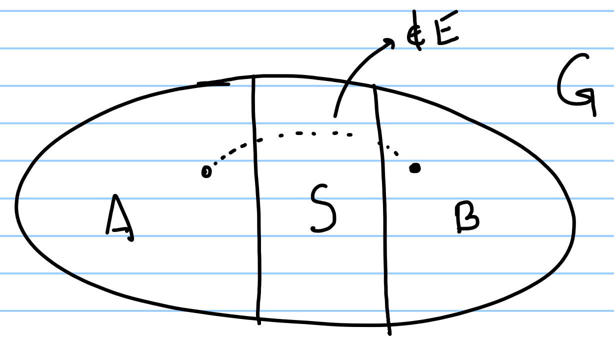
Note Title 13-06-2019

Local Search.

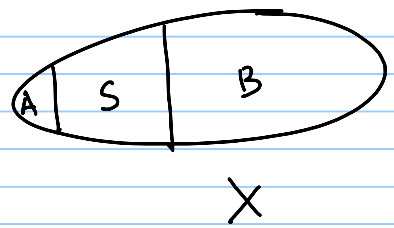
Separator: Given a graph $G=(V,E)$, a separator is a set $S \subset V$, such that: $V \setminus S$; G can be split into 2 groups A & B such that there is no edge between vertices in A & vertices in B.

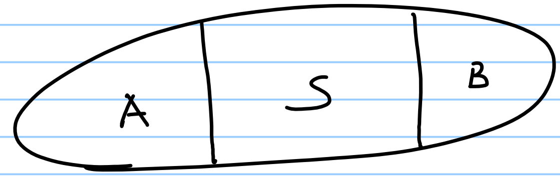


Note Title 13-06-2019

Balanced Separator: A separator is balanced if $\exists\ 0 < c < 1.$

$$|A|, |B| \leq c \cdot |V|.$$





By separator, we mean a balanced separator in this lecture.

Note Title 13-06-2019

- We are interested in hereditary classes of graphs that admit sub-linear size separators.

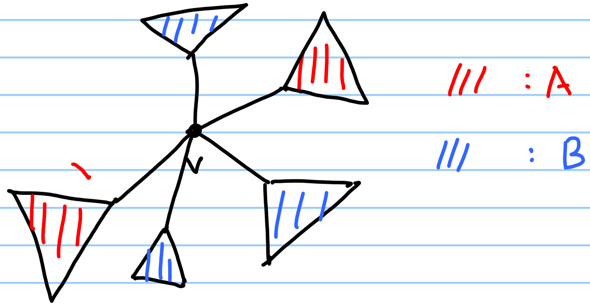
A graph class $\mathcal{C}$ is hereditary if every subgraph of any graph in $\mathcal{C}$ is also in $\mathcal{C}$.

Eg: · Acyclic graphs

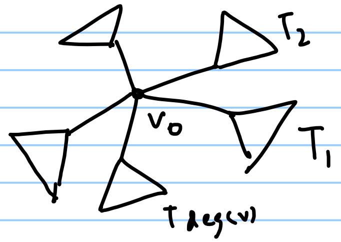
$\exists\, 1 > \delta > 0 : |S| \le |V|^{1-\delta}$

Note Title 13-06-2019

Example. For any tree $T=(V,E)$, there is a vertex whose removal separates T into 2 components A, B : $|A|, |B| \le 2/3 |V|$.



Proof: Start with an arbitrary vertex $v_0 \in V$



$T \setminus v$: breaks into $\le deg(v_0)$ sub-trees.

$|T_1| \ge |T_2| \ge \cdots \ge |T_{deg(v)}|$.

