Lecture hall graphs and the Askey scheme

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Outline

1. Basics on orthogonal polynomials.
2. Lecture hall partitions and little q-Jacoby poly.
3. Lecture hall graph.
4. Askey-Wilson polynomials
5. Main results

Orthogonal polynomials

Def) $\{p_n(x)\}_{n\geq 0}$ is a sequence of orthogonal polynomials (OPS) with respect to a linear functional $\mathcal{L}$ if

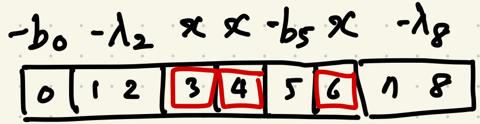
① $\deg p_n(x) = n$

② $\mathcal{L}(p_n(x)p_m(x)) = \begin{cases} 0 & \text{if } n\neq m \\ \text{nonzero} & \text{if } n=m. \end{cases}$

Def) The nth moment μ_n of OP is $\mathcal{L}(x^n)$.

Favard's thm $\{p_n(x)\}_{n\geq 0}$ is a monic OPS iff

$$p_{n+1}(x) = (x-b_n)p_n(x) - \lambda_n p_{n-1}(x). \qquad (\lambda_n \neq 0).$$

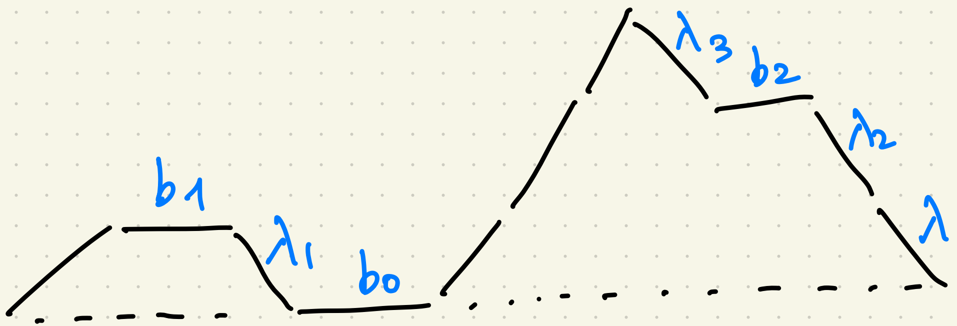
Cor $p_n(x) = \sum$ 

<u>Thm</u> (Viennot, 1983)

Let $\{p_n(x)\}$ be a monic OPS satisfying

$$p_{n+1}(x) = (x - b_n)\, p_n(x) - \lambda_n\, p_{n-1}(x).$$

$$\Rightarrow \quad \mu_n = \sum_{\pi \in \mathrm{Motz}_n} \mathrm{wt}(\pi).$$



$$\mathrm{wt}(\pi) = b_0 b_1 b_2 \lambda_1^2 \lambda_2 \lambda_3$$

Motzkin path

ex). Hermite polynomial

$$\mu_n = (2n-1)!! = 1 \cdot 3 \cdot \cdots (2n-1) = \#\text{ perfect matchings}$$

ex) Charlier polynomial

$$\mu_n = \#\text{ set partitions on } [n] = \{1, 2, \ldots, n\}.$$

ex) Laguerre polynomial

$$\mu_n = n! = \#\text{ permutations.}$$

Another way to define moments.

$P_n(x)$: OPS w.r.t. $\mathcal{L}$. $\quad (\mathcal{L}(1)=1, \; P_0(x)=1)$

Let $\quad x^n = \sum_{k=0}^{n} \mu_{n,k} P_k(x).$

Then $\quad \mathcal{L}(x^n) = \sum_{k=0}^{n} \mu_{n,k} \underbrace{\mathcal{L}(P_k(x))}_{=\mathcal{L}(P_k(x)P_0(x))} = \mu_{n,0}.$

Def). $P_n(x)$: OPS

Let $\quad x^n = \sum_{k=0}^{n} \mu_{n,k} P_k(x), \quad P_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k.$

$\mu_{n,k}$ is the mixed moment.

$\nu_{n,k}$ is the dual mixed moment.

(These are the entries of change of bases.)

Little q-Jacobi polynomials $p_n(x;a,b;q)$

$$p_n(x;a,b;q) = \frac{(ab;q)_n}{(-1)^n q^{-\binom{n}{2}} (abq^{n+1};q)_n} \, {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} ; q, qx\right),$$

$${}_2\phi_1\left(\begin{matrix} A, B \\ C \end{matrix} ; q, z\right) = \sum_{i\geq 0} \frac{(A;q)_i (B;q)_i}{(q;q)_i (C;q)_i} z^i,$$

$$(A;q)_i = (1-A)(1-Aq)\cdots(1-Aq^{i-1})$$

Corteel, K. 2020

There are combinatorial models for $\nu_{n,k}$, $\mu_{n,k}$.

using lecture hall partitions.

