Affine Gordon-Bender-Knuth identities and cylindric Young tableaux

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Outline

① Basic definitions & Robinson-Schensted algorithm

② Standard Young tableaux of bounded height

③ Noncrossing and nonnesting involutions

④ Cylindric tableaux, cylindric Schur ftns

⑤ Original Motivation

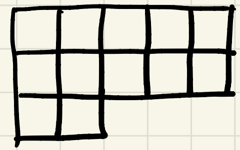
⑥ Affine Gordon-Bender-Knuth identities

Basic definitions

Def) $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a **partition** of n if

$$\lambda_1 \geq \cdots \geq \lambda_k > 0, \qquad \lambda_1 + \cdots + \lambda_k = n.$$

length of $\lambda = \ell(\lambda) = k = \#$ **parts**

The **Young diagram** of λ is  $\lambda = (5,5,2)$.

A **standard Young tableau** of shape λ is

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 7 \\ \hline 3 & 4 & 8 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$$

A **semistandard Young tableau** "

(column-strict tableau)

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

A **row-strict tableau** "

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array}$$

Connection with representation theory

- G : a finite group

$$\mathbb{C}G = \bigoplus_{i=1}^{m} V_i^{\dim V_i}$$

$\dim \mathbb{C}G = |G|$

$\mathbb{C}G$: regular representation of G

V_i : irreducible representations of G

$$\Rightarrow \quad |G| = \sum_{i=1}^{m} (\dim V_i)^2$$

- $G = S_n$: the symmetric group of order n
  (the group of bijections on $\{1, 2, \dots, n\}$)

  Fact : The irr. reps of S_n are V_λ, $\lambda \vdash n$.

  $\dim V_\lambda = f^\lambda = \#$ SYTs of shape λ.

- $$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

$$= \# \left\{ (P, Q) : \begin{array}{l} P, Q \text{ are SYTs of size } n \\ \mathrm{sh}(P) = \mathrm{sh}(Q) \end{array} \right\}$$

ex) $n = 3. \Rightarrow n! = 6$

$$(P, Q) = (1\,2\,3,\ 1\,2\,3),$$

$$\left(\begin{array}{ll} 1 & 2 \\ 3 & \end{array}, \begin{array}{ll} 1 & 2 \\ 3 & \end{array}\right), \left(\begin{array}{ll} 1 & 2 \\ 3 & \end{array}, \begin{array}{ll} 1 & 3 \\ 2 & \end{array}\right), \left(\begin{array}{ll} 1 & 3 \\ 2 & \end{array}, \begin{array}{ll} 1 & 2 \\ 3 & \end{array}\right)$$

$$\left(\begin{array}{ll} 1 & 3 \\ 2 & \end{array}, \begin{array}{ll} 1 & 3 \\ 2 & \end{array}\right), \left(\begin{array}{l} 1 \\ 2 \\ 3 \end{array}, \begin{array}{l} 1 \\ 2 \\ 3 \end{array}\right)$$

Robinson–Schensted algorithm

π : permutation of $[n]=\{1,2,\dots,n\}$.

$$\Big\updownarrow \ \text{1-1}$$

(P,Q) : pair of SYTs of size n and of same shape

ex) $\pi = 4\,1\,5\,6\,3\,2 \quad \overset{RS}{\longleftrightarrow} \quad P = \begin{array}{lll}1&2&6\\3&5\\4\end{array} \qquad Q = \begin{array}{lll}1&3&4\\2&5\\6\end{array}$

* insertion algorithm

$$P: \ \emptyset \leftarrow 4 \quad 4 \leftarrow 1 \quad \begin{array}{l}1\\4\end{array} \leftarrow 5 \quad \begin{array}{ll}1&5\\4\end{array} \leftarrow 6 \quad \begin{array}{lll}1&5&6\\4\end{array} \leftarrow 3 \quad \begin{array}{lll}1&3&6\\4&5\end{array} \leftarrow 2 \quad \begin{array}{lll}1&2&6\\3&5\\4\end{array}$$

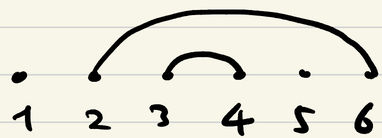
$$Q: \ \emptyset \quad 1 \quad \begin{array}{l}1\\2\end{array} \quad \begin{array}{ll}1&3\\2\end{array} \quad \begin{array}{lll}1&3&4\\2\end{array} \quad \begin{array}{lll}1&3&4\\2&5\end{array} \quad \begin{array}{lll}1&3&4\\2&5\\6\end{array}$$

Consequences of RS algorithm

- If $\pi \xleftrightarrow{RS} (P,Q)$, then $\pi^{-1} \xleftrightarrow{RS} (Q,P)$.

