (1-4c\varepsilon)|OPT| \leq \sum |K_i| =$ Solution returned by our algorithm.

$\blacksquare$

- It remains to show that if we repeatedly remove a separator from a connected component with $\geq 1/\varepsilon^2$ vertices, the total number of vertices removed is $\leq \varepsilon n$.

- Before we do this formally, let us use some rough calculation to build intuition.
- Assume that at each step, we get perfect separation, ie: on removing a separator the two components have size at most $n/2$.

- We build a recursion tree, where the node shows the size of the component & on the right, we list the total size of the separators we obtain from the components at a level:



We determine k such that: $\frac{n}{2^k} \leq \frac{1}{\varepsilon^2}$. This happens for:

$$k \geq \lceil \log_2 n\varepsilon^2 \rceil.$$

- The total size of the separators we have removed at all levels is therefore:

$$c\sqrt{n} + 2^{1/2} c\sqrt{n} + 2^{3/2} c\sqrt{n} + \cdots + 2^{k/2} c\sqrt{n}$$

$$= c\sqrt{n} \sum_{i=0}^{k} 2^{k/2} \leq c'\sqrt{n}\, 2^{k/2} = c'\sqrt{n}\, 2^{(\log n\varepsilon^2)/2}, \text{ for } c' > c$$

$$\leq c'\varepsilon n$$

Now, let us do this calculation more formally:

- Let $S(n)$ be the size of the separator.
- Then, the size of the separator is given by the following recurrence:

$$S(n) = \begin{cases} c\sqrt{n} + \max\limits_{\frac{1}{3} \leq \alpha \leq \frac{2}{3}} S(\alpha n) + S((1-\alpha)n - c\sqrt{n}), & n > 1/\varepsilon^2 \\ 0, & n \leq 1/\varepsilon^2 \end{cases}$$

- How do we solve this mess?

- Put $k = 1/\varepsilon^2$.

- We solve the recursion above by induction.

<u>Lemma</u>: $S(n) \leq \frac{\beta n}{\sqrt{\gamma}} - \gamma\sqrt{n}$; for some constants $\beta, \gamma > 0$.

<u>Proof</u>: • By induction on n.

- Let us assume the base case for now; and prove the induction step.

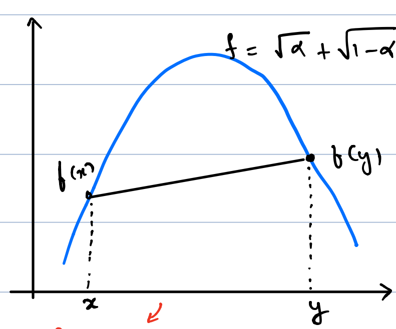
$$S(\alpha n) \leq \frac{\beta \alpha n}{\sqrt{k}} - \gamma\sqrt{\alpha n}\,;\quad S((1-\alpha)n) \leq \frac{\beta(1-\alpha)n}{\sqrt{k}} - \gamma\sqrt{(1-\alpha)n}. \quad [\text{by I.H}]$$

Thus,

$$S(n) \leq c\sqrt{n} + \frac{\beta n}{\sqrt{k}} - \gamma\sqrt{n}\left(\sqrt{\alpha} + \sqrt{1-\alpha}\right) \quad \text{—— (1)}$$

<u>Claim</u>: $\sqrt{\alpha} + \sqrt{1-\alpha}$ is a <u>concave</u> fn. for $\alpha \in [1/3, 2/3]$.

A function is concave in a domain D if $f(\mu x + (1-\mu)y) \geq \mu f(x) + (1-\mu) f(y)$ $\forall\, x, y \in D$ & $\mu \in [0,1]$.



f lies above the line segment connecting f(x) & f(y).

Eg: $\log x$ is concave in $[1, \infty)$

$\sqrt{x}$ is " in $(0, \infty)$

If $\frac{d^2 f}{dx^2} < 0$, $\forall x \in D$, f is concave in D.

- Let $f(\alpha) = \sqrt{\alpha} + \sqrt{1-\alpha}$.
- We can write $\alpha = \mu \cdot \frac{1}{3} + (1-\mu)\frac{2}{3}$, $\mu \in [0,1]$.
- $f(\alpha) = f\left(\mu \frac{1}{3} + (1-\mu)\frac{2}{3}\right) \geq \mu f\left(\frac{1}{3}\right) + (1-\mu) f\left(\frac{2}{3}\right)$

$$\geq \min\left\{f\left(\tfrac{1}{3}\right), f\left(\tfrac{2}{3}\right)\right\}.$$

$f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right) \approx 1.39$.

