Discussion of "On the Birnbaum Argument for the Strong Likelihood Principle"

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Abstract. In this discussion we demonstrate that fiducial distributions provide a natural example of an inference paradigm that does not obey Strong Likelihood Principle while still satisfying the Weak Conditionality Principle.

Key words and phrases: Generalized fiducial inference, strong likelihood principle violation, weak conditionality principle.

Professor Mayo should be congratulated on bringing new light into the veritable arguments about statistical foundations. It is well documented that p-values, confidence intervals and hypotheses tests do not satisfy the Strong Likelihood Principle (SLP). In the next section we will demonstrate that fiducial distributions provide a natural example of an inference paradigm that breaks SLP while still satisfying the Weak Conditionality Principle (WCP).

1. HISTORY OF FIDUCIAL INFERENCE

The origin of Generalized Fiducial Inference can be traced back to R. A. Fisher (Fisher, 1930, 1933, 1935) who introduced the concept of a fiducial distribution for a parameter and proposed the use of this fiducial distribution in place of the Bayesian posterior distribution. In the case of a one-parameter family of distributions, Fisher gave the following definition for a fiducial density $r(\theta)$ of the parameter based on a single observation x for the case where the cdf $F(x, \theta)$ is a decreasing function of θ:

$$(1.1) \qquad r(\theta) = -\frac{\partial F(x, \theta)}{\partial \theta}.$$

For multiparameter families of distributions Fisher did not give a formal definition. Moreover, the fiducial approach led to confidence sets whose frequentist coverage probabilities were close to the claimed confidence

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levels but they were not exact in the frequentist sense. Fisher's proposal led to major discussions among the prominent statisticians of the 1930s, 40s and 50s (e.g., Dempster, 1966, 1968; Fraser, 1961a, 1961b, 1966, 1968; Jeffreys, 1940; Lindley, 1958; Stevens, 1950). Many of these discussions focused on the nonexactness of the confidence sets and also on the nonuniqueness of fiducial distributions. The latter part of the 20th century has seen only a handful of publications (Barnard, 1995; Dawid, Stone and Zidek, 1973; Salome, 1998; Dawid and Stone, 1982; Wilkinson, 1977) as the fiducial approach fell into disfavor and became a topic of historical interest only.

Since the mid-2000s, there has been a true resurrection of interest in modern modifications of fiducial inference. These approaches have become known under the umbrella name of *distributional inference*. This increase of interest came both in the number of different approaches to the problem and the number of researchers working on these problems, and manifested itself in an increasing number of publications in premier journals. The common thread for these approaches is a definition of inferentially meaningful probability statements about subsets of the parameter space without the need for subjective prior information.

These modern approaches include the Dempster–Shafer theory (Dempster, 2008; Edlefsen, Liu and Dempster, 2009) and its recent extension called *inferential models* (Martin, Zhang and Liu, 2010; Zhang and Liu, 2011; Martin and Liu, 2013a, 2013b, 2013c, 2013d). A somewhat different approach termed *confidence distributions* looks at the problem of obtaining an inferentially meaningful distribution on the pa-

rameter space from a purely frequentist point of view (Xie and Singh, 2013). One of the main contributions of this approach is the ability to combine information from disparate sources with deep implications for meta analysis (Schweder and Hjort, 2002; Singh, Xie and Strawderman, 2005; Xie, Singh and Strawderman, 2011; Hannig and Xie, 2012; Xie et al., 2013). Another more mature approach is called *objective Bayesian inference* that aims at finding nonsubjective model-based priors. An example of a recent breakthrough in this area is the modern development of reference priors (Berger, 1992; Berger and Sun, 2008; Berger, Bernardo and Sun, 2009; 2012; Bayarri et al., 2012). Another related approach is based on higher order likelihood expansions and implied data dependent priors (Fraser, Fraser and Staicu, 2010; Fraser, 2004, 2011; Fraser and Naderi, 2008; Fraser et al., 2010; Fraser, Reid and Wong, 2005). There is also important initial work showing how some simple fiducial distributions that are not Bayesian posteriors naturally arise within the decision theoretical framework (Taraldsen and Lindqvist, 2013).