- π: involutions of $[n]$ $\xleftrightarrow{RS}$ SYTs of size n. (P,P).
  $(\pi^2 = id)$

- $\pi = (26)(34)$



$$\| \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 3 & 5 & 2 \end{pmatrix} \qquad \xleftrightarrow{RS} \qquad \begin{array}{lll} 1 & 2 & 5 \\ 3 & & \\ 4 & & \\ 6 & & \end{array}$$

involution $\longleftrightarrow$ matching

SYTs with bounded height.

Def) SYT_n = set of SYTs of size n.

$$ht=3\left\{\begin{array}{|c|c|c|c|}\hline 1&2&6&7\\\hline 3&4&8\\\cline{1-3} 5\\\cline{1-1}\end{array}\right.$$

$$SYT_n(h) = \{T \in SYT_n : \underbrace{ht(T)}_{= \#\text{ rows.}} \le h\}$$

ex). $|SYT_n(2)| = \binom{n}{\lfloor n/2 \rfloor}$

$|SYT_n(3)|$ = # Motzkin paths of length n.

$|SYT_n(4)| = C_{\lfloor \frac{n+1}{2} \rfloor} C_{\lceil \frac{n+1}{2} \rceil}$, $\quad C_n = \frac{1}{n+1}\binom{2n}{n}$ = nth Catalan number.

<u>Thm</u> (Gessel, 1990)

$$\sum_{n \ge 0} |SYT_n(2h+1)| \frac{x^n}{n!} = \exp(x) \det\left(I_{-i+j}(2x) - I_{i+j}(2x)\right)_{i,j=1}^{h}$$

$$I_\alpha(2x) = \sum_{\ell \ge 0} \frac{x^{2\ell+|\alpha|}}{\ell!(\ell+|\alpha|)!}$$ (Modified Bessel function).

Properties of RS-algorithm

① If $\pi \overset{RS}{\longleftrightarrow} (P, Q)$, then $\pi^{-1} \overset{RS}{\longleftrightarrow} (Q, P)$.

② max length of decreasing subsequence of π = $ht(P)$.

③ 〃 increasing 〃 = $width(P)$

ex) $\pi = 4156327 \overset{RS}{\longleftrightarrow}$

P	Q
1 2 6 7	1 3 4 7
3 5	2 5
4	6

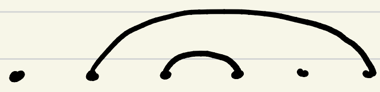
max dec 4 3 2

max inc 4 5 6 7

Consequences of RS algorithm

π : involutions of $[n]$ $\overset{RS}{\longleftrightarrow}$ SYTs of size n. (P,P).
$(\pi^2 = id)$

$\pi = (26)(34)$



1 2 3 4 5 6

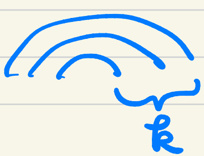
$\overset{RS}{\longleftrightarrow}$

$$\begin{matrix} 1 & 2 & 5 \\ 3 \\ 4 \\ 6 \end{matrix}$$

$\parallel$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \underline{6} & \underline{4} & \underline{3} & 5 & \underline{2} \end{pmatrix}$$

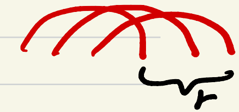
involutions of $[n]$ with no k-nesting $\overset{RS}{\longleftrightarrow}$ SYTs of size n with height $< 2k$



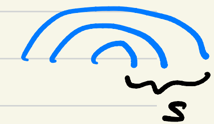
$\leftrightarrow$ dec seq of length $2k$

r-noncrossing and s-nonnesting involutions

Def) An involution π is r-noncrossing if π has no



" s-nonnesting "



$NCNN_n(r,s)$ = set of r-noncrossing and s-nonnesting involutions of $[n]$

<u>Thm</u> (Chen, Deng, Du, Stanley, Yan, 2007)

$$\# NCNN_n(r,s) = \# NCNN_n(s,r).$$

In particular,

$\#$ k-noncrossing involutions of $[n]$

$=$ $\#$ k-nonnesting " .

