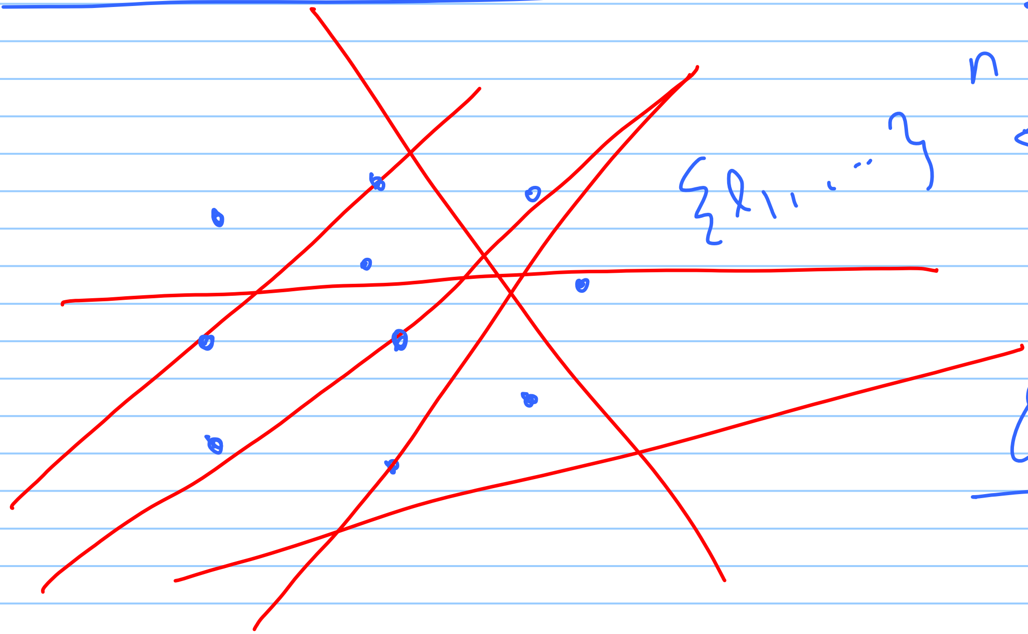
Note Title 04-06-2019

Incidence Geometry

- **Point line incidences**



$\{l_1, \dots\}$

$\{p_1, \dots\}$

$m = |P| \leftarrow$ points in $\mathbb{R}^2$

$n = |L| \leftarrow$ lines in $\mathbb{R}^2$

$$I(P, L) = \{(p_i, l_j) \mid p_i \in P,\ l_j \in L,\ p_i \in l_j\}$$

Q Estimate $I(P, L)$?

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Q2 $A \subseteq \mathbb{R}$

$A+A = \{a+b \mid a, b \in A\}$

$A \cdot A = \{ab \mid a, b \in A\}$

Estimate size $|A+A|$, $|A \cdot A|$ in terms of size of $|A|$.

Q3 Let P is a set of n points in $\mathbb{R}^2$. Estimate no. of unit area $\triangle$ that can be formed with vertices from P.

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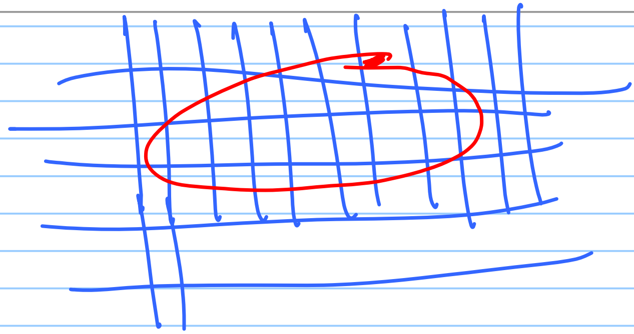
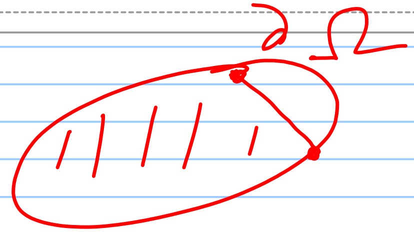
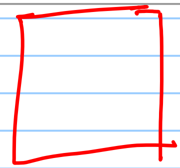
§4

$\mathbb{Z}^2$

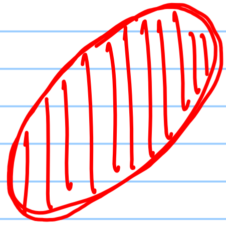


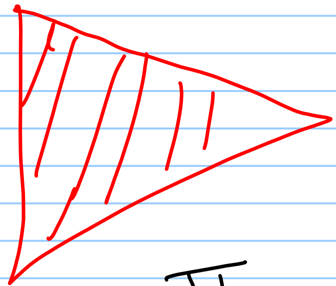




$\partial\Omega \cap \mathbb{Z}^2$ vs $\Omega \cap \mathbb{Z}^2$

Isoperimetric problems.





