THE INTUITION BEHIND BLACK-LITTERMAN MODEL PORTFOLIOS

by

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Abstract

In this article we demonstrate that the optimal portfolios generated by the Black-Litterman asset allocation model have a very simple, intuitive property. The unconstrained optimal portfolio in the Black-Litterman model is the scaled market equilibrium portfolio (reflecting the uncertainty in the equilibrium expected returns) plus a weighted sum of portfolios representing the investor's views. The weight on a portfolio representing a view is positive when the view is more bullish than the one implied by the equilibrium and the other views. The weight increases as the investor becomes more bullish on the view, and the magnitude of the weight also increases as the investor becomes more confident about the view.

1 Introduction

Since publication in 1990, the Black-Litterman asset allocation model has gained wide application in many financial institutions. As developed in the original paper, the Black-Litterman model provides the flexibility of combining the market equilibrium with additional market views of the investor. The Black-Litterman approach may be contrasted with the standard mean-variance optimization in which the user inputs a complete set of expected returns[1] and the portfolio optimizer generates the optimal portfolio weights. Because there is a complex mapping between expected returns and the portfolio weights, and because there is no natural starting point for the expected return assumptions, users of the standard portfolio optimizers often find their specification of expected returns produces output portfolio weights which do not seem to make sense. In the Black-Litterman model the user inputs any number of views, which are statements about the expected returns of arbitrary portfolios, and the model combines the views with equilibrium, producing both the set of expected returns of assets as well as the optimal portfolio weights.

Although Black and Litterman concluded in their 1992 article [Black and Litterman, 1992]:

> "...our approach allows us to generate optimal portfolios that start at a set of neutral weights and then tilt in the direction of the investor's views."

they did not discuss the precise nature of that phenomenon. As we demonstrate here, the optimal portfolio for an unconstrained investor is proportional to the market equilibrium portfolio plus a weighted sum of portfolios reflecting the investor's views.[2] Now the economic intuition becomes very clear. The investor starts by holding the scaled market equilibrium portfolio, reflecting her uncertainty on the equilibrium, then invests in portfolios representing her views. The Black-Litterman

[1] For simplicity, we use 'return' to refer to 'excess return over the one period risk free rate'.

[2] Please refer to Section 2 for precise definition of a view.

model computes the weight to put on the portfolio representing each view according to the strength of the view, the covariance between the view and the equilibrium, and the covariances among the views. We show the conditions for the weight on a view portfolio to be positive, negative, or zero. We also show that the weight on a view increases when the investor becomes more bullish on the view, and the magnitude of the weight increases when the investor becomes less uncertain about the view.

The rest of the article is organized as follows. In Section 2, we review the basics of the Black-Litterman asset allocation model. Then we present our main results in Section 3. Numeric examples are presented in Section 4 to illustrate our results. Finally, we conclude the article in Section 5.

2 The Black-Litterman Asset Allocation Model

The Black-Litterman asset allocation model uses the Bayesian approach to infer the assets' expected returns[Black and Litterman, 1990, 1992]. With the Bayesian approach, the expected returns are random variables themselves. They are not observable. One can only infer their probability distribution. The inference starts with a prior belief. Additional information is used along with the prior to infer the posterior distribution. In the Black-Litterman model, the CAPM equilibrium distribution is the prior, the investor's views are the additional information. The Bayesian approach is used to infer the probability distribution of the expected returns using both the CAPM prior and the additional views.

Let's assume that there are N-assets in the market, which may include equities, bonds, currencies, and other assets. The returns of these assets have a normal distribution with μ being the expected return and Σ the covariance matrix[3]. That is

$$r \sim N(\mu, \Sigma), \tag{1}$$

where r is the vector of asset returns. In equilibrium, all investors as a whole hold the market portfolio w_{eq}. The equilibrium risk premiums, Π, are such that if all investors hold the same view, the demand for these assets exactly equals to the outstanding supply[Black, 1989]. Assuming the average risk tolerance of the world is represented by the risk aversion parameter δ, the equilibrium risk premiums are given by

$$\Pi = \delta \Sigma w_{eq}. \tag{2}$$

The Bayesian prior is that the expected returns, μ, are centered at the equilibrium values, that is they are normally distributed with the mean of Π,

$$\mu = \Pi + \epsilon^{(e)}, \tag{3}$$

where $\epsilon^{(e)}$ is a normally distributed random vector with zero mean and covariance matrix $\tau\Sigma$, where τ is a scalar indicating the uncertainty of the CAPM prior.