SYTs of size n with height $\leq 2k+1$

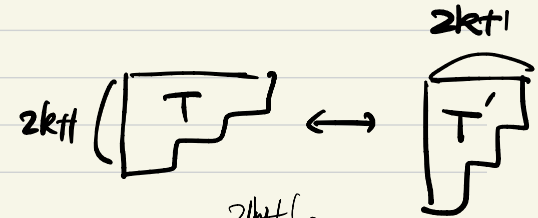
$\overset{\text{RS}}{=}$ # $(k+1)$-nonnesting involutions of $[n]$

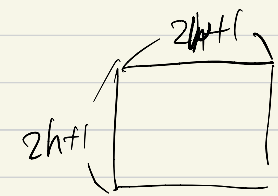
$\overset{\text{CDDSY}}{=}$ # $(k+1)$-noncrossing 〃

By taking transpose,

SYTs of size n with height $\leq 2k+1$

= # SYTs of size n with width $\leq 2k+1$





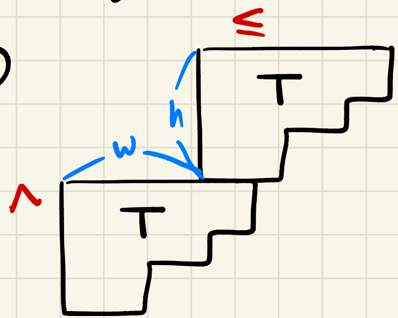
Question

SYTs of size n with height $\leq 2h+1$ and width $\leq 2w+1$

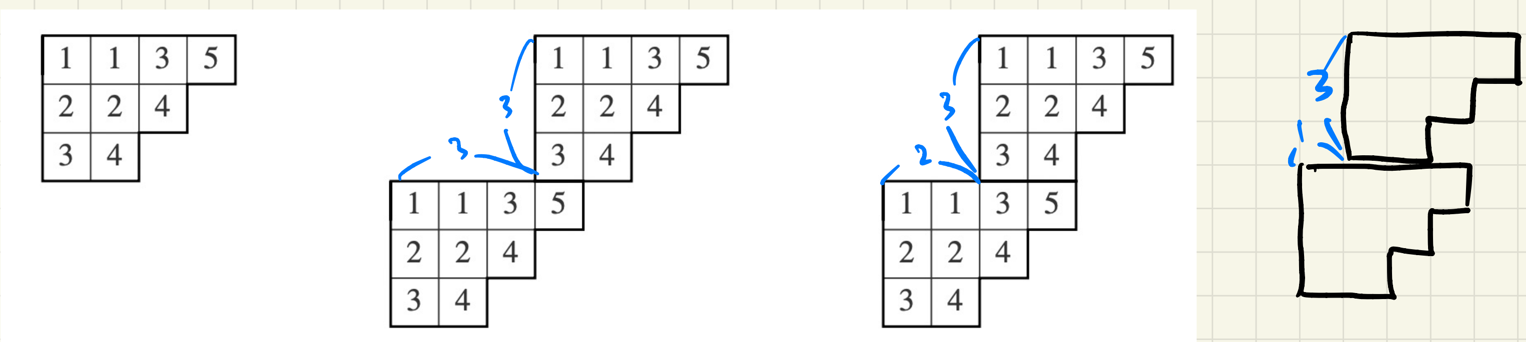
$\overset{?}{=}$ # $(h+1)$-nonnesting and $(w+1)$-noncrossing involutions of $[n]$

Def) **(h,w)-cylindric SSYT** is an SSYT T such that

① height of $T \leq h$

② 

is an SSYT of a valid skew shape.



(3,3)-cylindric

not (3,2)-cylindric

not (3,1)-cyl

Let $\mathrm{CSYT}_n(h,w) = \{ (h,w)\text{-cylindric SYTs of size } n \}$

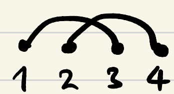
<u>Thm</u> (Huh, Kim, Krattenthaler, Okada)

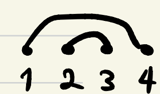
$$\# \mathrm{CSYT}_n(2h+1, 2w+1) = \# \mathrm{NCNN}_n(h+1, w+1)$$

ex) $n=4$, $h=1$, $w=1$.

There are 2 SYTs not counted in LHS : $\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$ $\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$

" 2 involutions " RHS :

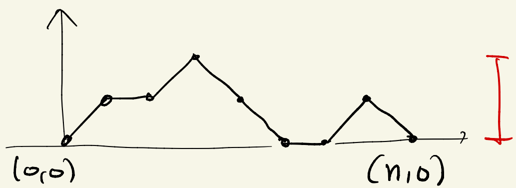




Open Problem : Find a bijective proof.

