§7.3. Hankel determinants of moments.

Let $\{p_n(x)\}_{n\geq 0}$ be a monic OPS with moments μ_n s.t.

$$p_{n+1} = (x-b_n)p_n - \lambda_n p_{n-1}.$$

Def) the <u>Hankel matrix</u> H of $\{\mu_n\}_{n\geq 0}$ is

$$H = (\mu_{i+j})_{i,j=0}^{\infty}$$

$$= \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots \\ \mu_1 & \mu_2 & & \\ \mu_2 & & \ddots & \\ \vdots & & & \end{pmatrix}$$

The <u>Hankel determinant</u>

$$\Delta_n = [H]_{\{0,\dots,n\},\{0,\dots,n\}}$$

$$= \det(\mu_{i+j})_{i,j=0}^{n}$$

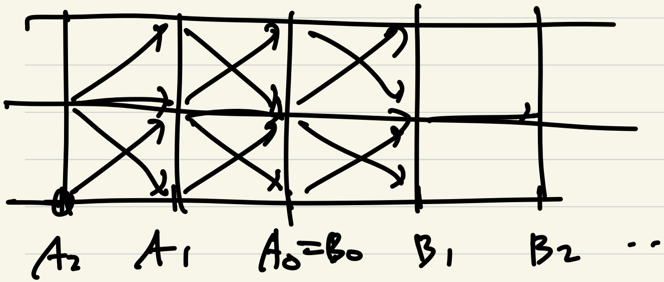
$$= \det\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}$$

$$\mu_{i+j} = \sum_{p\in \mathrm{Motz}((0,0)\to(i+j,0))} \mathrm{wt}(p)$$

$A = (A_0, \ldots, A_n)$

$B = (B_0, \ldots, B_n)$.

$A_i = (-i, 0)$, $B_i = (i, 0)$

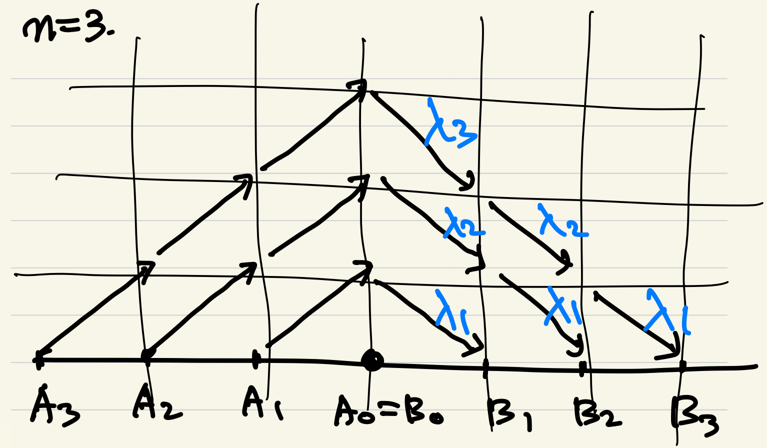


$$\mu_{i+j} = \sum_{p \in \mathrm{Motz}(A_i \to B_j)} \mathrm{wt}(p).$$

By LGV,

$$\Delta_n = \det(\mu_{ij})_0^n = \sum_{\mathbb{P} \in NI(A \to B)} \mathrm{sgn}(\mathbb{P}) \mathrm{wt}(\mathbb{P}).$$

$n = 3$.



$NI(A \to B)$ has a unique elt. $\mathbb{P}$

↳ $\mathrm{sgn} = 1$, $\mathrm{wt} = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1$

<u>Thm</u> $\Delta_n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1$

Define

$$\Delta_n' = [H]_{\{0,\dots,n\},\{0,\dots,n-1,n+1\}}$$

$$= \det(M_{ij})_{i,j=0}^{n}.$$

$M_{ij} = \mu_{i+j}$ if $j<n$

$M_{in} = \mu_{i+n+1}$.

$A = (A_0, \dots A_n)$

$B' = (B_0, \dots B_n)$

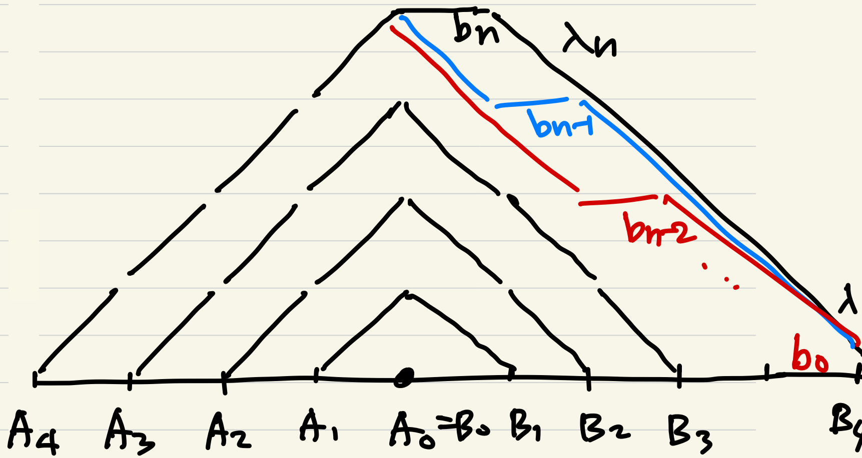
$A_i = (-i, 0)$.