In addition to the CAPM prior, the investor also has a number of views on the market returns. A view is expressed as a statement that the expected return of a portfolio p has a normal distribution with mean equal to q and a standard deviation given by ω.[4] Let K be the total number of the views, P be a $K \times N$ matrix whose rows are these portfolio weights and Q be a K-vector of the expected returns on these portfolios. That is

$$P' = \begin{pmatrix} p_1 & p_2 & \cdots & p_K \end{pmatrix} \tag{4}$$

$$Q' = \begin{pmatrix} q_1 & q_2 & \cdots & q_K \end{pmatrix}. \tag{5}$$

[3]The covariance matrix Σ is assumed to be a known constant matrix.

[4]We also refer to the portfolio p as a view portfolio.

Then the investor's views can be expressed as

$$P\mu = Q + \epsilon^{(v)}, \tag{6}$$

where $\epsilon^{(v)}$ is an unobservable normally distributed random vector with zero mean and a diagonal covariance matrix Ω.[5] It is further assumed that the $\epsilon^{(e)}$ and $\epsilon^{(v)}$ are independent

$$\begin{pmatrix} \epsilon^{(e)} \\ \epsilon^{(v)} \end{pmatrix} \sim N\left(0, \begin{bmatrix} \tau\Sigma & 0 \\ 0 & \Omega \end{bmatrix}\right). \tag{7}$$

These views then are combined with the CAPM prior in the Bayesian framework. The result is that the expected returns are distributed as $N(\bar{\mu}, \bar{M}^{-1})$, where the mean $\bar{\mu}$ is given by

$$\bar{\mu} = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q\right], \tag{8}$$

and the covariance matrix $\bar{M}^{-1}$ is given by[6]

$$\bar{M}^{-1} = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}. \tag{9}$$

3 The Unconstrained Optimal Portfolio

In practice, it is quite straight forward to apply equation (8) for calculating the mean of expected returns. However, it is often difficult to find the original economic intuitions of the views from these

[5]The assumption of a diagonal Ω matrix is not a restriction. When the matrix Ω is not diagonal, it can always be decomposed into the form $\Omega = V\hat{\Omega}V^{-1}$ where $\hat{\Omega}$ is diagonal. The original views can be transformed into $\hat{P}\mu = \hat{Q} + \hat{\epsilon}^{(v)}$, where $\hat{P} = V^{-1}P$, $\hat{Q} = V^{-1}Q$, and the covariance matrix of $\hat{\epsilon}^{(v)}$ is $\hat{\Omega}$ which is diagonal. In fact, the assumption of diagonal Ω matrix is not needed throughout this article. Only in the proof of Property 3.1, the Ω matrix needs to be block diagonal. See the proof for the details.

[6]As pointed out by S. Satchell and A Scowcroft[Satchell and Scowcroft, 1997], equation (9) is a well known result, despite that it is not displayed in Black and Litterman's original papers.

numbers, particularly when the number of assets are large. The trouble with trying to make sense out of the expected returns of different asset classes is that they are all related to each other through their relative volatilities and correlations. For example, it is easy to conjecture that the German equity market will outperform the rest of European markets, but much less easy to imagine what the implication of that conjecture is for the relative expected returns of the German and other markets. To increase the expected return on German equities and hold all other expected returns fixed does not, in the mathematics of an optimizer, suggest an overweight in German equities — rather, it suggests to the optimizer that by using the relative volatilities and correlations of the different markets it can create a much more complicated portfolio with higher expected return and lower risk than would be available simply by creating an overweight of the German market. In the Black-Litterman approach a view that German equities will outperform the rest of European equities is expressed as an expectation of a positive return on a portfolio consisting of a long position in German equities and market capitalization weighted short positions in the rest of the European markets. This view is then translated through equation (8), which appropriately takes volatilities and correlations into account, into adjustments to expected returns in all of the markets. As we show below, these adjustments to expected returns are exactly those needed to suggest to the optimizer that the best opportunity is a simple overweighting of the German equity market, financed by underweighting the rest of European markets. These types of complex transformations, from views on portfolios to the expected return vector, and from the expected return vector to the optimal portfolio, are generally difficult to understand. Thus, instead of looking at the expected returns directly, we examine the Black-Litterman optimal portfolio weights. We begin with the case of an unconstrained investor with representative risk aversion parameter equal to δ. This portfolio gives us some very interesting insights about the Black-Litterman asset allocation model.