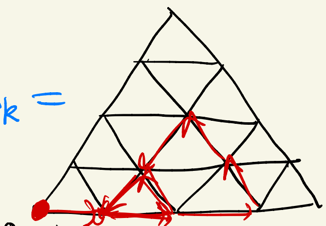
Original Motivation

Def) Motzkin path



height $= 2$

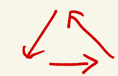
Def) Triangle lattice of size k

$\Delta_k =$ 

0

$k = 4$

steps :

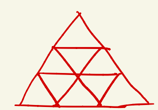


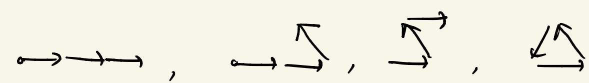
Thm (Mortimer & Prellberg, 2015)

Motzkin paths of length n with height $\leq w$

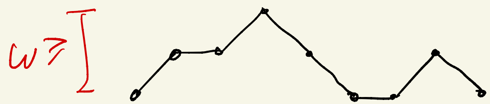
$=$ # lattice walks of length n from 0 in Δ_{2w+1}

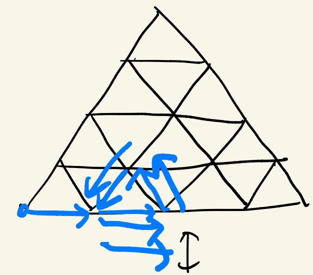
ex) $n=3,\ w=1$





Motzkin path $ht \le w$ $\longleftrightarrow$ paths in $\triangle_{2w+1}$





$\to$: add a cell in row 1

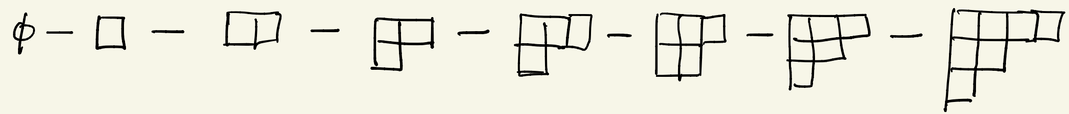
$\nwarrow$: " 2

$\swarrow$: " 3

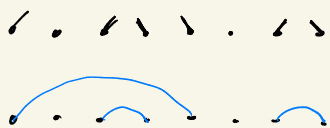
$\updownarrow$

involution

$\updownarrow$



$\updownarrow$



no 2-crossing

no $(w+1)$-nesting

1	2	4	7
3	5		
6			

$(3, 2w+1)$-cylindric SYT

1	2	4	7
3	5		
6			

(This was also observed by Elizalde.)

Mortimer & Prellberg's result is equivalent to

$$\mathrm{NCNN}_n(2, w+1) \quad = \quad \mathrm{CSYT}_n(3, 2w+1)$$

\# 2-noncrossing
(w+1)-nonnesting
involutions on $\{1, \ldots, n\}$

\# (3, 2w+1)-cylindric SYTs of size n.

This is a special case of our theorem:

<u>Thm</u> (Huh, Kim, Krattenthaler, Okada)

$$\mathrm{NCNN}_n(h+1, w+1) = \mathrm{CSYT}_n(2h+1, 2w+1).$$

Note: Courtiel, Elvey Price, Marcovici found a bijective proof of M-P result. (h = 1)

Thm (Huh, Kim, Krattenthaler, Okada)

$$\# CSYT_n(2h+1, 2w+1) = \# NCNN_n(h+1, w+1).$$

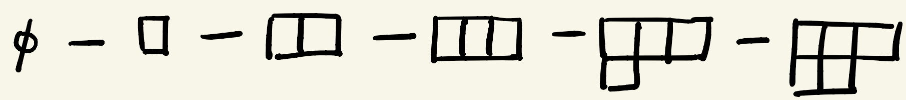
Idea of Proof

① Express each side as # lattice walks

② Express # lattice walks using determinants

③ Prove det = det.

* Lattice walks for $CSYT_n(2h+1, 2w+1)$.

CSYT $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \longleftrightarrow$ 

$\longleftrightarrow (0,0,0) - (1,0,0) - (2,0,0) - (3,0,0) - (3,1,0) - (3,2,0)$

a walk in the region $\{(x_1, x_2, x_3) : x_1 \geq x_2 \geq x_3 \geq 0\}$

with step set $\varepsilon_1 = (1,0,0)$, $\varepsilon_2 = (0,1,0)$, $\varepsilon_3 = (0,0,1)$.

