HOMEWORK

Contents

1. Homework 1 (Due: Sep 21)

Problem 1.1. Let $\mathcal{L}$ be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n\ge 0}$. Prove the following extremal property: for any monic real polynomial $\pi(x) \ne P_n(x)$ of degree n,

$$\mathcal{L}(P_n(x)^2) < \mathcal{L}(\pi(x)^2).$$

Solution. Let $\pi(x) = \sum_{k=0}^n a_k P_k(x)$. Since both $\pi(x)$ and $P_n(x)$ are monic, we have $a_n = 1$ [3 points]. Then

$$\begin{aligned}\mathcal{L}(\pi(x)^2) &= \sum_{k=0}^n a_k^2 \mathcal{L}(P_k(x)^2) \quad \text{[4 points]}\\ &\ge a_n^2 \mathcal{L}(P_n(x)^2) = \mathcal{L}(P_n(x)^2) \quad \text{[3 points]}.\end{aligned}$$

□

Problem 1.2. Let $\mathcal{L}$ be a linear functional such that $\Delta_n \ne 0$ for all $n \ge 0$. Prove that if $\pi(x)$ is a polynomial such that $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \ge 0$, then $\pi(x) = 0$.

Solution. Since $\Delta_n \ne 0$, there is a monic OPS $\{P_n(x)\}_{n\ge 0}$ for $\mathcal{L}$ [3 points]. Let $\pi(x) = \sum_{k=0}^n a_k P_k(x)$. Since $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \ge 0$, we have $\mathcal{L}(p(x)\pi(x)) = 0$ for any polynomial $p(x)$ [3 points]. Then, for each $0 \le k \le n$, we have $0 = \mathcal{L}(P_k(x)\pi(x)) = a_k \mathcal{L}(P_k(x)^2)$ [2 points]. Since $\mathcal{L}(P_k(x)^2) \ne 0$, we get $a_k = 0$ for all $0 \le k \le n$ [2 points]. Hence $\pi(x) = 0$.

A common mistake: It is not true in general that $\mathcal{L}(x^k P_n(x)) = 0$ for $k \ne n$. We can only say that $\mathcal{L}(x^k P_n(x)) = 0$ for $k < n$. □

Problem 1.3. The *Tchebyshev polynomials of the second kind* $U_n(x)$ are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta, \qquad n \ge 0.$$

(1) Prove that $U_n(x)$ is a polynomial of degree n.
(2) Prove that
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n \ge 0,$$
where $U_{-1}(x) = 0$ and $U_0(x) = 1$.
(3) Prove that
$$\int_{-1}^1 U_m(x)U_n(x)(1-x^2)^{1/2}dx = \frac{\pi}{2}\delta_{m,n}.$$
(4) Find the 3-term recurrence for the normalized Tchebyshev polynomials of the second kind. More precisely, find the numbers b_n and λ_n such that
$$\hat{U}_{n+1}(x) = (x - b_n)\hat{U}_n(x) - \lambda_n \hat{U}_{n-1}(x), \qquad n \ge 0,$$
where $\hat{U}_n(x)$ is the monic polynomial that is a scalar multiple of $U_n(x)$.

Solution. (1) This follows from (2) [2 points].

(2) By the addition rule for the sine function,

$$\sin(n+1)\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta,$$
$$\sin(n-1)\theta = \sin n\theta\cos\theta - \cos n\theta\sin\theta.$$

Adding the two equations and dividing both sides by $\sin\theta$, we get

$$U_n(x) + U_{n-2}(x) = 2xU_{n-1}(x), \qquad n \geq 1 \quad \text{[2 points]}.$$

This is equivalent to the recurrence in the problem.

