Generalized Fiducial Inference

Parts of this talk are joint work with
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[a]NSF support acknowledged

Outline

- Introduction
- Definition
- Theoretical Results
- Applications
- Conclusions

Outline

Fiducial?

- Oxford English Dictionary
  - adjective technical (of a point or line) used as a fixed basis of comparison.
  - Origin from Latin fiducia ‘trust, confidence’
- Merriam-Webster dictionary
  1. taken as standard of reference *a fiducial mark*
  2. founded on faith or trust
  3. having the nature of a trust : fiduciary

Long, long, long time ago...

- Bayes Theorem: $P(\theta|X) = \frac{f(X|\theta)\pi(\theta)}{\int_\Theta f(X|\theta)\pi(\theta)d\theta}$.

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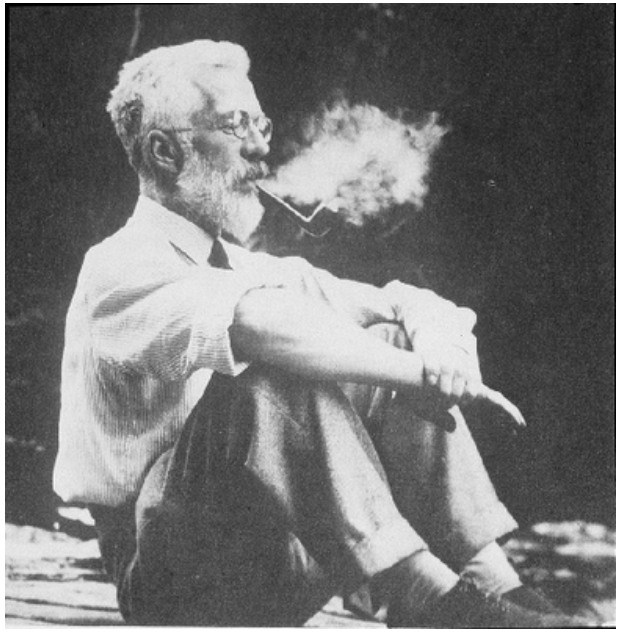
- Bayes Theorem: $P(\theta|X) = \frac{f(X|\theta)\pi(\theta)}{\int_\Theta f(X|\theta)\pi(\theta)d\theta}$.
- Bayes-Laplace postulate:

  When nothing is known about the parameter in advance, let the prior be so that all values of the parameter are equally likely.

Long, long, time ago...



"Not knowing the chance of mutually exclusive events and knowing the chance to be equal are two quite different states of knowledge" R. A. Fisher (1930)

Brief history of fiducial inference

- Fisher (1922, 1930, 1935) no formal definition

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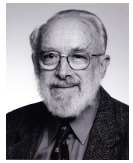

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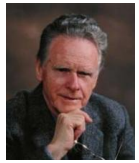

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- Barnard (1995) pivotal based methods.

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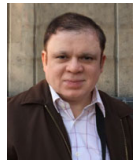

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- Weerahandi (1989, 1993) generalized inference.

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- Fiducial Inference H, Iyer & Patterson (2006), H (2009, 2013), H & Lee (2009), Taraldsen & Lindqvist (2013), Veronese & Melilli (2015), H, Iyer, Lai & Lee (2016)...

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- My point of view is frequentist
  - Justified using asymptotic theorems and simulations.
  - GFI shows very good repeated sampling performance in applications.

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- Likelihood is the function $f(\mathbf{x}, \xi)$, where ξ is variable and $\mathbf{x}$ is fixed.
  - Likelihood as a distribution?

Comparison to likelihood

- Data generating equation (DGE)

$$\mathbf{X} = \mathbf{G}(\mathbf{U}, \xi),$$

  - U is a random with known distribution (iid $U(0,1)$)
  - Parameter ξ is fixed.

  - Generate $\mathbf{X}$s by generating $\mathbf{U}$s and DGE.
    - This determines sampling distribution

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  - Generate ξ by generating $\mathbf{U}$s and inverting DGE.
    - This determines fiducial distribution
- Issues: Multiple solutions and no solutions.

Example -- Bernoulli trials

- Data generating equation

$$Y_i = 1\{U_i \leq p\},\ U_i \sim \text{Uniform}(0,1)$$

Generating U_i samples Bernoulli(p).

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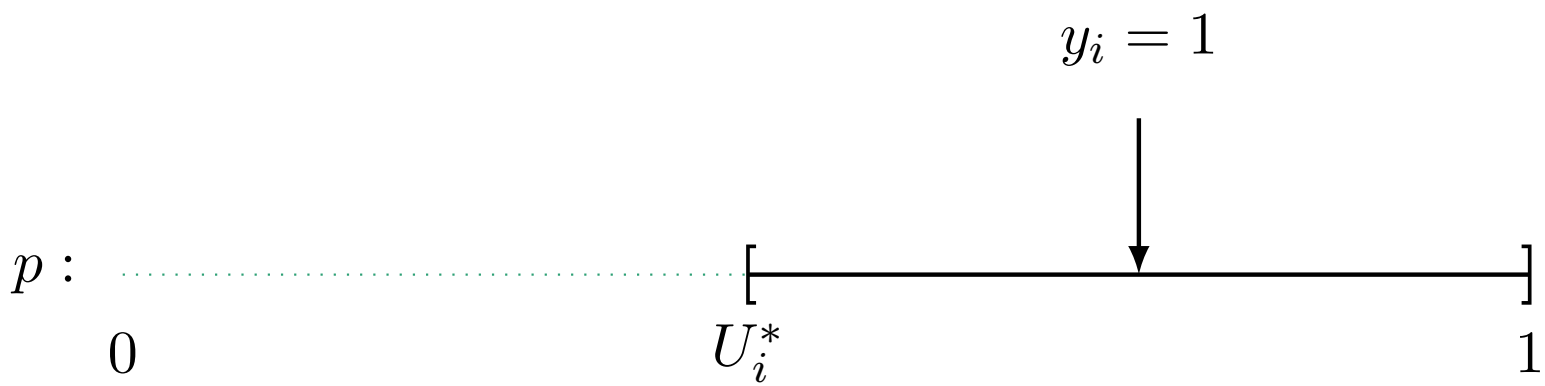
- If $y_i = 1$, then $p \in [U_i^\star, 1]$



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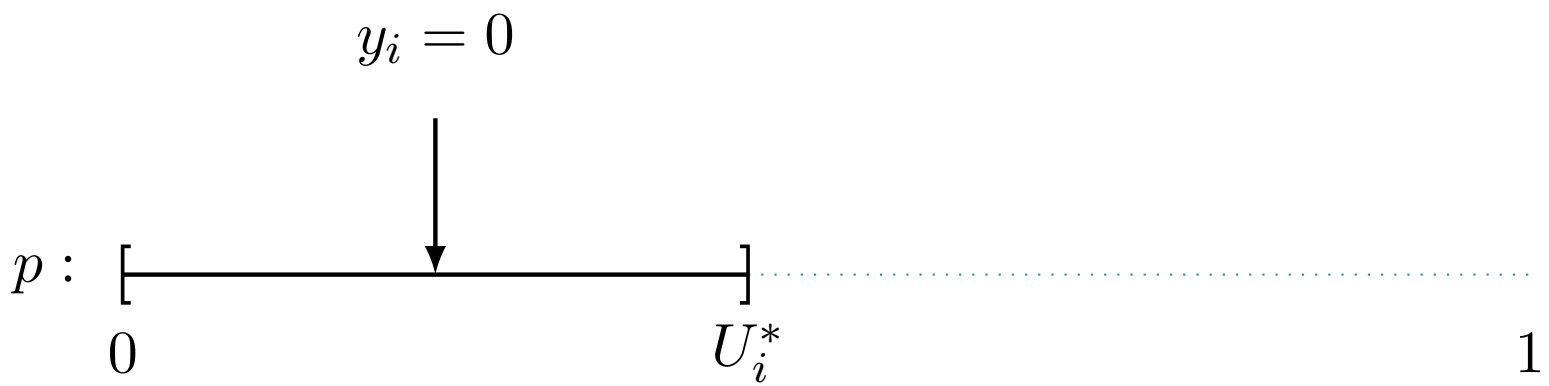
- If $y_i = 0$, then $p \in [0, U_i^\star]$



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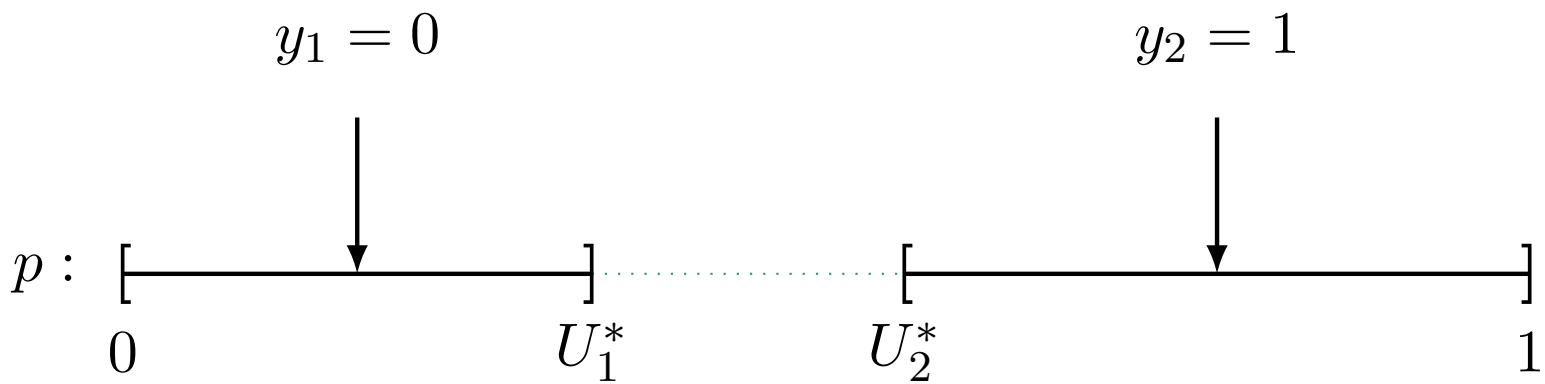
  $Y_1 = 1\{U_1 \leq p\}$, $Y_2 = 1\{U_2 \leq p\}$ $\quad$ U_1, U_2 i.i.d. Uniform(0,1)

- If $y_1 = 0, y_2 = 1$ and $U_1^\star < U_2^\star$



- No solution! Remove $(U_1^\star, U_2^\star)$ inconsistent with data.