Under this correspondence

$CSYT_n(2h+1, 2w+1) \longleftrightarrow$ walks from 0 of length n in

$$\{(x_1, \dots, x_{2h+1}) : x_1 \geq \dots \geq x_{2h+1} \geq 0, \quad x_1 - x_{2h+1} \leq 2w+1\}$$

$$= \{(x_1, \dots, x_{2h+1}) : x_1 \geq \dots \geq x_{2h+1} \geq x_1 - 2w - 1\}.$$

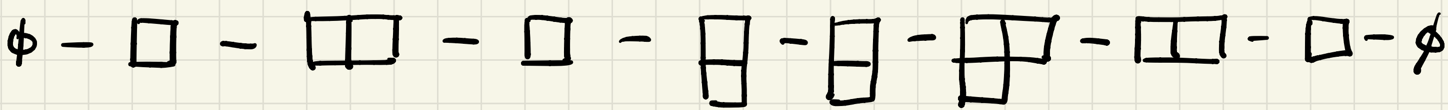
alcove of affine Weyl group of type $\tilde{A}_{2h}$

Def) A *vacillating tableau* is a sequence of partitions

$\phi = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)} = \phi$ such that

$\lambda^{(i)}$ and $\lambda^{(i+1)}$ differ by at most one cell.

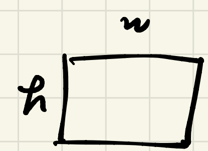
ex)



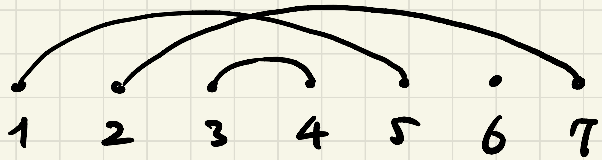
<u>Prop</u> involutions on $[n]$ $\xleftrightarrow{1-1}$ vacillating tableaux of length n

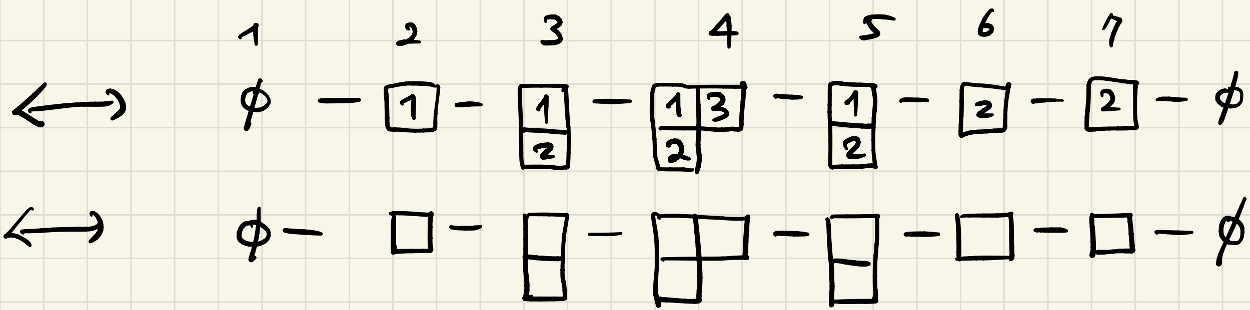
$NCNN_n(h+1, w+1)$ $\xleftrightarrow{1-1}$ "

with every partition contained in

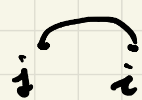


ex)



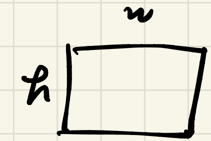


- Proceed from right to left.
- In vertex i, if $\overset{\bullet}{i}$ then $\lambda^{(i)} = \lambda^{(i+1)}$

  if  then $\lambda^{(i)} = \lambda^{(i+1)} \leftarrow j$

  if  then $\lambda^{(i)} = \lambda^{(i+1)} - \{i\}$.

$NCNN_n(h+1, w+1) \xleftrightarrow{1-1}$ vacillating tableaux of length n

with every partition contained in 

$\longleftrightarrow$ lattice walks from 0 to 0 of length n

in region $\{(x_1, \dots, x_n) : w \geqslant x_1 \geqslant \cdots \geqslant x_n \geqslant 0\}$

with step set $\{\pm \varepsilon_i\} \cup \{0\}$.

alcove of affine Weyl group of type $\tilde{C}_n$

Filaseta (1985) computed # lattice walks in alcove of affine Weyl groups.