Def) $\lambda \vdash n$ $\Leftrightarrow$ λ is a **partition** of n

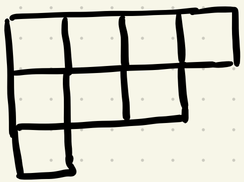
$\Leftrightarrow$ $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_1 + \dots + \lambda_k = n$

$\lambda_1 \geq \dots \geq \lambda_k > 0$

Each $\lambda_i > 0$ is called a **part**.

ex) $\lambda = (4,3,1,1)$ is a partition of $9 = 4+3+1+1$.

- **Young diagram** of $\lambda = (4,3,1)$ is

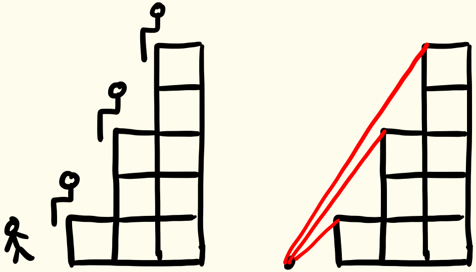


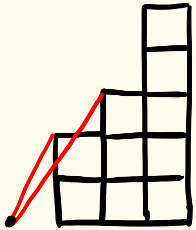
Lecture hall partitions

$\lambda = (\lambda_1, \ldots, \lambda_n)$ is a **lecture hall partition** if

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0$$

ex) $(5,3,1)$ is LHP $\qquad\qquad$ $(5,3,2)$ is NOT LHP





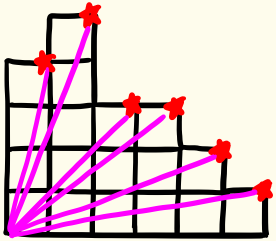
$\boxed{\text{Thm (Bousquet-Mélou, Eriksson, 1997)} \quad \displaystyle\sum_{\lambda \in L_n} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}}$

Anti-lecture hall compositions

$\alpha = (\alpha_1, \ldots, \alpha_n)$ is an anti-lecture hall composition (or a planetarium composition)

if $\quad \dfrac{\alpha_1}{1} \geq \dfrac{\alpha_2}{2} \geq \cdots \geq \dfrac{\alpha_n}{n} \geq 0.$

ex)



$(4, 5, 3, 3, 2, 1)$

Thm (Corteel, Savage, 2003)

$$\sum_{\alpha \in A_n} q^{|\alpha|} = \prod_{i=1}^{n} \frac{1+q^i}{1-q^{i+1}}$$

Thm (Corteel, K., 2020)

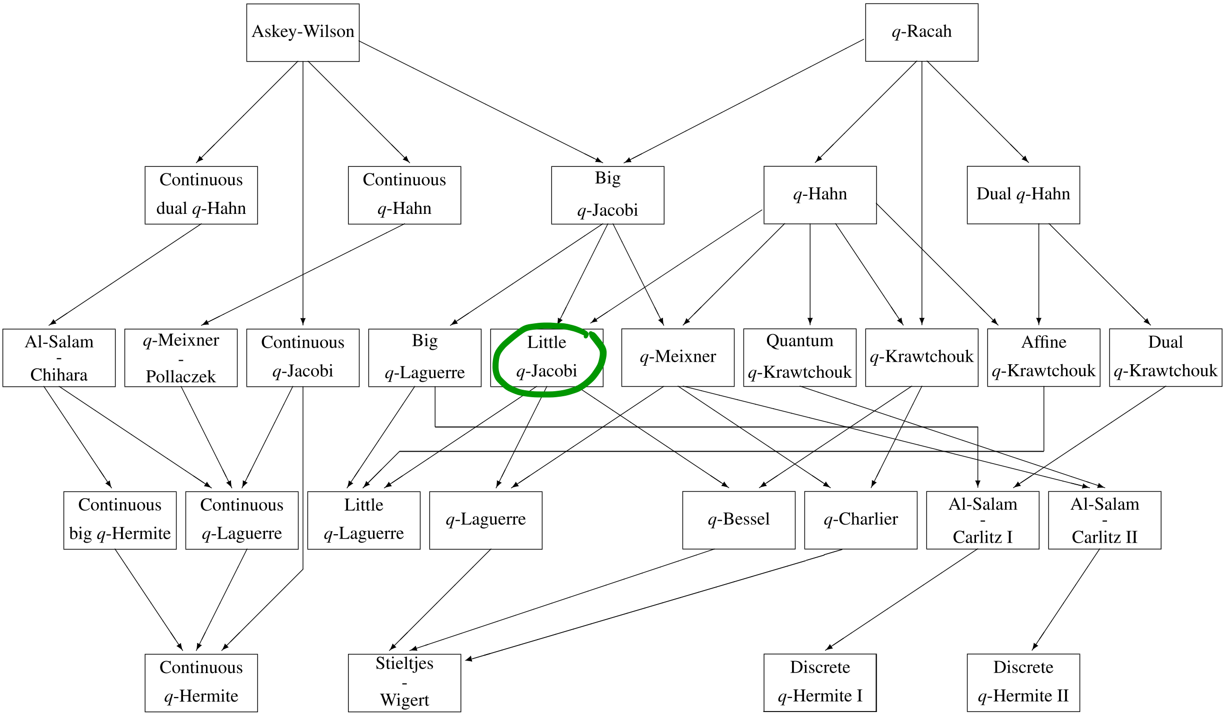
Letting $a=-uv$, $b=-u/v$ for little q-Jacobi,

$$\mu_{n,k} = \sum_{\frac{\alpha_1}{k+1} \geq \cdots \geq \frac{\alpha_{n-k}}{n} \geq 0} u^{\lfloor \alpha \rfloor} v^{o(\lfloor \alpha \rfloor)} q^{|\alpha|}$$

$$\nu_{n,k} = \sum_{\frac{\lambda_1}{n} > \cdots > \frac{\lambda_{n-k}}{k+1} > 0} u^{\lfloor \lambda \rfloor} v^{o(\lfloor \lambda \rfloor)} q^{|\lambda|}$$