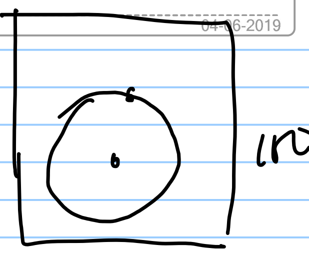
$<$

$\sim$

Thm 1 (Szemeredi - Trotter Thm 1983)

$$|I(P,L)| \lesssim \left((|P||L|)^{2/3} + |P| + |L| \right)$$

↳ ⟵ circles.



∘ Erdős Unit Distance Problem (1946)

<u>Q</u> How many unit distance pairs can there be in a set of "n" points in $\mathbb{R}^2$?

$u(n)$

$n \times 2^{\frac{100 \log n}{\log\log n}}$

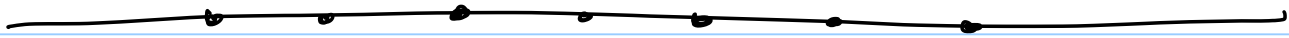
<u>Trivial Estimate</u> $\binom{n}{2}$

<u>Thm 2</u> (Erdős 1946) $n^{1+\frac{c}{\log\log n}} \lesssim \boxed{u(n) \lesssim n^{3/2}}$

Thm 3 (Spencer – Szemeredi – Trotter 84)

$$u(n) \lesssim n^{4/3} \longleftarrow \text{ST theorem}$$

$O(n)$

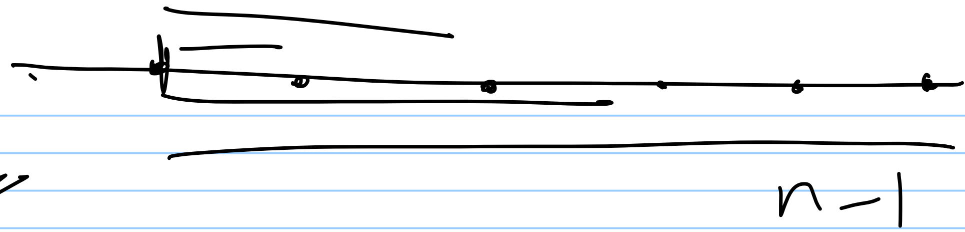


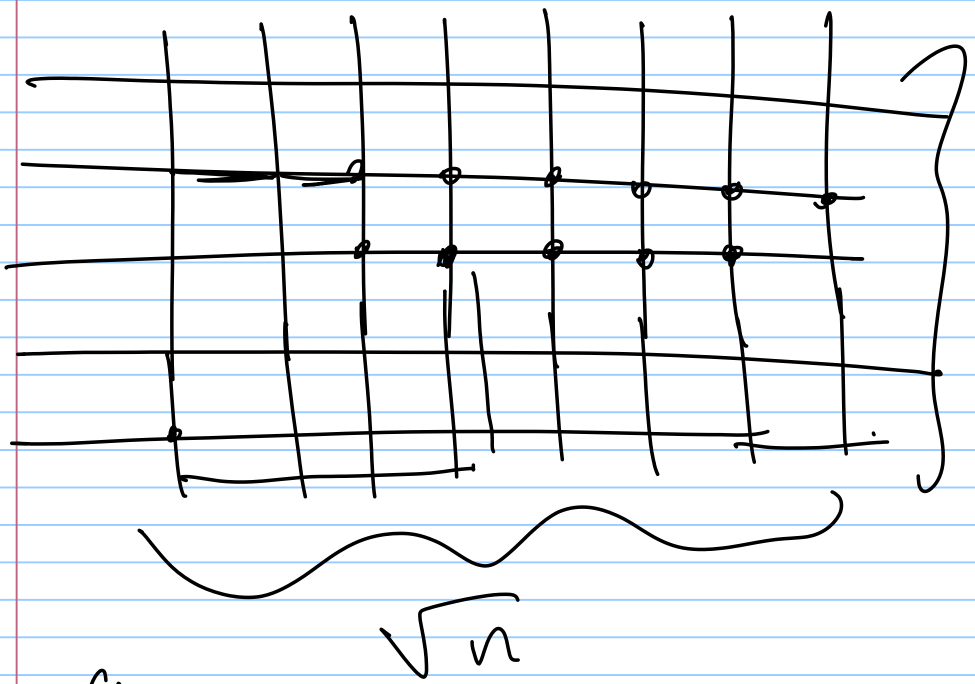
- **Erdös Distinct distance problem (1946)**