Applying Filaseta's result, we get

$$\left|CSYT_n(h,w)\right| = \sum_{\substack{\lambda \vdash n \\ \lambda_1 - \lambda_h \leq w}} \sum_{\substack{k_1+\cdots+k_h=0 \\ k_1,\ldots,k_h \in \mathbb{Z}}} n! \det\left(\frac{1}{(\lambda_i - i + j + (w+h)k_i)}\right)_{i,j=1}^{h}$$

and

$$\sum_{n \geq 0} \left|NCNN_n(h+1,w+1)\right| \frac{x^n}{n!}$$

$$= \exp(x) \sum_{k_1,\ldots,k_h \in \mathbb{Z}} \det\left(I_{-i+j+(2h+2w+2)k_i}(2x) - I_{i+j+(2h+2w+2)k_i}(2x)\right)_{i,j=1}^{h}$$

where $\quad I_k(2x) = \sum_{\ell \geq 0} \frac{x^{2\ell+k}}{\ell!(\ell+k)!} \qquad$ (Modified Bessel ftn).

So, to prove our theorem

$$|CSYT_n(2h+1, 2w+1)| = |NCNN_n(h+1, w+1)|$$

it suffices to prove

$$\sum_{\lambda : \lambda_1 - \lambda_h \leq w} \sum_{\substack{k_1 + \cdots + k_h = 0 \\ k_1, \ldots, k_h \in \mathbb{Z}}} x^n \det\left(\frac{1}{(\lambda_i - i + j + (w+h)k_i)}\right)_{i,j=1}^{h}$$

$$= \exp(x) \sum_{k_1, \ldots, k_h \in \mathbb{Z}} \det\left(I_{-i+j+(2h+2w+2)k_i}(2x) - I_{i+j+(2h+2w+2)k_i}(2x)\right)_{i,j=1}^{h}$$

We will show a symmetric function generalization of this.

Symmetric Polynomials

Def) A polynomial $f(x_1, \dots, x_n)$ is symmetric if

$f(x_{a_1}, \dots, x_{a_n}) = f(x_1, \dots, x_n)$ for any

rearrangement $x_{a_1}, \dots, x_{a_n}$ of $x_1, \dots, x_n$.

ex) $x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$, $x_1x_2x_3$

$x_1^2 + x_2^2 + x_3^2$, $x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$.

are symmetric.

ex) $x_1^2x_2 + x_2^2x_3 + x_3^2x_1$ is NOT symmetric since

$\neq x_2^2x_1 + x_1^2x_3 + x_3^2x_2$ ($x_1 \leftrightarrow x_2$ exchanged)

A symmetric function is a symmetric polynomial in infinitely many variables $x_1, x_2, x_3 \ldots$ (with bounded degree).

Def) The kth elementary symmetric function is

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

ex). $e_1 = x_1 + x_2 + x_3 + \cdots$

$e_2 = x_1 x_2 + x_2 x_3 + x_1 x_3 + x_1 x_4 + \cdots$

Def) The kth complete homogeneous symmetric function is

$$h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}$$

ex). $h_1 = e_1 = x_1 + x_2 + \cdots$

$h_2 = x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$$
$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$$

ex) $h_{(4,3,1)} = h_4 h_3 h_1$

$e_{(3,1,1)} = e_3 e_1 e_1$

Λ = the space of symmetric functions ($\mathbb{Q}$-algebra).

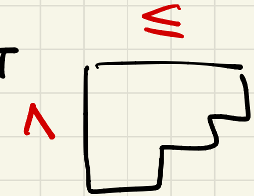
<u>Fundamental Thm of Symmetric functions</u>

$\{h_\lambda\}$ is a basis for Λ.

$\{e_\lambda\}$ " Λ.

Def) Schur function

Recall SSYT



$$s_\lambda = \sum_{T \in SSYT(\lambda)} x_1^{\#1's}\, x_2^{\#2's} \cdots$$

ex) $\lambda = (2,1)$

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau} \quad \cdots + \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} + \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} + \cdots$$

$$s_\lambda = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_3 + \cdots$$

Note: $[x_1 x_2 \cdots x_n]\, s_\lambda = \# SYT(\lambda)$, where $|\lambda| = n$.