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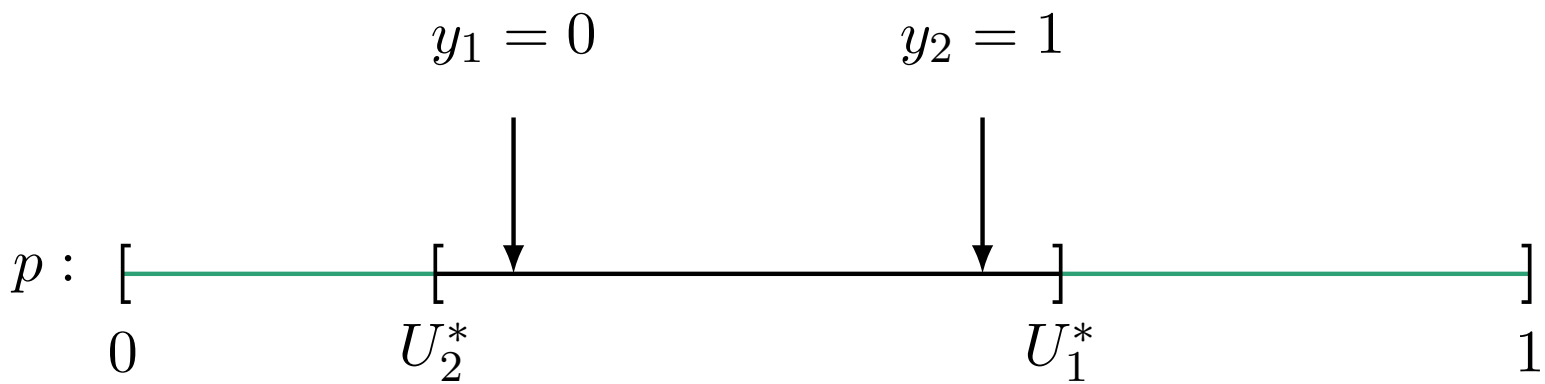

- $(U_1^\star, U_2^\star) \mid \{U_1^\star > U_2^\star\}$ estimates (u_1, u_2).

Example -- Binomial

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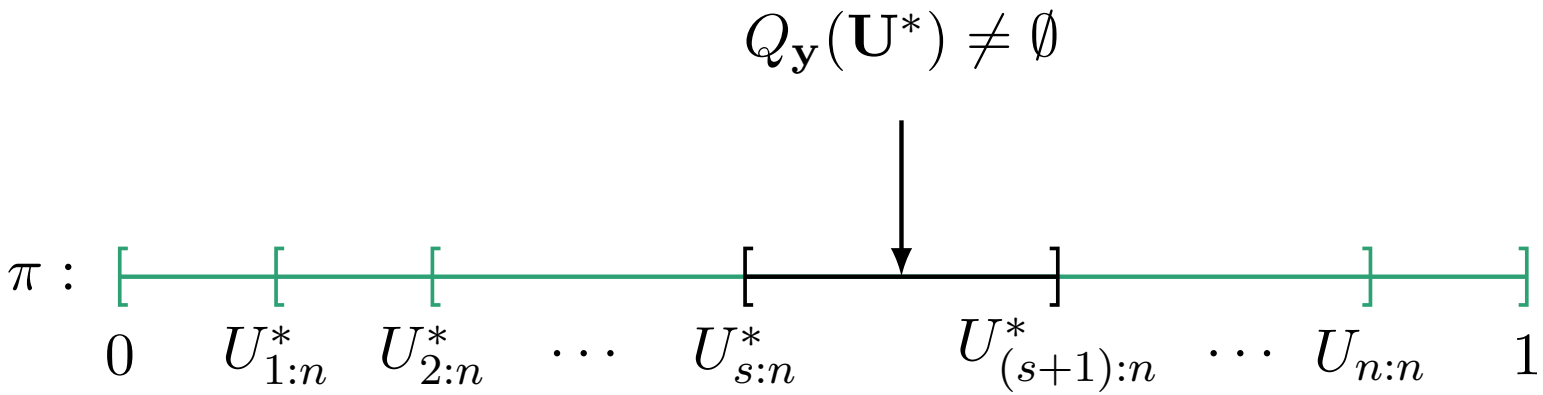

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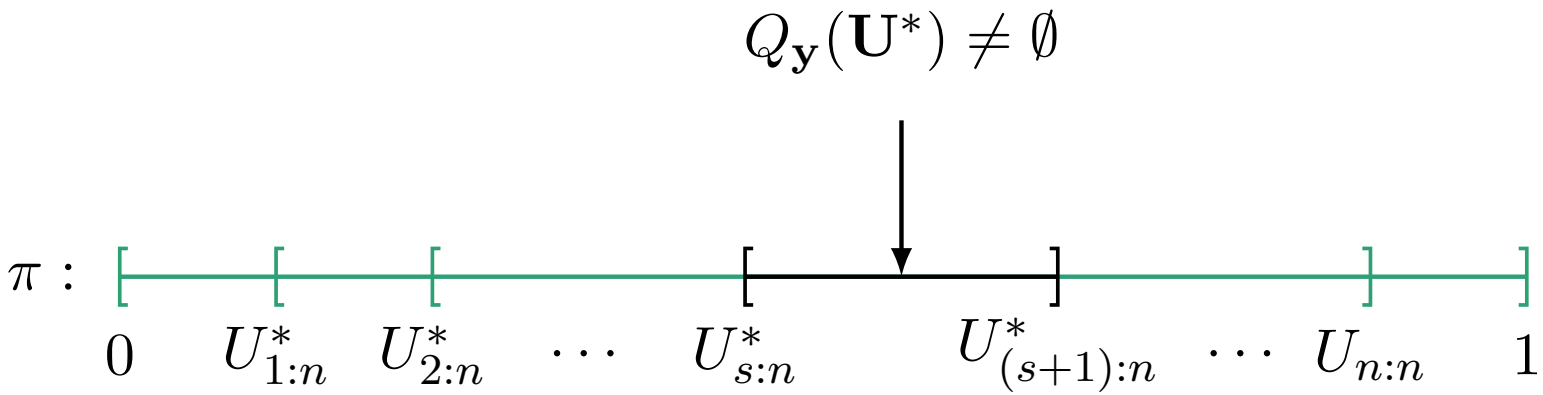

- May select a point in the interval; e.g. Beta$(s + 1/2, n - s + 1/2)$.

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$$Q_{\boldsymbol{x}}(\boldsymbol{u}) = \begin{cases} x_1 - u_1 & \text{if } x_2 - x_1 = u_2 - u_1^\star, \ldots, x_n - x_1 = u_n - u_1 \\ \emptyset & \text{otherwise} \end{cases}$$

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  - Location problem – same as posterior computed using Jeffreys prior

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- A distribution on the parameter space is Generalized Fiducial Distribution if it can be obtained as a limit (as $\varepsilon \downarrow 0$) of

$$\arg\min_{\xi} \|\mathbf{x} - \boldsymbol{G}(\mathbf{U}^\star, \xi)\| \mid \{\min_{\xi} \|\mathbf{x} - \boldsymbol{G}(\mathbf{U}^\star, \xi)\| \leq \varepsilon\} \quad (1)$$

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  - Similar to ABC; generating from prior replaced by $\min$.
  - Is this practical? Can we compute?

Explicit limit (1)

- Assume $\mathbf{X} \in \mathbb{R}^n$ is continuous; parameter $\xi \in \mathbb{R}^p$
- The limit in (1) has density (H, Iyer, Lai & Lee, 2016)

$$r(\xi|\mathbf{x}) = \frac{f_{\mathbf{X}}(\mathbf{x}|\xi) J(\mathbf{x}, \xi)}{\int_{\Xi} f_{\mathbf{X}}(\mathbf{x}|\xi') J(\mathbf{x}, \xi')\, d\xi'},$$

where $J(\mathbf{x}, \xi) = D\left(\nabla_{\xi} \mathbf{G}(\mathbf{u}, \xi)|_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x},\xi)}\right)$

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- $n = p$ gives $D(A) = |\det A|$
- $\|\bullet\|_2$ gives $D(A) = (\det A^{\top} A)^{1/2}$
- $\|\bullet\|_{\infty}$ gives $D(A) = \sum_{\mathbf{i}=(i_1,\ldots,i_p)} |\det(A)_{\mathbf{i}}|$

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  - $= n\frac{\bar{x}(2\theta-1)-\theta^2}{\theta^2-\theta}$ for L_∞.