$\Rightarrow$ $\mu_{n,k}$ = generating function for anti-lecture hall compositions

$\nu_{n,k}$ = " " " lecture hall partitions.



Askey-Wilson
q-Racah
Continuous dual q-Hahn
Continuous q-Hahn
Big q-Jacobi
q-Hahn
Dual q-Hahn
Al-Salam - Chihara
q-Meixner - Pollaczek
Continuous q-Jacobi
Big q-Laguerre
Little q-Jacobi
q-Meixner
Quantum q-Krawtchouk
q-Krawtchouk
Affine q-Krawtchouk
Dual q-Krawtchouk
Continuous big q-Hermite
Continuous q-Laguerre
Little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam - Carlitz I
Al-Salam - Carlitz II
Continuous q-Hermite
Stieltjes - Wigert
Discrete q-Hermite I
Discrete q-Hermite II

Askey-Wilson polynomials

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{2^n a^n (abcdq^{n-1};q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right),$$

where $x = \cos\theta = (e^{i\theta} + e^{-i\theta})/2$.

$$p_{n+1} = (x - b_n) p_n - \lambda_n p_{n-1}$$

$$b_n = \frac{1}{2}(a + a^{-1} - A_n - C_n), \quad \lambda_n = \frac{1}{4} A_{n-1} C_n$$

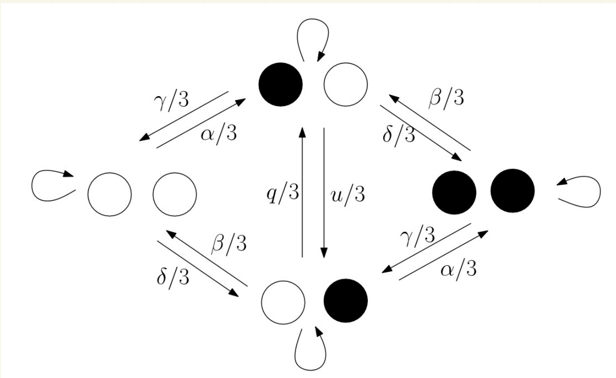
$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$

$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

Fact: $p_n(x;a,b,c,d|q)$ is symmetric in a, b, c, d.

Thm (Corteel, Williams, 2011)

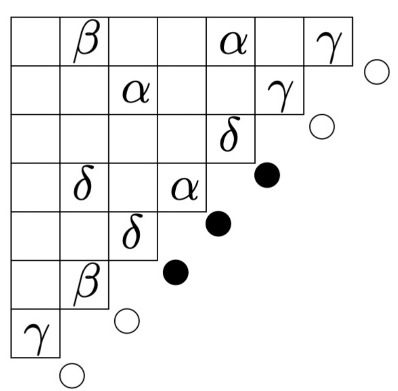
The Askey-Wilson moment μ_n is a partition function for ASEP.



state diagram of ASEP (asymmetric simple exclusion process)

Thm (Corteel, Stanley, Stanton, Williams. 2012)

$$
\begin{aligned}
&2^n (abcd;q)_n \mu_n(a,b,c,d;q) \\
&\quad = i^{-n} \sum_{T \in \mathcal{T}(n)} (-1)^{\mathrm{b}(T)} (1-q)^{A(T)+B(T)+C(T)+D(T)-n} q^{E(T)} \\
&\qquad \times (ac)^{C(T)} (bd)^{D(T)} \big((1+ai)(1+ci)\big)^{n-A(T)-C(T)} \big((1-bi)(1-di)\big)^{n-B(T)-D(T)}.
\end{aligned}
$$



staircase tableaux

After rescaling & reparametrization

μ_n is a polynomial in $\alpha, \beta, \gamma, \delta$.

$$
\alpha = \frac{1-q}{(1+ai)(1+ci)}, \qquad \beta = \frac{1-q}{(1-bi)(1-di)},
$$

$$
\gamma = \frac{ac(1-q)}{(1+ai)(1+ci)}, \qquad \delta = \frac{bd(1-q)}{(1-bi)(1-di)}.
$$

