COMBINATORICS OF ORTHOGONAL POLYNOMIALS

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Contents

1. Introduction to the lectures

Orthogonal polynomials are classical objects arising from the study of continued fractions. Due to the long history of orthogonal polynomials, they have now become important objects of study in many areas: classical analysis and PDE, mathematical physics, probability, random matrix theory, and combinatorics.

The combinatorial study of orthogonal polynomials was pioneered by Flajolet and Viennot in 1980s. In these lectures we will learn fascinating combinatorial properties of orthogonal polynomials.

We will first study basic properties of orthogonal polynomials based on Chihara's book, Chapter 1 [1]. We will then focus on the combinatorial approach of orthogonal polynomials, which will be based on Viennot's lecture notes [2]. We will also cover more recent developments in the combinatorics of orthogonal polynomials such as their connections with ASEP, staircase tableaux, lecture hall partitions, and orthogonal polynomials of type R_1.

The prerequisites of this course are Calculus 1, Linear Algebra, and Discrete Mathematics.

2. Elementary Theory of Orthogonal Polynomials

In this section we will cover the first chapter of Chihara's book [1].

2.1. Introduction.

Since

$$2\cos m\theta \cos n\theta = \cos(m+n)\theta + \cos(m-n)\theta,$$

for nonnegative integers m and n, we have

$$\int_0^{\pi} \cos m\theta \cos n\theta d\theta = 0, \qquad m \neq n. \tag{2.1}$$

In this situation we say that $\cos m\theta$ and $\cos n\theta$ are orthogonal over the interval $(0,\pi)$.

Note that $\cos n\theta$ is a polynomial in $\cos\theta$ of degree n. So we can write $\cos n\theta = T_n(\cos\theta)$ for a polynomial $T_n(x)$ of degree x.

By the change of variable $x = \cos\theta$, (2.1) can be rewritten as

$$\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-1/2}dx = 0, \qquad m \neq n.$$

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The polynomials $T_n(x)$, $n \geq 0$, are called the *Tchebyshev polynomials of the first kind*. The first few polynomials are:

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= \cos\theta = x, \\ T_2(x) &= \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x. \end{aligned}$$

Recall that in an inner product space V with inner product $\langle \cdot, \cdot \rangle$, a set of vectors $v_1, \dots, v_n$ are said to be orthogonal if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$. In this sense the Tchebyshev polynomials $T_n(x)$ are orthogonal, where $V = \mathbb{R}[x]$ is the space of polynomials with real coefficients with the inner product given by

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)^{-1/2} dx.$$

We say that $T_n(x)$ are *orthogonal polynomials* with respect to the *weight function* $(1-x^2)^{-1/2}$ on the interval $(-1, 1)$.

Definition 2.1. Suppose that $w(x)$ is a nonnegative and integrable function on (a, b) with $\int_a^b w(x)dx > 0$ and $\int_a^b x^n dx < \infty$ for all $n \geq 0$. A sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ is called an *orthogonal polynomial sequence (OPS)* with respect to the *weight function* (or *measure*) $w(x)$ on (a, b) if the following conditions hold:

(1) $\deg P_n(x) = n$, for $n \geq 0$,
(2) $\int_a^b P_m(x)P_n(x)w(x)dx = 0$ for $m \neq n$.

There is another way to define orthogonal polynomials without using the weight function. For a polynomial $f(x)$, if we define

$$\mathcal{L}(f(x)) = \int_a^b f(x)w(x)dx,$$

then $\mathcal{L}(f(x))$ is completely determined by the *moments* $\mu_n = \int_a^b x^n w(x)dx$. So, if we are only interested in polynomials, then we can define a linear functional $\mathcal{L}$ using a moment sequence $\mu_0, \mu_1, \dots$. Not every sequence $\mu_0, \mu_1, \dots$ gives rise to an OPS, though. We will see later a criterion for a sequence to be a moment sequence.

Definition 2.2. Let $\mathcal{L}$ be the linear functional defined on the space of polynomials in x. A sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ is called an *orthogonal polynomial sequence (OPS)* with respect to $\mathcal{L}$ if the following conditions hold:

(1) $\deg P_n(x) = n$, $n \geq 0$,
(2) $\mathcal{L}(P_m(x)^2) \neq 0$ for $m \geq 0$,
(3) $\mathcal{L}(P_m(x)P_n(x)) = 0$ for $m \neq n$.