Q Minimum number of distinct distances that can be determined ~~by~~ by "n" points in $\mathbb{R}^2$?

$d(n) \longleftarrow$ denotes this number.

Q $\quad d(n) \le n-1$





$$T = \{ d(p_i, p_j)^2 \mid p_i, p_j \in S \}$$

Thm (Landau-Ramanujan)

$$\Theta\left(\frac{n}{\sqrt{\log n}}\right)$$

Thm (Erdös 1946) $d(n) = O\left(\frac{n}{\sqrt{\log n}}\right)$

Conjectured $d(n) \geq C \frac{n}{\sqrt{\log n}}$

$$f(n) = \Theta(t(n))$$

$$c_1 t(n) \leq f(n) \leq c_2 t(n)$$

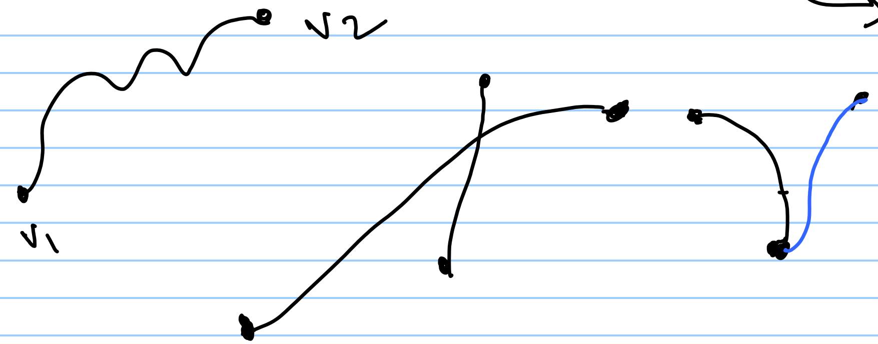
Thm (Guth–Katz 2010) $d(n) = \Omega\left(\frac{n}{\log n}\right)$

