ϕ^4 Theory with Path-Integral MC on the Lattice

Projects for Computational Physics

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ϕ^4 Theory

- real, scalar fields
- simplest interacting quantum field theory
- toy model for studying the principles of QFT
- describes self-interacting part the Higgs field

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) \underbrace{- \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4}_{-V(x)} \quad [7]$$

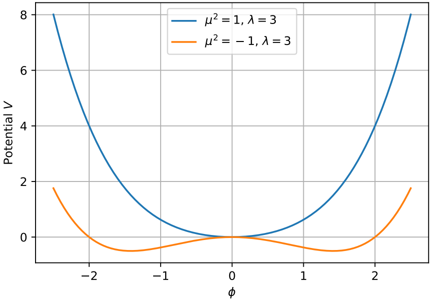
Spontaneous Symmetry Breaking

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

$\lambda < 0$: potential unbounded from below
$\rightarrow$ unphysical

$\lambda > 0, \mu^2 > 0$: $\mathcal{Z}_2$ symmetry preserved,
vacuum expectation value $\langle\phi\rangle = 0$

$\lambda > 0, \mu^2 < 0$: $\mathcal{Z}_2$ symmetry spontaneously broken,
field develops vev $\langle\phi\rangle \neq 0$



Example potentials

Correlation Functions in Minkowski Space-Time

$$C_n := \langle\Omega|\, T\left(\phi(x_1)\ldots\phi(x_n)\right)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\int\mathcal{D}[\phi]\,\phi(x_1)\ldots\phi(x_n)\exp\left(i\int\limits_{-t}^{+t}\mathrm{d}^4x\mathcal{L}\right)}{\int\mathcal{D}[\phi]\,\exp\left(i\int\limits_{-t}^{+t}\mathrm{d}^4x\mathcal{L}\right)} \quad [3]$$

- $|\Omega\rangle$ vacuum of the interacting theory
- the functional integral $\int\mathcal{D}[\phi]$ runs over all possible fields $\phi(x)$
- complex shift in time integration $\rightarrow$ exponential damping $\rightarrow$ vacuum at infinite $|t|$

Wick Rotations

Rotate time to imaginary axis

$$
\begin{aligned}
t &= -i\tau \\
(\partial_\mu \phi)(\partial^\mu \phi) = (\partial_t \phi)^2 - (\nabla \phi)^2 &= -\left((\partial_\tau \phi)^2 + (\nabla \phi)^2\right) =: -\left(\partial^E \phi\right)^2
\end{aligned}
$$

Reminder:

$$
\begin{aligned}
\text{Minkowski metric} \qquad s^2 &= t^2 - x^2 - y^2 - z^2 \\
\text{Euklidean metric} \quad (s^E)^2 &= \tau^2 + x^2 + y^2 + z^2
\end{aligned}
$$

Correlation Functions in Euklidean Space-Time

Define Euklidean action:

$$S^E[\phi] := -iS[\phi] = -i\int \mathrm{d}^4x\mathcal{L} = \int \mathrm{d}^4x^E \left(\frac{1}{2}\left(\partial^E\phi\right)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4\right) \quad [3]$$

Correlation function in Euklidean Space-Time:

$$C_n := \langle\Omega|\, T\left(\phi(x_1^E)\dots\phi(x_n^E)\right)|\Omega\rangle = \frac{\int\mathcal{D}[\phi]\,\phi(x_1^E)\dots\phi(x_n^E)\exp\left(-S^E[\phi]\right)}{\int\mathcal{D}[\phi]\,\exp\left(-S^E[\phi]\right)} \quad [3]$$

Partition function:

$$Z := \int\mathcal{D}[\phi]\, e^{-S^E[\phi]}$$

Classic Parametrization

Space-time lattice:

- periodic boundary conditions
- d dimensions, directions $\hat{e}_k$ with $k = 1 \dots d$
- $N = \prod_{k=1}^{d} n_k$ lattice sites x_i with $i = 1 \dots N$
- lattice spacings a_k, volume element $v = \prod_{k=1}^{d} a_k$