Note that the second condition above was not necessary in Definition 2.1 because it follows from the facts that $w(x)$ is nonnegative and $\int_a^b w(x)dx > 0$.

Remark 2.3. The moments of the Tchebyshev polynomials are

$$\mu_{2n} = \int_{-1}^{1} x^{2n}(1-x^2)^{-1/2} dx = \frac{\pi}{2^{2n}}\binom{2n}{n}, \qquad \mu_{2n+1} = 0.$$

This suggests that there could be some interesting combinatorics behind the scene. We will later find a combinatorial way to understand this situation.

Example 2.4 (Charlier polynomials)**.** The *Charlier polynomials* $P_n(x)$ are defined by

$$P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!},$$

where $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$. We will find a different type of orthogonality for $P_n(x)$.

The generating function for $P_n(x)$ is

$$G(x,w) = \sum_{n\ge 0} P_n(x) w^n = \sum_{n\ge 0}\left(\sum_{k=0}^{n}\binom{x}{k}\frac{(-a)^{n-k}}{(n-k)!}\right) w^n = \sum_{n\ge 0}\binom{x}{n} w^n \sum_{n\ge 0}\frac{(-a)^m}{m!} w^m,$$

which means

$$G(x,w) = e^{-aw}(1+w)^x.$$

Thus

$$a^x G(x,v)G(x,w) = e^{-a(v+w)}\left(a(1+v)(1+w)\right)^x.$$

We have

$$\sum_{k\ge 0}\frac{a^k G(k,v)G(k,w)}{k!} = \sum_{k\ge 0}\frac{e^{-a(v+w)}\left(a(1+v)(1+w)\right)^k}{k!} = e^{-a(v+w)}e^{a(1+v)(1+w)} = e^a e^{avw}.$$

Thus

$$\sum_{k\ge 0}\frac{a^k G(k,v)G(k,w)}{k!} = \sum_{n\ge 0}\frac{e^a (avw)^n}{n!}. \tag{2.2}$$

On the other hand

$$\begin{aligned}\sum_{k\ge 0}\frac{a^k G(k,v)G(k,w)}{k!} &= \sum_{k\ge 0}\frac{a^k}{k!}\sum_{m,n\ge 0} P_m(k)P_n(k) v^m w^n \\ &= \sum_{m,n\ge 0}\left(\sum_{k\ge 0} P_m(k)P_n(k)\frac{a^k}{k!}\right) v^m w^n. \qquad (2.3)\end{aligned}$$

Comparing the coefficients of $v^m w^n$ in (2.2) and (2.3) we obtain

$$\sum_{k\ge 0} P_m(k)P_n(k)\frac{a^k}{k!} = \frac{e^a a^n}{n!}\delta_{n,m}. \tag{2.4}$$

Therefore, if we define a linear functional $\mathcal{L}$ by

$$\mathcal{L}(x^n) = \sum_{k\ge 0} k^n \frac{a^k}{k!},$$

then $P_n(x)$ are orthogonal polynomials with respect to $\mathcal{L}$.

Note that we describe the orthogonality of $P_n(x)$ using only the linear functional $\mathcal{L}$ without referring to any weight function. However, we can also find a weight function in this case. Let $\psi(x)$ be the step function with a jump at $k = 0, 1, 2, \ldots$ of magnitude $a^k/k!$. Then the linear functional $\mathcal{L}$ can be written as the following Riemann–Stieltjes integral

$$\mathcal{L}(f(x)) = \int_{-\infty}^{\infty} f(x) d\psi(x).$$

We can also prove (2.4) in a combinatorial way, see Appendix A.

Remark 2.5. In the theory of orthogonal polynomials, finding an explicit weight function is an important problem. However, in these lectures, we will not pursue in this direction and we will be mostly satisfied with Definition 2.2.

2.2. **The moment functional and orthogonality.** We will consider the space $\mathbb{C}[x]$ polynomials with complex coefficients. A *linear function* on $\mathbb{C}[x]$ is a map $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$ such that $\mathcal{L}(af(x) + bg(x)) = a\mathcal{L}(f(x)) + b\mathcal{L}(g(x))$ for all $f(x), g(x) \in \mathbb{C}[x]$ and $a, b \in \mathbb{C}$.

