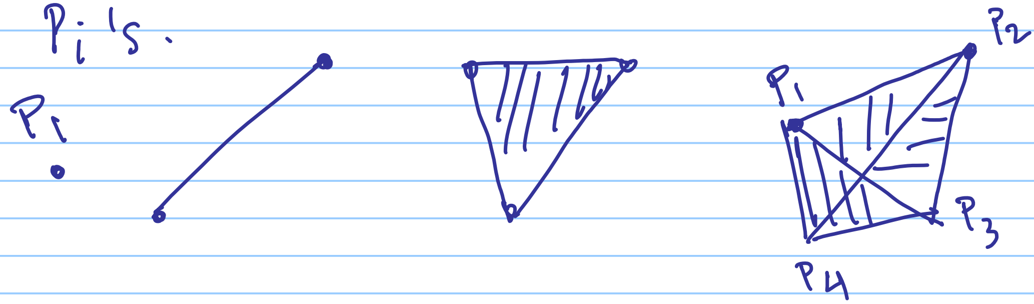
Simplicial Complex / Polytopes & Polyhedron / point hyperplane duality $\mathbb{R}^d$

Recallect
(1) Affine independence
(2) Convex hull

Def 1 (Geometric Simplex) $\{P_1, \ldots, P_k\} \subset \mathbb{R}^d$, s.t $k \le d+1$, and they affinely independent. $\sigma = [P_1, \ldots, P_k]$ be a convex hull of point P_i's.



$\mathbb{R}^2$

tetrahedrons ,

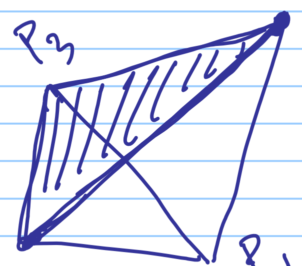
Dimension of a simplex $\sigma = [p_0, p_1, \dots, p_k]$, then its dimension $(k-1)$

Def (Faces of simplex) $\sigma = [p_0, \dots, p_k]$, let $I \subseteq \{0, \dots, k\}$

$A = \{p_i \mid i \in I\}$, $\operatorname{conv}(A)$ $\qquad A \subseteq B$

$\operatorname{conv}(A) \subseteq \operatorname{conv}(B)$

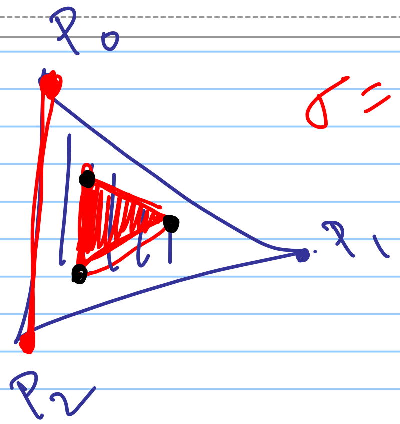
$$\Rightarrow \sum_{i \in I} \lambda_i p_i, \quad \text{s.t } \lambda_i \geq 0, \quad \sum_{i \in I} \lambda_i = 1$$



$I = \{0, \cancel{2}\}$ $\quad \{0, 2, 3\}$

Q | How faces does a k-simplex?

Note Title 05-06-2019



$\sigma = [P_0, P_1, P_2]$

Simple properties to verify

(1) $I \subseteq \{0, \ldots, k\}$ $A_I = \{P_i,\ i \in I\}$

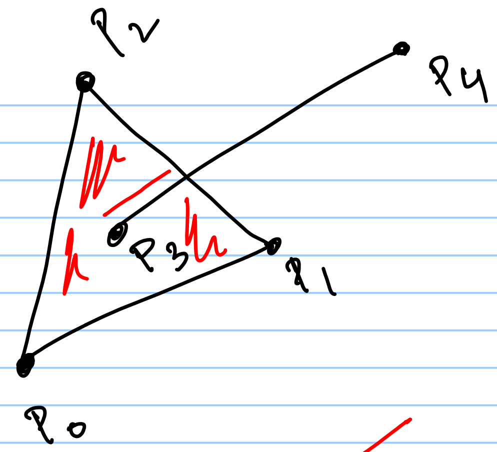
~~dom~~ conv (A_I) is also a simplex of dimension $|I| - 1$.

