HOMEWORK

Contents

1. Homework 1 (Due: Apr 5)

Problem 1.1. Let A be a nonempty set and let k be a positive integer with $k \leq |A|$. The symmetric group S_A acts on the set B consisting of all subsets of A of cardinality k by

$$\sigma \cdot \{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}.$$

(1) Prove that this is a group action.
(2) Describe explicitly how the elements (1 2) and (1 2 3) act on the six 2-element subsets of $\{1, 2, 3, 4\}$.

Solution.

(1) Since $\sigma \in S_A$ is bijective, $\sigma \cdot \{a_1, a_2, \cdots, a_k\} = \{\sigma(a_1), \sigma(a_2), \cdots, \sigma(a_k)\}$ is a subset of A with cardinality k so in B.

Since S_A is a group, $\sigma, \tau \in S_A$ implies that $\sigma\tau \in S_A$ and,

$$\sigma(\tau \cdot \{a_1, a_2, \cdots, a_k\}) = \sigma \cdot \{\tau(a_1), \tau(a_2), \cdots, \tau(a_k)\} = \{\sigma\tau(a_1), \sigma\tau(a_2), \cdots, \sigma\tau(a_k)\} = (\sigma\tau) \cdot \{a_1, a_2, \cdots, a_k\}$$

for all $\sigma, \tau \in S_A$ and all $\{a_1, a_2, \cdots, a_k\} \in B$.

Also, $1 = id \in S_A$ with $id \cdot \{a_1, a_2, \cdots, a_k\} = \{a_1, a_2, \cdots, a_k\}$ for all $\{a_1, a_2, \cdots, a_k\} \in B$.

Hence the acting of S_A on B is a group action.

(2)

acting (1,2)	acting (1 2 3)
$\{1,2\} \mapsto \{1,2\}$	$\{1,2\} \mapsto \{2,3\}$
$\{1,3\} \mapsto \{2,3\}$	$\{1,3\} \mapsto \{1,2\}$
$\{1,4\} \mapsto \{2,4\}$	$\{1,4\} \mapsto \{2,4\}$
$\{2,3\} \mapsto \{1,3\}$	$\{2,3\} \mapsto \{1,3\}$
$\{2,4\} \mapsto \{1,4\}$	$\{2,4\} \mapsto \{3,4\}$
$\{3,4\} \mapsto \{3,4\}$	$\{3,4\} \mapsto \{1,4\}$

□

Problem 1.2. Let H be a group acting on a set A. Prove that the relation $\sim$ on A defined by $a \sim b$ if and only if $a = hb$ for some $h \in H$ is an equivalence relation. (For each $x \in A$ the equivalence class of x under $\sim$ is called the orbit of x under the action of H. The orbits under the action of H partition the set A.)

Solution.

W.T.S $a \sim b$ defined by $a = hb$ for some $h \in H$ is an equiv. relation.

(reflexive) $a \sim a$ by 1_h $\because 1_h \cdot a = a$

(symmetric) $a \sim b$ if and only if $b \sim a$ by inverse element: $a = h \cdot b \leftrightarrow b = h^{-1} \cdot a$

(transitive) If $a \sim b$ and $b \sim c$, then $a \sim c$ by group axiom on H: $a = hb$, $b = h'c \rightarrow a = h \cdot (h'c) = (hh') \cdot c$, $hh' \in H$.

Hence $\sim$ is the equivalence relation.

□

Problem 1.3. In each of parts (1) to (5) give the number of nonisomorphic abelian groups of the specified order - do not list the groups:

(1) order 100
(2) order 576
(3) order 1155
(4) order 42875
(5) order 2704

Solution.

1.3 $P(n)$ is a partition number of n.

(1) $100 = 2^2 \cdot 5^2$, $P(2) \cdot P(2) = 2 \cdot 2 = 4$.
4.

(2) $576 = 2^6 \cdot 3^2$, $P(6) \cdot P(2) = 11 \cdot 2 = 22$
22.

(3) $1155 = 3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1$, $P(1)^4 = 1$.
1.

