Generalized Fiducial Inference

Parts of this short course are joint work with
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BFF 2018

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[a]NSF support acknowledged

Outline

- Introduction
- Definition
- Theoretical Results
- Applications
- Conclusions

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- Theoretical Results
- Applications
  - Distributed Data
  - Right Censored Data
  - High D Regression
- Conclusions

Fiducial?

- Oxford English Dictionary
  - adjective technical (of a point or line) used as a fixed basis of comparison.
  - Origin from Latin fiducia ‘trust, confidence’
- Merriam-Webster dictionary
  1. taken as standard of reference *a fiducial mark*
  2. founded on faith or trust
  3. having the nature of a trust : fiduciary

Long, long, long time ago...

 

- Bayes Theorem: $P(\theta|X) = \frac{f(X|\theta)\pi(\theta)}{\int_{\Theta} f(X|\theta)\pi(\theta)d\theta}$.

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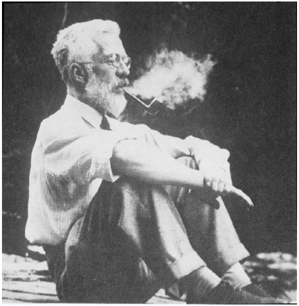
- Bayes Theorem: $P(\theta|X) = \frac{f(X|\theta)\pi(\theta)}{\int_\Theta f(X|\theta)\pi(\theta)d\theta}$.
- Bayes-Laplace postulate:

  When nothing is known about the parameter in advance, let the prior be so that all values of the parameter are equally likely.

Long, long, time ago...



"Not knowing the chance of mutually exclusive events and knowing the chance to be equal are two quite different states of knowledge" R. A. Fisher (1930)

Brief history of fiducial inference

- Fisher (1922, 1930, 1935) no formal definition

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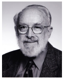

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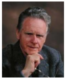

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- Weerahandi (1989, 1993) generalized inference.

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- Fiducial Inference H, Iyer & Patterson (2006), H (2009, 2013), H & Lee (2009), Taraldsen & Lindqvist (2013), Veronese & Melilli (2015), H, Iyer, Lai & Lee (2016)...

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- My point of view is frequentist
  - Justified using asymptotic theorems and simulations.
  - GFI shows very good repeated sampling performance in applications.

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- Likelihood is the function $f(\mathbf{x}, \xi)$, where ξ is variable and $\mathbf{x}$ is fixed.
  - Likelihood as a distribution?

Comparison to likelihood

- Data generating equation (DGE)

$$\mathbf{X} = \mathbf{G}(\mathbf{U}, \xi),$$

  - U is a random with known distribution (iid $U(0,1)$)
  - Parameter ξ is fixed.
  - Generate $\mathbf{X}$s by generating $\mathbf{U}$s and DGE.
    - This determines sampling distribution

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  - Generate ξ by generating $\mathbf{U}$s and inverting DGE.
    - This determines fiducial distribution
    - Denote the inverse $Q_{\boldsymbol{x}}(\boldsymbol{U}^*)$.

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  - Generate ξ by generating $\mathbf{U}$s and inverting DGE.
    - This determines fiducial distribution
    - Denote the inverse $Q_{\boldsymbol{x}}(\boldsymbol{U}^*)$.
- Issues: Multiple solutions and no solutions.

Example -- Bernoulli trials

- Data generating equation

$$X_i = 1\{U_i \leq p\},\ U_i \sim \text{Uniform}(0,1)$$

Generating U_i samples Bernoulli(p).

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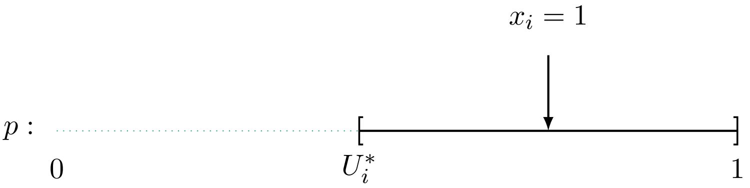
- If $x_i = 1$, then $p \in [U_i^\star, 1]$



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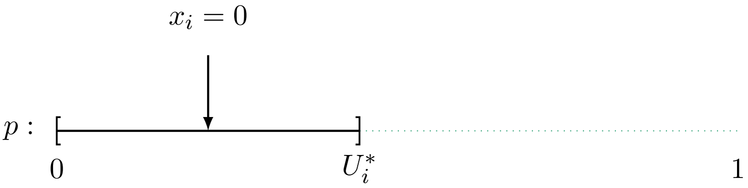
- If $x_i = 0$, then $p \in [0, U_i^\star]$



Example -- Binomial

- Data generating equation

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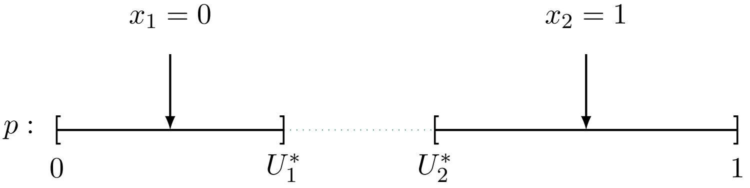
- Data generating equation

  $X_1 = 1\{U_1 \le p\},\ X_2 = 1\{U_2 \le p\}$ U_1, U_2 i.i.d. Uniform(0,1)

- If $X_1 = 0, X_2 = 1$ and $U_1^\star < U_2^\star$



- No solution! Remove $(U_1^\star, U_2^\star)$ inconsistent with data.

