Plane Sweep Algorithm

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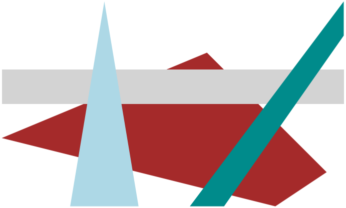
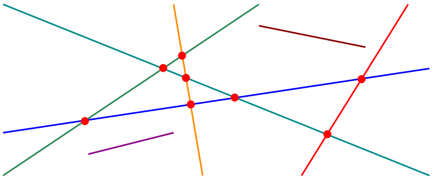
[1]Slide ideas borrowed from Marc van Kreveld and Subhash Suri

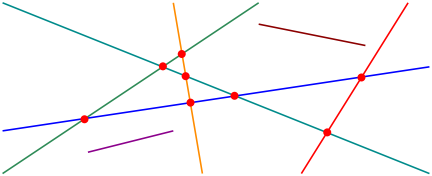
Intersection Detection



- Determine pairs of intersecting objects?
  - Collision detection in robotics and motion planning.
  - Visibility, occlusion, rendering in graphics.
  - Map overlay in GISs: e.g. road networks on county maps.



- Let's first look at the easiest version of the problem:
- Given a set of of n line segments in the plane, find all intersection points efficiently
- Naive algorithm?



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- Naive algorithm?Check all pairs. $O(n^2)$.

Algorithm 1 FindIntersections(S)

Input: A set S of line segments in the plane.

Output: The set of intersection points among the segments in S.

```
1: for each pair of line segments e_i, e_j ∈ S do
2:     if e_i and e_j intersect then
3:         report their intersection point
4:     end if
5: end for
```

Line Segment Intersection

Algorithm 2 FindIntersections(S)

Input: A set S of line segments in the plane.

Output: The set of intersection points among the segments in S.

for each pair of line segments $e_i, e_j \in S$ **do**
    if e_i and e_j intersect **then**
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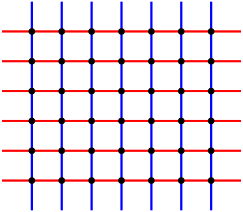
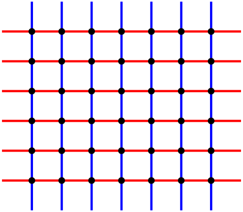
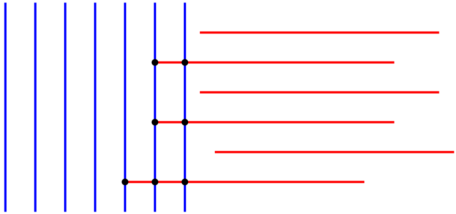
- Question: Why can we say that this algorithm is optimal?

Line Segment Intersection





- The asymptotic running time of an algorithm is always input-sensitive (depends on n)



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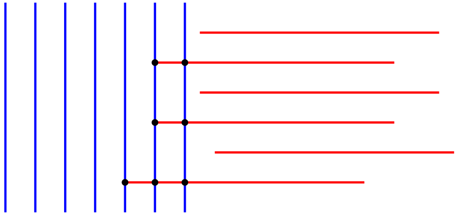

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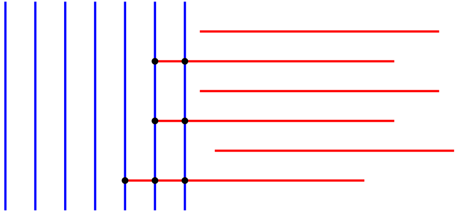

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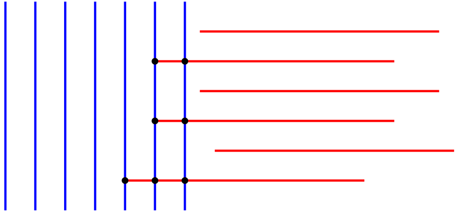

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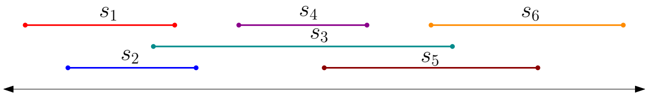
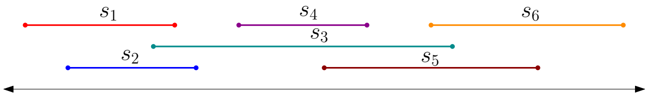
- We will describe a $O((n+k)logn)$ solution. Also introduce a new technique : PLANE SWEEP.

An Easier Problem:



- Given a set of intervals on the real line, find all overlapping pairs.