Suppose : $|T_1| \le \frac{1}{3}|V|$

$T_1 \cup T_2 \cdots \cup T_\ell$, where ℓ is the smallest index at which

$$\bigcup_{i=1}^{\ell} T_i > \frac{2}{3}$$

$$A = \bigcup_{i=1}^{\ell-1} T_i \quad , \quad B = \bigcup_{i=\ell}^{deg(v)} T_i$$

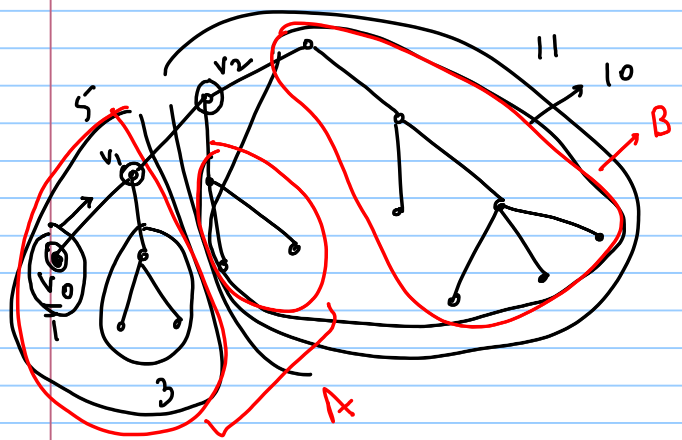
$$|A| \le \frac{2}{3}|V|, \quad |B| \le \frac{2}{3}|V|. \quad (\because \; |T_i| < 1/3)$$

Note Title 13-06-2019

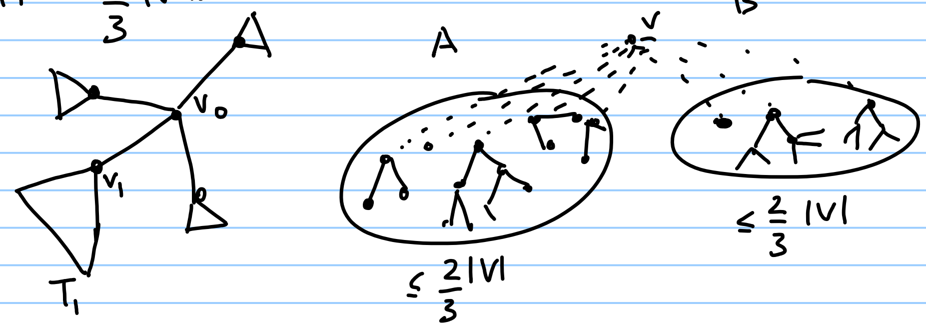
- Suppose $\frac{|V|}{3} < |T_1| \le \frac{2|V|}{3}$

$|T_1| \ge |T_2| \ge \dots \ge |T_{deg(v)}|$.

$A = T_1, \quad B = \bigcup_{i=2}^{deg(v)} T_i$



- Suppose $|T_1| > \frac{2}{3}|V|$.



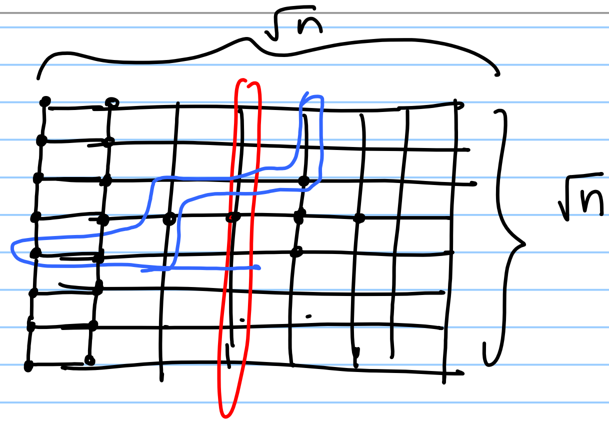
- Set v_1 to be the new candidate separator.

- The size of the largest component is monotonically decreasing.

$\Rightarrow$ We will eventually find a separator vertex.

Example 2 :

G =



G has n vertices.