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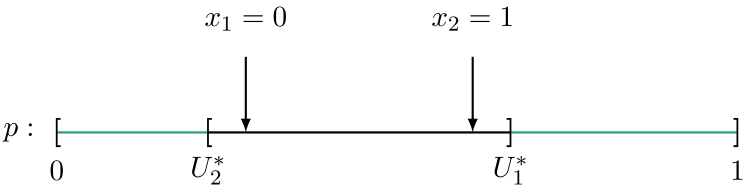

- $(U_1^\star, U_2^\star) \mid \{U_1^\star > U_2^\star\}$ estimates (u_1, u_2).

Example -- Binomial

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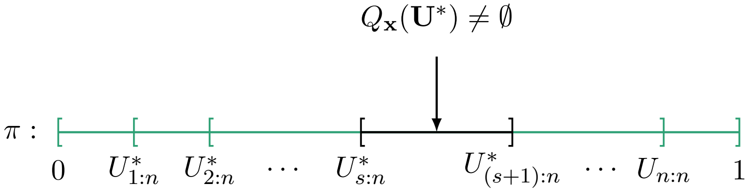

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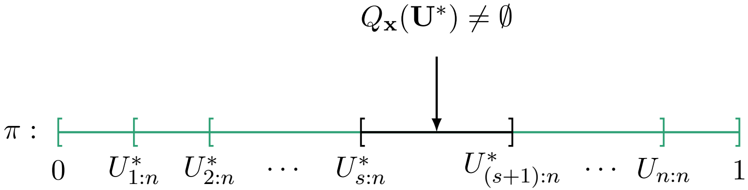

- May select a point in the interval; e.g. Beta$(s + 1/2, n - s + 1/2)$.

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$$Q_{\boldsymbol{x}}(\boldsymbol{u}) = \begin{cases} x_1 - u_1 & \text{if } x_2 - x_1 = u_2 - u_1^\star, \ldots, x_n - x_1 = u_n - u_1 \\ \emptyset & \text{otherwise} \end{cases}$$

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  - Location problem – same as posterior computed using Jeffreys prior

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$$\arg\min_{\xi} \|\mathbf{x} - \boldsymbol{G}(\mathbf{U}^\star, \xi)\| \mid \{\min_{\xi} \|\mathbf{x} - \boldsymbol{G}(\mathbf{U}^\star, \xi)\| \leq \varepsilon\} \tag{1}$$

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  - Similar to ABC; generating from prior replaced by $\min$.
  - Is this practical? Can we compute?

Explicit limit (1)

- Assume $\mathbf{X} \in \mathbb{R}^n$ is continuous; parameter $\xi \in \mathbb{R}^p$
- The limit in (1) has density (H, Iyer, Lai & Lee, 2016)

$$r(\xi|\mathbf{x}) = \frac{f_{\mathbf{X}}(\mathbf{x}|\xi) J(\mathbf{x}, \xi)}{\int_{\Xi} f_{\mathbf{X}}(\mathbf{x}|\xi') J(\mathbf{x}, \xi')\, d\xi'},$$

where $J(\mathbf{x}, \xi) = D\left(\nabla_\xi \mathbf{G}(\mathbf{u}, \xi)|_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x},\xi)} \right)$

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- $n = p$ gives $D(A) = |\det A|$
- $\|\bullet\|_2$ gives $D(A) = (\det A^\top A)^{1/2}$
- $\|\bullet\|_\infty$ gives $D(A) = \sum_{\mathbf{i}=(i_1,\ldots,i_p)} |\det(A)_{\mathbf{i}}|$

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- Jacobian

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  - $= n\frac{\bar{x}(2\theta-1)-\theta^2}{\theta^2-\theta}$ for L_∞.