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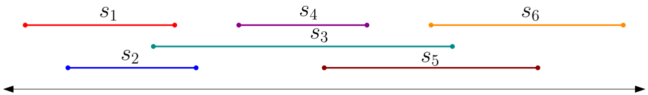

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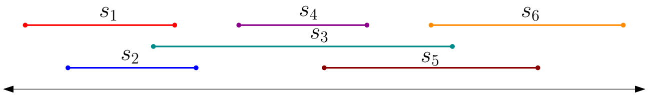

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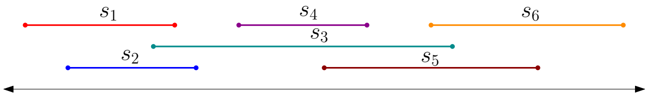

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- The sweep line stops and the algorithm computes at certain positions : EVENTS/ EVENT POINTS
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- The algorithm knows everything it needs to know before the sweep line, and found all intersection pairs.

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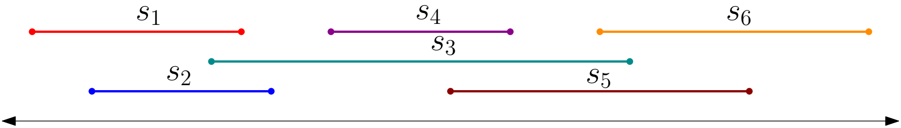

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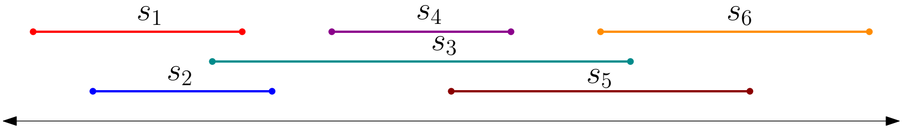

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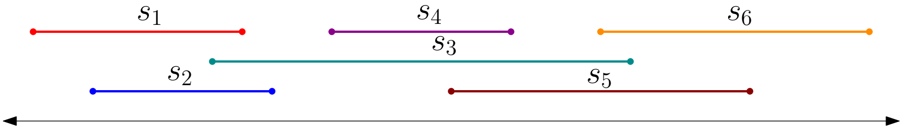

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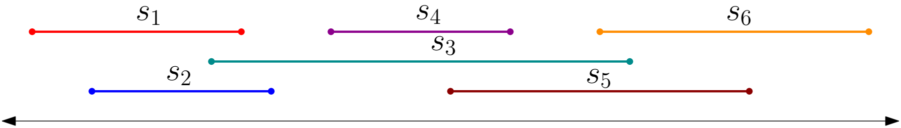

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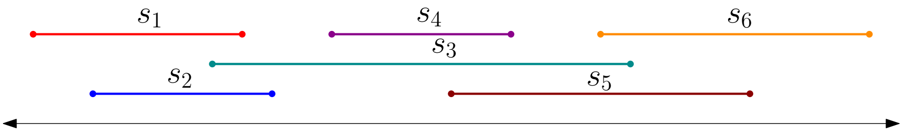

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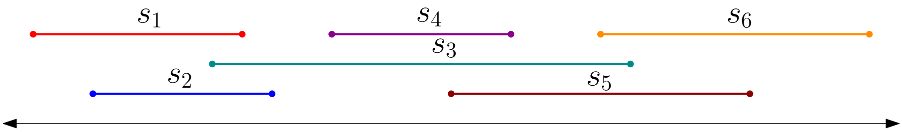

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- $2n * \log n + k$

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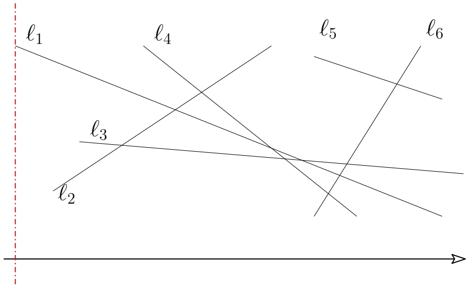

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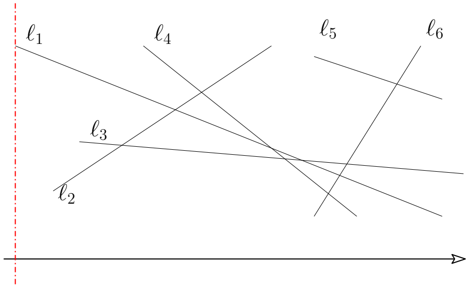

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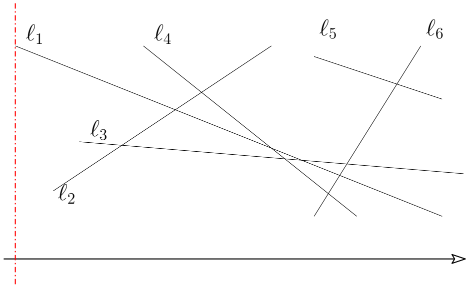

