## Homework 5 (Due: June 14)

- **Problem 1** (Section 14.1, Exercise 1). (1) Show that if the field K is generated over F by the elements  $\alpha_1, \ldots, \alpha_n$  then an automorphism  $\sigma$  of K fixing F is uniquely determined by  $\sigma(\alpha_1), \ldots, \sigma(\alpha_n)$ . In particular show that an automorphism fixes K if and only if it fixes a set of generators for K.
  - (2) Let  $G \leq \operatorname{Gal}(K/F)$  be a subgroup of the Galois group of the extension K/F and suppose  $\sigma_1, \ldots, \sigma_k$  are generators for G. Show that the subfield E/F is fixed by G if and only if it is fixed by the generators  $\sigma_1, \ldots, \sigma_k$ .

**Problem 2** (Section 14.1, Exercise 5). Determine the automorphisms of the extension  $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$  explicitly.

**Problem 3** (Section 14.1, Exercise 10). Let K be an extension of the field F. Let  $\varphi: K \to K'$  be an isomorphism of K with a field K' which maps F to the subfield F' of K'. Prove that the map  $\sigma \mapsto \varphi \sigma \varphi^{-1}$  defines a group isomorphism  $\operatorname{Aut}(K/F) \xrightarrow{\sim} \operatorname{Aut}(K'/F')$ .

**Problem 4** (Section 14.2, Exercise 3). Determine the Galois group of  $(x^2 - 2)(x^2 - 3)(x^2 - 5)$ . Determine all the subfields of the splitting field of this polynomial.

**Problem 5** (Section 14.2, Exercise 5). Prove that the Galois group of  $x^p - 2$  for p a prime is isomorphic to the group of matrices  $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$  where  $a, b \in \mathbb{F}_p, a \neq 0$ .

**Problem 6** (Section 14.2, Exercise 9). Give an example of fields  $F_1, F_2, F_3$  with  $\mathbb{Q} \subset F_1 \subset F_2 \subset F_3, [F_3 : \mathbb{Q}] = 8$  and each field is Galois over all its subfields with the exception that  $F_2$  is not Galois over  $\mathbb{Q}$ .

**Problem 7** (Section 14.2, Exercise 13). Prove that if the Galois group of the splitting field of a cubic over  $\mathbb{Q}$  is the cyclic group of order 3 then all the roots of the cubic are real.

**Problem 8** (Section 14.2, Exercise 15). (Biquadratic Extensions) Let F be a field of characteristic  $\neq 2$ .

- (1) If  $K = F(\sqrt{D_1}, \sqrt{D_2})$  where  $D_1, D_2 \in F$  have the property that none of  $D_1, D_2$  or  $D_1D_2$  is a square in F, prove that K/F is a Galois extension with Gal(K/F) isomorphic to the Klein 4-group.
- (2) Conversely, suppose K/F is a Galois extension with Gal(K/F) isomorphic to the Klein 4-group. Prove that  $K = F(\sqrt{D_1}, \sqrt{D_2})$  where  $D_1, D_2 \in F$  have the property that none of  $D_1, D_2$  or  $D_1D_2$  is a square in F.

**Problem 9** (Section 14.2, Exercise 16). (1) Prove that  $x^4 - 2x^2 - 2$  is irreducible over  $\mathbb{Q}$ .

(2) Show the roots of this quartic are

$$\alpha_1 = \sqrt{1 + \sqrt{3}}$$
  $\alpha_3 = -\sqrt{1 + \sqrt{3}}$   
 $\alpha_2 = \sqrt{1 - \sqrt{3}}$   $\alpha_4 = -\sqrt{1 - \sqrt{3}}$ .

- (3) Let  $K_1 = \mathbb{Q}(\alpha_1)$  and  $K_2 = \mathbb{Q}(\alpha_2)$ . Show that  $K_1 \neq K_2$ , and  $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3}) = F$ .
- (4) Prove that  $K_1, K_2$  and  $K_1K_2$  are Galois over F with  $Gal(K_1K_2/F)$  the Klein 4-group. Write out the elements of  $Gal(K_1K_2/F)$  explicitly. Determine all the subgroups of the Galois group and give their corresponding fixed subfields of  $K_1K_2$  containing F.
- Galois group and give their corresponding fixed subfields of  $K_1K_2$  containing F. (5) Prove that the splitting field of  $x^4 - 2x^2 - 2$  over  $\mathbb{Q}$  is of degree 8 with dihedral Galois group.

**Problem 10** (Section 14.2, Exercise 28). Let  $f(x) \in F[x]$  be an irreducible separable polynomial of degree n over the field F, let L be the splitting field of f(x) over F and let  $\alpha$  be a root of f(x) in L. If K is any Galois extension of F contained in L, show that the polynomial f(x) splits into a product of m irreducible polynomials each of degree d over K, where  $m = [F(\alpha) \cap K : F]$  and  $d = [K(\alpha) : K]$  (cf. also the generalization in Exercise 4 of Section 4). [If H is the subgroup of the Galois group of L over F corresponding to K then the factors of f(x) over K correspond to the orbits of H on the roots of f(x). Then use Exercise 9 of Section 4.1.]