Therefore, plugging into (1), we get:

$$S(n) \leq c\sqrt{n} + \frac{\beta n}{\sqrt{k}} - \gamma\sqrt{n} \cdot 1.39 \leq \frac{\beta n}{\sqrt{k}} - (1.39\gamma - c)\sqrt{n}.$$

If we choose γ such that: $1.39\gamma - c \leq \gamma$, or $\gamma \leq c/0.39$, then

$$S(n) \leq \frac{\beta n}{\sqrt{k}} - \gamma\sqrt{n},$$ as desired.

- We are left with the induction step:
- For $n \leq k$, $S(n) = 0$. ie: We require that:

$$\frac{\beta n}{\sqrt{k}} - \gamma\sqrt{n} \geqslant 0;$$

ie: $\beta \geq \frac{\gamma\sqrt{n}}{n} \cdot k = \frac{\gamma k}{\sqrt{n}}$. Choosing $\beta = \gamma k$ suffices for the base case.

$\square$.

- Thus, we have $S(n) \leq \frac{\beta n}{\sqrt{k}} - \gamma\sqrt{n}$, where $\beta = \gamma\sqrt{k}$.

$$\Rightarrow S(n) \leq \sqrt{k}\, n.$$

- Recall that $k = 1/\varepsilon^2$.

- This proves that the divide-and-conquer algorithm yields a PTAS for the MIS problem on planar graphs.

$\square$

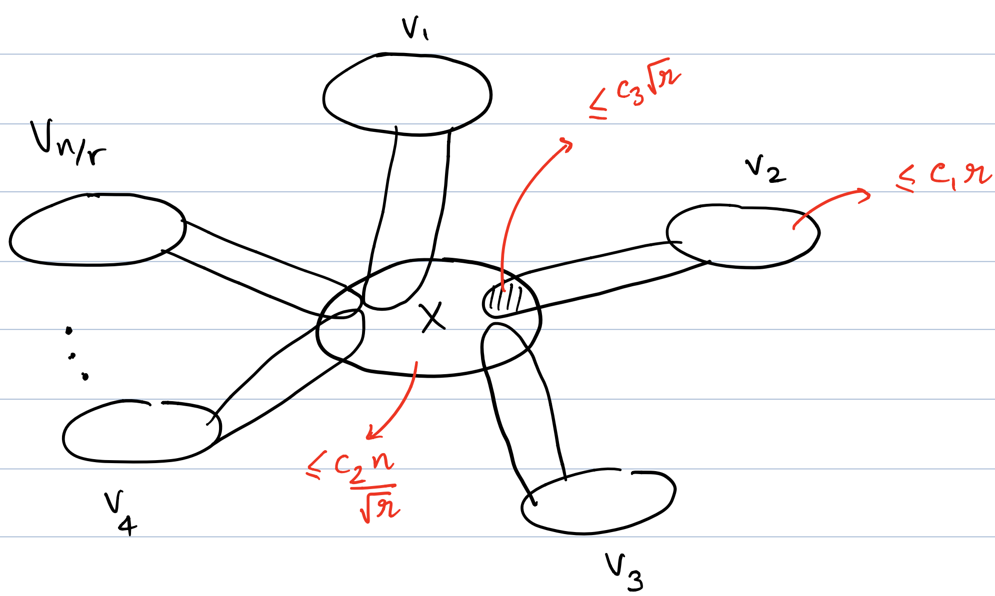
Running time: We did not describe it here, but there is an $O(n)$-time algorithm to compute a separator in a planar graph.

However, the brute-force algorithm dominates the running time for small values of ε. Hence, the running time is $O(n^{1/\varepsilon^2})$.

r-division: [Fredrickson '87]

For a parameter $r > 0$, there exist constants $c_1, c_2, c_3 > 0$ such that we can partition a planar graph $G = (V, E)$ into components: $V_1, \dots, V_{n/r}, X$ where $n = |V|$,

$$|V_i| \le c_1 r, \quad i = 1..n/r$$
$$|X| \le c_2 n/\sqrt{r}$$
$$|V_i \cap X| \le c_3 \sqrt{r}$$
$$(V_i \setminus X) \cap (V_j \setminus X) = \emptyset$$



(A recursive application of the separator theorem (with some additional work yields an r-division).

