STOR 655 – May 1, 2020 Name:_____

Final Exam

All problem parts have equal weight. In budgeting your time expect that some problems will take longer than others. You can use results of earlier parts in later parts.
Remember, answers without detailed justification will not receive full credit!

1. Prove or disprove: Let $b_n \to \infty$, $b_n(X_n - a) \xrightarrow{\mathcal{D}} X$ and g be a twice differentiable function satisfying $g'(a) = 0$. Then

$$b_n^2(g(X_n) - g(a)) \xrightarrow{\mathcal{D}} \frac{g''(a)}{2} X^2.$$

2. Let $X_1, \ldots, X_n$ i.i.d. with density $f(x|\theta) = e^{-(x-\theta)} I_{(\theta,\infty)}$. The unknown true parameter is θ_0,

   (a) Is this an exponential family?

   (b) Find $\hat{\theta}_n$ the MLE. Is it a consistent estimator of θ_0?

   (c) What is the asymptotic distribution of $\hat{\theta}_n$? Is it asymptotically normal? Hint: consider the limiting distribution of $n(\hat{\theta}_n - \theta_0)$

   (d) Find the Bayes posterior distribution using prior $\pi(\theta) = \frac{e^{-|\theta|}}{2}$.

   (e) Consider rescaling the posterior distribution by the change of variable $\zeta = n(\hat{\theta}_n - \theta)$. What does the rescaled posterior distribution converges to as $n \to \infty$? Is this convergence in L_1?

3. Assume $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \ldots \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$ are i.i.d. bivariate normal$\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \rho & \rho^2 \\ \rho^2 & \rho \end{pmatrix} \right)$, $0 < \rho < 1$.
Hint: $EX^4 = 3\rho^2, E(XY)^2 = \rho^2 + 2\rho^4, EX^3Y = 3\rho^3$

   (a) Consider $R_n = \frac{1}{n}\sum_{i=1}^n X_iY_i$ and $V_n = \frac{1}{n}\sum_{i=1}^n X_i^2$. Are $\sqrt{R_n}$ and V_n strongly consistent estimators of ρ?

   (b) What is the asymptotic distribution of these estimators? Are they asymptotically normal?

   (c) Is $\frac{R_n}{V_n}$ a strongly consistent estimator of ρ? What is its asymptotic distribution?

   (d) Prove that Fisher information is $\frac{4\rho^4 - 3\rho^2 + 1}{\rho^2(1-\rho^2)^2}$.

   (e) Is any of the three estimators above asymptotically efficient? If not, improve one of them by scoring.