Because the expected returns are random variables themselves in the Black-Litterman model, the distribution of the returns is no longer simply $N(\bar{\mu}, \Sigma)$. According to equation (1), (8), and (9), the distribution for the returns is

$$r \sim N(\bar{\mu}, \bar{\Sigma}), \tag{10}$$

where $\bar{\Sigma} = \Sigma + \bar{M}^{-1}$.

Given the mean $\bar{\mu}$ and the covariance matrix $\bar{\Sigma}$, the optimal portfolio can be constructed using the standard mean-variance optimization method. For an investor with the risk aversion parameter δ, the maximization problem can be written as

$$\max \quad w'\bar{\mu} - \frac{\delta}{2} w'\bar{\Sigma}w. \tag{11}$$

The first order condition yields that

$$\bar{\mu} = \delta\bar{\Sigma}w^*, \tag{12}$$

or equivalently,

$$w^* = \frac{1}{\delta}\bar{\Sigma}^{-1}\bar{\mu}, \tag{13}$$

where w^* is the vector of the optimal portfolio weights. Using equation (8), the optimal portfolio weights can be written as

$$w^* = \frac{1}{\delta}\bar{\Sigma}^{-1}\bar{M}^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q\right]. \tag{14}$$

Note the fact that

$$\bar{\Sigma}^{-1} = \left(\Sigma + \bar{M}^{-1}\right)^{-1} = \bar{M} - \bar{M}\left(\bar{M} + \Sigma^{-1}\right)^{-1}\bar{M} \tag{15}$$

the term $\bar{\Sigma}^{-1}\bar{M}^{-1}$ can be simplified as

$$\bar{\Sigma}^{-1}\bar{M}^{-1} = \frac{\tau}{1+\tau}\left(I - P'A^{-1}P\frac{\Sigma}{1+\tau}\right) \tag{16}$$

where matrix $A = \Omega/\tau + P\Sigma/(1+\tau)P'$. The optimal portfolio weights, $w^* = \bar{\Sigma}^{-1}\bar{\mu}/\delta$ now can be written as

$$w^* = \frac{1}{1+\tau}\left(w_{eq} + P' \times \Lambda\right) \tag{17}$$

where $w_{eq} = (\delta\Sigma)^{-1}\Pi$ is the market equilibrium portfolio and Λ is a vector defined as

$$\Lambda = \tau\Omega^{-1}Q/\delta - A^{-1}P\frac{\Sigma}{1+\tau}w_{eq} - A^{-1}P\frac{\Sigma}{1+\tau}P'\tau\Omega^{-1}Q/\delta. \tag{18}$$

Since each column of matrix P' is a view portfolio, equation (17) shows that the investor's optimal portfolio is the market equilibrium portfolio w_{eq}, plus a weighted sum of the portfolios forming the views, then scaled by a factor of $1/(1+\tau)$. The weight for each portfolio is given by the corresponding element in the vector Λ.

There are intuitive interpretations of the weights in equation (18). The first term shows that the stronger the view is (either with a higher expected return q_k, or a lower level of uncertainty, i.e. a higher value of the precision, ω_k/τ), the more weight it carries in the final optimal portfolio. The second term shows that the weight of a view is penalized for the covariance between the view portfolio and market equilibrium portfolio. Since the market equilibrium information is already presented by the prior, the covariance of a view portfolio with the market equilibrium portfolio indicates that the view carries less new information and the penalty makes sense. Similarly, the last term shows that the weight is penalized for the covariance of a view portfolio with other view portfolios. Again, since the covariance between these view portfolios indicates that the information is in a sense being double counted, it is intuitive that there should be a penalty on the weight associated with an increased covariance with other view portfolios.

For an investor with a different risk tolerance, the optimal portfolio $\hat{w}^*$ can be obtained by scaling the portfolio w^*

$$\hat{w}^* = (\delta/\hat{\delta})w^*, \tag{19}$$

where $\hat{\delta}$ is the risk aversion parameter for the investor. For an investor with a fixed risk (standard deviation) limit σ, the portfolio optimization is formulated as

$$\max_{w'\Sigma w \le \sigma^2} w'\bar{\mu}. \tag{20}$$

The optimal portfolio $\tilde{w}^*$ can also be obtained by scaling the portfolio w^*

$$\tilde{w}^* = \left(\sigma\delta/\sqrt{\bar{\mu}'\bar{\Sigma}\bar{\mu}}\right) w^*. \tag{21}$$

For an investor with other constraints on the portfolio, the optimal portfolio can be obtained by using the usual portfolio optimization package with $\bar{\mu}$ and $\bar{\Sigma}$ as inputs.