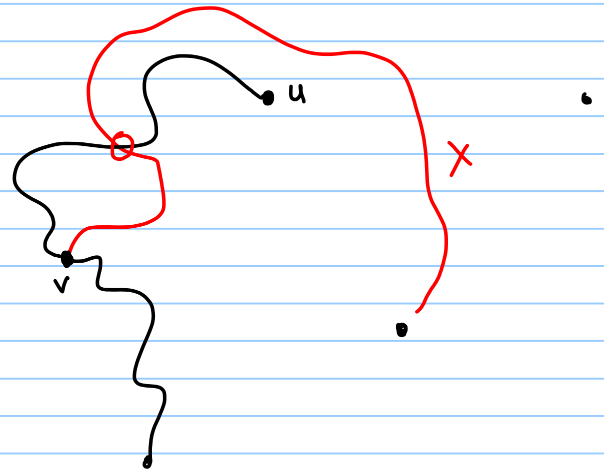
Theorem: There exists a separator of size $\leq \sqrt{n}$.

- Does there exist a sep. of size $o(\sqrt{n})$?
- Any separator has size $\Omega(\sqrt{n})$ for a $\sqrt{n} \times \sqrt{n}$ grid graph.

Note Title 13-06-2019

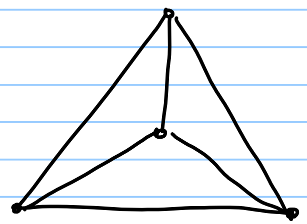
Planar Graphs:

A graph is planar if it can be drawn in the plane:

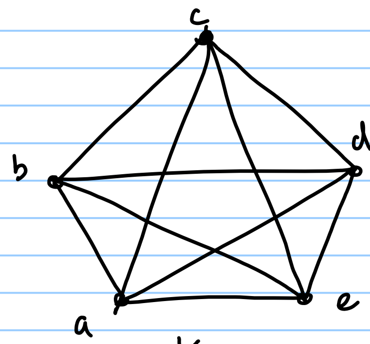


s.t. the vertices are points in $\mathbb{R}^2$.

- The edges are continuous curves between the vertices.
- No two edges share a point in their interior.

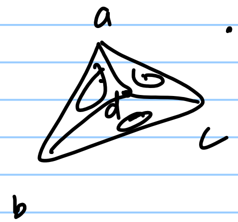


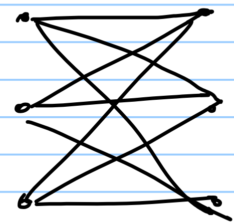
K_4



$K_{\underline{5}}$.

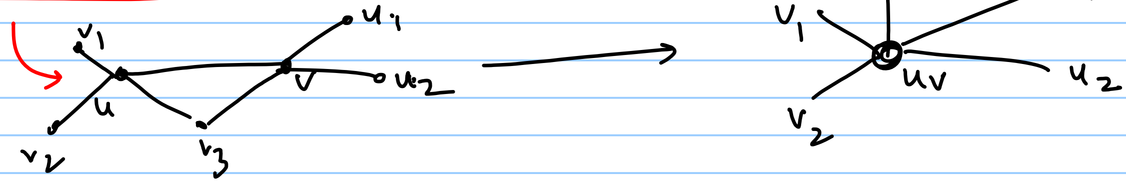
. Jordan Curve Theorem.





$K_{3,3}$

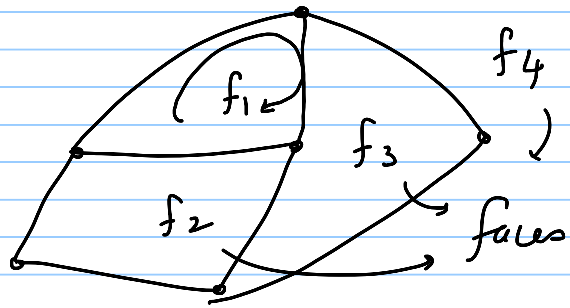
<u>Kuratowski's Theorem</u>: A graph is planar $\Leftrightarrow$ you cannot obtain $K_{\underline{5}}$, or $K_{3,3}$ via a seq. of <u>edge contractions</u>, vertex deletions.



