Provably Safe Real-Time Receding Horizon Trajectory Planning for Linear Time-Invariant Systems

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Introduction

- Motivation
  - Robust motion planning methods have been extensively studied to deploy mobile robots in environments with external disturbance.
  - Such planning algorithms are often overly conservative, or computationally inefficient.

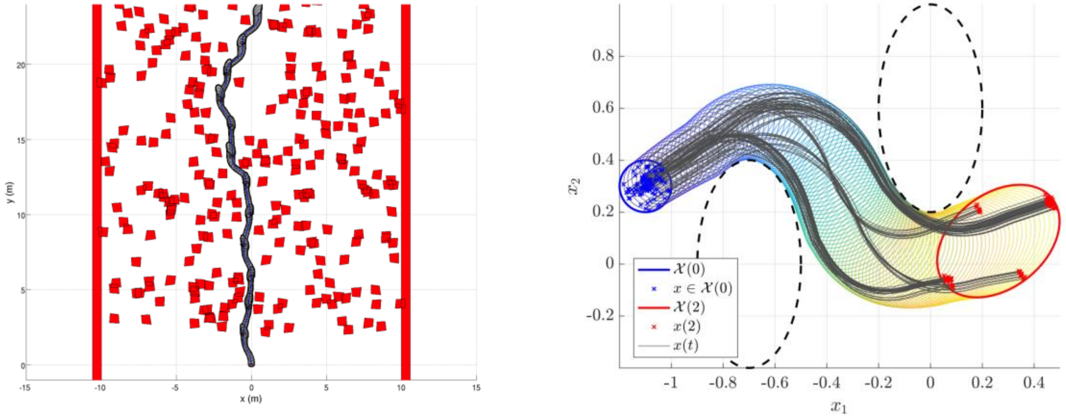


Fig. Funnels are sequentially composed to find a safe path to the goal. [1] (left) An aerial vehicle avoids ellipsoidal obstacles by utilizing Hamilton-Jacobi reachability analysis. [2] (right)

- Contributions
  1. We present a method to compute a polytopic outer approximation of the forward reachable set that can be used to ensure safety along the trajectory.
  2. We present a receding horizon trajectory planning algorithm for linear time invariant systems that is guaranteed to be safe under any sequence of disturbances, using the encompassing polytope of the forward reachable set.
  3. A Monte Carlo simulation shows that the proposed algorithm runs at real-time at faster than 100 Hz.

[1] Hoseong Seo et al. “Robust trajectory planning for a multirotor against disturbance based on hamilton-jacobi reachability analysis”. In: 2019 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS). IEEE. 2019, pp. 3150–3157.

[2] Anirudha Majumdar and Russ Tedrake. “Funnel libraries for real-time robust feedback motion planning”. In: The International Journal of Robotics Research 36.8 (2017), pp. 947–982.