Definition 2.6. Let $\{\mu_n\}_{n\ge 0}$ be a sequence of complex numbers. Let $\mathcal{L}$ be the linear functional on the space of polynomials defined by $\mathcal{L}(x^n) = \mu_n$, $n \ge 0$. In this case we say that $\mathcal{L}$ is the *moment functional* determined by the *moment sequence* $\{\mu_n\}$, and μ_n is called the *nth moment.*

We recall the definition of orthogonal polynomials.

Definition 2.7. Let $\mathcal{L}$ be the linear functional defined on the space of polynomials in x. A sequence of polynomials $\{P_n(x)\}_{n\ge 0}$ is called an *orthogonal polynomial sequence (OPS)* with respect to $\mathcal{L}$ if the following conditions hold:

(1) $\deg P_n(x) = n$, $n \ge 0$,
(2) $\mathcal{L}(P_m(x)P_n(x)) = K_n\delta_{m,n}$, for some $K_n \ne 0$.

We say that $P_n(x)$ are *orthonormal* if $\mathcal{L}(P_m(x)P_n(x)) = \delta_{m,n}$.

Theorem 2.8. *Let $\{P_n(x)\}$ be a sequence of polynomials and let $\mathcal{L}$ be a moment sequence. The following are equivalent:*

(1) *$\{P_n(x)\}$ is an OPS with respect to $\mathcal{L}$;*
(2) *$\mathcal{L}(\pi(x)P_n(x)) = 0$ if $\deg \pi(x) < n$ and $\mathcal{L}(\pi(x)P_n(x)) \ne 0$ if $\deg \pi(x) = n$;*
(3) *$\mathcal{L}(x^m P_n(x)) = K_n\delta_{m,n}$, $0 \le m \le n$, for some $K_n \ne 0$.*

Proof. (1) ⇒ (2): Suppose that $\deg \pi(x) \le n$. Since $\{P_n(x)\}$ is a basis of $\mathbb{C}[x]$, we can write

$$\pi(x) = c_0 + c_1P_1(x) + \cdots + c_nP_n(x).$$

Then

$$\mathcal{L}(\pi(x)P_n(x)) = \sum_{k=0}^{n} \mathcal{L}\left(c_kP_k(x)P_n(x)\right) = c_n\mathcal{L}(P_n(x)^2),$$

which is zero if $\deg \pi(x) < n$ and nonzero if $\deg \pi(x) = n$.

(2) ⇒ (3): Trivial. (2) ⇒ (3): Trivial. □

Theorem 2.9. *Suppose that $\{P_n(x)\}_{n\ge0}$ be an OPS with respect to $\mathcal{L}$. Then for any polynomial $\pi(x)$ of degree n,*

$$\pi(x) = \sum_{k=0}^{n} c_kP_k(x), \qquad c_k = \frac{\mathcal{L}(\pi(x)P_k(x))}{\mathcal{L}(P_k(x)^2)}.$$

Proof. Clearly, we can write

$$\pi(x) = \sum_{k=0}^{n} c_kP_k(x),$$

for some c_k. Multiplying $P_j(x)$ both sides and taking $\mathcal{L}$, we get

$$\mathcal{L}(\pi(x)P_j(x)) = \sum_{k=0}^{n} \mathcal{L}\left(c_kP_k(x)P_j(x)\right) = c_j\mathcal{L}(P_j(x)^2).$$

Dividing both sides by $\mathcal{L}(P_j(x)^2)$, we obtain the theorem. □

Theorem 2.10. *Suppose that $\{P_n(x)\}_{n\ge0}$ be an OPS with respect to $\mathcal{L}$. Then $P_n(x)$ is uniquely determined by $\mathcal{L}$ up to a nonzero factor. More precisely, if $\{Q_n(x)\}_{n\ge0}$ is an OPS with respect to $\mathcal{L}$, then there are constants $c_n \ne 0$ such that $Q_n(x) = c_nP_n(x)$ for all $n \ge 0$.*

