# LECTURE NOTES ON THE COMBINATORICS OF ORTHOGONAL POLYNOMIALS

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Contents



## 1. Introduction to the lectures

Orthogonal polynomials are classical objects arising from the study of continued fractions. Due to the long history of orthogonal polynomials, they have now become important objects of study in many areas: classical analysis and PDE, mathematical physics, probability, random matrix theory, and combinatorics.

The combinatorial study of orthogonal polynomials was pioneered by Flajolet and Viennot in 1980s. In these lectures we will learn fascinating combinatorial properties of orthogonal polynomials.

We will first study basic properties of orthogonal polynomials based on Chihara's book, Chapter 1 [1]. We will then focus on the combinatorial approach of orthogonal polynomials, which will be based on Viennot's lecture notes [2]. We will also cover more recent developments in the combinatorics of orthogonal polynomials such as their connections with ASEP, staircase tableaux, lecture hall partitions, and orthogonal polynomials of type $R_1$.

The prerequisites of this course are Calculus 1, Linear Algebra, and Discrete Mathematics.

In the first section we study elementary and classical results of orthogonal polynomials. Starting from the second section we focus on the combinatorics of orthogonal polynomials. Basic results in combinatorics are collected in the appendices.

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## 2. Elementary Theory of Orthogonal Polynomials

In this section we will cover the first chapter of Chihara's book [1].

### 2.1. **Introduction.**

Since

$$2\cos m\theta\cos n\theta = \cos(m+n)\theta + \cos(m-n)\theta,$$

for nonnegative integers $m$ and $n$, we have

$$\int_0^\pi \cos m\theta \cos n\theta d\theta = 0, \qquad m \ne n. \tag{2.1}$$

In this situation we say that $\cos m\theta$ and $\cos n\theta$ are orthogonal over the interval $(0,\pi)$.

Note that $\cos n\theta$ is a polynomial in $\cos\theta$ of degree $n$. So we can write $\cos n\theta = T_n(\cos\theta)$ for a polynomial $T_n(x)$ of degree $x$.

By the change of variable $x = \cos\theta$, (2.1) can be rewritten as

$$\int_{-1}^1 T_m(x)T_n(x)(1-x^2)^{-1/2}dx = 0, \qquad m \ne n.$$

The polynomials $T_n(x)$, $n \ge 0$, are called the **Tchebyshev polynomials of the first kind**. The first few polynomials are:

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= \cos\theta = x,\\
T_2(x) &= \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1,\\
T_3(x) &= 4x^3 - 3x.
\end{aligned}$$

Recall that in an inner product space $V$ with inner product $\langle\cdot,\cdot\rangle$, a set of vectors $v_1,\dots,v_n$ are said to be orthogonal if $\langle v_i, v_j\rangle = 0$ for all $i \ne j$. In this sense the Tchebyshev polynomials $T_n(x)$ are orthogonal, where $V = \mathbb{R}[x]$ is the space of polynomials with real coefficients with the inner product given by

$$\langle f(x), g(x)\rangle = \int_{-1}^1 f(x)g(x)(1-x^2)^{-1/2}dx.$$

We say that $T_n(x)$ are **orthogonal polynomials** with respect to the **weight function** $(1-x^2)^{-1/2}$ on the interval $(-1,1)$.

**Definition 2.1.** Suppose that $w(x)$ is a nonnegative and integrable function on $(a,b)$ with $\int_a^b w(x)dx > 0$ and $\int_a^b x^n dx < \infty$ for all $n \ge 0$. A sequence of polynomials $\{P_n(x)\}_{n\ge 0}$ is called an **orthogonal polynomial sequence (OPS)** with respect to the **weight function** (or **measure**) $w(x)$ on $(a,b)$ if the following conditions hold:

(1) $\deg P_n(x) = n$, for $n \ge 0$,
(2) $\int_a^b P_m(x)P_n(x)w(x)dx = 0$ for $m \ne n$.

There is another way to define orthogonal polynomials without using the weight function. For a polynomial $f(x)$, if we define

$$\mathcal{L}(f(x)) = \int_a^b f(x)w(x)dx,$$

then $\mathcal{L}(f(x))$ is completely determined by the **moments** $\mu_n = \int_a^b x^n w(x)dx$. So, if we are only interested in polynomials, then we can define a linear functional $\mathcal{L}$ using a moment sequence $\mu_0, \mu_1, \dots$. Not every sequence $\mu_0, \mu_1, \dots$ gives rise to an OPS, though. We will see later a criterion for a sequence to be a moment sequence.

**Definition 2.2.** Let $\mathcal{L}$ be a linear functional defined on the space of polynomials in $x$. A sequence of polynomials $\{P_n(x)\}_{n\ge 0}$ is called an **orthogonal polynomial sequence (OPS)** with respect to $\mathcal{L}$ if the following conditions hold:

(1) $\deg P_n(x) = n$, $n \ge 0$,

(2) $\mathcal{L}(P_m(x)^2) \neq 0$ for $m \geq 0$,
(3) $\mathcal{L}(P_m(x)P_n(x)) = 0$ for $m \neq n$.

Note that the second condition above was not necessary in Definition 2.1 because it follows from the facts that $w(x)$ is nonnegative and $\int_a^b w(x)dx > 0$.

**Remark 2.3.** The moments of the Tchebyshev polynomials are

$$\mu_{2n} = \int_{-1}^{1} x^{2n}(1-x^2)^{-1/2}dx = \frac{\pi}{2^{2n}}\binom{2n}{n}, \qquad \mu_{2n+1} = 0.$$

This suggests that there could be some interesting combinatorics behind the scene. We will later find a combinatorial way to understand this situation.

**Example 2.4** (Charlier polynomials)**.** The **Charlier polynomials** $P_n(x)$ are defined by

$$P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!},$$

where $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$. We will find a different type of orthogonality for $P_n(x)$.

The generating function for $P_n(x)$ is

$$G(x,w) = \sum_{n\geq 0} P_n(x)w^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{x}{k}\frac{(-a)^{n-k}}{(n-k)!}\right)w^n = \sum_{n\geq 0}\binom{x}{n}w^n \sum_{n\geq 0}\frac{(-a)^m}{m!}w^m,$$

which means

$$G(x,w) = e^{-aw}(1+w)^x.$$

Thus

$$a^x G(x,v)G(x,w) = e^{-a(v+w)}\left(a(1+v)(1+w)\right)^x.$$

We have

$$\sum_{k\geq 0}\frac{a^k G(k,v)G(k,w)}{k!} = \sum_{k\geq 0}\frac{e^{-a(v+w)}\left(a(1+v)(1+w)\right)^k}{k!} = e^{-a(v+w)}e^{a(1+v)(1+w)} = e^a e^{avw}.$$

Thus

$$\sum_{k\geq 0}\frac{a^k G(k,v)G(k,w)}{k!} = \sum_{n\geq 0}\frac{e^a(avw)^n}{n!}. \tag{2.2}$$

On the other hand

$$\begin{aligned}\sum_{k\geq 0}\frac{a^k G(k,v)G(k,w)}{k!} &= \sum_{k\geq 0}\frac{a^k}{k!}\sum_{m,n\geq 0}P_m(k)P_n(k)v^m w^n \\ &= \sum_{m,n\geq 0}\left(\sum_{k\geq 0}P_m(k)P_n(k)\frac{a^k}{k!}\right)v^m w^n. \end{aligned}\tag{2.3}$$

Comparing the coefficients of $v^m w^n$ in (2.2) and (2.3) we obtain

$$\sum_{k\geq 0}P_m(k)P_n(k)\frac{a^k}{k!} = \frac{e^a a^n}{n!}\delta_{n,m}. \tag{2.4}$$

Therefore, if we define a linear functional $\mathcal{L}$ by

$$\mathcal{L}(x^n) = \sum_{k\geq 0}k^n\frac{a^k}{k!},$$

then $P_n(x)$ are orthogonal polynomials with respect to $\mathcal{L}$.

Note that we describe the orthogonality of $P_n(x)$ using only the linear functional $\mathcal{L}$ without referring to any weight function. However, we can also find a weight function in this case. Let

$\psi(x)$ be the step function with a jump at $k = 0, 1, 2, \dots$ of magnitude $a^k/k!$. Then the linear functional $\mathcal{L}$ can be written as the following Riemann–Stieltjes integral

$$\mathcal{L}(f(x)) = \int_{-\infty}^{\infty} f(x) d\psi(x).$$

We can also prove (2.4) in a combinatorial way, see Appendix A.

**Remark 2.5.** In the theory of orthogonal polynomials, finding an explicit weight function is an important problem. However, in these lectures, we will not pursue in this direction and we will be mostly satisfied with Definition 2.2.

2.2. **The moment functional and orthogonality.** We will consider the space $\mathbb{C}[x]$ of polynomials with complex coefficients. A **linear functional** on $\mathbb{C}[x]$ is a map $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$ such that $\mathcal{L}(af(x) + bg(x)) = a\mathcal{L}(f(x)) + b\mathcal{L}(g(x))$ for all $f(x), g(x) \in \mathbb{C}[x]$ and $a, b \in \mathbb{C}$.

**Definition 2.6.** Let $\{\mu_n\}_{n\ge 0}$ be a sequence of complex numbers. Let $\mathcal{L}$ be the linear functional on the space of polynomials defined by $\mathcal{L}(x^n) = \mu_n$, $n \ge 0$. In this case we say that $\mathcal{L}$ is the **moment functional** determined by the **moment sequence** $\{\mu_n\}$, and $\mu_n$ is called the **$n$th moment**.