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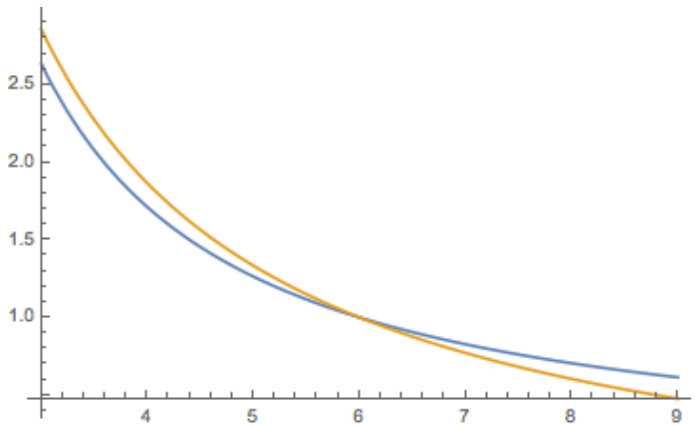
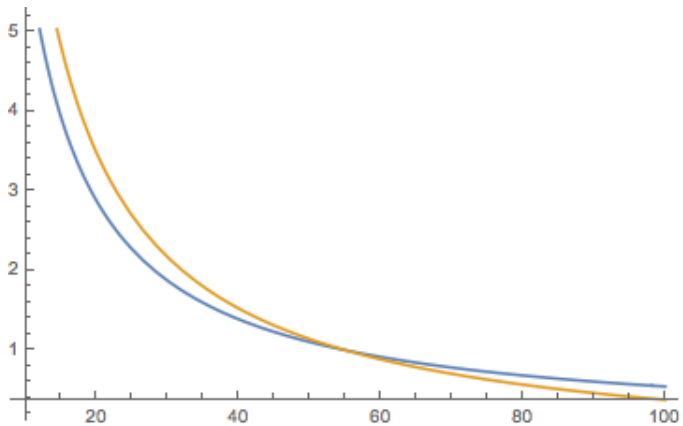
- reference prior vs fiducial Jacobian





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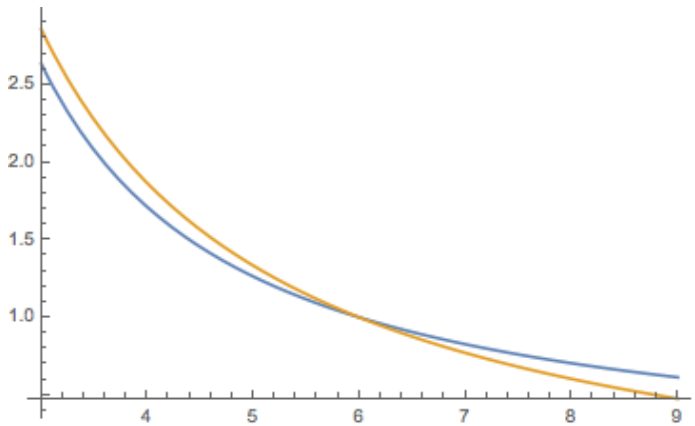

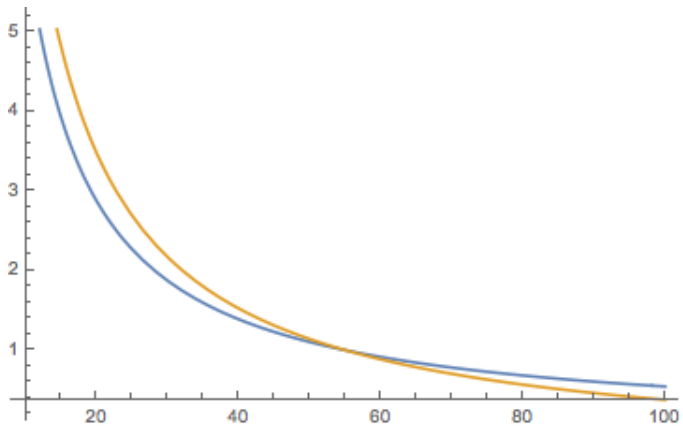

- In simulations fiducial was marginally better than reference prior which was much better than flat prior.

Example -- Linear Regression

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- Same as independence Jeffreys, *explicit* normalizing constant

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    - $= \sum_{i<j} \left| \frac{x_j\left(1+\frac{\gamma x_i}{\sigma}\right)\log\left(1+\frac{\gamma x_i}{\sigma}\right) - x_i\left(1+\frac{\gamma x_j}{\sigma}\right)\log\left(1+\frac{\gamma x_j}{\sigma}\right)}{\gamma^2} \right|$ for L_∞.

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Short course special - L_1 norm!

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- At minimum:
  - p coordinates equal to $\mathbf{x}$
  - KKT condition:

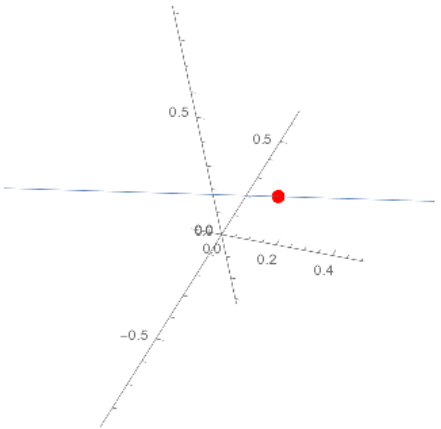
$$0 \in \partial \|\mathbf{x} - \boldsymbol{G}(\mathbf{U}^\star, \xi)\|_1,$$

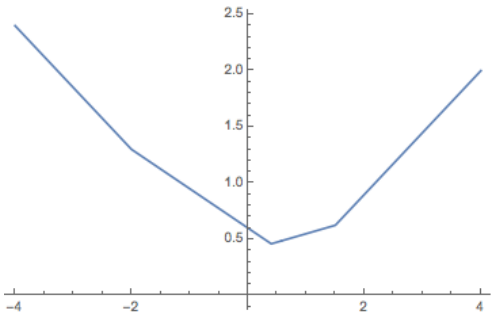
i.e., $0 = -\sum \lambda_i \nabla_x G_i(\mathbf{U}^\star, \xi)$, where

$$\lambda_i \in \begin{cases} \{1\} & x_i - G_i(\mathbf{U}^\star, \xi) > 0 \\ \{-1\} & x_i - G_i(\mathbf{U}^\star, \xi) < 0 \\ [-1, 1] & x_i - G_i(\mathbf{U}^\star, \xi) = 0 \end{cases}$$

$G = (.1, .25, -.2)^\top \xi + (.2, -.1, .3)^\top$

$\boldsymbol{x} = (0, 0, 0)^\top$





minimum at $\xi = .4,\ G = (0.24, 0, 0.22)$

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  - The KKT factor

$$k_{\boldsymbol{i}} = P(\exists \lambda \in [-1, 1]^p \ : \ \lambda \cdot \nabla_\xi \boldsymbol{G_i}(\boldsymbol{u}, \xi) + R \cdot \nabla_\xi \boldsymbol{G_{-i}}(\boldsymbol{u}, \xi) = 0),$$

where $\boldsymbol{u} = \mathbf{G}^{-1}(\boldsymbol{x}, \xi)$,
$R = (R_1, \ldots, R_{n-p})$ i.i.d. Rademacher.

Outline

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  - Adding a multiple of a column to another column does not alter $D(A)$. Row operations not allowed!

Classical Result (n=1, p=1; Frequentist)

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- If $S \sim \xi_0$ then $1 - F_S(S, \xi_0) \sim U(0, 1)$ – fiducial p-value.

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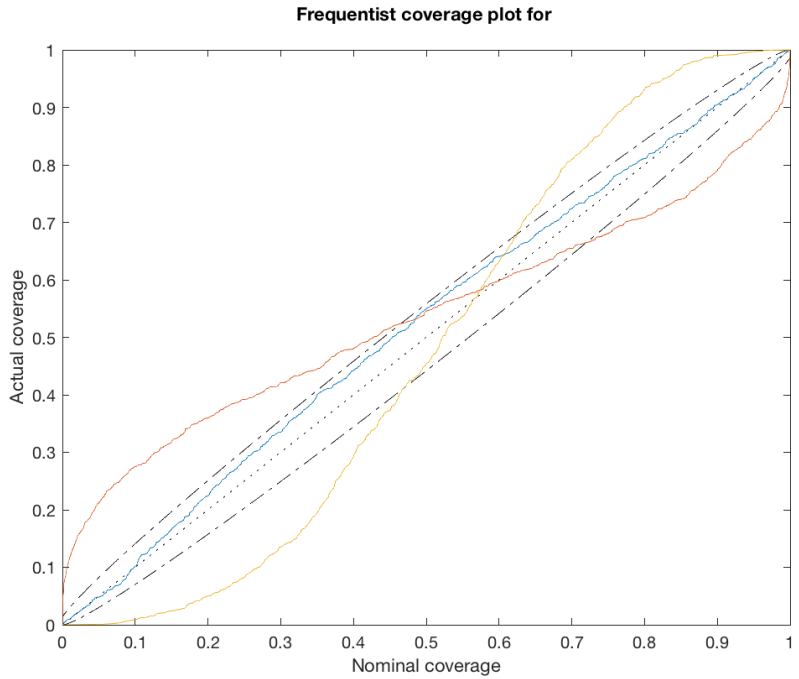
- This is general: simulate m fiducial p-values



exact
conservative
liberal

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  - Compare to pivotal method of deriving CIs
  - Basis of Inferential Models (Liu & Martin, 2015)

Ancillary Representation $(n > 1, p = 1)$

(4) Let $(S(\mathbf{X}), \mathbf{A}(\mathbf{X}))$ be a smooth 1-1 transformation of $\boldsymbol{X} = \boldsymbol{G}(\boldsymbol{U}, \xi)$.