Proof. Let us write $Q_n(x) = \sum_{k=0}^{n} c_kP_k(x)$. Then by Theorem 2.9, $c_k = \mathcal{L}(Q_n(x)P_k(x))/\mathcal{L}(P_k(x)^2)$. But by Theorem 2.8, $\mathcal{L}(Q_n(x)P_k(x)) = 0$ unless $k = n$. Thus $Q_n(x) = c_nP_n(x)$. □

Note that if $\{P_n(x)\}_{n\ge0}$ is an OPS for $\mathcal{L}$, then so is $\{c_nP_n(x)\}_{n\ge0}$ for any $c_n \ne 0$. Therefore there is a unique monic OPS, which is obtained by dividing each $P_n(x)$ by its leading coefficient. Note also that there is a unique orthonormal OPS as well given by $p_n(x) = P_n(x)/\mathcal{L}(P_n(x)^2)$. In summary we have the following corollary.

Corollary 2.11. *Suppose that $\mathcal{L}$ is a moment sequence such that there is an OPS for $\mathcal{L}$. Let K_n, $n \ge 0$, be a sequence of nonzero numbers.*

(1) *There is a unique monic OPS $\{P_n(x)\}_{n\ge0}$ for $\mathcal{L}$.*
(2) *There is a unique orthonormal OPS $\{P_n(x)\}_{n\ge0}$ for $\mathcal{L}$.*
(3) *There is a unique OPS $\{P_n(x)\}_{n\ge0}$ for $\mathcal{L}$ such that the leading coefficient of $P_n(x)$ is K_n.*
(4) *There is a unique OPS $\{P_n(x)\}_{n\ge0}$ for $\mathcal{L}$ such that $\mathcal{L}(x^nP_n(x)) = K_n$.*

Clearly, if $\{P_n(x)\}_{n\geq 0}$ is an OPS for $\mathcal{L}$, then it is also an OPS for $\mathcal{L}'$ given by $\mathcal{L}'(f(x)) = c\mathcal{L}(f(x))$ for some $c \neq 0$. Therefore, by dividing the linear functional by the value $\mathcal{L}(1)$, we may assume that $\mathcal{L}(1) = 1$.

2.3. **Existence of OPS.** The main question in this section is: for what moment functional $\mathcal{L}$ does there exist an OPS? To answer this question we need the following definition.

Definition 2.12. The *Hankel determinant* of a moment sequence $\{\mu_n\}$ is defined by

$$\Delta_n = \det(\mu_{i+j})_{i,j=0}^n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}.$$

Theorem 2.13. *Let $\mathcal{L}$ be a moment functional with moment sequence $\{\mu_n\}$. Then there is an OPS for $\mathcal{L}$ if and only if $\Delta_n \neq 0$ for all $n \geq 0$.*

Proof. Fix a sequence $\{K_n\}$ of nonzero real numbers K_n. By Corollary 2.11, if there is an OPS $\{P_n(x)\}_{n\geq 0}$ for $\mathcal{L}$, it is uniquely determined by the condition $\mathcal{L}(x^n P_n(x)) = K_n$, $n \geq 0$. In other words, using Theorem 2.8, there is an OPS for $\mathcal{L}$ if and only if there is a unique sequence $\{P_n(x)\}_{n\geq 0}$ of polynomials such that

$$\mathcal{L}(x^m P_n(x)) = K_n \delta_{m,n}, \qquad 0 \leq m \leq n. \tag{2.5}$$

Now let $P_n(x) = \sum_{k=0}^n c_{n,k} x^k$. Multiplying both sides by x^m and taking $\mathcal{L}$, we get

$$\mathcal{L}(x^m P_n(x)) = \sum_{k=0}^n c_{n,k} \mu_{n+k}.$$

Thus (2.5) can be written as the matrix equation

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix} \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ \vdots \\ c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ K_n \end{pmatrix}. \tag{2.6}$$

Then the uniqueness of the polynomials $P_n(x)$ satisfying (2.5) is equivalent to the uniqueness of the solution of the matrix equation (2.6) in $c_{n,0}, c_{n,1}, \ldots, c_{n,n}$. In order for (2.6) to have a unique solution, the Hankel determinant Δ_n must be nonzero for all $n \geq 0$. This proves the theorem. □

