HOMEWORK

Contents

1. Homework 1 (Due: Sep 21)

Problem 1.1. Let $\mathcal{L}$ be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n\geq 0}$. Prove the following extremal property: for any monic real polynomial $\pi(x) \neq P_n(x)$ of degree n,

$$\mathcal{L}(P_n(x)^2) < \mathcal{L}(\pi(x)^2).$$

Solution. Let $\pi(x) = \sum_{k=0}^n a_k P_k(x)$. Since both $\pi(x)$ and $P_n(x)$ are monic, we have $a_n = 1$ [**3 points**]. Then

$$\begin{aligned}\mathcal{L}(\pi(x)^2) &= \sum_{k=0}^n a_k^2 \mathcal{L}(P_k(x)^2) \quad \textbf{[4 points]}\\ &\geq a_n^2 \mathcal{L}(P_n(x)^2) = \mathcal{L}(P_n(x)^2) \quad \textbf{[3 points]}.\end{aligned}$$

□

Problem 1.2. Let $\mathcal{L}$ be a linear functional such that $\Delta_n \neq 0$ for all $n \geq 0$. Prove that if $\pi(x)$ is a polynomial such that $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \geq 0$, then $\pi(x) = 0$.

Solution. Since $\Delta_n \neq 0$, there is a monic OPS $\{P_n(x)\}_{n\geq 0}$ for $\mathcal{L}$ [**3 points**]. Let $\pi(x) = \sum_{k=0}^n a_k P_k(x)$. Since $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \geq 0$, we have $\mathcal{L}(p(x)\pi(x)) = 0$ for any polynomial $p(x)$ [**3 points**]. Then, for each $0 \leq k \leq n$, we have $0 = \mathcal{L}(P_k(x)\pi(x)) = a_k \mathcal{L}(P_k(x)^2)$ [**2 points**]. Since $\mathcal{L}(P_k(x)^2) \neq 0$, we get $a_k = 0$ for all $0 \leq k \leq n$ [**2 points**]. Hence $\pi(x) = 0$.

A common mistake: It is not true in general that $\mathcal{L}(x^k P_n(x)) = 0$ for $k \neq n$. We can only say that $\mathcal{L}(x^k P_n(x)) = 0$ for $k < n$. □

Problem 1.3. The *Tchebyshev polynomials of the second kind* $U_n(x)$ are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta, \qquad n \geq 0.$$

(1) Prove that $U_n(x)$ is a polynomial of degree n.
(2) Prove that
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n \geq 0,$$
where $U_{-1}(x) = 0$ and $U_0(x) = 1$.
(3) Prove that
$$\int_{-1}^1 U_m(x)U_n(x)(1-x^2)^{1/2}dx = \frac{\pi}{2}\delta_{m,n}.$$
(4) Find the 3-term recurrence for the normalized Tchebyshev polynomials of the second kind. More precisely, find the numbers b_n and λ_n such that
$$\hat{U}_{n+1}(x) = (x - b_n)\hat{U}_n(x) - \lambda_n \hat{U}_{n-1}(x), \qquad n \geq 0,$$
where $\hat{U}_n(x)$ is the monic polynomial that is a scalar multiple of $U_n(x)$.

Solution. (1) This follows from (2) [**2 points**].

(2) By the addition rule for the sine function,

$$\begin{aligned}\sin(n+1)\theta &= \sin n\theta\cos\theta + \cos n\theta \sin\theta,\\ \sin(n-1)\theta &= \sin n\theta\cos\theta - \cos n\theta\sin\theta.\end{aligned}$$

Adding the two equations and dividing both sides by $\sin\theta$, we get

$$U_n(x) + U_{n-2}(x) = 2xU_{n-1}(x), \qquad n \geq 1 \quad \textbf{[2 points]}.$$

This is equivalent to the recurrence in the problem.