Def (Geometric Simplicial Complex) It K will be a collection of simplices in $\mathbb{R}^d$, s.t the following two properties hold.

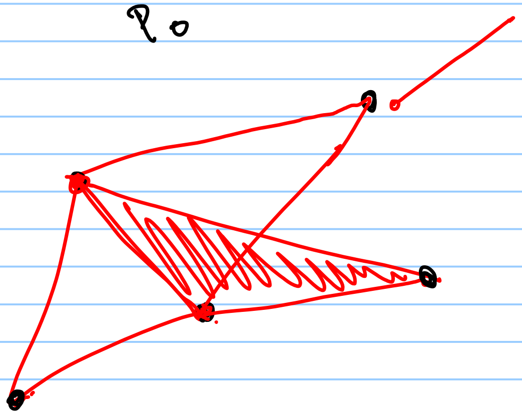
(i) Faces of a simplex in K is also contained in K

(ii) Intersection of two simplices in K is either empty or it's is also a simplex in K.

K

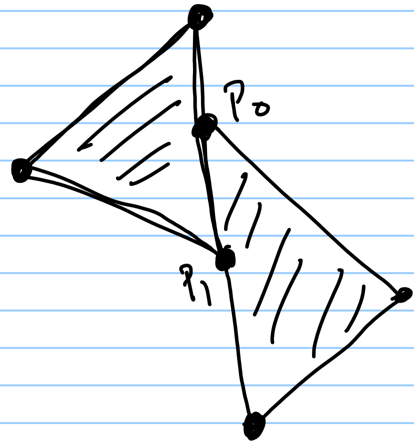


Collection $\left\{ [P_0],\ [P_1],\ [P_2],\ [P_3],\ [P_4],\ [P_0, P_1],\ [P_1, P_2],\ [P_0, P_2],\ [P_3, P_4],\ [P_0, P_1, P_2] \right.$



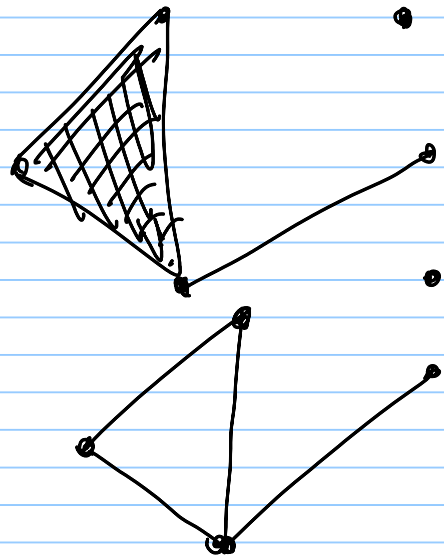
Simplicial complex

$\mathbb{R}^2$



$[P_0, P_1]$

Def 3 (Geom. Simplicial Complex Dim) ~~Dim~~ This is equal to the maximum dim of a simplex contained in the simplicial complex.



dim = 2.

1-skeleton

j-skeleton of complex.

of K $\quad K' \leftarrow$ j-skeleton

$$K' \subseteq K$$

Q2 Is the j-skeleton of K also simplicial complex?

