## 1. Homework 4 (Due: May 31)

**Problem 1.1** (Section 13.1, Exercise 2)**.** Show that $x^3 - 2x - 2$ is irreducible over $\mathbb{Q}$ and let $\theta$ be a root. Compute $(1+\theta)\left(1+\theta+\theta^2\right)$ and $\frac{1+\theta}{1+\theta+\theta^2}$ in $\mathbb{Q}(\theta)$.

**Problem 1.2** (Section 13.2, Exercise 7)**.** Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ [one inclusion is obvious, for the other consider $(\sqrt{2}+\sqrt{3})^2$, etc.]. Conclude that $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}] = 4$. Find an irreducible polynomial satisfied by $\sqrt{2}+\sqrt{3}$.

**Problem 1.3** (Section 13.2, Exercise 19)**.** Let $K$ be an extension of $F$ of degree $n$.

(1) For any $\alpha \in K$ prove that $\alpha$ acting by left multiplication on $K$ is an $F$-linear transformation of $K$.
(2) Prove that $K$ is isomorphic to a subfield of the ring of $n \times n$ matrices over $F$, so the ring of $n \times n$ matrices over $F$ contains an isomorphic copy of every extension of $F$ of degree $\leq n$.

**Problem 1.4** (Section 13.2, Exercise 20)**.** Show that if the matrix of the linear transformation "multiplication by $\alpha$" considered in the previous exercise is $A$ then $\alpha$ is a root of the characteristic polynomial for $A$. This gives an effective procedure for determining an equation of degree $n$ satisfied by an element $\alpha$ in an extension of $F$ of degree $n$. Use this procedure to obtain the monic polynomial of degree 3 satisfied by $\sqrt[3]{2}$ and by $1 + \sqrt[3]{2} + \sqrt[3]{4}$

**Problem 1.5** (Section 13.4, Exercise 5)**.** Let $K$ be a finite extension of $F$. Prove that $K$ is a splitting field over $F$ if and only if every irreducible polynomial in $F[x]$ that has a root in $K$ splits completely in $K[x]$. [Use Theorems 8 and 27.]

**Problem 1.6** (Section 13.4, Exercise 6)**.** Let $K_1$ and $K_2$ be finite extensions of $F$ contained in the field $K$, and assume both are splitting fields over $F$.

(1) Prove that their composite $K_1K_2$ is a splitting field over $F$.
(2) Prove that $K_1 \cap K_2$ is a splitting field over $F$. [Use the preceding exercise.]

**Problem 1.7.** Let $F$ be a field and let $E, E'$ be algebraic closures of $F$. Prove that there is an isomorphism $\sigma : E \to E'$ such that $\sigma|_F : F \to F$ is the identity map on $F$.

**Problem 1.8** (Section 13.5, Exercise 1)**.** Prove that the derivative $D_x$ of a polynomial satisfies $D_x(f(x) + g(x)) = D_x(f(x)) + D_x(g(x))$ and $D_x(f(x)g(x)) = D_x(f(x))g(x) + D_x(g(x))f(x)$ for any two polynomials $f(x)$ and $g(x)$.

**Problem 1.9** (Section 13.5, Exercise 7)**.** Suppose $K$ is a field of characteristic $p$ which is not a perfect field: $K \neq K^p$. Prove there exist irreducible inseparable polynomials over $K$. Conclude that there exist inseparable finite extensions of $K$.

**Problem 1.10** (Section 13.6, Exercise 6)**.** Prove that for $n$ odd, $n > 1$, $\Phi_{2n}(x) = \Phi_n(-x)$.