(4) $42875 = 5^3 \cdot 7^3$, $P(3)^2 = 3^2 = 9$.
9.

(5) $2704 = 2^4 \cdot 13^2$, $P(4) \cdot P(2) = 5 \cdot 2 = 10$
10.

□

Problem 1.4. In each of parts (1) to (5) give the lists of invariant factors for all abelian groups of the specified order:

(1) order 270
(2) order 9801
(3) order 320
(4) order 105
(5) order 44100

Solution.

(1) $270 = 2\times3^3\times5$ ⟶ $P(1)\times P(3)\times P(1) = 1\times3\times1 = 3$
(2) $9801 = 99^2 = 3^4\times11^2$ ⟶ $P(4)\times P(2) = 5\times2 = 10$
(3) $320 = 2^6\times5$ ⟶ $P(6)\times P(1) = 11\times1 = 11$
(4) $105 = 3\times5\times7$ ⟶ $P(1)\times P(1)\times P(1) = 1\times1\times1 = 1$
(5) $44100 = 210^2 = 2^2\times3^2\times5^2\times7^2$ ⟶ $P(2)\times P(2)\times P(2)\times P(2) = 2\times2\times2\times2 = 16$

Order	invariant factors		Order	invariant factors	
(1) 270	$2\cdot3^3\cdot5$	: $\mathbb{Z}_{270}$	(4) 105	$3\cdot5\cdot7$	: $\mathbb{Z}_{105}$
	$2\cdot3^2\cdot5,\ 3$	: $\mathbb{Z}_{90}\times\mathbb{Z}_3$			
	$2\cdot3\cdot5,\ 3,\ 3$	: $\mathbb{Z}_{30}\times\mathbb{Z}_3\times\mathbb{Z}_3$			
(2) 9801	$3^4\cdot11^2$	: $\mathbb{Z}_{9801}$	(5) 44100	$2^2\cdot3^2\cdot5^2\cdot7^2$	: $\mathbb{Z}_{44100}$
	$3^3\cdot11^2,\ 3$	: $\mathbb{Z}_{3267}\times\mathbb{Z}_3$		$2\cdot3^2\cdot5^2\cdot7^2,\ 2$	: $\mathbb{Z}_{22050}\times\mathbb{Z}_2$
	$3^2\cdot11^2,\ 3^2$	: $\mathbb{Z}_{1089}\times\mathbb{Z}_9$		$2^2\cdot3\cdot5^2\cdot7^2,\ 3$	: $\mathbb{Z}_{14700}\times\mathbb{Z}_3$
	$3^2\cdot11^2,\ 3,\ 3$	: $\mathbb{Z}_{1089}\times\mathbb{Z}_3\times\mathbb{Z}_3$		$2^2\cdot3^2\cdot5\cdot7^2,\ 5$	: $\mathbb{Z}_{8820}\times\mathbb{Z}_5$
	$3\cdot11^2,\ 3,\ 3,\ 3$	: $\mathbb{Z}_{363}\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$		$2^2\cdot3^2\cdot5^2\cdot7,\ 7$	: $\mathbb{Z}_{6300}\times\mathbb{Z}_7$
	$3^4\cdot11,\ 11$	: $\mathbb{Z}_{891}\times\mathbb{Z}_{11}$		$2\cdot3\cdot5^2\cdot7^2,\ 2\cdot3$	: $\mathbb{Z}_{1050}\times\mathbb{Z}_6$
	$3^3\cdot11,\ 3\cdot11$	: $\mathbb{Z}_{297}\times\mathbb{Z}_{33}$		$2\cdot3^2\cdot5\cdot7^2,\ 2\cdot5$	: $\mathbb{Z}_{490}\times\mathbb{Z}_{10}$
	$3^2\cdot11,\ 3^2\cdot11$	: $\mathbb{Z}_{99}\times\mathbb{Z}_{99}$		$2\cdot3^2\cdot5^2\cdot7,\ 2\cdot7$	: $\mathbb{Z}_{210}\times\mathbb{Z}_{14}$