Proved ST Theorem

Planar graphs

$G := (\underline{V}, E)$

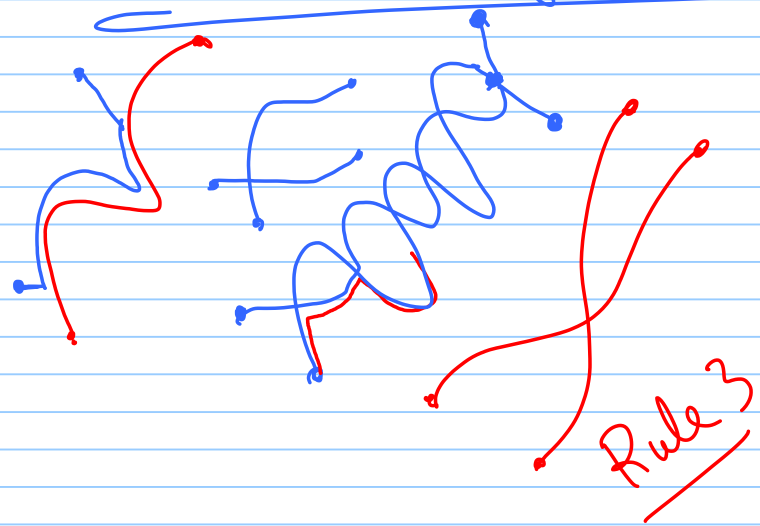
$E \hookrightarrow \{v_1, v_2\}$

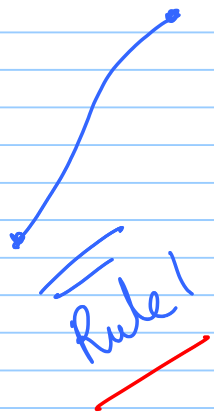


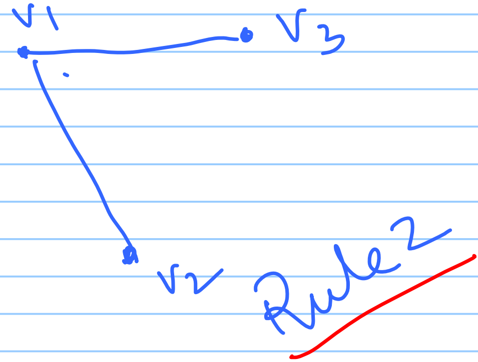
Lemma 1 (Bounds on the # of edges in planar graph) If G is planar then $|E| \leq 3|V| - 6$.

$n \log n$

Crossing no. of Graph $cr(G)$

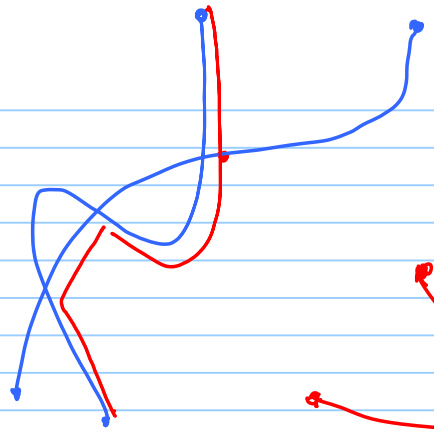


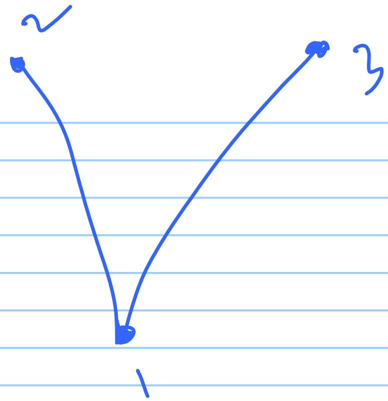


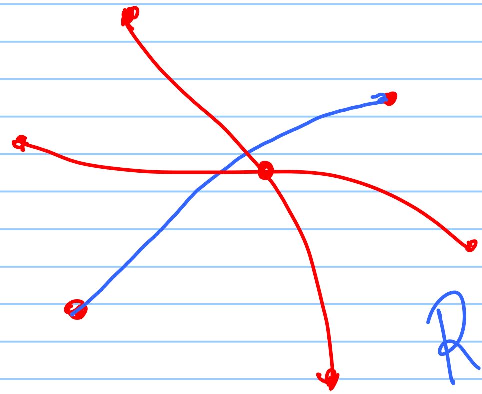


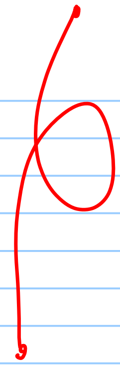
$\{v_1, v_2\}$

$\{v_1, v_3\}$









Rule 4

Gr(4) − 1

Lemma 2 (Loose crossing bound)

