Enumeration of multiplex juggling card sequences using generalized q-derivatives

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(Gyeonggi Science High School for the Gifted)

GIST

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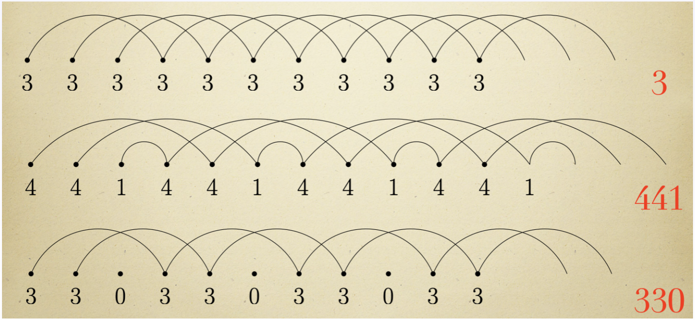

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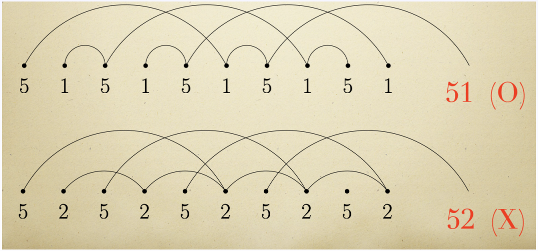

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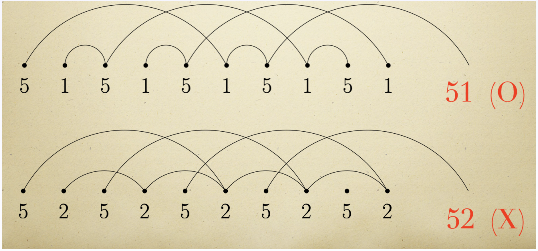

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Enumeration of simple juggling patterns

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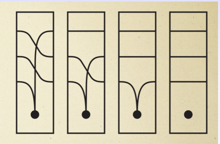

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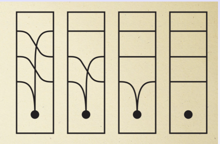

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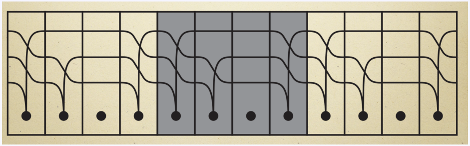

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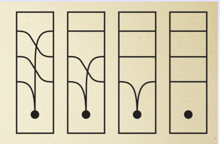

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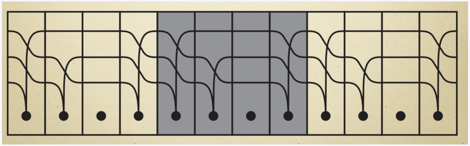

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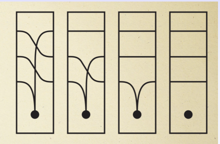

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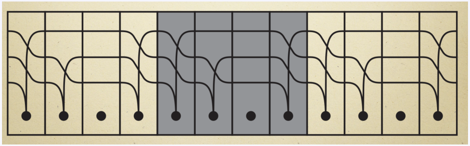

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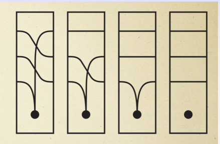

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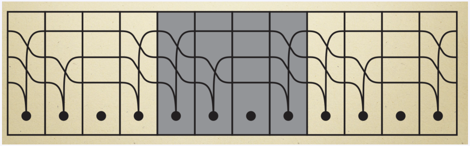

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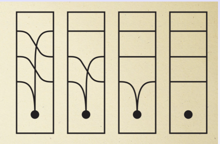

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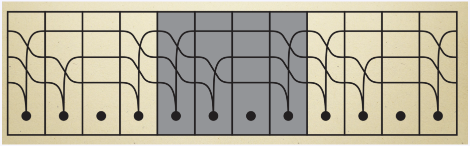

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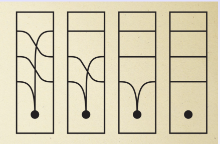

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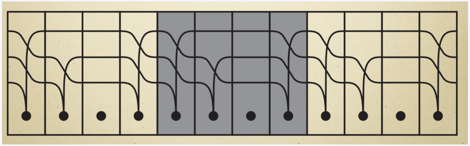

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$$\frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)((b+1)^d - b^d).$$