In the above example $[x_1 x_2 x_3]\, s_{(2,1)} = 2$

Thm (Jacobi-Trudi formula)

If $\lambda = (\lambda_1, \dots, \lambda_k)$, then

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{i,j=1}^{k}.$$

$$s_{\lambda'} = \det\left(e_{\lambda_i - i + j}\right)_{i,j=1}^{k}.$$

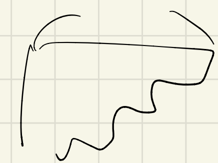
λ' : transpose of λ

$$\lambda = \ydiagram{4,3,1}$$

$$\lambda' = \ydiagram{3,2,2,1}$$

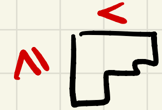
Thm (Gordon-Bender-Knuth, 1972)

$$\sum_{\ell(\lambda) \le 2m+1} s_{\lambda'} = \sum_{k \ge 0} e_k \det\left(f_{-i+j} - f_{i+j}\right)_{i,j=1}^{m}$$



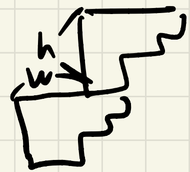
$$\sum_{\ell(\lambda) \le 2m} s_{\lambda'} = \det\left(f_{-i+j} + f_{i+j-1}\right)_{i,j=1}^{m}$$

where $f_k = \sum_{n \ge 0} e_n e_{n+k}$.

Def) A row-strict tableau is . (transpose of SSYT)

Def) (h,w)-cylindric RST is an RST T such that

$ht \leq h$ and



← This is also an RST.

Def) $CRST_\lambda(h,w) = \{ T : (h,w)\text{-cylindric RST of shape } \lambda \}$

<u>Thm</u> (Jacobi-Trudi for cylindric Schur ftn)

$$\sum_{T \in CRST_\lambda(h,w)} x_T = \sum_{\substack{k_1+\cdots+k_h=0 \\ k_1,\dots,k_h \in \mathbb{Z}}} \det\left(e_{\lambda_i - i + j + (h+w)k_i}\right)_{i,j=1}^{h}$$

Pf) Follows from Gessel-Krattenthaler 1997.

History of cylindric tableaux.

- Gessel, Krattenthaler (1997)

  g.f. of cylindric PP, $\hat{A}_r$ basic hypergeometric series

- Bertram, Ciocan-Fontanine, Fulton (1999)

  quantum Kostka number = certain cylindric SSYT.

- Postnikov (2005)

  cylindric skew Schur of toric shape = $\sum$ (3-pt Gromov-Witten inv) s_λ
  (with bounded # var).

- McNamara (2006)

  cylindric skew Schur is not Schur positive (infinite var)

  Conj: cylindric skew Schur is cylindric Schur positive

- S.J. Lee (2019) : McNamara's conj is true.

Thm (Gordon–Bender–Knuth, 1972)

$$\sum_{T \in RST(2h+1)} x_T = \sum_{k \geq 0} e_k \det\left(f_{-i+j} - f_{i+j}\right)_{i,j=1}^{h}$$

$$\sum_{T \in RST(2h)} x_T = \det\left(f_{-i+j} + f_{i+j-1}\right)_{i,j=1}^{h}$$

where $f_k = \sum_{n \geq 0} e_n e_{n+k}$.

Thm (Affine Gordon–Bender–Knuth)

$$\sum_{T \in CRST(2h+1, w)} x_T = \sum_{k \geq 0} e_k \sum_{k_1, \dots, k_h \in \mathbb{Z}} \det\left(f_{-i+j+(w+2h+1)k_i} - f_{i+j+(w+2h+1)k_i}\right)_{i,j=1}^{h}$$

$$\sum_{T \in CRST(2h, w)} x_T = \sum_{k_1, \dots, k_h \in \mathbb{Z}} (-1)^{k_1 + \dots + k_h} \det\left(f_{-i+j+(w+2h)k_i} + f_{i+j+(w+2h)k_i}\right)_{i,j=1}^{h}$$