Problem description

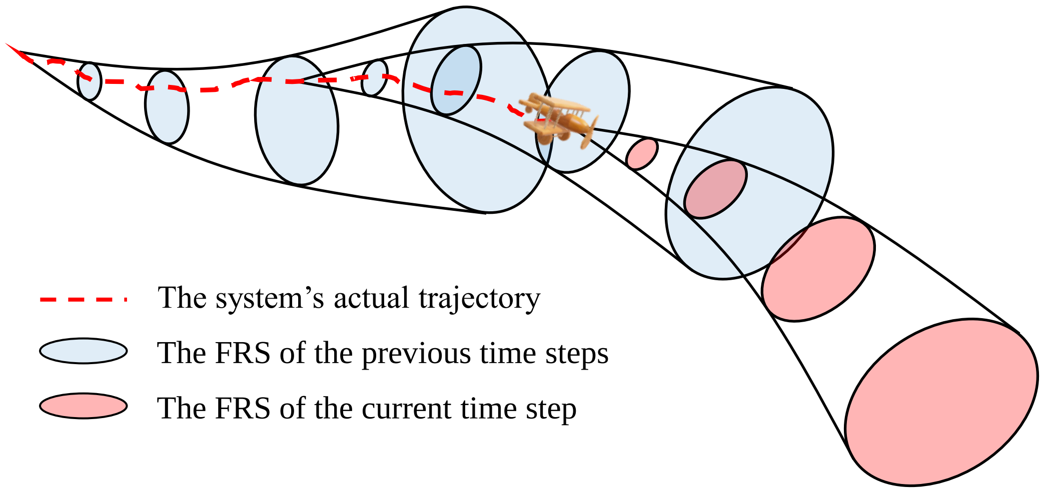


Fig. The toy plane is performing a receding horizon safe trajectory planning by iteratively calculating its forward reachable sets (FRS) under bounded disturbance $w_t \in W$.

- Objective
  - Given a linear dynamical system under disturbance $x_{t+1} = Ax_t + Bu_t + Dw_t$, iteratively find a receding horizon trajectory whose forward reachable set (FRS) is collision-free, i.e., the system is safe under any sequence of disturbance.
- Assumptions
  - The disturbance w_t is bounded by $w_t \in W$, where W is a convex bounded set.
  - The dynamical system is linear time invariant.
  - Time discretization step δt is small enough, so the system is close enough to a continuous-time system.

Forward reachability analysis

- Forward reachable set (FRS)

$$X_{t+k|t} = \left\{ x_{t+k} \;\middle|\; \begin{array}{c} \forall \tau \in \{t, t+1, \cdots, t+k-1\} \\ x_{\tau+1} = x_{\tau+1}^{ref} + A_c\left(x_\tau - x_\tau^{ref}\right) + Dw_\tau \\ x_t \in X_{t|t}, \quad w_\tau \in W \end{array} \right\}$$

- Recursive calculation of the FRS

$$X_{t+1|t_0} = x_{t+1}^{\mathrm{ref}} + A_c\left(X_{t|t_0} - x_t^{\mathrm{ref}}\right) + DW$$

- Closed-form solution for the FRS

$$X_{t+k|t} = x_{t+k}^{ref} + A_c^k\left(X_{t|t} - x_t^{ref}\right) + A_c^{k-1}DW + A_c^{k-2}DW + \cdots + A_cDW + DW$$

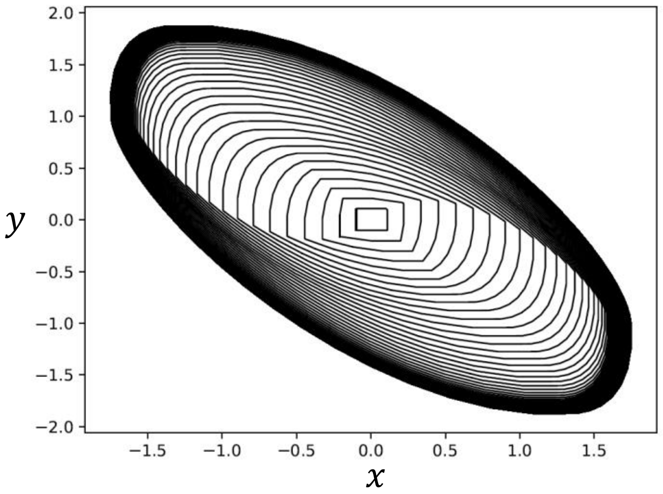


Fig. The FRS of the linear system $x_{t+1} = x_t + 0.1y_t + u_t$, $y_{t+1} = -0.2x_t + 0.8y_t + v_t$ with bounded disturbance $|u_t| \le 0.1$, $|v_t| \le 0.1$. At time step 250, it has more than 1000 vertices.

Equation details

- System evolution law: $x_{t+1} = Ax_t + Bu_t + Dw_t$
- The system is equipped with a controller $u_t = u_t^{\mathrm{ref}} - K\left(x_t - x_t^{\mathrm{ref}}\right)$.
- $A_c = A - BK$ is the matrix for the closed-loop dynamics.
- The plus sign (+) between sets denotes the Minkowski sum.
- $X_{t_1|t_0}$: The reachable set at time t_1 given information at t_0.
- $x^{\mathrm{ref}}, u^{\mathrm{ref}}$: Reference state and input trajectories.