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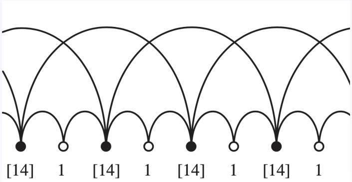

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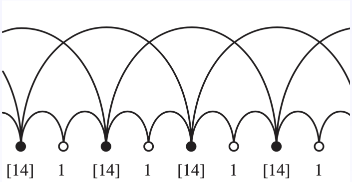

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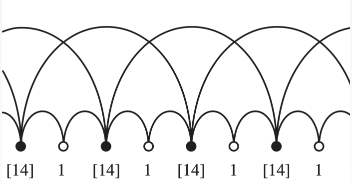

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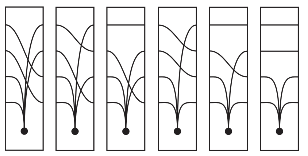

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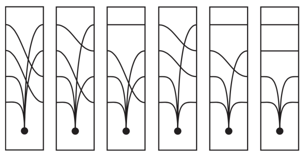

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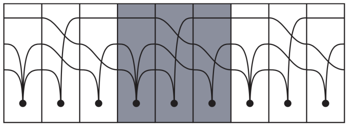

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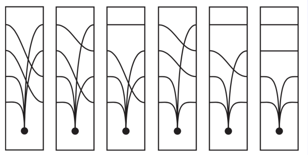

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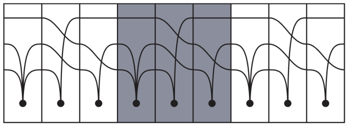

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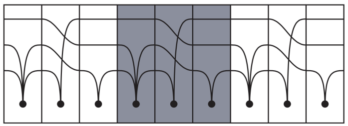

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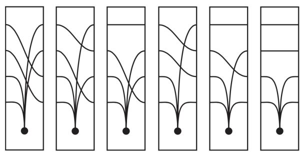

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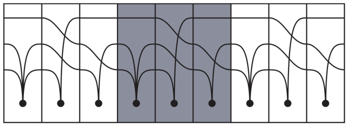

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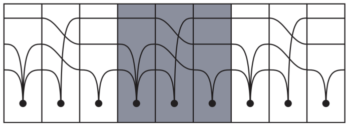

- All we can say is that the number of multiplex juggling patterns with at most b balls and length n is **at most** $\left(\binom{b}{0} + \binom{b}{1} + \cdots + \binom{b}{b}\right)^n = (2^b)^n = 2^{bn}$.

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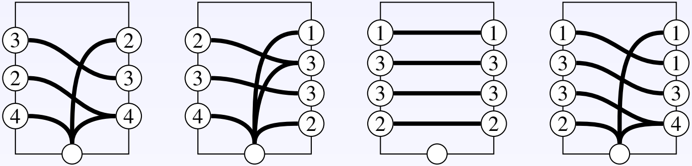

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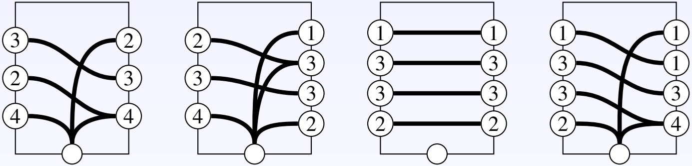

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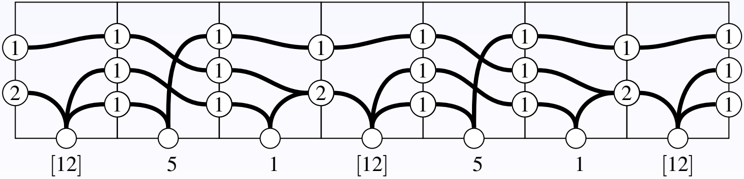

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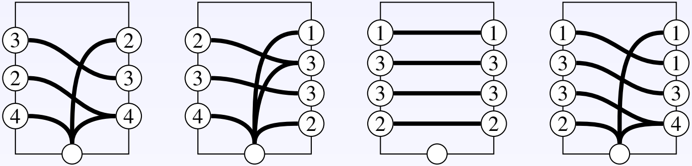

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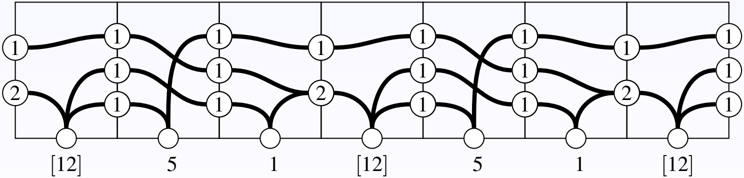

- However, the cards cannot be arranged independently.