- $S(\mathbf{X})$ is one dimensional satisfying 1, 2, 3.
- $\mathbf{A}(\mathbf{X})$ is a vector of functional ancillary statistics $(\frac{\partial}{\partial \xi}\mathbf{A} \circ \mathbf{G}(\boldsymbol{U}, \xi) = \mathbf{0})$.

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- Same argument works for $p > 1$.

Various Asymptotic Results

$r(\xi|\mathbf{x}) \propto f_{\mathbf{X}}(\mathbf{x}|\xi)J(\mathbf{x},\xi)$ where $J(\mathbf{x},\xi) = D\left(\frac{d}{d\xi}\mathbf{G}(\mathbf{u},\xi)\Big|_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x},\xi)}\right)$

- Most start with $C_n^{-1}J(\mathbf{x},\xi) \to J(\xi_0,\xi)$
- Bernstein-von Mises theorem for fiducial distributions provides asymptotic correctness of fiducial CIs H (2009, 2013), Sonderegger & H (2013) .
- Consistency of model selection H & Lee (2009), Lai, H & Lee (2015), H, Iyer, Lai & Lee (2016).
- Regular higher order asymptotics in Pal Majumdar & H (2016+).

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Sequence of data $\boldsymbol{X}_n$ generated using $\xi_{n,0}$ and GFD measures $R_{n,\boldsymbol{X}_n}$.

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1. $t_n(\boldsymbol{X}_n) \xrightarrow{\mathcal{D}} \boldsymbol{T} = (\boldsymbol{T}_1, \boldsymbol{T}_2)$
2. $\boldsymbol{T}_1 = \boldsymbol{H}_1(\boldsymbol{V}_1, \boldsymbol{\zeta}), \boldsymbol{T}_2 = \boldsymbol{H}_2(\boldsymbol{V}_2)$
   - $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are one-to-one
     $Q_{\boldsymbol{t}_1}(\boldsymbol{v}_1) = \zeta$ solves $\boldsymbol{t}_1 = \boldsymbol{H}_1(\boldsymbol{v}_1, \boldsymbol{\zeta})$
     GFD $R_{\boldsymbol{t}} \sim Q_{\boldsymbol{t}_1}(\boldsymbol{V}_1^\star) \mid \boldsymbol{H}_2(\boldsymbol{V}_2^\star) = \boldsymbol{t}_2$.
   - $R_{\boldsymbol{t}}(\partial C) = 0$ for open C
3. Homeomorphic injective $\boldsymbol{\Xi}_n$
   - $\Xi_n(\boldsymbol{\zeta}_{n,0}) = 0$
   - $t_n(\boldsymbol{x}_n) \to \boldsymbol{t}$
     implies $R_{n,\boldsymbol{y}_n}\boldsymbol{\Xi}_n^{-1} \xrightarrow{\mathcal{W}} R_{\boldsymbol{t}}$.

Bernstein-von Mises:

centered and scaled MLE

$\boldsymbol{T} = \boldsymbol{\zeta} + \boldsymbol{V},\ \boldsymbol{V} \sim N(0, I_{\boldsymbol{\xi}_0}^{-1})$

$N(\boldsymbol{t}, I_{\boldsymbol{\xi}_0}^{-1})$

$\boldsymbol{\Xi}_n(\boldsymbol{\xi}) = \sqrt{n}(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$

LARGE theorem

Theorem (H., Lai & Lee, 2016)

Assume LARGE, fix $0 < \alpha < 1$, *select open* $C_n(\boldsymbol{x}_n)$

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Then the sets $C_n(\boldsymbol{x}_n)$ are α asymptotic confidence sets.

LARGE theorem

Theorem (H., Lai & Lee, 2016)

Assume LARGE, fix $0 < \alpha < 1$, select open $C_n(\boldsymbol{x}_n)$

1. $R_{n,\boldsymbol{x}_n}(C_n(\boldsymbol{x}_n)) = \alpha$
2. $t_n(\boldsymbol{x}_n) \to \boldsymbol{t}$ *implies* $\boldsymbol{\Xi}_n(C_n(\boldsymbol{x}_n)) \to C(\boldsymbol{t})$
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Then the sets $C_n(\boldsymbol{x}_n)$ are α asymptotic confidence sets.

- If $\boldsymbol{T}_1 = \boldsymbol{H}_1(\boldsymbol{V}_1, \boldsymbol{\xi})$ has group structure – 3) means $C(\boldsymbol{t})$ is invariant in $\boldsymbol{t}_1$.

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- One sided credible sets have correct asymptotic coverage!

Outline

- Introduction
- Definition
- Theoretical Results
- Applications
  - Distributed Data
  - Right Censored Data
  - High D Regression
- Conclusions

Computational Issues

- When lucky: Direct Monte Carlo inversion of DGE

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- Fiducial Quirk – Interested in Frequentist Properties
  - m-number of MC samples vs k-number of replications:
    $k \sim Cm^2$

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  - merge results from different nodes

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- depending on the problem, $r(\boldsymbol{\theta}|\boldsymbol{x})$ and $r_k(\boldsymbol{\theta}|\boldsymbol{x}_k)$ may be known only up to normalizing constant.

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  - Target of order $n^{-1/2}$
  - Fiducial sample on each worker of order $n_k^{-1/2}$.
  - Most realizations get extremely small weights.

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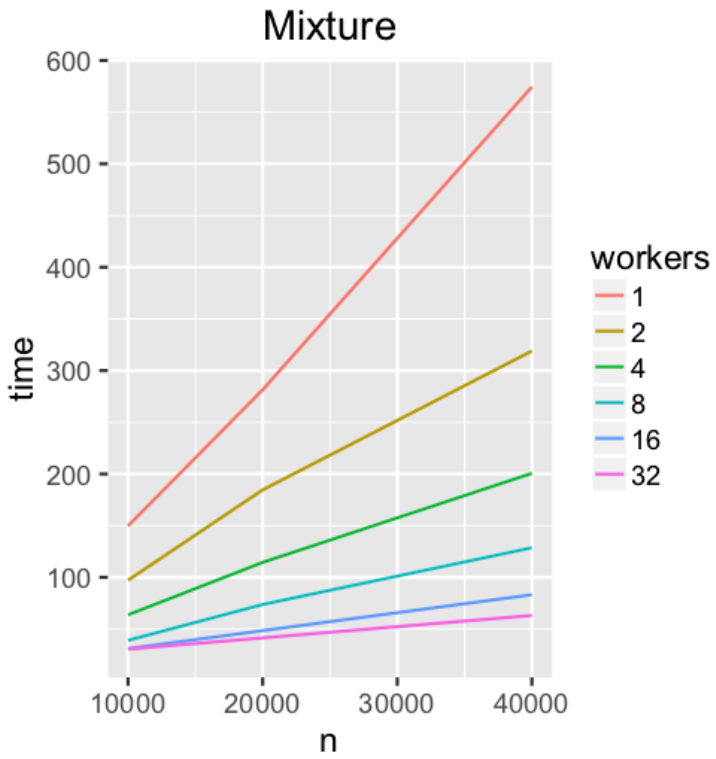
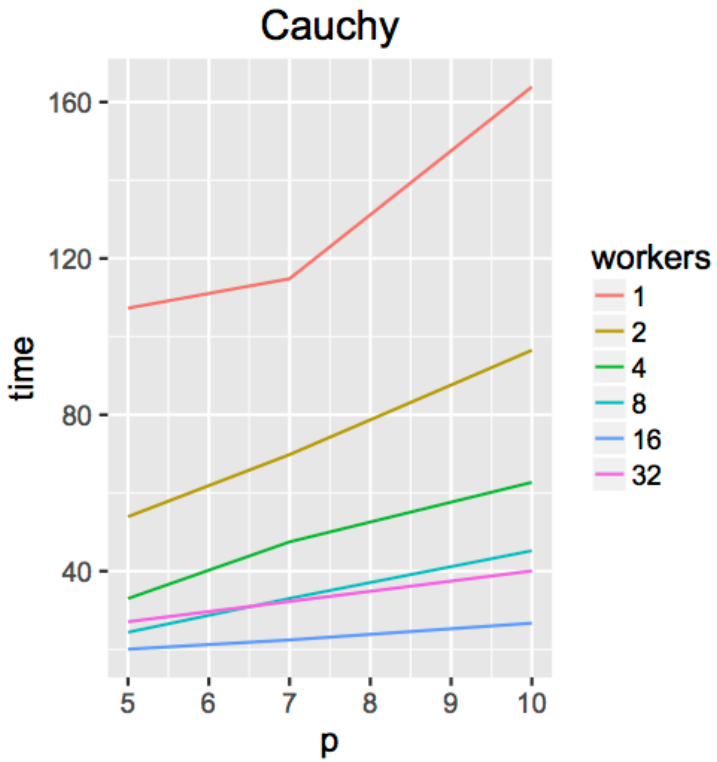
- We proved consistency and asymptotic normality of the approximation error.