We recall the definition of orthogonal polynomials.

**Definition 2.7.** Let $\mathcal{L}$ be the linear functional defined on the space of polynomials in $x$. A sequence of polynomials $\{P_n(x)\}_{n\ge 0}$ is called an **orthogonal polynomial sequence (OPS)** with respect to $\mathcal{L}$ if the following conditions hold:

(1) $\deg P_n(x) = n$, $n \ge 0$,
(2) $\mathcal{L}(P_m(x)P_n(x)) = K_n\delta_{m,n}$, for some $K_n \ne 0$.

We say that $P_n(x)$ are **orthonormal** if $\mathcal{L}(P_m(x)P_n(x)) = \delta_{m,n}$.

**Theorem 2.8.** *Let $\{P_n(x)\}$ be a sequence of polynomials and let $\mathcal{L}$ be a linear functional. The following are equivalent:*

(1) *$\{P_n(x)\}$ is an OPS with respect to $\mathcal{L}$;*
(2) *$\mathcal{L}(\pi(x)P_n(x)) = 0$ if $\deg \pi(x) < n$ and $\mathcal{L}(\pi(x)P_n(x)) \ne 0$ if $\deg \pi(x) = n$;*
(3) *$\mathcal{L}(x^m P_n(x)) = K_n\delta_{m,n}$, $0 \le m \le n$, for some $K_n \ne 0$.*

*Proof.* (1) $\Rightarrow$ (2): Suppose that $\deg \pi(x) \le n$. Since $\{P_n(x)\}$ is a basis of $\mathbb{C}[x]$, we can write

$$\pi(x) = c_0 + c_1P_1(x) + \cdots + c_nP_n(x).$$

Then

$$\mathcal{L}(\pi(x)P_n(x)) = \sum_{k=0}^{n} \mathcal{L}\left(c_kP_k(x)P_n(x)\right) = c_n\mathcal{L}(P_n(x)^2),$$

which is zero if $\deg \pi(x) < n$ and nonzero if $\deg \pi(x) = n$.

(2) $\Rightarrow$ (3): Trivial. (2) $\Rightarrow$ (3): Trivial. ∎

**Theorem 2.9.** *Suppose that $\{P_n(x)\}_{n\ge 0}$ be an OPS with respect to $\mathcal{L}$. Then for any polynomial $\pi(x)$ of degree $n$,*

$$\pi(x) = \sum_{k=0}^{n} c_kP_k(x), \qquad c_k = \frac{\mathcal{L}(\pi(x)P_k(x))}{\mathcal{L}(P_k(x)^2)}.$$

*Proof.* Clearly, we can write

$$\pi(x) = \sum_{k=0}^{n} c_kP_k(x),$$

for some $c_k$. Multiplying $P_j(x)$ both sides and taking $\mathcal{L}$, we get

$$\mathcal{L}(\pi(x)P_j(x)) = \sum_{k=0}^{n} \mathcal{L}\left(c_kP_k(x)P_j(x)\right) = c_j\mathcal{L}(P_j(x)^2).$$

Dividing both sides by $\mathcal{L}(P_j(x)^2)$, we obtain the theorem. ∎

**Theorem 2.10.** *Suppose that $\{P_n(x)\}_{n\ge0}$ be an OPS with respect to $\mathcal{L}$. Then $P_n(x)$ is uniquely determined by $\mathcal{L}$ up to a nonzero factor. More precisely, if $\{Q_n(x)\}_{n\ge0}$ is an OPS with respect to $\mathcal{L}$, then there are constants $c_n\ne0$ such that $Q_n(x)=c_nP_n(x)$ for all $n\ge0$.*

*Proof.* Let us write $Q_n(x)=\sum_{k=0}^n c_kP_k(x)$. Then by Theorem 2.9, $c_k=\mathcal{L}(Q_n(x)P_k(x))/\mathcal{L}(P_k(x)^2)$. But by Theorem 2.8, $\mathcal{L}(Q_n(x)P_k(x))=0$ unless $k=n$. Thus $Q_n(x)=c_nP_n(x)$. □

Note that if $\{P_n(x)\}_{n\ge0}$ is an OPS for $\mathcal{L}$, then so is $\{c_nP_n(x)\}_{n\ge0}$ for any $c_n\ne0$. Therefore there is a unique monic OPS, which is obtained by dividing each $P_n(x)$ by its leading coefficient. Note also that there is a unique orthonormal OPS as well given by $p_n(x)=P_n(x)/\mathcal{L}(P_n(x)^2)^{1/2}$. In summary we have the following corollary.

**Corollary 2.11.** *Suppose that $\mathcal{L}$ is a moment sequence such that there is an OPS for $\mathcal{L}$. Let $K_n$, $n\ge0$, be a sequence of nonzero numbers. Then the following hold.*

(1) *There is a unique monic OPS $\{P_n(x)\}_{n\ge0}$ for $\mathcal{L}$.*
(2) *There is a unique OPS $\{P_n(x)\}_{n\ge0}$ for $\mathcal{L}$ such that the leading coefficient of $P_n(x)$ is $K_n$.*
(3) *There is a unique OPS $\{P_n(x)\}_{n\ge0}$ for $\mathcal{L}$ such that $\mathcal{L}(x^nP_n(x))=K_n$.*

Clearly, if $\{P_n(x)\}_{n\ge0}$ is an OPS for $\mathcal{L}$, then it is also an OPS for $\mathcal{L}'$ given by $\mathcal{L}'(f(x))=c\mathcal{L}(f(x))$ for some $c\ne0$. Therefore, by dividing the linear functional by the value $\mathcal{L}(1)$, we may assume that $\mathcal{L}(1)=1$.

2.3. **Existence of OPS.** The main question in this section is: for what linear functional $\mathcal{L}$ does there exist an OPS? To answer this question we need the following definition.

**Definition 2.12.** The **Hankel determinant** of a moment sequence $\{\mu_n\}$ is defined by

$$\Delta_n=\det(\mu_{i+j})_{i,j=0}^n=\begin{vmatrix}\mu_0&\mu_1&\cdots&\mu_n\\ \mu_1&\mu_2&\cdots&\mu_{n+1}\\ \vdots&\vdots&\ddots&\vdots\\ \mu_n&\mu_{n+1}&\cdots&\mu_{2n}\end{vmatrix}.$$

**Theorem 2.13.** *Let $\mathcal{L}$ be a linear functional with moment sequence $\{\mu_n\}$. Then there is an OPS for $\mathcal{L}$ if and only if $\Delta_n\ne0$ for all $n\ge0$.*

*Proof.* Fix a sequence $\{K_n\}$ of nonzero real numbers $K_n$. By Corollary 2.11, if there is an OPS $\{P_n(x)\}_{n\ge0}$ for $\mathcal{L}$, it is uniquely determined by the condition $\mathcal{L}(x^nP_n(x))=K_n$, $n\ge0$. In other words, using Theorem 2.8, there is an OPS for $\mathcal{L}$ if and only if there is a unique sequence $\{P_n(x)\}_{n\ge0}$ of polynomials such that

$$\mathcal{L}(x^mP_n(x))=K_n\delta_{m,n},\qquad 0\le m\le n. \tag{2.5}$$

Now let $P_n(x)=\sum_{k=0}^n c_{n,k}x^k$. Multiplying both sides by $x^m$ and taking $\mathcal{L}$, we get

$$\mathcal{L}(x^mP_n(x))=\sum_{k=0}^n c_{n,k}\mu_{n+k}.$$

Thus (2.5) can be written as the matrix equation

$$\begin{pmatrix}\mu_0&\mu_1&\cdots&\mu_n\\ \mu_1&\mu_2&\cdots&\mu_{n+1}\\ \vdots&\vdots&\ddots&\vdots\\ \mu_n&\mu_{n+1}&\cdots&\mu_{2n}\end{pmatrix}\begin{pmatrix}c_{n,0}\\ c_{n,1}\\ \vdots\\ c_{n,n}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\\ K_n\end{pmatrix}. \tag{2.6}$$

Then the uniqueness of the polynomials $P_n(x)$ satisfying (2.5) is equivalent to the uniqueness of the solution of the matrix equation (2.6) in $c_{n,0},c_{n,1},\dots,c_{n,n}$. In order for (2.6) to have a unique solution, the Hankel determinant $\Delta_n$ must be nonzero for all $n\ge0$. Moreover, by Cramer's rule, $c_{n,n}=K_n\Delta_n/\Delta_{n-1}$ is nonzero iff $\Delta_n\ne0$. This proves the theorem. □

Applying Cramer's rule to (2.6) we can prove the following lemma, which will be used later.

**Lemma 2.14.** *Let $\{P_n(x)\}_{n\geq 0}$ be an OPS for $\mathcal{L}$. Then for a polynomial $\pi(x)$ of degree $n$ we have*

$$\mathcal{L}(\pi(x)P_n(x)) = \frac{ab\Delta_n}{\Delta_{n-1}},$$

*where $a$ and $b$ are the leading coefficients of $\pi(x)$ and $P_n(x)$, respectively. In particular, if $\{P_n(x)\}_{n\geq 0}$ is the monic OPS for $\mathcal{L}$, then*

$$\mathcal{L}(P_n(x)^2) = \frac{\Delta_n}{\Delta_{n-1}}.$$

*Proof.* We use the notation in the proof of Theorem 2.13. By solving (2.6) using Cramer's rule, we obtain that the leading coefficient of $P_n(x)$ is $b = c_{n,n} = K_n\Delta_{n-1}/\Delta_n$. Thus, if we let $\pi(x) = \sum_{k=0}^n a_k x^k$, we have

$$\mathcal{L}(\pi(x)P_n(x)) = \sum_{k=0}^n \mathcal{L}(a_k x^k P_n(x)) = a_n \mathcal{L}(x^n P_n(x)) = aK_n = \frac{ab\Delta_n}{\Delta_{n-1}},$$

as desired. □

Similarly every coefficient $c_{n,i}$ of $P_n(x)$ can be computed using (2.6). Thus we have an explicit determinant formula for $P_n(x)$.

