HOMEWORK

Contents

1. Homework 1 (Due: Sep 21)

Problem 1.1. Let $\mathcal{L}$ be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n\ge 0}$. Prove the following extremal property: for any monic real polynomial $\pi(x) \ne P_n(x)$ of degree n,

$$\mathcal{L}(P_n(x)^2) < \mathcal{L}(\pi(x)^2).$$

Problem 1.2. Let $\mathcal{L}$ be a linear functional such that $\Delta_n \ne 0$ for all $n \ge 0$. Prove that if $\pi(x)$ is a polynomial such that $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \ge 0$, then $\pi(x) = 0$.

Problem 1.3. The *Tchebyshev polynomials of the second kind* $U_n(x)$ are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta, \qquad n \ge 0.$$

(1) Prove that $U_n(x)$ is a polynomial of degree n.
(2) Prove that

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n \ge 0,$$

where $U_{-1}(x) = 0$ and $U_0(x) = 1$.
(3) Prove that

$$\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx = \frac{\pi}{2}\delta_{m,n}.$$

(4) Find the 3-term recurrence for the normalized Tchebyshev polynomials of the second kind. More precisely, find the numbers b_n and λ_n such that

$$\hat{U}_{n+1}(x) = (x-b_n)\hat{U}_n(x) - \lambda_n \hat{U}_{n-1}(x), \qquad n \ge 0,$$

where $\hat{U}_n(x)$ is the monic polynomial that is a scalar multiple of $U_n(x)$.

Problem 1.4. Let $\{P_n(x)\}_{n\ge 0}$ be the monic OPS for a linear functional $\mathcal{L}$ with three-term recurrence

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \ge 0.$$

(1) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & 1 & & 0 \\ \lambda_1 & x-b_1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & \lambda_{n-1} & x-b_{n-1} \end{vmatrix}.$$

(2) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & \sqrt{\lambda_1} & & 0 \\ \sqrt{\lambda_1} & x-b_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\lambda_{n-1}} \\ 0 & & \sqrt{\lambda_{n-1}} & x-b_{n-1} \end{vmatrix}.$$

(3) Using (2) prove that if $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all, then $P_n(x)$ has real roots only.

2. Homework 2 (Due: Oct 5)

Problem 2.1. Let id be the identity permutation.

(1) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^2 = id$.
(2) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^3 = id$.
(3) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^4 = id$.
(4) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^5 = id$.
(5) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^6 = id$.

Problem 2.2. Let $c_1, \dots, c_n$ be a sequence of nonnegative integers such that $\sum_{i=1}^n ic_i = n$. Show that the number of permutations $\pi \in \mathfrak{S}_n$ with c_i cycles of length i for all $i = 1, \dots, n$ is

$$\frac{n!}{\prod_{i=1}^n i^{c_i} c_i!}.$$

Problem 2.3. For $\pi \in \mathfrak{S}_n$, let $\ell(\pi)$ be the smallest number of simple transpositions whose product is π. Prove that $\ell(\pi) = \mathrm{inv}(\pi)$.

Problem 2.4. Prove that

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Proof. We proceed by induction on n. If $n = 1$, it is true. Let $n \geq 2$ and suppose the statement holds for $n-1$. Every $\pi \in \mathfrak{S}_n$ is obtained from $\sigma \in \mathfrak{S}_{n-1}$ by inserting n after j integers from the beginning for some $0 \leq j \leq n-1$ [3 points]. This construction gives $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma) + j$ [3 points]. Thus

$$\begin{aligned}\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} &= \sum_{\sigma \in \mathfrak{S}_{n-1}} \sum_{j=0}^{n-1} q^{\mathrm{inv}(\sigma)+j} = \sum_{\sigma \in \mathfrak{S}_{n-1}} q^{\mathrm{inv}(\sigma)}(1+q+\cdots+q^{n-1}) \quad \text{[2 points]} \\ &= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) \quad \text{[2 points]}.\end{aligned}$$

Thus the statement is also true for n and we are done. $\square$

3. Homework 3 (Due: Oct 19)

Problem 3.1. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS for a linear functional $\mathcal{L}$ with $\mathcal{L}(1)=1$ given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x) = (x-n)P_n(x) - nP_{n-1}(x).$$

Compute the following.

(1) $\mathcal{L}(x^3)$
(2) $\mathcal{L}(P_{10}(x)P_{10}(x))$
(3) $\mathcal{L}(x^3 P_{10}(x)P_{12}(x))$

Problem 3.2. A *left-to-right minimum* of a permutation $\pi = \pi_1 \cdots \pi_n$ is a number π_i such that $\pi_i = \min\{\pi_1, \dots, \pi_i\}$. Let $\mathrm{LRmin}(\pi)$ denote the number of left-to-right minima in π. For example, if $\pi = 6741372$, then the left-to-right minima are $6, 4, 1$, hence $\mathrm{LRmin}(\pi) = 3$. Prove that

$$\sum_{\pi\in\mathfrak{S}_n} \alpha^{\mathrm{cycle}(\pi)} = \sum_{\pi\in\mathfrak{S}_n} \alpha^{\mathrm{LRmin}(\pi)}.$$

Problem 3.3. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x) = (x-1)P_n(x) - nP_{n-1}(x).$$

Prove that μ_n is equal to the number of involutions in $\mathfrak{S}_n$. (An involution is a permutation π such that π^2 is the identity map.)

Problem 3.4. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

where $\lambda_n \neq 0$ for all $n \geq 1$.

Using the fact $\mu_n = \sum_{\pi\in\mathrm{Motz}_n} \mathrm{wt}(\pi)$, prove that $\mu_{2n+1} = 0$ for all $n\geq 0$ if and only if $b_n = 0$ for all $n \geq 0$.