Abstract Simplicial complexes

$[n] = \{1, \dots, n\}$

Def (Abstract Simplicial complexes) $\hat{K}$ Collection of subsets of $[n]$ which are downward closed.

ex $[4] = \{1, \dots, 4\}$ $\hat{K} = \{ \phi, \{1\}, \{2\}, \{1,2\}, \{3\}, \{3,1\} \}$

$\tilde{K}$ = Collection of subsets satis some property

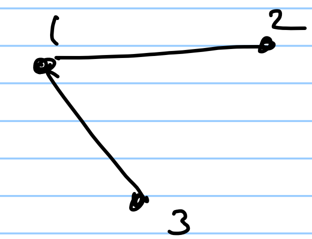
$\sigma \in \tilde{K}$

$\dim(\sigma) \overset{\text{def}}{=} |\sigma| - 1$

$\dim(\tilde{K}) = \ldots$

Thm (Well known) If the dim of $\tilde{K}$ is t, then it has rep. as a geometric simplicial complex in $\mathbb{R}^{2t+1}$.

Ex $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}$



$$\tilde{K} \xrightarrow{\pi} K$$

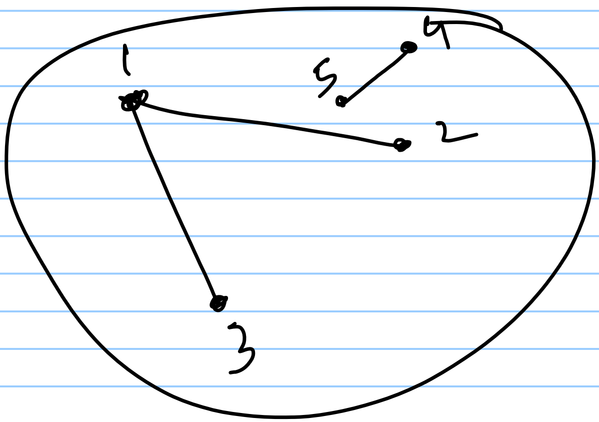
$$\sigma \in \tilde{K} \longrightarrow \underline{\pi(\sigma)} \in K$$