Note Title 13-06-2019

Four Color Theorem [Appel, Haken '71]. For every planar graph, we can color the vertices with ≤ 4 colors such that for every edge, the colors of its end-points are different

- 1800's . . . . Computer assisted proof.



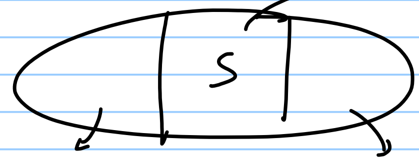
Euler's Theorem : $|V| - |E| + |F| = 2$

Planar Separator Theorem: [Lipton, Tarjan '79] Every planar graph admits a separator of size $O(\sqrt{n})$.

$\le 2\sqrt{2}\sqrt{n}$

$\sqrt{6}\sqrt{n}$

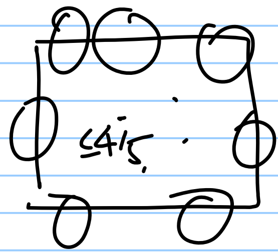
$\le 2\sqrt{2}\sqrt{n}$.



$|A| \le \frac{2}{3}|V|$ $\quad$ $|B| \le \frac{2}{3}|V|$

A separator theorem for disjoint disks in $\mathbb{R}^2$.

$\le \frac{4}{5}$



For a set $\mathcal{D}$ of n disjoint disks in $\mathbb{R}^2$; $\exists$ a square S s.t.

int(S), ext(S) $\le \frac{4}{5}n$ centers of the disks

& S intersects $O(\sqrt{n})$ disks on its boundary

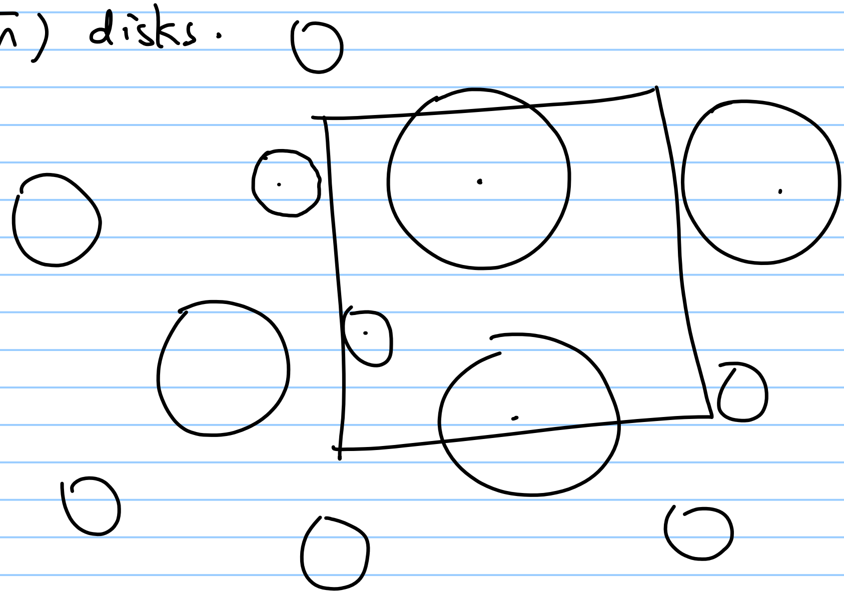
Note Title 13-06-2019

Theorem [Chan'03, Smith-Wormald'98] Let $\mathcal{D}$ be a set of n disjoint disks in $\mathbb{R}^2$.

$\exists$ a square S such that there are at most $\frac{4n}{5}$ disk centers inside S.

" " " " " outside S.

The boundary of S intersects $O(\sqrt{n})$ disks.