Derivative:

$$\partial_k \phi(x_i) = \frac{\phi(x_i + a_k \hat{e}_k) - \phi(x_i)}{a_k}$$

Action:

$$S[\phi] = v \cdot \sum_{i=1}^{N} \left(\frac{(\phi(x_i + a_k \hat{e}_k) - \phi(x_i))^2}{2a_k^2} + \frac{1}{2}\mu^2 \phi(x_i)^2 + \frac{\lambda}{4!}\phi(x_i)^2 \right)$$

Alternative Parametrization

Working on uniform, 4-dim. lattice ($a_k = a \,\forall k$, $d = 4$)
Introduce dimensionless field φ and dimensionless lattice parameters κ, α:

$$\begin{aligned} \varphi &= \frac{\sqrt{2}\kappa}{a}\phi \\ a^2\mu^2 &= \frac{1-2\alpha}{\kappa} - 8 \\ \lambda &= \frac{6\alpha}{\kappa} \end{aligned}$$

Action:

$$S[\varphi] = \sum_{i=1}^{N}\left(-2\kappa\sum_{k=1}^{4}\left[\varphi(x_i)\varphi(x_i + a\hat{e}_k)\right] + \varphi(x_i)^2 + \alpha\left[\varphi(x_i)^2 - 1\right]^2\right) \quad [5]$$

Discretization

Integration:

$$\int \mathcal{D}[\phi] \to \text{finite sum over field configurations}$$

Exponential varies rapidly $\to$ importance sampling from Boltzmann distribution:

$$P(\phi_\nu) = \frac{1}{Z} e^{-S[\phi_\nu]} = \frac{\exp\left(-S[\phi_\nu]\right)}{\int \mathcal{D}[\phi] \exp\left(-S[\phi]\right)}$$

Estimate for n-point correlation function:

$$\bar{C}_n = \frac{1}{M} \sum_{\nu=1}^{M} \left[\phi_\nu(x_1) \ldots \phi_\nu(x_n)\right]$$

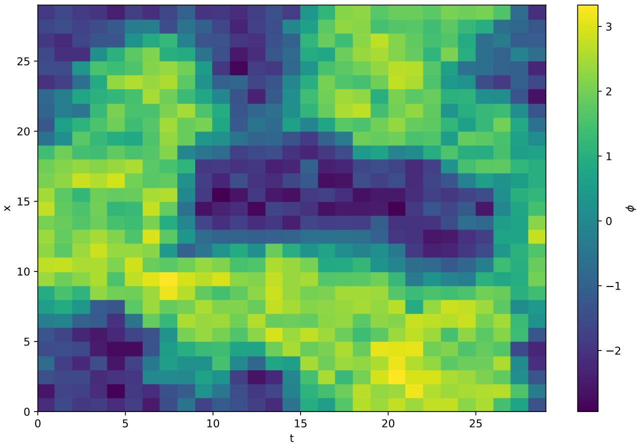
Metropolis-Hastings Algorithm

1. Initialize lattice with constant initial field strength ϕ_{ini} at all sites
2. Sweep over the lattice by performing the following steps at every lattice site x_j
   1. Sample from a uniform distribution $\Delta\phi \in [-\delta_{\mathrm{max}}, \delta_{\mathrm{max}}]$
   2. Propose variation of the field $\phi(x_j) \to \phi'(x_j) = \phi(x_j) + \Delta\phi$.
   3. Calculate change in the action ΔS
   4. Accept with probability $\rho = \min\left(1, e^{-\Delta S}\right)$, otherwise reject
3. Calculate an interesting quantity (e.g. product of fields $[\phi_\nu(x_1)\ldots\phi_\nu(x_n)]$)
4. Repeat the steps 2 and 3 several times
5. Average over stored values = estimate for the physical quantity

