- Local expansion in planar graphs:
- Let $G = (A \cup B, E)$ be a bipartite graph.
- Suppose for an integer k,

$$\forall\, S \subseteq A,\quad |S| \leq k, \quad \text{we have: } |S| \geq |N(S)|.$$

- Is it true that $|B| \geq \alpha(k)|A|$, for some $\alpha(k)$?
- Consider a complete bipartite graph, $|A| = n$, $|B| = k$.
- Then, $|N(S)| \geq k$, $\forall\, S \subseteq A$, $|S| \leq k$.
- But, $|B| \ll |A|$.

What if we restrict our attention to planar bipartite graphs?

- Then, we can prove that the claim is true.

Theorem: Let $G = (A \cup B, E)$ be a planar bipartite graph, and $k \geq 3$ such that $\forall\, S \subseteq A$, $|S| \leq k$, $|N(S)| \geq |S|$.

Then,

$$|A| \leq \left(1 + c/\sqrt{k}\right)|B|.$$

[Note: the theorem requires $k \geq 3$, since for $k = 1, 2$, the complete bipartite graph: $|A| = n$, $|B| = k$ is planar].

- Consider the case $k=3$, before we look at the case for arbitrary k:

  - For each $S \subseteq A$, $|S| \leq 3$, $|N(S)| \geq |S|$.

  - Adding edges does not weaken the property, and therefore let us assume that G is a maximally planar graph.

  - Then, $\deg(v) \geq 2 \;\; \forall\, v \in A$.

  - Let $A_2 \subseteq A$ be the vertices in A of degree 2.
  - Let $A_3 \subseteq A$ " " " " " ≥ 3.

  - Since G is a planar graph, $|E| \leq 2|A \cup B| - 4$.

Therefore,

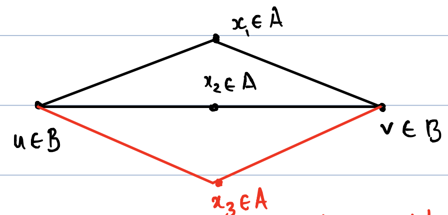
$$\sum_{v \in A} \deg(v) = \underbrace{\sum_{v \in A_2} \deg(v) + \sum_{v \in A_3} \deg(v)} \leq |E| \leq 2|A| + 2|B| - 4.$$

$$\Rightarrow \quad 2|A_2| + 3|A_3| \leq |E| \leq 2|A| + 2|B| - 4.$$

$$\Rightarrow \quad 3|A_3| \leq 2(|A| - |A_2|) + 2|B| - 4.$$

$$\Rightarrow \quad 3|A_3| \leq 2|A_3| + 2|B| - 4.$$

$$\Rightarrow \quad |A_3| \leq 2|B| \quad \text{—— (1)}$$

- Now consider the induced subgraph on $A_2 \cup B$.
- 

↳ not possible ∵ local expansion.

- Construct a multigraph G_B on the vertices in B, where $u, v \in B$ are adjacent, if they are at distance 2.

- Note that there are atmost 2 edges between $u, v \in G_B$, otherwise we violate the local expansion condition.
- The fact that G_B is planar follows from the fact that G is planar.
- Therefore, $|A_2| \leq 6|B|$. ——(2)

- Hence, from (1) & (2), $|A| \leq 8|B|$.

- This analysis is tight. There are examples where $|A| \leq 8|B|$.

Proof of theorem for general k: Follows directly from the r-division of the graph; and the analysis is essentially the same as the one we did for independent sets in planar graphs.

Local Search for Geometric problems:

- We will follow the same local search approach as we did for MIS in planar graphs.

- Recall: The local search paradigm is the following:

Local Search Schema:

1. Start with an arbitrary feasible solution. S
2. While possible try to find a feasible solution S',
   where $|S \triangle S'| \leq k$, and
   Symmetric difference: $(S \setminus S') \cup (S' \setminus S)$
   S' is better than S. specific to the problem.
   If such an S' is found, replace S by S'.
3. Return the solution S.

- Let L be a locally optimal solution, and OPT be an optimal solution
- For MIS in planar graphs, we saw that the induced bipartite graph on $L \cup OPT$ is k-locally expanding (for either OPT, or L)

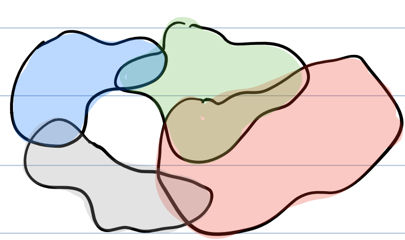
the local optimality condition.