(3) By the change of variables $x = \cos\theta$, $0 \leq \theta \leq \pi$, with $dx = -\sin\theta d\theta = -\sqrt{1-x^2}d\theta$,

$$\begin{aligned}
&\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx \\
&= \int_0^{\pi} \sin(m+1)\theta\sin(n+1)\theta d\theta \quad \text{[2 points]} \\
&= \frac{1}{2}\int_0^{\pi} (\cos(m-n)\theta + \cos(m+n)\theta)\, d\theta \quad \text{[2 points]} \\
&= \frac{\pi}{2}\delta_{m,n}.
\end{aligned}$$

(4) Since $\deg U_n(x) = 2^n$ for all $n \geq 0$, we have $\hat{U}_n(x) = 2^{-n}U_n(x)$. Dividing both sides of the recurrence in (2) by 2^{n+1}, we obtain $b_n = 0$ and $\lambda_n = 1/4$ [2 points]. □

Problem 1.4. Let $\{P_n(x)\}_{n\geq 0}$ be the monic OPS for a linear functional $\mathcal{L}$ with three-term recurrence

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \geq 0.$$

(1) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & 1 & & 0 \\ \lambda_1 & x-b_1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & \lambda_{n-1} & x-b_{n-1} \end{vmatrix}.$$

(2) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & \sqrt{\lambda_1} & & 0 \\ \sqrt{\lambda_1} & x-b_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\lambda_{n-1}} \\ 0 & & \sqrt{\lambda_{n-1}} & x-b_{n-1} \end{vmatrix}.$$

(3) Using (2) prove that if $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all, then $P_n(x)$ has real roots only.

Solution. (1) Let $Q_n(x)$ be the determinant on the right-hand side. Expanding the determinant along the last row, we obtain the recursion

$$Q_n(x) = (x-b_{n-1})Q_{n-1}(x) - \lambda_{n-1}Q_{n-2}(x) \quad \text{[2 points]}.$$

Since $P_n(x)$ and $Q_n(x)$ satisfy the same recurrence with with the initial conditions $Q_0(x) = 1$ and $Q_1(x) = x - b_0$, we obtain that $Q_n(x) = P_n(x)$.

(2) Let $A_n = (\alpha_{i,j})$ be the matrix in (1) and let B_n be the matrix in (2). Then it suffices to find an invertible diagonal matrix $D = \text{diag}(d_i)$ such that $B_n = DA_nD^{-1}$ [3 points]. To do this, observe that $DA_nD^{-1} = (d_i\alpha_{i,j}d_j^{-1})$. Since B_n and DA_nD^{-1} are tri-diagonal matrices, we have $B_n = DA_nD^{-1}$ if and only if the following hold:

$$\beta_{i,i} = d_i\alpha_{i,i}d_i^{-1}, \tag{1.1}$$

$$\beta_{i,i+1} = d_i\alpha_{i,i+1}d_{i+1}^{-1}, \tag{1.2}$$

$$\beta_{i+1,i} = d_{i+1}\alpha_{i+1,i}d_i^{-1}. \tag{1.3}$$

Since $\alpha_{i,i+1} = 1$ and $\beta_{i,i+1} = \sqrt{\lambda_{i+1}}$, (1.2) is equivalent to $d_{i+1} = d_i/\sqrt{\lambda_{i+1}}$. Indeed, if we set $d_0 = 1$ and $d_{i+1} = d_i/\sqrt{\lambda_{i+1}}$, then all three conditions above hold [3 points].

Note: Alternatively, it can be proved directly that the right-hand side of the equation satisfies the same recurrence as $P_n(x)$.

(3) Since the zeros of $P_n(x)$ are the eigenvalues of a real symmetric matrix, they are real [2 points]. □

2. Homework 2 (Due: Oct 5)

Problem 2.1. Let id be the identity permutation.

(1) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^2 = id$.
(2) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^3 = id$.
(3) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^4 = id$.
(4) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^5 = id$.
(5) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^6 = id$.