Finding an encompassing polytope of the FRS

- Problem

Given a vector c, system $x_{t+1} = Ax_t + Bu_t + Dw_t$, starting set $X_{t|t}$, and disturbance bound $w_t \in$ W, find

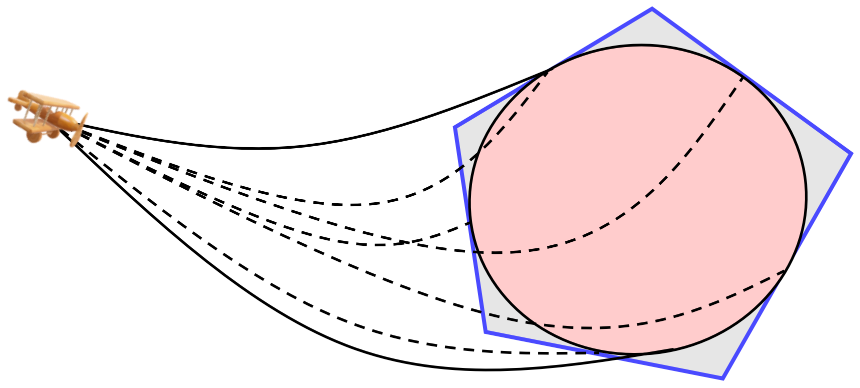
$$\sup_{\substack{w_\tau,\ x_t,\\ x\in X_{t+k|t}}} c^\top x\,.$$

- Decompose into linear programming (LP) subproblems

Using the closed form formula for the FRS, we get

$$\sup_{\substack{w_\tau,\ x_t,\\ x\in X_{t+k|t}}} c^\top x = c^\top x_{t+k}^{\mathrm{ref}} + \sup_{x\in X_{t|t}} A_c^k\left(x - x_t^{\mathrm{ref}}\right)$$
$$+ \sum_{j=0}^{k-1} \sup_{w\in W} c^\top A_c^j D w$$

Can be solved in real-time.



Trajectory under adversarial disturbance

The true reachable set

The polytopic outer approximation of the FRS found by the proposed method

Fig. The toy plane is simulating adversarial external disturbance to find an encompassing polytope of the FRS.

Guaranteeing safety using safe flight corridor

- Safe flight corridor (SFC)
  - Convex obstacle-free region around a path segment.
  - Frequently used to convexify trajectory optimization problems.