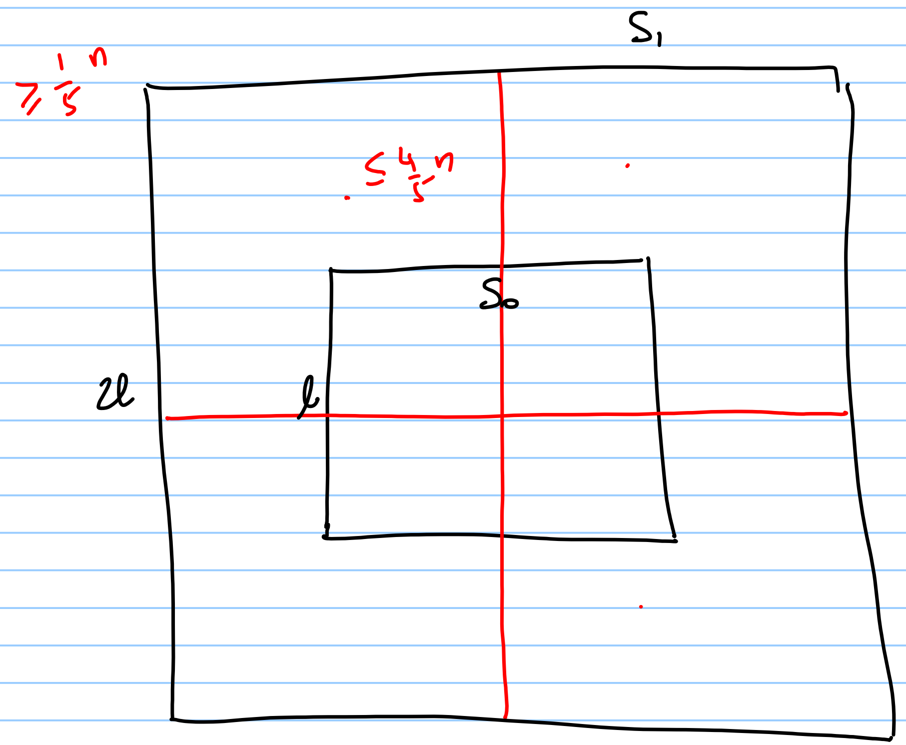


Note Title 13-06-2019

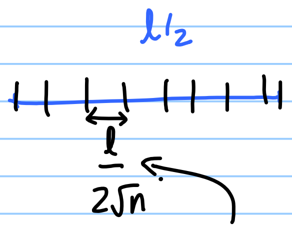
Proof: Let S_0 be the smallest square containing $\geq \frac{1}{5}$ of the disk centers.

Let S_1 be the square with the same center as S_0 & twice the size.

Obs 1: S_1 contains $\leq \frac{4}{5}$ of the disk centers.

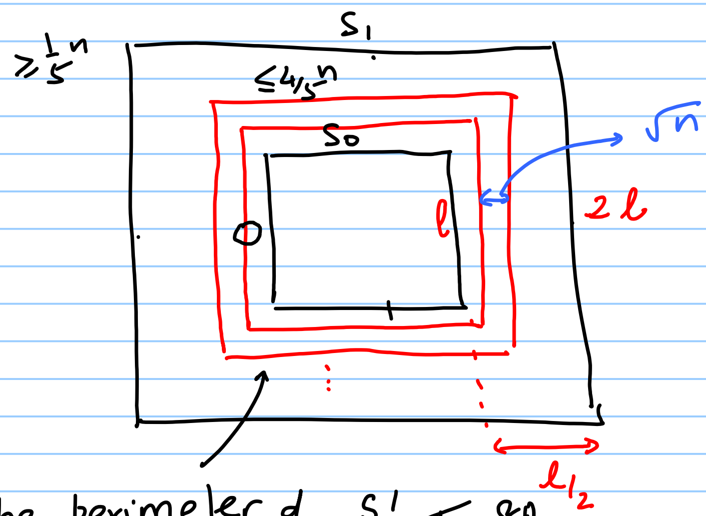


- Add $\sqrt{n}$ concentric squares between S_0 & S_1.



- A disk D is large if diam $(D) \geq \frac{\ell}{2\sqrt{n}}$
- otherwise a disk is small.