Motivation

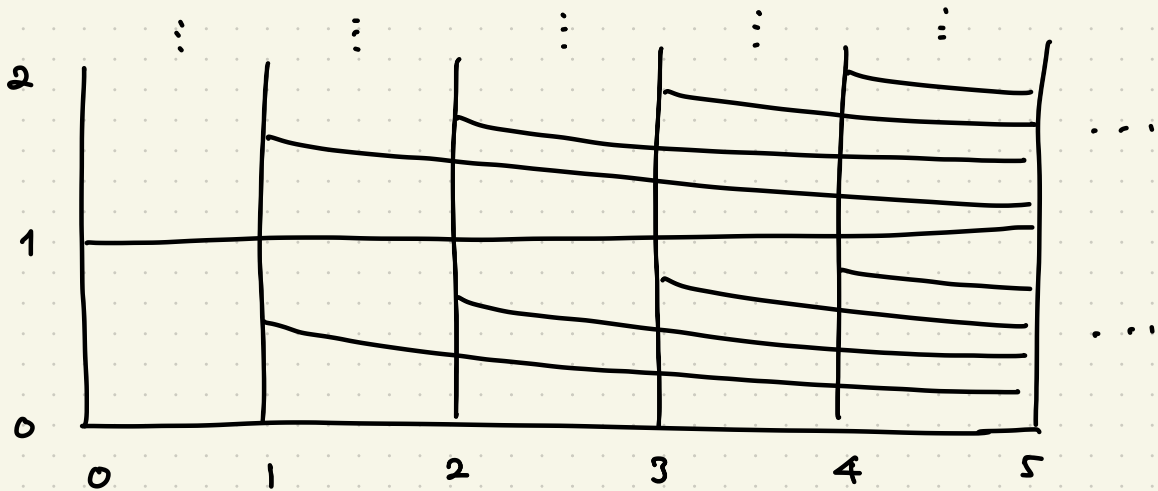
① Find a lecture hall partition model for Askey-Wilson.

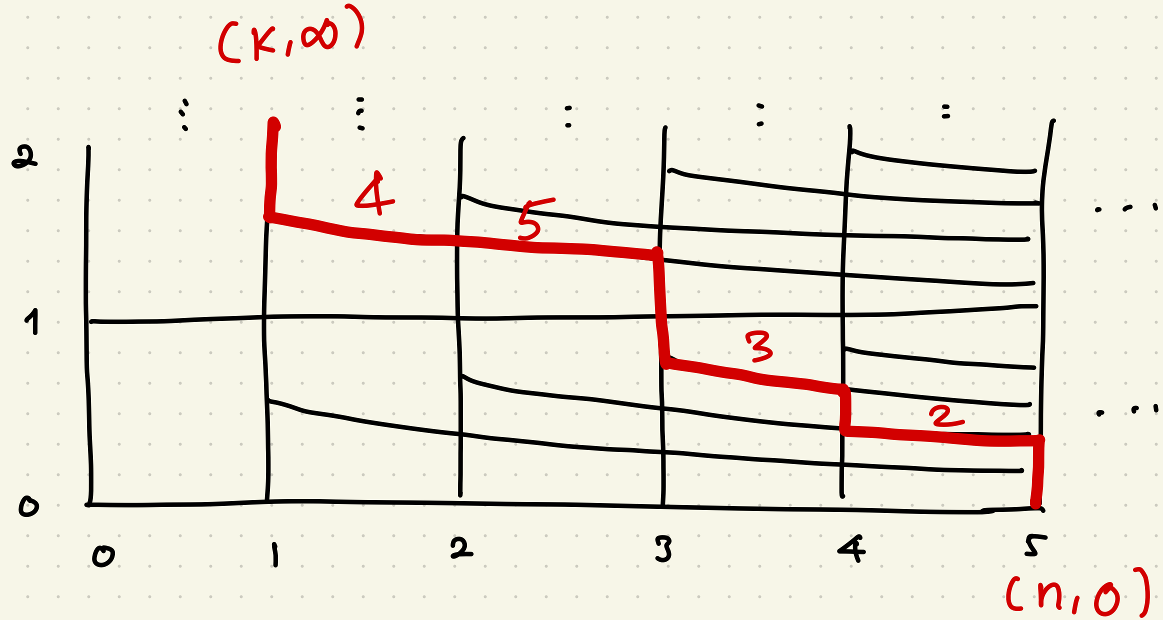
② Find a combinatorial proof of the symmetry in a, b, c, d.

Q. Why can't we generalize Corteel-Kim result on little q-Jacobi to Askey-Wilson?

→ We need a new approach to "guess" a combinatorial model.

Lecture hall graph (Introduced by Corteel–Kim 2020
Further studied by Corteel–Keating–Nicoletti 2021)





