HOMEWORK

Contents

1. Homework 1 (Due: Apr 5)

Problem 1.1. Let A be a nonempty set and let k be a positive integer with $k \leq |A|$. The symmetric group S_A acts on the set B consisting of all subsets of A of cardinality k by

$$\sigma \cdot \{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}.$$

(1) Prove that this is a group action.
(2) Describe explicitly how the elements (1 2) and (1 2 3) act on the six 2-element subsets of $\{1, 2, 3, 4\}$.

Problem 1.2. Let H be a group acting on a set A. Prove that the relation $\sim$ on A defined by $a \sim b$ if and only if $a = hb$ for some $h \in H$ is an equivalence relation. (For each $x \in A$ the equivalence class of x under $\sim$ is called the orbit of x under the action of H. The orbits under the action of H partition the set A.)

Problem 1.3. In each of parts (1) to (5) give the number of nonisomorphic abelian groups of the specified order - do not list the groups:

(1) order 100
(2) order 576
(3) order 1155
(4) order 42875
(5) order 2704

Problem 1.4. In each of parts (1) to (5) give the lists of invariant factors for all abelian groups of the specified order:

(1) order 270
(2) order 9801
(3) order 320
(4) order 105
(5) order 44100

Problem 1.5. In each of parts (1) to (5) give the lists of elementary divisors for all abelian groups of the specified order and then match each list with the corresponding list of invariant factors found in the preceding problem:

(1) order 270
(2) order 9801
(3) order 320
(4) order 105
(5) order 44100

Problem 1.6. Let R be a ring with identity and let S be a subring of R containing the identity. Prove that if u is a unit in S then u is a unit in R. Show by example that the converse is false.

Problem 1.7. Let R be a ring with $1 \neq 0$.

(1) Prove that if a is a zero divisor, then it is not a unit.
(2) Prove that if $ab = ac$ and $a \neq 0$ is not a zero divisor, then $b = c$.

Problem 1.8. Assume R is commutative with $1 \neq 0$. Prove that if P is a prime ideal of R and P contains no zero divisors then R is an integral domain.

Problem 1.9. Let R be a ring with $1 \neq 0$. Let $A = (a_1, a_2, \ldots, a_n)$ be a nonzero finitely generated ideal of R. Prove that there is an ideal B which is maximal with respect to the property that it does not contain A. [Use Zorn's Lemma.]

Problem 1.10. Let $n_1, n_2, \ldots, n_k$ be integers which are relatively prime in pairs: $(n_i, n_j) = 1$ for all $i \neq j$.

(1) Show that the Chinese Remainder Theorem implies that for any $a_1, \dots, a_k \in \mathbb{Z}$ there is a solution $x \in \mathbb{Z}$ to the simultaneous congruences

$$x \equiv a_1 \bmod n_1, \quad x \equiv a_2 \bmod n_2, \quad \dots, \quad x \equiv a_k \bmod n_k$$

and that the solution x is unique $\bmod n = n_1 n_2 \dots n_k$.

(2) Let $n_i' = n/n_i$ be the quotient of n by n_i, which is relatively prime to n_i by assumption. Let t_i be the inverse of $n_i' \bmod n_i$. Prove that the solution x in (a) is given by

$$x = a_1 t_1 n_1' + a_2 t_2 n_2' + \cdots + a_k t_k n_k' \bmod n.$$

Note that the elements t_i can be quickly found by the Euclidean Algorithm as described in Section 2 of the Preliminaries chapter (writing $an_i + bn_i' = (n_i, n_i') = 1$ gives $t_i = b$) and that these then quickly give the solutions to the system of congruences above for any choice of $a_1, a_2, \dots, a_k$.

(3) Solve the simultaneous system of congruences

$$x \equiv 1 \bmod 8, \quad x \equiv 2 \bmod 25, \quad \text{and } x \equiv 3 \bmod 81$$

and the simultaneous system

$$y \equiv 5 \bmod 8, \quad y \equiv 12 \bmod 25, \quad \text{and} \quad y \equiv 47 \bmod 81$$

2. Homework 2 (Due: Apr 19)

For all problems, suppose that R is a ring with $1 \neq 0$ and M is a left R-module.

Problem 2.1. An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(1) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the torsion submodule of M).
(2) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule. [Consider the torsion elements in the R-module R.]
(3) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

Problem 2.2. (1) If N is a submodule of M, the *annihilator of N in R* is defined to be $\{r \in R \mid rn = 0 \text{ for all } n \in N\}$. Prove that the annihilator of N in R is a 2-sided ideal of R.
(2) If I is a right ideal of R, the *annihilator of I in M* is defined to be $\{\, m \in M \mid am = 0$ for all $a \in I\}$. Prove that the annihilator of I in M is a submodule of M.
(3) Let M be the abelian group (i.e., $\mathbb{Z}$-module) $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$.
   (a) Find the annihilator of M in $\mathbb{Z}$ (i.e., a generator for this principal ideal).
   (b) Let $I = 2\mathbb{Z}$. Describe the annihilator of I in M as a direct product of cyclic groups.

Problem 2.3. (1) Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise about the origin by $\pi/2$ radians. Show that V and 0 are the only $F[x]$-submodules for this T.
(2) Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is projection onto the y-axis. Show that $V, 0$, the x-axis and the y-axis are the only $F[x]-$submodules for this T.
(3) Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise about the origin by π radians. Show that every subspace of V is an $F[x]$-submodule for this T.