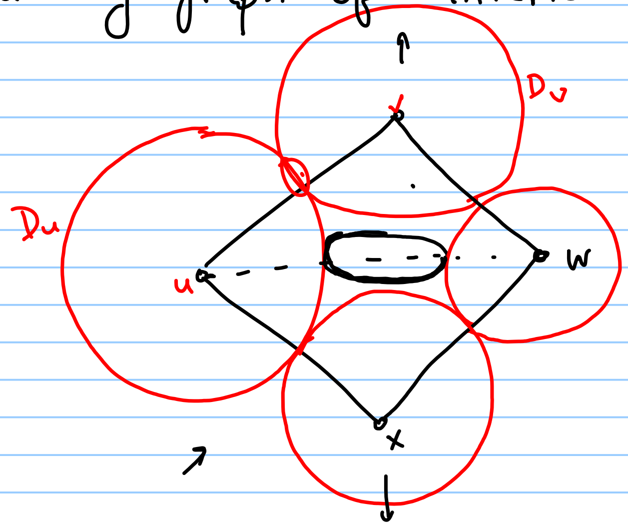
Claim 1: For any square S' between $S_0 .. S_1$,
$\#$ large disks intersecting the boundary of S' is $O(\sqrt{n})$.



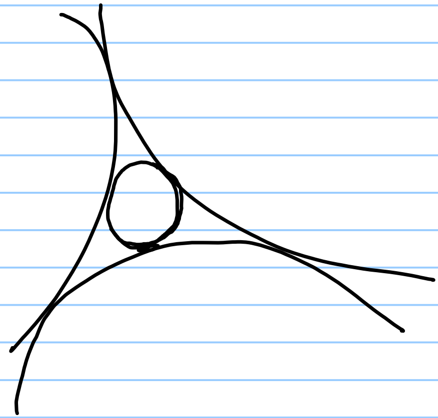
- The perimeter of $S' \leq 8\ell$.
- A large disk has radius $\geq \frac{\ell}{2\sqrt{n}}$.

Claim 2: Any small disk intersects the boundary of at most one concentric square.

Theorem: [Koebe - Andr'eer - Thurston] Any planar graph can be represented as the touching graph of interior disjoint disks in $\mathbb{R}^2$.



→ For each vertex, $v \in V$, $\exists$ a disk D_v s.t. D_u & D_v touch iff $\{u,v\} \in E$.



Theorem: Every planar graph has a separator of size $O(\sqrt{n})$ s.t. $|A|, |B| \leq \frac{4}{5}|V|$.

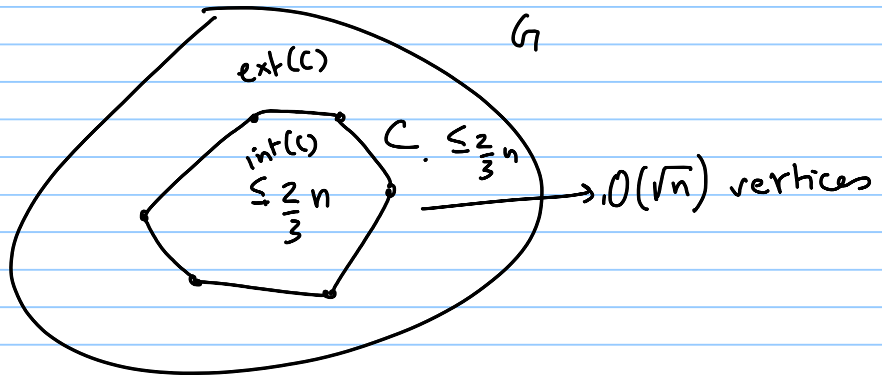
Pf: Use K.A.T thm $\longrightarrow$ Shrink disks slightly to obtain disjoint disks $\downarrow$ Use the sep. thm for disks.