	$3^2\cdot11,\ 3\cdot11,\ 3$	: $\mathbb{Z}_{99}\times\mathbb{Z}_{33}\times\mathbb{Z}_3$		$2^2\cdot3\cdot5\cdot7^2,\ 3\cdot5$	: $\mathbb{Z}_{420}\times\mathbb{Z}_{15}$
	$3\cdot11,\ 3\cdot11,\ 3,3$	: $\mathbb{Z}_{33}\times\mathbb{Z}_{33}\times\mathbb{Z}_3\times\mathbb{Z}_3$		$2^2\cdot3\cdot5^2\cdot7,\ 3\cdot7$	: $\mathbb{Z}_{300}\times\mathbb{Z}_{21}$
(3) 320	$2^6\cdot5$	: $\mathbb{Z}_{320}$		$2^2\cdot3^2\cdot5\cdot7,\ 5\cdot7$	: $\mathbb{Z}_{180}\times\mathbb{Z}_{35}$
	$2^5\cdot5,\ 2$	: $\mathbb{Z}_{160}\times\mathbb{Z}_2$		$2\cdot3\cdot5\cdot7^2,\ 2\cdot3\cdot5$	: $\mathbb{Z}_{1470}\times\mathbb{Z}_{30}$
	$2^4\cdot5,\ 2^2$	: $\mathbb{Z}_{80}\times\mathbb{Z}_4$		$2\cdot3\cdot5^2\cdot7,\ 2\cdot3\cdot7$	: $\mathbb{Z}_{1050}\times\mathbb{Z}_{42}$
	$2^3\cdot5,\ 2^3$	: $\mathbb{Z}_{40}\times\mathbb{Z}_8$		$2\cdot3^2\cdot5\cdot7,\ 2\cdot5\cdot7$	: $\mathbb{Z}_{630}\times\mathbb{Z}_{70}$
	$2^4\cdot5,\ 2,\ 2$	: $\mathbb{Z}_{80}\times\mathbb{Z}_2\times\mathbb{Z}_2$		$2^2\cdot3\cdot5\cdot7,\ 3\cdot5\cdot7$	: $\mathbb{Z}_{420}\times\mathbb{Z}_{105}$
	$2^3\cdot5,\ 2^2,\ 2$	: $\mathbb{Z}_{40}\times\mathbb{Z}_4\times\mathbb{Z}_2$		$2\cdot3\cdot5\cdot7,\ 2\cdot3\cdot5\cdot7$	: $\mathbb{Z}_{210}\times\mathbb{Z}_{210}$
	$2^2\cdot5,\ 2^2,\ 2^2$	: $\mathbb{Z}_{20}\times\mathbb{Z}_4\times\mathbb{Z}_4$			
	$2^3\cdot5,\ 2,\ 2,\ 2$	: $\mathbb{Z}_{40}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$			
	$2^2\cdot5,\ 2^2,\ 2,\ 2$	: $\mathbb{Z}_{20}\times\mathbb{Z}_{44}\times\mathbb{Z}_2\times\mathbb{Z}_2$			
	$2^2\cdot5,\ 2,\ 2,\ 2,\ 2$	: $\mathbb{Z}_{20}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$			
	$2\cdot5,\ 2,\ 2,\ 2,\ 2,2$	: $\mathbb{Z}_{10}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$			

□

Problem 1.5. In each of parts (1) to (5) give the lists of elementary divisors for all abelian groups of the specified order and then match each list with the corresponding list of invariant factors found in the preceding problem:

(1) order 270
(2) order 9801
(3) order 320
(4) order 105
(5) order 44100

Solution.