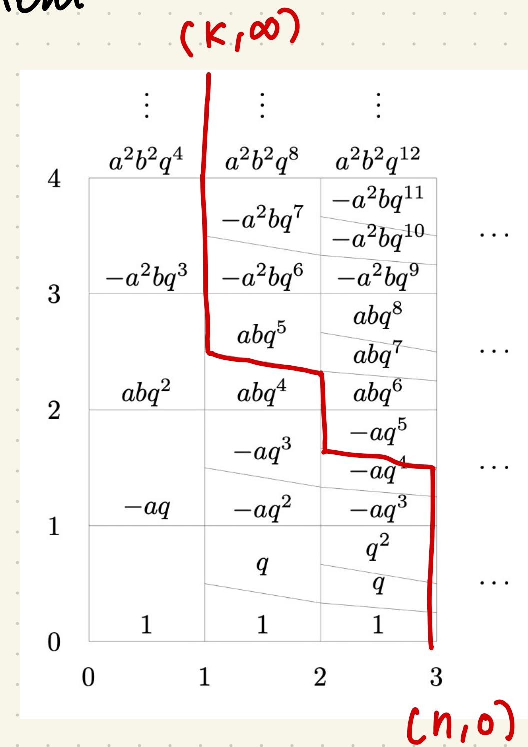
$$\frac{4}{2} \geqslant \frac{5}{3} \geqslant \frac{3}{4} \geqslant \frac{2}{5} \qquad (k=1,\ n=5).$$

$$\frac{\lambda_{k+1}}{k+1} \geqslant \frac{\lambda_{k+2}}{k+2} \geqslant \cdots \geqslant \frac{\lambda_n}{n} \quad \longleftrightarrow \quad \text{path from } (k,\infty) \text{ to } (n,0)$$

Thm (Corteel, K., 2020)

little q-Jacobi mixed moment

$$\mu_{n,k} = \sum_{p:(k,\infty)\to(n,0)} wt(p)$$



Thm (CJKK, 2023)

big q-Jacobi

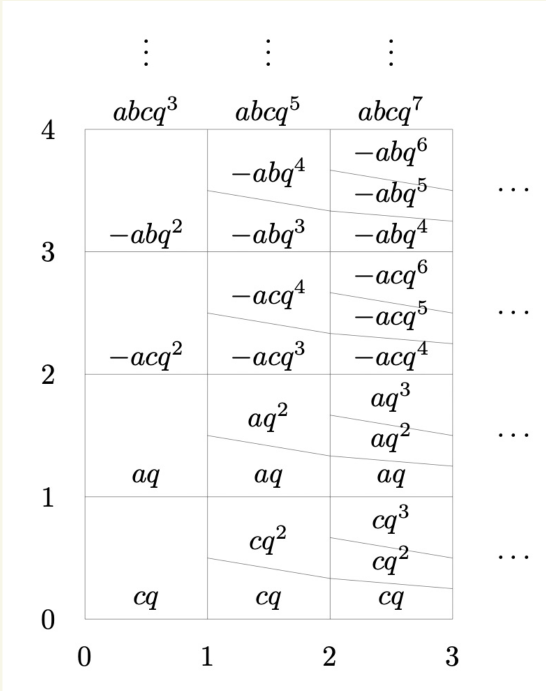
$$P_n(x;a,b,c|q) = \frac{(aq,cq;q)_n}{(abq^{n+1};q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}x \\ aq, cq \end{matrix}; q, q\right)$$

$$\mu_{n,k} = \sum_{p:(k,\infty)\to(n,0)} wt(p)$$

Cor Let $\tilde{\mu}_{n,k} = (-1)^{n-k}\mu_{n,k}(-a,-b,-c)$.

$\Rightarrow$ $(\tilde{\mu}_{n,k})_{n,k=0}^{\infty}$ is totally positive.

Equivalently multivariate big q-Jacobi is Schur positive.



Thm (CJKK, 2023)