**Theorem 2.15.** *Let $\mathcal{L}$ be a linear functional with moment sequence $\{\mu_n\}$ with $\Delta_n \neq 0$ for all $n \geq 0$. Then the monic OPS for $\mathcal{L}$ is given by*

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}.$$

*Proof.* This can be proved using (2.6). We can also prove directly that $\{P_n(x)\}_{n\geq 0}$ satisfies the conditions for an OPS. First, the coefficient of $x^n$ in $P_n(x)$ is 1, so $\deg P_n(x) = n$. For $0 \leq k \leq n$, we have

$$\mathcal{L}(x^k P_n(x)) = \frac{1}{\Delta_{n-1}} \mathcal{L}\left( \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ x^k & x^{k+1} & \cdots & x^{n+k} \end{vmatrix} \right) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_k & \mu_{k+1} & \cdots & \mu_{n+k} \end{vmatrix}.$$

If $k < n$, then the right-hand side of the above equation has two identical rows, hence zero. If $k = n$, the right-hand side is $\Delta_n/\Delta_{n-1} \neq 0$. This implies that $\{P_n(x)\}_{n\geq 0}$ is an OPS for $\mathcal{L}$. □

In many important cases of orthogonal polynomials there is a nonnegative weight function $w(x)$ representing the moment functional: $\mathcal{L}(x^n) = \int_a^b x^n w(x)dx$. In more general cases, $\mathcal{L}$ can be represented using the Riemann–Stieltjes integral $\mathcal{L}(x^n) = \int_a^b x^n d\psi(x)$, where $\psi(x)$ is a nondecreasing function such that $\{x : \psi(x+\epsilon) - \psi(x-\epsilon) > 0 \text{ for all } \epsilon > 0\}$ is an infinite set. It is known [1, Chapter 2] that there is such an expression if and only if $\mathcal{L}(\pi(x)) > 0$ for all nonzero polynomials $\pi(x)$ such that $\pi(x) \geq 0$ for all $x \in \mathbb{R}$.

A **nonnegative-valued** polynomial is a polynomial $\pi(x)$ such that $\pi(x) \geq 0$ for all $x \in \mathbb{R}$.

**Definition 2.16.** A linear functional $\mathcal{L}$ is **positive-definite** if $\mathcal{L}(\pi(x)) > 0$ for all nonzero nonnegative-valued polynomials $\pi(x)$.

If $\mathcal{L}$ is positive-definite, then it has a real OPS. We will see later that the converse is not true.

**Theorem 2.17.** *Let $\mathcal{L}$ be a positive-definite linear functional. Then $\mathcal{L}$ has real moments and there is a real OPS for $\mathcal{L}$.*

*Proof.* First, we show that the moments $\mu_n$ are real. Since $\mathcal{L}$ is positive-definite, $\mu_{2n} = \mathcal{L}(x^{2n}) > 0$ is real. Since $\mathcal{L}((x+1)^{2n}) = \sum_{k=0}^{2n} \binom{2n}{k}\mu_k$ is real, by induction, we obtain that $\mu_{2n-1}$ is also real.

Now, we construct a real OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$. Let $P_0(x) = 1$. Suppose that we have constructed real polynomials $P_0, \dots, P_n$ which are orthogonal with respect to $\mathcal{L}$, i.e., for $0 \le i, j \le n$, $\mathcal{L}(P_i(x)P_j(x))$ is zero if $i \ne j$ and nonzero if $i = j$. Now we need to find

$$P_{n+1}(x) = x^{n+1} + \sum_{k=0}^{n} a_k P_k(x) \tag{2.7}$$

such that $\mathcal{L}(P_k(x)P_{n+1}(x)) = 0$ for all $0 \le k \le n$. Multiplying $P_k(x)$ and taking $\mathcal{L}$ in (2.7) we get $\mathcal{L}(P_k(x)P_{n+1}(x)) = \mathcal{L}(x^{n+1}P_k(x)) + a_k\mathcal{L}(P_k(x)^2)$. Thus, if we set

$$a_k = -\frac{\mathcal{L}(x^{n+1}P_k(x))}{\mathcal{L}(P_k(x)^2)},$$

which is real, then $P_{n+1}(x)$ is orthogonal to $P_0(x), \dots, P_n(x)$. In this way we can construct a real OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$. □

Note that if $\mathcal{L}$ is positive-definite, then $\mathcal{L}(P_n(x)^2) > 0$. Thus in this case we can construct a real orthonormal OPS $\{p_n(x)\}_{n\ge 0}$ by rescaling: $p_n(x) = P_n(x)/\sqrt{\mathcal{L}(P_n(x)^2)}$.

Nonnegative-valued polynomials have the following useful property.

**Lemma 2.18.** *Let $\pi(x)$ be a nonnegative-valued polynomial. Then $\pi(x) = p(x)^2 + q(x)^2$ for some real polynomials $p(x)$ and $q(x)$.*

*Proof.* Since $\pi(x)$ is real for all real $x$, the coefficients of $\pi(x)$ are real. This can be seen inductively by observing that if $\deg \pi(x) = n$, then the leading coefficient of $\pi(x)$ is equal to

$$\lim_{x\to\infty} \frac{\pi(x)}{x^n}.$$

Since $\pi(x)$ is a real polynomial such that $\pi(x) \ge 0$, every real zero of $\pi(x)$ has even multiplicity and complex roots appear in conjugate pairs. Thus we can write

$$\pi(x) = r(x)^2 \prod_{k=1}^{m} (x - \alpha_k - \beta_k i)(x - \alpha_k + \beta_k i),$$

where $r(x)$ is a real polynomial and $\alpha_k, \beta_k \in \mathbb{R}$. If we write $\prod_{k=1}^{m}(x - \alpha_k - \beta_k i) = A(x) + iB(x)$, then $\prod_{k=1}^{m}(x - \alpha_k + \beta_k i) = A(x) - iB(x)$. Thus $\pi(x) = r(x)^2(A(x)^2 + B(x)^2)$ as desired. □

By Lemma 2.18, we have the following criterion for linear functionals.

**Corollary 2.19.** *A linear functional $\mathcal{L}$ is positive-definite if and only if $\mathcal{L}(p(x)^2) > 0$ for every nonzero real polynomial $p(x)$.*

You may wonder why $\mathcal{L}$ is called "positive-definite". To see this recall that a real $n \times n$ matrix $A$ is positive definite if $u^T A u > 0$ for every nonzero vector $u \in \mathbb{R}^n$. Sylvester's criterion says that $A$ is positive definite if and only if every principal minor of $A$ is positive. The following theorem justifies the terminology "positive-definite" for $\mathcal{L}$.

**Theorem 2.20.** *A linear functional $\mathcal{L}$ is positive-definite if and only if every moment $\mu_n$ is real and $\Delta_n > 0$ for all $n \ge 0$. In other words, $\mathcal{L}$ is positive-definite if and only if the Hankel matrix $(\mu_{i+j})_{i,j=0}^n$ is positive-definite for all $n \ge 0$.*

*Proof.* ($\Rightarrow$) By Theorem 2.17, the moments are real and there is a real OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$. By Lemma 2.14, $\Delta_n/\Delta_{n-1} = \mathcal{L}(P_n(x)^2) > 0$ for $n \ge 0$, where $\Delta_{-1} = 1$. Thus by induction we obtain $\Delta_n > 0$ for all $n \ge 0$.

($\Leftarrow$) Since $\Delta_n > 0$, by Theorem 2.13, there is an OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$. By Corollary 2.19, it suffices to show that $\mathcal{L}(p(x)^2) > 0$ for any nonzero real polynomial $p(x)$. To do this let $p(x) = \sum_{k=0}^{n} a_k P_k(x)$. Then by the orthogonality,

$$\mathcal{L}(p(x)^2) = \sum_{k=0}^{n} a_k^2 \mathcal{L}(P_k(x)^2).$$

Since $\Delta_n > 0$, we have $\mathcal{L}(P_k(x)^2) > 0$ by Lemma 2.14. Thus $\mathcal{L}(p(x)^2) > 0$ as desired. $\square$

2.4. **The fundamental recurrence formula.** One important property of orthogonal polynomials is that they satisfy a 3-term recurrence relation.

**Theorem 2.21.** *Let $\mathcal{L}$ be a linear functional with monic OPS $\{P_n(x)\}_{n\ge 0}$. Then these monic orthogonal polynomials satisfy the following 3-term recurrence relation:*

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \ge 0, \tag{2.8}$$

*with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$ for some sequences $\{b_n\}_{n\ge 0}$ and $\{\lambda_n\}_{n\ge 1}$ such that $\lambda_n \ne 0$.*

*Proof.* Since $P_n(x)$ are monic polynomials, $P_{n+1}(x) - xP_n(x)$ has degree at most $n$. Thus we can write

$$P_{n+1}(x) - xP_n(x) = \sum_{k=0}^{n} a_k P_k(x).$$

By Theorem 2.8, multiplying both sides by $P_j(x)$ for $0 \le j \le n-2$ and taking $\mathcal{L}$ gives

$$0 = \mathcal{L}(P_j(x)P_{n+1}(x) - xP_j(x)P_n(x)) = \sum_{k=0}^{n} a_k \mathcal{L}(P_j(x)P_k(x)) = a_j \mathcal{L}(P_j(x)^2).$$

Since $\mathcal{L}(P_j(x)^2) \ne 0$, we obtain $a_j = 0$ for all $0 \le j \le n-2$. Then we alway have $P_{n+1}(x) - xP_n(x) = a_n P_n(x) + a_{n-1}P_{n-1}(x)$ for some constants $a_n$ and $a_{n-1}$. This implies that the polynomials $P_n(x)$ satisfy the 3-term recurrence relation (2.8).