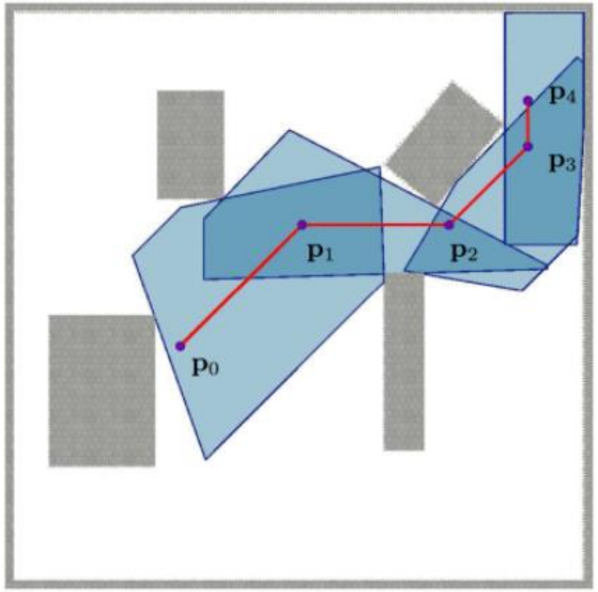
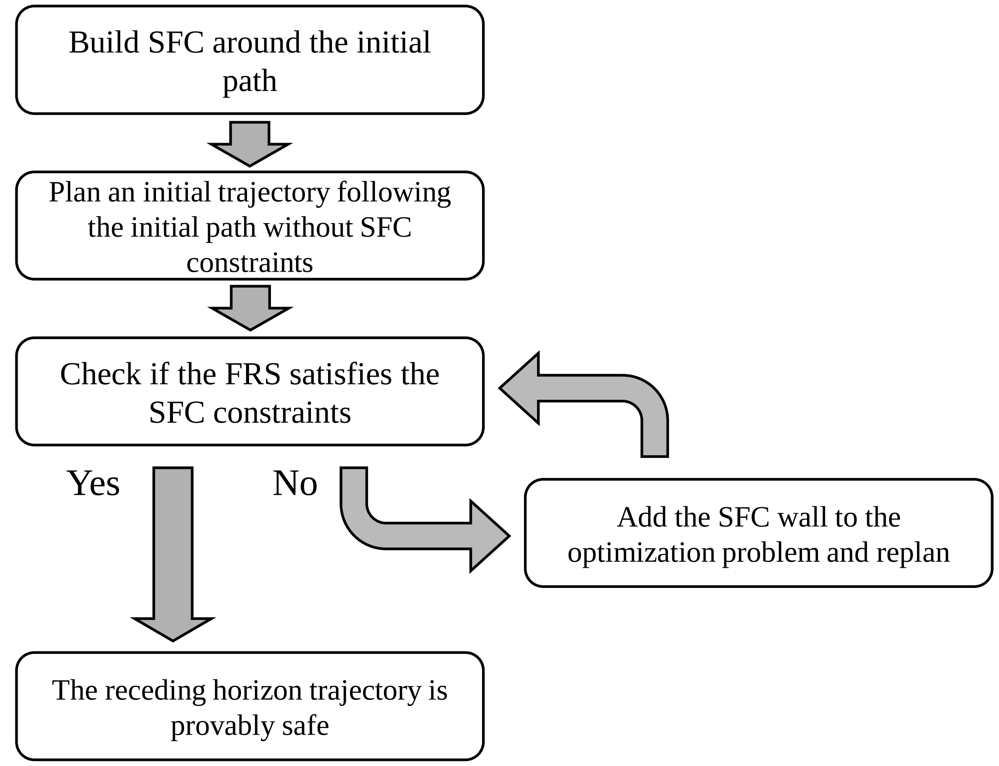


Fig. Polytopic safe flight corridor (SFC) found along a path from $\mathbf{p}_0$ to $\mathbf{p}_4$. Figure taken from [1].

- Planning strategy



[1] S. Liu *et al.*, "Planning Dynamically Feasible Trajectories for Quadrotors Using Safe Flight Corridors in 3-D Complex Environments," in *IEEE Robotics and Automation Letters*, vol. 2, no. 3, pp. 1688-1695, July 2017, doi: 10.1109/LRA.2017.2663526.