simplex in K

$$\sigma', \sigma \in \tilde{K}$$

$\pi(\underline{\sigma \cap \sigma'})$ simplex in K

Proof



$\{1\} \ldots \{5\}$

$\{1,2\}, \{1,3\}, \{5,4\}$

Proof $\tilde{K}$ $[n]$ $\xrightarrow{}$ t

K ← all t-dim simplex that I can make with

$\mathbb{R}^{d = 2t+1}$ $P_1, \ldots, P_n$

$P_1, \ldots, P_n$ in $\mathbb{R}^d$

$\uparrow \quad \uparrow \quad \quad \uparrow$

$1 \quad 2 \quad \quad n$

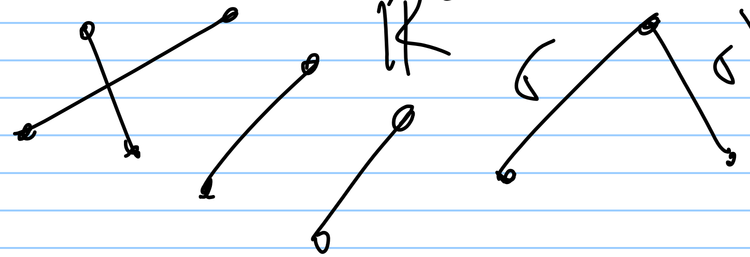
$\pi(i) \longmapsto P_i$

$\{P_1, \ldots, P_{t+1}\}$ K

$\sigma_1 = [P_0, \ldots, P_t]$

$\sigma_2 = [P_{t+1}, \ldots, P_{2t+1}]$

$\mathbb{R}^2$



$\{P_0, \ldots, P_t\}$ $\{P_{t+1}, \ldots, P_{2t+1}\}$ $d = 2t+1$

$\sigma' = [P_0, \ldots, P_t, \ldots P_{2t+1}]$

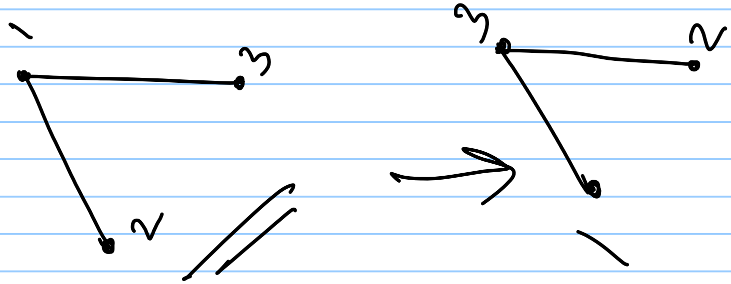
$\boxed{\sigma} = \cancel{K}$

$\boxed{K_\sigma} = \{\text{all faces of } \sigma\}$

Def (Geom, Abs Simplicial Complex)

$$\underline{K} \qquad\qquad \underline{K'} \qquad \Big| \quad \phi : V \longrightarrow V'$$
$$V \qquad\qquad V' \qquad \Big| \quad \text{st } [v_1, \ldots, v_t] \in K$$

iff $[\phi(v_1), \ldots \phi(v_t)] \in K'$.



Nerve

$$S = \{ \underbrace{S_1, \quad \dots \quad, S_\ell} \} \qquad S_i \subseteq \Omega$$

$$v_i \longleftrightarrow S_i$$

Def $\underline{\text{Nerve}(S)}$ (i) $\underline{\{v_1, \dots, v_\ell\}}$

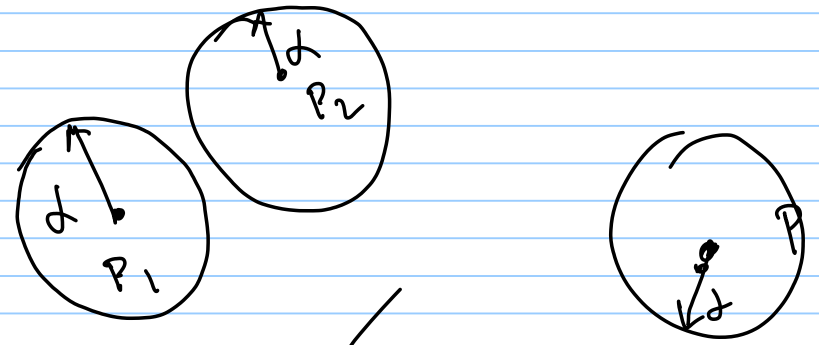
(ii) $\sigma \subseteq \{v_1, \dots, v_\ell\} \in \underline{\text{Nerve}(S)}$

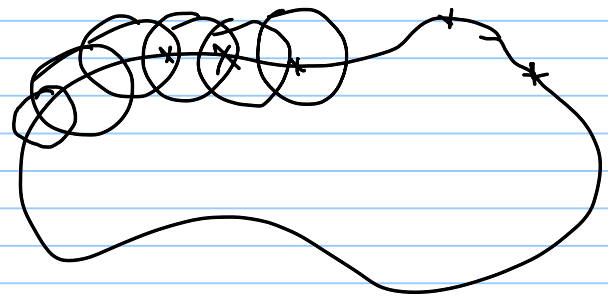
iff $\bigcap_{v_i \in \sigma} S_i \neq \phi$.

Topological Data Analysis

$\mathbb{R}^d$

$\mathcal{F}$





$\subseteq \mathbb{R}^d$

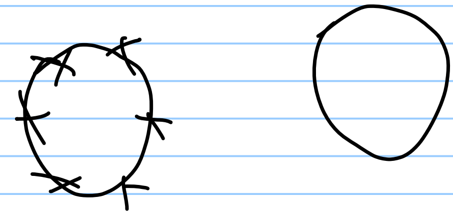
$\sigma \subseteq \{P_1, \dots, P_n\}$

$$\check{C}ech^{\alpha}(\mathcal{F}) \ni \sigma \text{ iff } \bigcap_{P_i \in \sigma} B(P_i, \alpha) \neq \phi.$$

Thm | $Rips^{\alpha}(\mathcal{F})$

Thm Dese point sample on "nice" manifold M.