$B_i = (i + \delta_{n,i}, 0)$.

$$\Rightarrow \Delta_n' = \det(M_{ij})_0^n$$

By LGV,

$$\Delta_n' = \sum_{P \in NI(A \to B')} \operatorname{sgn}(p)\, wt(p).$$



There are $n+1$ elts in $NI(A \to B')$ with $\operatorname{sgn} = 1$ wts $= \Delta_n b_k$ $0 \le k \le n$.

Thm $\Delta_n' = \Delta_n (b_0 + \cdots + b_n)$.

$$\left(\Delta_n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1 \right)$$

$$\lambda_1 \cdots \lambda_n = \frac{\Delta_n}{\Delta_{n-1}}$$

$$\Rightarrow \quad \lambda_n = \frac{\Delta_n}{\Delta_{n-1}} \left(\frac{\Delta_{n-1}}{\Delta_{n-2}} \right)^{-1}$$

$$= \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}$$

$$b_0 + \cdots + b_n = \frac{\Delta_n'}{\Delta_n}$$

$$b_n = \frac{\Delta_n'}{\Delta_n} - \frac{\Delta_{n-1}'}{\Delta_{n-1}}.$$

Cor $\lambda_n = \dfrac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}$

$$b_n = \frac{\Delta_n'}{\Delta_n} - \frac{\Delta_{n-1}'}{\Delta_{n-1}}.$$

Cor $\{\mu_n\}$: a seq of numbers.

$\exists$ OPS with moments μ_n

$\iff \Delta_n \neq 0 \quad \forall n \geq 0$.

pf) Recall $\{P_n\}$ OPS $\iff \lambda_n \neq 0$.

$(\Rightarrow)$ P_n OPS $\Rightarrow \lambda_n \neq 0 \Rightarrow \Delta_n \neq 0$.

$(\Leftarrow)$ Define λ_n, b_n using Cor.

Define P_n using λ_n, b_n.

$\Rightarrow$ moment of $P_n = \mu_n$. $\square$.

Now suppose $b_n = 0$. $\forall n$.

<u>Recall</u> $b_n = 0 \iff \mu_{2n+1} = 0$.

$$\Delta_3 = \det \begin{pmatrix} \boxed{\mu_0} & & \boxed{\mu_2} & \\ & \mu_2 & & \mu_4 \\ \boxed{\mu_2} & & \boxed{\mu_4} & \\ & \mu_4 & & \mu_6 \end{pmatrix}$$

$$= \det \begin{pmatrix} \mu_0 & \mu_2 \\ \mu_2 & \mu_4 \end{pmatrix} \cdot \det \begin{pmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{pmatrix}.$$

Let $\Delta_n(2) = \det \left(\mu_{2i+2j} \right)_{i,j=0}^{n}$

$$\Delta_n^+(2) = \det \left(\mu_{2i+2j+2} \right)_{i,j=0}^{n}$$

$$\Delta_4 = \det \begin{pmatrix} \boxed{\mu_0} & 0 & \boxed{\mu_2} & 0 & \boxed{\mu_4} \\ 0 & \boxed{\mu_2} & 0 & \boxed{\mu_4} & 0 \\ \boxed{\mu_2} & 0 & \boxed{\mu_4} & 0 & \boxed{\mu_6} \\ 0 & \boxed{\mu_4} & 0 & \boxed{\mu_6} & 0 \\ \boxed{\mu_4} & 0 & \boxed{\mu_6} & 0 & \boxed{\mu_8} \end{pmatrix}$$

$$= \det \begin{pmatrix} \mu_0 & \mu_2 & \mu_4 \\ \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \end{pmatrix} \det \begin{pmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{pmatrix}$$

In general

$$\Delta_{2n} = \Delta_n(2)\, \Delta_{n-1}^+(2)$$

$$\Delta_{2n+1} = \Delta_n(2)\, \Delta_n^+(2).$$

Since $b_n = 0$,

$$\mu_{2n} = \sum_{p \in \mathrm{Dyck}_{2n}} \mathrm{wt}(p)$$

Let $A = (A_0, \dots A_n)$
$B = (B_0, \dots B_n)$

$A_i = (-2i, 0)$, $B_i = (2i, 0)$.