Arguably, Generalized Fiducial Inference has been on the forefront of the modern fiducial revival. Starting in the early 1990s, the work of Tsui and Weerahandi (1989, 1991) and Weerahandi (1993, 1994, 1995) on *generalized confidence intervals* and the work of Chiang (2001) on the *surrogate variable method* for obtaining confidence intervals for variance components led to the realization that there was a connection between these new procedures and fiducial inference. This realization evolved through a series of works in the early 2000s (Hannig, 2009; Hannig, Iyer and Patterson, 2006; Iyer, Wang and Mathew, 2004; Patterson, Hannig and Iyer, 2004). The strengths and limitations of the fiducial approach are starting to be better understood; see especially Hannig (2009, 2013). In particular, the asymptotic exactness of fiducial confidence sets, under fairly general conditions, was established in Hannig (2013); Hannig, Iyer and Patterson (2006); Sonderegger and Hannig (2014). Generalized fiducial inference has also been extended to prediction problems in Wang, Hannig and Iyer (2012). Computational issues were discussed in Cisewski and Hannig (2012), Hannig, Lai and Lee (2014), and model selection in the context of Generalized Fiducial Inference has been studied in Hannig and Lee (2009); Lai, Hannig and Lee (2013).

2. GENERALIZED FIDUCIAL DISTRIBUTION AND THE WEAK CONDITIONALITY PRINCIPLE

Most modern incarnations of fiducial inference begin with expressing the relationship between the data, $\mathbf{X}$, and the parameters, ξ, as

$$(2.1) \qquad \mathbf{X} = \mathbf{G}(\mathbf{U}, \xi),$$

where $\mathbf{G}(\cdot, \cdot)$ is termed the *data generating equation* (also called the association equation or structural equation) and $\mathbf{U}$ is the random component of this data generating equation whose distribution is free of parameters and completely known.

After observing the data $\mathbf{x}$ the next step is to use the known distribution of $\mathbf{U}$ and the inverse of the data (2.1) to define probabilities for the subsets of the parameter space. In particular, Generalized Fiducial Inference defines a distribution on the parameter space as the weak limit as $\varepsilon \to 0$ of the conditional distribution

$$\arg\min_{\xi} \|\mathbf{x} - \mathbf{G}(\mathbf{U}^{\star}, \xi)\| \mid \Big\{ \min_{\xi} \|\mathbf{x} - \mathbf{G}(\mathbf{U}^{\star}, \xi)\| \le \varepsilon \Big\},$$
(2.2)

where $\mathbf{U}^{\star}$ has the same distribution as $\mathbf{U}$. If there are multiple values minimizing the norm, the operator $\arg\min_{\xi}$ selects one of them (possibly at random). We stress at this point that the Generalized Fiducial Distribution is not unique. For example, different data generating equations can give a somewhat different Generalized Fiducial Distribution. Notice also that if $P(\min_{\xi} \|\mathbf{x} - \mathbf{G}(\mathbf{U}^{\star}, \xi)\| = 0) > 0$, which is the case for discrete distributions, the limit in (2.2) is the conditional distribution evaluated at $\varepsilon = 0$.

The conditional form of (2.2) immediately implies the Weak Conditional Principle for the limiting Generalized Fiducial Distribution. To demonstrate this, let us consider the two-instrument example of (Cox, 1958) (see also Section 4.1 of the discussed article). The data generating equation can be written in a hierarchical form:

$$M = 1 + I_{(0,1/2)}(U),$$
$$X = \theta + \sigma_M Z,$$

where $U \sim U(0, 1)$ and $Z \sim N(0, 1)$ are independent and the precisions $\sigma_1 << \sigma_2$ are known. If both $X = x$ the measurement made and $M = m$ the instrument used ($m = 1, 2$ for machine 1 and 2 respectively) are observed, the conditional distribution (2.2) is $N(x, \sigma_m^2)$, only taking into account the experiment actually performed. On the other hand, if only M is unobserved, then the conditional distribution (2.2) is the mixture $0.5 N(\theta, \sigma_1^2) + 0.5 N(\theta, \sigma_2^2)$. As claimed, the Generalized Fiducial Distribution follows WCP in this example.