Experiments

- All good performance ($K = 1, 2, 4, 8, 16, 32$)
  - Gaussian mixture: $0.6N(-1, 1) + 0.4N(1, 1)$ ($n = 4 \times 10^5$)
  - Linear regression with Cauchy errors ($n = 10^5$, $p_T = 4, p = 6, 8, 11$)

Computational time





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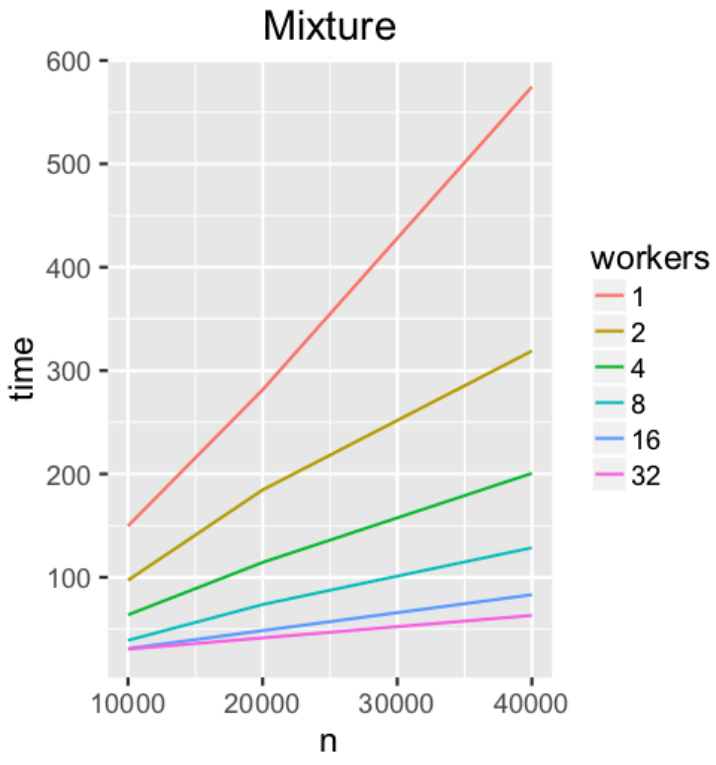
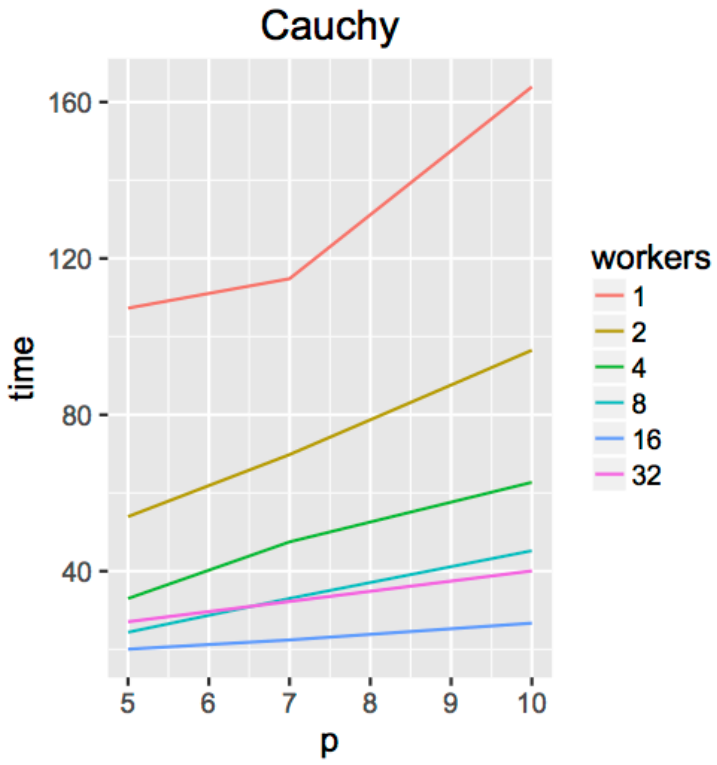
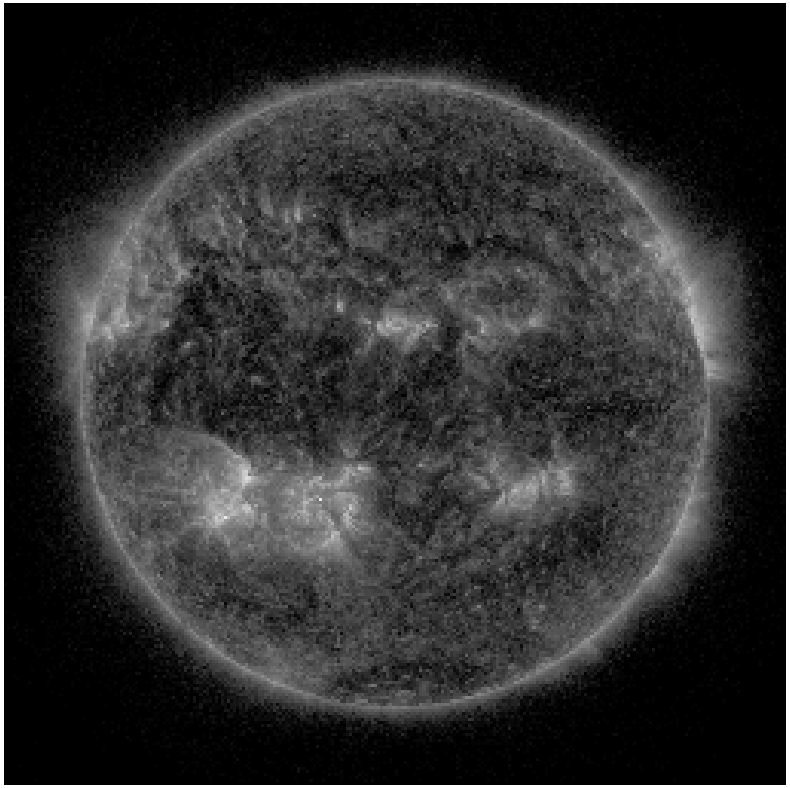


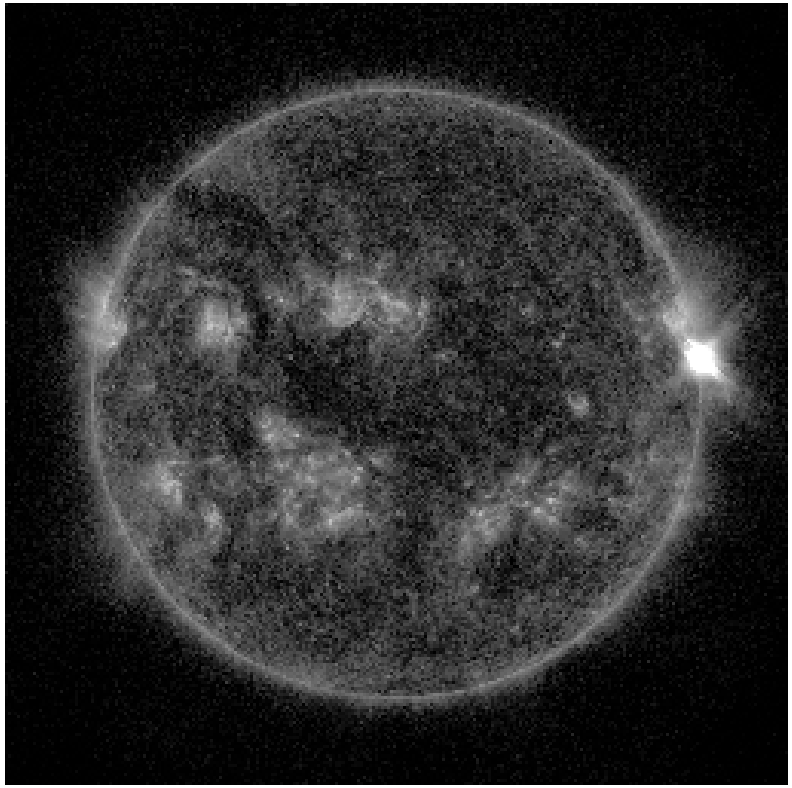


- For Cauchy: speed improves until $K = 16$ then deteriorates (Cheng & Shang, 2015)