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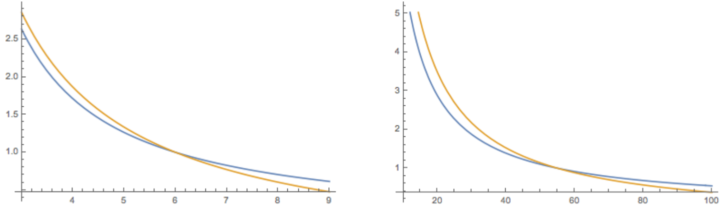

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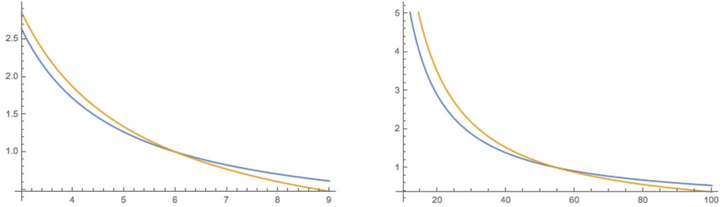

- In simulations fiducial was marginally better than reference prior which was much better than flat prior.

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- Same as independence Jeffreys, *explicit* normalizing constant

Example -- Generalized Pareto

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- Likelihood $f(\mathbf{x}, \gamma, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma\left(1 + \frac{\gamma x_i}{\sigma}\right)^{1+1/\gamma}}$.
- Jacobian evaluated at $u_i = \left(1 + \frac{\gamma x_i}{\sigma}\right)^{-1/\gamma}$
  - $\frac{d}{d\sigma} G(u_i, \gamma, \sigma) = \frac{x_i}{\sigma}$.

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Short course special - L_1 norm!

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- At minimum:
  - p coordinates equal to $\mathbf{x}$
  - KKT condition:

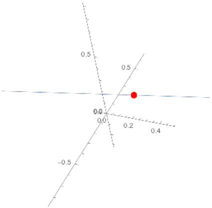
$$0 \in \partial \|\mathbf{x} - \boldsymbol{G}(\mathbf{U}^\star, \xi)\|_1,$$

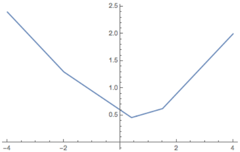
i.e., $0 = -\sum \lambda_i \nabla_x G_i(\mathbf{U}^\star, \xi)$, where

$$\lambda_i \in \begin{cases} \{1\} & x_i - G_i(\mathbf{U}^\star, \xi) > 0 \\ \{-1\} & x_i - G_i(\mathbf{U}^\star, \xi) < 0 \\ [-1, 1] & x_i - G_i(\mathbf{U}^\star, \xi) = 0 \end{cases}$$

$G = (.1, .25, -.2)^\top \xi + (.2, -.1, .3)^\top$

$\boldsymbol{x} = (0, 0, 0)^\top$





minimum at $\xi = .4,\ G = (0.24, 0, 0.22)$

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  - The KKT factor

$$k_{\boldsymbol{i}} = P(\exists \lambda \in [-1, 1]^p \ : \ \lambda \cdot \nabla_\xi \boldsymbol{G_i}(\boldsymbol{u}, \xi) + R \cdot \nabla_\xi \boldsymbol{G_{-i}}(\boldsymbol{u}, \xi) = 0),$$

where $\boldsymbol{u} = \mathbf{G}^{-1}(\boldsymbol{x}, \xi)$,
$R = (R_1, \ldots, R_{n-p})$ i.i.d. Rademacher.

Outline

- Introduction
- Definition
- Theoretical Results
- Applications
  - Distributed Data
  - Right Censored Data
  - High D Regression
- Conclusions

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  - Adding a multiple of a column to another column does not alter $D(A)$. Row operations not allowed!

Classical Result (n=1, p=1; Frequentist)

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- If $S \sim \xi_0$ then $1 - F_S(S, \xi_0) \sim U(0, 1)$ – fiducial p-value.

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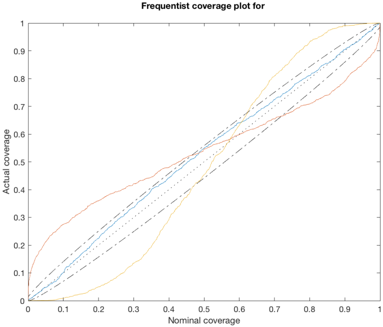
- This is general: simulate m fiducial p-values



exact
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  - Reverse: Map $C(S)$ of fiducial probability $1-\alpha$ to $\mathcal{U}$.
    If invariant in $\boldsymbol{X}$ then exact coverage.