(3) By the change of variables $x = \cos\theta$, $0 \le \theta \le \pi$, with $dx = -\sin\theta d\theta = -\sqrt{1-x^2}d\theta$,

$$\begin{aligned}
&\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx \\
&= \int_0^\pi \sin(m+1)\theta \sin(n+1)\theta d\theta \quad \textbf{[2 points]} \\
&= \frac{1}{2}\int_0^\pi \left(\cos(m-n)\theta + \cos(m+n)\theta\right) d\theta \quad \textbf{[2 points]} \\
&= \frac{\pi}{2}\delta_{m,n}.
\end{aligned}$$

(4) Since $\deg U_n(x) = 2^n$ for all $n \ge 0$, we have $\hat{U}_n(x) = 2^{-n}U_n(x)$. Dividing both sides of the recurrence in (2) by 2^{n+1}, we obtain $b_n = 0$ and $\lambda_n = 1/4$ **[2 points]**. □

Problem 1.4. Let $\{P_n(x)\}_{n\ge 0}$ be the monic OPS for a linear functional $\mathcal{L}$ with three-term recurrence

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \ge 0.$$

(1) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & 1 & & 0 \\ \lambda_1 & x-b_1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & \lambda_{n-1} & x-b_{n-1} \end{vmatrix}.$$

(2) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & \sqrt{\lambda_1} & & 0 \\ \sqrt{\lambda_1} & x-b_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\lambda_{n-1}} \\ 0 & & \sqrt{\lambda_{n-1}} & x-b_{n-1} \end{vmatrix}.$$

(3) Using (2) prove that if $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all, then $P_n(x)$ has real roots only.

Solution. (1) Let $Q_n(x)$ be the determinant on the right-hand side. Expanding the determinant along the last row, we obtain the recursion

$$Q_n(x) = (x-b_{n-1})Q_{n-1}(x) - \lambda_{n-1}Q_{n-2}(x) \quad \textbf{[2 points]}.$$

Since $P_n(x)$ and $Q_n(x)$ satisfy the same recurrence with with the initial conditions $Q_0(x) = 1$ and $Q_1(x) = x - b_0$, we obtain that $Q_n(x) = P_n(x)$.

(2) Let $A_n = (\alpha_{i,j})$ be the matrix in (1) and let B_n be the matrix in (2). Then it suffices to find an invertible diagonal matrix $D = \mathrm{diag}(d_i)$ such that $B_n = DA_nD^{-1}$ **[3 points]**. To do this, observe that $DA_nD^{-1} = (d_i\alpha_{i,j}d_j^{-1})$. Since B_n and DA_nD^{-1} are tri-diagonal matrices, we have $B_n = DA_nD^{-1}$ if and only if the following hold:

$$\beta_{i,i} = d_i\alpha_{i,i}d_i^{-1}, \tag{1.1}$$

$$\beta_{i,i+1} = d_i\alpha_{i,i+1}d_{i+1}^{-1}, \tag{1.2}$$

$$\beta_{i+1,i} = d_{i+1}\alpha_{i+1,i}d_i^{-1}. \tag{1.3}$$

Since $\alpha_{i,i+1} = 1$ and $\beta_{i,i+1} = \sqrt{\lambda_{i+1}}$, (1.2) is equivalent to $d_{i+1} = d_i/\sqrt{\lambda_{i+1}}$. Indeed, if we set $d_0 = 1$ and $d_{i+1} = d_i/\sqrt{\lambda_{i+1}}$, then all three conditions above hold **[3 points]**.

Note: Alternatively, it can be proved directly that the right-hand side of the equation satisfies the same recurrence as $P_n(x)$.

(3) Since the zeros of $P_n(x)$ are the eigenvalues of a real symmetric matrix, they are real **[2 points]**. □

2. Homework 2 (Due: Oct 5)

Problem 2.1. Let id be the identity permutation.

(1) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^2 = id$.
(2) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^3 = id$.
(3) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^4 = id$.
(4) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^5 = id$.
(5) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^6 = id$.

Solution. We have $\pi^k = id$ if and only if every cycle of π is of length divisible by k. For example, if $\pi^6 = id$, then the decreasing sequence of the lengths of cycles of π must be (6), $(3,3)$, $(3,2,1)$, $(3,1,1,1)$, $(2,2,2)$, $(2,2,1,1)$, $(2,1,1,1,1)$, $(1,1,1,1,1,1)$. The number of such permutations is $5!$, $\binom{6}{3}\frac{1}{2}2^2$, $\binom{6}{3}\binom{3}{2}\cdot 2$, $\binom{6}{3}\cdot 2$, $5\cdot 3$, $\binom{6}{4}\cdot 3$, $\binom{6}{2}$, 1, respectively. In this way we get the answers as follows.