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- Note that $J(b, k, 1)$ is the number of multiplex juggling cards with b balls and capacity k.

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- For the case $k=2$, Butler et al. (2019) conjectured the following generating function formula:

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$$\sum_{b\geq 0} J(b,k,1)x^b.$$

- We also find an expression for

$$\sum_{b\geq 0} J(b,k,\ell)x^b \tag{1}$$

  for any capacity k and length ℓ by introducing a generalization of the q-derivative operator.

Our results

- We prove Butler et al.'s conjecture:

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  for any capacity k and length ℓ by introducing a generalization of the q-derivative operator.
- As a consequence, we show that (1) is a rational function in x.

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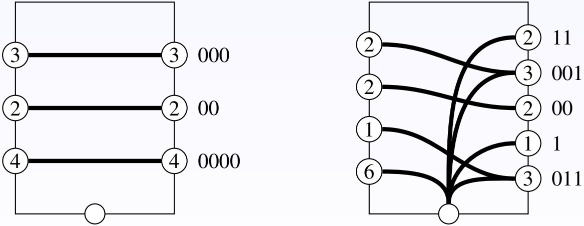
Example

The left card corresponds to the embedding $(0000, 00, 000)$. The right card corresponds to the embedding $(011, 1, 00, 001, 11)$.



The generating function for the number of cards

Proposition

For a positive integer k, we have

$$\sum_{b\geq 0} J(b,k,1)x^b = [z^k]\left(\frac{1}{1-z}\cdot\frac{1}{1-\sum_{i=1}^{k} x^i \sum_{j=0}^{i} z^j}\right).$$

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- Let W be the set of words 0^u1^v such that $u,v\geq 0$ and $1\leq u+v\leq k$.
- For a word $w=0^u1^v\in W$, let $\ell(w)=u+v$ and $\ell_1(w)=v$. By definition,

$$\sum_{w\in W} x^{\ell(w)} z^{\ell_1(w)} = \sum_{i=1}^{k} x^i \sum_{j=0}^{i} z^j.$$

Proof continued

- Thus

$$\frac{1}{1-\sum_{i=1}^{k} x^i \sum_{j=0}^{i} z^j} = 1 + \sum_{r\geq 1} \sum_{(w_1,\ldots,w_r)\in W^r} x^{\ell(w_1)+\cdots+\ell(w_r)} z^{\ell_1(w_1)+\cdots+\ell_1(w_r)}. \quad (2)$$

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- On the other hand, for $b \geq 1$, $\mathcal{E}(b,k)$ is the set of $\gamma = (\gamma_1,\ldots,\gamma_r) \in W^r$, $r \geq 1$, such that $\ell(\gamma_1)+\cdots+\ell(\gamma_r) = b$ and $0 \leq \ell_1(\gamma_1)+\cdots+\ell_1(\gamma_r) \leq k$.

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- Thus

$$\begin{aligned}\sum_{b\geq 0} J(b,k,1)x^b &= 1 + \sum_{b\geq 1} |\mathcal{E}(b,k)| x^b \\ &= 1 + \sum_{r\geq 1} \sum_{\substack{(w_1,\ldots,w_r)\in W^r \\ 0\leq \ell_1(\gamma_1)+\cdots+\ell_1(\gamma_r)\leq k}} x^{\ell(w_1)+\cdots+\ell(w_r)} \\ &= \sum_{s=0}^{k} [z^s] \left(1 + \sum_{r\geq 1} \sum_{(w_1,\ldots,w_r)\in W^r} x^{\ell(w_1)+\cdots+\ell(w_r)} z^{\ell_1(w_1)+\cdots+\ell_1(w_r)}\right).\end{aligned}$$

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- Thus

$$\begin{aligned}
\sum_{b\geq 0} J(b,k,1)x^b &= 1 + \sum_{b\geq 1} |\mathcal{E}(b,k)| x^b \\
&= 1 + \sum_{r\geq 1} \sum_{\substack{(w_1,\ldots,w_r)\in W^r \\ 0\leq \ell_1(\gamma_1)+\cdots+\ell_1(\gamma_r)\leq k}} x^{\ell(w_1)+\cdots+\ell(w_r)} \\
&= \sum_{s=0}^k [z^s] \left(1 + \sum_{r\geq 1} \sum_{(w_1,\ldots,w_r)\in W^r} x^{\ell(w_1)+\cdots+\ell(w_r)} z^{\ell_1(w_1)+\cdots+\ell_1(w_r)}\right).
\end{aligned}$$

- By (2), this is equal to

$$\sum_{s=0}^k [z^s] \left(\frac{1}{1-\sum_{i=1}^k x^i \sum_{j=0}^i z^j}\right) = [z^k] \left(\frac{1}{1-z} \cdot \frac{1}{1-\sum_{i=1}^k x^i \sum_{j=0}^i z^j}\right),$$

as desired.