(1) $210 = 2\times3\times5\times7$
(2) $9801 = 3^4\times11^2$
(3) $320 = 2^6\times5$
(4) $105 = 3\times5\times7$
(5) $44100 = 2^2\times3^2\times5^2\times7^2$

Order	elementary divisors			invariant factors
(1) 210	$2, 3, 5, 7$	$\cong \mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_7$	$\cong \mathbb{Z}_{210}$	$2\cdot3\cdot5\cdot7$
	$2, 3, 5, 7$	$\cong \mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_7$	$\cong \mathbb{Z}_{30}\times\mathbb{Z}_7$	$2\cdot3\cdot5, 7$
	$2, 3, 5, 7$	$\cong \mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_7$	$\cong \mathbb{Z}_{30}\times\mathbb{Z}_7$	$2\cdot3\cdot5, 7$
(2) 9801	$3^4, 11^2$	$\cong \mathbb{Z}_{81}\times\mathbb{Z}_{121}$	$\cong \mathbb{Z}_{9801}$	$3^4\cdot11^2$
	$3^3, 3, 11^2$	$\cong \mathbb{Z}_{27}\times\mathbb{Z}_3\times\mathbb{Z}_{121}$	$\cong \mathbb{Z}_{3267}\times\mathbb{Z}_3$	$3^3\cdot11^2, 3$
	$3^2, 3^2, 11^2$	$\cong \mathbb{Z}_9\times\mathbb{Z}_9\times\mathbb{Z}_{121}$	$\cong \mathbb{Z}_{1089}\times\mathbb{Z}_9$	$3^2\cdot11^2, 3^2$
	$3^2, 3, 3, 11^2$	$\cong \mathbb{Z}_9\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{121}$	$\cong \mathbb{Z}_{1089}\times\mathbb{Z}_3\times\mathbb{Z}_3$	$3^2\cdot11^2, 3, 3$
	$3, 3, 3, 3, 11^2$	$\cong \mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{121}$	$\cong \mathbb{Z}_{363}\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	$3\cdot11^2, 3, 3, 3$
	$3^4, 11, 11$	$\cong \mathbb{Z}_{81}\times\mathbb{Z}_{11}\times\mathbb{Z}_{11}$	$\cong \mathbb{Z}_{891}\times\mathbb{Z}_{11}$	$3^4\cdot11, 11$
	$3^3, 3, 11, 11$	$\cong \mathbb{Z}_{27}\times\mathbb{Z}_3\times\mathbb{Z}_{11}\times\mathbb{Z}_{11}$	$\cong \mathbb{Z}_{297}\times\mathbb{Z}_{33}$	$3^3\cdot11, 3\cdot11$
	$3^2, 3^2, 11, 11$	$\cong \mathbb{Z}_9\times\mathbb{Z}_9\times\mathbb{Z}_{11}\times\mathbb{Z}_{11}$	$\cong \mathbb{Z}_{99}\times\mathbb{Z}_{99}$	$3^2\cdot11, 3^2\cdot11$
	$3^2, 3, 3, 11, 11$	$\cong \mathbb{Z}_9\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{11}\times\mathbb{Z}_{11}$	$\cong \mathbb{Z}_{99}\times\mathbb{Z}_{33}\times\mathbb{Z}_3$	$3^2\cdot11, 3\cdot11, 3$
	$3, 3, 3, 3, 11, 11$	$\cong \mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_{11}\times\mathbb{Z}_{11}$	$\cong \mathbb{Z}_{33}\times\mathbb{Z}_{33}\times\mathbb{Z}_3\times\mathbb{Z}_3$	$3\cdot11, 3\cdot11, 3, 3$
(3) 320	$2^6, 5$	$\cong \mathbb{Z}_{64}\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{320}$	$2^6\cdot5$
	$2^5, 2, 5$	$\cong \mathbb{Z}_{32}\times\mathbb{Z}_2\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{160}\times\mathbb{Z}_2$	$2^5\cdot5, 2$