(1) 76 **[2 points]**
(2) 81 **[2 points]**
(3) 256 **[2 points]**
(4) 145 **[2 points]**
(5) 396 **[2 points]**

$\square$

Problem 2.2. Let $c_1, \dots, c_n$ be a sequence of nonnegative integers such that $\sum_{i=1}^n ic_i = n$. Show that the number of permutations $\pi \in \mathfrak{S}_n$ with c_i cycles of length i for all $i = 1, \dots, n$ is

$$\frac{n!}{\prod_{i=1}^n i^{c_i} c_i!}.$$

Solution. Let X be the set of such permutations. We construct a map $\phi : \mathfrak{S}_n \to X$ as follows. Given $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, let $\phi(\sigma)$ be the permutation whose cycle notation is obtained from the word $\sigma_1 \cdots \sigma_n$ by placing parentheses so that the first c_1 cycles are of length 1, the next c_2 cycles are of length 2, and so on **[3 points]**. By the construction, this gives a map $\phi : \mathfrak{S}_n \to X$.

For any $\pi \in X$, there are $c_i!$ ways to arrange its c_i cycles and i ways to cyclically shift each each of these cycles. Therefore, there are $\prod_{i=1}^n i^{c_i} c_i!$ permutations $\sigma \in \mathfrak{S}_n$ whose image under ϕ is π **[4 points]**. This shows that $|X| = |\mathfrak{S}_n| / \prod_{i=1}^n i^{c_i} c_i!$ as desired **[3 points]**. $\square$

Problem 2.3. For $\pi \in \mathfrak{S}_n$, let $\ell(\pi)$ be the smallest number of simple transpositions whose product is π. Prove that $\ell(\pi) = \operatorname{inv}(\pi)$.

Solution. Suppose that $\pi = s_1 \cdots s_r$ for some simple transpositions s_i's. Since multipling a simple transposition increases or decreases the number of inversions by 1, we have $r \geq \operatorname{inv}(\pi)$ **[3 points]**. Hence $\ell(\pi) \geq \operatorname{inv}(\pi)$ **[2 points]**.

On the other hand, we can find an expression $\pi = s_1 \cdots s_r$ with $r = \operatorname{inv}(\pi)$ by sorting $\pi = \pi_1 \cdots \pi_n$ **[3 points]** because multiplying the simple transposition $(i, i+1)$ to the right of $\pi = \pi_1 \cdots \pi_n$ gives

$$\pi(i, i+1) = \pi_1 \cdots \pi_{i-1} \pi_{i+1} \pi_i \pi_{i+1} \cdots \pi_n.$$

This implies $\ell(\pi) \leq \operatorname{inv}(\pi)$ **[2 points]**. Thus, $\ell(\pi) = \operatorname{inv}(\pi)$. $\square$

Problem 2.4. Prove that

$$\sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{inv}(\pi)} = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}).$$

Solution. We proceed by induction on n. If $n = 1$, it is true. Let $n \geq 2$ and suppose the statement holds for $n-1$. Every $\pi \in \mathfrak{S}_n$ is obtained from $\sigma \in \mathfrak{S}_{n-1}$ by inserting n after j integers from the beginning for some $0 \leq j \leq n-1$ **[3 points]**. This construction gives $\operatorname{inv}(\pi) = \operatorname{inv}(\sigma) + j$

[3 points]. Thus

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = \sum_{\sigma \in \mathfrak{S}_{n-1}} \sum_{j=0}^{n-1} q^{\mathrm{inv}(\sigma)+j} = \sum_{\sigma \in \mathfrak{S}_{n-1}} q^{\mathrm{inv}(\sigma)}(1+q+\cdots+q^{n-1}) \quad \textbf{[2 points]}$$
$$= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) \quad \textbf{[2 points]}.$$

Thus the statement is also true for n and we are done. $\square$

3. Homework 3 (Due: Oct 19)

Problem 3.1. Suppose that $\{P_n(x)\}_{n\ge 0}$ is a monic OPS for a linear functional $\mathcal{L}$ with $\mathcal{L}(1) = 1$ given by $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \ge 0$,

$$P_{n+1}(x) = (x - n)P_n(x) - nP_{n-1}(x).$$

Compute the following.

