

Generalized Fiducial Inference

Parts of this talk are joint work with

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^aNSF support acknowledged

Outline

- Introduction
- Definition
- Theoretical Results
- Applications
- Conclusions

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 - Distributed Data
 - Right Censored Data
 - High D Regression
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Fiducial?

▶ Oxford English Dictionary

- ▶ adjective technical (of a point or line) used as a fixed basis of comparison.
- ▶ Origin from Latin fiducia 'trust, confidence'

▶ Merriam-Webster dictionary

1. taken as standard of reference *a fiducial mark*
2. founded on faith or trust
3. having the nature of a trust : fiduciary

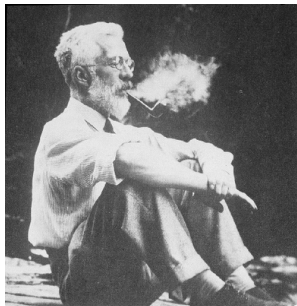
Long, long, long time ago...



- ▶ Bayes Theorem: $P(\theta|X) = \frac{f(X|\theta)\pi(\theta)}{\int_{\Theta} f(X|\theta)\pi(\theta)d\theta}$.
- ▶ Bayes-Laplace postulate:

When nothing is known about the parameter in advance, let the prior be so that all values of the parameter are equally likely.

Long, long, time ago...



"Not knowing the chance of mutually exclusive events and knowing the chance to be equal are two quite different states of knowledge" R. A. Fisher (1930)

Brief history of fiducial inference



- ▶ Fisher (1922, 1930, 1935) no formal definition
- ▶ Lindley (1958) fiducial vs Bayes
- ▶ Fraser (1966) structural inference
- ▶ Dempster (1967) upper and lower probabilities
- ▶ Dawid and Stone (1982) theoretical results for simple cases.
- ▶ Barnard (1995) pivotal based methods.
- ▶ Weerahandi (1989, 1993) generalized inference.

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Fiducial Inspired Inference since 2000

- ▶ Dempster-Shafer calculus; Dempster (2008), Edlefsen, Liu & Dempster (2009)
- ▶ Inferential Models; Liu & Martin (2015)
- ▶ Confidence Distributions; Xie, Singh & Strawderman (2011), Schweder & Hjort (2016)
- ▶ Higher order likelihood, tangent exponential family, r^* , Reid & Fraser (2010)
- ▶ Objective Bayesian inference, e.g., reference prior Berger, Bernardo & Sun (2009, 2012).
- ▶ Fiducial Inference H, Iyer & Patterson (2006), H (2009, 2013), H & Lee (2009), Taraldsen & Lindqvist (2013), Veronese & Melilli (2015), H, Iyer, Lai & Lee (2016)...

Aims

- ▶ Explain the definition of generalized fiducial distribution
- ▶ Discuss theoretical results
- ▶ Show successful applications

- ▶ My point of view is frequentist
 - ▶ Justified using asymptotic theorems and simulations.
 - ▶ GFI shows very good repeated sampling performance in applications.

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Comparison to likelihood

- ▶ **Density** is the function $f(\mathbf{x}, \xi)$, where ξ is fixed and \mathbf{x} is variable.
- ▶ **Likelihood** is the function $f(\mathbf{x}, \xi)$, where ξ is variable and \mathbf{x} is fixed.
 - ▶ Likelihood as a distribution?

Comparison to likelihood

- ▶ Data generating equation (DGE)

$$\mathbf{X} = \mathbf{G}(\mathbf{U}, \xi),$$

- ▶ U is a random with known distribution (iid $U(0, 1)$)
- ▶ Parameter ξ is fixed.
- ▶ Generate \mathbf{X} s by generating \mathbf{U} s and DGE.
 - ▶ This determines sampling distribution
- ▶ Issues: Multiple solutions and no solutions.

Comparison to likelihood

- ▶ Data generating equation (DGE)

$$\mathbf{x} = \mathbf{G}(\mathbf{U}^*, \xi),$$

- ▶ U is a random with known distribution (iid $U(0, 1)$)
- ▶ Data \mathbf{x} is fixed
- ▶ Generate ξ by generating \mathbf{U} s and inverting DGE.
 - ▶ This determines fiducial distribution
- ▶ Issues: Multiple solutions and no solutions.