$$\boxed{Cr(G) \geq |E| - 3|V| + 6}$$

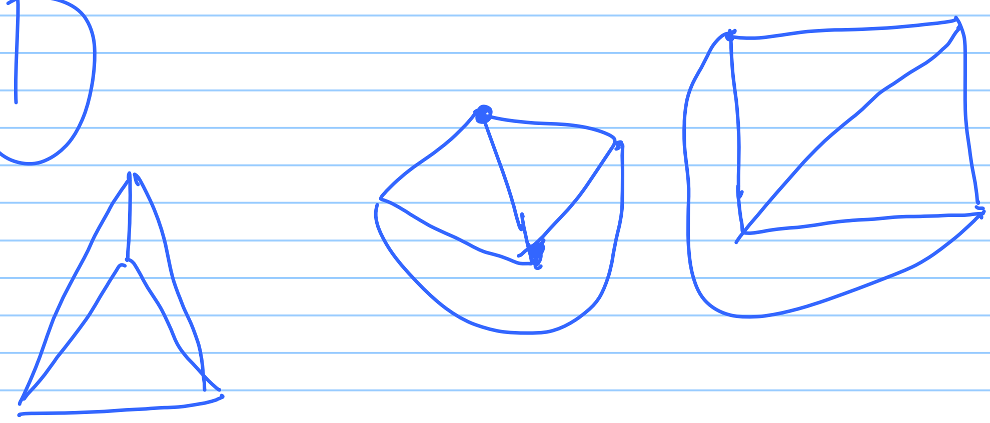
$-6+6$

$|V| \geq 3$

$- = (3V - 6)$

K_5 & $Cr(K_5) = 1$

≥ 0



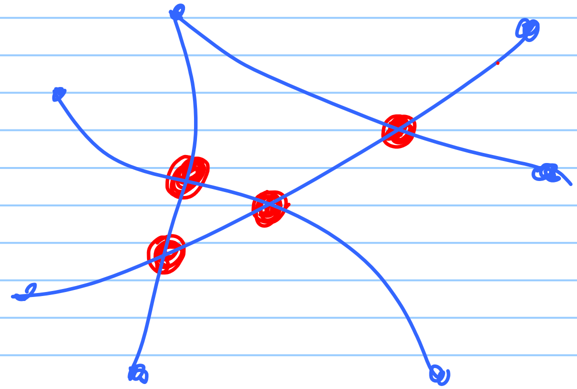
Proof $G := (V, E)$ $\quad Cr(G)$

Let $E' \subseteq E$ be a maximal set of edges that does not cross in the drawing. $|E| - |E \setminus E'| \leq 3|V| - 6$

$G := (V, E')$ $\quad |E'| \leq 3|V| - 6$

Looking at edges in $E \setminus E'$ $\quad Cr(G) \geq |E \setminus E'|$

$$Cr(G) \geq |E \setminus E'| \geq |E| - 3|V| + 6 \qquad \square$$



$$(2n-1) \times (3a-1)$$

$$G \to H \to |V| + Cr(G)$$

$$H \to |E| + 2\,Cr(G)$$

$$\boxed{Cr(G) \geq |E| - 3|V| + 6}$$

$$|E| \geq 4|V| \quad \square$$

$$\Omega\left(\frac{|E|^3}{|V|^2}\right)$$

Thm (Crossing Lemma) $G := (V, E)$ If $|E| \geq 4|V|$, then

$$Cr(G) = \Omega\left(\frac{|E|^3}{|V|^2}\right).$$

$|V| - |E| + |F| = 2$

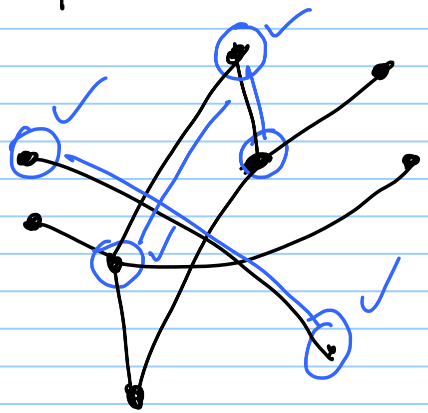
$2|E| \geq 3|F|$

Proof $G = (V, E)$, drawn with exactly $Cr(G)$ many crossings. Let $0 \leq p \leq 1$,

$V_p = \{$ set of vertices selected from V, where each vertex is picked with probability $p\}$ & $G_p := (V_p, E_p)$

$\mathbb{E}[|V_p|] = np$ | $\mathbb{E}[|E_p|] = |E| p^2$

$$G_p := (V_p, E_p)$$



$$\mathbb{E}[C'(G_p)] = Cr(G)\, p^4$$

- $\boxed{Cr(G_p) \geq |E_p| - 3|V_p|}$
- $C'(G_p) \geq |E_p| - 3|V_p|$

$$\mathbb{E}[C'(G_p) - |E_p| - 3|V_p|] \geq 0$$

$$\mathbb{E}[C'(G_p)] - \mathbb{E}[|E_p|] - 3\,\mathbb{E}[|V_p|] \geq 0$$

$$\mathbb{E}[c'(G_p)] \geq \mathbb{E}[|E_p|] - 3\,\mathbb{E}[|V_p|]$$

$$\boxed{Cr(G) \geq \left(\frac{|E|}{p^2} - \frac{3|V|}{p^3} \right)}$$

$$p = \frac{4|V|}{|E|}$$

$$= \Omega\left(|E|^3 / |V|^2 \right). \qquad \square$$

Thm (ST Theorem) $I(P, L) \lesssim (|P||L|)^{2/3} + |P| + |L|$

$$(mn)^{2/3} + m + n$$

Proof $L = \{ \ell_1, \ldots, \ell_n \}$

$P = \{ p_1, \ldots, p_m \}$

$I(P, L) \overset{\text{def}}{=} \{ (p, \ell) \mid p \in P, \ell \in L, p \in \ell \}$.

