

HOMEWORK

CONTENTS

1. Homework 1 (Due: Sep 21)	2
2. Homework 2 (Due: Oct 5)	3
3. Homework 3 (Due: Oct 19)	4

1. HOMEWORK 1 (DUE: SEP 21)

Problem 1.1. Let \mathcal{L} be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n \geq 0}$. Prove the following extremal property: for any monic real polynomial $\pi(x) \neq P_n(x)$ of degree n ,

$$\mathcal{L}(P_n(x)^2) < \mathcal{L}(\pi(x)^2).$$

Problem 1.2. Let \mathcal{L} be a linear functional such that $\Delta_n \neq 0$ for all $n \geq 0$. Prove that if $\pi(x)$ is a polynomial such that $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \geq 0$, then $\pi(x) = 0$.

Problem 1.3. The *Tchebyshev polynomials of the second kind* $U_n(x)$ are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad x = \cos \theta, \quad n \geq 0.$$

- (1) Prove that $U_n(x)$ is a polynomial of degree n .
- (2) Prove that

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n \geq 0,$$

where $U_{-1}(x) = 0$ and $U_0(x) = 1$.

- (3) Prove that

$$\int_{-1}^1 U_m(x)U_n(x)(1-x^2)^{1/2}dx = \frac{\pi}{2}\delta_{m,n}.$$

- (4) Find the 3-term recurrence for the normalized Tchebyshev polynomials of the second kind. More precisely, find the numbers b_n and λ_n such that

$$\hat{U}_{n+1}(x) = (x - b_n)\hat{U}_n(x) - \lambda_n\hat{U}_{n-1}(x), \quad n \geq 0,$$

where $\hat{U}_n(x)$ is the monic polynomial that is a scalar multiple of $U_n(x)$.

Problem 1.4. Let $\{P_n(x)\}_{n \geq 0}$ be the monic OPS for a linear functional \mathcal{L} with three-term recurrence

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \quad n \geq 0.$$

- (1) Prove that

$$P_n(x) = \begin{vmatrix} x - b_0 & 1 & & 0 \\ \lambda_1 & x - b_1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & \lambda_{n-1} & x - b_{n-1} \end{vmatrix}.$$

- (2) Prove that

$$P_n(x) = \begin{vmatrix} x - b_0 & \sqrt{\lambda_1} & & 0 \\ \sqrt{\lambda_1} & x - b_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\lambda_{n-1}} \\ 0 & & \sqrt{\lambda_{n-1}} & x - b_{n-1} \end{vmatrix}.$$

- (3) Using (2) prove that if $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all, then $P_n(x)$ has real roots only.

2. HOMEWORK 2 (DUE: OCT 5)

Problem 2.1. Let id be the identity permutation.

- (1) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^2 = id$.
- (2) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^3 = id$.
- (3) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^4 = id$.
- (4) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^5 = id$.
- (5) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^6 = id$.

Problem 2.2. Let c_1, \dots, c_n be a sequence of nonnegative integers such that $\sum_{i=1}^n ic_i = n$. Show that the number of permutations $\pi \in \mathfrak{S}_n$ with c_i cycles of length i for all $i = 1, \dots, n$ is

$$\frac{n!}{\prod_{i=1}^n i^{c_i} c_i!}.$$

Problem 2.3. For $\pi \in \mathfrak{S}_n$, let $\ell(\pi)$ be the smallest number of simple transpositions whose product is π . Prove that $\ell(\pi) = \text{inv}(\pi)$.

Problem 2.4. Prove that

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}(\pi)} = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}).$$

3. HOMEWORK 3 (DUE: OCT 19)

Problem 3.1. Suppose that $\{P_n(x)\}_{n \geq 0}$ is a monic OPS for a linear functional \mathcal{L} with $\mathcal{L}(1) = 1$ given by $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \geq 0$,

$$P_{n+1}(x) = (x - n)P_n(x) - nP_{n-1}(x).$$

Compute the following.

- (1) $\mathcal{L}(x^3)$
- (2) $\mathcal{L}(P_{10}(x)P_{10}(x))$
- (3) $\mathcal{L}(x^3P_{10}(x)P_{12}(x))$

Problem 3.2. A *left-to-right minimum* of a permutation $\pi = \pi_1 \cdots \pi_n$ is a number π_i such that $\pi_i = \min\{\pi_1, \dots, \pi_i\}$. Let $\text{LRmin}(\pi)$ denote the number of left-to-right minima in π . For example, if $\pi = 6741352$, then the left-to-right minima are 6, 4, 1, hence $\text{LRmin}(\pi) = 3$. Prove that

$$\sum_{\pi \in \mathfrak{S}_n} \alpha^{\text{cycle}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} \alpha^{\text{LRmin}(\pi)}.$$

Problem 3.3. Suppose that $\{P_n(x)\}_{n \geq 0}$ is a monic OPS given by $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \geq 0$,

$$P_{n+1}(x) = (x - 1)P_n(x) - nP_{n-1}(x).$$

Prove that μ_n is equal to the number of involutions in \mathfrak{S}_n . (An involution is a permutation π such that π^2 is the identity map.)

Problem 3.4. Suppose that $\{P_n(x)\}_{n \geq 0}$ is a monic OPS given by $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \geq 0$,

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

where $\lambda_n \neq 0$ for all $n \geq 1$.

Using the fact $\mu_n = \sum_{\pi \in \text{Motz}_n} \text{wt}(\pi)$, prove that $\mu_{2n+1} = 0$ for all $n \geq 0$ if and only if $b_n = 0$ for all $n \geq 0$.