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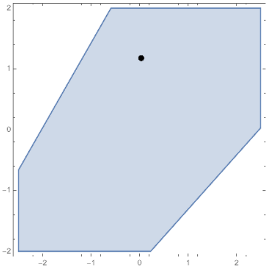

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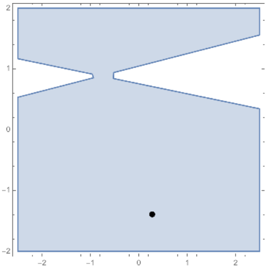

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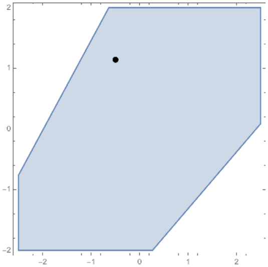

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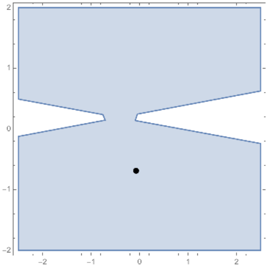

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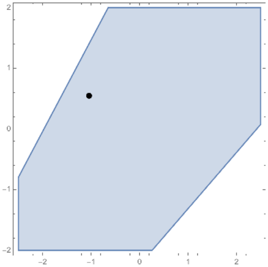

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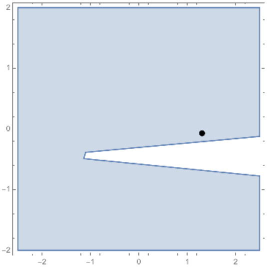

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- GFD1 $\approx$ GFD2 if $|y| >> 0$.

Ancillary Representation $(n > 1, p = 1)$

(4) Let $(S(\mathbf{X}), \mathbf{A}(\mathbf{X}))$ be a smooth 1-1 transformation of $\boldsymbol{X} = \boldsymbol{G}(\boldsymbol{U}, \xi)$.

- $S(\mathbf{X})$ is one dimensional satisfying 1, 2, 3.
- $\mathbf{A}(\mathbf{X})$ is a vector of functional ancillary statistics $(\frac{\partial}{\partial \xi}\mathbf{A} \circ \mathbf{G}(\boldsymbol{U}, \xi) = \mathbf{0})$.

Theorem (Majumder, 2015)

If (4) is satisfied GFI derived from $(S, \mathbf{A})$ is exact.

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Theorem (Majumder, 2015)

If (4) is satisfied GFI derived from $(S, \mathbf{A})$ is exact.

- Idea: The GFD is the same as FD based on $S = G_{S|\mathbf{a}}(\mathbf{U}_\mathbf{a}, \xi)$
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Ancillary Representation $(n > 1, p = 1)$

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- Same argument works for $p > 1$.

Various Asymptotic Results

$r(\xi|\mathbf{x}) \propto f_{\mathbf{X}}(\mathbf{x}|\xi)J(\mathbf{x},\xi)$ where $J(\mathbf{x},\xi) = D\left(\frac{d}{d\xi}\mathbf{G}(\mathbf{u},\xi)\Big|_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x},\xi)}\right)$

- Most start with $C_n^{-1}J(\mathbf{x},\xi) \to J(\xi_0,\xi)$
- Bernstein-von Mises theorem for fiducial distributions provides asymptotic correctness of fiducial CIs H (2009, 2013), Sonderegger & H (2013) .
- Consistency of model selection H & Lee (2009), Lai, H & Lee (2015), H, Iyer, Lai & Lee (2016).
- Regular higher order asymptotics in Pal Majumdar & H (2016+).

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  - Linkage of credible sets across all potential data.

LARGE

Data $\boldsymbol{X}_n$ generated using $\xi_{n,0}$ and GFD measures $R_{n,\boldsymbol{X}_n}$.

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Bernstein-von Mises Thm: centered and scaled MLE

1. $t_n(\boldsymbol{X}_n) \xrightarrow{\mathcal{D}} \boldsymbol{T} = (\boldsymbol{T}_1, \boldsymbol{T}_2)$
2. $\boldsymbol{T}_1 = \boldsymbol{H}_1(\boldsymbol{V}_1, \boldsymbol{\zeta}), \boldsymbol{T}_2 = \boldsymbol{H}_2(\boldsymbol{V}_2)$
   - $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are one-to-one
     $Q_{\boldsymbol{t}_1}(\boldsymbol{v}_1) = \zeta$ solves $\boldsymbol{t}_1 = \boldsymbol{H}_1(\boldsymbol{v}_1, \boldsymbol{\zeta})$
     GFD $R_{\boldsymbol{t}} \sim Q_{\boldsymbol{t}_1}(\boldsymbol{V}_1^\star) \mid \boldsymbol{H}_2(\boldsymbol{V}_2^\star) = \boldsymbol{t}_2$.
   - $R_{\boldsymbol{t}}(\partial C) = 0$ for open C
3. Homeomorphic injective $\boldsymbol{\Xi}_n$
   - $\Xi_n(\boldsymbol{\zeta}_{n,0}) = 0$
   - $t_n(\boldsymbol{x}_n) \to \boldsymbol{t}$

   implies $R_{n,\boldsymbol{x}_n}\boldsymbol{\Xi}_n^{-1} \xrightarrow{\mathcal{W}} R_{\boldsymbol{t}}$.