L' = those lines that contains at least 2 points

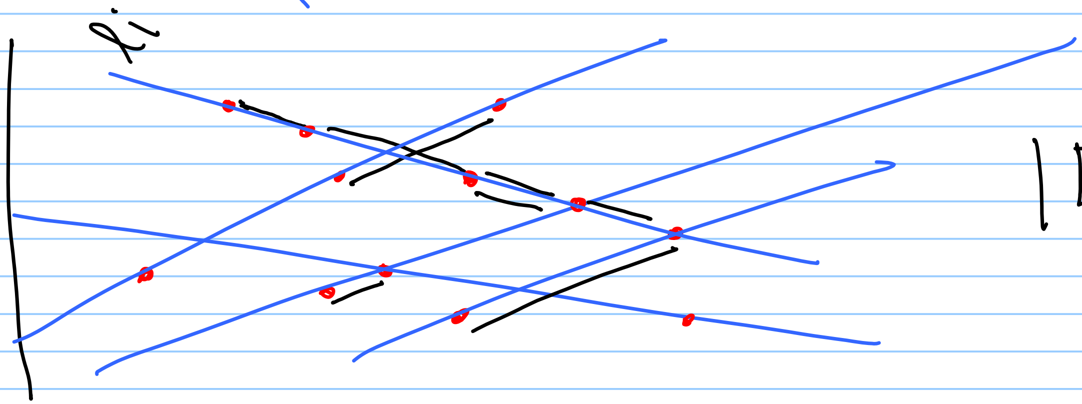
$I(P, L) \leq \underline{I(P, L')} + n$

$\ell_i \leftarrow m_i$

$\sum m_i = I(P, L)$

$|E| \leq 4m$

$I(P, L) < 4m + n$



$$|E| = \sum_{i=1}^{n} (m_i - 1)$$

$$= \underline{I(P, L) - n}$$

(2) $|E| \gtrsim 4m$ $\qquad f \underset{\lesssim t}{\leq} g \qquad f \leq g+t$

$$n^2 \geq Cr(G) \gtrsim \frac{|E|^3}{|V|^2} = \frac{(I(P,L)-n)^3}{m^2}$$

$$\frac{(I(P,L)-n)^3}{m^2} \lesssim n^2 \qquad I(P,L) \lesssim m+n$$