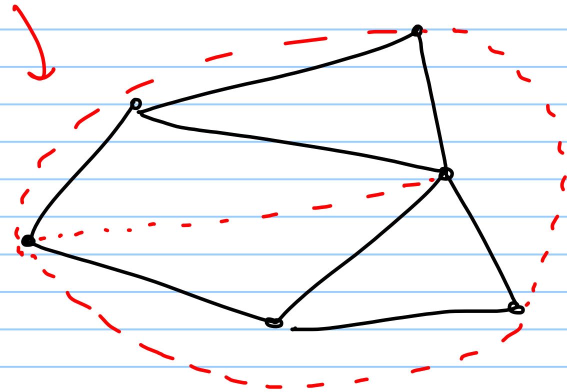
- Only need to show that the sep. for the disks is indeed a sep. for the graph.

∎

[Miller '80's]

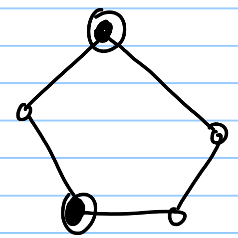
Theorem: Let $G=(V,E)$ be a triangulated planar graph, then $\exists$ a sep. that is a simple cycle in G.





Independent Sets in planar graphs.

- For a graph $G=(V,E)$, a set $K \subseteq V$ is an independent set if $\forall\ u,v \in K$, $\{u,v\} \notin K$.



- Maximum Independent set: Find K of largest size. (MIS)

  - NP-hard even for planar graphs.
  - For general graphs, we cannot obtain a better than $n^{1-\varepsilon}$-approx unless $P=NP$.

Theorem: There is a PTAS for MIS problem for planar graphs.

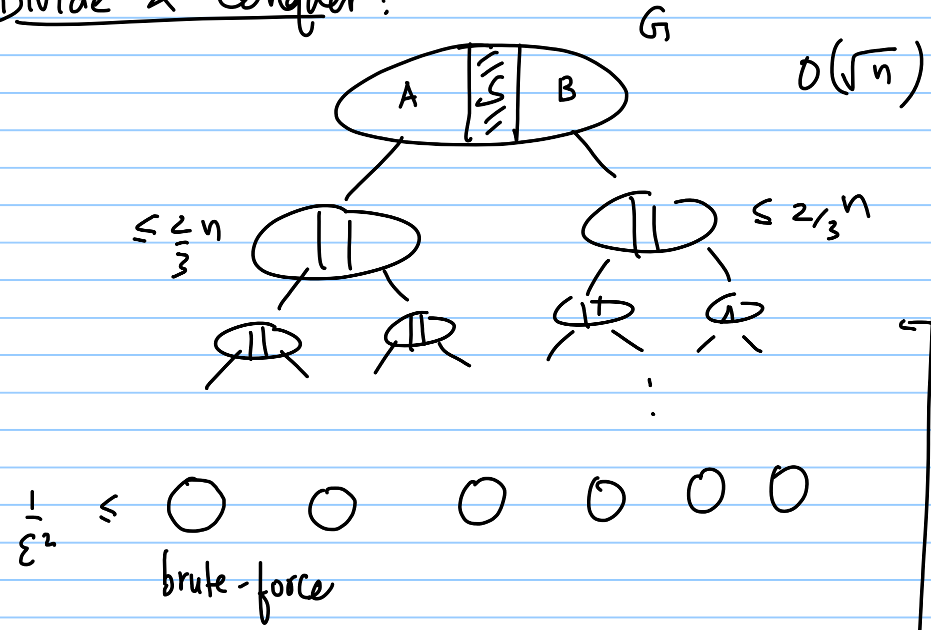
Polynomial time approximation scheme.

An alg. is a PTAS if $\forall \varepsilon > 0$, the alg. returns a soln. of value $\geq (1-\varepsilon)\,\mathrm{OPT}$ (max. problem)

$\leq (1+\varepsilon)\,\mathrm{OPT}$ (min. problem).