$\boldsymbol{T} = \boldsymbol{\zeta} + \boldsymbol{V},\ \boldsymbol{V} \sim N(0, I_{\boldsymbol{\xi}_0}^{-1})$

$N(\boldsymbol{t}, I_{\boldsymbol{\xi}_0}^{-1})$

$\boldsymbol{\Xi}_n(\boldsymbol{\xi}) = \sqrt{n}(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$

LARGE theorem

Theorem (H., Lai & Lee, 2016)

Assume LARGE, fix $0 < \alpha < 1$, select open $C_n(\boldsymbol{x}_n)$

1. $R_{n,\boldsymbol{x}_n}(C_n(\boldsymbol{x}_n)) = \alpha$
2. $t_n(\boldsymbol{x}_n) \to \boldsymbol{t}$ *implies* $\boldsymbol{\Xi}_n(C_n(\boldsymbol{x}_n)) \to C(\boldsymbol{t})$
3. *the set* $\mathcal{V}_{\boldsymbol{t}_2} = \{\boldsymbol{v} : \boldsymbol{t} = \boldsymbol{H}(\boldsymbol{v}, \xi)$ *for* $\xi \in C(\boldsymbol{t})\}$ *depends on* $\boldsymbol{t} = (\boldsymbol{t}_1, \boldsymbol{t}_2)$ *only through ancillary* $\boldsymbol{t}_2$.

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Then the sets $C_n(\boldsymbol{x}_n)$ are α asymptotic confidence sets.

- If $\boldsymbol{T}_1 = \boldsymbol{H}_1(\boldsymbol{V}_1, \boldsymbol{\xi})$ has group structure – 3) means $C(\boldsymbol{t})$ is invariant in $\boldsymbol{t}_1$.

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- One sided credible sets have correct asymptotic coverage!

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- Fiducial Quirk – Interested in Frequentist Properties
  - m-number of MC samples vs k-number of replications:
    $k \sim Cm^2$

Outline

- Introduction
- Definition
- Theoretical Results
- Applications
  - Distributed Data
  - Right Censored Data
  - High D Regression
- Conclusions

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  - merge results from different nodes

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- depending on the problem, $r(\boldsymbol{\theta}|\boldsymbol{x})$ and $r_k(\boldsymbol{\theta}|\boldsymbol{x}_k)$ may be known only up to normalizing constant.

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- Not feasible and very inefficient!
  - Target of order $n^{-1/2}$
  - Fiducial sample on each worker of order $n_k^{-1/2}$.
  - Most realizations get extremely small weights.

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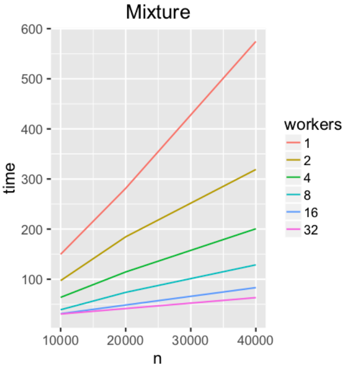
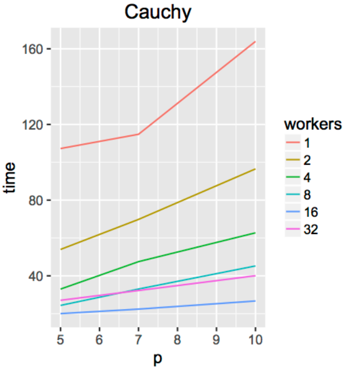
- We proved consistency and asymptotic normality of the approximation error.

Experiments

- All good performance ($K = 1, 2, 4, 8, 16, 32$)
  - Gaussian mixture: $0.6N(-1, 1) + 0.4N(1, 1)$ ($n = 4 \times 10^5$)
  - Linear regression with Cauchy errors ($n = 10^5$, $p_T = 4, p = 6, 8, 11$)

Computational time





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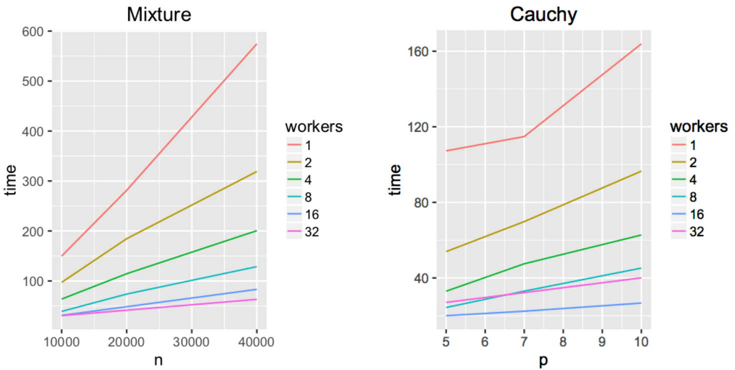
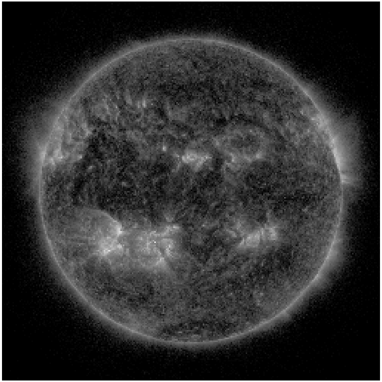
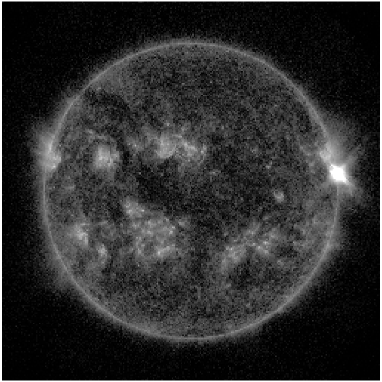