Guaranteeing safety using safe flight corridor

- Planning strategy

Build SFC around the initial path

Plan an initial trajectory following the initial path without SFC constraints

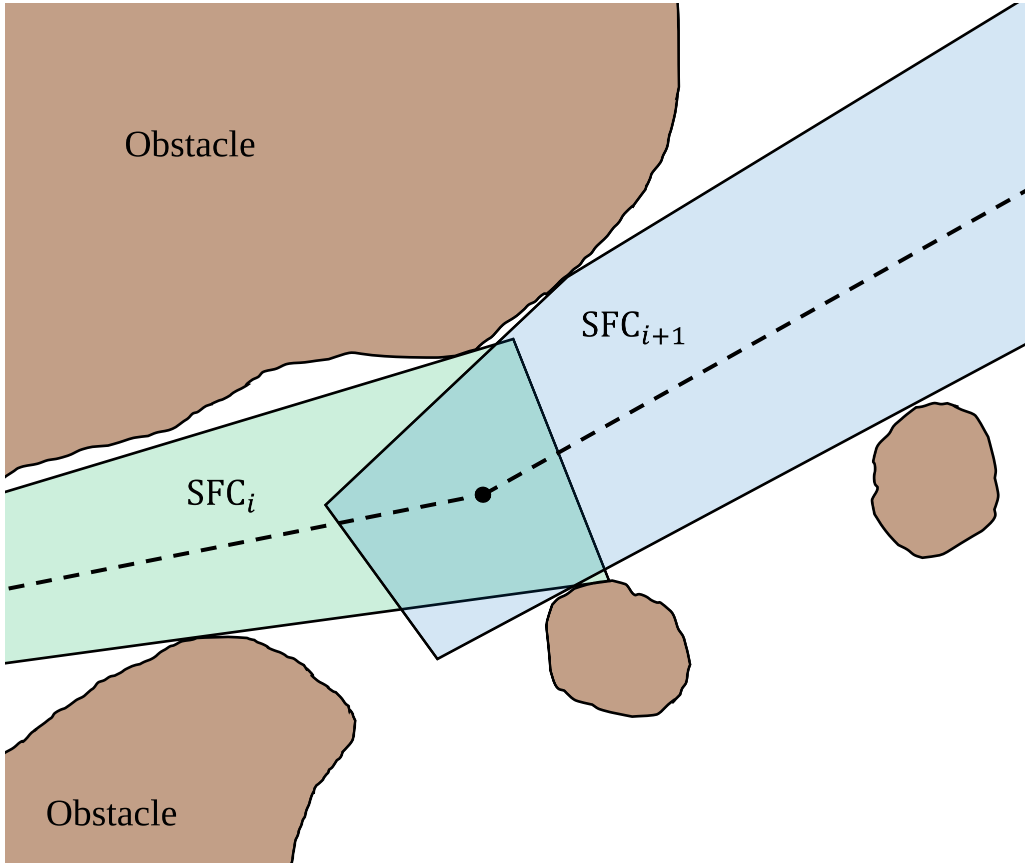
Check if the FRS satisfies the SFC constraints

Yes

No

Add the SFC wall to the optimization problem and replan

The receding horizon trajectory is provably safe



Polytopic SFC is created along the initial path.

Guaranteeing safety using safe flight corridor

- Planning strategy

Build SFC around the initial path

Plan an initial trajectory following the initial path without SFC constraints

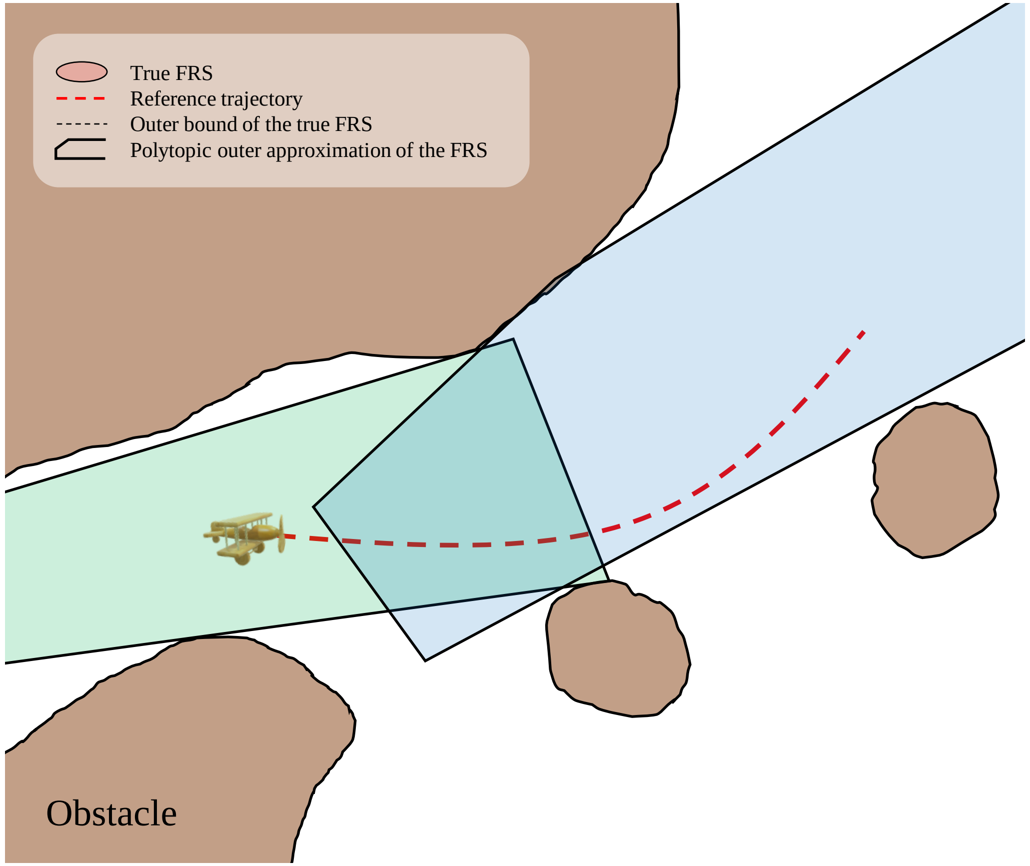
Check if the FRS satisfies the SFC constraints