Solution. We have $\pi^k = id$ if and only if every cycle of π is of length divisible by k. For example, if $\pi^6 = id$, then the decreasing sequence of the lengths of cycles of π must be (6), (3, 3), (3, 2, 1), (3, 1, 1, 1), (2, 2, 2), (2, 2, 1, 1), (2, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1). The number of such permutations is $5!$, $\binom{6}{3}\frac{1}{2}2^2$, $\binom{6}{3}\binom{3}{2}\cdot 2$, $\binom{6}{3}\cdot 2$, $5\cdot 3$, $\binom{6}{4}\cdot 3$, $\binom{6}{2}$, 1, respectively. In this way we get the answers as follows.

(1) 76 [2 points]
(2) 81 [2 points]
(3) 256 [2 points]
(4) 145 [2 points]
(5) 396 [2 points]

□

Problem 2.2. Let $c_1, \dots, c_n$ be a sequence of nonnegative integers such that $\sum_{i=1}^n ic_i = n$. Show that the number of permutations $\pi \in \mathfrak{S}_n$ with c_i cycles of length i for all $i = 1, \dots, n$ is

$$\frac{n!}{\prod_{i=1}^n i^{c_i} c_i!}.$$

Solution. Let X be the set of such permutations. We construct a map $\phi : \mathfrak{S}_n \to X$ as follows. Given $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, let $\phi(\sigma)$ be the permutation whose cycle notation is obtained from the word $\sigma_1 \cdots \sigma_n$ by placing parentheses so that the first c_1 cycles are of length 1, the next c_2 cycles are of length 2, and so on [3 points]. By the construction, this gives a map $\phi : \mathfrak{S}_n \to X$.

For any $\pi \in X$, there are $c_i!$ ways to arrange its c_i cycles and i ways to cyclically shift each each of these cycles. Therefore, there are $\prod_{i=1}^n i^{c_i} c_i!$ permutations $\sigma \in \mathfrak{S}_n$ whose image under ϕ is π [4 points]. This shows that $|X| = |\mathfrak{S}_n| / \prod_{i=1}^n i^{c_i} c_i!$ as desired [3 points]. □

Problem 2.3. For $\pi \in \mathfrak{S}_n$, let $\ell(\pi)$ be the smallest number of simple transpositions whose product is π. Prove that $\ell(\pi) = \mathrm{inv}(\pi)$.

Solution. Suppose that $\pi = s_1 \cdots s_r$ for some simple transpositions s_i's. Since multipling a simple transposition increases or decreases the number of inversions by 1, we have $r \geq \mathrm{inv}(\pi)$ [3 points]. Hence $\ell(\pi) \geq \mathrm{inv}(\pi)$ [2 points].

On the other hand, we can find an expression $\pi = s_1 \cdots s_r$ with $r = \mathrm{inv}(\pi)$ by sorting $\pi = \pi_1 \cdots \pi_n$ [3 points] because multiplying the simple transposition $(i, i+1)$ to the right of $\pi = \pi_1 \cdots \pi_n$ gives

$$\pi(i, i+1) = \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i\pi_{i+1} \cdots \pi_n.$$

This implies $\ell(\pi) \leq \mathrm{inv}(\pi)$ [2 points]. Thus, $\ell(\pi) = \mathrm{inv}(\pi)$. □

Problem 2.4. Prove that

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Proof. We proceed by induction on n. If $n = 1$, it is true. Let $n \geq 2$ and suppose the statement holds for $n-1$. Every $\pi \in \mathfrak{S}_n$ is obtained from $\sigma \in \mathfrak{S}_{n-1}$ by inserting n after j integers from the beginning for some $0 \leq j \leq n-1$ [3 points]. This construction gives $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma) + j$ [3 points]. Thus

$$\begin{aligned}\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} &= \sum_{\sigma \in \mathfrak{S}_{n-1}} \sum_{j=0}^{n-1} q^{\mathrm{inv}(\sigma)+j} = \sum_{\sigma \in \mathfrak{S}_{n-1}} q^{\mathrm{inv}(\sigma)}(1 + q + \cdots + q^{n-1}) \quad \text{[2 points]} \\ &= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) \quad \text{[2 points]}.\end{aligned}$$

Thus the statement is also true for n and we are done. □