Note that by solving (2.6) using Cramer's rule, we have $c_{n,n} = K_n \Delta_{n-1}/\Delta_n$, which is the leading coefficient of $P_n(x)$. In particular, we have

$$\mathcal{L}(P_n(x)^2) = \sum_{k=0}^n \mathcal{L}(c_{n,k} x^k P_n(x)) = c_{n,n} \mathcal{L}(x^n P_n(x)) = c_{n,n} K_n = \frac{\Delta_n}{\Delta_{n-1}}. \tag{2.7}$$

In many important cases of orthogonal polynomials there is a nonnegative weight function $w(x)$ representing the moment functional: $\mathcal{L}(x^n) = \int_a^b x^n w(x)dx$. In more general cases, $\mathcal{L}$ can be represented using the Riemann–Stieltjes integral $\mathcal{L}(x^n) = \int_a^b x^n d\psi(x)$, where $\psi(x)$ is a nondecreasing function such that $\{x : \psi(x+\epsilon) - \psi(x-\epsilon) > 0 \text{ for all } \epsilon > 0\}$ is an infinite set. It is known [1, Chapter 2] that there is such an expression if and only if $\mathcal{L}(\pi(x)) > 0$ for all nonzero polynomials $\pi(x)$ such that $\pi(x) \geq 0$ for all $x \in \mathbb{R}$.

Definition 2.14. A moment functional $\mathcal{L}$ is *positive-definite* if $\mathcal{L}(\pi(x)) > 0$ for all nonzero polynomials $\pi(x)$ such that $\pi(x) \geq 0$ for all $x \in \mathbb{R}$.

If $\mathcal{L}$ is positive-definite, then it has a real OPS. We will see later that the converse is not true.

Theorem 2.15. *Let $\mathcal{L}$ be a positive-definite moment functional. Then $\mathcal{L}$ has real moments and there is a real OPS for $\mathcal{L}$.*

Proof. First, we show that the moments μ_n are real. Since $\mathcal{L}$ is positive-definite, $\mu_{2n} = \mathcal{L}(x^{2n}) > 0$ is real. Since $\mathcal{L}((x+1)^{2n}) = \sum_{k=0}^{2n} \binom{2n}{k}\mu_k$ is real, by induction, we obtain that μ_{2n-1} is also real.

Now, we construct a real OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$. Let $P_0(x) = 1$. Suppose that we have constructed $P_0, \dots, P_n$ which are orthogonal with respect to $\mathcal{L}$, i.e., $\mathcal{L}(P_i(x)P_j(x)) = 0$ for $0 \le i, j \le n$ with $i \ne j$. Now we need to find

$$P_{n+1}(x) = x^{n+1} + \sum_{k=0}^{n} a_k P_k(x) \tag{2.8}$$

such that $\mathcal{L}(P_k(x)P_{n+1}(x)) = 0$ for all $0 \le k \le n$. Multiplying $P_k(x)$ and taking $\mathcal{L}$ in (2.8) we get $\mathcal{L}(P_k(x)P_{n+1}(x)) = \mathcal{L}(x^{n+1}P_k(x)) + a_k\mathcal{L}(P_k(x)^2)$. Thus, if we set

$$a_k = -\frac{\mathcal{L}(x^{n+1}P_k(x))}{\mathcal{L}(P_k(x)^2)},$$

which is real, then $P_{n+1}(x)$ is orthogonal to $P_0(x), \dots, P_n(x)$. In this way we can construct a real OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$. □

Note that if $\mathcal{L}$ is positive-definite, then $\mathcal{L}(P_n(x)^2) > 0$. Thus in this case we can construct a real orthonormal OPS $\{p_n(x)\}_{n\ge 0}$ by rescaling: $p_n(x) = P_n(x)/\sqrt{\mathcal{L}(P_n(x)^2)}$.

You may wonder why $\mathcal{L}$ is called "positive-definite". To see this recall that a real $n \times n$ matrix A is positive definite if $u^T A u > 0$ for every nonzero vector $u \in \mathbb{R}^n$. Sylvester's criterion says that A is positive definite if and only if every principal minor of A is positive. The following theorem justifies the terminology "positive-definite" for $\mathcal{L}$.