Since the weight λ on a view plays a very important role in the portfolio construction process, we need to know when the weight is positive, and how the weight changes. The following two properties show just what we wanted.

Property 3.1 *Let P, Q, and Ω represent the K views held by the investor initially, $\bar{\mu}$ be the mean of expected returns by using these views in the Black-Litterman model, Λ be the weight vector defined by equation (18). Assume the investor now has one additional view, represented by p, q, and ω. For the case of $K+1$ views, the new weight vector $\hat{\Lambda}$ is given by*

$$\hat{\Lambda} = \begin{pmatrix} \Lambda - \hat{\lambda}_{K+1}A^{-1}b \\ \hat{\lambda}_{K+1} \end{pmatrix} \tag{22}$$

where $b = P\frac{\Sigma}{1+\tau}p$, $c = \omega/\tau + p'\frac{\Sigma}{1+\tau}p$, and $\hat{\lambda}_{K+1}$, the weight on the additional view is

$$\hat{\lambda}_{K+1} = \frac{q - p'\Sigma\bar{\Sigma}^{-1}\bar{\mu}}{(c - b'A^{-1}b)\delta}. \tag{23}$$

Let

$$\tilde{\mu} = \Sigma\bar{\Sigma}^{-1}\bar{\mu}, \tag{24}$$

since $w^* = (\delta\bar{\Sigma})^{-1}\bar{\mu}$ is the unconstrained optimal portfolio of the Black-Litterman model with the first K views, $\tilde{\mu} = \delta\Sigma w^*$ can be seen as the implied *expected returns of the first K views, even though $\tilde{\mu}$ usually is different from $\bar{\mu}$, due to the difference between Σ and $\bar{\Sigma}$. In the limit as τ and Ω go to zero while Ω/τ stays finite, $\tilde{\mu}$ and $\bar{\mu}$ become identical.*

Since $c - b'A^{-1}b > 0$, equation (23) shows that λ_{K+1}, the weight on the additional view will have the same sign as the expression $q - p'\tilde{\mu}$. This means that the weight $\hat{\lambda}_{K+1}$ is positive (negative) when the strength of the view on the portfolio p is more bullish (bearish) than the portfolio's implied *expected return $p'\tilde{\mu}$. The additional view will have a zero weight if $q = p'\tilde{\mu}$. In this case, the weights on the first K views are identical to the weights generated in the Black-Litterman model with the first K views.*

Proof: See Appendix.

Property 3.2 *For a particular view k, its weight λ_k is an increasing function of its expected return q_k. The absolute value of λ_k is an increasing function of its confidence level ω_k^{-1}.*

Proof: See Appendix.

4 Numeric Examples

In this section, we present several numeric examples to illustrate our results. In all examples the market consists of the equity indexes of seven major industrial countries.[7] The correlations between the index returns are listed in Table 1, and the index return volatilities, the market capitalization

[7] We ignore the bond markets, currency markets, and other equity markets for simplicity.

weights, and equilibrium risk premiums are listed in Table 2. The δ parameter, representing the world average risk aversion is assumed to be 2.5.

First, let us examine the example of a view on the German equity market presented in Section 3 with the traditional mean-variance approach. Suppose the investor is not familiar with the Black-Litterman model. Nevertheless, let us assume that she uses the CAPM as her starting point. When she does not have other views, she assumes that the expected returns equal the equilibrium risk premiums as shown in Table 2. As a result, she holds the market portfolio. With a view that the German equity market will outperform the rest of European markets by 5% a year, she needs to modify the expected returns. She might proceed as follows. To be precise in expressing her view, she sets the expected return for Germany 5% higher than the (market capitalization) weighted average of the expected returns of France and the United Kingdom. She keeps the market weighted average expected returns for the European countries and the spread between France and the United Kingdom unchanged from their equilibrium values. She also keeps the expected returns for non-European countries unchanged from their equilibrium values. Since the equilibrium already implies that Germany will outperform the rest of Europe, the changes in expected returns are actually quite small, as shown in Table 3. However, the optimal portfolio is quite different from what one would have expected: *although the increased weight in the German market and the decreased weights in the United Kingdom and France markets are expected, the size of the changes are quite dramatic. Even more puzzling are the reduced weights in Australia and Canada as well as the increased weights in Japan and the United States.* Since the investor does not have any view on these countries, why should she adjust the weight in these countries? Presumably it is because of the way the views are being translated into expected returns. As we can see, the investor has already taken extra care in her approach to translating the view into expected returns. Can she possibly do any better?

