

# Affine Gordon-Bender-Knuth identities and cylindric Young tableaux

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## Outline

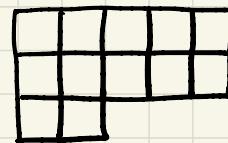
- ① Basic definitions & Robinson-Schensted algorithm
- ② Standard Young tableaux of bounded height
- ③ Noncrossing and nonnesting involutions
- ④ Cylindric tableaux, cylindric Schur ftns
- ⑤ Affine Gordon-Bender-Knuth identities

## Basic definitions

Def)  $\lambda = (\lambda_1, \dots, \lambda_k)$  is a **partition** of  $n$  if  
 $\lambda_1 \geq \dots \geq \lambda_k > 0$ ,  $\lambda_1 + \dots + \lambda_k = n$ .

length of  $\lambda = l(\lambda) = k = \# \text{ parts}$

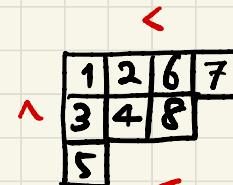
The Young diagram of  $\lambda$  is



$$\lambda = (5, 5, 2).$$

A standard Young tableau of shape  $\lambda$  is

1	2	6	7
3	4	8	
5			



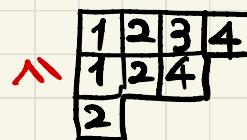
A semistandard Young tableau "

1	1	2	3
2	3	3	
4			



A row-strict tableau "

1	2	3	4
1	2	4	
2			



## Connection with representation theory

- $G$ : a finite group  $\rightarrow \dim \mathbb{C}G = |G|$
- $\mathbb{C}G = \bigoplus_{i=1}^m V_i$   $\mathbb{C}G$ : regular representation of  $G$   
 $V_i$ : irreducible representations of  $G$
- $\Rightarrow |G| = \sum_{i=1}^m (\dim V_i)^2$
- $G = S_n$ : the symmetric group of order  $n$   
(the group of bijections on  $\{1, 2, \dots, n\}$ )  
Fact: The irr. reps of  $S_n$  are  $V_\lambda$ ,  $\lambda \vdash n$ .  
 $\dim V_\lambda = f^\lambda = \# \text{SYTs of shape } \lambda$ .
- $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

$$= \# \left\{ (P, Q) : P, Q \text{ are SYTs of size } n \atop sh(P) = sh(Q) \right\}$$

ex)  $n=3 \Rightarrow n! = 6$

$$(P, Q) = (1 \ 2 \ 3, 1 \ 2 \ 3),$$

$$\left( \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}, \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \right)$$

$$\left( \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}, \begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix} \right), \left( \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \right)$$

## Robinson-Schensted algorithm

$\pi$  : permutation of  $[n] = \{1, 2, \dots, n\}$ .

↓ 1-1

$(P, Q)$  : pair of SYTs of size  $n$  and of same shape

ex) $\pi = 4 1 5 6 3 2$	$\leftrightarrow$	$P$	$Q$
	$\xrightarrow{\text{RS}}$	$1 2 6$ $3 5$ $4$	$1 3 4$ $2 5$ $6$

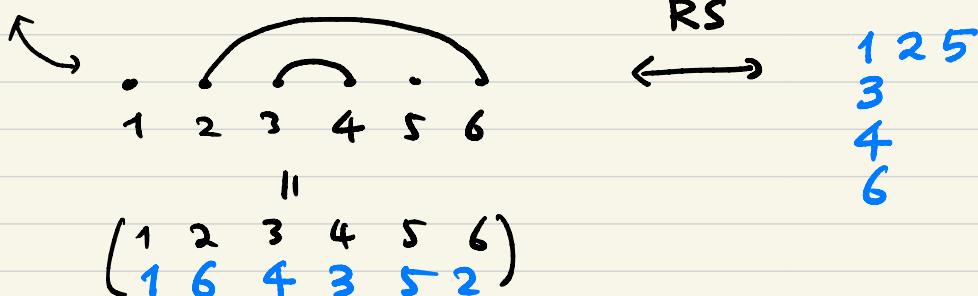
\* insertion algorithm

$P : \emptyset \leftarrow 4$	$4 \leftarrow 1$	$1 \leftarrow 5$	$1 5 \leftarrow 6$	$1 5 6 \leftarrow 3$	$1 3 6 \leftarrow 2$	$1 2 6$ $3 5$ $4$
						$1 3 4$ $2 5$ $6$