Imagine a horizontal line passing over the plane from top to bottom, solving the problem as it moves

- Question: What are the event points?
- Maintain vertical order of segments intersecting the sweep line;

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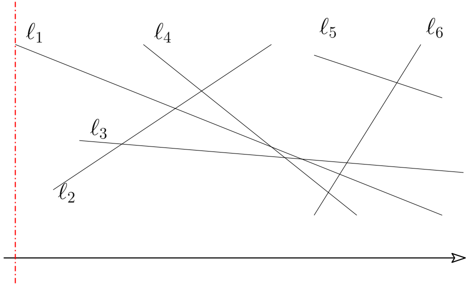

- Insert ℓ_1, add the end point of ℓ_1 to the event queue

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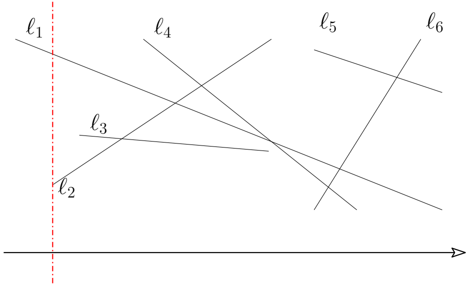

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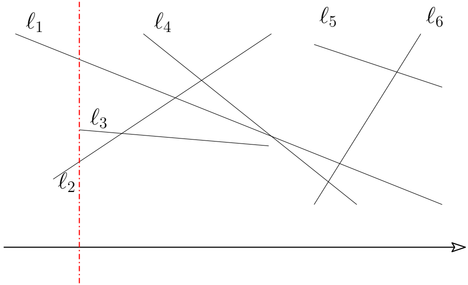
- Insert ℓ_2, Current order ℓ_1, ℓ_2



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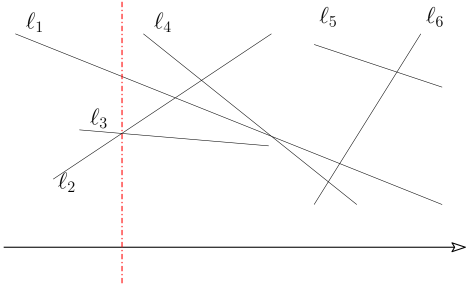

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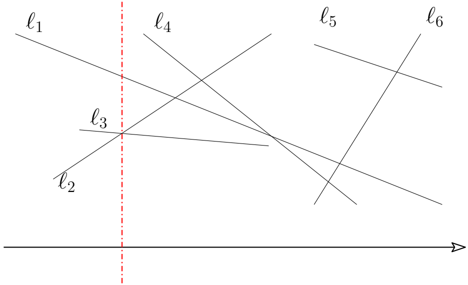

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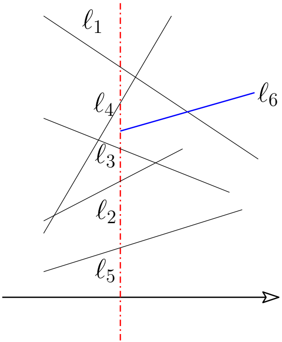
. . . and so on . . .

Events

When do the events happen? When the sweep line is at

- a left endpoint of a line segment
- a right endpoint of a line segment
- an intersection point of a line segment

A left endpoint of a line segment



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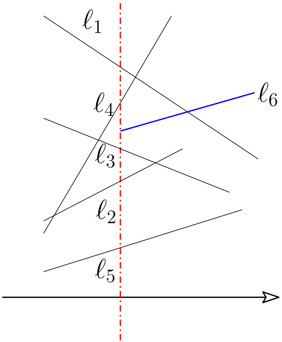
- Search and insert.

A left endpoint of a line segment



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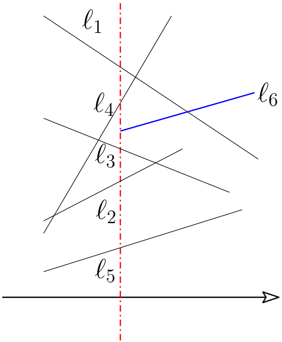

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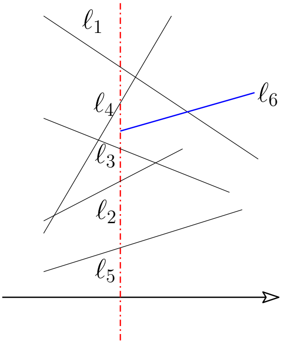

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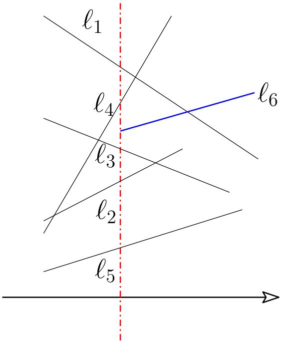
A left endpoint of a line segment