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Theorem

For a positive integer k, we have

$$\sum_{b \geq 0} J(b, k, 1)x^b = \sum_{\alpha \in \mathrm{Comp}(k)} \frac{(-1)^{\ell_2(\alpha)} x^{k-\ell(\alpha)}}{(1 - x - \cdots - x^k)^{1+\ell(\alpha)}},$$

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Corollary

For a positive integer k, we have

$$\sum_{b\geq 0} J(b,k,1)x^b = \sum_{r=1}^{k}\sum_{s=0}^{r} \frac{(-1)^{r-s}\binom{r}{s}\binom{k-r-1}{r-s-1}x^{k-r}}{(1-x-\cdots-x^k)^{1+r}}.$$

Example

If $k = 1$, then $\mathrm{Comp}(k) = \{(1)\}$. Thus

$$\sum_{b \geq 0} J(b, 1, 1)x^b = \frac{1}{(1-x)^2}.$$

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If $k = 2$, then $\mathrm{Comp}(k) = \{(2), (1, 1)\}$. Thus

$$\sum_{b \geq 0} J(b, 2, 1)x^b = \frac{-x}{(1-x-x^2)^2} + \frac{1}{(1-x-x^2)^3} = \frac{1-x+x^2+x^3}{(1-x-x^2)^3}.$$

This proves Butler et al. 's conjecture.

Example

If $k = 1$, then $\mathrm{Comp}(k) = \{(1)\}$. Thus

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Example

If $k = 2$, then $\mathrm{Comp}(k) = \{(2),(1,1)\}$. Thus

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This proves Butler et al. 's conjecture.

Example

If $k = 3$, then $\mathrm{Comp}(k) = \{(3),(2,1),(1,2),(1,1,1)\}$. Thus

$$\sum_{b\geq 0} J(b,3,1)x^b = \frac{-x^2}{(1-x-x^2-x^3)^2} + \frac{-2x}{(1-x-x^2-x^3)^3} + \frac{1}{(1-x-x^2-x^3)^4}$$
$$= \frac{1-2x+x^2+4x^3+3x^4-3x^6-2x^7-x^8}{(1-x-x^2-x^3)^4}.$$

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Juggling card sequences with fixed capacity

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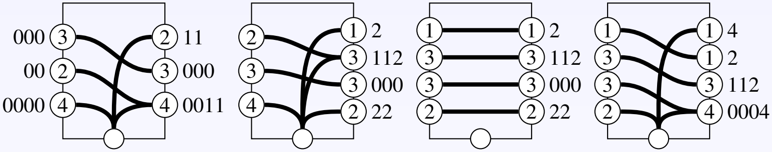
Embeddings

- We can extend the notion of an embedding of a card to a sequence of cards.



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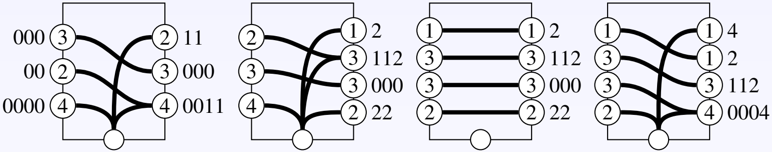

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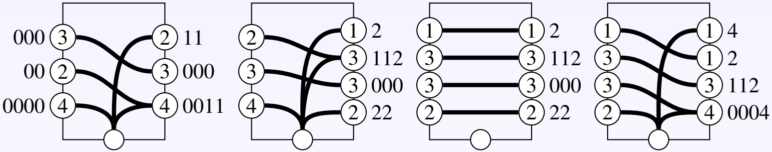

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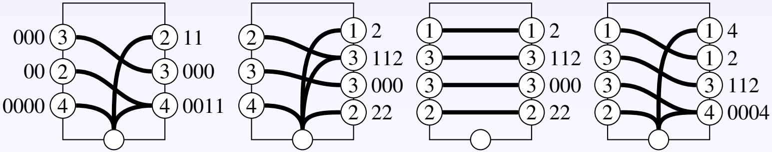

Observation

- In the ith card, if there are r i's then r balls are thrown at beat i.
- The r thrown balls have have an associated word of integers in $0, 1, \ldots, i-1$.

