Homework set #3

Based on lectures 3 – 4.

1. (a) Prove that if $EX^2 < \infty$ then $P(X - EX \geq t) \leq \frac{\operatorname{Var} X}{\operatorname{Var} X + t^2}$ for all $t > 0$. (Hint: $t \leq E\{[(t - (X - EX)]I_{\{X - EX < t\}}\}$ might be useful).

   (b) Then show that this inequality cannot be improved. In particular show that for any fixed $t \geq 0$,

$$\sup_X \left(P(X - EX \geq t) \Big/ \frac{\operatorname{Var} X}{\operatorname{Var} X + t^2} \right) = 1,$$

   where the supremum goes over all possible random variables satisfying $EX^2 < \infty$.

2. (a) Find the moment generating function of the following distributions. Poisson(λ), Exp(λ) and $N(0,1)$

   (b) Use the series expansion of the MGF of the standard normal to find the moments EZ^{2k} for $Z \sim \mathcal{N}(0,1)$.

3. Is it possible for X, Y, Z to have the same distribution and satisfy $X = U(Y + Z)$, where $U \sim U(0,1)$, and Y, Z are independent of U and each other?

4. Let $X \sim \text{Bin}(n, p)$. Find the MGF of X. Show that for $\epsilon > p$

$$P(X/n \geq \epsilon) \; \leq \; \exp\left\{ -n(1-\epsilon)\log\left(\frac{1-\epsilon}{1-p}\right) - n\epsilon \log\left(\frac{\epsilon}{p}\right) \right\}$$