Theorem (Informal) For an optimization problem Π, if we can construct a k-locally expanding planar bipartite graph on $L \cup OPT$, such that each local exchange is feasible, then local search yields a PTAS.

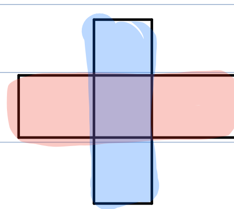
a $(1 \pm O(1/\sqrt{k}))$-approximation.

MIS for intersection graph of (pseudo-) disks:

Pseudodisks: Regions defined by simple jordan curves, such that for any pair, their boundaries either do not intersect, or intersect exactly twice.

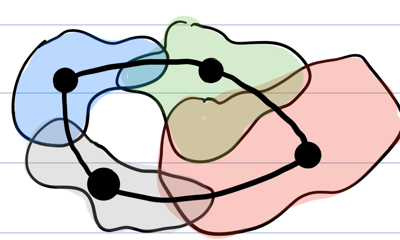


Pseudo disks.



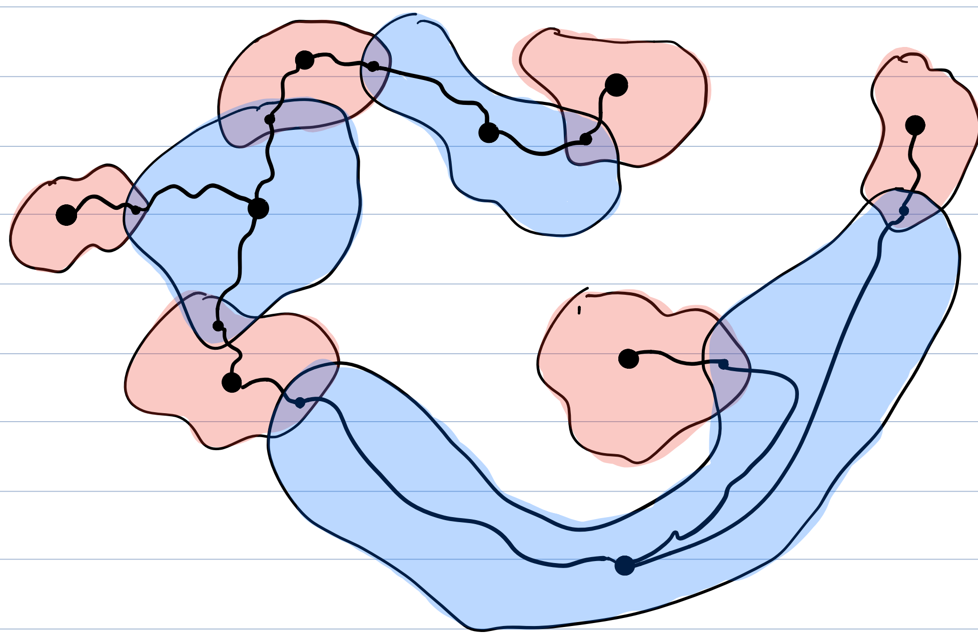
Not Pseudodisks.

Intersection graph: A graph whose vertex set is the set of regions, and two vertices are adjacent if and only if they intersect.



Theorem: [Char, Har-Peled '09] Local Search with parameter k yields a $(1 - O(1/\sqrt{k}))$-approximation for MIS in intersection graphs of pseudodisks.

Proof:
- Let L be the locally optimal solution returned by the algorithm (colored RED)
- Let OPT be an optimal solution (colored BLUE).



- **Claim:** The intersection graph of $L \cup OPT$ is planar.

  Proof:
  - We can assume no pseudo-disk is contained in another. Otherwise, we can remove the larger pseudodisk from the input before running the local search algorithm.
  - Since L & OPT are independent sets, every point in the plane is in at most 2 pseudo-disks.

- Since there is no containment, every pseudo-disk contains a point not in any other pseudo-disk; and this region is connected.
  - Place a vertex for each pseudo-disk in such a region.
- Put a dummy node at the intersection region of 2 pseudo-disks.
- We can add curves from the vertex for each pseudo-disk to the dummy nodes, without any edges crossing.