	$2^4, 2^2, 5$	$\cong \mathbb{Z}_{16}\times\mathbb{Z}_4\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{80}\times\mathbb{Z}_4$	$2^4\cdot5, 2^2$
	$2^3, 2^3, 5$	$\cong \mathbb{Z}_8\times\mathbb{Z}_8\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{40}\times\mathbb{Z}_8$	$2^3\cdot5, 2^3$
	$2^4, 2, 2, 5$	$\cong \mathbb{Z}_{16}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{80}\times\mathbb{Z}_2\times\mathbb{Z}_2$	$2^4\cdot5, 2, 2$
	$2^3, 2^2, 2, 5$	$\cong \mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{40}\times\mathbb{Z}_4\times\mathbb{Z}_2$	$2^3\cdot5, 2^2, 2$
	$2^2, 2^2, 2^2, 5$	$\cong \mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{20}\times\mathbb{Z}_4\times\mathbb{Z}_4$	$2^2\cdot5, 2^2, 2^2$
	$2^3, 2, 2, 2, 5$	$\cong \mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{40}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$2^3\cdot5, 2, 2, 2$
	$2^2, 2^2, 2, 2, 5$	$\cong \mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{20}\times\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2$	$2^2\cdot5, 2^2, 2, 2$
	$2^2, 2, 2, 2, 2, 5$	$\cong \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{20}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$2^2\cdot5, 2, 2, 2, 2$
	$2, 2, 2, 2, 2, 2, 5$	$\cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5$	$\cong \mathbb{Z}_{10}\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$2\cdot5, 2, 2, 2, 2, 2$
(4) 105	$3, 5, 7$	$\cong \mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_7$	$\cong \mathbb{Z}_{105}$	$3\cdot5\cdot7$
(5) 44100	$2^2, 3^2, 5^2, 7^2$	$\cong \mathbb{Z}_{49}\times\mathbb{Z}_{25}\times\mathbb{Z}_9\times\mathbb{Z}_4$	$\cong \mathbb{Z}_{44100}$	$2^2\cdot3^2\cdot5^2\cdot7^2$
	$2, 3^2, 5^2, 7^2, 2$	$\cong \mathbb{Z}_{49}\times\mathbb{Z}_{25}\times\mathbb{Z}_9\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\cong \mathbb{Z}_{22050}\times\mathbb{Z}_2$	$2\cdot3^2\cdot5^2\cdot7^2, 2$
	$2^2, 3, 5^2, 7^2, 3$	$\cong \mathbb{Z}_{49}\times\mathbb{Z}_{25}\times\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_3$	$\cong \mathbb{Z}_{14700}\times\mathbb{Z}_3$	$2^2\cdot3\cdot5^2\cdot7^2, 3$
	$2^2, 3^2, 5, 7^2, 5$	$\cong \mathbb{Z}_{49}\times\mathbb{Z}_9\times\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_4$	$\cong \mathbb{Z}_{8820}\times\mathbb{Z}_5$	$2^2\cdot3^2\cdot5\cdot7^2, 5$
	$2^2, 3^2, 5^2, 7, 7$	$\cong \mathbb{Z}_{25}\times\mathbb{Z}_9\times\mathbb{Z}_7\times\mathbb{Z}_7\times\mathbb{Z}_4$	$\cong \mathbb{Z}_{4200}\times\mathbb{Z}_7$	$2^2\cdot3\cdot5^2\cdot7, 7$