Theorem 2.16. *A moment functional $\mathcal{L}$ is positive-definite if and only if every moment μ_n is real and $\Delta_n > 0$ for all $n \ge 0$. In other words, $\mathcal{L}$ is positive-definite if and only if the Hankel matrix $(\mu_{i+j})_{i,j\ge 0}$ is positive-definite.*

Proof. ($\Rightarrow$) By Theorem 2.15, the moments are real and there is a real OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$. By (2.7), $\Delta_n/\Delta_{n-1} = \mathcal{L}(P_n(x)^2) > 0$ for $n \ge 0$, where $\Delta_{-1} = 1$. Thus by induction we obtain $\Delta_n > 0$ for all $n \ge 0$.

($\Leftarrow$) Since $\Delta_n > 0$, by Theorem 2.13, there is an OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$. We need to show that $\mathcal{L}(\pi(x)) > 0$ for any nonzero polynomial $\pi(x)$ with $\pi(x) \ge 0$ for all $x \in \mathbb{R}$. Such a polynomial $\pi(x)$ can be written as $\pi(x) = p(x)^2 + q(x)^2$ for some real polynomials $p(x)$ and $q(x)$, see Lemma 2.17 below. Thus it suffices to show that $\mathcal{L}(p(x)^2) = 0$ for a polynomial $p(x)$. To do this let $p(x) = \sum_{k=0}^{n} a_k P_k(x)$. Then by the orthogonality

$$\mathcal{L}(p(x)^2) = \sum_{k=0}^{n} a_k^2 \mathcal{L}(P_k(x)^2)$$

Since $\Delta_n > 0$, we have $\mathcal{L}(P_k(x)^2) > 0$ by (2.7). Thus $\mathcal{L}(p(x)^2) > 0$ as desired. □

Lemma 2.17. *Let $\pi(x)$ be a nonzero polynomial such that $\pi(x) \ge 0$ for all $x \in \mathbb{R}$. Then $\pi(x) = p(x)^2 + q(x)^2$ for some polynomials $p(x)$ and $q(x)$.*

Proof. Since $\pi(x)$ is real for all real x, the coefficients of $\pi(x)$ are real. This can be seen inductively by observing that if $\deg \pi(x) = n$, then the leading coefficient of $\pi(x)$ is equal to

$$\lim_{x\to\infty} \frac{\pi(x)}{x^n}.$$

Since $\pi(x)$ is a real polynomial such that $\pi(x) \ge 0$ every real zero has even multiplicity and complex roots appear in conjugate pairs. Thus we can write

$$\pi(x) = r(x)^2 \prod_{k=1}^{m} (x - \alpha_k - \beta_k i)(x - \alpha_k + \beta_k i),$$

where $r(x)$ is a real polynomial and $\alpha_k, \beta_k \in \mathbb{R}$. If we write $\prod_{k=1}^{m}(x - \alpha_k - \beta_k i) = A(x) + iB(x)$, then $\prod_{k=1}^{m}(x - \alpha_k + \beta_k i) = A(x) - iB(x)$. Thus $\pi(x) = r(x)^2(A(x)^2 + B(x)^2)$ as desired. □

Definition 2.18. We say that $\mathcal{L}$ is *quasi-definite* if $\Delta_n \ne 0$ for all $n \ge 0$.

Appendix A. Sign-reversing involutions

Definition A.1. A *sign* of a set X is a function $\operatorname{sgn} : X \to \{+1, -1\}$. A *sign-reversing involution* on X is an involution $\phi : X \to X$ such that $\operatorname{sgn}(x) = 1$ for $x \in \operatorname{Fix}(\phi)$ and $\operatorname{sgn}(\phi(x)) = -\operatorname{sgn}(x)$ for all $x \in X \setminus \operatorname{Fix}(\phi)$, where $\operatorname{Fix}(\phi)$ is the set of *fixed points* of ϕ, i.e., $\operatorname{Fix}(\phi) = \{x \in X : \phi(x) = x\}$.

It is easy to see that if ϕ is a sign-reversing involution on X, then

$$\sum_{x\in X} \operatorname{sgn}(X) = |\operatorname{Fix}(\phi)|. \tag{A.1}$$

Example A.2. Let's prove the following identity using sign-reversing involutions:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0. \tag{A.2}$$

To this end we need to construct a set X and a sign-reversing involution ϕ on X such that (A.1) becomes (A.2).