It remains to show that $\lambda_n \ne 0$. Multiplying $x^{n-1}$ both sides of (2.8) and taking $\mathcal{L}$ gives

$$0 = \mathcal{L}(x^{n-1}P_{n+1}(x)) = \mathcal{L}(x^n P_n(x)) - b_n \mathcal{L}(x^{n-1}P_n(x)) - \lambda_n \mathcal{L}(x^{n-1}P_{n-1}(x)). \tag{2.9}$$

By Lemma 2.14, we have $\mathcal{L}(x^n P_n(x)) = \mathcal{L}(P_n(x)P_n(x))$. Thus (2.9) implies

$$\lambda_n = \frac{\mathcal{L}(P_n(x)^2)}{\mathcal{L}(P_{n-1}(x)^2)}. \tag{2.10}$$

Since $\mathcal{L}(P_n(x)^2) \ne 0$, we get $\lambda_n \ne 0$. $\square$

**Theorem 2.22.** *Following the notation in Theorem 2.21, we have*

$$\lambda_n = \frac{\mathcal{L}(P_n(x)^2)}{\mathcal{L}(P_{n-1}(x)^2)} = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}, \tag{2.11}$$

$$b_n = \frac{\mathcal{L}(xP_n(x)^2)}{\mathcal{L}(P_n(x)^2)}, \tag{2.12}$$

$$\mathcal{L}(P_n(x)^2) = \lambda_1 \cdots \lambda_n \mathcal{L}(1) = \frac{\Delta_n}{\Delta_{n-1}}, \tag{2.13}$$

$$\Delta_n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1 \mathcal{L}(1)^{n+1}. \tag{2.14}$$

*Proof.* By Lemma 2.14, we have $\mathcal{L}(P_n(x)^2) = \Delta_n/\Delta_{n-1}$. Thus the first identity (2.11) follows from (2.10).

Multiplying $P_n(x)$ both sides of (2.8) and taking $\mathcal{L}$ gives

$$\begin{aligned} 0 &= \mathcal{L}(P_n(x)P_{n+1}(x)) = \mathcal{L}(xP_n(x)^2) - b_n\mathcal{L}(P_n(x)^2) - \lambda_n\mathcal{L}(P_nP_{n-1}(x)) \\ &= \mathcal{L}(xP_n(x)^2) - b_n\mathcal{L}(P_n(x)^2), \end{aligned}$$

which implies (2.12).

The identity (2.13) is an immediate consequence of (2.11). The identity (2.14) follows from (2.13). $\square$

**Corollary 2.23.** *Following the notation in Theorem 2.21, the linear functional $\mathcal{L}$ is positive-definite if and only if $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all $n$ and $\mathcal{L}(1) > 0$.*

*Proof.* Suppose that $\mathcal{L}$ is positive-definite. Then by Theorem 2.17 the polynomials $P_n(x)$ are real, hence the recurrence coefficients $b_n$ and $\lambda_n$ are real. By Theorem 2.20, we have $\Delta_n > 0$, which together with (2.11) implies $\lambda_n > 0$.

Now suppose that $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all $n$. By (2.11) and (2.12), one can easily check by induction that all the moments are real. By (2.14), we have $\Delta_n > 0$. Thus by Theorem 2.20, $\mathcal{L}$ is positive-definite. □

Oftentimes non-monic orthogonal polynomials are used in the literature. We can always make them monic by dividing each polynomial by its leading coefficient. This allows us to convert a 3-term recurrence of monic orthogonal polynomials to that of non-monic orthogonal polynomials and vice versa.

Suppose that $\{P_n(x)\}_{n\ge0}$ is an OPS for $\mathcal{L}$, which is not monic. If $k_n$ is the leading coefficient of $P_n(x)$, then the monic OPS for $\mathcal{L}$ is given by $\{\hat{p}_n(x)\}_{n\ge0}$, where $\hat{p}_n(x) = P_n(x)/k_n$. Then, by Theorem 2.21, we have

$$\hat{p}_{n+1}(x) = (x - b_n)\hat{p}_n(x) - \lambda_n \hat{p}_{n-1}(x), \quad n \ge 0; \quad \hat{p}_{-1}(x) = 0, \hat{p}_0(x) = 1. \tag{2.15}$$

Substituting $\hat{p}_n(x) = P_n(x)/k_n$ in the above formula, we get

$$P_{n+1}(x) = (A_n x - B_n)P_n(x) - C_n P_{n-1}(x), \quad n \ge 0; \quad P_{-1}(x) = 0, P_0(x) = k_0, \tag{2.16}$$

where $A_n = k_{n+1}/k_n$, $B_n = b_n k_{n+1}/k_n$, and $C_n = \lambda_n k_{n+1}/k_{n-1}$.

Conversely, from the recurrence (2.16), the leading coefficient of $P_n(x)$ is $k_n = A_{n-1}A_{n-2}\cdots A_0 k_0$. Hence

$$\hat{p}_n(x) = (A_{n-1}A_{n-2}\cdots A_0 k_0)^{-1} P_n(x),$$

and we can obtain the recurrence (2.15) by dividing (2.16) by $A_n A_{n-1}\cdots A_0 k_0$.

**Example 2.24.** Since

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta, \qquad n \ge 1,$$

we have

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \ge 1.$$

Since $T_0(x) = 1$ and $T_1(x) = x$, we have

$$T_{n+1}(x) = A_n x T_n(x) - T_{n-1}(x), \qquad n \ge 0, \tag{2.17}$$

where $T_{-1}(x) = 0$, $A_0 = 1$ and $A_n = 2$ for $n \ge 1$. Thus the monic Tchebyshev polynomials are given by $\hat{T}_n(x) = 2^{1-n}T_n(x)$ for $n \ge 1$. Dividing (2.17) by $2^n$ gives

$$\hat{T}_{n+1}(x) = x\hat{T}_n(x) - \lambda_n \hat{T}_{n-1}(x), \qquad n \ge 0, \tag{2.18}$$

where $\lambda_1 = 1/2$ and $\lambda_n = 1/4$ for $n \ge 2$.

Note that in the recurrence (2.18) for the (monic) Tchebyshev polynomials, $b_n = 0$. This, in fact, implies that $T_{2n}(x)$ is an even function and $T_{2n+1}(x)$ is an odd function. It also turns out that the odd moments are all zero.

**Definition 2.25.** A linear functional $\mathcal{L}$ is **symmetric** if all of its odd moments are zero.

**Theorem 2.26.** *Let $\mathcal{L}$ be a linear functional with monic OPS $\{P_n(x)\}_{n\ge0}$. The following are equivalent:*

1. *$\mathcal{L}$ is symmetric.*
2. *$P_n(-x) = (-1)^n P_n(x)$ for $n \ge 0$.*
3. *In the 3-term recurrence (2.8), $b_n = 0$ for $n \ge 0$.*

*Proof.* (1) ⇒ (2): Since $\mathcal{L}$ is symmetric, $\mathcal{L}(\pi(-x)) = \mathcal{L}(\pi(x))$ for all polynomials $\pi(x)$. Thus $\mathcal{L}(P_m(-x)P_n(-x)) = \mathcal{L}(P_m(x)P_n(x)) = K_n\delta_{m,n}$. By the uniqueness of orthogonal polynomials, Theorem 2.10, we have $P_n(-x) = c_n P_n(x)$ for some $c_n \ne 0$. Comparing their leading coefficients, we obtain $c_n = (-1)^n$.

(2) ⇒ (1): Since $P_{2n+1}(-x) = -P_{2n+1}(x)$, $P_{2n+1}(x)$ is an odd polynomial. Thus $\mathcal{L}(P_{2n+1}(x)) = 0$ is a sum of odd moments. This shows by induction that all odd moments are zero.

(2) $\Leftrightarrow$ (3): Let $Q_n(x) = (-1)^n P_n(-x)$. Then the condition in (2) is the same as $P_n(x) = Q_n(x)$. By Theorem 2.21, we have

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x),$$
$$Q_{n+1}(x) = (x + b_n)Q_n(x) - \lambda_n Q_{n-1}(x),$$

where the second recurrence is obtained from the first by replacing $x$ by $-x$ and multiplying both sides by $(-1)^{n+1}$. Clearly, the condition $P_n(x) = Q_n(x)$ is equivalent to $b_n = 0$, $n \geq 0$. □

Recall Theorem 2.21, which states that orthogonal polynomials satisfy a 3-term recurrence. The converse of this theorem is also true.

