

HOMEWORK SET #4  
Based on lectures 5 - 6

1. Let  $X_1, \dots, X_n$  be an i.i.d. sample from uniform(0, 1) distribution. Find the  $EX_{(k)}$ .
2. Let  $X_1, \dots, X_{4n}$  be an i.i.d. sample from exponential( $\beta$ ) distribution. Find the joint distribution of  $(\frac{X_{(n)} + X_{(3n)}}{2}, X_{(3n)} - X_{(n)})$ .
3. Let  $X_1, \dots, X_n$  be an i.i.d. sample from a distribution with density  $\frac{1}{x^2}I_{(1,\infty)}(x)$ . Is  $EX_k$  finite? What about  $EX_{(k)}$ ? (Hint: The answer might be different for different  $k$ .)
4. Let  $X_1, \dots, X_n$  be i.i.d. sample from a distribution with density  $f(x)$ . Fix  $i < j < k$  and find the formula for the joint density  $f_{X_{(i)}, X_{(j)}, X_{(k)}}(s, u, v)$ .
5. Let  $X_1, \dots, X_n$  be i.i.d. from Geometric( $p$ ) distribution.
  - (a) Set (for this problem part only)  $n = 10, p = \frac{1}{4}$ . Find  $P(X_{(3)} = X_{(5)} = 3)$ .
  - (b) Fix  $1 \leq i < j \leq n$ . Find the probability mass function  $f_{X_{(i)}, X_{(j)}}(a, b)$ .
6. Let  $\bar{X}_n$  and  $S_n^2$  be the sample mean and variance based on  $X_1, \dots, X_n$ . Suppose a new observation  $X_{n+1}$  becomes available. Show
  - (a)  $\bar{X}_{n+1} = \frac{X_{n+1} + n\bar{X}_n}{n+1}$ ;
  - (b)  $nS_{n+1}^2 = (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \bar{X}_n)^2$ .
7. What is the probability that a larger of two continuous i.i.d. random variables exceeds the population median? Generalize the result to sample of size  $n$ .
8. Let  $X_1, \dots, X_n$  be i.i.d.  $U(0, \theta)$ . Show that  $X_{(1)}/X_{(n)}$  and  $X_{(n)}$  are independent.
9. Show that each of the following are exponential family and describe the natural parameter space.
  - (a) Gamma( $\alpha, \beta$ ) with both  $\alpha, \beta$  unknown.

- (b) Beta( $\alpha, \beta$ ) with both  $\alpha, \beta$  unknown.
  - (c) Negative Binomial( $r, p$ ) with  $r$  known and  $p$  unknown.
10. For each of the following families: (i) verify it is an exponential family, (ii) describe whether this is a curved family (iii) describe and sketch the graph of which the curved parameter space.
- (a)  $N(\theta, \theta)$ , with  $\theta > 0$  unknown/
  - (b)  $N(\theta, a\theta^2)$ , with  $a > 0$  known.
  - (c) Gamma( $\alpha, \alpha^{-1}$ ) with  $\alpha > 0$  unknown.
  - (d)  $f(x|\theta) \propto e^{-(x-\theta)^4}$  with  $\theta$  unknown.