Basics of Convex Geometry

$\mathbb{R}^d$, $d = 2, 3$

Basic facts (I) $v_1, \ldots, v_{d+1} \in \mathbb{R}^d$

$\exists\ c_1, \ldots, c_{d+1}$, not all "0"

s.t. $\sum c_i v_i = 0$

(II) $p_1, \ldots, p_k$ in $\mathbb{R}^d$, $k \leq d$

Note Title 04-06-2019

Def 1 (Affine combination) Let $p_1, \ldots, p_n$ be in n points in $\mathbb{R}^d$, we define affine comb. of the these points in the following way

$$\sum_{i=1}^{n} c_i p_i \quad \text{where} \quad \sum_{i=1}^{n} c_i = 1$$

smallest dim flat containing these points

Unique?

$\boxed{c_i \geq 0}$.

Def 2 (Convex Comb.)

Convex hull of the set $\{p_1, \ldots, p_n\}$ Unique?

Note Title 04-06-2019

Def 3 (Affine independence) $Aff(P)$, $P = \{p_1, \ldots, p_n\}$

affinely dependent

$$x \in Aff(P)$$

$$\sum c_i p_i \qquad c_1 \neq c_1'$$

$$\sum c_i' p_i \qquad c_i \neq c_i'$$

Goal: (1) Radon's Theorem ✓

(2) Helly's Theorem

(3) Caratheodory Theorem

Note Title 04-06-2019

Theorem 1 (Radon's Theorem) Given a set ~~of~~ A of $d+2$ points in $\mathbb{R}^d$, there exists two disjoint subsets ~~of~~ A_1, A_2 of A such $\mathrm{Conv}(A_1) \cap \mathrm{Conv}(A_2) \neq \phi$.

$A_1 \to \phi$, $A_2 \to A$

Proof $A = \{a_1, \dots, a_{d+2}\}$, $\exists\, c_1, \dots, c_{d+2}$ (not all zeros) s.t $\sum_{i=1}^{d+2} c_i a_i = 0$, and $\sum_{i=1}^{d+2} c_i = 0$

Note Title 04-06-2019

$x \in Aff(A)$ s.t $x = \sum_{i=1}^{d+2} c_i a_i$, and $\sum c_i = 1$

$= \sum c_i' a_i$, and $\sum c_i' = 1$

$$\Rightarrow \sum_{i=1}^{d+2} \left((c_i - c_i') a_i \right) = 0$$

$$\sum_{i=1}^{d+2} (c_i - c_i') = 0 \rightarrow \underbrace{\sum c_i}_{1} - \underbrace{\sum c_i'}_{1}$$

$$\sum_{i=1}^{d+2} c_i a_i = 0, \quad \text{and} \quad \sum_{i=1}^{d+2} c_i = 0$$

$$A_1 = \{ a_i \mid c_i > 0 \} \qquad A_2 = \{ a_i \mid c_i < 0 \}$$

$$\underbrace{\sum_{a_i \in A_1} c_i}_{S} > 0 \qquad \underbrace{\sum_{a_i \in A_2} c_i}_{-S} < 0$$

$$\sum_{a_i \in A_2} \left(\frac{-c_i}{S} \right) a_i = X = \sum_{a_i \in A_1} \frac{c_i}{S} a_i \in \mathrm{Conv}(A_1)$$

$$\sum_{a_i \in A_2} \left(\frac{-c_i}{s} \right) a_i \rightarrow \mathrm{Conv}(A_2)$$

> 0

claim

$$= x = \sum_{a_i \in A_1} \left(\frac{c_i}{s} \right) a_i \in \mathrm{Conv}(A_1)$$

$$\sum_{a_i \in A_1} \frac{c_i}{s} = 1$$

witness

$$\sum_{i=1}^{d+2} c_i a_i = 0$$

$$= \sum_{a_i \in A_1} c_i a_i + \sum_{a_i \in A_2} c_i a_i + \cancel{\sum_{a_i \in A \setminus \{A_1 \cup A_2\}} c_i a_i}$$