A local search algorithm:

- Here, we present a different, arguably simpler algorithm for the independent set problem in planar graphs.
- The algorithm takes a parameter $k \in \mathbb{N}$, the size of the local search. The algorithm is the following:

1. Start with an arbitrary independent set K.
2. While true:
3. If there exists an independent set $S \subseteq V \setminus K$, s.t.

   (i) $|S| \leq k$

   (ii) $(K \setminus N(S)) \cup S$ is an independent set, and

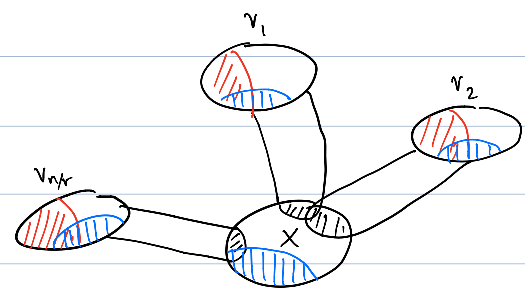
   (iii) $|(K \setminus N(S)) \cup S| > |K|$
4. $K \leftarrow (K \setminus N(S)) \cup S$.
5. Else break.
6. Return K.

- The algorithm starts with an arbitrary independent set.
- If we can obtain a larger independent set by adding at most k vertices from outside K & removing their neighbours in K, we do this.

- We will apply this algorithmic paradigm to many problems.

<u>Theorem</u>: The local search algorithm with parameter k is a $(1 - 1/\sqrt{k})$-approximation for the MIS problem in planar graphs.

<u>Proof Idea</u>:



- The parameter for the r-division is chosen so that
  $k > c_1 r + c_3\sqrt{r}$.

- Let the red parts denote the local search solution L, and let the blue parts denote an optimal solution OPT.

- For each V_i, let $OPT_i = OPT \cap V_i$. Let $L_i = V_i \cap L$.

<u>Obs1</u>: If in a component, $|OPT_i| > |L_i|$, there is a local exchange that can improve the size of L, contradicting the fact that L is a locally optimal solution. Therefore;

$$|L_i| \geq |OPT_i| \quad \forall\, i = 1 \ldots n/r.$$

<u>Obs 2</u>: $|X|$ is not too large. $\Rightarrow$ $|OPT \cap X|$ is not too large.

Hence, $|L| \approx |OPT|$.

Proof: · Let L be the solution returned by the local search algorithm, and let OPT denote an optimal solution.

- We can assume wlog that $L \cap OPT = \emptyset$.

- Consider the induced graph on $L \cup OPT$: $H = G[L \cup OPT]$
- Since L & OPT are both independent sets, H is a bipartite graph.
- Further, H is planar, since it is an induced subgraph of G.

- Therefore, H admits an r-division:

$$V_1, V_2 \dots, V_{n/r}; X.$$

- Let $OPT_i = OPT \cap V_i$, $L_i = OPT \cap L_i$.

- Choose $r \leq k/(c_1+c_3)$.

Claim: If the local search parameter $k > c_1 r + c_2\sqrt{r}$, then $|L_i| \geq |OPT_i| \;\; \forall i = 1.. n/r$.

- Otherwise there is a locally improving move: $(L \setminus N(OPT_i)) \cup (OPT_i)$.

$$|OPT| = \sum_{i=1}^{n/r} |OPT_i| + |OPT \cap X|.$$

$$\leq \sum_{i=1}^{n/r} |L_i| + |X|. \quad \text{---}(1)$$

$$|X| \leq \frac{c_2 OPT}{\sqrt{r}} \leq \frac{\gamma\, OPT}{\sqrt{k}}, \quad \text{for some constant } \gamma > 0.$$

Plugging into (1), we obtain:

$$\left(1 - \frac{\gamma}{\sqrt{k}}\right) |OPT| \leq \sum_{i=1}^{n/r} |L_i| \leq |L|.$$

Therefore, the local search algorithm yields a PTAS for MIS if we choose $k = \frac{1}{\varepsilon^2}$.

∎

Running time: • Each local move involves trying each of the at most $\binom{n}{1/\varepsilon^2}$ sets for a local improvement,

• At each step, we improve the solution size by at least 1. Therefore, there are at most n iterations of the algorithm.

The running time of the algorithm is $O(n^{1/\varepsilon^2})$.

∎