Let X be the set of all subsets of $[n] := \{1, \dots, n\}$ and for $A \in X$, define $\operatorname{sgn}(A) = (-1)^{|A|}$. Then it suffices to construct a sign-reversing involution on X with no fixed points. This can be done by letting $\phi(A) = A\Delta\{1\}$, where $A\Delta B := (A \cup B) \setminus (A \cap B)$.

Example A.3. Recall that we proved the following identitiy, which was stated in (2.4), using generating functions:

$$\sum_{k\geq 0} P_m(k)P_n(k)\frac{a^k}{k!} = \frac{e^a a^n}{n!}\delta_{n,m}, \tag{A.3}$$

where $P_n(x)$ are the Charlier polynomials defined by

$$P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!}.$$

We will prove this identity using sign-reversing involutions. To do this, we will consider (A.3) as a power series in a. Note that

$$\begin{aligned}
\sum_{k\geq 0} P_m(k)P_n(k)\frac{a^k}{k!} &= \sum_{k\geq 0}\sum_{i=0}^{m} \binom{k}{i}\frac{(-a)^{m-i}}{(m-i)!}\sum_{j=0}^{n}\binom{k}{j}\frac{(-a)^{n-j}}{(n-j)!}\frac{a^k}{k!} \\
&= \sum_{k\geq 0}\sum_{i=0}^{m}\sum_{j=0}^{n} \binom{k}{m-i}\frac{(-a)^i}{i!}\binom{k}{n-j}\frac{(-a)^j}{j!}\frac{a^k}{k!} \\
&= \sum_{N\geq 0}\frac{a^N}{N!}\sum_{i+j+k=N}(-1)^{i+j}\frac{N!}{i!j!k!}\binom{k}{m-i}\binom{k}{n-j},
\end{aligned}$$

where $\binom{r}{s} = 0$ if $s < 0$.

For a fixed N,

$$\sum_{i+j+k=N}(-1)^{i+j}\binom{N}{i,j,k}\binom{k}{m-i}\binom{k}{n-j} = \sum_{(A,B,C)\in X}(-1)^{|B\setminus A|+|C\setminus A|},$$

where X is the set of triples (A, B, C) such that $A\cup B\cup C = \{1, \dots, N\}$, $|A| = k, |B| = m, |C| = n$, $(B \cap C) \setminus A = \emptyset$. Define $\operatorname{sgn}(A, B, C) = (-1)^{|B\setminus A|+|C\setminus A|}$. We will find a sign-reversing involution on X toggling the smallest integer in regions 1 and 2 or in regions 3 and 4 in Figure 1.

To be precise, for $(A, B, C) \in X$, define $\phi(A, B, C)$ as follows.

- **Case 1:** The regions $1, 2, 3, 4$ are all empty. In this case we define $\phi(A, B, C) = (A, B, C)$.
- **Case 2:** At least one of the regions $1, 2, 3, 4$ is nonempty. Let s be the smallest integer in $(B \cap C) \setminus A$. If s is in region 1 (respectively 2, 3, 4), then move this integer to region 2 (respectively 1, 4, 3). Then let $\phi(A, B, C) = (A', B', C')$, where A', B', C' are the resulting sets.

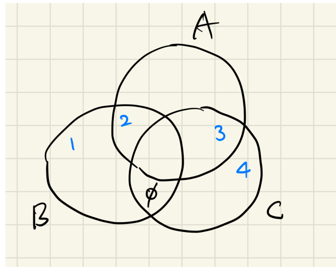

FIGURE 1. The triple (A, B, C).

By the construction, ϕ is a sign-reversing involution on X whose fixed points are the triples (A, B, C) such that the regions $1, 2, 3, 4$ are all empty, that is, $B = C \subseteq A$. If $B = C \subseteq A$, then $A = [N]$, so the number of such triples (A, B, C) is $\binom{N}{n}$ if $m = n$ and 0 otherwise. Thus

$$\sum_{(A,B,C)\in X} (-1)^{|B\setminus A|+|C\setminus A|} = |\operatorname{Fix}(\phi)| = \delta_{m,n}\binom{N}{n}.$$

This implies

$$\sum_{k\geq 0} P_m(k)P_n(k)\frac{a^k}{k!} = \delta_{m,n}\sum_{N\geq 0}\frac{a^N}{N!}\binom{N}{n} = \frac{e^a a^n}{n!}\delta_{n,m}.$$

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