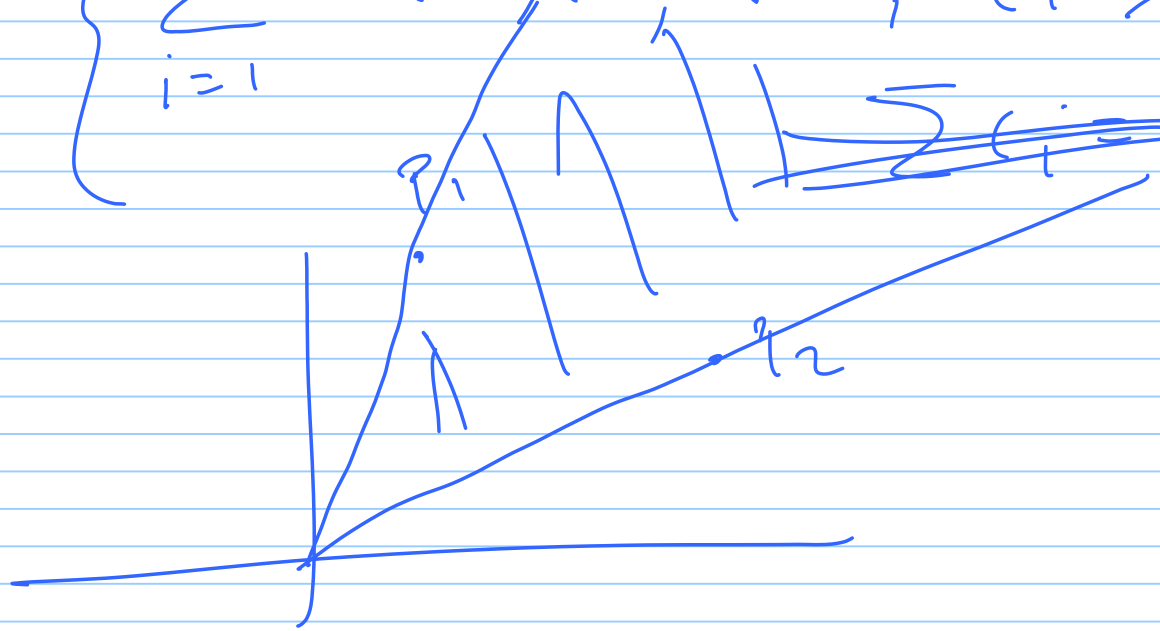
$0 \quad 0$

Complete the proof.

Can you think of other generalisation?

$A = \{a_1, \ldots, a_n\}$

Cones $\left\{ \sum_{i=1}^{n} c_i a_i,\ \forall i,\ c_i \geq 0,\ \cancel{\sum c_i = 1} \right\}$



$$\left\{ a_0, \dots, a_n \;\middle|\; \exists\, c_i \quad \sum_{i=0}^{n} c_i a_i = 0, \text{ and } \sum_{i=0}^{n} c_i = 0 \right\}$$