(1) $\mathcal{L}(x^3)$
(2) $\mathcal{L}(P_{10}(x)P_{10}(x))$
(3) $\mathcal{L}(x^3P_{10}(x)P_{12}(x))$

Solution. We can compute these quantities using

$$\mathcal{L}(x^nP_r(x)P_s) = \lambda_1 \cdots \lambda_s \sum_{\pi\in\mathrm{Motz}((0,r)\to(n,s))} \mathrm{wt}(\pi).$$

(1) $\mathcal{L}(x^3) = 1$ [**3 points**]
(2) $\mathcal{L}(P_{10}(x)P_{10}(x)) = 10!$ [**3 points**]
(3) $\mathcal{L}(x^3P_{10}(x)P_{12}(x)) = 33 \cdot 12!$ [**4 points**]

□

Problem 3.2. A *left-to-right minimum* of a permutation $\pi = \pi_1 \cdots \pi_n$ is a number π_i such that $\pi_i = \min\{\pi_1, \dots, \pi_i\}$. Let $\mathrm{LRmin}(\pi)$ denote the number of left-to-right minima in π. For example, if $\pi = 6741352$, then the left-to-right minima are $6, 4, 1$, hence $\mathrm{LRmin}(\pi) = 3$. Prove that

$$\sum_{\pi\in\mathfrak{S}_n} \alpha^{\mathrm{cycle}(\pi)} = \sum_{\pi\in\mathfrak{S}_n} \alpha^{\mathrm{LRmin}(\pi)}.$$

Solution. We can uniquely write the cycles of a permutation $\pi \in \mathfrak{S}_n$ so that each cycle starts with its smallest element and the cycles are listed in the decreasing order of their smallest elements [**3 points**]. For example,

$$\pi = (5, 11)(3)(1, 4, 2, 9, 10, 7, 6, 8).$$

Let $\widehat{\pi}$ be the permutation obtained from this list of cycles by deleting the parentheses [**4 points**]. In the example above,

$$\widehat{\pi} = 5\,11\,3\,1\,4\,2\,9\,10\,7\,6\,8.$$

Then the first elements of the cycles of π are the left-to-right minima of $\widehat{\pi}$ [**3 points**]. Since $\pi \mapsto \widehat{\pi}$ is a bijection we have

$$\sum_{\pi\in\mathfrak{S}_n} \alpha^{\mathrm{cycle}(\pi)} = \sum_{\pi\in\mathfrak{S}_n} \alpha^{\mathrm{LRmin}(\pi)}.$$ □

Problem 3.3. Suppose that $\{P_n(x)\}_{n\ge 0}$ is a monic OPS given by $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \ge 0$,

$$P_{n+1}(x) = (x - 1)P_n(x) - nP_{n-1}(x).$$

Prove that μ_n is equal to the number of involutions in $\mathfrak{S}_n$. (An involution is a permutation π such that π^2 is the identity map.)

Solution. Recall the bijection $\phi : \mathrm{CH}_n \to \Pi_n$ between the Charlier histories of length n and the set partitions of $[n]$ for the case $b_n = n + 1$ and $\lambda_n = n$ [**3 points**]. If $b_n = 1$ and $\lambda_n = n$, then by restricting this map to the Charlier histories with 0 label for every horizontal step, the images are the set partitions in which every block is of size 1 or 2 [**4 points**]. Then we can identify such a set partition as an involution [**3 points**]. This implies the statement in the problem. □

Problem 3.4. Suppose that $\{P_n(x)\}_{n\ge 0}$ is a monic OPS given by $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \ge 0$,

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_nP_{n-1}(x),$$

where $\lambda_n \ne 0$ for all $n \ge 1$.

Using the fact $\mu_n = \sum_{\pi\in\mathrm{Motz}_n} \mathrm{wt}(\pi)$, prove that $\mu_{2n+1} = 0$ for all $n \ge 0$ if and only if $b_n = 0$ for all $n \ge 0$.

Solution. ($\Leftarrow$): Suppose $b_n = 0$ for all $n \geq 0$. Then μ_n is the generating function for Dyck paths, hence $\mu_{2n+1} = 0$ **[3 points]**.

($\Rightarrow$): Suppose that $\mu_{2n+1} = 0$ for all $n \geq 0$. Then we prove $b_n = 0$ for all $n \geq 0$ by induction on n. Since $\mu_1 = b_0$, we have $b_0 = 0$ **[3 points]**. Suppose that $b_i = 0$ for all $0 \leq i < n$. Then

$$\mu_{2n+1} = \sum_{\mathrm{Motz}_{2n+1}} \mathrm{wt}(\pi) = b_n \lambda_1 \cdots \lambda_n \quad \textbf{[4 points]}$$

because all Motzkin paths in Motz_{2n+1} except $U^n H D^n$ has weight 0. Since $\mu_{2n+1} = 0$ and $\lambda_i \leq 0$ for all i, we obtain $b_n = 0$. By induction, $b_n = 0$ for all $n \geq 0$. $\square$