	$2, 3, 5^2, 7^2, 2, 3$	$\cong \mathbb{Z}_{49}\times\mathbb{Z}_{25}\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\cong \mathbb{Z}_{3150}\times\mathbb{Z}_6$	$2\cdot3\cdot5^2\cdot7^2, 2\cdot3$
	$2, 3^2, 5, 7^2, 2, 5$	$\cong \mathbb{Z}_{49}\times\mathbb{Z}_9\times\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\cong \mathbb{Z}_{4410}\times\mathbb{Z}_{10}$	$2\cdot3^2\cdot5^2\cdot7^2, 2\cdot5$
	$2, 3^2, 5^2, 7, 2, 7$	$\cong \mathbb{Z}_{25}\times\mathbb{Z}_9\times\mathbb{Z}_7\times\mathbb{Z}_7\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\cong \mathbb{Z}_{3150}\times\mathbb{Z}_{14}$	$2\cdot3^2\cdot5^2\cdot7, 2\cdot7$
	$2^2, 3, 5, 7^2, 3, 5$	$\cong \mathbb{Z}_{49}\times\mathbb{Z}_4\times\mathbb{Z}_9\times\mathbb{Z}_{25}\times\mathbb{Z}_3\times\mathbb{Z}_3$	$\cong \mathbb{Z}_{2940}\times\mathbb{Z}_{15}$	$2^2\cdot3\cdot5\cdot7^2, 3\cdot5$
	$2^2, 3, 5^2, 7, 3, 7$	$\cong \mathbb{Z}_{25}\times\mathbb{Z}_4\times\mathbb{Z}_9\times\mathbb{Z}_7\times\mathbb{Z}_4\times\mathbb{Z}_3\times\mathbb{Z}_3$	$\cong \mathbb{Z}_{2100}\times\mathbb{Z}_{21}$	$2^2\cdot3\cdot5^2\cdot7, 3\cdot7$
	$2^2, 3^2, 5, 7, 5, 7$	$\cong \mathbb{Z}_4\times\mathbb{Z}_9\times\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_7\times\mathbb{Z}_7\times\mathbb{Z}_4$	$\cong \mathbb{Z}_{2100}\times\mathbb{Z}_{35}$	$2^2\cdot3\cdot5\cdot7^2, 5\cdot7$
	$2, 3, 5, 7^2, 2, 3, 5$	$\cong \mathbb{Z}_{49}\times\mathbb{Z}_{25}\times\mathbb{Z}_9\times\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\cong \mathbb{Z}_{44100}\times\mathbb{Z}_{30}$	$2\cdot3\cdot5\cdot7^2, 2\cdot3\cdot5$
	$2, 3, 5^2, 7, 2, 3, 7$	$\cong \mathbb{Z}_{25}\times\mathbb{Z}_9\times\mathbb{Z}_7\times\mathbb{Z}_7\times\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\cong \mathbb{Z}_{44100}\times\mathbb{Z}_{42}$	$2\cdot3\cdot5^2\cdot7^2, 2\cdot3\cdot7$
	$2, 2^2, 5^2, 7, 2, 5, 7$	$\cong \mathbb{Z}_9\times\mathbb{Z}_4\times\mathbb{Z}_{25}\times\mathbb{Z}_7\times\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\cong \mathbb{Z}_{4200}\times\mathbb{Z}_{210}$	$2^2\cdot3\cdot5^2\cdot7, 2\cdot5\cdot7$
	$2^2, 3, 5^2, 7^2, 3, 5, 7$	$\cong \mathbb{Z}_4\times\mathbb{Z}_9\times\mathbb{Z}_{25}\times\mathbb{Z}_{49}\times\mathbb{Z}_5\times\mathbb{Z}_7\times\mathbb{Z}_3$	$\cong \mathbb{Z}_{44100}\times\mathbb{Z}_{105}$	$2^2\cdot3\cdot5^2\cdot7^2, 3\cdot5\cdot7$
	$2, 3, 5^2, 7^2, 2, 3, 5, 7$	$\cong \mathbb{Z}_4\times\mathbb{Z}_9\times\mathbb{Z}_{25}\times\mathbb{Z}_{49}\times\mathbb{Z}_5\times\mathbb{Z}_7\times\mathbb{Z}_2\times\mathbb{Z}_3$	$\cong \mathbb{Z}_{210}\times\mathbb{Z}_{210}$	$2\cdot3\cdot5\cdot7, 2\cdot3\cdot5\cdot7$