Setup

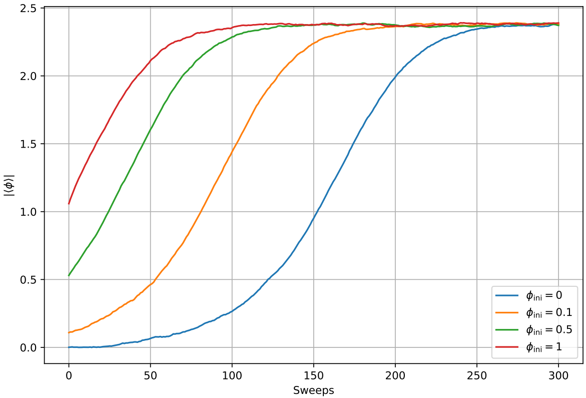
- Markov chain Monte Carlo simulation using Metropolis-Hastings algorithm
- Change of field per sweep limited to $\Delta\phi \in [-\delta_{\mathrm{max}}, \delta_{\mathrm{max}}] \rightarrow$ acceptance $\sim 70\,\%$
- Lattice size $n = 10 \rightarrow$ relative uncertainty of $0.7\,\%$

A Colourful Image



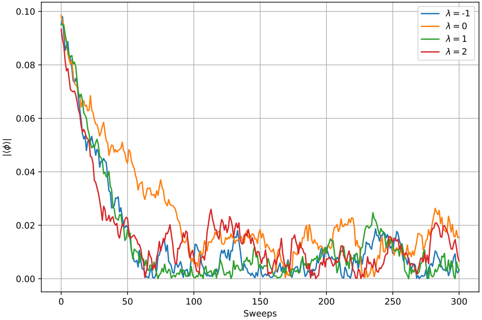
Plot of ϕ in dependence of t and x on a 2 dimensional lattice with $\mu^2 = -1$ and $\lambda = 1$

Classic Parametrization

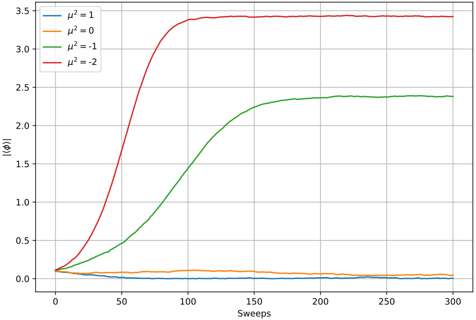


Plot of $|\langle\phi\rangle|$ in dependence of the amount of sweeps and varying ϕ_{ini} with $\mu^2 = -1$ and $\lambda = 1$

Classic Parametrization



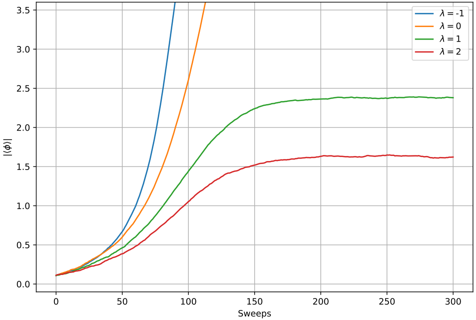
Varying λ with $\mu^2 = 1$ and $\phi_{\text{ini}} = 0.1$



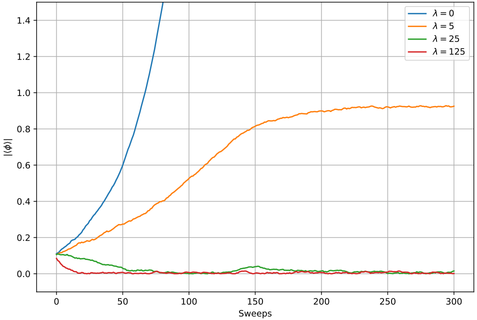
Varying μ^2 with $\lambda = 1$ and $\phi_{\text{ini}} = 0.1$

Plots of $|\langle\phi\rangle|$ in dependence of the amount of sweeps and varying parameters

Classic Parametrization



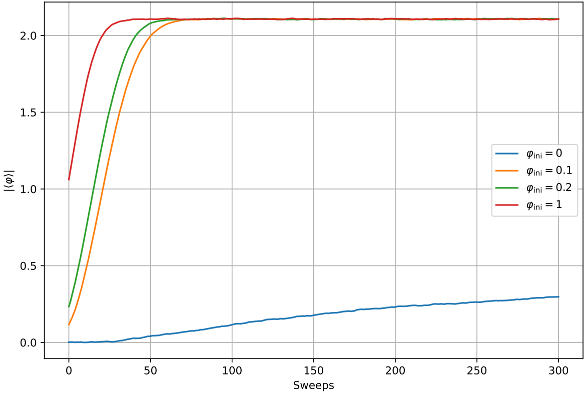
Varying λ with $\mu^2 = -1$ and $\phi_{\text{ini}} = 0.1$