With the Black-Litterman approach,[8] the same view is expressed as an expectation of a 5% return on a portfolio consisting of a long position in German equity and market capitalization weighted short position in rest of the European markets. The Black-Litterman expected returns are calculated by equation (8), and the optimal portfolio weights are calculated by $(\delta\bar{\Sigma})^{-1}\bar{\mu}$. The view portfolio, the expected returns, the optimal portfolio, and the deviation from her initial portfolio weights are shown in Table 4. The optimal weight vector is very intuitive in this case. Being a Bayesian, the investor has started with a scaled market portfolio $w_{eq}/(1+\tau)$, reflecting her uncertainty on the CAPM.[9] Since she expressed a view on the portfolio of Germany versus the rest of Europe, she simply adds an exposure to the view portfolio in addition to her initial portfolio, the scaled market equilibrium portfolio. The size of the exposure is given by $\lambda/(1+\tau)$. The weight λ is calculated by the Black-Litterman model with equation (18). In this example, the value of λ is 0.302. One can easily verify in the Table 4, the optimal deviation is exactly the view portfolio multiplied by λ.

Now, let's assume that the investor has one more view. In additional to the first view of Germany versus the rest of Europe, she also believes that the Canadian equities will outperform the US equity by 3% a year. This view is expressed as an expected 3% annualized returns for the portfolio of long Canadian equities and short US equities. Table 5 shows the views, the expected returns, and the optimal portfolio. Again, the results clearly show that the optimal deviation is a weighted sum of the view portfolios. The corresponding weights $\lambda_1/(1+\tau)$, $\lambda_2/(1+\tau)$ are equal to 0.284 and 0.398, respectively. Again, the result is very easy to understand, the investor has two views, so she adds exposures to these portfolios. The Black-Litterman model gives the optimal weights.

Assume now that the investor becomes more bullish on the Canada/US view. Instead of the

[8]The parameter τ is assumed to be 0.05, which corresponds to the confidence level of the CAPM prior mean if it was estimated using 20 years of data.

[9]In the case of no additional views, the quantities $\bar{\mu}$ and $\bar{\Sigma}$ are reduced to Π and $(1+\tau)\Sigma$.

3% outperformance, she believes that the Canadian equities will outperform the US market by 4%. According to Property 3.2, the weight λ_2 on the second portfolio should increase. As shown in Table 6, the value of λ_2 increased from the value of 0.418 in the Table 5 to 0.538.

Assume the investor now becomes less confident on the view of Germany versus the rest of Europe. She is only 50% as confident in the view now. The is reflected by a change of ω/τ from 0.021 to 0.043. Her view on Canada/US is unchanged at 4%. According to Property 3.2, the absolute value of weight λ_1 on the first portfolio should decrease. Table 7 shows the value decreased from 0.292 to 0.193.

Our last example illustrates Property 3.1. In addition to the two views in Table 7, the investor has a third view. The third view is that the Canadian equities will outperform the Japanese equities by 4.12%. But following equation (24), the implied expected return of the portfolio of long Canada and short Japan is exactly 4.12% when the investor had two views. The expected returns of third view portfolio matches exactly the expected returns implied by the Black-Litterman model using only two views. According to Property 3.1, the weights on the first two views should not be changed and the weight on the third view should be zero. Table 8 does show that the weight λ_3 on the third view is zero and the expected returns, optimal portfolio weights are exactly the same as in Table 7.

5 Conclusion

Instead of treating the Black-Litterman asset allocation model as a rocket science black box which generates expected returns in some mysterious way, we presented a method to understand the intuition of the model. With the new method, investing using the Black-Litterman model becomes very intuitive. The investor would invest in the scaled market portfolio first, then add the portfolios representing the views. The Black-Litterman model gives the optimal weights for the portfolios. When there are constraints, or the risk tolerance is different from the world average, one can always use the expected returns, $\bar{\mu}$ and the covariance matrix $\bar{\Sigma}$ in a portfolio optimization package to obtain the optimal portfolio. Now, the mean of expected returns $\bar{\mu}$ is simply the matrix product of $\bar{\Sigma}$ and the optimal portfolio $w^* = (w_e q + P'\Lambda)/(1+\tau)$ scaled by the average risk aversion parameter δ. Unlike a standard mean-variance optimization, the Black-Litterman model, if properly implemented, will always generate an optimal portfolio whose weights are relatively easy to understand.