- This yields a planar graph.

- Since $L \cup OPT$ is planar, there exists an r-division.
- Further for any set S of size $\leq k$ in L, we have:

  $|S| \geq |N(S)|$ (otherwise, we have a local improvement)
- For any set $O \subseteq OPT$, $(L \setminus N(O)) \cup O$ is an independent set, where $N(O)$ is the set of neighbours of O in the graph constructed above.

Now, the result follows exactly along the lines of MIS for planar graphs.

$\square$

Theorem: Local-search yields a PTAS for the minimum hitting set problem for disks.

- Given a set of disks $\mathcal{D}$, and a set P of points, find the smallest number of points that hit each disk.

Proof: We show that there is a k-locally expanding graph for the hitting set problem.

- We can assume that $L \cap OPT = \emptyset$.
- Let G' be the delaunay triangulation on $L \cup OPT$.
- G' is clearly planar.
- Remove the edges in G' between two vertices in OPT, or between two vertices in L. Let G be the resulting graph.
- We claim that G is k-locally expanding, and that it satisfies the locality condition

$\hookrightarrow$ $(L \setminus S) \cup N(S)$ is a feasible hitting set for any $S \subseteq L$.

Claim: For any disk $D \in \mathcal{D}$, the subgraph $G[D]$ induced by the points in D is connected.

Since every disk contains atleast one point of L & one point of OPT, it follows that:

$(L \setminus S) \cup N(S)$ is a feasible hitting set.

- Since G is k-locally expanding, and satisfies the locality condition, we obtain a PTAS by an argument similar to that of MIS for planar graphs.

For completeness, we give the argument here:

- By the theorem on r-divisions, applied to the delaunay triangulation $G=(V,E)$ of $L \cup OPT$, we know that for any parameter $r>0$, $\exists$ constants $c_1, c_2, c_3 > 0$ such that:

  we can partition V into sets: $V_1, \ldots, V_{n/r}, X$ such that:

  (i) $|V_i| \leq c_1 r$

  (ii) $|X| \leq c_2 n/\sqrt{r}$.

  (iii) $|V_i \cap X| \leq c_3 \sqrt{r}$.

  (iv) $(V_i \setminus X) \cap (V_j \setminus X) \neq \phi$, $i \neq j$.

- We require $k > c_1 r + c_2\sqrt{r} \Rightarrow$ choose $r = \gamma k \leq k/(c_1+c_2)$.

- Let $OPT_i = V_i \cap OPT$. Let $L_i = L \cap V_i$.

- By local optimality, & the fact that G satisfies the locality condition, $|L_i| \leq |OPT_i|$. ——(1)

$$\sum_{i=1}^{n/r} |L_i| \leq \sum_i |OPT|_i .$$

$$\Rightarrow \quad \sum |L_i| + |L \cap X| \leq \sum |OPT_i| + |L \cap X|$$

$$L \leq \sum |OPT_i| + |L \cap X| + |OPT \cap X|$$

$$= |OPT| + |L \cap X|.$$

$$\leq |OPT| + |X|$$

$$\leq |OPT| + c_2 \frac{|OPT| + |L|}{\sqrt{r}}$$

$$L\left(1 - \frac{c_2}{\sqrt{r}}\right) \leq |OPT| \left(1 + \frac{c_2}{\sqrt{r}}\right).$$

$$\Rightarrow \quad L \leq |OPT| \left(\frac{1 + c_2/\sqrt{r}}{1 - c_2/\sqrt{r}}\right).$$

$$\leq |OPT| \frac{\left(1 + \frac{c_2\sqrt{\gamma}}{\sqrt{k}}\right)}{\left(1 - c_2\sqrt{\gamma}/\sqrt{k}\right)}$$

$\leq |OPT|(1+\varepsilon)$, for $k \leq d/\varepsilon$, where d is a sufficiently large constant.

- The key step in proving that Local Search is a PTAS for an optimization problem is the construction of a graph on $L \cup OPT$ such that:

  (i) G is from a hereditary class of graphs that admit a sub-linear size separator.

  (ii) G is k-locally expanding, and

  (iii) G satisfies the locality construction.

- If the three conditions are met, then we obtain a PTAS by using the structure of an r-division.