## Consequences of RS algorithm

- If  $\pi \xleftrightarrow{\text{RS}} (P, Q)$ , then  $\pi^{-1} \xleftrightarrow{\text{RS}} (Q, P)$ .
- $\pi$ : involutions of  $[n]$   $\xleftrightarrow{\text{RS}}$  SYTs of size  $n$ .  $(P, P)$ .

- $\pi = (26)(34)$



involution  $\longleftrightarrow$  matching

## SYTs with bounded height.

Def)  $\text{SYT}_n = \text{set of SYTs of size } n.$

1	2	6	7
3	4	8	
5			

$$\text{SYT}_n(h) = \left\{ T \in \text{SYT}_n : \underbrace{\text{ht}(T)}_{= \# \text{ rows}} \leq h \right\}$$

ex).  $|\text{SYT}_n(2)| = \binom{n}{\lfloor n/2 \rfloor}$

$|\text{SYT}_n(3)| = \# \text{ Motzkin paths of length } n.$

$$|\text{SYT}_n(4)| = C_L \lfloor \frac{n+1}{2} \rfloor C_R \lceil \frac{n+1}{2} \rceil, \quad C_n = \frac{1}{n+1} \binom{2n}{n} = \text{nth Catalan number.}$$

Thm (Gessel, 1990)

$$\sum_{n \geq 0} |\text{SYT}_n(2h+1)| \frac{x^n}{n!} = \exp(x) \det \left( I_{-i+j}(2x) - I_{i+j}(2x) \right)_{i,j=1}^h$$

$$I_\alpha(2x) = \sum_{l \geq 0} \frac{x^{2l+|\alpha|}}{l! (l+|\alpha|)!} \quad (\text{Modified Bessel function})$$

## Properties of RS-algorithm

- ① IF  $\pi \xleftrightarrow{\text{RS}} (P, Q)$ , then  $\pi' \xleftrightarrow{\text{RS}} (Q, P)$ .
- ② max length of decreasing subsequence of  $\pi$  =  $ht(P)$ .
- ③ " Increasing " = width( $P$ )

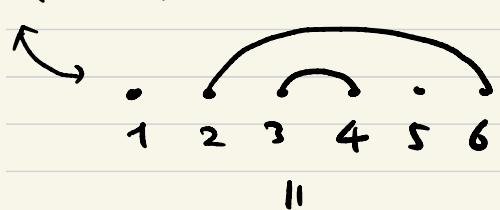
$\pi = 4156327$	$\xleftrightarrow{\text{RS}}$	$P$	$Q$
		1267 35 4	1347 25 6
max dec	432		
max inc	4567		

## Consequences of RS algorithm

$$\pi: \text{involutions of } [n] \xleftrightarrow{\text{RS}} \text{SYTs of size } n. \quad (\rho, \rho).$$

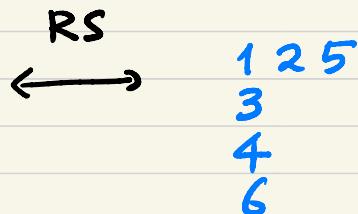
$\pi(\pi^2 = \text{id})$

$$\tau_l = (26)(34)$$



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \textcolor{blue}{6} & 4 & \textcolor{blue}{3} & 5 & \textcolor{blue}{2} \end{pmatrix}$$

involutions of  $[n]$   
with no  $k$ -nesting



SYTs of size  $n$   
with height  $< 2k$



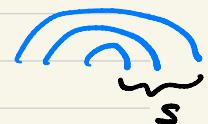
$\leftrightarrow$  dec seq of  
length  $2k$