Appendix

Proof of Property 3.1

In the case where the investor hold the initial K views, the weight vector Λ can be written as

$$\Lambda = \tau\Omega^{-1}Q/\delta - A^{-1}P\frac{\Sigma}{1+\tau}\left(w_{eq} + P'\tau\Omega^{-1}Q/\delta\right). \tag{25}$$

Similarly, in the case where the investor hold an additional view, the weight vector $\hat{\Lambda}$ is

$$\hat{\Lambda} = \tau\hat{\Omega}^{-1}\hat{Q}/\delta - \hat{A}^{-1}\hat{P}\frac{\Sigma}{1+\tau}\left(w_{eq} + \hat{P}'\tau\hat{\Omega}^{-1}\hat{Q}/\delta\right). \tag{26}$$

where $\hat{P}$, $\hat{Q}$, $\hat{\Omega}$, and $\hat{A}$ are the counterpart of P, Q, Ω, and A in the case of $K+1$ views. They can be written as[10]

$$\hat{P} = \begin{pmatrix} P \\ p' \end{pmatrix}, \tag{27}$$

$$\hat{Q} = \begin{pmatrix} Q \\ q \end{pmatrix}, \tag{28}$$

$$\hat{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & \omega \end{pmatrix}, \tag{29}$$

and

$$\hat{A} = \begin{pmatrix} A & b \\ b' & c \end{pmatrix}. \tag{30}$$

Using equation (8), after a few steps of algebra, equation (26) can be written as

$$\hat{\Lambda} = \frac{1}{\delta}\left[\begin{pmatrix} \tau\Omega^{-1}Q \\ \tau\omega^{-1}q \end{pmatrix} - \frac{\hat{A}^{-1}}{1+\tau}\begin{pmatrix} P\bar{\mu} + P\Sigma P'\tau\Omega^{-1}P\bar{\mu} + P\Sigma p\omega^{-1}q \\ p'\bar{\mu} + p'\Sigma P'\tau\Omega^{-1}P\bar{\mu} + p'\Sigma p\omega^{-1}q \end{pmatrix}\right]. \tag{31}$$

[10] The assumption of diagonal matrix $\hat{\Omega}$ is not needed here. It only needs it to be block diagonal so equation (29) holds.

The inverse of $\hat{A}$ is

$$\hat{A}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}bb'A^{-1}/d & -A^{-1}b/d \\ -b'A^{-1}/d & 1/d \end{pmatrix}. \tag{32}$$

where $d = (c - b'A^{-1}b)$. Applying the last row of $\hat{A}^{-1}$ in equation (31) gives the last element of $\hat{\Lambda}$ as

$$\hat{\lambda}_{K+1} = \frac{\tau\omega^{-1}q}{\delta} + \frac{\begin{pmatrix} b'A^{-1} & -1 \end{pmatrix}}{(1+\tau)d\delta} \begin{pmatrix} P\bar{\mu} + P\Sigma P'\tau\Omega^{-1}P\bar{\mu} + P\Sigma p\omega^{-1}q \\ p'\bar{\mu} + p'\Sigma P'\tau\Omega^{-1}P\bar{\mu} + p'\Sigma p\omega^{-1}q \end{pmatrix}. \tag{33}$$

After expanding the matrix multiplication and a few steps of algebra, the above equation directly leads to

$$\hat{\lambda}_{K+1} = \frac{q - p'\Sigma\bar{\Sigma}^{-1}\bar{\mu}}{(c - b'A^{-1}b)\delta}, \tag{34}$$

which is exactly equation (23). Similarly, applying the first K rows of $\hat{A}^{-1}$ in equation (31) gives the first K elements of $\hat{\Lambda}$ as

$$\begin{aligned} \tilde{\Lambda} &= \tau\Omega^{-1}Q - \begin{pmatrix} A^{-1} + \frac{A^{-1}bb'A^{-1}}{c-b'A^{-1}b} & -\frac{A^{-1}b}{c-b'A^{-1}b} \end{pmatrix} \begin{pmatrix} P \\ p' \end{pmatrix} \\ &\quad \times \frac{\Sigma}{1+\tau}\left(\delta w_{eq} + P'\tau\Omega^{-1}Q + p\tau\omega^{-1}q\right). \end{aligned} \tag{35}$$