- For Cauchy: speed improves until $K = 16$ then deteriorates (Cheng & Shang, 2015)

Sun Spots



low activity



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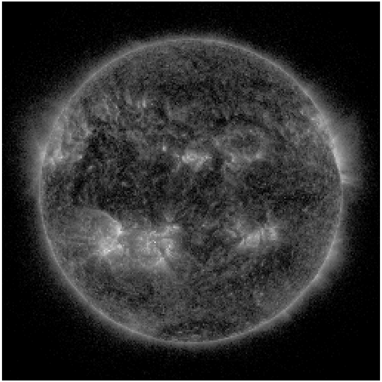

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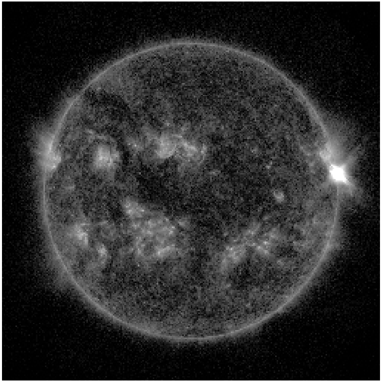

high activity

- The bright flare on the right has value 253. Is this high?

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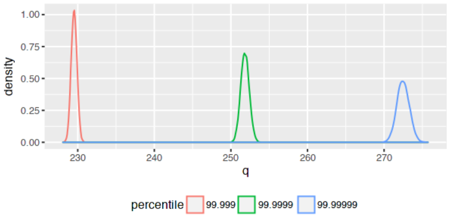
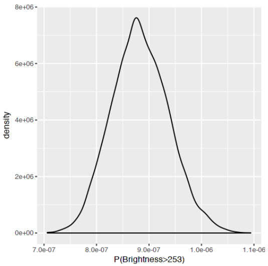
- Tool: GFD for Generalized Pareto (Wandler & H, 2012)

GFD for extreme quantiles



Large Quantiles



Fiducial Excedance Probability

Outline

Right Censored Data

- Data generating equation:

  $T_i = F^{-1}(U_i),\ C_i = G_i^{-1}(W_i|F^{-1}(U_i)), \quad U_i, W_i$ i.i.d. Uniform(0,1)

  observe only $X_i = \min(T_i, C_i), \delta_i = I_{\{T_i \leq C_i\}}$.

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- Solvie for F and G:

$$\begin{array}{lll} \delta_i = 1 & F^*(x_i - \epsilon) < U_i^\star \le F^*(x_i) & G_i^*(x_i - \epsilon|x_i) \le W_i^\star \\ \delta_i = 0 & F^*(x_i) < U_i^\star & G_i^*(x_i - \epsilon|F^{-1}(U_i^\star)) < W_i^\star \le G_i^*(x_i|F^{-1}(U_i^\star)) \end{array}$$

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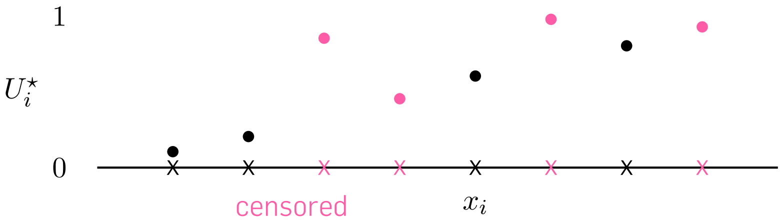
- Generate U^* conditional on existence of F^* solving the equations.