$\exists \alpha > 0, \rightarrow f(M)$ s.t if take the union of balls of radius α on the point sample then they have same top as manifold.

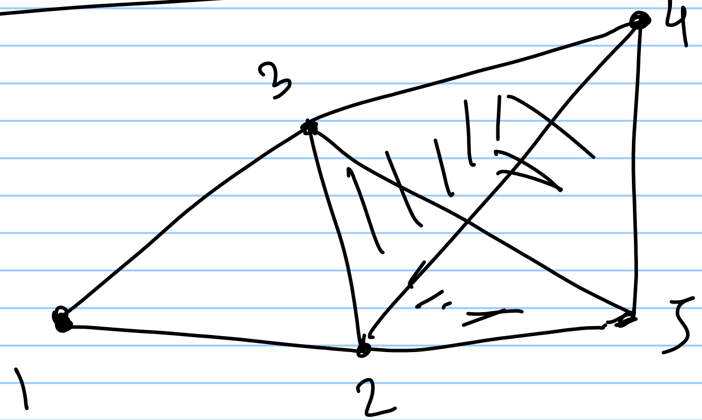


Ex2 $P = \{P_1, \ldots, P_n\} \in \mathbb{R}^d$

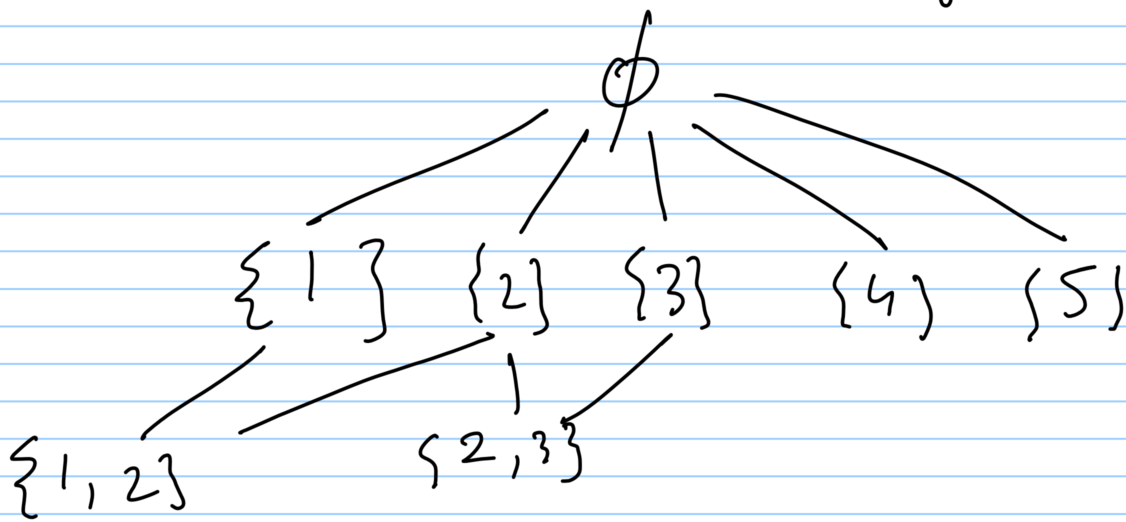
$$Vor(P_i) = \{ x \in \mathbb{R}^d \mid d(x, P_i) \leq d(x, P_j), \forall P_j \in P \}$$

Nerve of the Voronoi cells of p_i's

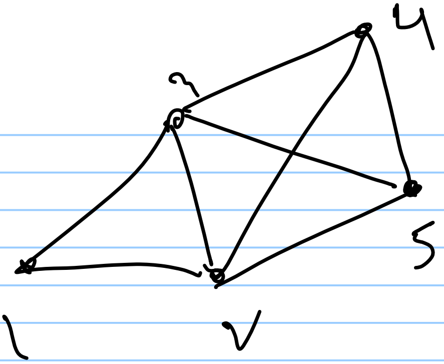
How simplices are stored



Hasse diagram



Simplex-Tree :