$$\Rightarrow \boxed{I(P,L) \lesssim (nm)^{2/3} + n}$$

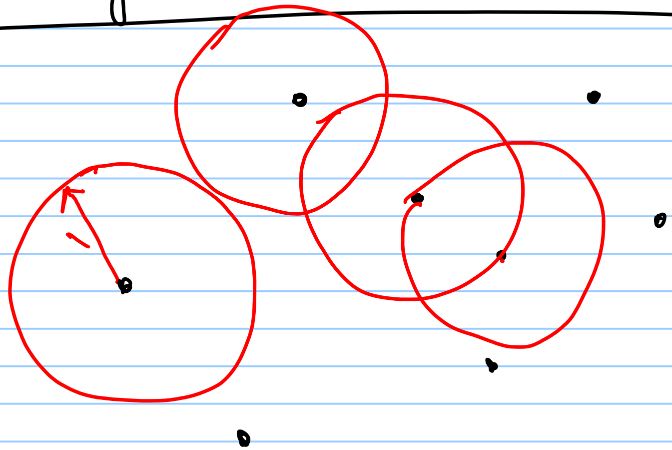
$\square$

Thm (Point Circle Incidences) $L \leftarrow$ set n circles in plane

$P \leftarrow$ set ~~of~~ m points in the plane

$$I(P,L) \lesssim (mn)^{2/3} + m + n$$

Revisiting Erdős ~~distinct~~ no. of unit distance problem



$P \leftarrow n$ circles

$L \leftarrow n$ points

$$I(P,L) \lesssim n^{4/3}$$

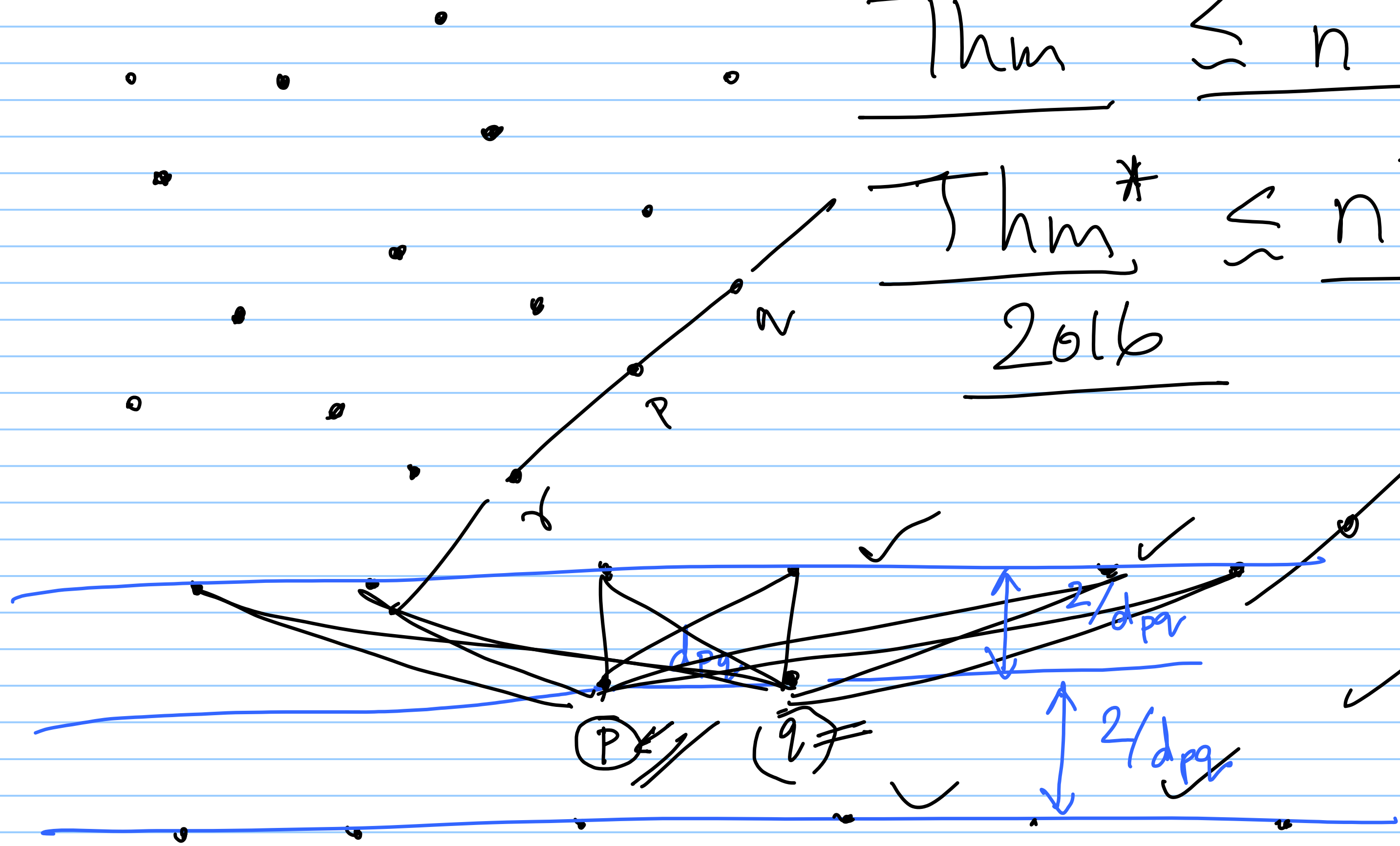
$$u(n) = \frac{I(P,L)}{2} \lesssim n^{4/3}$$

