Paper: [Tensor-tensor products for invertible linear transforms] (Kilmer, 2015)

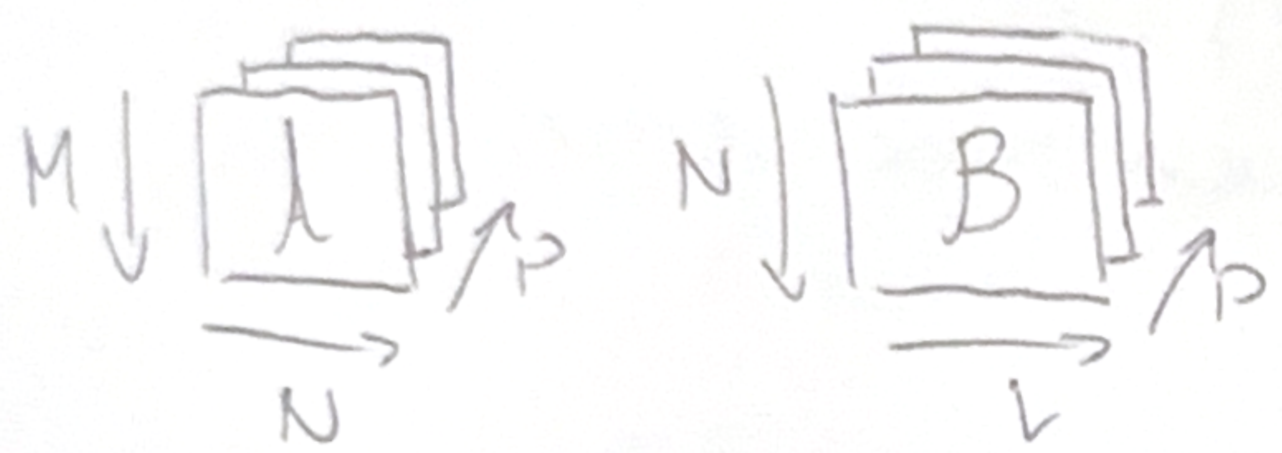
Topic: $*_M$-product for Tensor Multiplication

Outline:

i) short review of t-product
ii) structure for $*_M$-product
iii) some Algebraic properties
iv) Eckart-Young optimality

Problem / Notation

Find $*_M : \mathbb{C}^{M\times N\times P} \times \mathbb{C}^{N\times L\times P} \to \mathbb{C}^{M\times L\times P}$



Frontal Face: $(A)_k$, $k \in (1,\dots,P)$

Tube Fiber: $(a)_{ij}$, $i \in (1,\dots,M)$
$j \in (1,\dots,N)$

Recall: t-product

$$DFT^{-1}(DFT(A) \vartriangle DFT(B))$$

where $\hat{A}_{ijk} := DFT(A_{ijk}) = \sum_{p=1}^{P} A_{ijp} \cdot e^{-i2\pi(\frac{kp}{P})}$

Notice: requires complex arithmetic
↳ Not optimal for real-valued tensors

IDEA: Substitute some other invertible linear transform that better suits the problem.

Want: $*_L$ s.t. $A *_L B = L^{-1}(L(A) \vartriangle L(B))$

Similar structure to t-product, more flexible

We call this the $*_M$ product for some linear transform L.

<u>DEF</u> (mode-3 product)

$\times_3 : \mathbb{C}^{M\times N\times P} \times \mathbb{C}^{P\times P} \to \mathbb{C}^{M\times N\times P}$

by $(A \times_3 M)_{ijk} = \sum_{p=1}^{P} A_{ijp} M_{kp}$

Q: Can we generalize this to any invertible linear transform?

DEF| Let $L: \mathbb{C}^n \to \mathbb{C}^n$ be invertible, linear defined by

$$L((y)_{ij}) = M \operatorname{vec}[(y)_{ij}]$$

for some invertible $P \times P$ matrix M and length-P tube fiber $(y)_{ij}$

Then,

DEF| $*_L : \mathbb{C}^{M \times N \times P} \times \mathbb{C}^{N \times L \times P} \to \mathbb{C}^{M \times L \times P}$ so that

$$L(A *_L B) = L(A) \Delta L(B)$$

performs linear transform across each tube vector then takes face-wise product.

Note: we observe that for $A \in \mathbb{C}^{M \times N \times P}$, $M \in \mathbb{C}^{P \times P}$

$$L(A) = A \times_3 M \quad , \quad L^{-1}(A) = A \times_3 M^{-1}$$

e.g.| $*_c$ Cosine Transform Product

Def| $$\operatorname{mat}(A) := \begin{bmatrix} (A)_1 & (A_2) & \cdots & (A)_P \\ (A)_2 & \ddots & & \vdots \\ \vdots & & \ddots & (A)_2 \\ (A)_P & \cdots & (A)_2 & (A)_1 \end{bmatrix} + \begin{bmatrix} (A)_2 & (A_3) & \cdots & (A)_P & 0 \\ (A)_3 & & & \cdots & (A)_P \\ \vdots & & 0 & & \vdots \\ (A)_P & & & & \\ 0 & (A)_P & & \cdots & (A)_2 \end{bmatrix}$$

$$\operatorname{ten}(\operatorname{mat}(A)) = A$$

$$\Rightarrow A *_c B := \operatorname{ten}(\operatorname{mat}(A)\operatorname{mat}(B))$$

or let $M = \operatorname{dct}(\operatorname{eye}(P))$ in Matlab (requires toolbox)

such that

$$A *_c B = L^{-1}(L(A) \Delta L(B))$$

can be computed efficiently in the transform domain, similarly to t-product, using DCT instead of FFT

↳ doesn't require complex arithmetic

↳ optimized for tasks like image processing

ECKART - YOUNG OPTIMALITY

$*_L$ evokes important propeties like

i) operator norms
ii) Hermitian Transposes
iii) inner products

$\Big\}$ allows us to have unitary operators

with these properties, we can produce:

Tensor SVD induced by $*_L$

↳ Given $\mathcal{A}$, compute $\hat{\mathcal{A}} = L(\mathcal{A})$

$[\hat{\mathcal{U}}^{(k)}, \hat{S}^{(k)}, \hat{V}^{(k)H}] \leftarrow \mathrm{svd}(\hat{\mathcal{A}}^{(k)}) \quad \forall k \in (1, \ldots, P)$

compute $\mathcal{U} \leftarrow L^{-1}(\hat{\mathcal{U}})$, $S \leftarrow L^{-1}(\hat{S})$, $V \leftarrow L^{-1}(\hat{V})$

$\Rightarrow \mathcal{A} = \mathcal{U} *_L S *_L V^H$, with $\mathcal{U}, V$ $*_L$-unitary