Homework set #8

1. Assume that $X_1, \dots, X_n$ are i.i.d. Gamma$(r, 1/\lambda)$, where r is known, i.e., $f(x|r,\lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$. Find the posterior Bayes estimator of λ for the $\Gamma(l, k)$ prior.

2. Let $X_1, \dots, X_n$ be i.i.d. $N(\theta, \sigma^2)$, where σ^2 is known and θ is unknown.

   (a) Find the posterior Bayes estimator for prior $\pi \sim N(a, b^2)$.

   (b) Find the posterior Bayes estimator for prior $\pi(\theta) = 1,\ \theta \in \mathbb{R}$. (This is called an "improper prior".)

3. Let $X_1, \dots, X_n$ be i.i.d. Bernoulli(p). In what follows we will consider the loss function $L(p, a) = \frac{(p-a)^2}{p}$. (This is not the squared loss!)

   (a) Find the Bayes rule for the loss function $L(p, a)$ and Beta(α, β) prior. (Hint: Calculate the posterior risk $r = \int_0^1 L(p, a)\pi(p|\mathbf{x})dp$ and notice that this is a quadratic function in a. Then find a that minimizes r.)

   (b) Find the MINIMAX estimator for this loss function.

4. Let $X_1, \dots, X_n$ be i.i.d. Exp(λ). (Use the parametrization where $EX = \lambda^{-1}$). Find the MINIMAX estimator for the loss $L(a, \lambda) = \lambda^2(a - \lambda^{-1})^2$.

5. (a) Prove or disprove: Any admissible estimator with a flat risk is MINIMAX.

   (b) Does a MINIMAX estimator have to be admissible?

6. Let S^2 be the sample variance based on sample of size n from a normal population. We know that $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Consider the inverse gamma, IG(α, β) prior, given by

$$\pi(\sigma^2) = \frac{e^{-\frac{1}{\beta\sigma^2}}}{\Gamma(\alpha)\beta^\alpha(\sigma^2)^{\alpha+1}} I_{(0,\infty)}(\sigma^2).$$

   Find the posterior. Is this prior conjugate? What is the posterior Bayes estimator of σ^2?

7. Let $X_1 \dots, X_n$ be iid Poisson(λ), and consider the prior $\lambda \sim \Gamma(\alpha, \beta)$.

(a) Find the posterior distribution.

(b) Calculate the posterior mean and variance.

8. Let $X_1, \ldots, X_n$ be iid $\mathrm{N}(\theta, \sigma^2)$, with $\sigma^2 > 0$ known. Consider the double exponential prior $\pi(\theta) = e^{-|\theta|/a}/(2a)$ with $a > 0$ known. Find the mean of the posterior distribution of θ. (Hint: The calculations in this problem are complicated. Use of symbolic math program is recommended.)

9. A loss function investigated by Zellner (1986) is the LINEX loss given by

$$L(\theta, a) = e^{c(a-\theta)} - c(a-\theta) - 1,$$

where $c > 0$ is a constant.

(a) Plot $L(\theta, a)$ as a function of $a - \theta$ for $c = 0.2, 0.5, 1$.

(b) Show that the Bayes rule is $-\frac{\log E(e^{-c\theta}|X)}{c}$.

(c) Let $X_1, \ldots, X_n$ be iid $\mathrm{N}(\theta, \sigma^2)$ with $\sigma^2 > 0$ known, and consider the improper prior $\pi(\theta) = 1$. Find δ the Bayes estimator using the LINEX loss.

(d) Calculate the posterior risk for δ and $\bar{X}_n$ using the LINEX loss.

(e) Calculate the posterior risk for δ and $\bar{X}_n$ using the square loss.