## $r$ -noncrossing and $s$ -nonnesting involutions

Def) An involution  $\pi$  is  $r$ -noncrossing if  $\pi$  has no



"  $s$ -nonnesting "  $s$



$NCNN_n(r,s)$  = set of  $r$ -noncrossing and  $s$ -nonnesting  
involutions of  $[n]$

Thm (Chen, Deng, Du, Stanley, Yan, 2007)

$$\# NCNN_n(r,s) = \# NCNN_n(s,r).$$

In particular,

$\# r$ -noncrossing involutions of  $[n]$

$= \# s$ -nonnesting " .

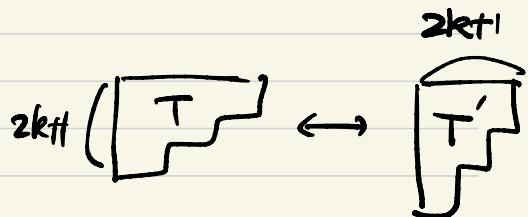
# SYTs of size  $n$  with height  $\leq 2k+1$

**RS**  
= #  $(k+1)$ -nonnesting involutions of  $[n]$

**CDDSY**  
= #  $(k+1)$ -noncrossing      "

By taking transpose,

# SYTs of size  $n$  with height  $\leq 2k+1$



= # SYTs of size  $n$  with width  $\leq 2k+1$

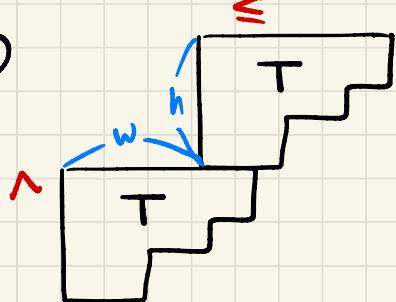
## Question

? # SYTs of size  $n$  with height  $\leq 2h+1$  and width  $\leq 2w+1$   
= #  $(h+1)$ -nonnesting and  $(w+1)$ -noncrossing involutions  
of  $[n]$

Def)  $(h, w)$ -cylindric SSYT is an SSYT  $T$  such that

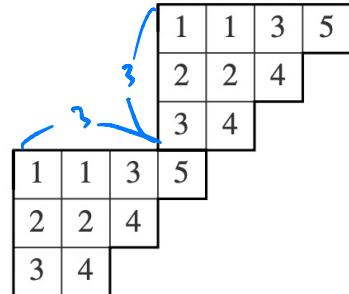
① height of  $T \leq h$

②

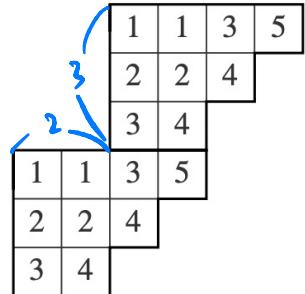


is an SSYT of a valid skew shape.

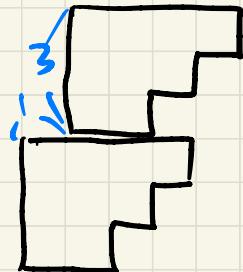
1	1	3	5
2	2	4	
3	4		



$(3,3)$ -cylindric



not  $(3,2)$ -cylindric



not  
 $(3,1)$ -cyl

Let  $\text{CSYT}_n(h, w) = \{ (h, w)\text{-cylindric SYTs of size } n \}$

Thm (Huh, Kim, Krattenthaler, Okada)