Sun Spots



low activity



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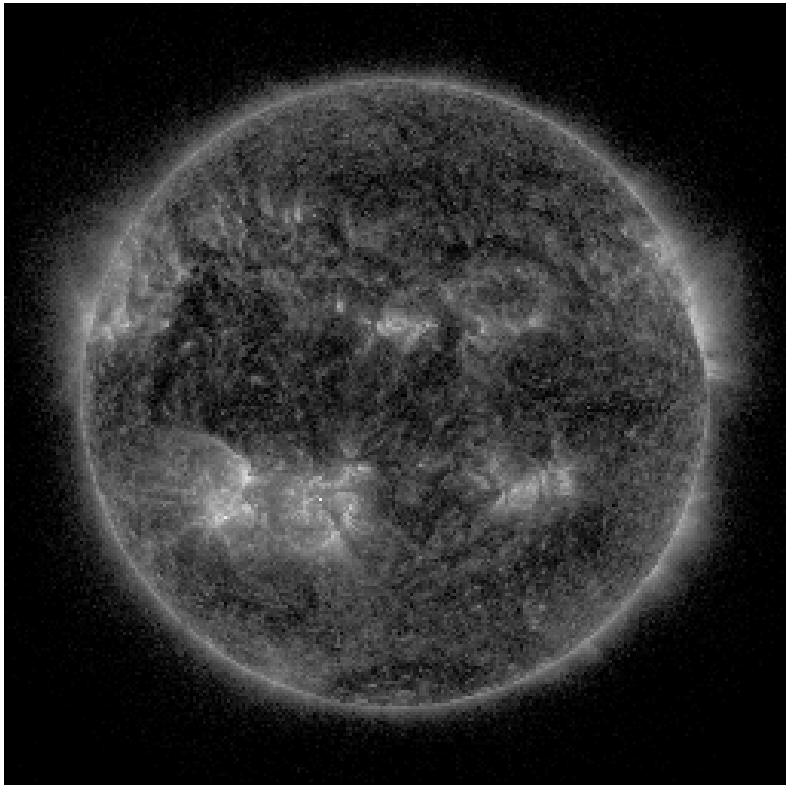
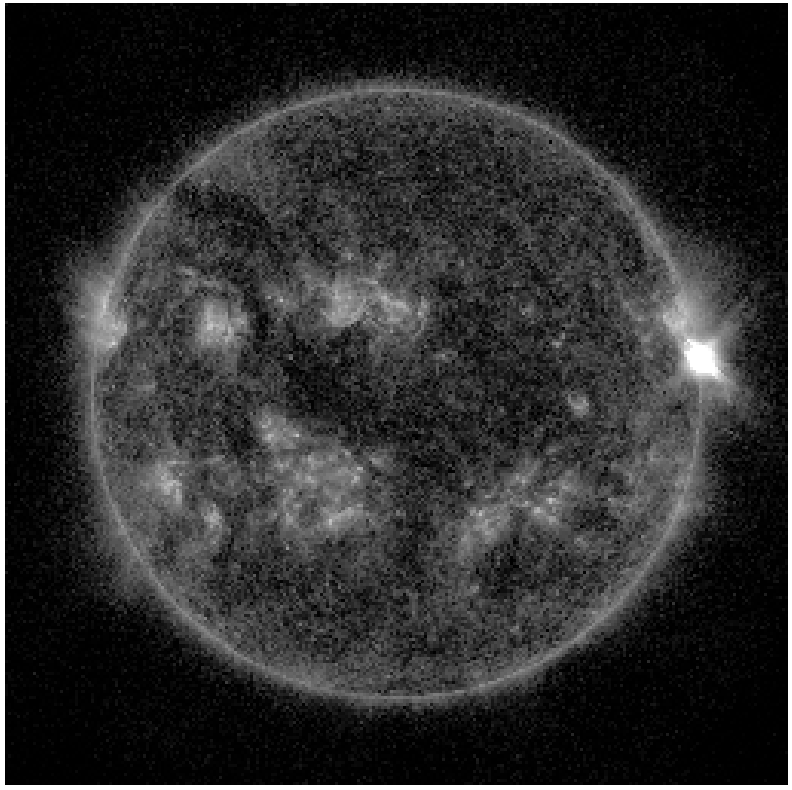


low activity



high activity

- The bright flare on the right has value 253. Is this high?

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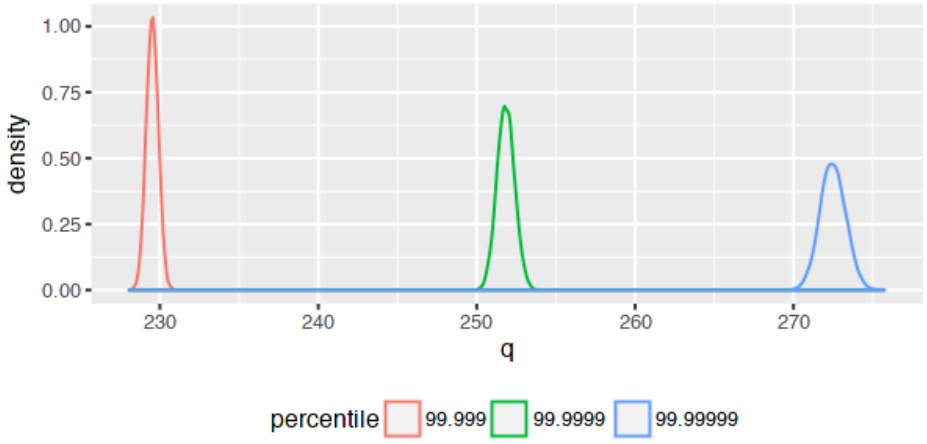
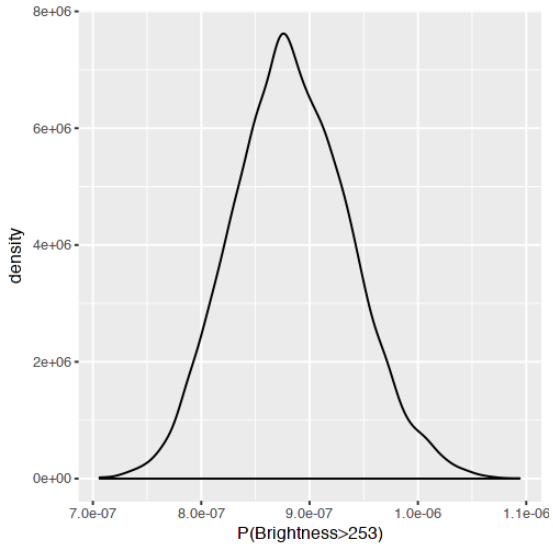
- Tool: GFD for Generalized Pareto (Wandler & H, 2012)

GFD for extreme quantiles



Large Quantiles



Fiducial Excedance Probability

Outline

Right Censored Data

- Data generating equation:

$$T_i = F^{-1}(U_i),\ C_i = G_i^{-1}(W_i|F^{-1}(U_i)), \quad U_i, W_i \text{ i.i.d. Uniform(0,1)}$$

observe only $X_i = \min(T_i, C_i), \delta_i = I_{\{T_i \leq C_i\}}$.

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- Inverting (solving for H and G) we get

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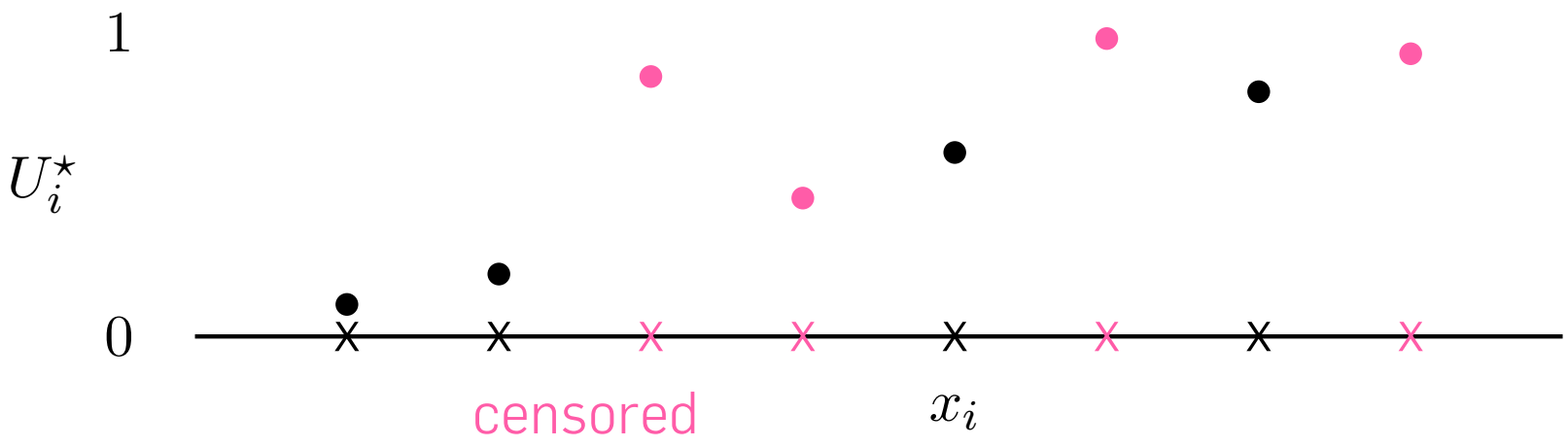
- Generate U^* conditional on existence of F^* solving the equations.