Example -- Bernoulli trials

- Data generating equation

$$Y_i = 1\{U_i \leq p\}, U_i \sim \text{Uniform}(0,1)$$

Generating U_i samples Bernoulli(p).

Example -- Bernoulli trials

- Data generating equation

$$Y_i = 1\{U_i^* \leq p\}, U_i^* \sim \text{Uniform}(0,1)$$

Estimating U_i by U_i^* defines fiducial distribution

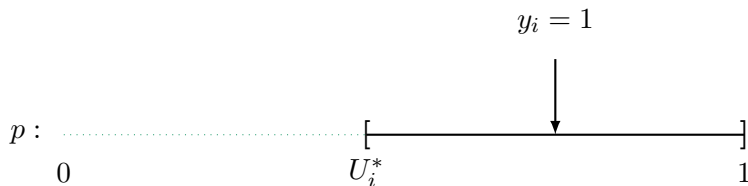
Example -- Bernoulli trials

- Data generating equation

$$Y_i = 1\{U_i^* \leq p\}, U_i^* \sim \text{Uniform}(0,1)$$

Estimating U_i by U_i^* defines fiducial distribution

- If $y_i = 1$, then $p \in [U_i^*, 1]$



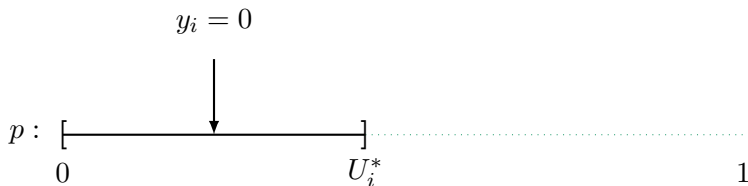
Example -- Bernoulli trials

- Data generating equation

$$Y_i = 1\{U_i^* \leq p\}, U_i^* \sim \text{Uniform}(0,1)$$

Estimating U_i by U_i^* defines fiducial distribution

- If $y_i = 0$, then $p \in [0, U_i^*]$



Example -- Binomial

- Data generating equation

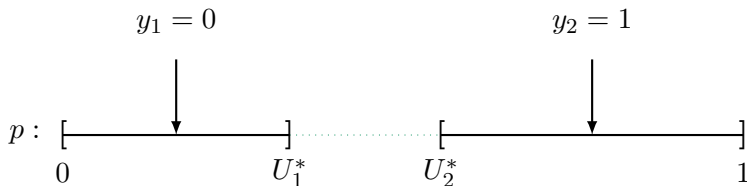
$$Y_1 = 1\{U_1 \leq p\}, Y_2 = 1\{U_2 \leq p\} \quad U_1, U_2 \text{ i.i.d. Uniform}(0,1)$$

Example -- Binomial

- Data generating equation

$$Y_1 = 1\{U_1 \leq p\}, Y_2 = 1\{U_2 \leq p\} \quad U_1, U_2 \text{ i.i.d. Uniform}(0,1)$$

- If $y_1 = 0, y_2 = 1$ and $U_1^* < U_2^*$



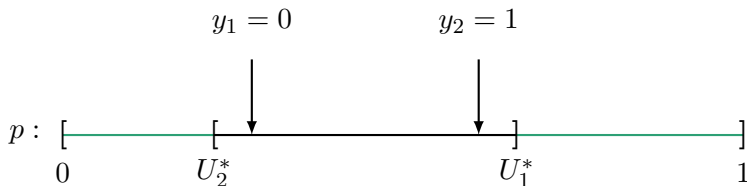
- No solution! Remove (U_1^*, U_2^*) inconsistent with data.

Example -- Binomial

- Data generating equation

$$Y_1 = 1\{U_1 \leq p\}, Y_2 = 1\{U_2 \leq p\} \quad U_1, U_2 \text{ i.i.d. Uniform}(0,1)$$

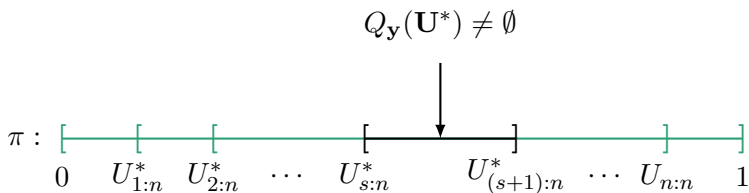
- If $y_1 = 0, y_2 = 1$ and $U_1^* > U_2^*$



- $(U_1^*, U_2^*) \mid \{U_1^* > U_2^*\}$ estimates (u_1, u_2) .