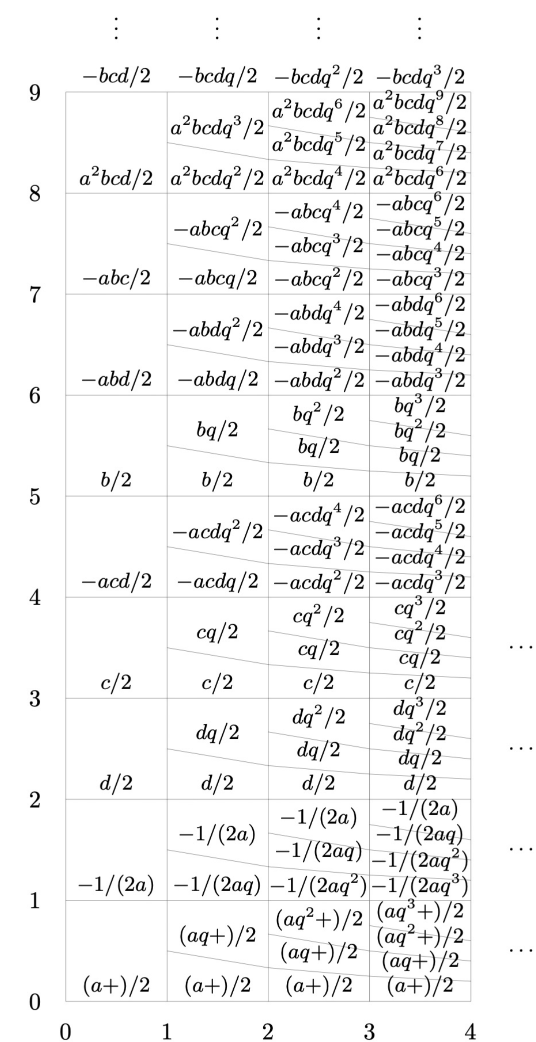
Askey-Wilson mixed moment

$$\mu_{n,k} = \sum_{p:(k,\infty)\to(n,0)} wt(p)$$

$$\nu_{n,k} = \sum_{p:(k,0)\to(n,\infty)} wt(p)$$

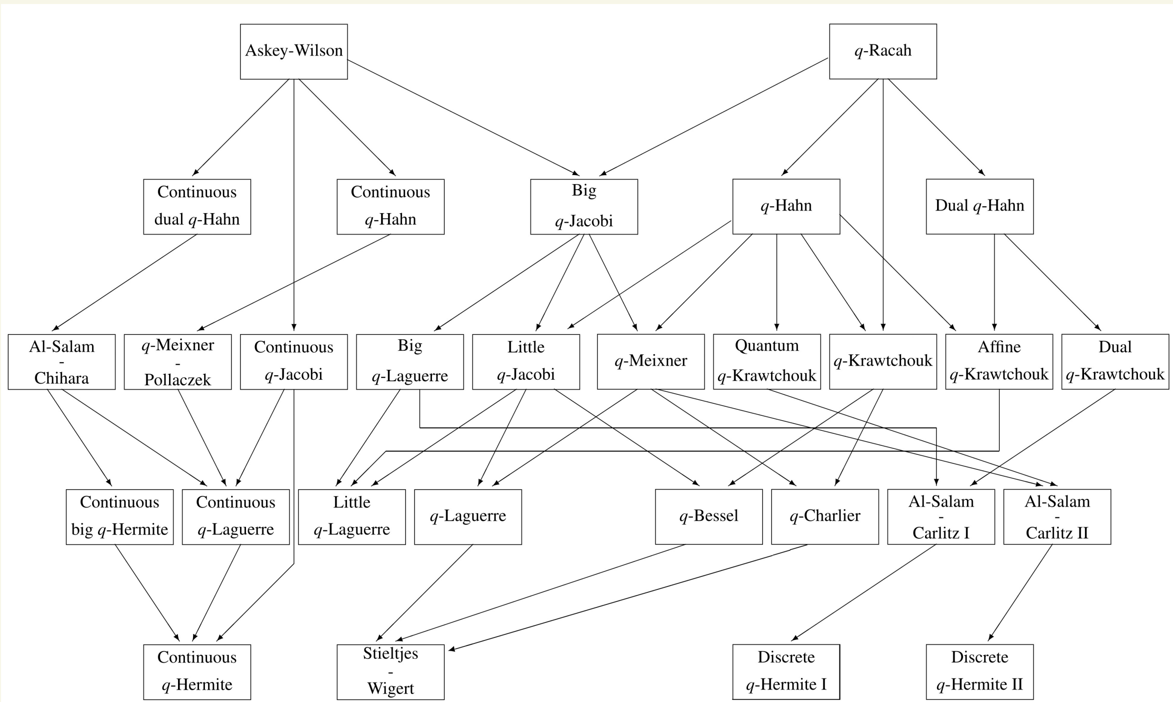
no consecutive east step.

$(x+) = x + x^{-1}$



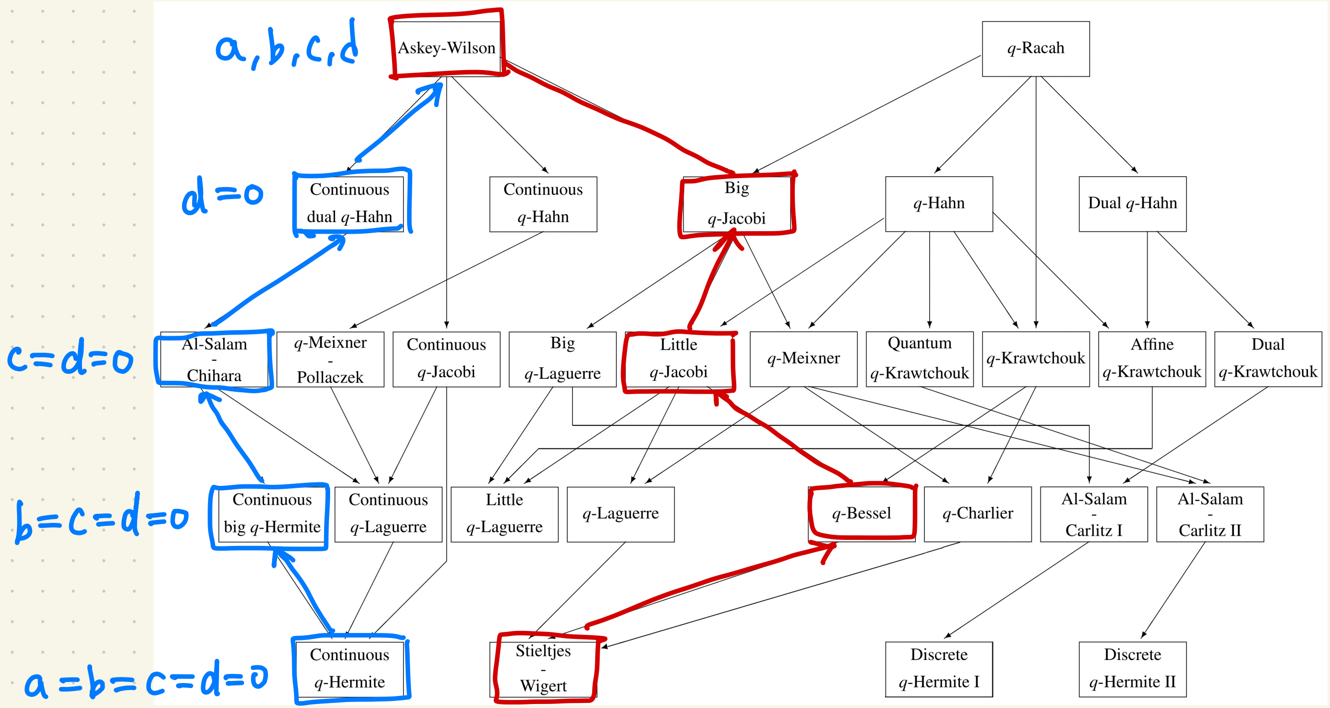
Thm (CJKK, 2023).