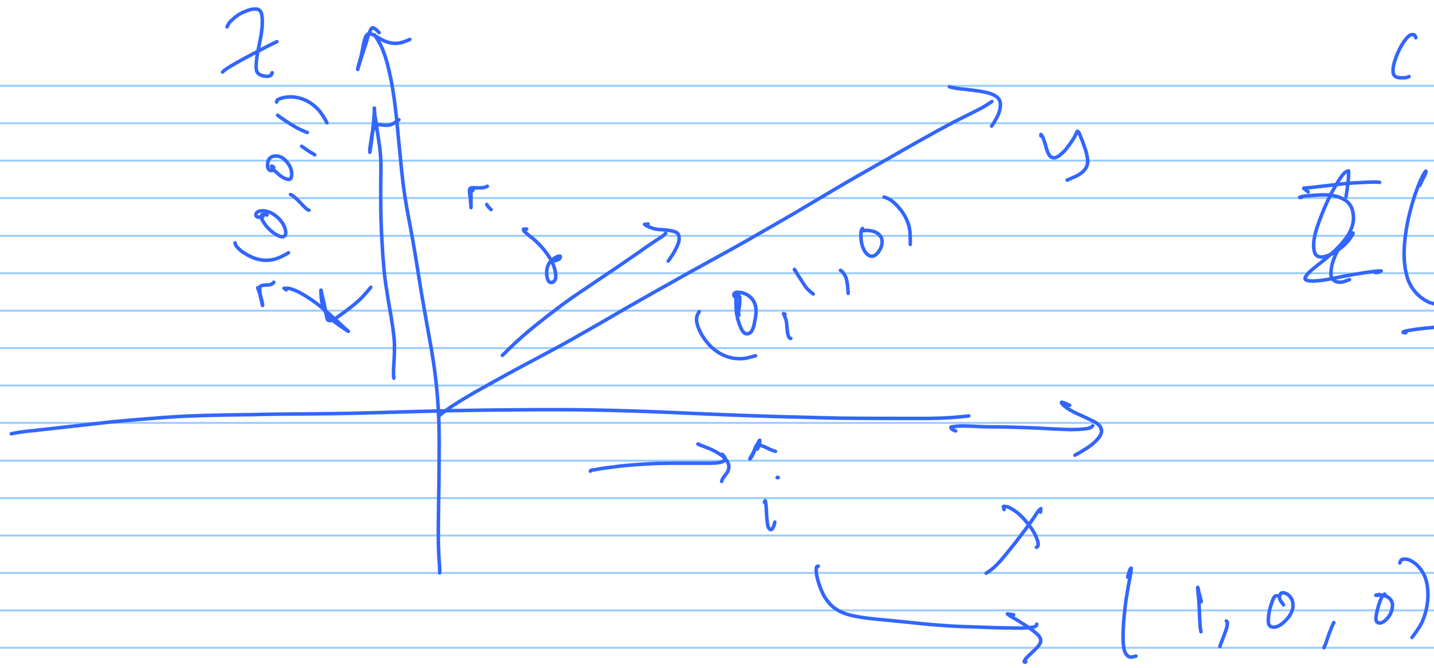
$$v_i = a_i - a_0$$

$$\left. v_1, \dots, v_n \right\} \text{dependent}$$

$$\sum_{i=1}^{n} d_i v_i = 0 \;\Big|\; \sum_{i=1}^{n} d_i (a_i - a_0) = 0$$

$$c_i = d_i \qquad i = \{1, \dots, n\}$$

$$c_0 = -\sum d_i$$



$c_1, \quad c_2, \quad c_3$

$$\nexists \left(c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k} \right) = 0$$

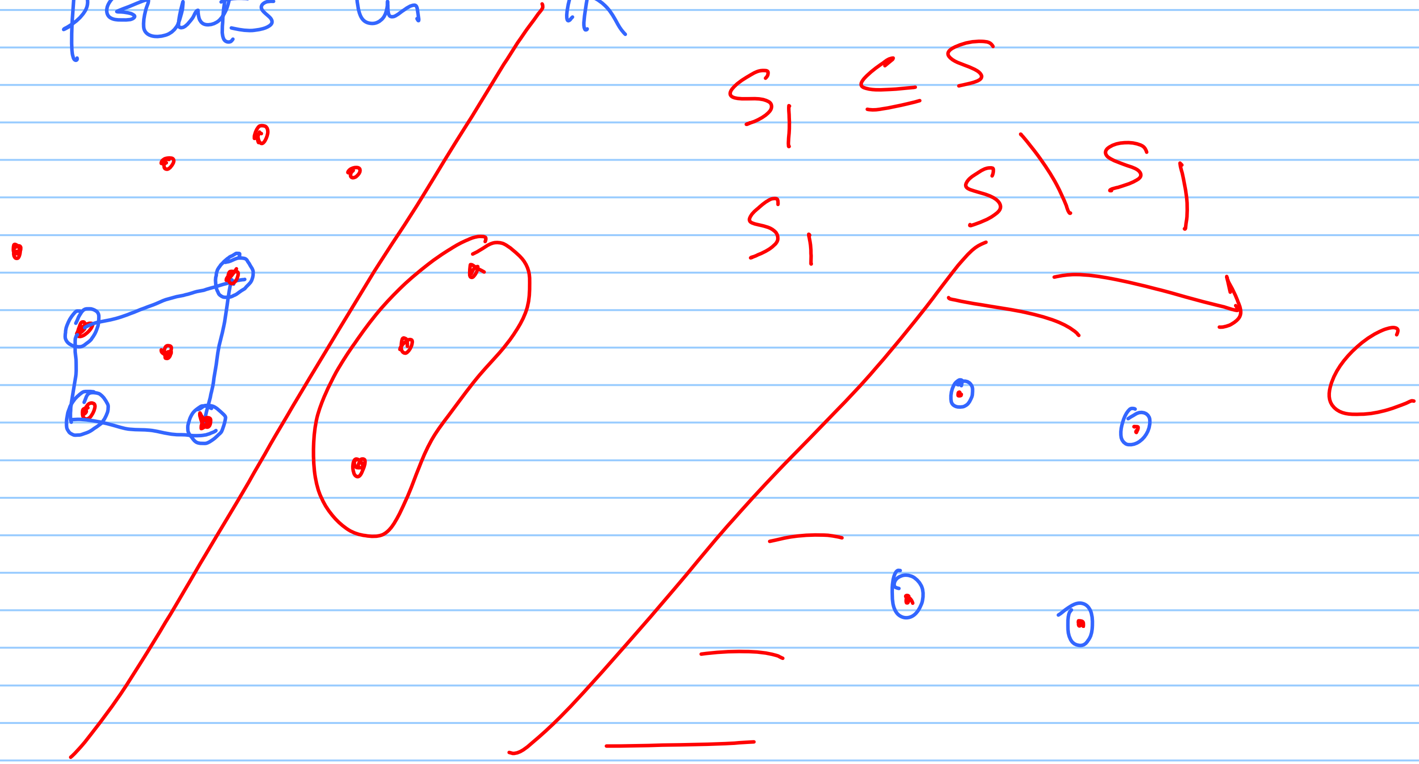
(c_1, c_2, c_3)