**Theorem 2.27** (Favard's theorem)**.** *Suppose $\{b_n\}_{n\geq 0}$ and $\{\lambda_n\}_{n\geq 1}$ be sequences of complex numbers such that $\lambda_n \neq 0$ for all $n \geq 1$. Let $\{P_n(x)\}_{n\geq 0}$ be the polynomials defined by $P_{-1}(x) = 0$, $P_0(x) = 1$, and*

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \geq 0. \tag{2.19}$$

*Then there is a (unique) linear functional $\mathcal{L}$ with $\mathcal{L}(1) = 1$ for which $\{P_n(x)\}_{n\geq 0}$ is an OPS if and only if $\lambda_n \neq 0$ for all $n \geq 1$.*

*Moreover, $\mathcal{L}$ is positive-definite if and only if $\lambda_n > 0$ for all $n \geq 1$.*

*Proof.* The "only if" part is done in Theorem 2.21. To prove the "if" part, we assume $\lambda_n \neq 0$ for all $n \geq 1$. Note that if $\{P_n(x)\}_{n\geq 0}$ is an OPS for $\mathcal{L}$, then we must have $\mathcal{L}(P_n(x)) = 0$ for $n \geq 1$. This together with $\mathcal{L}(1) = 1$ completely determines the moments of $\mathcal{L}$. Thus we define $\mathcal{L}$ to be the unique linear functional such that $\mathcal{L}(1) = 1$ and $\mathcal{L}(P_n(x)) = 0$ for $n \geq 1$. We need to show that $\{P_n(x)\}_{n\geq 0}$ is indeed an OPS for $\mathcal{L}$. By Theorem 2.8, it suffices to show that, for $n \geq 1$,

$$\mathcal{L}(x^k P_n(x)) = K_n \delta_{k,n}, \qquad 0 \leq k \leq n, \tag{2.20}$$

where $K_n \neq 0$.

We will prove (2.20) by induction on $k \geq 0$. More precisely, we claim that for every $k \geq 0$,

$$\mathcal{L}(x^k P_n(x)) = \lambda_1 \cdots \lambda_n \delta_{k,n}, \qquad n \geq \max(k, 1). \tag{2.21}$$

By the constriction of $\mathcal{L}$, (2.21) is true when $k = 0$. Let $k \geq 1$ and suppose that (2.21) holds for $k - 1$. To prove (2.21) for $k$, consider an integer $n \geq k$. Multiplying $x^{k-1}$ to (2.19), we get

$$x^k P_n(x) = x^{k-1}P_{n+1}(x) + b_n x^{k-1}P_n(x) + \lambda_n x^{k-1}P_{n-1}(x).$$

By the induction hypothesis, taking $\mathcal{L}$ in the above formula gives

$$\mathcal{L}(x^k P_n(x)) = \begin{cases} 0 & \text{if } 1 \leq k \leq n-1, \\ \lambda_n \mathcal{L}(x^{n-1}P_{n-1}(x)) & \text{if } k = n. \end{cases}$$

Thus (2.21) also holds for $k$, and the claim is established.

The "moreover" statement follows from Corollary 2.23. □

**Theorem 2.28** (The Christoffel–Darboux identities)**.** *Let $\{P_n(x)\}_{n\geq 0}$ be given by the 3-term recurrence* (2.8)*. For $n \geq 0$, we have*

$$\sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\lambda_1 \cdots \lambda_k} = \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{\lambda_1 \cdots \lambda_n (x - y)}, \tag{2.22}$$

$$\sum_{k=0}^{n} \frac{P_k(x)^2}{\lambda_1 \cdots \lambda_k} = \frac{P'_{n+1}(x)P_n(x) - P_{n+1}(x)P'_n(x)}{\lambda_1 \cdots \lambda_n}. \tag{2.23}$$

*Proof.* Multiply $P_n(y)$ to (2.8) to get

$$P_{n+1}(x)P_n(y) = (x - b_n)P_n(x)P_n(y) - \lambda_n P_{n-1}(x)P_n(y). \tag{2.24}$$

Interchanging $x$ and $y$ in (2.24) gives

$$P_{n+1}(y)P_n(x) = (y - b_n)P_n(x)P_n(y) - \lambda_n P_{n-1}(y)P_n(x). \tag{2.25}$$

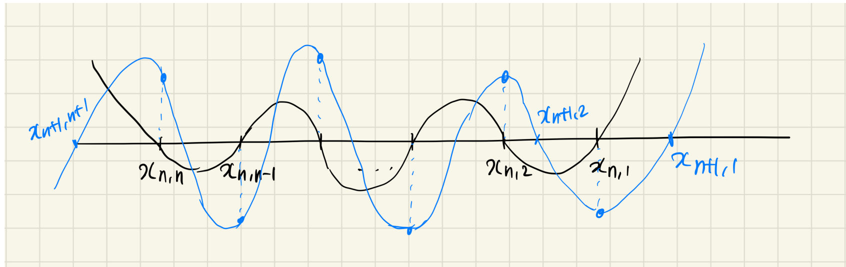


Figure 1. The interchanging zeros of $P_n(x)$ and $P_{n+1}(x)$.

Subtracting (2.25) from (2.24), we have

$$P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x) = (x-y)P_n(x)P_n(y) - \lambda_n(P_{n-1}(x)P_n(y) - P_{n-1}(y)P_n(x)).$$

Let $f_k = P_{k+1}(x)P_k(y) - P_{k+1}(y)P_k(x)$. Then we can rewrite the above equation (with $n$ replaced by $k$) as

$$(x-y)P_k(x)P_k(y) = f_k - \lambda_k f_{k-1}.$$

Dividing both sides by $\lambda_1 \cdots \lambda_k(x-y)$ gives

$$\frac{P_k(x)P_k(y)}{\lambda_1 \cdots \lambda_k} = \frac{f_k}{\lambda_1 \cdots \lambda_k(x-y)} - \frac{f_{k-1}}{\lambda_1 \cdots \lambda_{k-1}(x-y)}.$$

Summing the equation for $k = 0, \dots, n$, we obtain (2.22).

Rewriting (2.22) as

$$\sum_{k=0}^{n} \frac{P_k(x)P_k(y)}{\lambda_1 \cdots \lambda_k} = \frac{(P_{n+1}(x) - P_{n+1}(y))P_n(y) - P_{n+1}(y)(P_n(x) - P_n(y))}{\lambda_1 \cdots \lambda_n(x-y)}$$

and taking the limit $y \to x$ gives (2.23). $\square$

The Christoffel–Darboux identities have an interesting application on the zeros of orthogonal polynomials. We first show that orthogonal polynomials have distinct real zeros if $\mathcal{L}$ is positive-definite.

**Lemma 2.29.** *Let $\mathcal{L}$ be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n\ge 0}$. Then $P_n(x)$ has $n$ distinct real roots for all $n \ge 1$.*

*Proof.* Since $\mathcal{L}(P_n(x)) = 0$, $P_n(x)$ must have a root of odd multiplicity. (Because otherwise $P_n(x) \ge 0$ for all $x \in \mathbb{R}$, which in turn implies $\mathcal{L}(P_n(x)) > 0$ by the assumption that $\mathcal{L}$ is positive-definite.) Let $x_1, \dots, x_k$ be the distinct roots of $P_n(x)$ with odd multiplicities. Then $(x-x_1)\cdots(x-x_k)P_n(x) \ge 0$ for all $x \in \mathbb{R}$. Therefore $\mathcal{L}((x-x_1)\cdots(x-x_k)P_n(x)) > 0$. But by Theorem 2.8 this implies $k \ge n$. Clearly, $k \le n$ and we obtain $k = n$. This means that $P_n(x)$ has $n$ distinct roots. $\square$

**Theorem 2.30.** *Let $\mathcal{L}$ be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n\ge 0}$. Then $P_n(x)$ has $n$ distinct real roots for all $n \ge 1$ and the zeros of $P_n(x)$ and $P_{n+1}(x)$ interlace. More precisely, if $x_{n,1} > x_{n,2} > \cdots > x_{n,n}$ are the zeros of $P_n(x)$, then*

$$x_{n+1,1} > x_{n,1} > x_{n+1,2} > x_{n,2} > \cdots > x_{n+1,n} > x_{n,n} > x_{n+1,n+1}. \tag{2.26}$$

*Proof.* The first part is proved in Lemma 2.29. For the second part, we substitute $x = x_{n,j}$ in (2.23) to get

$$0 < \sum_{k=0}^{n} \frac{P_k(x_{n,j})^2}{\lambda_1 \cdots \lambda_k} = \frac{P'_{n+1}(x_{n,j})P_n(x_{n,j}) - P_{n+1}(x_{n,j})P'_n(x_{n,j})}{\lambda_1 \cdots \lambda_n} = \frac{-P_{n+1}(x_{n,j})P'_n(x_{n,j})}{\lambda_1 \cdots \lambda_n}.$$

This implies that the sign of $P_{n+1}(x_{n,j})$ is the opposite of the sign of $P'_n(x_{n,j})$. Considering the graph of $y = P_n(x)$, the sign of $P'_n(x_{n,j})$ is $(-1)^{j-1}$, see Figure 1. Thus the sign of $P_{n+1}(x_{n,j})$, for $j = 1, 2, \dots, n$, is $(-1)^j$ as indicated by the blue dots in Figure 1. This means that $P_{n+1}(x)$

has a root between each interval $(x_{n,j+1}, x_{n,j})$ for $j = 1, \dots, n-1$. Considering the limits $\lim_{x\to\infty} P_{n+1}(x) = \infty$ and $\lim_{x\to-\infty} P_{n+1}(x) = (-1)^{n+1}\infty$, we can see that $P_{n+1}(x)$ also has one root in $(x_{n,1}, \infty)$ and one root in $(-\infty, x_{n,n})$. Thus we obtain (2.26). □

## 3. Combinatorial interpretations for orthogonal polynomials and their moments

From now one we will focus on the combinatorial approaches to orthogonal polynomials in Viennot's lecture notes [2]. Part of this section will have some overlaps with the previous sections.