Visual Demonstation



U_i^* generated so that there is a solution

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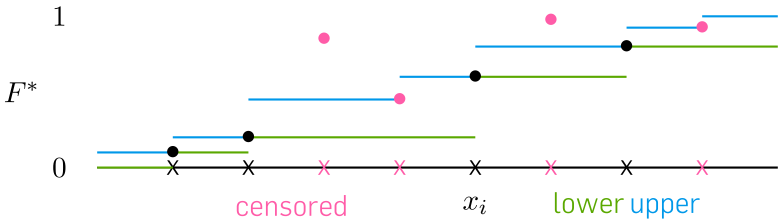

F^* is any cdf between bounds

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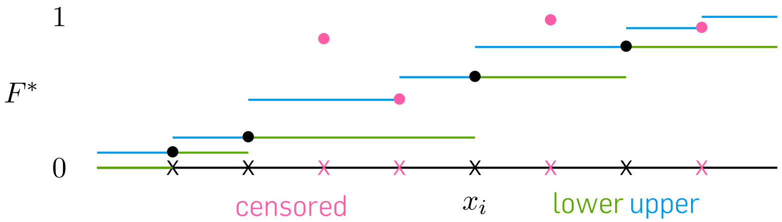

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Fact: For any failure time $E^*[F_L^\star(s)] < \hat{F}(s) < E^*[F_U^\star(s)]$.

Theoretical Results

- Known: KM estimator

$$\sqrt{n}[\hat{F}(t) - F(t)] \xrightarrow{\mathcal{D}} W_t \text{ in D[0,T]}$$

(W – mean zero Gaussian process with known covariance depending on censoring)

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- Theorem (Cui, Hannig): Under similar assumptions

$$\sqrt{n}[F^{\star}(t) - \hat{F}(t)]|\mathbf{X} \xrightarrow{\mathcal{D}} W_t \text{ almost surely.}$$

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- Simultaneous CI
  - Center on mean (or Kaplan Mayer).
  - Find L_∞ ball containing $1-\alpha$ fiducial sample curves.

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- Quantiles of fiducial at given x – approximate pointwise CIs.
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  - Center on mean (or Kaplan Mayer).
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- For difference of two populations – use difference of fiducial distributions

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- Pointwise simulations show good coverage
- Simultaneous CI also show good/conservative coverage
- Two sample testing
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  - Compared to log rank tests and sup-log rank test: competitive.
- Room for improvement:
  - Use something else than L_∞ ball – half-region depth?

Gastric Tumor Study Group (1982)

- Clinical trial of chemotherapy vs. chemotherapy combined with radiotherapy.

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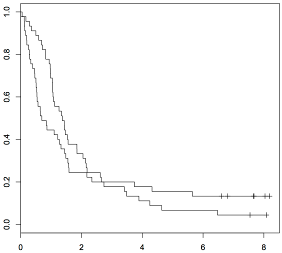

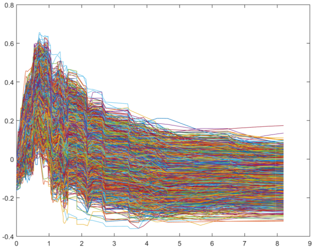

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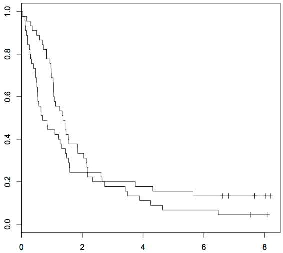

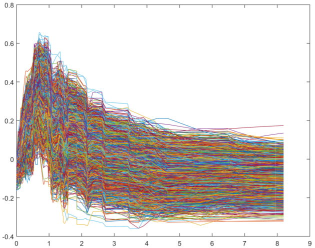

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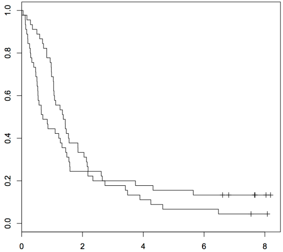

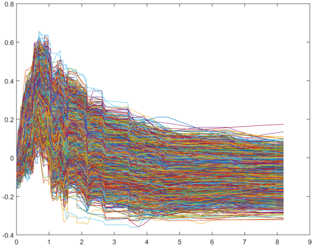

- Different: Log-rank statistics p-value 0.63 (hazard crossing); Fiducial p-value 0.003.
- Simulation with estimated hazard shows fiducial more powerful than competitors.

Outline

Model Selection

- $\boldsymbol{X} = \boldsymbol{G}(M, \boldsymbol{\xi}_M, \boldsymbol{U}), \qquad M \in \mathcal{M},\ \boldsymbol{\xi}_M \in \boldsymbol{\xi}_M$

Theorem: (H, Iyer, Lai, Lee 2016) Under assumptions

$$r(M|\boldsymbol{y}) \propto q^{|M|} \int_{\boldsymbol{\xi}_M} f_M(\boldsymbol{y}, \boldsymbol{\xi}_M) J_M(\boldsymbol{y}, \boldsymbol{\xi}_M)\, d\boldsymbol{\xi}_M$$

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  - Default value $q = n^{-1/2}$ (motivated by MDL)

Alternative to penalty - see poster!