<u>Thm 2</u> (Equivalent statement) $A = \{a_1, \ldots, a_{d+2}\}$ in $\mathbb{R}^d$

$\exists$ a partition of $A = A_1 \sqcup A_2$ s.t $\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \phi$.

Example (Using Radon)

S points in $\mathbb{R}^2$



$\rightarrow \mathbb{R}^d$

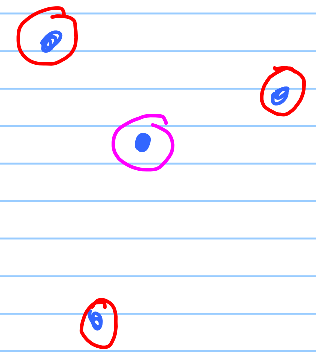
$d+2$

Hyperplanes

S

$\cancel{\mathbb{R}^1}$

$\mathbb{R}^2$



A_1

A_2

ϕ

$S = S_1 \sqcup S_2$

$\mathrm{Conv}(S_1) \cap \mathrm{Conv}(S_2) \neq \phi$

S $\cancel{A}$ Q

Size = $\underline{\underline{d+2}}$

<u>Thm (Ver 3)</u> $|A| \geq d+2, \quad A \subseteq \mathbb{R}^d \quad \ldots$

$$|A'| = d+2$$

$$A_1' \sqcup A_2' = A'$$

$$A_1 = \cancel{A_1' \cup} (A \setminus A_2') \supseteq A_1'$$

$$A_2 = A_2'$$

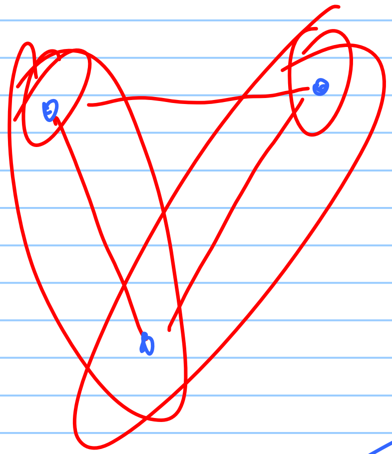
$B \subseteq C$

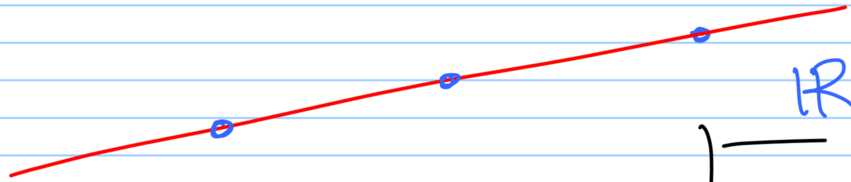
$\operatorname{conv}(B) \subseteq \operatorname{conv}(C)$

What about $d+1$?