There is a lecture hall graph model for $\mu_{n,k}$ and $\nu_{n,k}$ for every orthogonal polynomial in Askey scheme



Idea of proof

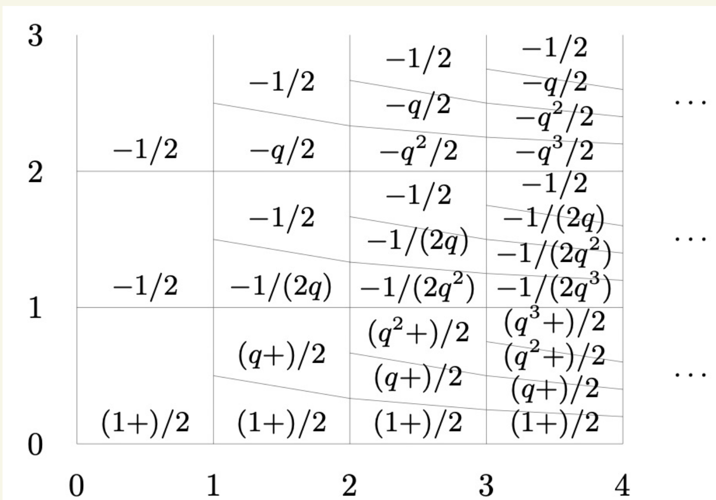
Two bootstrapping methods



Thm (CJKK)

Continuous q-Hermite

$$\mu_{n,k} = \sum_{p:(k,3)\to(n,0)} wt(p)$$



Thm (CJKK)

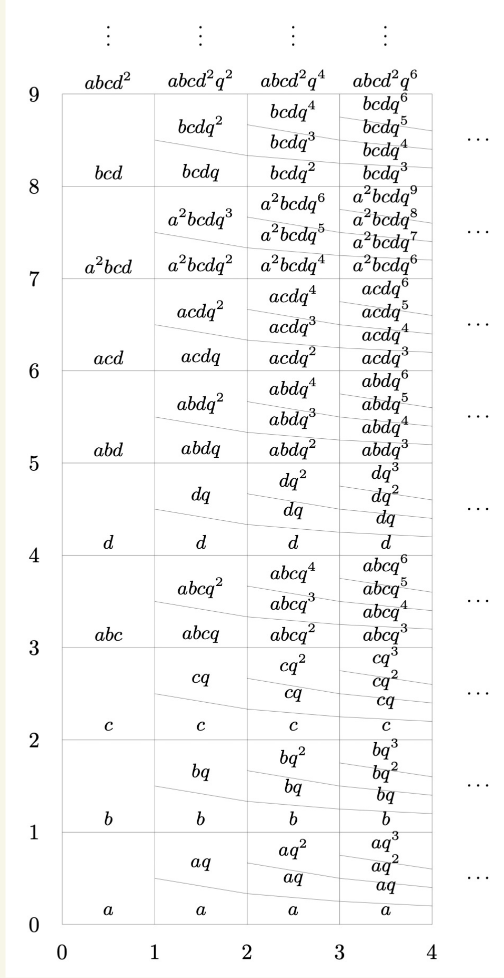
mixed moment of AW with respect to q-Hermite

$$H_n(x) = \sum_k \tilde{\mu}_{n,k} \, p_k^{AW}(x)$$

$$\Rightarrow \frac{2^{n-k}}{i} \tilde{\mu}_{n,k}(ai, bi, ci, di)$$

$$= \sum_{p:(k,\infty)\to(n,0)} wt(p)$$

Cor Askey-Wilson is symetric in a, b, c, d.



Advantages of lecture hall graph models.

1. a model for $\mu_{n,k}$ gives a model for $\nu_{n,k}$.

2. multivariate moments can be obtained by Lindström–Gessel–Viennot Lemma

3. Total positivity

4. Applicable to all OPS in Askey scheme.

Thank You!

