- Geometric Optimization problems
- NP - hard.
- Approximation algorithms
  - Linear Programming
  - Local Search

Linear Programming

- Maximizing / Minimizing a linear fn. s.t. linear constraints.

Eg:

Objective fn.

maximize $\underbrace{x_1 + x_2}$

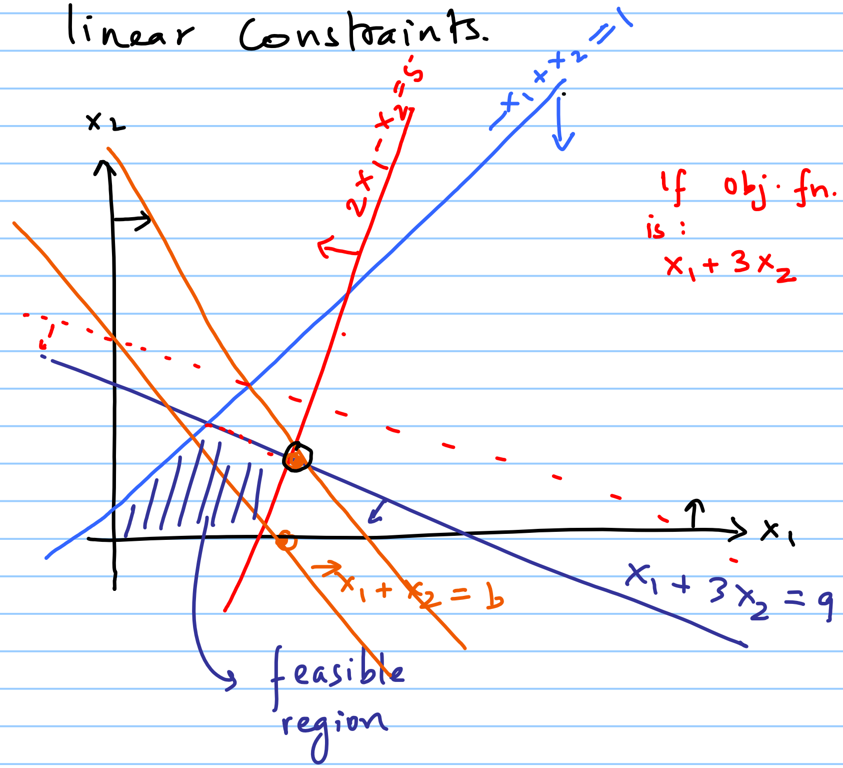
subject to:

$$\left\{\begin{array}{l} -x_1 + x_2 \le 1 \\ 2x_1 - x_2 \le 5 \\ x_1 + 3x_2 \le 9 \end{array}\right.$$

Constraints

Non-neg. Constr.

$$\left\{\begin{array}{l} x_1 \ge 0 \\ x_2 \ge 0 \end{array}\right.$$



Linear Program (LP)

maximize $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

s.t.

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$$

$$x_1, x_2, \ldots x_n \geq 0.$$

$$\max \quad c^T x.$$

$$Ax \leq b.$$

$$x \geq 0.$$

$x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$.

$b \in \mathbb{R}^m$.

Linear Prog:

- Unique Solution.
- Infinitely many solutions →
- No Solutions. →

$$\max \; x_1 + x_2$$

s.t.

$$\left.\begin{aligned} x_1 &> x_2 + 1 \\ x_1 &< x_2 + 1 \end{aligned}\right\}$$

Solving Linear Programs

- Solve LPs in polynomial time.

  \# constraints

  \# Variables.

  largest value in A, b

- Khachiyan '79 Ellipsoid algorithm
  - first poly time alg.
- Simplex alg: Dantzig ~48
  - Exponential time
- Karmarkar'84: Interior pt. method ✓
  poly. + fast

Integer (linear) programs (ILP/IP)

- Discrete Optimization

Max / Min. a linear Obj fn.

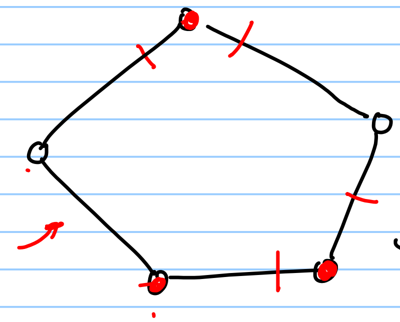
s.t. linear Constraints

\+

Variables take integer values.

- Given $G = (V, E)$
- Minimum Vertex Cover :

$S \subseteq V$ is a vertex cover if $\forall \{u, v\} \in E$, either u, or v (or both) are in our set $S \subseteq V$.



→ vertex Cover of size 3

Let x_v, for $v \in V$

$$x_v = \begin{cases} 1, & \text{if } v \in S \\ 0, & \text{o.w} \end{cases}$$

$$\text{minimize} \sum_{v \in V} x_v$$

$$x_u + x_v \geq 1 \quad \forall \{u,v\} \in E$$

$$x_u \in \{0,1\} \quad \forall u \in V$$

ILP:

- Solving ILP: is NP-hard

ILP for VC: is NP-hard

$$\min \sum_{v \in V} x_v$$

$$x_u + x_v \geq 1 \quad \forall \{u,v\} \in E$$

$$x_u \in \{0,1\} \quad \forall u \in V$$

Approximation Algorithms.

- An alg that runs in poly time

  &

  produces a soln. whose value is guaranteed to be "close" to OPT

  OPT ↳ optimal solution.