Example -- Binomial

- ▶ $(Y_1, \dots, Y_n) \stackrel{iid}{\sim} \text{Bernoulli}(p)$, $S = \sum_{i=1}^n Y_i \sim \text{Binomial}(n, p)$
- ▶ Condition \mathbf{U}^* on having a solution for p



- ▶ May select a point in the interval; e.g.
 $\text{Beta}(s + 1/2, n - s + 1/2)$.

Example -- Location Cauchy

- ▶ Consider $X_i = \mu + U_i$ where U_i are i.i.d. standard Cauchy.
- ▶ Solve:

$$Q_{\mathbf{x}}(\mathbf{u}) = \begin{cases} x_1 - u_1 & \text{if } x_2 - x_1 = u_2 - u_1^*, \dots, x_n - x_1 = u_n - u_1^* \\ \emptyset & \text{otherwise} \end{cases}$$

- ▶ Estimate \mathbf{u} by conditional

$$\mathbf{U}^* \mid x_2 - x_1 = U_2^* - U_1^*, \dots, x_n - x_1 = U_n^* - U_1^*;$$

- ▶ Fiducial density $r(\mu|\mathbf{x}) \propto \prod_{i=1}^n (1 + (\mu - x_i)^2)^{-1}$.
 - ▶ Location problem – same as posterior computed using Jeffreys prior

General Definition

- ▶ Data generating equation $\mathbf{X} = \mathbf{G}(\mathbf{U}, \xi)$.
 - ▶ e.g. $X_i = \mu + \sigma U_i$
- ▶ A distribution on the parameter space is **Generalized Fiducial Distribution** if it can be obtained as a limit (as $\varepsilon \downarrow 0$) of

$$\arg \min_{\xi} \|\mathbf{x} - \mathbf{G}(\mathbf{U}^*, \xi)\| \mid \left\{ \min_{\xi} \|\mathbf{x} - \mathbf{G}(\mathbf{U}^*, \xi)\| \leq \varepsilon \right\} \quad (1)$$

- ▶ Similar to ABC; generating from prior replaced by **min**.
- ▶ Is this practical? Can we compute?

Explicit limit (1)

- ▶ Assume $\mathbf{X} \in \mathbb{R}^n$ is continuous; parameter $\xi \in \mathbb{R}^p$
- ▶ The limit in (1) has density (H, Iyer, Lai & Lee, 2016)

$$r(\xi|\mathbf{x}) = \frac{f_{\mathbf{X}}(\mathbf{x}|\xi)J(\mathbf{x}, \xi)}{\int_{\Xi} f_{\mathbf{X}}(\mathbf{x}|\xi')J(\mathbf{x}, \xi') d\xi'},$$

where $J(\mathbf{x}, \xi) = D \left(\nabla_{\xi} \mathbf{G}(\mathbf{u}, \xi) |_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x}, \xi)} \right)$

- ▶ $n = p$ gives $D(A) = |\det A|$
- ▶ $\|\cdot\|_2$ gives $D(A) = (\det A^{\top} A)^{1/2}$
- ▶ $\|\cdot\|_{\infty}$ gives $D(A) = \sum_{\mathbf{i}=(i_1, \dots, i_p)} |\det(A)_{\mathbf{i}}|$

Example -- Uniform(θ, θ^2)

- ▶ X_i i.i.d. $U(\theta, \theta^2)$, $\theta > 1$
 - ▶ Data generating equation $X_i = \theta + (\theta^2 - \theta)U_i$, $U_i \sim U(0, 1)$.
- ▶ $\frac{d}{d\theta}[\theta + (\theta^2 - \theta)U_i] = 1 + (2\theta - 1)U_i$, with $U_i = \frac{X_i - \theta}{\theta^2 - \theta}$.
- ▶ Jacobian

$$J(\mathbf{x}, \theta) = D \begin{pmatrix} 1 + \frac{(2\theta-1)(x_1-\theta)}{\theta^2-\theta} \\ \vdots \\ 1 + \frac{(2\theta-1)(x_n-\theta)}{\theta^2-\theta} \end{pmatrix} = \frac{1}{\theta^2 - \theta} D \begin{pmatrix} x_1(2\theta - 1) - \theta^2 \\ \vdots \\ x_n(2\theta - 1) - \theta^2 \end{pmatrix}$$