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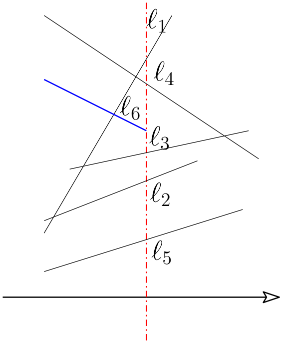

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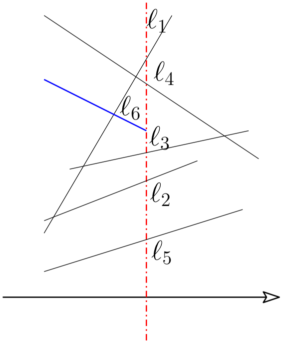
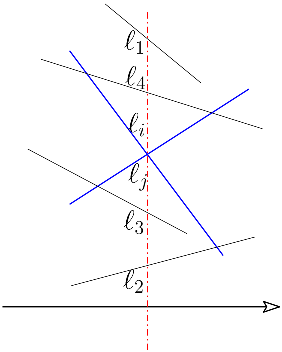
A right endpoint of a line segment



- Sweep line reaches right endpoint of a line segment: delete the line segment
- After deletion of ℓ_6, ℓ_3 and ℓ_4 becomes adjacent.
- If ℓ_3 and ℓ_4 intersects insert the intersection point into the event queue.



Sweep line reaches an intersection point of ℓ_i and ℓ_j

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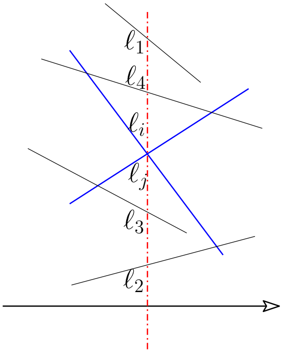

Sweep line reaches an intersection point of ℓ_i and ℓ_j

- Exchange ℓ_i and ℓ_j in the order list.
- If ℓ_i and its new left neighbor intersects, then insert this intersection point in the event queue
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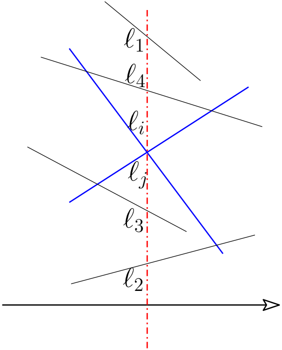
Sweep line reaches an intersection point



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- Report the intersection point.

Finding events

- Before the sweep algorithm starts, we know all upper endpoint events and all lower endpoint events
- But: How do we know intersection point events??? (those we were trying to find . . .)
- Observe: Two line segments can only intersect if they are horizontal neighbors

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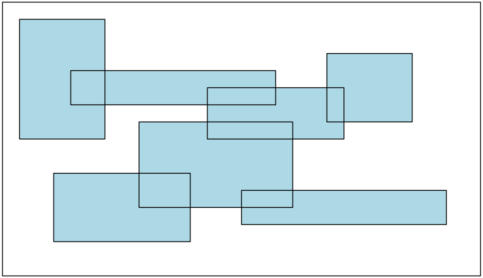
- Note that if k is really large, the brute force $O(n^2)$ time algorithm is more efficient



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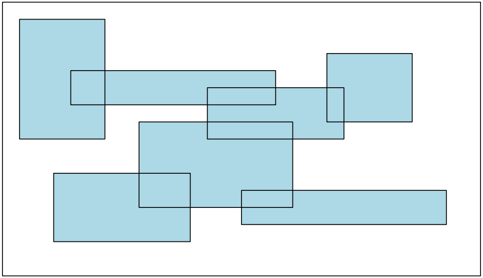
Area of Union of rectangles



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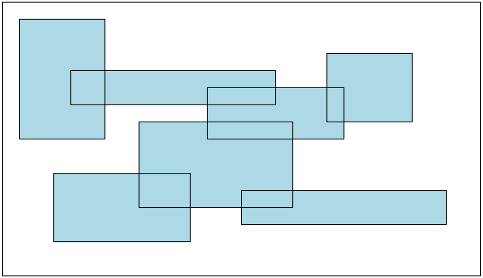

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- PLANE SWEEP: Keep track of the area that being sweeped.
- EVENT POINTS: Left and right end point of the rectangeles.
- What is the area between any two sweep lines?

