Homework set #6
Based on lectures 9 - 10

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. with density $f(x) = \lambda e^{-\lambda(x-\beta)} I_{(\beta,\infty)}(x)$, where $\lambda > 0,\ \beta \in \mathbb{R}$ are unknown parameters.

   (a) Is this exponential family? Would the answer change if we assumed $\beta = 0$?

   (b) Find a minimal sufficient statistic.

   (c) Is $S(\mathbf{X}) = \frac{X_{(n)} - X_{(1)}}{X_{(n)} - X_n}$ an ancillary statistics? Justify!
   (Hint: If $Y \sim \text{Exp}(1)$ and $X = \frac{Y}{\lambda} + \beta$, then the density of $X$ is $f(x) = \lambda e^{-\lambda(x-\beta)} I_{(\beta,\infty)}(x)$.)

   (d) Assume $\beta = 0$. Calculate $E\left(\frac{X_{(1)}}{\bar{X}_n}\right)$. (Hint: Use Basu's theorem.)

   (e) Do not assume $\beta = 0$. Calculate $E\left(\frac{X_{(n)} - X_{(1)}}{\bar{X}_n - \beta}\right)$.

2. For each of the following location parameter families let $X_1, \ldots, X_n$ be iid with density $f(x - \theta)$. Find a minimal sufficient statistics.

   (a) $f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$;

   (b) $f(x) = e^{-x} I_{(0,\infty)}(x)$;

   (c) $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}$;

   (d) $f(x) = \frac{1}{\pi(1+x^2)}$;

   (e) $f(x) = \frac{e^{-|x|}}{2}$.

   (Hint: parts c and d use Fundamental Theorem of Algebra in the proof.)

3. Let $x$ be **one** observation from

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x = -1, 0, 1, \quad 0 \le \theta \le 1.$$

   (a) Is $X$ a complete sufficient statistic?

   (b) Is $|X|$ a complete sufficient statistic?

(c) Does $f(x|\theta)$ form an exponential family for $\theta$?

4. Let $X_1, \ldots, X_n$ be iid from inverse Gaussian distribution, i.e.,

$$f(x|\mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}} I_{(0,\infty)}(x).$$

(a) Show that $\bar{X}_n$ and $T_n = n/(\sum_{i=1}^n (1/X_i - 1/\bar{X})$ are sufficient and complete.

(b) For $n = 2$, show that $\bar{X}_n$ has an inverse Gaussian distribution and $n\lambda T_n$ has a chi-square distribution with $n-1$ degrees of freedom. and they are independent.