After some simple but lengthy algebra, $\tilde{\Lambda}$ can be simplified into the following form,

$$\tilde{\Lambda} = \Lambda - \hat{\lambda}_{K+1}A^{-1}b. \tag{36}$$

Combining equation (34) and equation (36), it is easy to see that

$$\hat{\Lambda} = \begin{pmatrix} \Lambda - \hat{\lambda}_{K+1}A^{-1}b \\ \hat{\lambda}_{K+1} \end{pmatrix}. \tag{37}$$

Since matrix $\hat{\Omega}$ is positive definite and $\hat{P}\Sigma\hat{P}'$ is positive semi-definite, the sum of the two, matrix A is positive definite, so is A^{-1}. And since $c - b'A^{-1}b$ is a diagonal element of $\hat{A}^{-1}$, it must be positive.

Therefore, $\hat{\lambda}_{K+1}$, the weight on the last view has the same sign as the numerator in equation (34). In the limit when τ and $\hat{\Omega}$ go to zero and $\hat{\Omega}/\tau$ stay constant, the numerator becomes $q - p'\bar{\mu}$.

Proof of Property 3.2

Notice that equation (18) can be rearranged as

$$\Lambda = A^{-1}Q/\delta - A^{-1}P\frac{\Sigma}{1+\tau}w_{eq}. \tag{38}$$

It is clear that

$$\frac{\partial \lambda_i}{\partial q_k} = \frac{1}{\delta}\left(A^{-1}\right)_{ik}, \tag{39}$$

in particular

$$\frac{\partial \lambda_k}{\partial q_k} = \frac{1}{\delta}\left(A^{-1}\right)_{kk}. \tag{40}$$

Because $(A^{-1})_{kk}$ is a diagonal element of a positive definite matrix, it must be positive. Therefore, λ_k is a increasing function of q_k.

Using equation (38), the partial derivative $\partial\Lambda/\partial\omega_k^{-1}$ can be written as

$$\frac{\partial \Lambda}{\partial \omega_k^{-1}} = \frac{\partial A^{-1}}{\partial \omega_k^{-1}}\left(Q/\delta - P\frac{\Sigma}{1+\tau}w_{eq}\right). \tag{41}$$

Because $\partial A^{-1}/\partial\omega_k^{-1} = -A^{-1}(\partial A/\partial\omega_k^{-1})A^{-1}$ and $\partial A/\partial\omega_k^{-1} = -\omega_k^2\iota_{kk}/\tau$, equation (41) leads to

$$\frac{\partial \Lambda}{\partial \omega_k^{-1}} = (\omega_k^2/\tau)A^{-1}\iota_{kk}\Lambda. \tag{42}$$

where ι_{kk} is a matrix with 1 as the k-th diagonal element and zero everywhere else. The k-th element of the above partial derivative is

$$\frac{\partial \lambda_k}{\partial \omega_k^{-1}} = (\omega_k^2/\tau)\left(A^{-1}\right)_{kk}\lambda_k. \tag{43}$$

It has the same sign as λ_k. For a positive (negative) λ_k, increasing ω_k^{-1} would cause λ_k to increase (decrease). In other words, the absolute value of λ_k is an increasing function of ω_k^{-1}.

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Table 1: Correlations among the equity market index returns of the seven countries.

	Australia	Canada	France	Germany	Japan	UK
Canada	0.488					
France	0.478	0.664				
Germany	0.515	0.655	0.861			
Japan	0.439	0.310	0.355	0.354		
UK	0.512	0.608	0.783	0.777	0.405	
USA	0.491	0.779	0.668	0.653	0.306	0.652

Table 2: Annualized volatilities, market capitalization weights, and equilibrium risk premiums for the equity markets.

	σ	w_{eq}	Π
Australia	16.0	1.6	3.9
Canada	20.3	2.2	6.9
France	24.8	5.2	8.4
Germany	27.1	5.5	9.0
Japan	21.0	11.6	4.3
UK	20.0	12.4	6.8
USA	18.7	61.5	7.6

Table 3: The expected returns, optimal portfolio weights, and their changes from equilibrium values in the traditional mean-variance approach. The expected returns are adjusted to reflect the view that Germany equities will outperform the rest of European equities by 5%.