Varying λ with $\mu^2 = -1$ and $\phi_{\text{ini}} = 0.1$

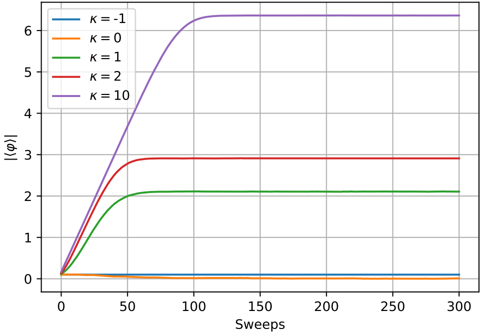
Plots of $|\langle \phi \rangle|$ in dependence of the amount of sweeps and varying λ

Alternative Parametrization

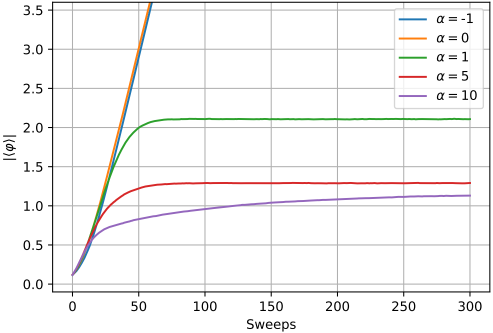


Plot of $|\langle\varphi\rangle|$ in dependence of the amount of sweeps and varying ϕ_{ini} with $\kappa = \alpha = 1$

Alternative Parametrization



Varying κ with $\alpha = 1$ and $\varphi_{\text{ini}} = 0.1$



Varying α with $\kappa = 1$ and $\varphi_{\text{ini}} = 0.1$

Plots of $|\langle\varphi\rangle|$ in dependence of the amount of sweeps and varying parameters

2-point Correlators and Effective Mass

Sum over correlation functions of 2 points, separated by time t

$$\tilde{C}_2(t) = \sum_{i=1}^{N} \langle\Omega| \phi(t, \vec{x}_i)\phi(0, \vec{x}_i) |\Omega\rangle$$

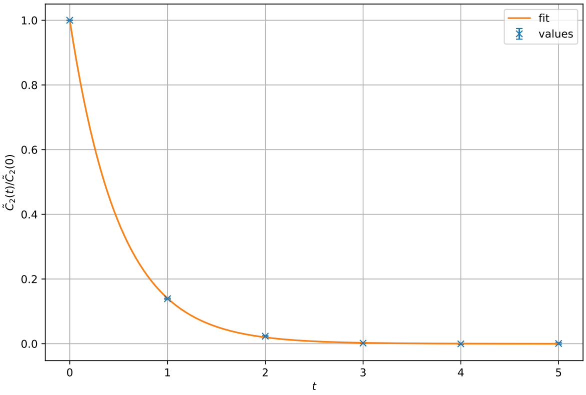
Relation to effective (renormalized) mass of the field:

$$\tilde{C}_2(t) = A \cdot e^{-m_{\mathrm{eff}}|t|} + \dots \quad [6]$$

For large $t \gg 1$:

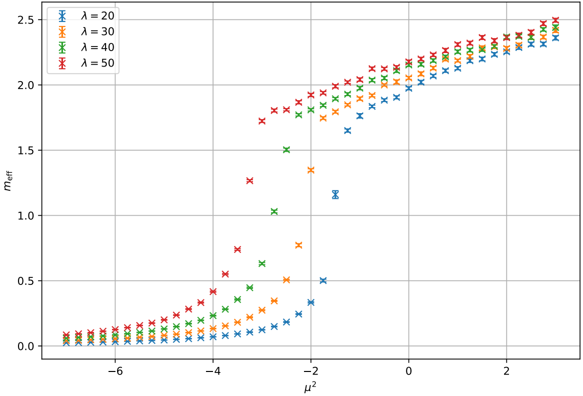
$$m_{\mathrm{eff}} \approx \log\left(\frac{\tilde{C}_2(t)}{\tilde{C}_2(t+1)}\right)$$