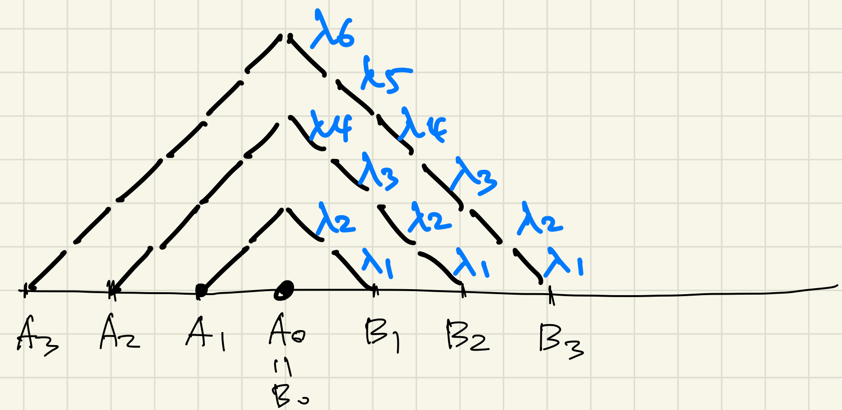
$$\mu_{2i+2j} = \sum_{p \in \mathrm{Dyck}(A_i \to B_j)} \mathrm{wt}(p)$$

By LGV.

$$\Delta_n(2) = \det\left(\mu_{2i+2j}\right)_0^n$$

$$\Delta_n(2) = \sum_{P \in NI(A \to B)} \mathrm{sgn}(P)\, \mathrm{wt}(P).$$

$$= (\lambda_1 \lambda_2)^n (\lambda_3 \lambda_4)^{n-1} \cdots (\lambda_{2n-1} \lambda_{2n})^1$$



Since $b_n=0$,

$$\mu_{2n} = \sum_{p\in \mathrm{Dyck}_{2n}} wt(p)$$

Let $A=(A_0,\dots\dots A_n)$

$B^+=(B_0^+,\dots B_n^+)$

$A_i=(-2i,0)$, $B_i^+=(2i+2,0)$

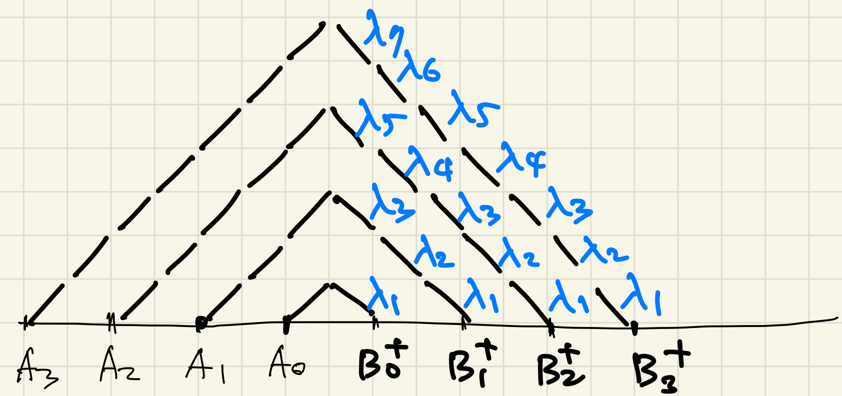
$$\mu_{2i+2j+2} = \sum_{p\in \mathrm{Dyck}(A_i\to B_j^+)} wt(p)$$

By LGV.

$$\Delta_n^+(2)=\det\left(\mu_{2i+2j+2}\right)_0^n$$

$$\Delta_n^+(2) = \sum_{P\in NI(A\to B^+)} sgn(P)\, wt(P).$$

$$= \lambda_1^{n+1}(\lambda_2\lambda_3)^n(\lambda_4\lambda_5)^{n-1}\cdots(\lambda_{2n}\lambda_{2n+1})^1$$



Cor If $b_n=0$ for all $n\geq 0$

then

$$\lambda_{2n} = \frac{\Delta_n(2)\,\Delta^{+}_{n-2}(2)}{\Delta_{n-1}(2)\,\Delta^{+}_{n-1}(2)}, \quad \lambda_{2n+1} = \frac{\Delta^{+}_n(2)\,\Delta_{n-1}(2)}{\Delta_n(2)\,\Delta^{+}_{n-1}(2)}.$$

pf)

$$\frac{\Delta^{+}_n(2)}{\Delta_n(2)} = \lambda_1\lambda_3\cdots\lambda_{2n+1}. \qquad \frac{\Delta_n(2)}{\Delta^{+}_{n-1}(2)} = \lambda_2\lambda_4\cdots\lambda_{2n}.$$