$\mathbb{R}^2$

$S \quad S_1 \quad S_2$

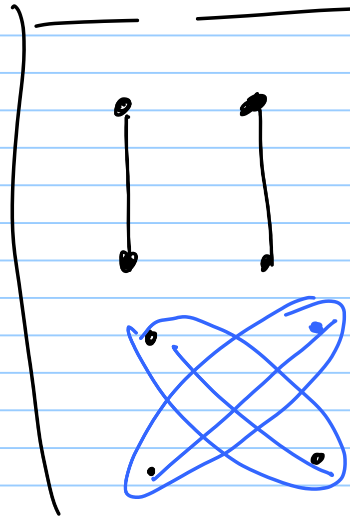




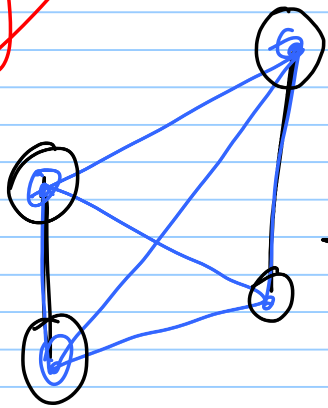
$\mathbb{R}^1$

$\mathbb{R}^2$

$S_1 \quad S_2$



$\mathbb{R}^3$

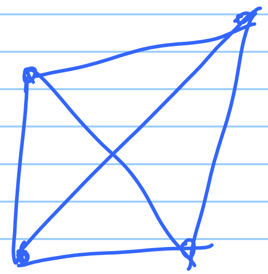


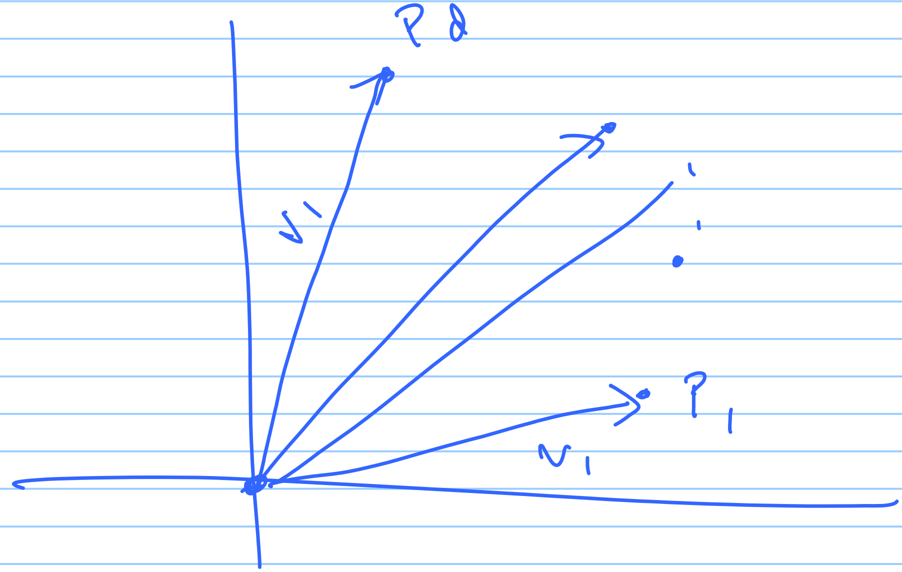
$\longrightarrow$ Simplex

Thm (ver ~~4~~ 3) A is a finite ~~A~~ point set in some Euclidean Sp.

If $|A| \geq \dim(\mathrm{Aff}(A)) + 2$ the . . . .

$\mathbb{R}^d$





Thm 2 (Helly's Thm) Given a collection $\{C_1, \dots, C_n\}$ of $n \geq d+1$ convex sets in $\mathbb{R}^d$, if it satisfies $(d+1)$-property, then

$$\bigcap_{i=1}^{n} C_i \neq \phi .$$

Def (q-Property) $C_1, \dots, C_n$ (assum $n \geq q$) satisfies q-property then any q convex bodies from the set have a common point.

$c_{i_1}, \ldots, c_{i_q}$

$\exists x$ s.t.

Proof By induction on "n".

Base case $n = d+1$

Ind hyp holds $n-1$

Ind. step holds for $n \geq d+2$ (to prove).

$S = \{c_1, \ldots, c_n\}$ $S_i = \{c_j \mid i \neq j\}$

Can we apply ind hyp. on S_i's.

$\exists\ p_i \in \mathbb{R}^d$ s.t p_i belong all convex bodies contained in S_i $\forall i$

$A = \{p_1, \ldots, p_n\}$

$\hookrightarrow A_1 \sqcup A_2$ s.t $\operatorname{conv}(A_1) \cap \operatorname{Con}(A_2) \neq \phi$.