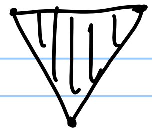
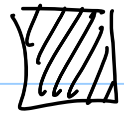


1, 2, 3, 4, 5,
12, 23, . . .
, 2345

Convex Polytope / Polyhedron

Def (Convex Polytope) Conv hull of a finite set in $\mathbb{R}^d$.

- dim of Polytope ?

$$h_{a,b} = \{ x \mid \langle x, a \rangle + b = 0 \}$$

$a \in \mathbb{R}^d$

$\langle a, x \rangle$ dot product in $\mathbb{R}^d$

$b \in \mathbb{R}$

$$h_{a,b}(x) = \langle x, a \rangle + b$$

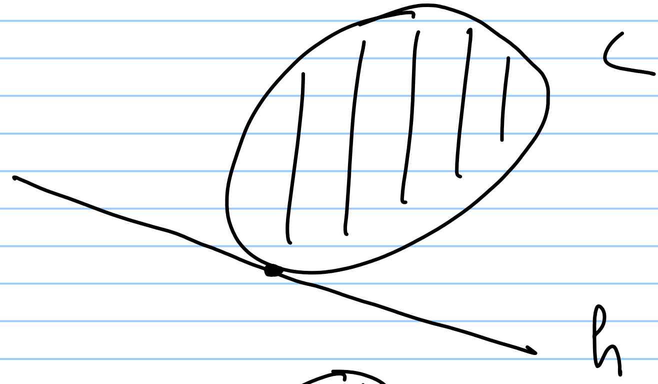
$$h^{\pm}_{a,b} = \left\{ x \mid h_{a,b}(x) \begin{matrix} > 0 \\ < 0 \end{matrix} \right\}$$

Def (Faces of Polytopes) ~~Separating~~ supporting hyperplane of convex body C.

h is sep. hyp for C

if (1) $h \cap C \neq \phi$

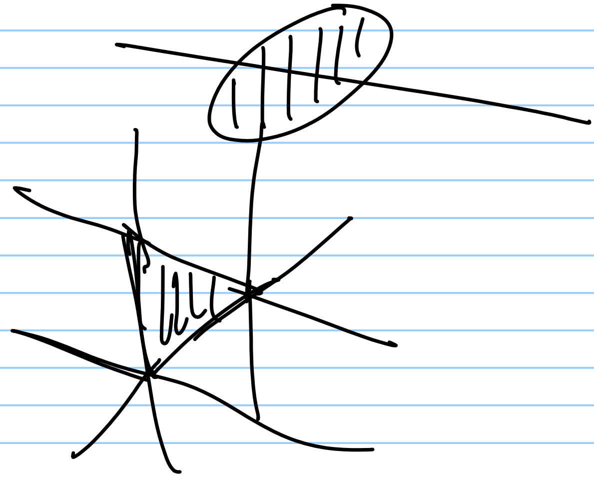
(2) $C \subseteq h \cup h^{\pm}$



$f \subseteq P = \text{conv}(\{p_1, \ldots, p_n\})$

is a face iff $\exists h$ s.t h is a

supp hyp of P, and $P \cap h = f$.



Lemma ~~Faces of Polytope~~ $\mathbb{P}$. a ~~Polytope~~ which is a convex hull of some

Lemma Faces of $\mathbb{P} = \mathrm{conv}(A)$ $\to = \{P_1, \dots, P_n\}$ is also a polytope and it will a conv hull of some subset of A.

Proof $f \subseteq \mathbb{P}$, $h \cap \mathbb{P} = f$. $\mathbb{P} \subseteq h \cup h^+$

To prove $f = \mathrm{conv}(h \cap A)$.