□

Problem 1.6. Let R be a ring with identity and let S be a subring of R containing the identity. Prove that if u is a unit in S then u is a unit in R. Show by example that the converse is false.

Solution.

If u is a unit in S, then by the definition, $\exists v \in S$, s.t. $uv = vu = 1$. Since $u, v \in S$ and S is a subset of R, we have $u, v \in R$. Thus for $u \in R$, $\exists v \in R$, s.t. $uv = vu = 1$, which means u is a unit in R.

Converse: If u is a unit in R then u is a unit in S.

Consider the ring $R = \mathbb{Q}$ with identity 1. $S = \mathbb{Z}$ is a subring of $\mathbb{Q}$ containing the identity 1.

Notice 2 is a unit in R, but it is not a unit in S since $\mathbb{Z}^\times = \{\pm 1\}$. $\Rightarrow$ The converse is false.

□

Problem 1.7. Let R be a ring with $1 \neq 0$.

(1) Prove that if a is a zero divisor, then it is not a unit.
(2) Prove that if $ab = ac$ and $a \neq 0$ is not a zero divisor, then $b = c$.

Solution.

(1) Since a is a zero divisor, $\exists b \in R \setminus \{0\}$ s.t. $ab = 0$ or $ba = 0$, WOLG $ab = 0$.
Sps that a is a unit i.e. $\exists a^{-1} \in \mathbb{Q}$ s.t. $aa^{-1} = a^{-1}a = 1$.
$ab = 0 \Rightarrow a^{-1}(ab) = a^{-1} \cdot 0 = 0$ so that $b = 0$. However, $b \neq 0$.
$(a^{-1}a)b$ Therefore a is not a unit.

(2) $ab = ac \Rightarrow ab - ac = 0$ by distributive law.
$a(b-c) = 0$ Since a is not a zero divisor.
$b - c = 0 \Rightarrow b = c$.

□

Problem 1.8. Assume R is commutative with $1 \neq 0$. Prove that if P is a prime ideal of R and P contains no zero divisors then R is an integral domain.

Solution.

Sps that R is not an integral domain. i.e. $\exists a, b \in R$ s.t. $ab = 0$ ... (*)
$\neq 0$
Since P is an abelian group in addition., $0 \in P$. and $ab = 0 \in P$.
from the fact that P is an ideal.
Also, since P is a prime ideal, $a \in P$ or $b \in P$. WOLG let $a \in P$.
However, a is a zero divisor that not in P. $\not\in$.
Therefore R is an integral domain.

□

Problem 1.9. Let R be a ring with $1 \neq 0$. Let $A = (a_1, a_2, \ldots, a_n)$ be a nonzero finitely generated ideal of R. Prove that there is an ideal B which is maximal with respect to the property that it does not contain A. [Use Zorn's Lemma.]

Solution.

Let J be ideals s.t. $A \nsubseteq J$.
Let S be a collection of all J's.
Note that $(S, \subseteq)$: poset.
Also, since $\{0\}$ is ideal of R, $\{0\} \in S$. $\therefore S$ is nonempty.
Let B_i's are subset of S and $B = \bigcup_{i \in I} B_i$.
s.t. $B_1 \subseteq B_2 \subseteq \cdots \subseteq B$

To show ① : B is ideal.
Suppose that B is not ideal.
then $\exists\, a \in R$ s.t. $aB \nsubseteq B$ or $Ba \nsubseteq B$.
WLOG, $aB \nsubseteq B$. then $\exists b \in B$ s.t. $ab \notin B$.
Since $b \in B$, then $\exists i$ s.t. $b \in B_i$.
Since B_i is ideal, $ab \in B_i \subseteq B$. ↯ contradiction.
∴ B is ideal. $\Rightarrow B \subseteq S$.
Hence, B is upperbound in S.

To show ② : $A \nsubseteq B$
Suppose that $A \subseteq B$.
Let $a_j \in B_{i_j}$. for $j = 1, 2, \cdots, n$.
let $M := \max\{i_1, i_2, \cdots, i_n\}$.
Then $a_1, \cdots, a_n \in B_M \subseteq S$. →← contradiction.
∴ $A \nsubseteq B$.

Therefore by Zorn's Lemma,
There is an maximal ideal that does not contain A.

□

Problem 1.10. Let $n_1, n_2, \ldots, n_k$ be integers which are relatively prime in pairs: $(n_i, n_j) = 1$ for all $i \neq j$.