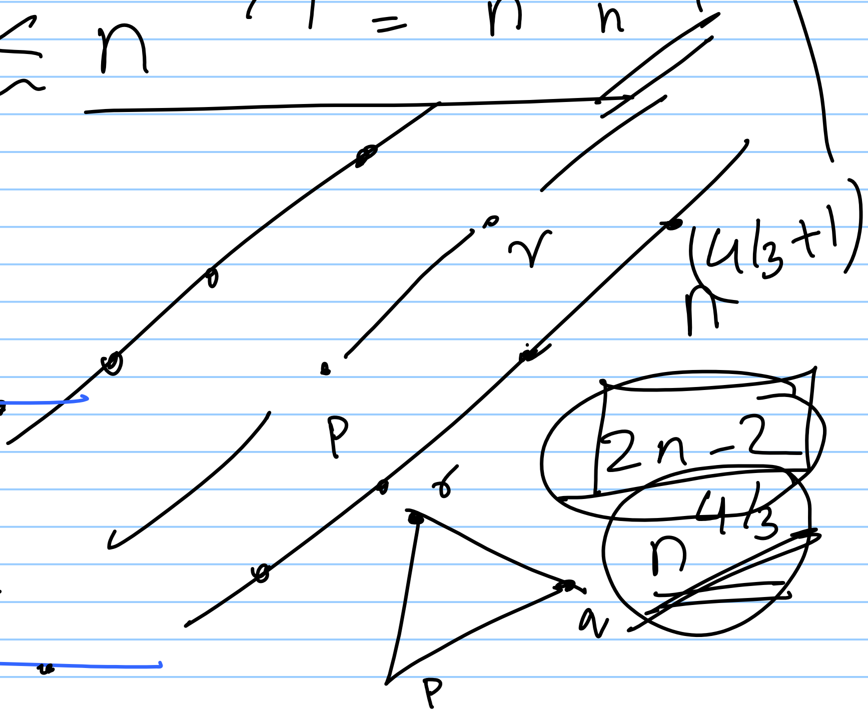
No. of unit triangle problem

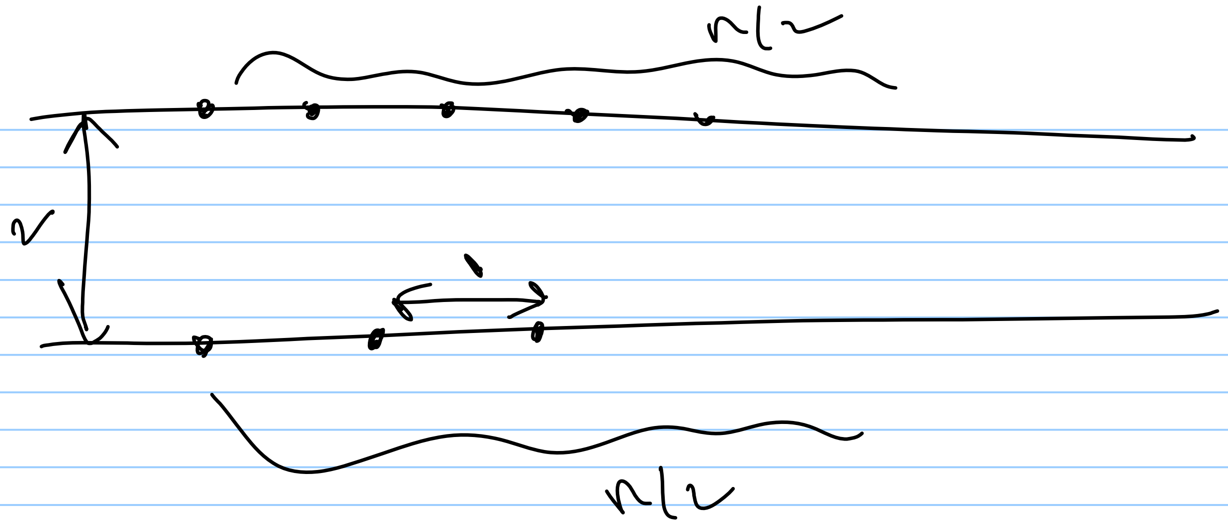
$$\underline{\text{Thm}} \lesssim n^{7/3} = n^2 \times n^{1/3}$$

$$\underline{\text{Thm}^*} \lesssim n^{20/9} = n^2\, n^{2/9}$$

2016







$n^2 \to \boxed{n^2 \log n}$

Erdös–Purdy

$n^{20/9}$