$\Rightarrow \mathrm{conv}(h \cap A) \underset{\supseteq}{\overset{?}{\subseteq}} f \ni X$

$X = \sum_{i=1}^{n} \lambda_i P_i$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$

$h(X) = 0$

$= \sum_{i=1}^{n} \lambda_i h(P_i)$

0

$$X = \sum_{i=1}^{n} \lambda_i P_i = \cancelto{0}{\sum_{P_i \in h \cap A} \lambda_i h(P_i)} + \sum_{P_i \in A \setminus \hat{A}} \lambda_i h(P_i)$$

$\underbrace{h \cap A}_{\hat{A}}$

$h(x) = 0$

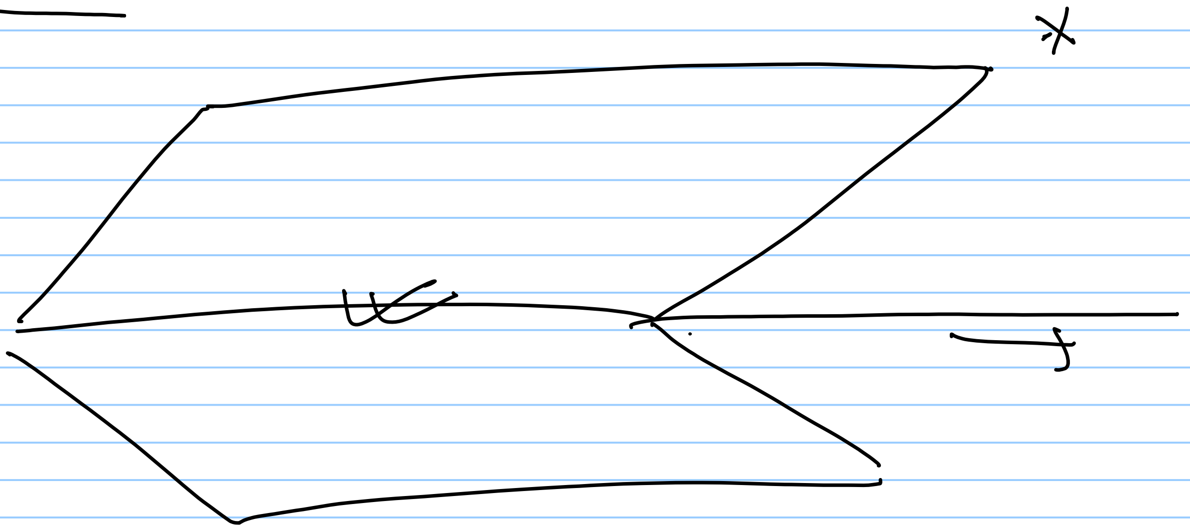
$x \in f = P \cap h$

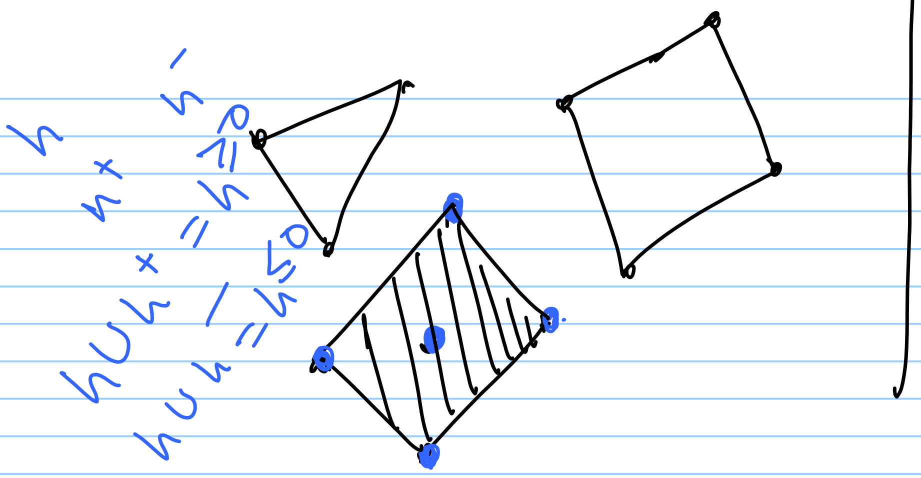
$h($

$P_i \in h \cap A$ then $h(P_i) = 0$

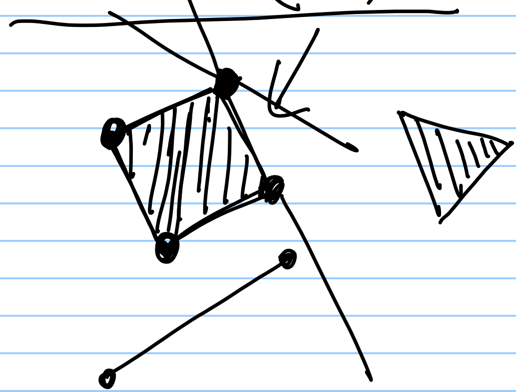
if $P_i \in A \setminus \hat{A}$ $h(P_i) > 0$

$\square$





Conv(A) ← gen. pos $\quad \mathbb{R}^{d.10}$



Simplicial Polytopes.

Lemma (1) $\mathbb{P}$ is a convex hull of its vertices.

(2) Any Polytope can be represented by intersection of finite number of half planes,

