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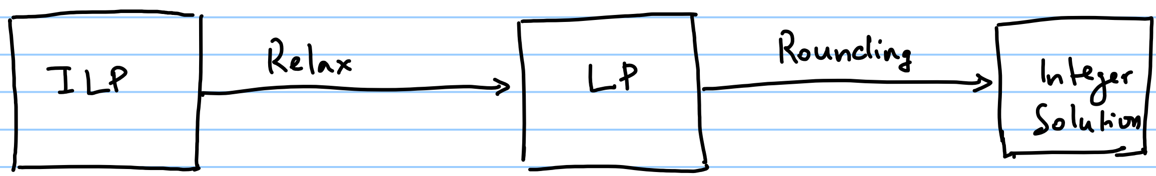
- Linear Programs : They can be solved in polynomial time.

- Many discrete optimization problems: We can formulate the problem as an ILP.

- Approximation algorithms. That run in poly. time & guarantee a quality of solution for all instances of the problem.



Vertex Cover: $G = (V, E)$, $S \subseteq V$: $\forall \{u, v\} \in E$,

$u \in S$, $v \in S$ (or both)

LP: $\min \sum_{v \in V} x_v$

s.t.

$x_u + x_v \geq 1 \quad \forall \{u, v\} \in E$

$x_u \in [0, 1] \quad \forall u \in V.$

$\longrightarrow$ **Fractional Solution**

$(x_1 \cdots \qquad x_n)$

$\longrightarrow$ **Rounding**

$$\underbrace{x_1 \cdots \cdots x_i}_{1} \; \underbrace{x_{i+1} \cdots x_k}_{\leq \frac{1}{2}} \; \underbrace{x_{k+1} \cdots x_n}_{\frac{1}{2} >}$$

$$\updownarrow$$

$$\overbrace{y_1 \cdots \quad y_i}\;\overbrace{y_{i+1} \cdots y_k}\;\overbrace{y_{k+1} \cdots y_n}$$

$$\underbrace{y_1 \cdots y_k}_{1} \; \underbrace{y_{k+1} \cdots y_n}_{0}$$

Correctness.

- Look at the constraints & observe that $x_u + x_v \geq 1$, either x_u, or $x_v \geq 1/2$.
- Therefore, if all $x_u \geq 1/2$ are rounded up to 1, then y is a feasible solution

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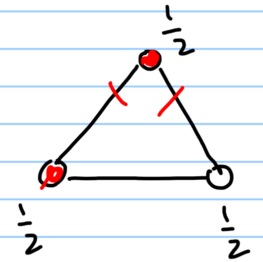
Theorem: The rounding algorithm yields a 2-approximation for VC.

Pf: $S = \{ v : y_v = 1 \}$

$$\text{cost}(S) = \sum_{v \in V} y_v \le 2 \cdot \sum_{v \in V} x_v = 2 \cdot OPT_{LP} \le 2 \cdot OPT \qquad \blacksquare$$



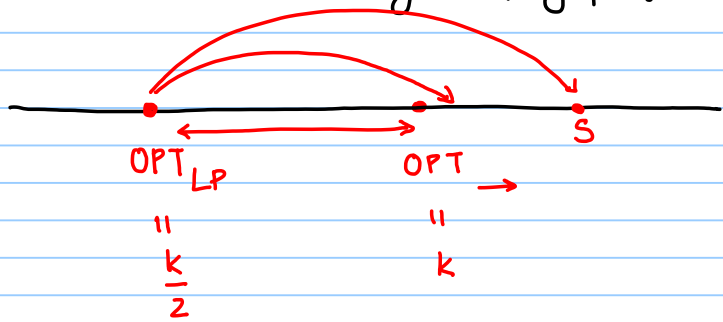
$OPT_{LP} = 3/2$

$OPT = 2$

Questions: Is the analysis tight?

Yes.

The LP has an integrality gap of 2.