In this section we will give combinatorial interpretations for orthogonal polynomials and their moments. Using these combinatorial interpretations, we can reprove the orthogonality of orthogonal polynomials using combinatorics only.

3.1. **Orthogonal polynomials and 3-term recurrences.** In this subsection we give basic definitions and prove simple but useful lemmas. We then state the 3-term recurrence of orthogonal polynomials and Favard's theorem.

Let $K$ be a field. We denote by $K[x]$ the ring of polynomials in $x$ with coefficients in $K$. A **linear functional** is a linear transformation $\mathcal{L} : K[x] \to K$, i.e., a function satisfying $\mathcal{L}(af(x)+bg(x)) = a\mathcal{L}(f(x)) + b\mathcal{L}(g(x))$ for all $f(x), g(x) \in K[x]$ and $a, b \in K$. The $n$th **moment** of $\mathcal{L}$ is defined to be $\mu_n = \mathcal{L}(x^n)$.

**Definition 3.1.** Let $\mathcal{L}$ be a linear functional defined on the space of polynomials in $x$. A sequence of polynomials $\{P_n(x)\}_{n\ge 0}$ is called an **orthogonal polynomial sequence (OPS)** with respect to $\mathcal{L}$ if the following conditions hold:

(1) $\deg P_n(x) = n$, $n \ge 0$,
(2) $\mathcal{L}(P_m(x)P_n(x)) = 0$ for $m \ne n$,
(3) $\mathcal{L}(P_m(x)^2) \ne 0$ for $m \ge 0$.

We also say that $\{P_n(x)\}_{n\ge 0}$ is orthogonal for the moments $\{\mu_n\}_{n\ge 0}$.

Orthogonal polynomials in the above definition are called "formal" or "general" orthogonal polynomials because the field $K$ can be anything. For instance, it may contain arbitrary formal variables such as $a, b, c, d$. Then the polynomials $P_n(x)$ and the moments $\mu_n$ can be treated as polynomials (or more complicated objects such as formal power series or rational functions) in these formal variables.

**Proposition 3.2.** *Suppose that $\{P_n(x)\}_{n\ge 0}$ is an OPS for $\mathcal{L}$.*

(1) *$\{P_n(x)\}_{n\ge 0}$ is also orthogonal with respect to $\mathcal{L}'$ for any $\mathcal{L}' = a\mathcal{L}$ for $a \ne 0$.*
(2) *$\mathcal{L}$ is uniquely determined up to nonzero scalar multiplication.*
(3) *If we set $\mathcal{L}(1) = 1$, then $\mathcal{L}$ is uniquely determined.*
(4) *$\{a_nP_n(x)\}_{n\ge 0}$ is an OPS with respect to $\mathcal{L}$ for any sequence $\{a_n\}_{n\ge 0}$ with $a_n \ne 0$.*

*Proof.* All statements are easy to check. For example, (2) can be seen by noticing that once the 0th moment $\mu_0 = \mathcal{L}(1)$ is determined, then the $n$th moment $\mu_n$, for $n \ge 1$, is uniquely determined by the condition $\mathcal{L}(P_n(x)) = 0$. □

By the above proposition we may assume that $\mathcal{L}(1) = 1$. **From now on we will always assume that $\deg P_n(x) = n$ and $\mathcal{L}(1) = 1$ unless otherwise stated.**

Recall from Theorem 2.21 that every OPS satisfies a 3-term recurrence.

**Theorem 3.3** (3-term recurrence)**.** *Let $\mathcal{L}$ be a linear functional with monic OPS $\{P_n(x)\}_{n\ge 0}$. Then there are sequences $\{b_n\}_{n\ge 0}$ and $\{\lambda_n\}_{n\ge 1}$ such that $\lambda_n \ne 0$ and*

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \ge 0,$$

*where $P_{-1}(x) = 0$ and $P_0(x) = 1$.*

The inverse of the above theorem is also true, which is one of the most important results in the theory of classical orthogonal polynomials.

**Theorem 3.4** (Favard's theorem)**.** *Let $\{P_n(x)\}_{n\ge0}$ be a sequence of monic polynomials. Then $\{P_n(x)\}_{n\ge0}$ is an OPS for some linear functional $\mathcal{L}$ if and only if there are sequences $\{b_n\}_{n\ge0}$ and $\{\lambda_n\}_{n\ge1}$ such that $\lambda_n \ne 0$ and*

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \ge 0, \tag{3.1}$$

*where $P_{-1}(x) = 0$ and $P_0(x) = 1$.*

The main goal of this section is to give combinatorial interpretations for the orthogonal polynomials $P_n(x)$ and their moments $\mu_n$. Using these combinatorial interpretations we will prove Favard's theorem bijectively.

3.2. **A model for orthogonal polynomials using Favard tilings.** In this subsection we define Favard paths and Favard tilings. These are equivalent combinatorial objects, which give a combinatorial meaning to orthogonal polynomials.

See Appendix B for the basics on generating functions.

**Definition 3.5.** A **path** is a sequence $\alpha = (s_0, s_1, \dots, s_n)$ of points $s_i = (x_i, y_i) \in \mathbb{Z}^2$. Each point $s_i$ is called a **vertex** of $\alpha$. We say that $\alpha$ travels from the **initial point** (or **starting point**) $s_0$ to the **final point** (or **ending point**) $s_n$. The pairs $(s_i, s_{i+1})$ are called the **elementary steps** of $\alpha$. The **length** of $\alpha$, denoted $|\alpha|$, is defined to be $n$.

Let $\alpha = (s_0, s_1, \dots, s_n)$ and $\beta = (t_0, t_1, \dots, t_m)$ be two paths. If $s_n = t_0$, then we define $\alpha\beta$ to be the path $(s_0, s_1, \dots, s_n, t_1, \dots, t_m)$.

Note that an elementary step $(s_i, s_{i+1})$ is also a path. In this point of view, we can consider $\alpha$ as the product of its elementary steps: $\alpha = (s_0, s_1)(s_1, s_2)\cdots(s_{n-1}, s_n)$. A **factorization** of $\alpha$ is an expression $\alpha = \alpha_1 \cdots \alpha_k$, where $\alpha_1, \dots, \alpha_k$ are paths such that the initial point of $\alpha_i$ is equal to the final point of $\alpha_{i+1}$ for $i = 1, 2, \dots, n-1$. Each $\alpha_i$ is called a **factor** of $\alpha$. In particular, we say that $\alpha_1$ is a **prefix** of $\alpha$ and $\alpha_n$ is a **suffix** of $\alpha$. We also consider an empty path $(s, s)$.

A **north step** is a step of the form $((x, y), (x, y+1))$. We define a **south step**, an **east step**, and a **west step** similarly. In addition, a **northeast step** is a step of the form $((x, y), (x+1, y+1))$ and a **north-north step** is a step of the form $((x, y), (x, y+2))$.

The **level** of a vertex $s = (x, y)$ is defined to be $y$.

A **(step) weight** is a function $w : \mathbb{Z}^2 \times \mathbb{Z}^2 \to K$. For a path $\alpha = (s_0, s_1, \dots, s_n)$, define

$$w(\alpha) = w(s_0, s_1)w(s_1, s_2)\cdots w(s_{n-1}, s_n).$$

**Definition 3.6.** A **Favard path** is a path $\alpha = (s_0, s_1, \dots, s_n)$ from $s_0 = (0, 0)$ with 3 types of elementary steps: a north step (N), a north-north step (NN), and a northeast step (NE). Given two sequences $b = \{b_n\}_{n\ge0}$ and $\lambda = \{\lambda_n\}_{n\ge1}$, the **Favard-weight** $w_F$ is defined as follows:

$$w_F((i,k),(i,k+1)) = -b_k,$$
$$w_F((i,k-1),(i,k+1)) = -\lambda_k,$$
$$w_F((i,k),(i+1,k+1)) = x.$$

For a step $s$ other than a north step, a north-north step, and a northeast step, we define $w_F(s) = 0$.

Let $F_n$ denote the set of Favard paths from $(0, 0)$ to a vertex of level $n$.

See Figure 2.

**Lemma 3.7.** *Suppose that $\{P_n(x)\}_{n\ge0}$ is a sequence of polynomials satisfying* (3.1)*. Then*

$$P_n(x) = \sum_{\alpha\in F_n} w_F(\alpha).$$

We can identify a Favard path with a Favard tiling.

**Definition 3.8.** A **Favard tiling of size $n$** is a tiling of a $1 \times n$ square board with tiles where each tile is a domino or a monomino and each monomino is colored black or red. We label the squares in the $1 \times n$ board by $1, 2, \dots, n$ from left to right. The set of Favard tilings of size $n$ is denoted by $\mathrm{FT}_n$. We define $\mathrm{FT}_0$ to be the set consisting of the empty tiling.

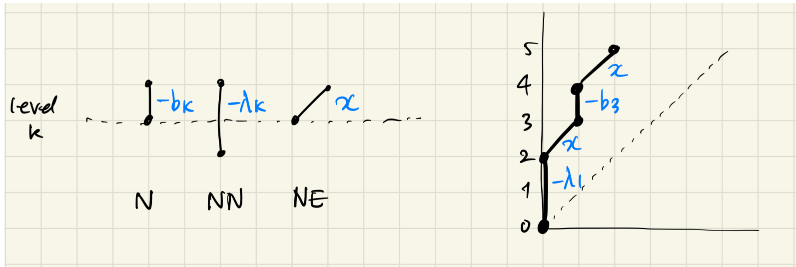


FIGURE 2. The Favard-weight and a Favard path.