Outline of Proof of AGB identity

① Reformulation

$$\sum_{\substack{\mu_1,\dots,\mu_m\in\mathbb{Z}\\ \mu_1>\cdots>\mu_m\\ \mu_1-\mu_m<N}} \sum_{\substack{k_1,\dots,k_m\in\mathbb{Z}\\ k_1+\cdots+k_m=0}} \det_{1\le i,j\le m}\left(e_{\mu_i+Nk_i+j}\right) = \sum_{\substack{\alpha_1,\dots,\alpha_m\in\mathbb{Z}\\ R_N(\alpha_1)>\cdots>R_N(\alpha_m)}} \det_{1\le i,j\le m}\left(e_{\alpha_i+j}\right).$$

$$\sum_{\substack{\alpha_1,\dots,\alpha_{2h+1}\in\mathbb{Z}\\ R_N(\alpha_1)>\cdots>R_N(\alpha_{2h+1})}} \det_{1\le i,j\le 2h+1}\left(e_{\alpha_i+j}\right) = \sum_{k\ge 0} e_k \det_{1\le i,j\le h}\left(F_{-i+j,N} - F_{i+j,N}\right)$$

② Use Pfaffians.

$$\sum_{\substack{\alpha_1,\dots,\alpha_{2h+1}\in\mathbb{Z}\\ R_N(\alpha_1)>\cdots>R_N(\alpha_{2h+1})}} \det_{1\le i,j\le 2h+1}\left(e_{\alpha_i+j}\right) = \operatorname{Pf}\begin{pmatrix} 0 & E \\ -E^t & D_N(2h+1)\end{pmatrix},$$

$$\operatorname{Pf}\begin{pmatrix} 0 & E \\ -E^t & D_N(2h+1)\end{pmatrix} = e \operatorname*{Pf}_{1\le i,j\le 2h}\left(F_{j-i-1,N} - F_{j-i+1,N}\right),$$

$$\operatorname*{Pf}_{1\le i,j\le 2h}\left(F_{j-i-1,N} - F_{j-i+1,N}\right) = \det_{1\le i,j\le h}\left(F_{-i+j,N} - F_{i+j,N}\right).$$

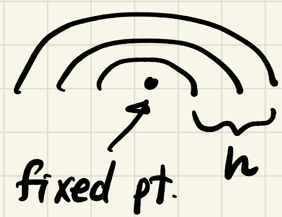
Cor

① $\# \mathrm{CSYT}_n(2h+1, 2w+1) = \# \mathrm{NCNN}_n(h+1, w+1)$

② $\# \mathrm{CSYT}_n(2h+1, 2w) = \# \mathrm{NCNN}_n(h+1, w+\frac{1}{2})$

③ $\# \mathrm{CSYT}_n(2h, 2w+1) = \# \mathrm{NCNN}_n(h+\frac{1}{2}, w+1)$

④ $$\# \mathrm{CSYT}_n(2h, 2w) = \sum_{M \in \mathrm{NCNN}'_n(h+1, w+1)} (-1)^{\mathrm{fix}_1(M)}.$$

$(h+\frac{1}{2})$-nonnesting $\Longleftrightarrow$ no 

If $h=1$, then ①,② reduce to Mortimer–Prellberg result.

If $h=1$, then ③, ④ become

Cor

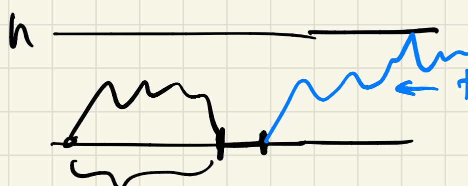
③ # Dyck prefixes of length n with height $\leq 2h+1$

$=$ # Motzkin paths of length n with height $\leq h$ and every horizontal step is on x-axis.

④ # Dyck prefixes of length n with height $\leq 2h$

$=$ # Motzkin paths of length n with height $\leq h$ and every horizontal step is on x-axis.

s.t.



← there is a Dyck prefix from ht 0 to ht h.

even # horizontal steps

We found bijective proofs using recent results of Gu–Prodinger and Dershowitz.

Open Problems

1. Find a bijective proof of

① $\# \, CSYT_n(2h+1, 2w+1) = \# \, NCNN_n(h+1, w+1)$

② $\# \, CSYT_n(2h+1, 2w) = \# \, NCNN_n(h+1, w+\frac{1}{2})$

③ $\# \, CSYT_n(2h, 2w+1) = \# \, NCNN_n(h+\frac{1}{2}, w+1)$

④ $\# \, CSYT_n(2h, 2w) = \sum_{M \in NCNN'_n(h+1, w+1)} (-1)^{fix_1(M)}$.

2. Find a formula for g.f. for $NCNN_n(h+\frac{1}{2}, w+\frac{1}{2})$.

3. Find a sign-free expression for ④.

Thank You

for your attention!