Definition

For a nonnegative integer n, the **homogeneous symmetric function** $h_n(x_1, \ldots, x_m)$ is defined by

$$h_n(x_1, \ldots, x_m) = \sum_{i_1 + \cdots + i_m = n} x_1^{i_1} \cdots x_m^{i_m},$$

where the sum is over all m-tuples $(i_1, \ldots, i_m)$ of nonnegative integers such that $i_1 + \cdots + i_m = n$.

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For indeterminates $z_1, \ldots, z_k$, we denote by $D_{z_1,\ldots,z_k}$ the linear operator on the space of formal power series in z_k defined by

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Remark

Note that this can be seen as a generalization of the **q-derivative operator** $(\frac{d}{dz})_q$, which is the linear operator on the space of formal power series in z defined by

$$\left(\frac{d}{dz}\right)_q z^n = (1 + q + \cdots + q^{n-1}) z^{n-1}.$$

Hence $D_{q,z}$ is equal to the operator $(\frac{d}{dz})_q z$, which multiplies z and then takes the q-derivative.

Main result

Theorem

For fixed positive integers k and ℓ, we have

$$\sum_{b\geq 0} J(b,k,\ell)x^b$$
$$= [z_1^k \cdots z_\ell^k]\left(\frac{1}{1-z_1}\cdots\frac{1}{1-z_\ell}D_{z_1,z_2}D_{z_1,z_2,z_3}\cdots D_{z_1,\ldots,z_\ell}\,\frac{1}{2-h_k(1,x,xz_1,\ldots,xz_\ell)}\right).$$

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- By the quotient rule in calculus, one can easily deduce that the derivative of a rational function is also a rational function.

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we have

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  we have

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- Hence, if $f(z_2)$ is a rational function in z_2, then so is $D_{z_1,z_2} f(z_2)$.

Lemma

For integers $n \geq 0$ and $m \geq 2$, we have

$$h_n(1, z_1, \ldots, z_m) = \frac{z_{m-1} h_n(1, z_1, \ldots, z_{m-1}) - z_m h_n(1, z_1, \ldots, z_{m-2}, z_m)}{z_{m-1} - z_m}.$$

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Lemma

For an integer $m \geq 2$, we have

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Lemma

For integers $n \geq 0$ and $m \geq 2$, we have

$$h_n(1, z_1, \dots, z_m) = \frac{z_{m-1} h_n(1, z_1, \dots, z_{m-1}) - z_m h_n(1, z_1, \dots, z_{m-2}, z_m)}{z_{m-1} - z_m}.$$

Lemma

For an integer $m \geq 2$, we have

$$D_{z_1, \dots, z_{m+1}} = \frac{z_{m-1} D_{z_1, \dots, z_{m-1}, z_{m+1}} - z_m D_{z_1, \dots, z_{m-2}, z_m, z_{m+1}}}{z_{m-1} - z_m}.$$

Proposition

Suppose that ℓ and m are integers with $2 \leq m \leq \ell$ and let $z_1, \dots, z_\ell$ be indeterminates. If $f(z_1, \dots, z_\ell)$ is a formal power series in $z_1, \dots, z_\ell$ that is a rational function in $z_1, \dots, z_\ell$, then $D_{z_1, \dots, z_m} f(z_1, \dots, z_\ell)$ is a rational function in $z_1, \dots, z_\ell$.

Corollary

For fixed positive integers k and ℓ, the generating function

$$\sum_{b \geq 0} J(b, k, \ell) x^b$$

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- We proved that

$$\sum_{b \geq 0} J(b, k, \ell) x^b = [z_1^k \cdots z_\ell^k] f(z_1, \ldots, z_\ell, x) \tag{4}$$

  for a formal power series $f(z_1, \ldots, z_\ell, x)$ in $z_1, \ldots, z_\ell, x$ that is a rational function in these indeterminates.

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- If $g(z)$ is a formal power series in z that is a rational function in z and some other indeterminates, say $u_1,\ldots,u_r$, then by the quotient rule, $[z^k]g(z) = k!g^{(k)}(0)$ is a rational function in $u_1,\ldots,u_r$.

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- Therefore the right-hand side of (4) is a rational function x as desired.

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Problem

Determine whether the following is a rational function in x, y, and z:

$$\sum_{b\geq 0}\sum_{k\geq 0}\sum_{\ell\geq 0} J(b,k,\ell)x^b y^k z^\ell.$$

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Thank you!