HOMEWORK

Contents

1. Homework 1 (Due: Apr 5)

Problem 1.1. Let A be a nonempty set and let k be a positive integer with $k \leq |A|$. The symmetric group S_A acts on the set B consisting of all subsets of A of cardinality k by

$$\sigma \cdot \{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}.$$

(1) Prove that this is a group action.
(2) Describe explicitly how the elements (1 2) and (1 2 3) act on the six 2-element subsets of $\{1, 2, 3, 4\}$.

Problem 1.2. Let H be a group acting on a set A. Prove that the relation $\sim$ on A defined by $a \sim b$ if and only if $a = hb$ for some $h \in H$ is an equivalence relation. (For each $x \in A$ the equivalence class of x under $\sim$ is called the orbit of x under the action of H. The orbits under the action of H partition the set A.)

Problem 1.3. In each of parts (1) to (5) give the number of nonisomorphic abelian groups of the specified order - do not list the groups:

(1) order 100
(2) order 576
(3) order 1155
(4) order 42875
(5) order 2704

Problem 1.4. In each of parts (1) to (5) give the lists of invariant factors for all abelian groups of the specified order:

(1) order 270
(2) order 9801
(3) order 320
(4) order 105
(5) order 44100

Problem 1.5. In each of parts (1) to (5) give the lists of elementary divisors for all abelian groups of the specified order and then match each list with the corresponding list of invariant factors found in the preceding problem:

(1) order 270
(2) order 9801
(3) order 320
(4) order 105
(5) order 44100

Problem 1.6. Let R be a ring with identity and let S be a subring of R containing the identity. Prove that if u is a unit in S then u is a unit in R. Show by example that the converse is false.

Problem 1.7. Let R be a ring with $1 \neq 0$.

(1) Prove that if a is a zero divisor, then it is not a unit.
(2) Prove that if $ab = ac$ and $a \neq 0$ is not a zero divisor, then $b = c$.

Problem 1.8. Assume R is commutative with $1 \neq 0$. Prove that if P is a prime ideal of R and P contains no zero divisors then R is an integral domain.

Problem 1.9. Let R be a ring with $1 \neq 0$. Let $A = (a_1, a_2, \ldots, a_n)$ be a nonzero finitely generated ideal of R. Prove that there is an ideal B which is maximal with respect to the property that it does not contain A. [Use Zorn's Lemma.]

Problem 1.10. Let $n_1, n_2, \ldots, n_k$ be integers which are relatively prime in pairs: $(n_i, n_j) = 1$ for all $i \neq j$.

(1) Show that the Chinese Remainder Theorem implies that for any $a_1, \ldots, a_k \in \mathbb{Z}$ there is a solution $x \in \mathbb{Z}$ to the simultaneous congruences

$$x \equiv a_1 \bmod n_1, \quad x \equiv a_2 \bmod n_2, \quad \ldots, \quad x \equiv a_k \bmod n_k$$

and that the solution x is unique mod$n = n_1 n_2 \ldots n_k$.

(2) Let $n_i' = n/n_i$ be the quotient of n by n_i, which is relatively prime to n_i by assumption. Let t_i be the inverse of n_i' mod n_i. Prove that the solution x in (a) is given by

$$x = a_1 t_1 n_1' + a_2 t_2 n_2' + \cdots + a_k t_k n_k' \bmod n.$$

Note that the elements t_i can be quickly found by the Euclidean Algorithm as described in Section 2 of the Preliminaries chapter (writing $an_i + bn_i' = (n_i, n_i') = 1$ gives $t_i = b$) and that these then quickly give the solutions to the system of congruences above for any choice of $a_1, a_2, \ldots, a_k$.

(3) Solve the simultaneous system of congruences

$$x \equiv 1 \bmod 8, \quad x \equiv 2 \bmod 25, \quad \text{and } x \equiv 3 \bmod 81$$

and the simultaneous system

$$y \equiv 5 \bmod 8, \quad y \equiv 12 \bmod 25, \quad \text{and} \quad y \equiv 47 \bmod 81$$