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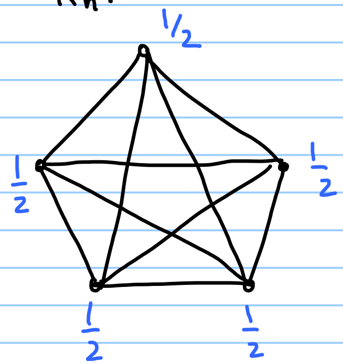
Integrality Gap Example for VC:

Graph is K_n.



$$x_u + x_v \geq 1 \quad \forall \{u,v\} \in E$$

$$\sum_{u \in V} x_u = \frac{n}{2}.$$

$$OPT_{LP} \leq \frac{n}{2}.$$

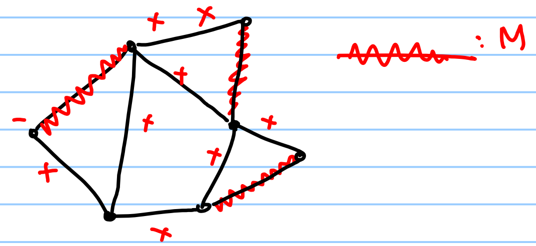
$\underline{OPT} = n-1$ for K_n.

For each pair $\{u,v\}$, there is an edge $\Rightarrow$ at least one end-point must be chosen in S.

$$\frac{OPT}{OPT_{LP}} \geq \frac{2(n-1)}{n} \rightarrow 2 \text{ as } n \rightarrow \infty.$$

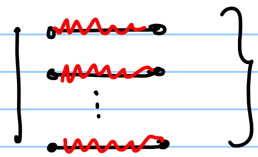
Other Algorithms?

- A matching $M \subseteq E$ in a graph $G=(V,E)$ is a set of edges that do not share a vertex.



A matching is <u>maximal</u> if it cannot be extended.

<u>Lemma</u>. $OPT \geq k$, where k = size of a maximal matching.



each edge requires a different vertex to cover it.

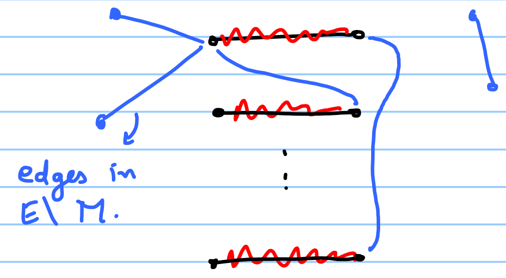
0. $S = \emptyset$

<u>Alg</u>: 1. Pick a maxl. matching M.

2. Add both end-points of all edges in M into S.

3. Return S.

<u>Claim</u>. S is a vertex cover.



<u>Thm</u>. Alg is a 2-approx.

$$|S| = 2|M| \leq 2\,OPT.$$

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Qn: Is the analysis tight?

Qn: Better algorithms?

The algorithms presented are from the 1970's

- The best approx. alg. for VC guarantee an approximation factor of 2.

Dinur & Safra '04 : Unless $P = NP$, $\nexists$ a better than ~ 1.36 - approx for V.C.

Subhash Khot, Regev '08 : There is no $(2-\varepsilon)$-approx for VC unless Unique Games Conjecture is False.

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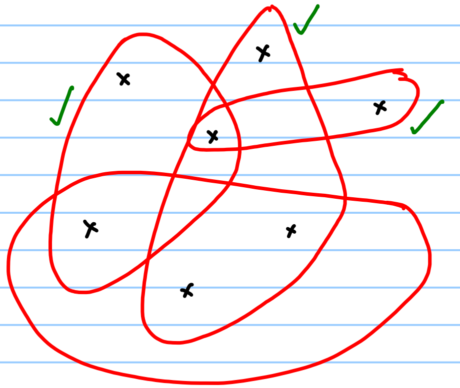
Set Cover:

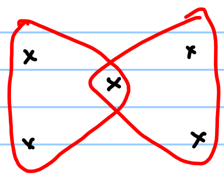
Given a set system (hypergraph, range spaces)

$\mathcal{H} = (X, \mathcal{S})$, $\mathcal{S} = \{S_1 \ldots S_m\}$

$|X| = n$ $S_i \subseteq X$

$\bigcup_{i=1}^{m} S_i = X$.