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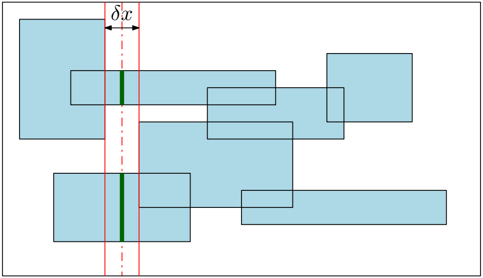
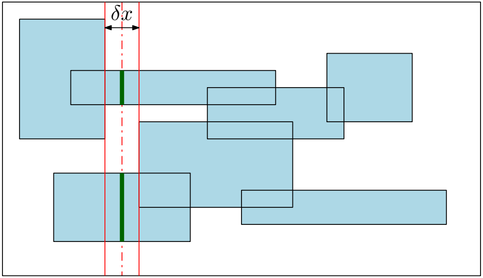
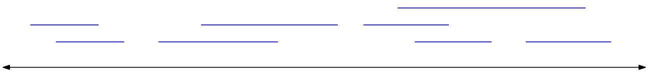


- Given a layout in which objects are orthogonal polygons with sides parallel to the axises. The task is to the area covered by all the objects.
- PLANE SWEEP: Keep track of the area that being sweeped.
- EVENT POINTS: Left and right end point of the rectangeles.
- What is the area between any two sweep lines?
- $\delta x \times y$ where y is the length of the intersection of the the rectangels with the sweep line.



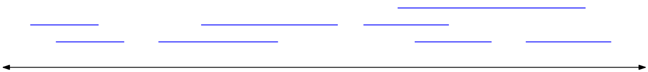
- Intersection of the the rectangels with the sweep line is a set of intervals.
- Thus the problem at hand becomes to maintain the intercepts. The y can change only at
  - The beginning of a rectangle.
  - The end of the rectangle.

Sum of the union of the intervals



- Naïve Method:

Sum of the union of the intervals



- Naïve Method:
- At each event point find out y by a sweepline method.
- Complexity $O(2n \cdot n \log n) = O(n^2 \log n)$

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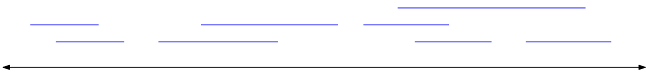
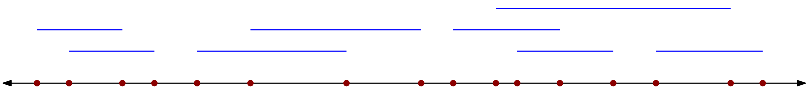


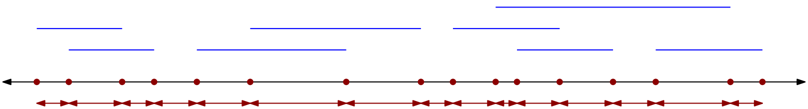
- Naïve Method:
- At each event point find out y by a sweepline method.
- Complexity $O(2n \cdot n \log n) = O(n^2 \log n)$
- How to maintain sum of the union of the intervals with respect to insertion and deletion.

Sum of the union of the intervals



- Sort the end points of the intervals.
- This will create a set of elementary intervals.

Sum of the union of the intervals



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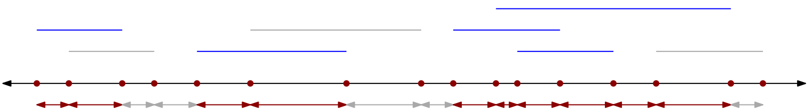
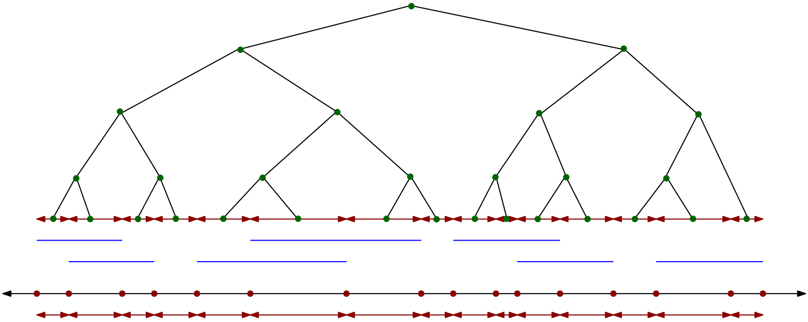
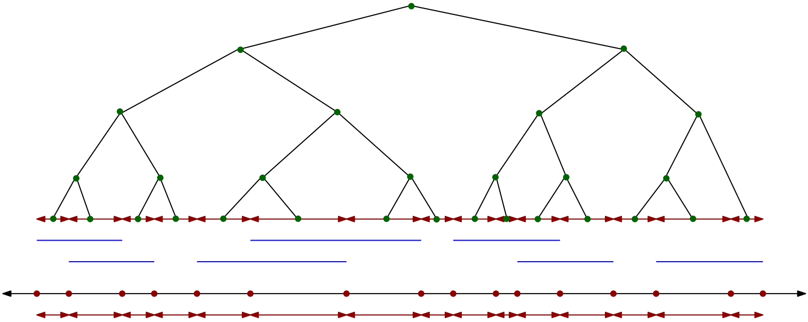


- Sort the end points of the intervals.
- This will create a set of elementary intervals.
- Depending on which intervals are ACTIVE , a set of elementary intervals will be ACTIVE .