3. GENERALIZED FIDUCIAL DISTRIBUTION AND THE STRONG LIKELIHOOD PRINCIPLE

In general, the Generalized Fiducial Distribution does not satisfy the Strong Likelihood principle. We first demonstrate this on inference for geometric distribution. To begin, we perform some preliminary calculations. Let X be a random variable with discrete distribution function $F(x, \xi)$. Let us assume for simplicity of presentation that for each fixed x, $F(x, \xi)$ is monotone in ξ and spans the whole $[0, 1]$. The inverse distribution function $F^{-1}(u, \xi) = \inf\{x : F(x, \xi) \geq u\}$ forms a natural data generating equation

$$X = F^{-1}(U, \xi), \quad U \sim (0, 1).$$

The minimizer in (2.2) is not unique, but any fiducial distribution will have a distribution function satisfying $1 - F(x, \xi) \leq H(\xi) \leq 1 - F(x_-, \xi)$ if $F(x, \xi)$ is decreasing in ξ and $F(x_-, \xi) \leq H(\xi) \leq F(x, \xi)$ if $F(x, \xi)$ is increasing. To resolve this nonuniqueness, Hannig (2009) and Efron (1998) recommend using the half correction which is the mixture distribution with distribution functions $H(\xi) = 1 - (F(x, \xi) + F(x_-, \xi))/2$ if $F(x, \xi)$ is decreasing in ξ or $H(\xi) = (F(x, \xi) + F(x_-, \xi))/2$ if $F(x, \xi)$ increasing.

Let us now consider observing a random variable $N = n$ following the Geometric(p) distribution. SLP implies that the inference based on observing $N = n$ should be the same as inference based on observing $X = 1$ where X is Binomial(n, p). However, the Geometric based Generalized Fiducial Distribution has a distribution function between $1 - (1 - p)^{n-1} \leq H_G(p) \leq 1 - (1 - p)^n$. The binomial based Generalized Fiducial Distribution uses bounds $1 - (1 - p)^n - np(1 - p)^{n-1} \leq H_B(p) \leq 1 - (1 - p)^n$. Thus, the effect of the stopping rule demonstrates itself in the Generalized Fiducial Inference through the lower bound that is much closer to the upper bound in the case of geometric distribution. (We remark that one cannot ignore the lower bound, as the upper bound is used to form upper confidence intervals and the lower bound is used for lower confidence intervals on p.) To conclude, the fiducial distribution in this example depends on both the distribution function of x and also on the distribution function of $x - 1$.

Let us now turn our attention to continuous distributions. In particular, assume that the parameter $\xi \in \Theta \subset \mathbb{R}^p$ is p-dimensional and that the inverse to (2.1) $G^{-1}(x, \xi) = u$ exists. Then under some differentiability assumptions, Hannig (2013) shows that the generalized fiducial distribution is absolutely continuous with density

$$(3.1) \qquad r(\xi) = \frac{f(\mathbf{x}, \xi) J(\mathbf{x}, \xi)}{\int_\Theta f(\mathbf{x}, \xi') J(\mathbf{x}, \xi') \, d\xi'},$$

where $f(\mathbf{x}, \xi)$ is the likelihood and the function $J(\mathbf{x}, \xi)$ is

$$
\begin{aligned}
(3.2) \quad & J(\mathbf{x}, \xi) \\
& = \sum_{\substack{\mathbf{i} = (i_1, \ldots, i_p) \\ 1 \leq i_1 < \cdots < i_p \leq n}} \left| \det\left(\frac{d}{d\xi} \mathbf{G}(\mathbf{u}, \xi) \Big|_{\mathbf{u} = \mathbf{G}^{-1}(\mathbf{x}, \xi)} \right)_{\mathbf{i}} \right|,
\end{aligned}
$$