Given $\mathcal{H} = (X, \mathcal{S})$, find $\mathcal{T} \subseteq \mathcal{S}$;

$|\mathcal{T}|$ as small as possible such that:

$$\underline{\bigcup_{S \in \mathcal{T}} S = X.}$$

Set Cover is a generalization of Vertex Cover

ILP.

Variables : x_S ; $S \in \mathcal{S}$ $x_S \begin{cases} 1, & S \in \mathcal{T} \\ 0, & S \notin \mathcal{T} \end{cases}$

Constraints: $\underline{\sum_{S \in \mathcal{S} : x \in S} x_S \geq 1 \quad \forall x \in X}$

$x_S \in \{0, 1\}$

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ILP → LP → Rounding.

ILP formulation:

$$\min \sum_{S \in \mathcal{S}} x_S .$$

$$\boxed{\sum_{S : x \in S} x_S \geq 1 \quad \forall\, x \in X}$$

$$\cancel{x_S \in \{0,1\} \quad \forall\, S \in \mathcal{S}}$$

$$\underbrace{x_S \in [0,1]}_{x_S \geq 0} \quad \forall\, S \in \mathcal{S}.$$

Let $x = (x_{S_1} \ldots x_{S_m})$ be an optimal LP solution.

- Rounding is randomized.
- Each x_S is in $[0,1]$.
- If x_S is high, then the LP suggests that S is a good set to pick into our solution.
- Interpret x_S as the probability of picking S.

Rounding : Attempt 1.

- Pick each set S with probability x_S.

- We get a collection T_0 of sets.

$$\mathbb{E}[\text{cost}(T_0)] = \mathbb{E}[|T_0|] =$$

$$= \sum_{S \in \mathcal{S}} \Pr[S \text{ is picked in } T_0]$$

$$= \sum_{S \in \mathcal{S}} x_S = OPT_{LP}.$$

- The expected cost of T_0 is OPT_{LP}.

- But, T_0 may not be a feasible soln.

Rounding:

- For $i = 0 \ldots t$.
  - Let T_i = sets chosen w/ prob x_S.
- Return $\bigcup_{i \geq 1}^{t} T_i$

$P[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B]$.

- Fix an $x \in X$.
- $\mathbb{P}r[x \text{ is not covered in } \mathcal{P}_0] = \prod_{S: x \in S} (1 - x_S)$

$\sum_{S: x \in S} x_S \geq 1$ ($\because$ x is feasible for the LP)

$$\leq \exp\Big(-\underbrace{\sum_{S: x \in S} x_S}\Big) \quad [e^y \geq 1 + y \ \forall y \in \mathbb{R}]$$

$$\leq e^{-1} = 1/e.$$

- $\mathbb{P}[x \text{ is not covered in } \mathcal{P} = \bigcup_{i=0}^{t} \mathcal{P}_i] \leq (1/e)^t$.

Choose $t = 2 \ln n$.

$$\leq \frac{1}{n^2}$$

$\mathbb{P}r[\exists \text{ an } x \in X \text{ that is uncovered by } \mathcal{P} = \bigcup \mathcal{P}_i]$

$$\leq \sum_{x \in X} \mathbb{P}r[x \text{ is uncovered by } \mathcal{P}] \quad [\text{Union bound}]$$

$$\leq n\left(\frac{1}{n^2}\right) = \frac{1}{n}$$

$\therefore$ $\mathcal{P}$ is a set cover w/ prob $\geq 1 - 1/n$.

$\mathbb{E}\,|\mathcal{P}| = \sum \mathbb{E}[\mathcal{P}_i] \leq OPT_{LP} \cdot (2 \ln n) \Rightarrow$ We have a $2 \ln n$-approx for set cov.