- We maintain a special data structure called the INTERVAL-TREE

Interval-Tree



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- It is a balanced binary tree of the ELEMENTORY INTERVALS.

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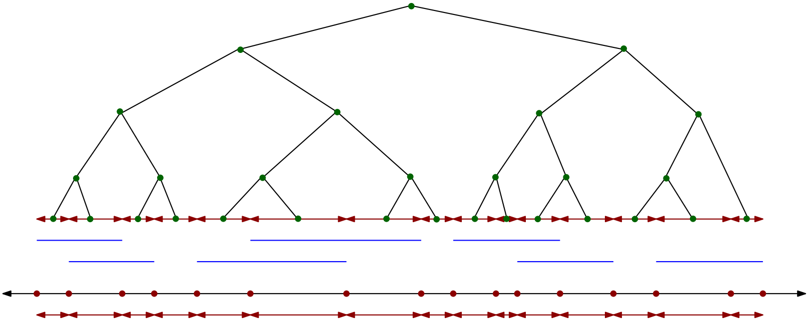
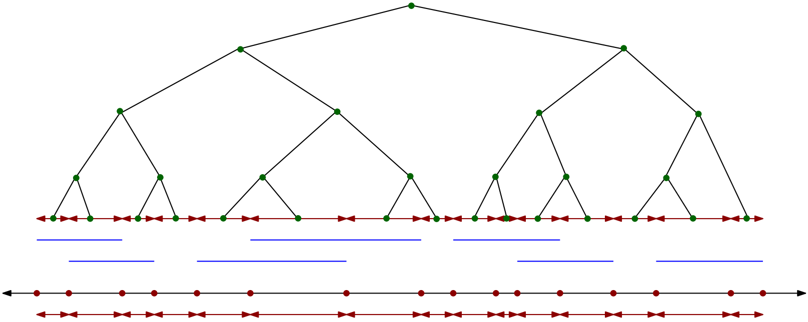


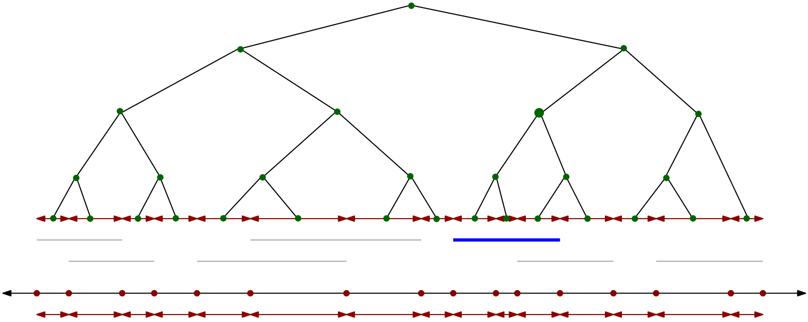
- We maintain a special data structure called the INTERVAL-TREE
- It is a balanced binary tree of the ELEMENTORY INTERVALS.
- Each node represents an interval.

Interval-Tree

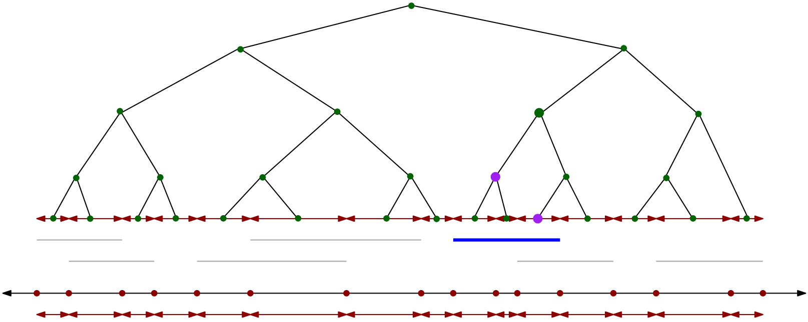


- How an interval is stored in the tree?

Interval-Tree



- How an interval is stored in the tree?
- Start with the root and find a split node.