Yes

The receding horizon trajectory is provably safe

No

Add the SFC wall to the optimization problem and replan



Initial trajectory is generated within the receding time window.

Guaranteeing safety using safe flight corridor

- Planning strategy

Build SFC around the initial path

Plan an initial trajectory following the initial path without SFC constraints

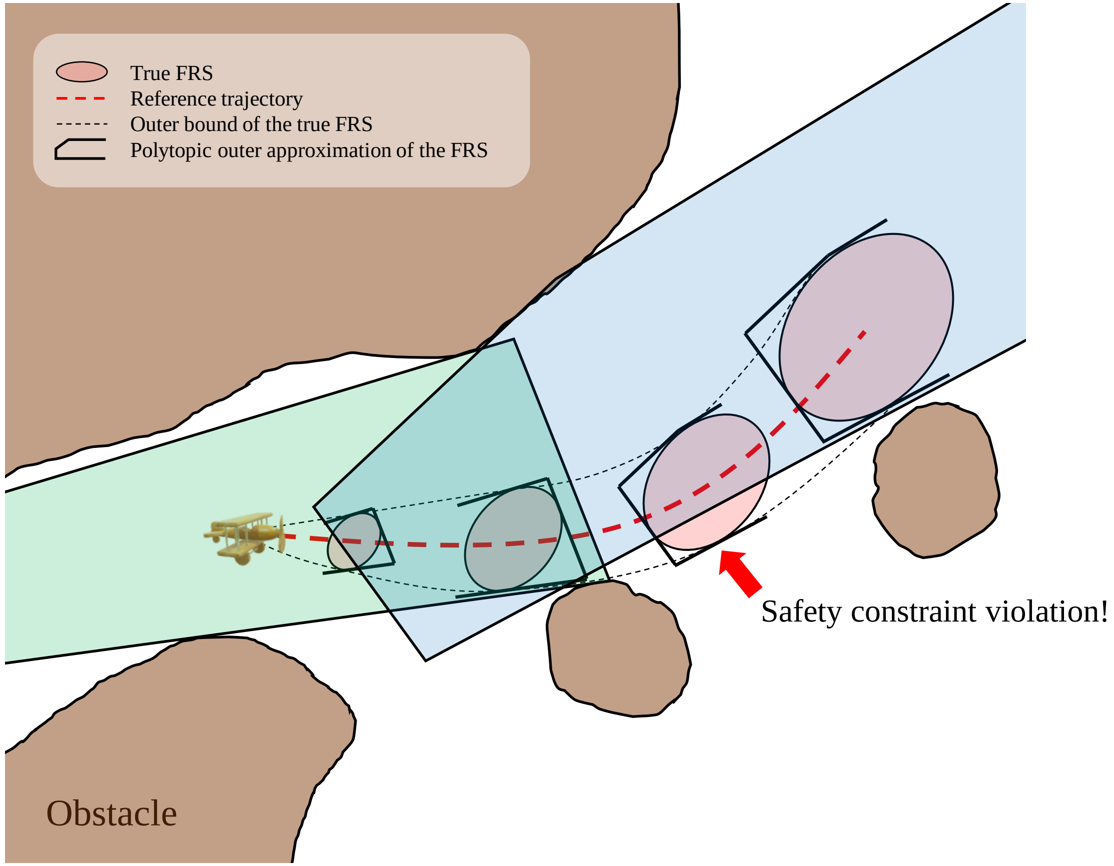
Check if the FRS satisfies the SFC constraints