Remember we will apply Radon's Theorem.

Consider the convex set C_i, and wlog assume $p_i \in A_1$.

$\forall p_j$, with $i \neq j$, $p_j \in C_i$.

$p_i \in A_1$

$\Rightarrow \forall p_j \in A_2 \quad , \quad p_j \in C_i$

$$\mathrm{Conv}(A_2) \subseteq C_i \Rightarrow \underbrace{\mathrm{Conv}(A_1) \cap \mathrm{Conv}(A_2)}_{\neq \phi} \subseteq \underline{\underline{C_i}}$$

$$\bigcap_{i=1}^{n} \underline{\underline{C_i}} \supseteq \mathrm{Conv}(A_1) \cap \mathrm{Conv}(A_2) \qquad \square$$

Issues to add:

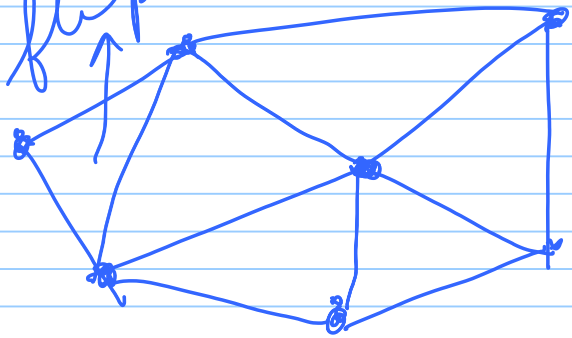
Thm 3 (Caratheodory theorem) Any convex ~~set~~ combination of a set of points $A = \{a_1, \dots, a_n\}$ (gen. pos.) $\subseteq \mathbb{R}^d$ is a convex combination of $d+1$ points in A.

Convex comb.

$$\sum_{i=1}^{n} c_i a_i \,, \quad c_i \geq 0 \qquad \sum c_i = 1$$

Proof $d=2$

flips

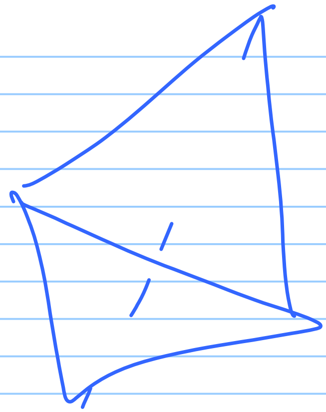


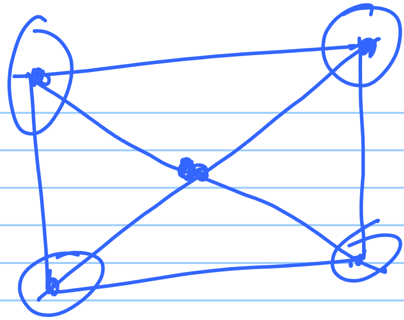
Delaunay tria..

Triangulates the convex hull of the point.

$\mathbb{R}^3$

$\mathbb{R}^d$





$\{a_1, \ldots, a_n\}$

Proof without Delaunay. Let $x \in \operatorname{conv}(A)$, and let

$$x = \sum_{i=1}^{k} c_i a_i, \text{ where } c_i \geq 0, \sum_{i=1}^{k} c_i = 1$$

If $k \leq d+1$.

Minimal $c_i > 0$.

(i) $k \leq d+1$ (Nothing to prove)

(ii) $k \geq d+2$

$\exists\ d_i$, (Not all zeros) s.t.

$$\sum_{i=1}^{k} d_i a_i = 0, \quad \text{and} \quad \sum_{i=1}^{k} d_i = 0$$

$d_1, d_2, \ldots, d_k$.

WLOG $d_k > 0$, and satisfy that among all $d_i > 0$

$$\cancel{\emptyset} \quad \frac{c_k}{d_k} \le \frac{c_i}{d_i} \quad \forall i, \qquad \frac{d_i > 0}{\frac{c_i}{d_i}} \quad \forall\, d_i > 0$$

Not look at the following thing

$$e_i = c_i - \frac{c_k}{d_k} d_i$$

Are they ≥ 0 ?

Now consider $\left\| \sum_{i=1}^{k-1} e_i a_i \right\| \Rightarrow \frac{e_k = 0}{1} = X$

$e_i \geq 0$, $\quad \sum_{i=1}^{k-1} e_i$ -

$\square$

Q Center point ://