Homework set #2

Based on lectures 1 – 4.

1. Let $X_1 \sim \Gamma(\alpha_1, 1)$ and $X_2 \sim \Gamma(\alpha_2, 1)$ be independent. Use the two-dimensional change of variables formula to show that $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$ are independent with $Y_1 \sim \Gamma(\alpha_1 + \alpha_2, 1)$ and $Y_2 \sim \beta(\alpha_1, \alpha_2)$.

2. (a) Using integration by parts, show that the gamma function $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$ satisfies the relation $\Gamma(t+1) = t\Gamma(t)$ for $t > 0$.

   (b) Use the Hölder's inequality to show that $g(x) = \log \Gamma(x)$ is convex for $x \in (0, \infty)$.

3. Prove or disprove: If X has a cdf F and $a \geq 0$ then $P(F(X) \leq a) \leq a$. Under what condition on F will you get $P(F(X) \leq a) = a$?

4. Use Jensen's inequality to show that for $a, b > 0$ and $p \geq 1$,

$$(a+b)^p \leq 2^{p-1}[a^p + b^p].$$

   Verify this inequality in case $p = 2$ by a direct calculation.

5. Let $X_1, \ldots, X_n$ be a random sample from $N(0,1)$ population. Define

$$Y_1 = \left| \frac{1}{n} \sum_{i=1}^n X_i \right|, \quad Y_2 = \frac{1}{n} \sum_{i=1}^n |X_i| \,.$$

   Calculate EY_1 and EY_2. Which one is bigger?

6. (a) Prove that if $EX^2 < \infty$ then $P(X - EX \geq t) \leq \frac{\operatorname{Var} X}{\operatorname{Var} X + t^2}$ for all $t > 0$. (Hint: $t \leq E\{[(t - (X - EX)] I_{\{X - EX < t\}}\}$ might be useful).

   (b) Then show that this inequality cannot be improved. In particular show that for any fixed $t \geq 0$,

$$\sup_X \left(P(X - EX \geq t) \Big/ \frac{\operatorname{Var} X}{\operatorname{Var} X + t^2} \right) = 1,$$

   where the supremum goes over all possible random variables satisfying $EX^2 < \infty$.