Yes

No

Add the SFC wall to the optimization problem and replan

The receding horizon trajectory is provably safe



The safety constraint is checked using the polytopic outer approximation of the FRS.

Guaranteeing safety using safe flight corridor

- Planning strategy

Build SFC around the initial path

Plan an initial trajectory following the initial path without SFC constraints

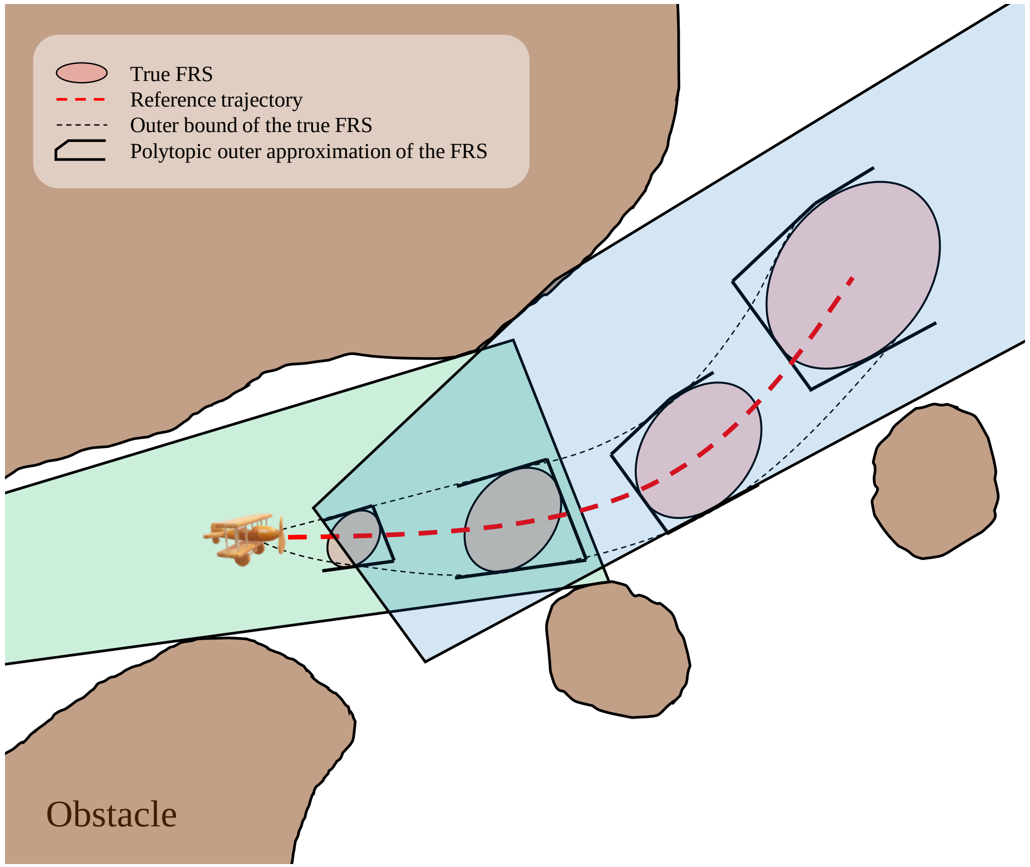
Check if the FRS satisfies the SFC constraints

Yes

No

Add the SFC wall to the optimization problem and replan

The receding horizon trajectory is provably safe



If safety constraint violation occurred, the corresponding wall is added to the constraint set. Then, the trajectory is replanned.

