HOMEWORK SET #7

1. (a) Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{N}(\mu, \sigma^2)$. Find $c > 0$, possibly depending on n, that minimizes $MSE(cS_n^2)$.

   (b) What implications does this have for sample variance?

2. Let $X_1, \ldots, X_n$ be i.i.d. Exponential(λ), i.e., $f(x|\lambda) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$. Find the UMVUE for estimating $e^{-10\lambda}$.

3. Let us assume that $X_1, X_2, \ldots, X_{2n}$ be i.i.d. from some distribution with parameter θ and $S(X_1, \ldots, X_{2n})$ is minimal sufficient complete statistics. Further assume that $T(X_1, \ldots, X_n)$ is an unbiased estimator of $q(\theta)$. Is there a UMVUE of $q(\theta)^2$? Justify!

4. Let $X_1, \ldots, X_n$ be an i.i.d. sample from geometric(p) distribution. (i.e. $f(x,p) = p(1-p)^{x-1} I_{\{1,2\ldots\}}$, where $p \in (0,1)$.)

   (a) Argue exponential family and deduce complete sufficient statistics.

   (b) Is there UMVUE of $\tau(p) = E_p X_1$? If so, find it.

   (c) Find the UMVUE of $1/p^2$.

   (d) Is there UMVUE of p? If so, find it.

5. Let $X_1, \ldots, X_n$ be an i.i.d. sample from Pareto distribution. (i.e. $f(x;\theta) = \frac{\theta}{(1+x)^{\theta+1}} I_{(0,\infty)}(x)$).

   (a) Argue exponential family and find complete sufficient statistics.

   (b) Is there UMVUE of θ? If so, find it.

   (c) Is there UMVUE of $\tau(\theta) = 2^{-\theta}$? If so, find it.

6. Let $X, \ldots, X_n$ be i.i.d. $N(\theta, \theta^2)$, $\theta > 0$. For this model both $\bar{X}_n$ and $Y_n = \frac{\sqrt{n-1}\Gamma((n-1)/2)}{\sqrt{2}\Gamma(n/2)} S_n$ are unbiased estimators of θ.

   (a) Prove that any $a\bar{X}_n + (1-a)Y_n$ is an unbiased estimator of θ.

   (b) Find the value of a that produces the estimator with minimum variance?

(c) Show that $(\bar{X}_n, Y_n)$ is minimal sufficient but NOT complete sufficient.

7. Let $X_1, \ldots, X_n$ be an i.i.d. sample from Uniform(θ_1, θ_2) distribution. $(n > 10$ and $\theta_1 < \theta_2)$

   (a) Find the MM of (θ_1, θ_2).

   (b) Consider the estimators as two one-dimensional estimators. Are the estimators found in part a) unbiased?

   (c) Find the MLE of (θ_1, θ_2).

   (d) Consider the estimators as two one-dimensional estimators. Are the estimators found in part c) unbiased? What are their MSE's?

8. Let $X_1, \ldots, X_n$ be an i.i.d. sample from Beta(a, b) distribution.

   (a) Find MM of (a, b) using the first and second raw moments.

   (b) Find MLE of (a, b). (Hint: If you are not able to get a formula. Explain what would you do.)

   (c) You observe: $0.88, 0.66, 0.74, 0.65, 0.39, 0.87, 0.93, 0.11, 0.97, 0.99$. Find the MLE and MM of (a, b).

9. Assume that $Y_1, \ldots, Y_m$ are i.i.d. Uniform$(0, \theta_1]$ and $Z_1, \ldots, Z_n$ are i.i.d. Uniform$[\theta_1, \theta_2]$. (The Y's and Z's are mutually independent).

   (a) Using $\bar{Y}$ and $\bar{Z}$ find MME of θ_1, θ_2. Find also the MSE's of the estimators.

   (b) Find MLE of θ_1, θ_2. Find also the MSE's of the estimators.

   (c) Does MLE seem to be better then MME in this case?

10. Let $X_1, \ldots, X_n$ be an i.i.d. sample from $N(0, \sigma^2)$. Find MLE of σ^2.

11. Let $X_1, \ldots, X_n$ be an i.i.d. sample from Pareto distribution. (i.e. $f(x; \theta) = \frac{\theta}{(1+x)^{\theta+1}} I_{(0,\infty)}(x)$). Find MLE of θ.

12. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a distribution with density. $f(x; a, b) = \frac{1}{b} e^{-\left(\frac{x-a}{b}\right)} I_{(a,\infty)}(x)$.

(a) Find MLE of (a, b).

(b) Find the conditional distribution of $X_1 - a, X_2 - X_1, \ldots, X_n - X_1$ given $X_2 > X_1, \cdots, X_n > X_1$.

(c) Consider the MLEs as two one-dimensional estimators. Find their MSE. (Hint: Carefully consider what the result of part (b) mean for your calculations.)