Problem 2.4. (1) For any left ideal I of R define

$$IM = \left\{ \sum_{\text{finite}} a_i m_i \mid a_i \in I, m_i \in M \right\}$$

to be the collection of all finite sums of elements of the form am where $a \in I$ and $m \in M$. Prove that IM is a submodule of M.
(2) Let $A_1, A_2, \ldots, A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, 2, \ldots, n$. Prove that

$$\left(A_1 \times \cdots \times A_n\right) / \left(B_1 \times \cdots \times B_n\right) \cong \left(A_1/B_1\right) \times \cdots \times \left(A_n/B_n\right).$$

(3) Let I be a left ideal of R and let n be a positive integer. Prove that

$$R^n/IR^n \cong R/IR \times \cdots \times R/IR \quad (n \text{ times}) \ .$$

(4) Assume R is commutative. Prove that $R^n \cong R^m$ if and only if $n = m$, i.e., two free R-modules of finite rank are isomorphic if and only if they have the same rank. [Apply the previous problem with I a maximal ideal of R. You may use the fact that if F is a field, then $F^n \cong F^m$ if and only if $n = m$.]

Problem 2.5. Let I be a nonempty index set and for each $i \in I$ let M_i be an R-module. The direct product of the modules M_i is defined to be their direct product as abelian groups (cf. Exercise 15 in Section 5.1) with the action of R componentwise multiplication. The direct sum of the modules M_i is defined to be the restricted direct product of the abelian groups M_i (cf. Exercise 17 in Section 5.1) with the action of R componentwise multiplication. In other words, the direct sum of the M_i's is the subset of the direct product, $\prod_{i \in I} M_i$, which consists of all elements $\prod_{i \in I} m_i$ such that only finitely many of the components m_i are nonzero; the action of R on the direct product or direct sum is given by $r \prod_{i \in I} m_i = \prod_{i \in I} r m_i$ (cf. Appendix I for the definition of Cartesian products of infinitely many sets). The direct sum will be denoted by $\oplus_{i \in I} M_i$.

(1) Prove that the direct product of the M_i's is an R-module and the direct sum of the M_i's is a submodule of their direct product.
(2) Show that if $R = \mathbb{Z}, I = \mathbb{Z}^+$and M_i is the cyclic group of order i for each i, then the direct sum of the M_i 's is not isomorphic to their direct product. [Look at torsion.]

Problem 2.6. (1) Show that the element "$2 \otimes 1$" is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.
(2) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.
(3) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]
(4) If R is any integral domain with quotient field Q, prove that $(Q/R) \otimes_R (Q/R) = 0$.
(5) Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

Problem 2.7. Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \ldots, e_n$.

(1) For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.
(2) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R linearly independent then it is not necessarily true that all the m_i are 0 . [Consider $R = \mathbb{Z}, n = 1, M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.

Problem 2.8. Suppose that

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\alpha"'] & B \arrow[r, "\varphi"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\ A' \arrow[r, "\psi'"] & B' \arrow[r, "\varphi'"] & C' \end{tikzcd}$$

is a commutative diagram of groups and that the rows are exact. Prove that

(1) if φ and α are surjective, and β is injective then γ is injective. [If $c \in \ker \gamma$, show there is a $b \in B$ with $\varphi(b) = c$. Show that $\varphi'(\beta(b)) = 0$ and deduce that $\beta(b) = \psi'(a')$ for some $a' \in A'$. Show there is an $a \in A$ with $\alpha(a) = a'$ and that $\beta(\psi(a)) = \beta(b)$. Conclude that $b = \psi(a)$ and hence $c = \varphi(b) = 0$.]
(2) if ψ', α, and γ are injective, then β is injective,
(3) if φ, α, and γ are surjective, then β is surjective,
(4) if β is injective, α and γ are surjective, then γ is injective,
(5) if β is surjective, γ and ψ' are injective, then α is surjective.

Problem 2.9. (1) Let P_1 and P_2 be R-modules. Prove that $P_1 \oplus P_2$ is a projective R-module if and only if both P_1 and P_2 are projective.
(2) Let Q_1 and Q_2 be R-modules. Prove that $Q_1 \oplus Q_2$ is an injective R-module if and only if both Q_1 and Q_2 are injective.
(3) Let A_1 and A_2 be R-modules. Prove that $A_1 \oplus A_2$ is a flat R-module if and only if both A_1 and A_2 are flat. More generally, prove that an arbitrary direct sum $\sum A_i$ of R-modules is flat if and only if each A_i is flat. [Use the fact that tensor product commutes with arbitrary direct sums.]

Problem 2.10. Let $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ be a sequence of R-modules.

(1) Prove that the associated sequence

$$0 \longrightarrow \operatorname{Hom}_R(D, L) \xrightarrow{\psi'} \operatorname{Hom}_R(D, M) \xrightarrow{\varphi'} \operatorname{Hom}_R(D, N) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence. [To show the sequence splits, take $D = N$ and show the lift of the identity map in $\mathrm{Hom}_R(N, N)$ to $\mathrm{Hom}_R(N, \boldsymbol{M})$ is a splitting homomorphism for φ.]

(2) Prove that the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence.