HOMEWORK SET #4

1. Let $X_1, \ldots, X_n$ be an i.i.d. sample from uniform$(0,1)$ distribution. Find the $EX_{(k)}$.

2. Let $X_1, \ldots, X_{4n}$ be an i.i.d. sample from exponential(β) distribution. Find the joint distribution of $(\frac{X_{(n)}+X_{(3n)}}{2}, X_{(3n)} - X_{(n)})$.

3. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a distribution with density $\frac{1}{x^2} I_{(1,\infty)}(x)$. Is EX_k finite? What about $EX_{(k)}$? (Hint: The answer might be different for different k.)

4. Let $X_1, \ldots, X_n$ be i.i.d. sample from a distribution with density $f(x)$. Fix $i < j < k$ and find the formula for the joint density $f_{X_{(i)},X_{(j)},X_{(k)}}(s,u,v)$.

5. Let $X_1, \ldots, X_n$ be i.i.d. from Geometric(p) distribution.
   (a) Set (for this problem part only) $n = 10, p = \frac{1}{4}$. Find $P(X_{(3)} = X_{(5)} = 3)$.
   (b) Fix $1 \le i < j \le n$. Find the probability mass function $f_{X_{(i)},X_{(j)}}(a,b)$.

6. Let $\bar{X}_n$ and S_n^2 be the sample mean and variance based on $X_1, \ldots, X_n$. Suppose a new observation X_{n+1} becomes available. Show
   (a) $\bar{X}_{n+1} = \frac{X_{n+1} + n\bar{X}_n}{n+1}$;
   (b) $nS_{n+1}^2 = (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \bar{X}_n)^2$.

7. What is the probability that a larger of two continuous i.i.d. random variables exceeds the population median? Generalize the result to sample of size n.

8. Let $X_1, \ldots, X_n$ be i.i.d. U$(0,\theta)$. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent.

9. Show that each of the following are exponential family and describe the natural parameter space.
   (a) Gamma(α,β) with both α,β unknown.
   (b) Beta(α,β) with both α,β unknown.
   (c) Negative Binomial(r,p) with r known and p unknown.