Visual Demonstation



U_i^* generated so that there is a solution

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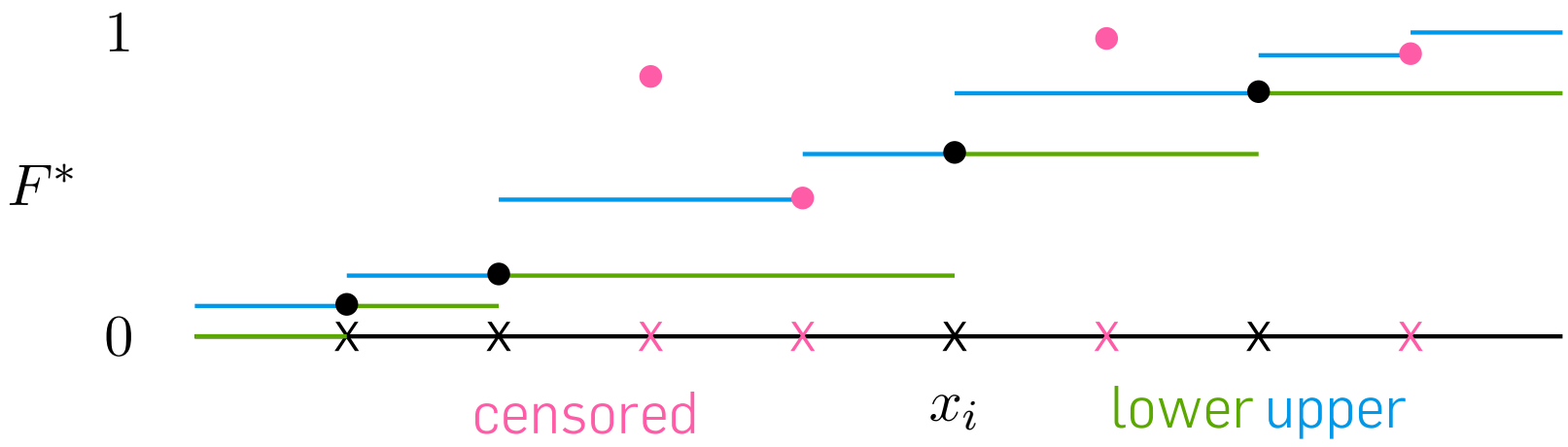

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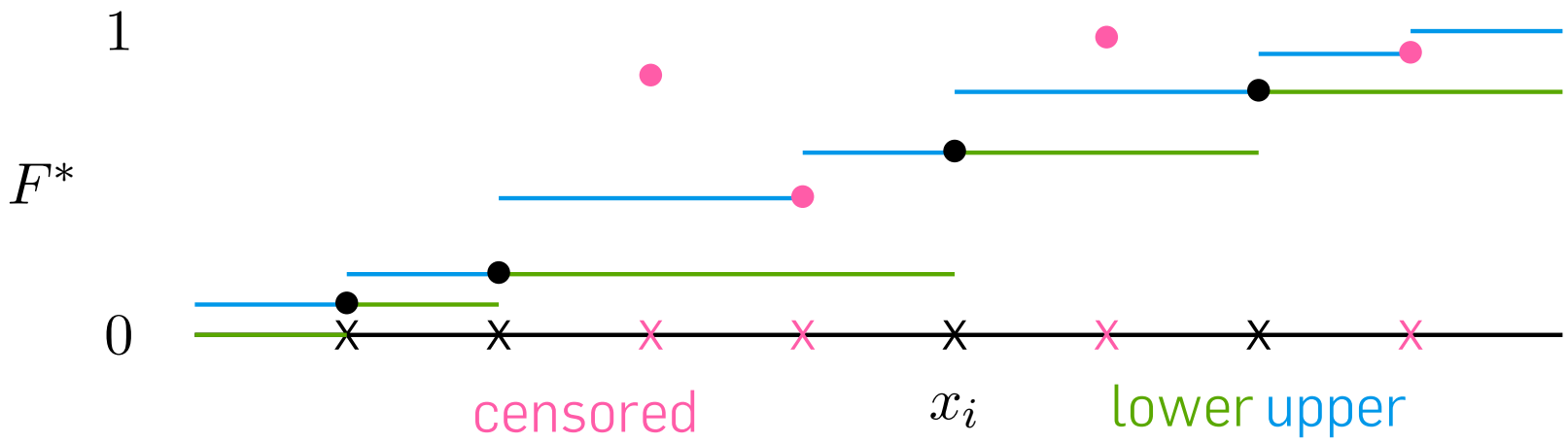

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Fact: For any failure time $EF_L^\star(s) < \hat{F}(s) < EF_U^\star(s)$.

Theoretical Results

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$$\sqrt{n}[\hat{F}(t) - F(t)] \xrightarrow{\mathcal{D}} W_t \text{ in D[0,T]}$$

(W – mean zero Gaussian process with known covariance depending on censoring)

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- Theorem (Cui, Hannig): Under similar assumptions

$$\sqrt{n}[F^\star(t) - \hat{F}(t)]|\mathbf{X} \xrightarrow{\mathcal{D}} W_t \text{ almost surely.}$$

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- For difference of two populations – use difference of fiducial distributions

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- Simultaneous CI also show good/conservative coverage
- Two sample testing
  - Good type 1 error
  - Compared to log rank tests and sup-log rank test: competitive.
- Room for improvement:
  - Use something else than L_∞ ball – half-region depth?

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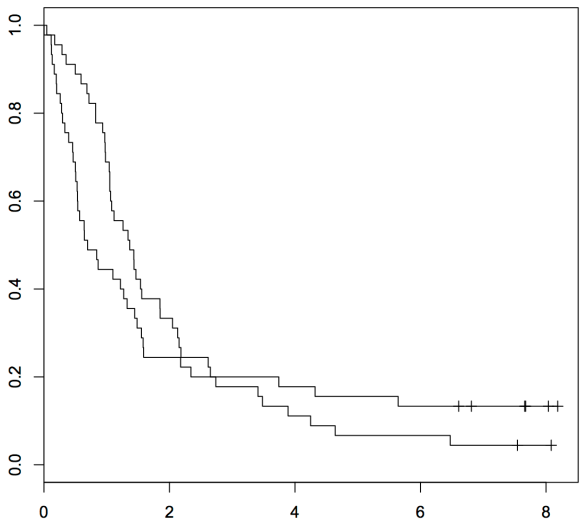

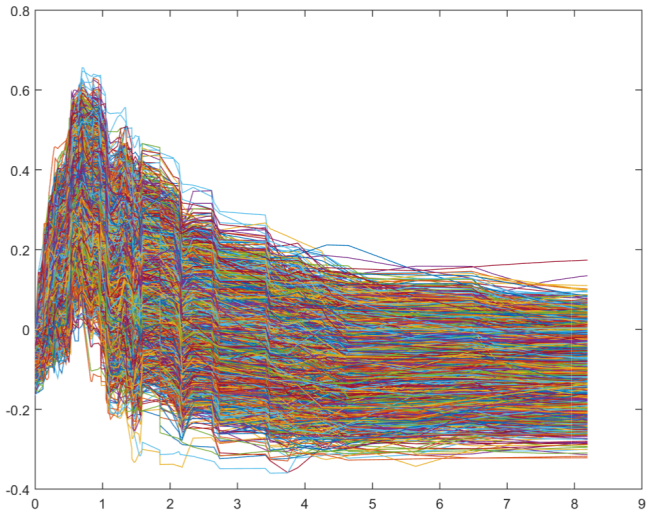

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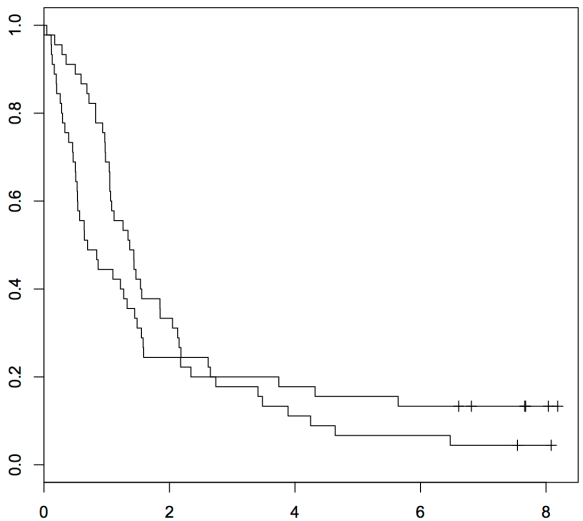

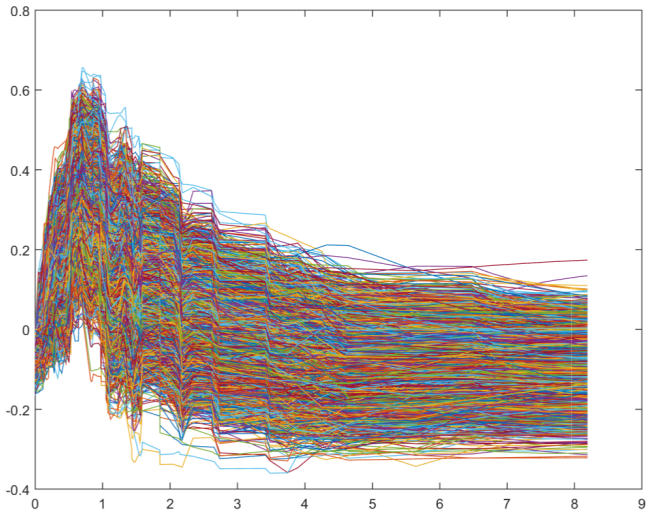

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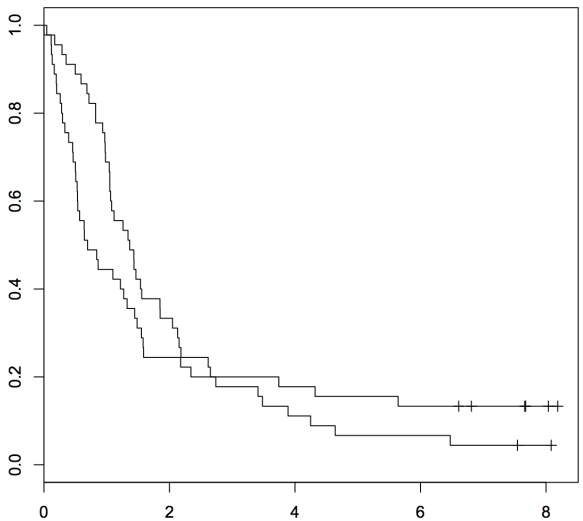

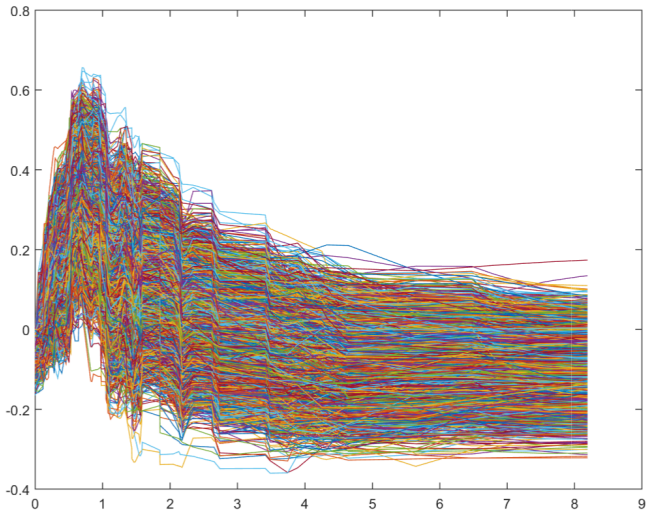

- Different: Log-rank statistics p-value 0.63 (hazard crossing); Fiducial p-value 0.003.
- Simulation with estimated hazard shows fiducial more powerful than competitors.

