1. Homework 4 (Due: May 31)

Problem 1 (Section 13.1, Exercise 2). Show that $x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$ and let θ be a root. Compute $(1+\theta)\left(1+\theta+\theta^2\right)$ and $\frac{1+\theta}{1+\theta+\theta^2}$ in $\mathbb{Q}(\theta)$.

Problem 2 (Section 13.2, Exercise 7). Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$ [one inclusion is obvious, for the other consider $(\sqrt{2}+\sqrt{3})^2$, etc.]. Conclude that $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}] = 4$. Find an irreducible polynomial satisfied by $\sqrt{2}+\sqrt{3}$.

Problem 3 (Section 13.2, Exercise 19). Let K be an extension of F of degree n.

(1) For any $\alpha \in K$ prove that α acting by left multiplication on K is an F-linear transformation of K.
(2) Prove that K is isomorphic to a subfield of the ring of $n \times n$ matrices over F, so the ring of $n \times n$ matrices over F contains an isomorphic copy of every extension of F of degree $\leq n$.

Problem 4 (Section 13.2, Exercise 20). Show that if the matrix of the linear transformation "multiplication by α" considered in the previous exercise is A then α is a root of the characteristic polynomial for A. This gives an effective procedure for determining an equation of degree n satisfied by an element α in an extension of F of degree n. Use this procedure to obtain the monic polynomial of degree 3 satisfied by $\sqrt[3]{2}$ and by $1+\sqrt[3]{2}+\sqrt[3]{4}$

Problem 5 (Section 13.4, Exercise 5). Let K be a finite extension of F. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[x]$ that has a root in K splits completely in $K[x]$. [Use Theorems 8 and 27.]

Problem 6 (Section 13.4, Exercise 6). Let K_1 and K_2 be finite extensions of F contained in the field K, and assume both are splitting fields over F.

(1) Prove that their composite K_1K_2 is a splitting field over F.
(2) Prove that $K_1 \cap K_2$ is a splitting field over F. [Use the preceding exercise.]

Problem 7. Let F be a field and let E, E' be algebraic closures of F. Prove that there is an isomorphism $\sigma : E \to E'$ such that $\sigma|_F : F \to F$ is the identity map on F.

Problem 8 (Section 13.5, Exercise 1). Prove that the derivative D_x of a polynomial satisfies $D_x(f(x)+g(x)) = D_x(f(x)) + D_x(g(x))$ and $D_x(f(x)g(x)) = D_x(f(x))g(x) + D_x(g(x))f(x)$ for any two polynomials $f(x)$ and $g(x)$.

Problem 9 (Section 13.5, Exercise 7). Suppose K is a field of characteristic p which is not a perfect field: $K \neq K^p$. Prove there exist irreducible inseparable polynomials over K. Conclude that there exist inseparable finite extensions of K.

Problem 10 (Section 13.6, Exercise 6). Prove that for n odd, $n > 1$, $\Phi_{2n}(x) = \Phi_n(-x)$.