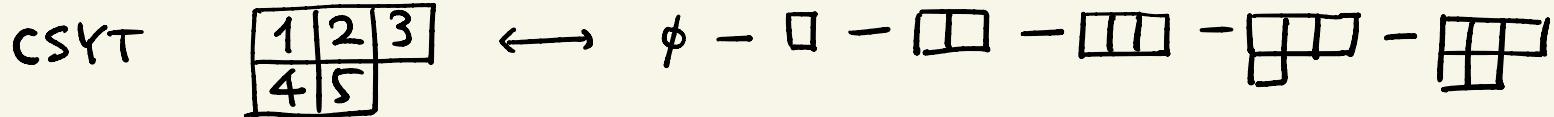
$$\# \text{CSYT}_n(2h+1, 2w+1) = \# \text{NCNN}_n(h+1, w+1)$$

Idea of Proof

- ① Express each side as # lattice walks
- ② Express # lattice walks using determinants
- ③ Prove  $\det = \det.$

Open Problem : Find a bijective proof.

\* Lattice walks for  $\text{CSYT}_n(2h+1, 2w+1)$ .



$$\longleftrightarrow (0,0,0) - (1,0,0) - (2,0,0) - (3,0,0) - (3,1,0) - (3,2,0)$$

a walk in the region  $\{(x_1, x_2, x_3) : x_1 \geq x_2 \geq x_3 \geq 0\}$

with step set  $\varepsilon_1 = (1, 0, 0)$ ,  $\varepsilon_2 = (0, 1, 0)$ ,  $\varepsilon_3 = (0, 0, 1)$ .

Under this correspondence

$\text{CSYT}_n(2h+1, 2w+1) \longleftrightarrow$  walks from 0 of length  $n$  in

$$\{(x_1, \dots, x_{2h+1}) : x_1 \geq \dots \geq x_{2h+1} \geq 0, \\ x_1 - x_{2h+1} \leq 2w+1\}$$

$$= \{(x_1, \dots, x_{2h+1}) : x_1 \geq \dots \geq x_{2h+1} \geq x_1 - 2w-1\}$$

$\hookrightarrow$  alcove of affine Weyl group of type  $\widehat{A}_{2h}$

Def) A *vacillating tableau* is a sequence of partitions

$$\phi = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)} = \phi \quad \text{such that}$$

$\lambda^{(i)}$  and  $\lambda^{(i+1)}$  differ by at most one cell.

ex)

$$\phi - \square - \boxed{\square} - \square - \boxed{\square} - \boxed{\square} - \boxed{\square} - \boxed{\square} - \square - \phi$$

Prop involutions on  $[n]$   $\xleftrightarrow{1-1}$  vacillating tableaux  
of length  $n$

$$NCNN_n(h+1, w+1) \xleftrightarrow{1-1} "$$

with every partition  
contained in  $h \boxed{ }$

$\text{NCNN}_n(h+1, w+1) \xleftrightarrow{1-1}$  vacillating tableaux  
of length  $n$

with every partition  
contained in  $h \begin{array}{c} w \\ \boxed{\phantom{0}} \end{array}$

$\longleftrightarrow$  lattice walks from 0 to 0 of length  $n$

in region  $\{(x_1, \dots, x_n) : w \geq x_1 \geq \dots \geq x_n \geq 0\}$

with step set  $\{\pm e_i\} \cup \{0\}$ .

alove of  
affine Weyl group of type  $\tilde{C}_n$

Filaseta (1985) computed # lattice walks in alcove of affine Weyl groups.

Applying Filaseta's result, we get

$$|\text{CSYT}_n(h, w)| = \sum_{\substack{\lambda \vdash n \\ \lambda_1 - \lambda_n \leq w}} \sum_{\substack{k_1 + \dots + k_h = 0 \\ k_1, \dots, k_h \in \mathbb{Z}}} n! \det \left( \frac{1}{(\lambda_i - i + j + (w+h)k_i)} \right)_{i,j=1}^h$$

and

$$\sum_{n \geq 0} |\text{NCNN}_n(h+1, w+1)| \frac{x^n}{n!}$$

$$= \exp(x) \sum_{k_1, \dots, k_h \in \mathbb{Z}} \det \left( I_{-i+j+(2h+2w+2)k_i}(2x) - I_{i+j+(2h+2w+2)k_i}(2x) \right)_{i,j=1}^h$$

where

$$I_k(x) = \sum_{l \geq 0} \frac{x^{2l+k}}{l!(l+k)!}$$

(Modified Bessel ftn).