$$\mathbb{P} = \bigcap_{i=1}^{t} h_i^{\geq 0}$$

Def (Polyhedron) $\mathbb{P} = \bigcap_{i=1}^{n} h_i^{\geq 0}$

Lemma A bounded Polyhedron is a polytope.

Def (Gen. pos. for hyperplanes)

$H = \{h_1, \ldots, h_n\}$

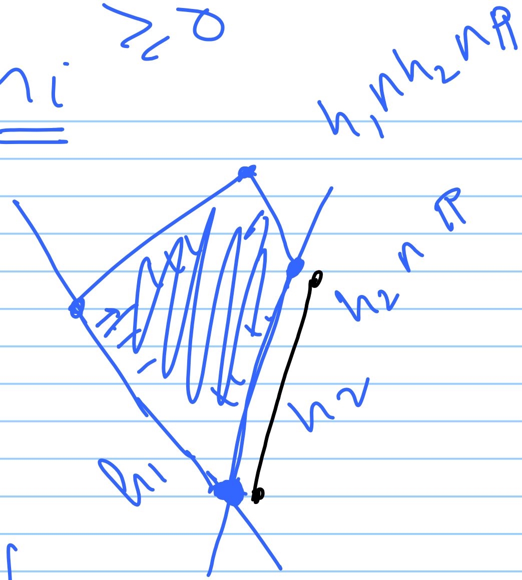
$h_1, \ldots, h_{d+1}$, $k \leq d$

$\bigcap_{i=1}^{k} h_i$ ← affine space with

↳ $= \phi$ dim $d-k$

Def (Simple Poly.) $\underline{\underline{\mathbb{P}}} = \bigcap \underline{\underline{h_i}}^{\geq 0}$

Simple



Lemma (Simple Poly). Simple Polyhedron $\mathbb{P} = \bigcap_{i=1}^{n} h_i^{\geq 0}$

Let $I \subseteq \{1, \dots, n\}$ $F_I = \bigcap_{i \in I} h_i$, if $F_I \cap \mathbb{P} \neq \emptyset$ then $F_I \cap \mathbb{P}$ is a face of $\mathbb{P}$. ||

(dim $(F_I \cap \mathbb{P}) = \underline{d - |I|}$) //. And, any face of $\mathbb{P}$ can be obtained by some set $I \subseteq \{1, \dots, n\}$.)