Outline

Model Selection

- $\boldsymbol{X} = \boldsymbol{G}(M, \boldsymbol{\xi}_M, \boldsymbol{U}), \qquad M \in \mathcal{M},\ \boldsymbol{\xi}_M \in \boldsymbol{\xi}_M$

Theorem: (H, Iyer, Lai, Lee 2016) Under assumptions

$$r(M|\boldsymbol{y}) \propto q^{|M|} \int_{\boldsymbol{\xi}_M} f_M(\boldsymbol{y}, \boldsymbol{\xi}_M) J_M(\boldsymbol{y}, \boldsymbol{\xi}_M)\, d\boldsymbol{\xi}_M$$

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  - Default value $q = n^{-1/2}$ (motivated by MDL)

Alternative to penalty - see poster!

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- Motivated by non-local priors of Johnson & Rossell (2009)

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where b_{min} solves

$$\min_{b \in R^p} \frac{1}{2}\|X^T(X_M\beta_M - Xb)\|_2^2 \quad \text{subject to} \quad \|b\|_0 \leq |M| - 1.$$

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- Call this: ε-admissible subset

GFD

$$r(M|\boldsymbol{y}) \propto \pi^{\frac{|M|}{2}} \Gamma\Big(\frac{n-|M|}{2}\Big) RSS_M^{-(\frac{n-|M|-1}{2})} E[h_M^{\varepsilon}(\beta_M^{\star})]$$

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- Implemented using Grouped Independence Metropolis Hastings (Andrieu & Roberts, 2009).

Main Result

Theorem Williams & H (2017+)
Suppose the true model is given by M_T. Then under certain conditions, for a fixed positive constant $\alpha < 1$,

$$r(M_T|y) = \frac{r(M_T|y)}{\sum_{j=1}^{n^\alpha} \sum_{M:|M|=j} r(M|y)} \xrightarrow{P} 1 \text{ as } n, p \to \infty.$$

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- For a large model $|M| > p_T$ and large enough n or p,

$$\frac{9}{2}\|X^T(H_M - H_{M(-1)})\mu_T\|_2^2 < \varepsilon_M,$$

where H_M and $H_{M(-1)}$ are the projection matrix for M and M with a covariate removed respectively.

Default ε

$$\varepsilon = \Lambda_M \widehat{\sigma}_M^2 \left(\frac{n^{0.51}}{9} + |M| \frac{\log(p\pi)^{1.1}}{9} - p_T \right)_+,$$

- $\Lambda_M := \mathrm{tr}\left((H_M X)' H_M X \right)$ with $H_M := X_M (X_M' X_M)^{-1} X_M'$
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- Tuning parameter p_T represents belief about true $|M_T|$.

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- Generate 1000 data vectors y from linear model with $\beta^0_{M_o} = (-1.5, -1, -.8, -.6, .6, .8, 1, 1.5)'$, and $\sigma^0_{M_o} = 1$.

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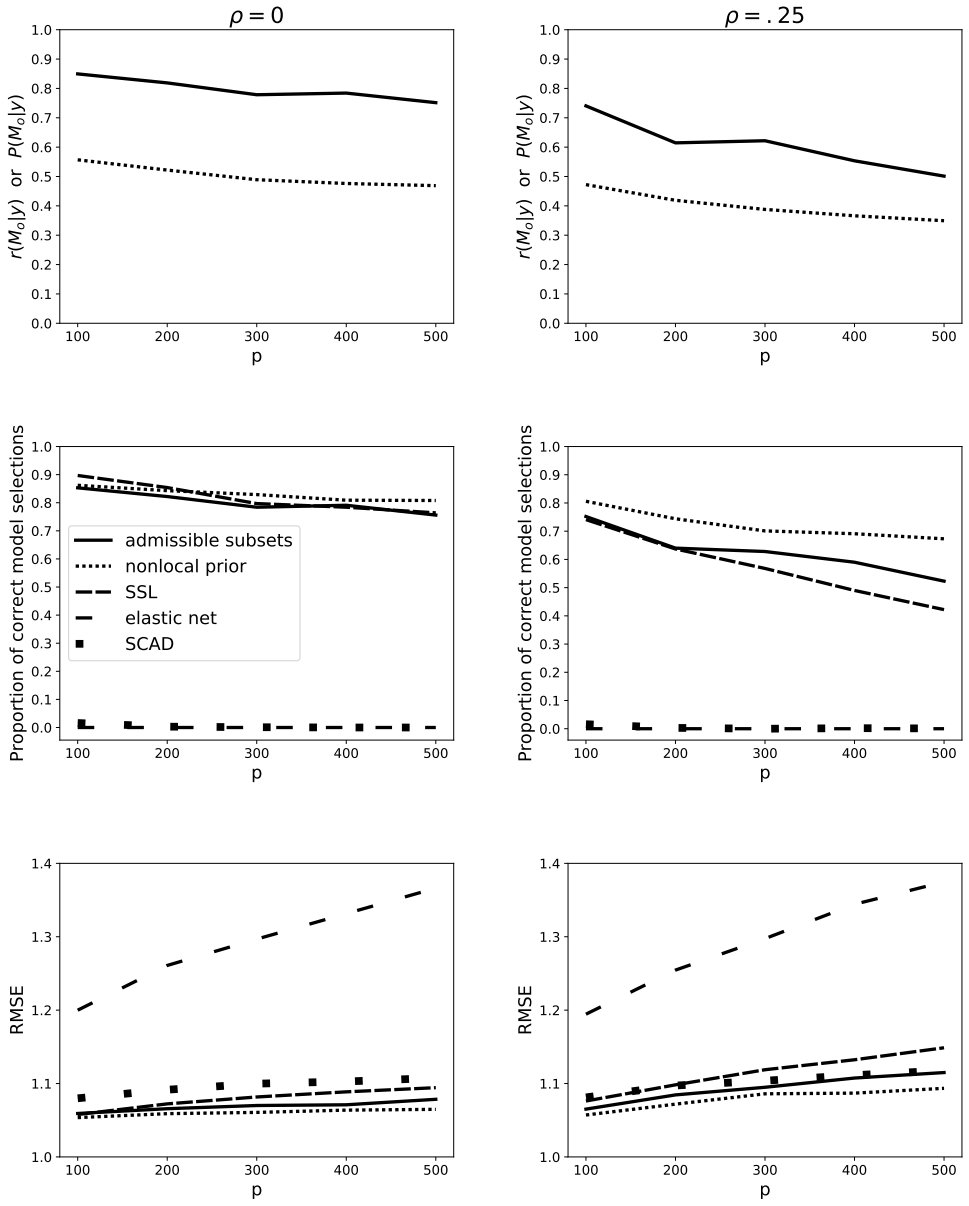
- Set $n = 100$, and consider $p = 100, 200, 300, 400, 500$.

Simulation results 1



Simulation setup 2

To illustrate the difference from the nonlocal prior approach, for $n = 30$, generate data from the following model.

$$Y \sim \mathsf{N}_n\Big(1 \cdot x^{(1)} + 1 \cdot x^{(2)} + \cdots + 1 \cdot x^{(9)}, I_n\Big),$$

where $x^{(1)}, x^{(2)}, x^{(3)} \overset{\text{iid}}{\sim} \mathsf{N}_n(0, I_n)$, and

$$\begin{aligned}
x^{(4)} &\sim \mathsf{N}_n\Big(.25 \cdot x^{(1)} &&&&&&&& , .1^2 I_n \Big) \\
x^{(5)} &\sim \mathsf{N}_n\Big(&& .5 \cdot x^{(2)} &&&&&& , .1^2 I_n \Big) \\
x^{(6)} &\sim \mathsf{N}_n\Big(&&&&&& - && .75 \cdot x^{(3)} , .1^2 I_n \Big) \\
x^{(7)} &\sim \mathsf{N}_n\Big(&& x^{(1)} &&&& + && x^{(3)} , .1^2 I_n \Big) \\
x^{(8)} &\sim \mathsf{N}_n\Big(&&&& x^{(2)} && - && x^{(3)} , .1^2 I_n \Big) \\
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\end{aligned}$$

Simulation results 2

	MAP size	RMSE	$P(M_{\text{MAP}}\|y)$
ε-$admissible$ subsets	3.476	1.138	.365
nonlocal prior	8.997	1.197	.333

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- Nonlocal prior procedure typically includes all 9 covariates even though the y can be mostly explained by 3.

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Thank you!