So, to prove our theorem

$$|CSYT_n(2h+1, 2w+1)| = |NCNN_n(h+1, w+1)|$$

it suffices to prove

$$\sum_{\lambda: \lambda_1 - \lambda_n \leq w} \sum_{\substack{k_1 + \dots + k_h = 0 \\ k_1, \dots, k_h \in \mathbb{Z}}} x^n \det \left( \frac{1}{(\lambda_i - i + j + (w+h)k_i)} \right)_{i,j=1}^h$$

$$= \exp(x) \sum_{k_1, \dots, k_h \in \mathbb{Z}} \det \left( I_{-i+j+(2h+2w+2)k_i}(2x) - I_{i+j+(2h+2w+2)k_i}(2x) \right)_{i,j=1}^h$$

We will show a symmetric function generalization of this.

A symmetric function is a symmetric polynomial in infinitely many variables  $x_1, x_2, x_3, \dots$  (with bounded degree).

Def) The  $k$ th elementary symmetric function is

$$e_k = \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k}$$

ex).  $e_1 = x_1 + x_2 + x_3 + \dots$

$$e_2 = x_1 x_2 + x_1 x_3 + \dots$$

Def) The  $k$ th complete homogeneous symmetric function is

$$h_k = \sum_{i_1 \leq \dots \leq i_k} x_{i_1} \cdots x_{i_k}$$

ex).  $h_1 = e_1 = x_1 + x_2 + \dots$

$$h_2 = x_1^2 + x_2^2 + \dots + x_1 x_2 + x_1 x_3 + \dots$$

For a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$ , define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$$

ex)  $h_{(4,3,1)} = h_4 h_3 h_1$

$$e_{(3,1,1)} = e_3 e_1 e_1$$

$\Lambda$  = the space of symmetric functions ( $\mathbb{Q}$ -algebra).

### Fundamental Thm of Symmetric functions

$\{h_\lambda\}$  is a basis for  $\Lambda$ .

$\{e_\lambda\}$  "  $\Lambda$ .

Def) Schur function

$$S_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x_1^{\#1's} x_2^{\#2's} \dots$$

Recall SSYT 

ex)  $\lambda = (2, 1)$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array} + \dots + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} + \dots$$

$$S_\lambda = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + \dots + x_1 x_2 x_3 + x_1 x_2 x_3 + \dots$$

Note:  $[x_1 x_2 \dots x_n] S_\lambda = \# \text{SYT}(\lambda)$ , where  $|\lambda| = n$ .

In the above example  $[x_1 x_2 x_3] S_{(2, 1)} = 2$

Thm (Jacobi-Trudi formula)

If  $\lambda = (\lambda_1, \dots, \lambda_k)$ , then

$$s_\lambda = \det (h_{\lambda_i - i + j})_{i,j=1}^k.$$

$$s_{\lambda'} = \det (e_{\lambda_i - i + j})_{i,j=1}^k.$$

$\lambda'$ : transpose of  $\lambda$

$$\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

$$\lambda' = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

Thm (Gordon-Bender-Knuth, 1972)

$$\sum_{l(\lambda) \leq 2m+1} s_{\lambda'} = \sum_{k \geq 0} e_k \det (f_{-i+j} - f_{i+j})_{i,j=1}^m$$

$$\sum_{l(\lambda) \leq 2m} s_{\lambda'} = \det (f_{-i+j} + f_{i+j-1})_{i,j=1}^m$$

where  $f_k = \sum_{n \geq 0} e_n e_{n+k}.$

Def) A **row-strict tableau** is  $\nwarrow \begin{smallmatrix} & & \\ & & \\ & & \end{smallmatrix}$ . (transpose of SSYT).