	μ	w_{opt}	$\mu - \Pi$	$w_{opt} - w_{eq}$
Australia	3.9	-5.1	0.0	-6.7
Canada	6.9	-2.3	0.0	-4.5
France	7.6	-50.1	-0.8	-55.3
Germany	11.5	83.6	2.4	78.1
Japan	4.3	14.9	0.0	3.3
UK	6.0	-22.8	-0.8	-35.2
USA	7.6	66.6	0.0	5.1

Table 4: The first column is the portfolio used by the investor to express the view on Germany versus the rest of Europe in the Black-Litterman approach. The expected return of this portfolio is 5%. The relative uncertainty of the view is represented by the value of ω/τ, which equals 0.021. The rest of table shows the expected returns calculated by equation (8), the optimal portfolio $(\delta\bar{\Sigma})^{-1}\bar{\mu}$, and the deviation of the portfolio from the initial portfolio $w_{eq}/(1+\tau)$.

	p	$\bar{\mu}$	w^*	$w^* - \frac{w_{eq}}{1+\tau}$
Australia	0.0	4.3	1.5	0.0
Canada	0.0	7.6	2.1	0.0
France	-29.5	9.3	-4.0	-8.9
Germany	100.0	11.0	35.4	30.2
Japan	0.0	4.5	11.0	0.0
UK	-70.5	7.0	-9.5	-21.3
USA	0.0	8.1	58.6	0.0

Table 5: The first two columns are the two views held by the investor. The rest of table shows the expected returns calculated by equation (8), the optimal portfolio $(\delta\bar{\Sigma})^{-1}\bar{\mu}$, and the deviation of the optimal portfolio from the scaled equilibrium weights.

	P'		$\bar{\mu}$	w^*	$w^* - \frac{w_{eq}}{1+\tau}$
Australia	0.0	0.0	4.4	1.5	0.0
Canada	0.0	100.0	8.7	41.9	39.8
France	-29.5	0.0	9.5	-3.4	-8.4
Germany	100.0	0.0	11.2	33.6	28.4
Japan	0.0	0.0	4.6	11.0	0.0
UK	-70.5	0.0	7.0	-8.2	-20.0
USA	0.0	-100.0	7.5	18.8	-39.8
q	5.00	3.00			
ω/τ	0.021	0.017			
λ	0.298	0.418			

Table 6: The Canada/US view is more bullish than the previous case. The expected returns on the view is 4% instead of 3% shown in Table 5.

	P'		$\bar{\mu}$	w^*	$w^* - \frac{w_{eq}}{1+\tau}$
Australia	0.0	0.0	4.4	1.5	0.0
Canada	0.0	100.0	9.1	53.3	51.3
France	-29.5	0.0	9.5	-3.3	-8.2
Germany	100.0	0.0	11.3	33.1	27.8
Japan	0.0	0.0	4.6	11.0	0.0
UK	-70.5	0.0	7.0	-7.8	-19.6
USA	0.0	-100.0	7.3	7.3	-51.3
q	5.00	4.00			
ω/τ	0.021	0.017			
λ	0.292	0.538			

Table 7: In this case, the investor becomes less certain about the view on the Germany versus the rest of the Europe. It is represented by the value of ω/τ being double from the value in Table 6.

	P'		$\bar{\mu}$	w^*	$w^* - \frac{w_{eq}}{1+\tau}$
Australia	0.0	0.0	4.3	1.5	0.0
Canada	0.0	100.0	8.9	53.9	51.8
France	-29.5	0.0	9.3	-0.5	-5.4
Germany	100.0	0.0	10.6	23.6	18.4
Japan	0.0	0.0	4.6	11.0	0.0
UK	-70.5	0.0	6.9	-1.1	-13.0
USA	0.0	-100.0	7.2	6.8	-51.8
q	5.00	4.00			
ω/τ	0.043	0.017			
λ	0.193	0.544			

Table 8: In this case, the investor has three views. However, since the third view is already implied by the equilibrium and the first two views as shown in Table 7, the weights on the first two views are the same as the previous case. The optimal portfolio weights in this case is identical to the values of the previous case.

		P'		$\bar{\mu}$	w^*	$w^* - \frac{w_{eq}}{1+\tau}$
Australia	0.0	0.0	0.0	4.3	1.5	0.0
Canada	0.0	100.0	100.0	8.9	53.9	51.8
France	-29.5	0.0	0.0	9.3	-0.5	-5.4
Germany	100.0	0.0	0.0	10.6	23.6	18.4
Japan	0.0	0.0	-100.0	4.6	11.0	0.0
UK	-70.5	0.0	0.0	6.9	-1.1	-13.0
USA	0.0	-100.0	0.0	7.2	6.8	-51.8
q	5.00	4.00	4.12			
ω/τ	0.043	0.017	0.059			
λ	0.193	0.544	0.000			