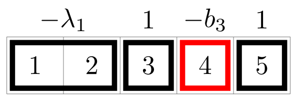


FIGURE 3. A Favard tiling $T \in \mathrm{FT}_5$ with $\mathrm{wt}(T) = \lambda_1 b_3$.

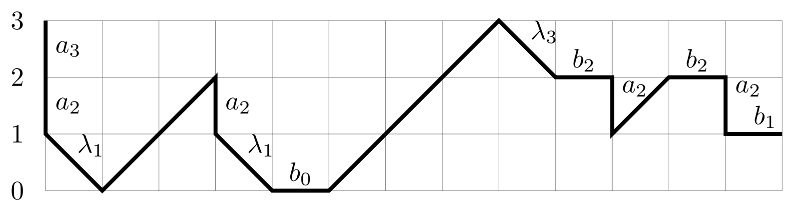


FIGURE 4. A Motzkin-Schröder path $\pi$ from $(0, 3)$ to $(13, 1)$ with $\mathrm{wt}(\pi) = a_2^4 a_3 b_0 b_1 b_2^2 \lambda_1^2 \lambda_3$.

**Lemma 3.9.** *Suppose that $\{P_n(x)\}_{n\geq 0}$ is a sequence of polynomials satisfying* (3.1). *Then*

$$P_n(x) = \sum_{\alpha \in \mathrm{FT}_n} \mathrm{wt}(\alpha).$$

**Remark 3.10.** Note that the number $u_n$ of Favard tilings of size $n$ satisfies $u_{n+1} = 2u_n + u_{n-1}$ with $u_0 = 1$ and $u_1 = 2$. These numbers are called the Pell numbers.

3.3. **How to find a combinatorial model for moments.** Moments are important quantities of a linear functional $\mathcal{L}$ because they have all the information of $\mathcal{L}$. In this subsection we will find a combinatorial interpretation for the moments of orthogonal polynomials.

Suppose that $\mathcal{L}$ is a linear functional with monic OPS $\{P_n(x)\}_{n\geq 0}$, which satisfies the 3-term recurrence (3.1). Let's assume $\mathcal{L}(1) = 1$. Then, using the orthogonality, we have

$$\mathcal{L}(P_n(x)) = \delta_{n,0}. \tag{3.2}$$

This relation in fact completely determines the moments $\mu_n$. For example, since

$$\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= (x - b_0)P_0(x) - \lambda_0 P_{-1}(x) = x - b_0,\\
P_2(x) &= (x - b_1)P_1(x) - \lambda_1 P_0(x) = x^2 - (b_1 + b_0)x + b_0 b_1 - \lambda_1,
\end{aligned}$$

we have

$$\begin{aligned}
\mu_0 &= \mathcal{L}(1) = 1,\\
\mu_1 &= \mathcal{L}(x) = \mathcal{L}(P_1(x) + b_0) = b_0,\\
\mu_2 &= \mathcal{L}(x^2) = \mathcal{L}(P_2(x) + (b_0 + b_1)x - b_0 b_1 + \lambda_1) = (b_0 + b_1)b_0 - b_0 b_1 + \lambda_1 = b_0^2 + \lambda_1.
\end{aligned}$$

In this way, we can compute a few more moments:

$$\mu_3 = b_0^3 + 2b_0\lambda_1 + b_1\lambda_1,$$
$$\mu_4 = b_0^4 + 3b_0^2\lambda_1 + 2b_0b_1\lambda_1 + b_1^2\lambda_1 + \lambda_1^2 + \lambda_1\lambda_2,$$
$$\mu_5 = b_0^5 + 4b_0^3\lambda_1 + 3b_0^2b_1\lambda_1 + 2b_0b_1^2\lambda_1 + b_1^3\lambda_1 + 3b_0\lambda_1^2 + 2b_1\lambda_1^2 + 2b_0\lambda_1\lambda_2 + 2b_1\lambda_1\lambda_2 + b_2\lambda_1\lambda_2.$$

The above experiments clearly suggest that $\mu_n$ would be a polynomial in $b_i$'s and $\lambda_i$'s with nonnegative integer coefficients. How can we prove such a conjecture? A satisfying answer to this question is to find combinatorial objects whose generating function is $\mu_n$. That is to find a set $X$ of combinatorial objects and a weight $\mathrm{wt}(A)$ of each element $A \in X$ such that

$$\mu_n = \sum_{A\in X} \mathrm{wt}(A),$$

and $\mathrm{wt}(A)$ is a polynomial (preferably a monomial) in $b_i$'s and $\lambda_i$'s with nonnegative integer coefficients.

But how can we find such combinatorial objects? Suppose that such combinatorial objects exist with monomial weight $\mathrm{wt}(A)$ for each $A \in X$. Then if we set $b_i = \lambda_i = 1$ for all $i$ then $\mu_n$ would be the number of elements in $X$. If we compute $\mu_n$ with this substitution for $n = 0, 1, 2, \dots$, then we obtain the following sequence:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, \dots.$$

There is a very useful webpage `https://oeis.org/` where you can search integer sequences. If you search the above sequence, the webpage will tell you that this is the sequence of the number of Motzkin paths. So we can guess that there must be a close connection with the moments of orthogonal polynomials and Motzkin paths.

3.4. **Motzkin paths.** After spending enough time of trials and errors, we can come up with the following combinatorial model for $\mu_n$.

**Theorem 3.11.** *We have*

$$\mu_n = \sum_{\pi\in \mathrm{Motz}_n} \mathrm{wt}(\pi).$$

3.5. **A bijective proof of Favard's theorem.**

## 4. Moments of particular orthogonal polynomials

In this section we consider Tchebyshev polynomials of the 1st and 2nd kinds, Laguerre polynomials. Hermite polynomials, Charlier polynomials, and Meixner polynomials of the 1st and 2nd kinds.

Note that an OPS $\{P_n(x)\}_{n\ge0}$ can be defined in many ways, namely, one of the following determines the orthogonal polynomials:

(1) the coefficients $a_{n,k}$ of $P_n(x)$,
(2) the generating function $\sum_{n\ge0} P_n(x)t^n$ or $\sum_{n\ge0} P_n(x)t^n/n!$,
(3) the moments $\{\mu_n\}_{n\ge0}$,
(4) the 3-term recurrence coefficients $\{b_n\}_{n\ge0}$ and $\{\lambda_n\}_{n\ge1}$.

For each OPS, we will show bijectively the equivalence of (3) and (4).

For example, in the case that $b_k$ and $\lambda_k$ are integers, we interprete $\mathrm{wt}(\alpha)$ as a certain number of "histories". To each history we associate bijectively a certain combinatorial object $\xi$ of a finite set $M_n$. Each of $b_k$ and $\lambda_k$ is considered as the number of possible choices in a stage of a construction of the object, where each stage corresponds to an elementary step of $\alpha$. Then it remains to show that $|M_n| = \mu_n$.

If $P_n(x)$ depend on some parameters, it will be sufficient to consider the histories and the combinatorial objects in $M_n$.

4.1. **Tchebycheff polynomials.**

## Appendix A. Sign-reversing involutions

**Definition A.1.** A **sign** of a set $X$ is a function $\mathrm{sgn} : X \to \{+1, -1\}$. A **sign-reversing involution** on $X$ is an involution $\phi : X \to X$ such that

(1) $\mathrm{sgn}(x) = 1$ for all $x \in \mathrm{Fix}(\phi)$;
(2) $\mathrm{sgn}(\phi(x)) = -\mathrm{sgn}(x)$ for all $x \in X \setminus \mathrm{Fix}(\phi)$,

where $\mathrm{Fix}(\phi)$ is the set of **fixed points** of $\phi$, i.e., $\mathrm{Fix}(\phi) = \{x \in X : \phi(x) = x\}$.

It is easy to see that if $\phi$ is a sign-reversing involution on $X$, then

$$\sum_{x\in X} \mathrm{sgn}(X) = |\,\mathrm{Fix}(\phi)|. \tag{A.1}$$

**Example A.2.** Let's prove the following identity using sign-reversing involutions:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0. \tag{A.2}$$

To this end we need to construct a set $X$ and a sign-reversing involution $\phi$ on $X$ such that (A.1) becomes (A.2).

Let $X$ be the set of all subsets of $[n] := \{1, \dots, n\}$ and for $A \in X$, define $\mathrm{sgn}(A) = (-1)^{|A|}$. Then it suffices to construct a sign-reversing involution on $X$ with no fixed points. This can be done by letting $\phi(A) = A\Delta\{1\}$, where $A\Delta B := (A \cup B) \setminus (A \cap B)$.

**Example A.3.** Recall that we proved the following identitiy, which was stated in (2.4), using generating functions:

$$\sum_{k\ge 0} P_m(k)P_n(k)\frac{a^k}{k!} = \frac{e^a a^n}{n!}\delta_{n,m}, \tag{A.3}$$

where $P_n(x)$ are the Charlier polynomials defined by

$$P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!}.$$

We will prove this identity using sign-reversing involutions. To do this, we will consider (A.3) as a power series in $a$. Note that

$$\begin{aligned}
\sum_{k\ge 0} P_m(k)P_n(k)\frac{a^k}{k!} &= \sum_{k\ge 0}\sum_{i=0}^{m}\binom{k}{i}\frac{(-a)^{m-i}}{(m-i)!}\sum_{j=0}^{n}\binom{k}{j}\frac{(-a)^{n-j}}{(n-j)!}\frac{a^k}{k!}\\
&= \sum_{k\ge 0}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{k}{m-i}\frac{(-a)^{i}}{i!}\binom{k}{n-j}\frac{(-a)^{j}}{j!}\frac{a^k}{k!}\\
&= \sum_{N\ge 0}\frac{a^N}{N!}\sum_{i+j+k=N}(-1)^{i+j}\frac{N!}{i!j!k!}\binom{k}{m-i}\binom{k}{n-j},
\end{aligned}$$

where $\binom{r}{s} = 0$ if $s < 0$.