Def)  $(h, w)$ -cylindric RST is an RST  $T$  such that

$$ht \leq h \text{ and}$$



← This is also an RST.

Def)  $CRST_{\lambda}(h, w) = \{T : (h, w)\text{-cylindric RST of shape } \lambda\}$

Thm (Jacobi-Trudi for cylindric Schur ftn)

$$\sum_{T \in CRST_{\lambda}(h, w)} \chi_T = \sum_{\substack{k_1 + \dots + k_n = 0 \\ k_1, \dots, k_n \in \mathbb{Z}}} \det \left( e_{\lambda_i - i + j + (ht+w)k_i} \right)_{i,j=1}^h$$

Pf) Follows from Gessel-Krattenthaler 1997.

# History of cylindric tableaux.

- Gessel, Krattenthaler (1997)  
g.f. of cylindric PP ,  $\widehat{A}_r$  basic hypergeometric series
- Bertram, Ciocan-Fontanine, Fulton (1999)  
quantum Kostka number = certain cylindric SSYT.
- Postnikov (2005)  
cylindric skew Schur of toric shape =  $\sum$  (3-pt Gromov-Witten inv)  $S_\lambda$   
(with bounded # van).
- NcNamara (2006)  
cylindric skew Schur is not Schur positive (infinite van)  
Conj: cylindric skew Schur is cylindric Schur positive
- S.J. Lee (2019) : NcNamara's conj is true.

Thm (Gordon-Bender-Knuth, 1972)

$$\sum_{T \in RST(2h+1)} x_T = \sum_{k \geq 0} e_k \det(f_{-i+j} - f_{i+j})_{i,j=1}^h$$

$$\sum_{T \in RST(2h)} x_T = \det(f_{-i+j} + f_{i+j-1})_{i,j=1}^h$$

where  $f_k = \sum_{n \geq 0} e_n e_{n+k}$ .

Thm (Affine Gordon-Bender-Knuth)

$$\sum_{T \in CRST(2h+1, w)} x_T = \sum_{k \geq 0} e_k \sum_{k_1, \dots, k_h \in \mathbb{Z}} \det(f_{-i+j+(w+2h+1)k_i} - f_{i+j+(w+2h+1)k_i})_{i,j=1}^h$$

$$\sum_{T \in CRST(2h, w)} x_T = \sum_{k_1, \dots, k_h \in \mathbb{Z}} (-1)^{k_1+\dots+k_h} \det(f_{-i+j+(w+2h)k_i} + f_{i+j+(w+2h)k_i})_{i,j=1}^h$$

# Outline of Proof of AGB identity.

## ① Reformulation

$$\sum_{\substack{\mu_1, \dots, \mu_m \in \mathbb{Z} \\ \mu_1 > \dots > \mu_m \\ \mu_1 - \mu_m < N}} \sum_{\substack{k_1, \dots, k_m \in \mathbb{Z} \\ k_1 + \dots + k_m = 0}} \det_{1 \leq i, j \leq m} (e_{\mu_i + Nk_i + j}) = \sum_{\substack{\alpha_1, \dots, \alpha_m \in \mathbb{Z} \\ R_N(\alpha_1) > \dots > R_N(\alpha_m)}} \det_{1 \leq i, j \leq m} (e_{\alpha_i + j}).$$

$$\sum_{\substack{\alpha_1, \dots, \alpha_{2h+1} \in \mathbb{Z} \\ R_N(\alpha_1) > \dots > R_N(\alpha_{2h+1})}} \det_{1 \leq i, j \leq 2h+1} (e_{\alpha_i + j}) = \sum_{k \geq 0} e_k \det_{1 \leq i, j \leq h} (F_{-i+j, N} - F_{i+j, N})$$