Formally: An alg. $\mathcal{A}$ is an α-approx. for Π if $\forall$ instance I of Π, produces a soln. S:

$$\text{cost}(S) \leq \alpha \cdot \text{OPT}(I) \quad (\text{for min.})$$

α need not be a constant.

Approx alg:

- Prove correctness
- Approximation guarantee : cost(S) $\le \alpha$ OPT(I) $\forall\ I \in \Pi$

$$\cancel{\min \sum_{v \in V} x_v}$$

$$\cancel{x_u + x_v \ge 1 \quad \forall \{u,v\} \in E}$$

$$\cancel{x_u \in \{0,1\} \quad \forall\ u \in V}$$

$$x_u \in [0,1]$$

LP - relaxation

Obs: Any feasible integer soln. is also a soln. to the ~~LP~~ LP-relaxation.

Eg: $x_u = 1 \quad \forall\ u \in V$

is feasible for ILP & LP.

However,

OPT_{LP} : Optimal soln. to the LP-relax

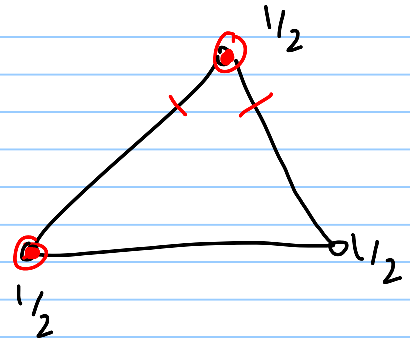
$$\boxed{OPT_{LP} \le OPT}$$

$$\min \sum x_u$$

$x_u + x_v \geq 1 \leftarrow \checkmark$

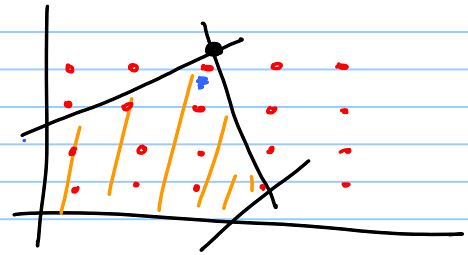
~~$x_u \in \{0,1\}$~~

$x_u \in [0,1]$



There is a solution of value $\leq 3/2$ for the LP.

OPT



$OPT_{LP} \rightsquigarrow S$

$cost(S) \leq \alpha \cdot OPT_{LP} \leq \alpha \cdot OPT$

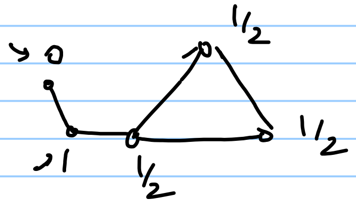
LP-rounding:

1. Solve the linear Program (LP. relaxation)
2. From $OPT_{LP} \rightsquigarrow S$ without losing mush.

fractional

Integral & feasible

$\mathcal{A}$: A 2-approx. for Vertex Cover.



$\min \sum x_v$

$x_u + x_v \geq 1$

$x_u \in [0,1]$

1. Solve the LP-relaxation.
2. Let $x\,(x_1, x_2, \ldots, x_n) = OPT_{LP}$.
3. Pick all vertices s.t. $x_u \geq 1/2$

$\rightarrow (y_1, y_2 \ldots y_n)$

$\in \{0,1\}$

Thm : $\mathcal{A}$ is a 2-approx. for Vertex Cover.

Pf: Since $\boxed{x_u + x_v \geq 1 \quad \forall (u,v) \in E}$

$\Rightarrow$ either $x_u \geq 1/2$, or $x_v \geq 1/2$.

Obj: $\underbrace{x_1 + x_2 + \cdots}_{1} \overset{\geq \frac{1}{2}}{} \,|\, \overset{< \frac{1}{2}}{\cdots} + \underbrace{x_n}_{0}$. $\rightarrow$ $\underbrace{y_1 + y_2 + \cdots}_{1} + \underbrace{y_n}_{0}$

OPT_{LP} $\quad$ 1 $\quad$ [0,1] $\quad$ 0

S

$y_i \leq 2x_i \quad (\because x_i \geq \frac{1}{2})$

$\Rightarrow \sum y_i \leq 2 \sum x_i = 2\,OPT_{LP} \leq 2\,OPT.$

Vertex Cover:

Algorithm: ($\mathcal{A}$)

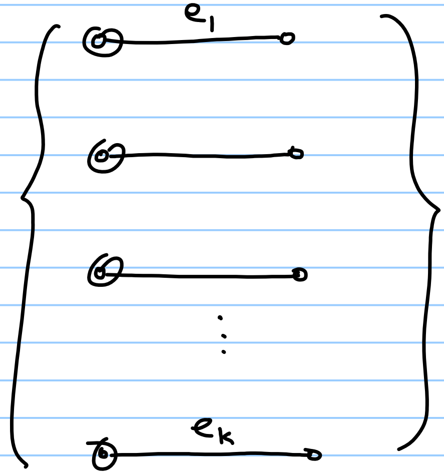
1. $S = \emptyset$.
2. While $\exists \{u,v\}$: uncovered

   $S \leftarrow S \cup \{u,v\}$ → pick both end-points!

3. Return S.

Claim: $\mathcal{A}$ is a 2-approx.

Pf:



If G has k disjoint edges.

$\Rightarrow OPT \geq k$

□