Homework set #5
Based on lectures 7 - 8

1. For each of the following families: (i) verify it is an exponential family, (ii) describe whether this is a curved family (iii) describe and sketch the graph of which the curved parameter space.

   (a) $N(\theta, \theta)$, with $\theta > 0$ unknown/

   (b) $N(\theta, a\theta^2)$, with $a > 0$ known.

   (c) $\text{Gamma}(\alpha, \alpha^{-1})$ with $\alpha > 0$ unknown.

   (d) $f(x|\theta) \propto e^{-(x-\theta)^4}$ with θ unknown.

2. Let $X_1, \ldots, X_n$ be an i.i.d. sample from $\text{Uniform}(-\theta, \theta^2)$ distribution, $\theta > 0$. Find a one-dimensional sufficient statistics.

3. Let $X_1, \ldots, X_n$ be an i.i.d. sample from $\mathrm{N}(0, \sigma^2)$. Find a one-dimensional sufficient statistics.

4. Let $X_1, \ldots, X_n$ be an i.i.d. sample from Pareto distribution. (i.e. $f(x; \theta) = \frac{\theta}{(1+x)^{\theta+1}} I_{(0,\infty)}(x)$). Find a one-dimensional sufficient statistics.

5. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a distribution with density. $f(x; a, b) = \frac{1}{b} e^{-\left(\frac{x-a}{b}\right)} I_{(a,\infty)}(x)$.

   (a) Show that $(X_{1:n}, \sum_{i=1}^n X_i)$ is sufficient.

   (b) Show that $(X_{1:n}, \sum_{i=2}^n (X_{i:n} - X_{1:n}))$ is sufficient.

6. Let $X_1, \ldots, X_n$ be an i.i.d. sample from exponential(β) distribution *censored* at $a > 0$. Find a sufficient statistics. (Hint: Use the "mixed" density $f(x; a, \beta) = \frac{1}{\beta} e^{-x/\beta} I_{(0,a)}(x) + e^{-a/\beta} I_{\{a\}}(x)$.)

7. A famous example in genetics modeling is a genetic linkage multinomial model, where we observe the multinomial vector (X_1, X_2, X_3, X_4) with probabilities given by $\left(\frac{1}{2} + \frac{\theta}{4}, \frac{1}{4} - \frac{\theta}{4}, \frac{1}{4} - \frac{\theta}{4}, \frac{\theta}{4}\right)$.

   (a) Show that this is a curved exponential family.

   (b) Find a minimal sufficient statistics for θ.