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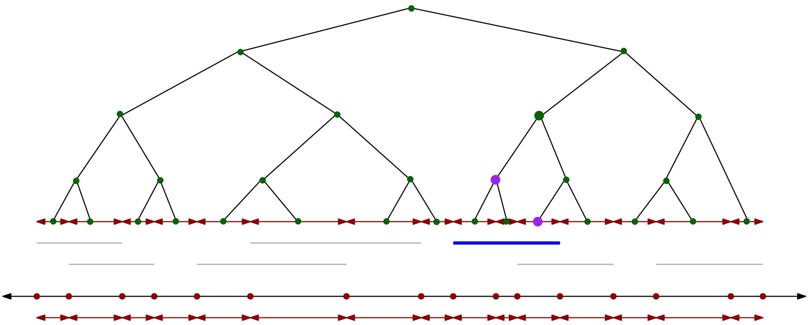
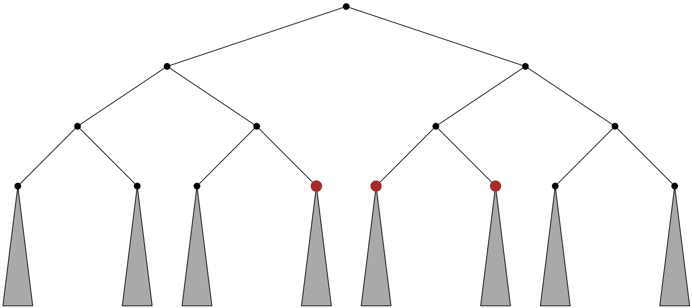


- Claim: Each interval is stored in $O(\log n)$ nodes.

Interval-Tree



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- All of them have to be consecutive.

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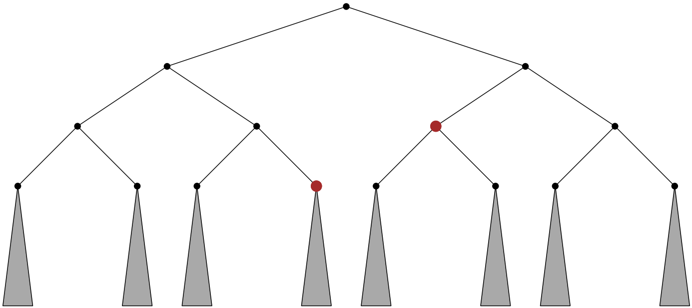

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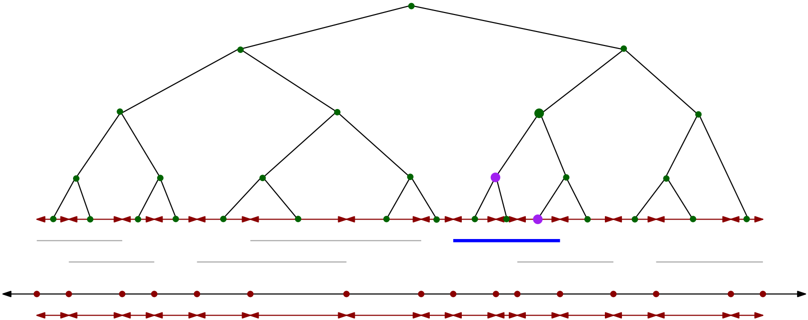
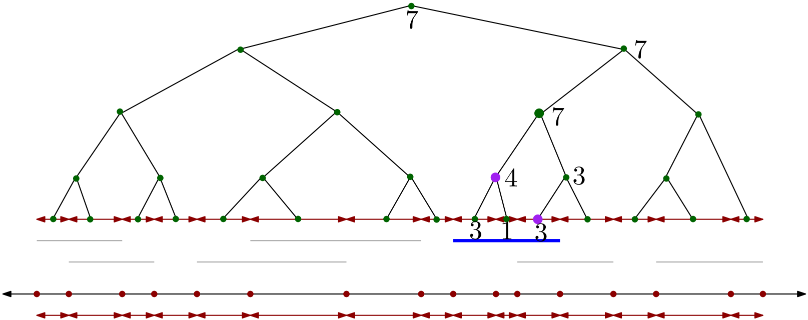


- Each interval can be inserted and deleted in $O(\log n)$ time.
- At each node we maintain the length of the active elementary intervals.



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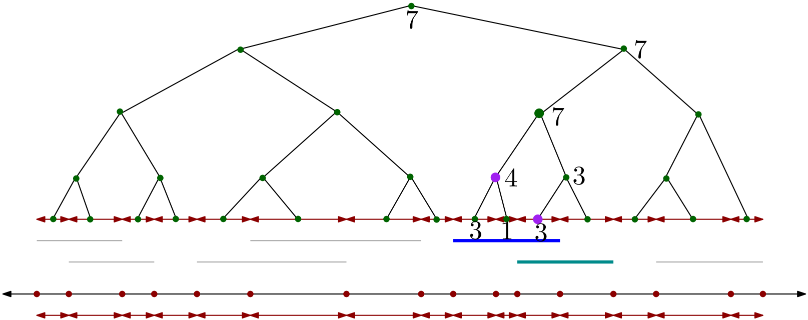