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- Motivated by non-local priors of Johnson & Rossell (2009)

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where b_{min} solves

$$\min_{b\in R^p} \frac{1}{2}\|X^T(X_M\beta_M - Xb)\|_2^2 \quad \text{subject to} \quad \|b\|_0 \leq |M| - 1.$$

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    different norm and target
  - Call this: ε-admissible subset

GFD

$$r(M|\boldsymbol{y}) \propto \pi^{\frac{|M|}{2}} \Gamma\left(\frac{n-|M|}{2}\right) RSS_M^{-\left(\frac{n-|M|-1}{2}\right)} E[h_M^{\varepsilon}(\beta_M^{\star})]$$

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- Implemented using Grouped Independence Metropolis Hastings (Andrieu & Roberts, 2009).

Main Result

Theorem Williams & H (2017+)

Suppose the true model is given by M_T. Then under certain conditions, for a fixed positive constant $\alpha < 1$,

$$r(M_T|y) = \frac{r(M_T|y)}{\sum_{j=1}^{n^\alpha} \sum_{M:|M|=j} r(M|y)} \xrightarrow{P} 1 \text{ as } n, p \to \infty.$$

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- For a large model $|M| > p_T$ and large enough n or p,

$$\frac{9}{2}\|X^T(H_M - H_{M(-1)})\mu_T\|_2^2 < \varepsilon_M,$$

where H_M and $H_{M(-1)}$ are the projection matrix for M and M with a covariate removed respectively.

Default ε

$$\varepsilon = \Lambda_M \widehat{\sigma}^2_M \left(\frac{n^{0.51}}{9} + |M| \frac{\log(p\pi)^{1.1}}{9} - p_T \right)_+,$$

- $\Lambda_M := \operatorname{tr}\left((H_M X)' H_M X\right)$ with $H_M := X_M (X_M' X_M)^{-1} X_M'$
- $\widehat{\sigma}^2_M := \mathsf{RSS}_M / (n - |M|)$

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- Tuning parameter p_T represents belief about true $|M_T|$.

Simulation setup 1

- Generate 1000 data vectors y from linear model with $\beta^0_{M_o} = (-1.5, -1, -.8, -.6, .6, .8, 1, 1.5)'$, and $\sigma^0_{M_o} = 1$.

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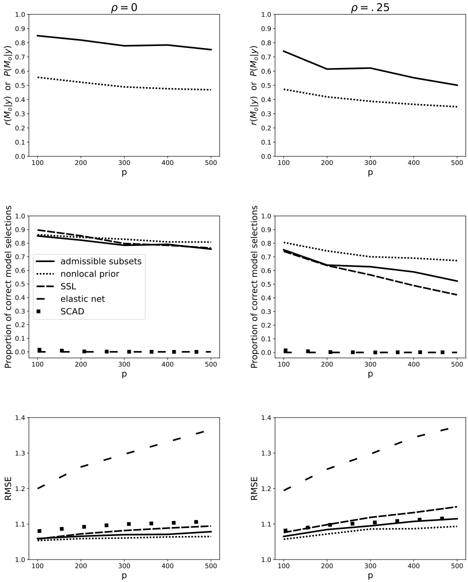
- Set $n = 100$, and consider $p = 100, 200, 300, 400, 500$.

Simulation results 1



Simulation setup 2

To illustrate the difference from the nonlocal prior approach, for $n = 30$, generate data from the following model.

$$Y \sim \mathsf{N}_n\Big(1 \cdot x^{(1)} + 1 \cdot x^{(2)} + \cdots + 1 \cdot x^{(9)}, I_n\Big),$$

where $x^{(1)}, x^{(2)}, x^{(3)} \overset{\text{iid}}{\sim} \mathsf{N}_n(0, I_n)$, and

$$\begin{array}{ccccccccc}
x^{(4)} & \sim & \mathsf{N}_n\Big(& .25 \cdot x^{(1)} & & & & & , .1^2 I_n\Big) \\
x^{(5)} & \sim & \mathsf{N}_n\Big(& & & .5 \cdot x^{(2)} & & & , .1^2 I_n\Big) \\
x^{(6)} & \sim & \mathsf{N}_n\Big(& & & & - & .75 \cdot x^{(3)} & , .1^2 I_n\Big) \\
x^{(7)} & \sim & \mathsf{N}_n\Big(& x^{(1)} & & & + & x^{(3)} & , .1^2 I_n\Big) \\
x^{(8)} & \sim & \mathsf{N}_n\Big(& & & x^{(2)} & - & x^{(3)} & , .1^2 I_n\Big) \\
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	MAP size	RMSE	$P(M_{\mathsf{MAP}}\|y)$
ε-$admissible$ subsets	3.476	1.138	.365
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- Nonlocal prior procedure typically includes all 9 covariates even though the y can be mostly explained by 3.

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Can Bayesian, Fiducial and Frequentist
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