For a fixed $N$,

$$\sum_{i+j+k=N}(-1)^{i+j}\binom{N}{i,j,k}\binom{k}{m-i}\binom{k}{n-j} = \sum_{(A,B,C)\in X}(-1)^{|B\setminus A|+|C\setminus A|},$$

where $X$ is the set of triples $(A, B, C)$ such that $A\cup B\cup C = \{1, \dots, N\}$, $|A| = k$, $|B| = m$, $|C| = n$, $(B \cap C) \setminus A = \emptyset$. Define $\mathrm{sgn}(A, B, C) = (-1)^{|B\setminus A|+|C\setminus A|}$. We will find a sign-reversing involution on $X$ toggling the smallest integer in regions 1 and 2 or in regions 3 and 4 in Figure 5.

To be precise, for $(A, B, C) \in X$, define $\phi(A, B, C)$ as follows.

**Case 1:** The regions $1, 2, 3, 4$ are all empty. In this case we define $\phi(A, B, C) = (A, B, C)$.

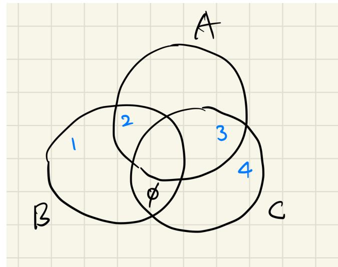


FIGURE 5. The triple $(A, B, C)$.

**Case 2:** At least one of the regions $1, 2, 3, 4$ is nonempty. Let $s$ be the smallest integer in $(B \cap C) \setminus A$. If $s$ is in region 1 (respectively 2, 3, 4), then move this integer to region 2 (respectively 1, 4, 3). Then let $\phi(A, B, C) = (A', B', C')$, where $A', B', C'$ are the resulting sets.

By the construction, $\phi$ is a sign-reversing involution on $X$ whose fixed points are the triples $(A, B, C)$ such that the regions $1, 2, 3, 4$ are all empty, that is, $B = C \subseteq A$. If $B = C \subseteq A$, then $A = [N]$, so the number of such triples $(A, B, C)$ is $\binom{N}{n}$ if $m = n$ and 0 otherwise. Thus

$$\sum_{(A,B,C)\in X} (-1)^{|B\setminus A|+|C\setminus A|} = |\operatorname{Fix}(\phi)| = \delta_{m,n}\binom{N}{n}.$$

This implies

$$\sum_{k\geq 0} P_m(k)P_n(k)\frac{a^k}{k!} = \delta_{m,n}\sum_{N\geq 0}\frac{a^N}{N!}\binom{N}{n} = \frac{e^a a^n}{n!}\delta_{n,m}.$$

## APPENDIX B. FORMAL POWER SERIES AND GENERATING FUNCTIONS

In this section, we study basics of formal power series and generating functions.

A **power series** is a series of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

If $a_i$'s are real numbers, then $f(x)$ may be considered as a function on $x$ whose domain is the set of real numbers $x$ such that the above infinite series converges.

For example, if

$$f(x) = 1 + x + x^2 + \cdots ,$$

then we have $f(x) = 1/(1-x)$ for $|x| < 1$. Thus we can write, for $|x| < 1$,

$$1 + x + x^2 + \cdots = \frac{1}{1-x}. \tag{B.1}$$

This, however, does not make sense if $|x| > 1$. Hence, in calculus, whenever we consider a power series we always have to mention for what values of $x$ the series converges. But in formal power series the convergence is not needed.

Let $R$ be a commutative ring with identity. Recall that $R[x]$ denotes the ring of polynomials in $x$ with coefficients in $R$.

**Definition B.1.** Let $R$ be a field. The **ring of formal power series** in $x$ with coefficients in $R$ is the set

$$R[[x]] = \{a_0 + a_1 x + a_2 x^2 + \cdots : a_0, a_1, a_2, \ldots \in R\},$$

with addition

$$\left(\sum_{i=0}^{\infty} a_i x^n\right) + \left(\sum_{i=0}^{\infty} b_i x^n\right) = \sum_{i=0}^{\infty}(a_i + b_i)x^n,$$

and multiplication

$$\left(\sum_{i=0}^{\infty} a_i x^n\right)\left(\sum_{i=0}^{\infty} b_i x^n\right) = \sum_{i=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n.$$

So, roughly speaking, a formal power series is a polynomial of infinite degree.

The multiplicative identity of $R[[1]]$ is 1, that is, $1 + 0x + 0x^2 + \cdots$. For $f(x), g(x) \in R[[1]]$, if $f(x)g(x) = 1$, then we say that $f(x)$ is the **inverse** of $g(x)$ and write $f(x) = g(x)^{-1} = 1/g(x)$.

In the language of formal power series, (B.1) is a perfectly valid identity without any convergence considered because

$$(1 + x + x^2 + \cdots)(1 - x) = (1 + x + x^2 + \cdots) - x(1 + x + x^2 + \cdots) = 1.$$

An important aspect of a formal power series is that the coefficient of $x^n$ must be computed using a finitely many additions and multiplications in $R$.

**Example B.2.** The series

$$e^{1+x} = \sum_{n\geq 0} \frac{(1+x)^n}{n!}$$

is not a formal power series in $\mathbb{R}[[x]]$ because the constant term (the coefficient of $x^0$) is $\sum_{n\geq 0} 1/n!$, which cannot be computed by a finite number of additions and multiplications in $\mathbb{R}$ (although we know that it is equal to $e$). On the other hand,

$$e \cdot e^x = \sum_{n\geq 0} \frac{e x^n}{n!}$$

is a formal power series in $\mathbb{R}[[x]]$.

We can also consider formal power series in more than one variable.

**Definition B.3.** Let $R$ be a field. Let $\mathbf{x} = (x_1, x_2, \dots)$ be a sequence of variables. Let $Z$ denote the set of sequences $I = (i_1, i_2, \dots) \in \mathbb{Z}_{\geq 0}^{\infty}$ such that $i_1 + i_2 + \cdots < \infty$. For $I = (i_1, i_2, \dots) \in Z$, we write $\mathbf{x}^I = x_1^{i_1} x_2^{i_2} \cdots$. The **ring of formal power series** in $x_1, x_2, \dots$ with coefficients in $R$ is the set

$$R[[\mathbf{x}]] = \left\{ \sum_{I\in Z} a_I \mathbf{x}^I : a_I \in R \right\},$$

with addition

$$\left(\sum_{I\in Z} a_I \mathbf{x}^I\right) + \left(\sum_{I\in Z} b_I \mathbf{x}^I\right) = \left(\sum_{I\in Z} (a_I + b_I) \mathbf{x}^I\right),$$

and multiplication

$$\left(\sum_{I\in Z} a_I \mathbf{x}^I\right)\left(\sum_{I\in Z} b_I \mathbf{x}^I\right) = \sum_{I\in Z}\left(\sum_{I_1, I_2 \in Z, I_1 + I_2 = I} a_{I_1} b_{I_2}\right) \mathbf{x}^I.$$

Now we define the notion of generating functions. Usually, the generating function for a sequences $\{a_n\}_{n\geq 0}$ is defined to be the formal power series

$$a_0 + a_1 x + a_2 x^2 + \cdots .$$

This definition can easily be extended to the generating function for an array $\{a_I\}_{I\in Z}$ of elements $a_I \in R$. More generally, we will consider generating functions for arbitrary (combinatorial) objects.

**Definition B.4.** Let $A$ be a set of objects. A **weight** on $A$ is a function $\mathrm{wt} : A \to R[[\mathbf{x}]]$. The **generating function** for $A$ with respect to the weight function wt is the formal power series

$$\sum_{a\in A} \mathrm{wt}(a),$$

provided that this is a valid formal power series in $R[[\mathbf{x}]]$.

**Example B.5.** Let $A = \{0, 1, 2, \dots\}$ and define a weight of $A$ by $\mathrm{wt}(a) = x^a$. Then the generating function for $A$ (with this weight) is

$$\sum_{a \in A} \mathrm{wt}(a) = \sum_{n=0}^{n} \mathrm{wt}(n) = \sum_{n=0}^{n} x^n = \frac{1}{1-x}.$$

**Example B.6.** Let $A$ be the set of subsets of $[n]$ and define a weight of $A$ by $\mathrm{wt}(a) = x^{|a|}$. Then the generating function for $A$ (with this weight) is

$$\sum_{a \in A} \mathrm{wt}(a) = \sum_{a \subseteq [n]} x^{|a|} = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

**Example B.7.** Let $A$ be the set $S_n$ of permutations of $[n]$ and define a weight of $A$ by $\mathrm{wt}(a) = x^{\mathrm{cycle}(a)}$. Then it can be proved that the generating function for $A$ (with this weight) is

$$\sum_{a \in A} \mathrm{wt}(a) = \sum_{\pi \in S_n} x^{\mathrm{cycle}(a)} = x(x+1)\cdots(x+n-1).$$

We will often use the term "generating function" in a flexible manner. For example, the generating function for the number of permutations would mean the generating function for the sequence $\{a_n = n!\}_{n \geq 0}$, that is, $\sum_{n \geq 0} n! x^n$.

## Appendix C. Catalan numbers

## Appendix D. The Lindström–Gessel–Viennot lemma

## References

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