Homework 5 (Due: June 14)

Problem 1 (Section 14.1, Exercise 1). (1) Show that if the field K is generated over F by the elements $\alpha_1, \ldots, \alpha_n$ then an automorphism σ of K fixing F is uniquely determined by $\sigma(\alpha_1), \ldots, \sigma(\alpha_n)$. In particular show that an automorphism fixes K if and only if it fixes a set of generators for K.

(2) Let $G \leq \mathrm{Gal}(K/F)$ be a subgroup of the Galois group of the extension K/F and suppose $\sigma_1, \ldots, \sigma_k$ are generators for G. Show that the subfield E/F is fixed by G if and only if it is fixed by the generators $\sigma_1, \ldots, \sigma_k$.

Problem 2 (Section 14.1, Exercise 5). Determine the automorphisms of the extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ explicitly.

Problem 3 (Section 14.1, Exercise 10). Let K be an extension of the field F. Let $\varphi : K \to K'$ be an isomorphism of K with a field K' which maps F to the subfield F' of K'. Prove that the map $\sigma \mapsto \varphi\sigma\varphi^{-1}$ defines a group isomorphism $\mathrm{Aut}(K/F) \xrightarrow{\sim} \mathrm{Aut}(K'/F')$.

Problem 4 (Section 14.2, Exercise 3). Determine the Galois group of $(x^2-2)(x^2-3)(x^2-5)$. Determine all the subfields of the splitting field of this polynomial.

Problem 5 (Section 14.2, Exercise 5). Prove that the Galois group of $x^p - 2$ for p a prime is isomorphic to the group of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{F}_p, a \neq 0$.

Problem 6 (Section 14.2, Exercise 9). Give an example of fields F_1, F_2, F_3 with $\mathbb{Q} \subset F_1 \subset F_2 \subset F_3$, $[F_3 : \mathbb{Q}] = 8$ and each field is Galois over all its subfields with the exception that F_2 is not Galois over $\mathbb{Q}$.

Problem 7 (Section 14.2, Exercise 13). Prove that if the Galois group of the splitting field of a cubic over $\mathbb{Q}$ is the cyclic group of order 3 then all the roots of the cubic are real.

Problem 8 (Section 14.2, Exercise 15). (Biquadratic Extensions) Let F be a field of characteristic $\neq 2$.

(1) If $K = F\left(\sqrt{D_1}, \sqrt{D_2}\right)$ where $D_1, D_2 \in F$ have the property that none of D_1, D_2 or D_1D_2 is a square in F, prove that K/F is a Galois extension with $\mathrm{Gal}(K/F)$ isomorphic to the Klein 4-group.

(2) Conversely, suppose K/F is a Galois extension with $\mathrm{Gal}(K/F)$ isomorphic to the Klein 4-group. Prove that $K = F\left(\sqrt{D_1}, \sqrt{D_2}\right)$ where $D_1, D_2 \in F$ have the property that none of D_1, D_2 or D_1D_2 is a square in F.

Problem 9 (Section 14.2, Exercise 16). (1) Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$.

(2) Show the roots of this quartic are

$$\begin{aligned} \alpha_1 &= \sqrt{1+\sqrt{3}} & \alpha_3 &= -\sqrt{1+\sqrt{3}} \\ \alpha_2 &= \sqrt{1-\sqrt{3}} & \alpha_4 &= -\sqrt{1-\sqrt{3}}. \end{aligned}$$

(3) Let $K_1 = \mathbb{Q}(\alpha_1)$ and $K_2 = \mathbb{Q}(\alpha_2)$. Show that $K_1 \neq K_2$, and $K_1 \cap K_2 = \mathbb{Q}(\sqrt{3}) = F$.

(4) Prove that K_1, K_2 and K_1K_2 are Galois over F with $\mathrm{Gal}(K_1K_2/F)$ the Klein 4-group. Write out the elements of $\mathrm{Gal}(K_1K_2/F)$ explicitly. Determine all the subgroups of the Galois group and give their corresponding fixed subfields of K_1K_2 containing F.

(5) Prove that the splitting field of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$ is of degree 8 with dihedral Galois group.

Problem 10 (Section 14.2, Exercise 28). Let $f(x) \in F[x]$ be an irreducible separable polynomial of degree n over the field F, let L be the splitting field of $f(x)$ over F and let α be a root of $f(x)$ in L. If K is any Galois extension of F contained in L, show that the polynomial $f(x)$ splits into a product of m irreducible polynomials each of degree d over K, where $m = [F(\alpha) \cap K : F]$ and $d = [K(\alpha) : K]$ (cf. also the generalization in Exercise 4 of Section 4). [If H is the subgroup of the Galois group of L over F corresponding to K then the factors of $f(x)$ over K correspond to the orbits of H on the roots of $f(x)$. Then use Exercise 9 of Section 4.1.]