Simulation results

- Monte Carlo simulation on a triple integrator system under additive velocity disturbance w_t.

$$r_{t+1} = r_t + v_t\delta t + \frac{1}{2}a_t\delta t^2 + \frac{1}{6}j_t\delta t^3 + w_t\delta t$$
$$v_{t+1} = v_t + a_t\delta t + \frac{1}{2}j_t\delta t^2$$
$$a_{t+1} = a_t + j_t\delta t$$

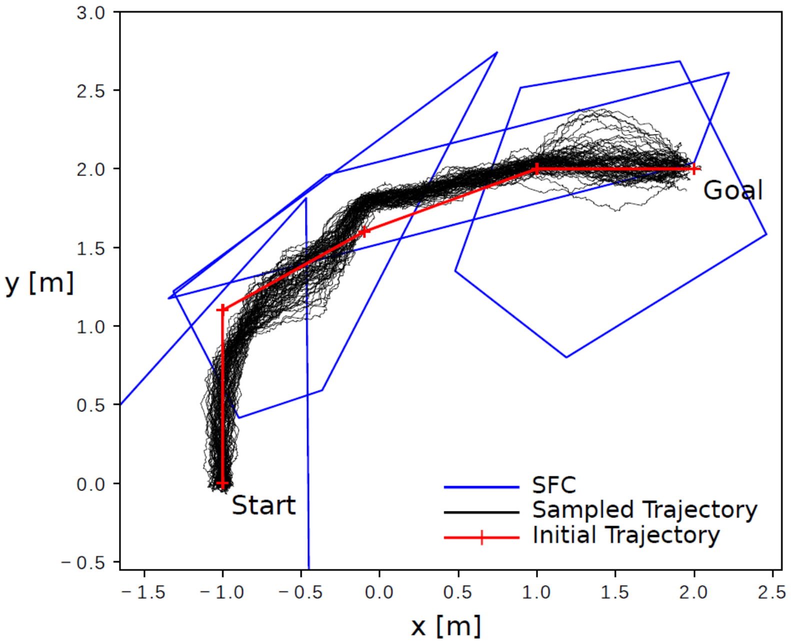


Fig. Simulation results

- The proposed method generated provably safe trajectories in real time.
  - The overall algorithm runs at 100 ~ 200 Hz depending on the environment, on a computer equipped with 3.2 GHz CPU and 16 GB RAM.

Table. Simulation parameters.

Parameter	**Value**
Dimension (n)	2
Degrees of freedom	6
Length of the time window (N)	100
Length of the collision checking time window (N_c)	40
Number of via points (N_s)	5
Time step (δt)	0.01 s
Control gains (K)	$k_r = 400$, $k_v = 120$, $k_a = 10$
Control limits (L)	$\lvert a\rvert \preceq 10\ \mathrm{m/s^2}$
Cost weights	$w_r = 1000$, $w_v = 0$, $w_a = 0$, $w_j = 1$
Disturbance bound (W)	$\lvert w\rvert \preceq 0.7\ \mathrm{m/s}$
Reference velocity (v_{ref})	$0.9\ \mathrm{m/s}$

Conclusion

- We present a provably safe receding horizon trajectory planning algorithm for linear time invariant systems.
- We guarantee safety by constraining polytopic outer approximation of the FRS to be contained within an SFC.
- A Monte Carlo simulation showed that the proposed algorithm runs at real-time and generates collision-free trajectories under bounded disturbance.

- Future works may include
  - extending the algorithm to time-varying or nonlinear systems,
  - resolving the infeasibility issue of the algorithm,
  - or estimating the disturbance at real-time and use it as the disturbance bound.

Thank you!

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