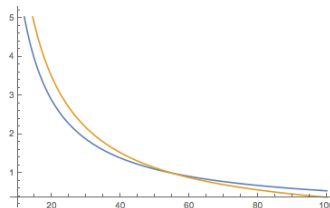
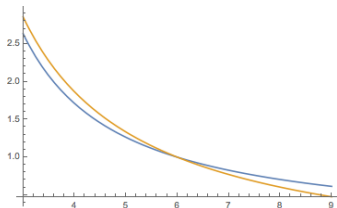
$$\text{▶ } = n \frac{\bar{x}(2\theta-1) - \theta^2}{\theta^2 - \theta} \text{ for } L_\infty.$$

Example -- Uniform(θ, θ^2)

- ▶ Reference prior (Berger, Bernardo & Sun, 2009)

$$\pi(\theta) = \frac{e^{\psi\left(\frac{2\theta}{2\theta-1}\right)}(2\theta-1)}{\theta^2-\theta}.$$

- ▶ reference prior vs fiducial Jacobian



- ▶ In simulations fiducial was marginally better than reference prior which was much better than flat prior.

Example -- Linear Regression

- ▶ Data generating equation $Y = X\beta + \sigma Z$
- ▶ $\frac{d}{d\theta} Y = (X, Z)$ and $Z = (Y - X\beta)/\sigma$.
- ▶ Jacobian $J(\mathbf{y}, \beta, \sigma) = D\left(\mathbf{X}, \frac{\mathbf{y} - \mathbf{X}\beta}{\sigma}\right) = \sigma^{-1} D(\mathbf{X}, \mathbf{y})$
 - ▶ $= \sigma^{-1} |\det(X^T X)|^{1/2} (RSS)^{1/2}$ for L_2 .
- ▶ Same as independence Jeffreys, *explicit* normalizing constant

Example -- Generalized Pareto

- ▶ $X_i = G(U_i, \gamma, \sigma) = \sigma \frac{U_i^{-\gamma} - 1}{\gamma}$
 - ▶ Models excedances over a large threshold.
- ▶ Likelihood $f(\mathbf{x}, \gamma, \sigma) = \prod_{i=1}^n \frac{1}{\sigma \left(1 + \frac{\gamma x_i}{\sigma}\right)^{1+1/\gamma}}$.
- ▶ Jacobian evaluated at $u_i = \left(1 + \frac{\gamma x_i}{\sigma}\right)^{-1/\gamma}$
 - ▶ $\frac{d}{d\sigma} G(u_i, \gamma, \sigma) = \frac{x_i}{\sigma}$.
 - ▶ $\frac{d}{d\gamma} G(u_i, \gamma, \sigma) = -\frac{x_i}{\gamma} + \frac{\sigma \left(1 + \frac{\gamma x_i}{\sigma}\right) \log \left(1 + \frac{\gamma x_i}{\sigma}\right)}{\gamma^2}$.
 - ▶ $J(\mathbf{x}, \gamma, \sigma) = \gamma^{-2} D \begin{pmatrix} x_1 & \left(1 + \frac{\gamma x_1}{\sigma}\right) \log \left(1 + \frac{\gamma x_1}{\sigma}\right) \\ \vdots & \vdots \\ x_n & \left(1 + \frac{\gamma x_n}{\sigma}\right) \log \left(1 + \frac{\gamma x_n}{\sigma}\right) \end{pmatrix}$
 - ▶ $= \sum_{i < j} \left| \frac{x_j \left(1 + \frac{\gamma x_i}{\sigma}\right) \log \left(1 + \frac{\gamma x_i}{\sigma}\right) - x_i \left(1 + \frac{\gamma x_j}{\sigma}\right) \log \left(1 + \frac{\gamma x_j}{\sigma}\right)}{\gamma^2} \right|$ for L_∞ .

Jacobian Formula

Short course special - L_1 norm!

