Homework set #3

1. Prove or disprove: Let $b_n \to \infty$, $b_n(X_n - a) \xrightarrow{\mathcal{D}} X$ and g be a twice differentiable function satisfying $g'(a) = 0$. Then

$$b_n^2(g(X_n) - g(a)) \xrightarrow{\mathcal{D}} \frac{g''(a)}{2} X^2.$$

   What about $b_n^3(g(X_n) - g(a))$ if $g'(a) = g''(a) = 0$?

2. Suppose that $X_n, Y_n = O_p(1)$ and $U_n = o_p(1)$. Prove the following relations.

   (a) $X_n Y_n = O_p(1)$

   (b) $X_n U_n = o_p(1)$

   (c) If $W_n = o_p(X_n)$ then $W_n = o_p(1)$

   (d) If $a_n X_n = O_p(1)$ then $X_n = O_p(a_n^{-1})$

3. Assume $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \ldots$ are i.i.d. random vectors with finite fourth order moments.

   (a) Show $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n) \xrightarrow{P} \mathrm{Cov}(X, Y)$.

   (b) Show $\frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \cdot \sum_{i=1}^n (Y_i - \bar{Y}_n)^2}} \xrightarrow{P} \rho_{X,Y}$.

   (c) Show

$$\sqrt{n}[\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n) - \frac{1}{n}\sum_{i=1}^n (X_i - EX)(Y_i - EY)] \xrightarrow{P} 0$$

   (d) Find the limiting distribution of

$$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{i=1}^n (X_i - EX)^2 \\ \frac{1}{n}\sum_{i=1}^n (Y_i - EY)^2 \\ \frac{1}{n}\sum_{i=1}^n (X_i - EX)(Y_i - EY) \end{pmatrix} - \begin{pmatrix} \mathrm{Var}\, X \\ \mathrm{Var}\, Y \\ \mathrm{Cov}(X, Y) \end{pmatrix}\right).$$

(e) Find the limiting distribution of

$$\sqrt{n}\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X}_n)(Y_i-\bar{Y}_n)}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\cdot\sum_{i=1}^{n}(Y_i-\bar{Y}_n)^2}}-\rho_{X,Y}\right)$$

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4. Assume that X_n is multinomial$(n, p_1, \ldots, p_k)$ random variable. Notice that X_n is a k dimensional random vector.

   (a) Find the approximate distribution of X_n. (Hint: There is a random vector μ and matrix Σ such that $\sqrt{n}(X_n/n-\mu)\xrightarrow{d}\mathrm{N}(0,\Sigma)$).

   (b) Find the approximate distribution of $(X_n)^2$. (Hint: Use Delta method.)

5. Assume that $X_1, X_2, \ldots$ are i.i.d. k-dimensional random vectors with finite fourth moments and $\mu=(0,\ldots,0)'$ and $\Sigma=I$, where I is an identity matrix (i.e. it has 1 on the diagonal and 0 everywhere else). Find the approximate distribution of $||\bar{X}_n||^2=\sum_{j=1}^{k}(\bar{X}_n^j)^2$. (Hint: Use multivariate CLT and Slutzky's theorem!)