The running time: $O\left(\underline{n}^{\mathrm{poly}(1/\varepsilon)}\right)$

Divide & Conquer:



Algorithm: ① while $\exists$ a conn. component C of size $> 1/\varepsilon^2$, use the planar sep. thm on C.

② For each component, compute MIS by brute force in $O(2^{1/\varepsilon^2})$ time.

③ Return the union of the vertices of the independent sets in each component.

Obs 1: The solution returned is an IS.

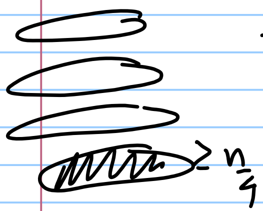
(By separator property)

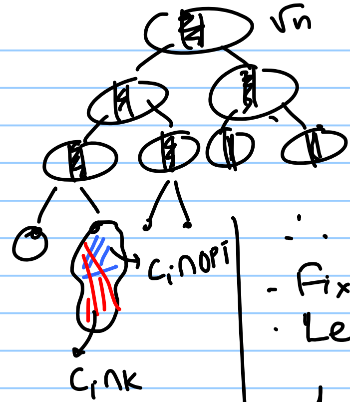
Assume that the total # of vertices in the separators over all levels $\leq \varepsilon n$.

Theorem: The alg. is a $(1-\varepsilon)$-approx for MIS.

Pf: (1) By the 4-Color thm, G can be colored with ≤ 4 colors.

- Each color class is an independent set.
- $\therefore$ The largest color class has size $\geq \frac{n}{4}$.





$\therefore$ $OPT \geq n/4$. — (1)

- Fix an optimal soln. OPT.
- Let $C_1 \dots C_\ell$ be the components of size $\leq 1/\varepsilon^2$ we obtain.

Let $OPT_i = C_i \cap OPT$.

- Let K be the soln. returned by our alg.

Let $K_i = C_i \cap K$

Claim: $|K_i| \geq |OPT_i|$.

$$\sum_{i=1}^{\ell} |K_i| \geq \sum_{i=1}^{\ell} |OPT_i|.$$

the separators at all levels.

$$|OPT| = \sum_{i=1}^{\ell} |OPT_i| + |OPT \cap S|$$

$$\leq \sum_{i=1}^{\ell} |K_i| + |OPT \cap S|$$

$$\leq \sum_{i=1}^{\ell} |K_i| + |S|$$

(We assumed $|S| \leq \varepsilon n$)

$$|OPT| \leq |K| + \varepsilon n \quad (\text{we know } OPT \geq \frac{n}{4} \Rightarrow 4OPT \geq n)$$

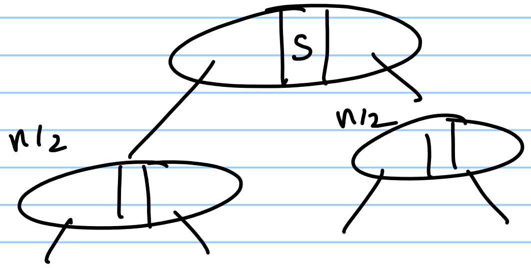
$$\leq |K| + 4\varepsilon |OPT|$$

$$|OPT| (1 - 4\varepsilon) \leq |K|.$$

∎

$|S| \leq \varepsilon n$.

Lemma: $|S| \leq \varepsilon n$.



$\sqrt{n}$

$\rightarrow \quad 2 \cdot \sqrt{\frac{n}{2}} = \sqrt{2}\, n.$

$2^2 \sqrt{\frac{n}{2^2}} = 2\sqrt{n}$

$\vdots$

$$|S| = \sqrt{n} + \sqrt{2}\sqrt{n} + 2\sqrt{n} + \cdots$$

$$= \sum_{i=0}^{\log(n\varepsilon^2)} \sqrt{2}^{\,i} \sqrt{n}$$

$\vdots$

$$\leq \varepsilon n$$

$\square$

We stop when $\frac{n}{2^i} = \frac{1}{\varepsilon^2} \Rightarrow i = \log(n\varepsilon^2)$