2-point Correlators



Normalized 2-point correlators for the classic parametrization, $\mu^2 = \lambda = 1$

Effective Mass



Plot of m_{eff} in dependence of μ^2 for varying λ

Effective Potential

Through renormalization effective potential becomes:

$$V_{\mathrm{eff}}(\phi) = \frac{1}{2} m_{\mathrm{eff}}^2 Z_\phi^{-1} \phi^2 + \frac{\lambda_{\mathrm{r}}}{4!} Z_\phi^{-2} \phi^4 \quad [2]$$

Trick: add external source J

$$U(\phi) := V_{\mathrm{eff}}(\phi) - J\phi$$

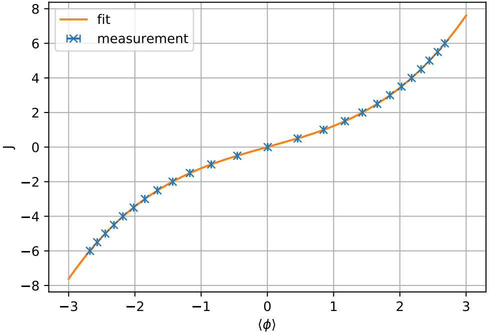
$$\langle\phi\rangle = \langle\phi\rangle(J) \to J = J\left(\langle\phi\rangle\right)$$

vev found at minimum of potential

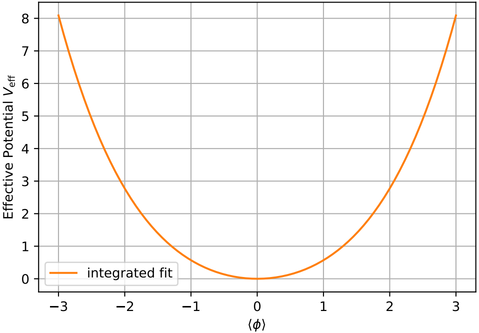
$$\frac{\partial U}{\partial \phi}\left(\langle\phi\rangle\right) = 0 = \frac{\partial V_{\mathrm{eff}}}{\partial \phi}\left(\langle\phi\rangle\right) - J\left(\langle\phi\rangle\right)$$

$$V_{\mathrm{eff}} = \int_0^\phi \mathrm{d}\langle\phi\rangle J\left(\langle\phi\rangle\right)$$

Unbroken Symmetry



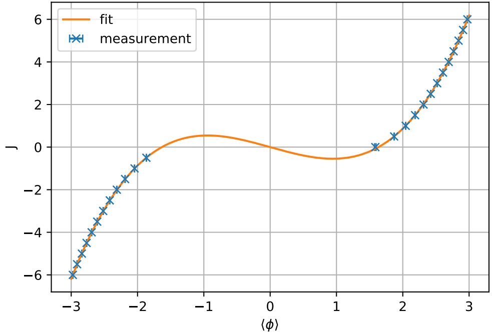
$J(\langle\phi\rangle)$



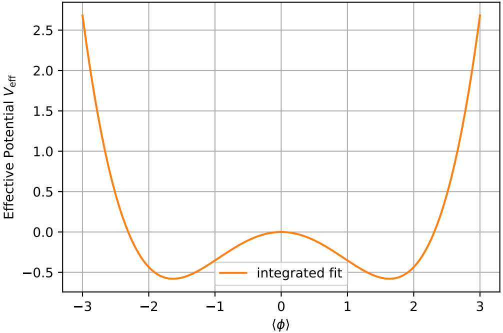
Effective potential V_{eff}

Measurements for the effective potential with $\mu^2 = \lambda = 1$

Broken Symmetry



$J(\langle\phi\rangle)$



Effective potential V_{eff}

Measurements for the effective potential with $\mu^2 = -1$ and $\lambda = 2$

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