HOMEWORK SET #4

Based on lectures 5 - 6

1. Prove that for all $x > 0$
$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) < 1 - \Phi(x) < \frac{1}{x}\phi(x).$$

2. Let $Z_1, \ldots, Z_n$ be i.i.d. $N(0, \sigma^2)$ and write $U_i = \max(\min(Z_i, 1), -1)$. In what follows $t > 0$.
   (a) Prove $P(\sum_{i=1}^n Z_i > nt) \leq \frac{\sigma \exp(-\frac{nt^2}{2\sigma^2})}{\sqrt{2\pi n}\ t}$.
   (b) Prove $P(\sum_{i=1}^n U_i > nt) \leq \exp(-\frac{nt^2}{2})$.
   (c) Find EU_i and prove $\operatorname{Var} U_i \leq \sigma^2$.
   (d) Prove $P(\sum_{i=1}^n U_i > nt) \leq \exp(-\frac{nt^2}{2\sigma^2 + 4t/3})$.

3. Let $X_1, \ldots, X_n$ be any sample (not necessarily independent) from a distribution with common mean μ and variance σ^2. Define $\bar{X}_n = \sum_{i=1}^n X_i/n$.
   (a) Find $E(\bar{X}_n)$.
   (b) Prove or disprove: $Var\bar{X}_n \to 0$ as $n \to \infty$.

4. Let $X_1, \ldots, X_n$ be an i.i.d. sample from uniform$(0, 1)$ distribution. Find the $EX_{(k)}$.

5. Let $X_1, \ldots, X_{4n}$ be an i.i.d. sample from exponential(β) distribution. Find the joint distribution of $(\frac{X_{(n)} + X_{(3n)}}{2}, X_{(3n)} - X_{(n)})$.

6. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a distribution with density $\frac{1}{x^2} I_{(1,\infty)}(x)$. Is EX_k finite? What about $EX_{(k)}$? (Hint: The answer might be different for different k.)

7. Let $X_1, \ldots, X_n$ be i.i.d. sample from a distribution with density $f(x)$. Fix $i < j < k$ and find the formula for the joint density $f_{X_{(i)}, X_{(j)}, X_{(k)}}(s, u, v)$.

8. Let $X_1, \ldots, X_n$ be i.i.d. from Geometric(p) distribution.

(a) Set (for this problem part only) $n = 10, p = \frac{1}{4}$. Find $P(X_{(3)} = X_{(5)} = 3)$.

(b) Fix $1 \leq i < j \leq n$. Find the probability mass function $f_{X_{(i)},X_{(j)}}(a,b)$.

9. From the book 5.15, 5.21, 5.24.