HOMEWORK

Contents

1. Homework 1 (Due: Sep 21)

Problem 1.1. Let $\mathcal{L}$ be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n\geq 0}$. Prove the following extremal property: for any monic real polynomial $\pi(x) \neq P_n(x)$ of degree n,

$$\mathcal{L}(P_n(x)^2) < \mathcal{L}(\pi(x)^2).$$

Solution. Let $\pi(x) = \sum_{k=0}^{n} a_k P_k(x)$. Since both $\pi(x)$ and $P_n(x)$ are monic, we have $a_n = 1$ [3 points]. Then

$$\begin{aligned}\mathcal{L}(\pi(x)^2) &= \sum_{k=0}^{n} a_k^2 \mathcal{L}(P_k(x)^2) \quad \text{[4 points]}\\ &\geq a_n^2 \mathcal{L}(P_n(x)^2) = \mathcal{L}(P_n(x)^2) \quad \text{[3 points]}.\end{aligned}$$

□

Problem 1.2. Let $\mathcal{L}$ be a linear functional such that $\Delta_n \neq 0$ for all $n \geq 0$. Prove that if $\pi(x)$ is a polynomial such that $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \geq 0$, then $\pi(x) = 0$.

Solution. Since $\Delta_n \neq 0$, there is a monic OPS $\{P_n(x)\}_{n\geq 0}$ for $\mathcal{L}$ [3 points]. Let $\pi(x) = \sum_{k=0}^{n} a_k P_k(x)$. Since $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \geq 0$, we have $\mathcal{L}(p(x)\pi(x)) = 0$ for any polynomial $p(x)$ [3 points]. Then, for each $0 \leq k \leq n$, we have $0 = \mathcal{L}(P_k(x)\pi(x)) = a_k \mathcal{L}(P_k(x)^2)$ [2 points]. Since $\mathcal{L}(P_k(x)^2) \neq 0$, we get $a_k = 0$ for all $0 \leq k \leq n$ [2 points]. Hence $\pi(x) = 0$.

A common mistake: It is not true in general that $\mathcal{L}(x^k P_n(x)) = 0$ for $k \neq n$. We can only say that $\mathcal{L}(x^k P_n(x)) = 0$ for $k < n$. □

Problem 1.3. The *Tchebyshev polynomials of the second kind* $U_n(x)$ are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta, \qquad n \geq 0.$$

(1) Prove that $U_n(x)$ is a polynomial of degree n.
(2) Prove that
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n \geq 0,$$
where $U_{-1}(x) = 0$ and $U_0(x) = 1$.
(3) Prove that
$$\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx = \frac{\pi}{2}\delta_{m,n}.$$
(4) Find the 3-term recurrence for the normalized Tchebyshev polynomials of the second kind. More precisely, find the numbers b_n and λ_n such that
$$\hat{U}_{n+1}(x) = (x - b_n)\hat{U}_n(x) - \lambda_n \hat{U}_{n-1}(x), \qquad n \geq 0,$$
where $\hat{U}_n(x)$ is the monic polynomial that is a scalar multiple of $U_n(x)$.

Solution. (1) This follows from (2) [2 points].

(2) By the addition rule for the sine function,

$$\sin(n+1)\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta,$$
$$\sin(n-1)\theta = \sin n\theta\cos\theta - \cos n\theta\sin\theta.$$

Adding the two equations and dividing both sides by $\sin\theta$, we get

$$U_n(x) + U_{n-2}(x) = 2xU_{n-1}(x), \qquad n \ge 1 \quad \text{[2 points]}.$$

This is equivalent to the recurrence in the problem.

(3) By the change of variables $x = \cos\theta$, $0 \le \theta \le \pi$, with $dx = -\sin\theta d\theta = -\sqrt{1-x^2}d\theta$,

$$\begin{aligned}
&\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx \\
&= \int_0^\pi \sin(m+1)\theta\sin(n+1)\theta d\theta \quad \text{[2 points]} \\
&= \frac{1}{2}\int_0^\pi (\cos(m-n)\theta + \cos(m+n)\theta)\, d\theta \quad \text{[2 points]} \\
&= \frac{\pi}{2}\delta_{m,n}.
\end{aligned}$$

(4) Since $\deg U_n(x) = 2^n$ for all $n \ge 0$, we have $\hat{U}_n(x) = 2^{-n}U_n(x)$. Dividing both sides of the recurrence in (2) by 2^{n+1}, we obtain $b_n = 0$ and $\lambda_n = 1/4$ [2 points]. □

Problem 1.4. Let $\{P_n(x)\}_{n\ge0}$ be the monic OPS for a linear functional $\mathcal{L}$ with three-term recurrence

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \ge 0.$$

(1) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & 1 & & 0 \\ \lambda_1 & x-b_1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & \lambda_{n-1} & x-b_{n-1} \end{vmatrix}.$$

(2) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & \sqrt{\lambda_1} & & 0 \\ \sqrt{\lambda_1} & x-b_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\lambda_{n-1}} \\ 0 & & \sqrt{\lambda_{n-1}} & x-b_{n-1} \end{vmatrix}.$$

(3) Using (2) prove that if $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all, then $P_n(x)$ has real roots only.

Solution. (1) Let $Q_n(x)$ be the determinant on the right-hand side. Expanding the determinant along the last row, we obtain the recursion

$$Q_n(x) = (x - b_{n-1})Q_{n-1}(x) - \lambda_{n-1}Q_{n-2}(x) \quad \text{[2 points]}.$$

Since $P_n(x)$ and $Q_n(x)$ satisfy the same recurrence with with the initial conditions $Q_0(x) = 1$ and $Q_1(x) = x - b_0$, we obtain that $Q_n(x) = P_n(x)$.

(2) Let $A_n = (\alpha_{i,j})$ be the matrix in (1) and let B_n be the matrix in (2). Then it suffices to find an invertible diagonal matrix $D = \mathrm{diag}(d_i)$ such that $B_n = DA_nD^{-1}$ [3 points]. To do this, observe that $DA_nD^{-1} = (d_i\alpha_{i,j}d_j^{-1})$. Since B_n and DA_nD^{-1} are tri-diagonal matrices, we have $B_n = DA_nD^{-1}$ if and only if the following hold:

$$\beta_{i,i} = d_i\alpha_{i,i}d_i^{-1}, \tag{1.1}$$

$$\beta_{i,i+1} = d_i\alpha_{i,i+1}d_{i+1}^{-1}, \tag{1.2}$$

$$\beta_{i+1,i} = d_{i+1}\alpha_{i+1,i}d_i^{-1}. \tag{1.3}$$

Since $\alpha_{i,i+1} = 1$ and $\beta_{i,i+1} = \sqrt{\lambda_{i+1}}$, (1.2) is equivalent to $d_{i+1} = d_i/\sqrt{\lambda_{i+1}}$. Indeed, if we set $d_0 = 1$ and $d_{i+1} = d_i/\sqrt{\lambda_{i+1}}$, then all three conditions above hold [3 points].

(3) Since the zeros of $P_n(x)$ are the eigenvalues of a real symmetric matrix, they are real [2 points]. $\square$