STOR 655 – February 27, 2020 Name:________________________

Midterm

All problem parts have equal weight. In budgeting your time expect that some problems will take longer than others.
Remember, answers without proper justification will not receive full credit!

1. Assume that $U \sim U(0,1)$, V is independent of U and $\beta(x) \leq x$. Prove or disprove: The dependency criterion DC(β) is satisfied for (U, V).

2. Let $X_1, \ldots$ be i.i.d. Geometric(p), i.e.,

$$P(X_i = k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots.$$

(a) Find the asymptotic distribution of $\log(\bar{X}_n)$.

(b) Find the variance stabilizing distribution for $\bar{X}_n$.
(Hint: if $r > 1$ then $\frac{d}{dr} \sinh^{-1}\left(\sqrt{r-1}\right) = \frac{1}{2\sqrt{r(r-1)}}$.)

3. (a) Show that if $\frac{E(X_n - Y_n)^2}{\operatorname{Var} X_n} \to 0$, then $\frac{(EX_n - EY_n)^2}{\operatorname{Var} X_n} \to 0$, $\frac{\operatorname{Var} Y_n}{\operatorname{Var} X_n} \to 1$, and $\operatorname{Corr}(X_n, Y_n) \to 1$.

   (b) Show that if

$$\frac{X_n - EX_n}{\sqrt{\operatorname{Var} X_n}} \xrightarrow{\mathcal{D}} X \quad \text{and} \quad \frac{E(X_n - Y_n)^2}{\operatorname{Var} X_n} \to 0$$

   then

$$\frac{Y_n - EY_n}{\sqrt{\operatorname{Var} Y_n}} \xrightarrow{\mathcal{D}} X.$$

   (Hint: You can use part (a) even if you did not prove it.)

4. (a) Let A be a symmetric matrix. Show that if $\mathbf{X} \sim N_p(\mu, \Sigma)$ and $\mathbf{Y} = \mathbf{X}^T A \mathbf{X}$ then $EY = \mathrm{tr}(A\Sigma) + \mu^T A \mu$.

   (b) Show that if $Y_1 \sim \chi^2_{p_1}(\lambda_1)$ and $Y_2 \sim \chi^2_{p_2}(\lambda_2)$ are independent, then $Y_1 + Y_2 \sim \chi^2_{p_1+p_2}(\lambda_1 + \lambda_2)$.