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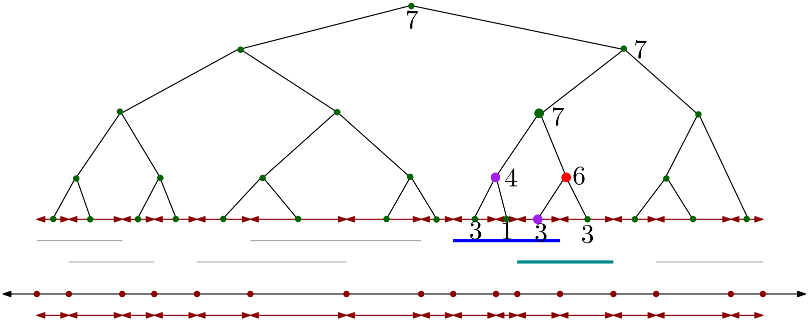

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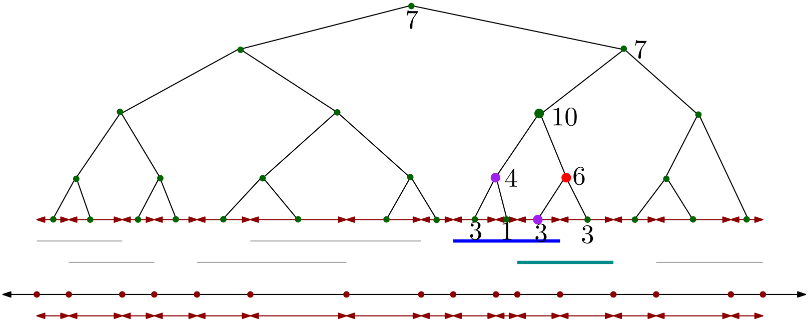

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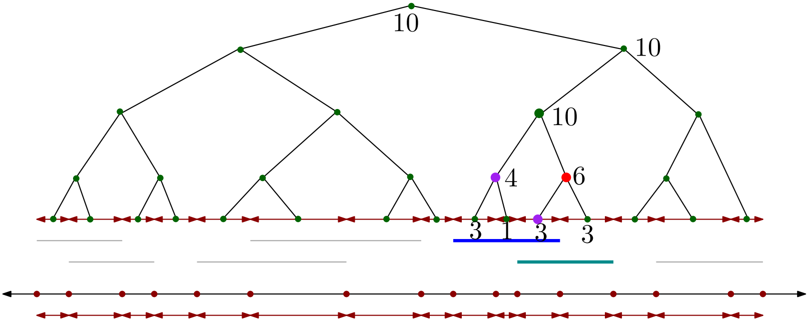

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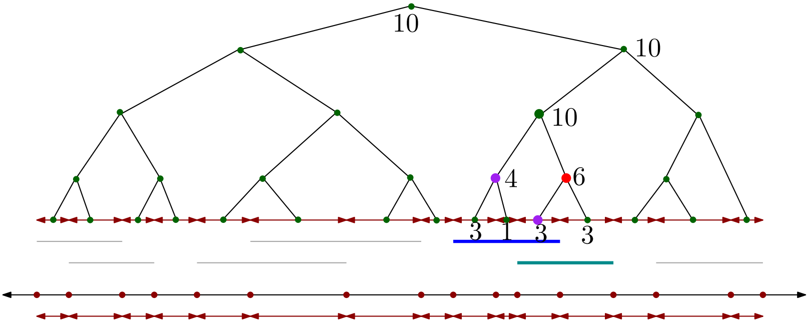

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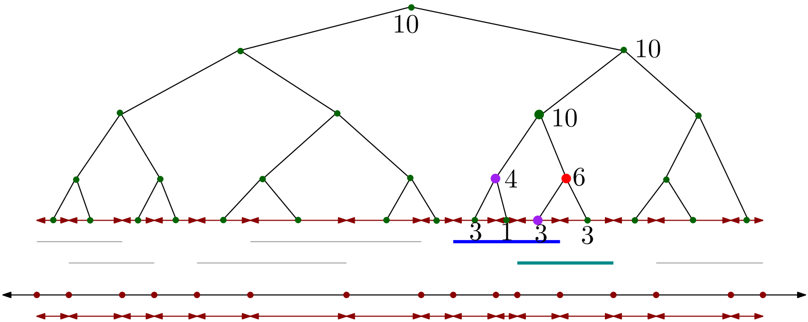
- In time $O(\log^2 n)$ we can perform the updates.



- $O(\log n)$ insert each takes $O(\log n)$ time.
- In time $O(\log^2 n)$ we can perform the updates.
- Can be done in time $O(\log n)$ Homework

Sum of the union of the intervals

- In time $O(n \log^2 n)$ we can find out the area of the union of n rectangles.