(1) Show that the Chinese Remainder Theorem implies that for any $a_1, \ldots, a_k \in \mathbb{Z}$ there is a solution $x \in \mathbb{Z}$ to the simultaneous congruences

$$x \equiv a_1 \bmod n_1, \quad x \equiv a_2 \bmod n_2, \quad \ldots, \quad x \equiv a_k \bmod n_k$$

and that the solution x is unique $\bmod n = n_1 n_2 \dots n_k$.

(2) Let $n_i' = n/n_i$ be the quotient of n by n_i, which is relatively prime to n_i by assumption. Let t_i be the inverse of n_i' mod n_i. Prove that the solution x in (a) is given by

$$x = a_1 t_1 n_1' + a_2 t_2 n_2' + \cdots + a_k t_k n_k' \bmod n.$$

Note that the elements t_i can be quickly found by the Euclidean Algorithm as described in Section 2 of the Preliminaries chapter (writing $an_i + bn_i' = (n_i, n_i') = 1$ gives $t_i = b$) and that these then quickly give the solutions to the system of congruences above for any choice of $a_1, a_2, \dots, a_k$.

(3) Solve the simultaneous system of congruences

$$x \equiv 1 \bmod 8, \quad x \equiv 2 \bmod 25, \quad \text{and } x \equiv 3 \bmod 81$$

and the simultaneous system

$$y \equiv 5 \bmod 8, \quad y \equiv 12 \bmod 25, \quad \text{and} \quad y \equiv 47 \bmod 81$$

Solution.

(a) Let $R = \mathbb{Z}$ and $A_i = n_i\mathbb{Z}$. Then A_i and A_j are comaximal for $i \neq j$.

So the following natural map $\varphi : \mathbb{Z} \to \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ defined by

$$x \mapsto \left(x \pmod{n_1},\ x \pmod{n_2},\ \cdots,\ x \pmod{n_k}\right)$$

is surjective. Also $\mathbb{Z}_n \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ by 1st isomorphism theorem.

Let $(a_1, a_2, \cdots, a_k) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$. Then there is $x \in \mathbb{Z}$ s.t.

$$x \equiv a_1 \pmod{n_1},\ x \equiv a_2 \pmod{a_2},\ \cdots,\ x \equiv a_k \pmod{n_k}$$

Then x is unique up to mod n by isomorphism theorem.

(b) Since $(a_1, a_2, \cdots, a_k) = \sum_{i=1}^{k} a_i e_i$ for $e_i = (0, 0, \cdots, \overset{\text{ith coordinate}}{\overset{\downarrow}{1}}, 0, \cdots, 0)$, $\varphi\left(t_i \frac{n}{n_i}\right) = e_i$ for the inverse t_i of $\frac{n}{n_i} \pmod{n_i}$

we can find that

$$\varphi\left(\sum_{i=1}^{k} a_i t_i n_i'\right) = \varphi\left(\sum_{i=1}^{k} a_i t_i \frac{n}{n_i}\right) = \sum_{i=1}^{k} a_i\, \varphi\left(t_i \frac{n}{n_i}\right) = \sum_{i=1}^{k} a_i e_i$$

Hence $\sum_{i=1}^{k} a_i t_i n_i' \pmod{n}$ is the desired solution. ∎

(c) $\begin{cases} n_1 = 2^3 \\ n_2 = 5^2 \\ n_3 = 3^4 \end{cases}$ $\begin{cases} a_1 = 1 \\ a_2 = 2 \\ a_3 = 3 \end{cases}$

$(25 \times 81, 8) = 1 \to t_1 = 1$, $(8 \times 81, 25) = 1 \to t_2 = 12$, $(8 \times 25, 81) = 1 \to t_3 = 32$

$\therefore x = 1 \times 1 \times 2025 + 2 \times 12 \times 648 + 3 \times 32 \times 200 \pmod{16200}$

$\boxed{x \equiv 4377 \pmod{16200}}$

(d) Using (c) to get t_i's,

$x = 5 \times 1 \times 2025 + 12 \times 12 \times 648 + 47 \times 32 \times 200 \pmod{16200}$

$\boxed{\therefore x \equiv -763 \pmod{16200}}$

□