## ② Use Pfaffians.

$$\sum_{\substack{\alpha_1, \dots, \alpha_{2h+1} \in \mathbb{Z} \\ R_N(\alpha_1) > \dots > R_N(\alpha_{2h+1})}} \det_{1 \leq i, j \leq 2h+1} (e_{\alpha_i + j}) = \text{Pf} \begin{pmatrix} 0 & E \\ -E^t & D_N(2h+1) \end{pmatrix},$$

$$\text{Pf} \begin{pmatrix} 0 & E \\ -E^t & D_N(2h+1) \end{pmatrix} = e \det_{1 \leq i, j \leq 2h} (F_{j-i-1, N} - F_{j-i+1, N}),$$

$$\det_{1 \leq i, j \leq 2h} (F_{j-i-1, N} - F_{j-i+1, N}) = \det_{1 \leq i, j \leq h} (F_{-i+j, N} - F_{i+j, N}).$$

Cor

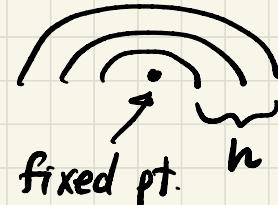
$$\textcircled{1} \quad \# \text{CSYT}_n(2h+1, 2w+1) = \# \text{NCNN}_n(h+1, w+1)$$

$$\textcircled{2} \quad \# \text{CSYT}_n(2h+1, 2w) = \# \text{NCNN}_n(h+1, w+\frac{1}{2})$$

$$\textcircled{3} \quad \# \text{CSYT}_n(2h, 2w+1) = \# \text{NCNN}_n(h+\frac{1}{2}, w+1)$$

$$\textcircled{4} \quad \# \text{CSYT}_n(2h, 2w) = \sum_{M \in \text{NCNN}'_n(h+1, w+1)} (-1)^{\text{fix}_1(M)}.$$

$(h+\frac{1}{2})$ -nonnesting  $\iff$  no



If  $h=1$ , then  $\textcircled{1}, \textcircled{2}$  reduce to Mortimer–Prellberg result.

If  $h=1$ , then ③, ④ become

Cor

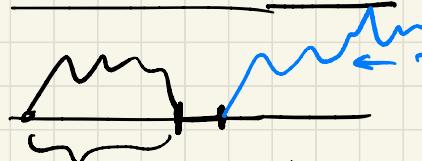
③ # Dyck prefixes of length  $n$  with height  $\leq 2ht1$

= # Motzkin paths of length  $n$  with height  $\leq h$   
and every horizontal step is on  $x$ -axis.

④ # Dyck prefixes of length  $n$  with height  $\leq 2h$

= # Motzkin paths of length  $n$  with height  $\leq h$   
and every horizontal step is on  $x$ -axis.

s.t.  $h$



even # horizontal steps

there is a Dyck prefix  
from ht 0 to ht h.

We found bijective proofs using recent results of Gu-Prodinger  
and Dershowitz.

## Open Problems

1. Find a bijective proof of

$$\textcircled{1} \quad \# \text{CSYT}_n(2h+1, 2w+1) = \# \text{NCNN}_n(h+1, w+1)$$

$$\textcircled{2} \quad \# \text{CSYT}_n(2h+1, 2w) = \# \text{NCNN}_n(h+1, w+\frac{1}{2})$$

$$\textcircled{3} \quad \# \text{CSYT}_n(2h, 2w+1) = \# \text{NCNN}_n(h+\frac{1}{2}, w+1)$$

$$\textcircled{4} \quad \# \text{CSYT}_n(2h, 2w) = \sum_{M \in \text{NCNN}'_n(h+1, w+1)} (-1)^{\text{fix}_i(M)}.$$

2. Find a formula for g.f. for  $\text{NCNN}_n(h+\frac{1}{2}, w+\frac{1}{2})$ .

3. Find a sign-free expression for  $\textcircled{4}$ .

Thank You  
for your attention !