where $\frac{d}{d\xi} \mathbf{G}(\mathbf{u}, \xi)$ is the $n \times p$ Jacobian matrix of partial derivatives computed with respect of components of ξ. The sum in (3.2) spans over all p-tuples of indexes $\mathbf{i} = (1 \leq i_1 < \cdots < i_p \leq n) \subset \{1, \ldots, n\}$. Additionally, for any $n \times p$ matrix J, the sub-matrix $(J)_{\mathbf{i}}$ is the $p \times p$ matrix containing the rows $\mathbf{i} = (i_1, \ldots, i_p)$ of A. The form of (3.1) suggests that as long as the Jacobian $J(\mathbf{x}, \xi)$ does not separate into $J(\mathbf{x}, \xi) = f(\mathbf{x}) g(\xi)$, in which case the Generalized Fiducial Distribution is the same as the Bayes posterior with $g(\xi)$ used as a prior, the Generalized Fiducial Distribution does not satisfy SLP due to the dependance on $d\mathbf{G}(\mathbf{u}, \xi)/d\xi$.

4. GENERALIZED FIDUCIAL DISTRIBUTION AND SUFFICIENCY PRINCIPLE

Whether the Generalized Fiducial Distribution satisfies the sufficiency principle depends entirely on what data generating equation is chosen. For example, let us assume that $\mathbf{Y} = (\mathbf{S}(\mathbf{X}), \mathbf{A}(\mathbf{X}))'$, where $\mathbf{S}$ is a p-dimensional sufficient and $\mathbf{A}$ is ancillary and $\mathbf{X}$ satisfies (2.1). Because $d\mathbf{A}/d\xi = 0$, the sum in (3.2) contains only one nonzero term:

$$(4.1) \quad J(\mathbf{x}, \xi) = \left| \det\left(\frac{d}{d\xi} \mathbf{S}(\mathbf{G}(\mathbf{u}, \xi)) \Big|_{\mathbf{u} = \mathbf{G}^{-1}(\mathbf{x}, \xi)} \right) \right|.$$

Let $\mathbf{s} = \mathbf{S}(\mathbf{x})$ and $\mathbf{a} = \mathbf{A}(\mathbf{x})$ be the observed values of the sufficient and ancillary statistics respectively. To interpret the Generalized Fiducial Distribution assume that there is a unique ξ solving $\mathbf{s} = \mathbf{S}(\mathbf{G}(\mathbf{u}, \xi))$ for every $\mathbf{u}$ and denote this solution $Q_\mathbf{s}(\mathbf{u}) = \xi$. Also assume that the ancillary data generating equation $\mathbf{A}(\mathbf{G}(\mathbf{u}, \xi)) = \mathbf{A}(\mathbf{u})$ is not a function of ξ. A straightforward calculation shows that the fiducial density (3.1) with (4.1) is the conditional distribution of $Q_\mathbf{s}(\mathbf{U}^\star) \mid \mathbf{A}(\mathbf{U}^\star) = \mathbf{a}$. We conclude that this choice of data generating equation leads to inference based on sufficient statistics conditional on the ancillary. However, we still do not expect the SLP to hold in general even for this data generating equation.

Heuristically, this is because GFI is using not only the data observed, but also the data that based on the data generating equation could have been observed in the neighborhood of the observed data.

5. FINAL REMARKS

Let us close with discussing the example of Section 3.1. While the paper is not very clear on the exact specification of the events, it appears that for experiment 1 we observe the event

$$O_1 = \{\bar{y}_{169} > 1.96\sigma/\sqrt{169}\},$$

while for the experiment 2 we observe

$$O_2 = \{\bar{y}_k \le 1.96\sigma/\sqrt{k}, k = 1, \dots, 168,$$
$$\bar{y}_{169} > 1.96\sigma/\sqrt{169}\}.$$

Since $O_2 \subset O_1$, we see that the likelihood

$$P_\theta(O_2) = P_\theta(O_2 \mid O_1) P(O_1).$$

Consequently, we would have an SLP pair if and only if $P_\theta(O_2 \mid O_1)$ was a constant as a function of θ. However, this is not the case, as clearly $P_0(O_2 \mid O_1) > 0$ and $\lim_{\theta \to \infty} P_\theta(O_2 \mid O_1) = 0$. Consequently, we do not have an SLP pair.

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