Moments of multivariate orthogonal polynomials

Symmetric polynomials

Def) $F(x_1, \dots, x_n)$ is symmetric if

$$F(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = F(x_1, \dots, x_n) \quad \text{for all } \sigma \in S_n.$$

When we have a basis $\{p_m(x)\}_{m \geq 0}$ of univariate poly, we can construct a basis $\{p_\lambda(\mathbb{x})\}_\lambda$ of sym. poly.

by

$\mathbb{x} = (x_1, \dots, x_n)$

$$p_\lambda(\mathbb{x}) = \frac{\det(p_{\lambda_i + n - i}(x_j))}{\det(p_{n-i}(x_j))}.$$

Def) The Schur function $s_\lambda(x_1, \dots, x_n)$ is defined in this way with basis $\{x^m\}_{m \geq 0}$:

$$s_\lambda(\mathbb{x}) = \frac{\det(x_j^{\lambda_i + n - i})}{\det(x_j^{n-i})}.$$

Multivariate little q-Jacobi poly.

Def) $$P_\lambda(\mathbf{x};a,b;q) = \frac{\det\left(P_{\lambda_i+n-i}(x_j;a,b;q)\right)}{\det\left(P_{n-i}(x_j;a,b;q)\right)}.$$

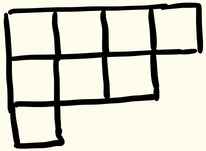
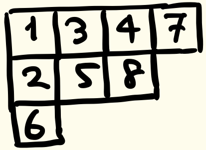
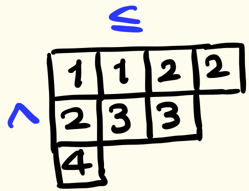
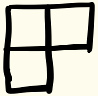
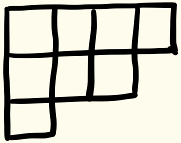
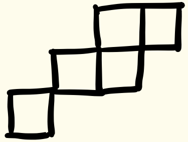
Since both P_λ and s_λ are bases for the space of symmetric polynomials, we can expand

$$s_\lambda = \sum_\mu \underbrace{M_{\lambda\mu}}_{\text{mixed moment}} P_\mu, \qquad P_\lambda = \sum_\mu \underbrace{N_{\lambda\mu}}_{\text{dual mixed moment}} s_\mu.$$

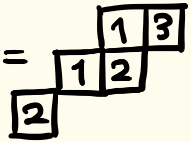
Multivariate little q-Jacobi polynomials

& Lecture hall tableaux

Basic definitions

- $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a partition of n if $\lambda_1 \geq \dots \geq \lambda_\ell > 0$ and $\lambda_1 + \dots + \lambda_\ell = n$
- Young diagram of $\lambda = (4,3,1)$ is 
- Standard Young tableau 
- semistandard Young tableau 
- $\mu =$  $\subseteq \lambda =$  skew shape $\lambda/\mu =$ 
- Schur function

$$s_{\lambda/\mu}(x) = \sum_{T \in SSYT(\lambda/\mu)} x^T ,$$

$T =$  , $x^T = x_1^2 x_2^2 x_3^1$

Lecture hall tableaux

T : a filling of λ with nonnegative integers

T is a **lecture hall tableau** of type $(n, \geq, >)$ if

$$\frac{T(i,j)}{n+c(i,j)} \geq \frac{T(i,j+1)}{n+c(i,j+1)}, \qquad \frac{T(i,j)}{n+c(i,j)} > \frac{T(i+1,j)}{n+c(i+1,j)}$$

$c(i,j) = j - i$ (content)

ex)

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 8 & 9 & 10 & 9 & 4 & 3 \\ \hline 4 & 5 & 6 & 4 & 3 & 1 \\ \hline 2 & 2 & 1 & 0 & & \\ \cline{1-4} 1 & 0 & 0 & & & \\ \cline{1-3} \end{array}$$

LHT of type $(5, \geq, >)$

$$\geq$$

$$> \quad \begin{array}{|c|c|c|c|c|c|} \hline 8/5 & 9/6 & 10/7 & 9/8 & 4/9 & 3/10 \\ \hline 4/4 & 5/5 & 6/6 & 4/7 & 3/8 & 1/9 \\ \hline 2/3 & 2/4 & 1/5 & 0/6 & & \\ \cline{1-4} 1/2 & 0/3 & 0/4 & & & \\ \cline{1-3} \end{array}$$

If $n \to \infty$, LHT is (reverse) SSYT.

The multivariate little q-Jacobi polynomials (with $t=q$)

$$P_\lambda(x) = P_\lambda(x_1,\ldots,x_n; a,b; q)$$

$$s_\lambda(x) = \sum_{\mu \subseteq \lambda} M_{\lambda,\mu} P_\mu(x), \qquad P_\lambda(x) = \sum_{\mu \subseteq \lambda} N_{\lambda,\mu} s_\mu(x)$$

Thm (Corteel, K., 2018)

$M_{\lambda,\mu}$ = g.f. for LHT of shape λ/μ of type $(n, \geq, >)$

$N_{\lambda,\mu}$ = " " " $(n, <, \leq)$

The "moment" $M_{\lambda,\emptyset}$ has a product formula.

The "dual moment" $N_{\lambda,\emptyset}$ " .