► Recall:

$$\arg \min_{\xi} \|\mathbf{x} - \mathbf{G}(\mathbf{U}^*, \xi)\| \mid \{\min_{\xi} \|\mathbf{x} - \mathbf{G}(\mathbf{U}^*, \xi)\| \leq \varepsilon\}$$

- Optimization problem: $\min_{\xi} \sum_i |x_i - G_i(\mathbf{U}^*, \xi)|$
- $\mathbf{G}(\mathbf{U}^*, \xi)$ is nearly linear in ξ on near \mathbf{x}

L_1 minimum

- ▶ Objective function: locally linear, locally convex
- ▶ At minimum:
 - ▶ p coordinates equal to \mathbf{x}
 - ▶ KKT condition:

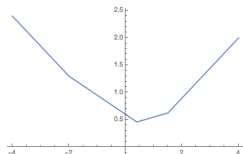
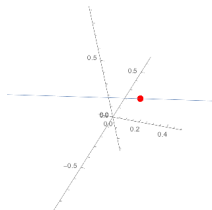
$$0 \in \partial \|\mathbf{x} - \mathbf{G}(\mathbf{U}^*, \xi)\|_1,$$

i.e., $0 = -\sum \lambda_i \nabla_x G_i(\mathbf{U}^*, \xi)$,
where

$$\lambda_i \in \begin{cases} \{1\} & x_i - G_i(\mathbf{U}^*, \xi) > 0 \\ \{-1\} & x_i - G_i(\mathbf{U}^*, \xi) < 0 \\ [-1, 1] & x_i - G_i(\mathbf{U}^*, \xi) = 0 \end{cases}$$

$$\mathbf{G} = (.1, .25, -.2)^\top \xi + (.2, -.1, .3)^\top$$

$$\mathbf{x} = (0, 0, 0)^\top$$



minimum at $\xi = .4$, $G = (0.24, 0, 0.22)$

L_1 Jacobian

- ▶ Select p equations $\mathbf{i} = (i_1, \dots, i_p)$ and solve $\mathbf{x}_{\mathbf{i}} = \mathbf{G}_{\mathbf{i}}(U, \xi)$
 - ▶ Implicit function theorem: $\left| \det(\nabla_{\xi} \mathbf{G}_{\mathbf{i}}(\mathbf{u}, \xi))_{\mathbf{u}=\mathbf{G}_{\mathbf{i}}^{-1}(\mathbf{x}_{\mathbf{i}}, \xi)} \right| f(\mathbf{x}_{\mathbf{i}}|\xi)$
- ▶ Condition on remaining of the equations and KKT condition
- ▶ Final formula $r(\xi|\mathbf{x}) \propto J(\mathbf{x}, \xi) f(\mathbf{x}|\xi),$

$$J(\mathbf{x}, \xi) = \sum_{\mathbf{i}} k_{\mathbf{i}} \left| \det(\nabla_{\xi} \mathbf{G}_{\mathbf{i}}(\mathbf{u}, \xi))_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x}, \xi)} \right|.$$

- ▶ The KKT factor

$$k_{\mathbf{i}} = P(\exists \lambda \in [-1, 1]^p : \lambda \cdot \nabla_{\xi} \mathbf{G}_{\mathbf{i}}(\mathbf{u}, \xi) + R \cdot \nabla_{\xi} \mathbf{G}_{-\mathbf{i}}(\mathbf{u}, \xi) = 0),$$

where $\mathbf{u} = \mathbf{G}^{-1}(\mathbf{x}, \xi),$

$R = (R_1, \dots, R_{n-p})$ i.i.d. Rademacher.

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Important Observations (Bayesian)

- ▶ GFD is always proper
- ▶ GFD is invariant to re-parametrizations (same as Jeffreys)
- ▶ GFD is *not* invariant to smooth transformation of the data if $n > p$
- ▶ Consequently:
 - ▶ GFD does not satisfy likelihood principle.
 - ▶ Adding a multiple of a column to another column does not alter $D(A)$. Row operations not allowed!

Classical Result ($n=1$, $p=1$; Frequentist)

Data generating equation $S = G_S(\mathbf{U}, \xi)$ (1-dimensional statistic)

1. $G_S(\mathbf{u}, \xi)$ is **non-decreasing** in ξ for all \mathbf{u}
2. For all \mathbf{u} and s the inverse $Q_s(\mathbf{u}) = \{\xi : s = G_S(\mathbf{u}, \xi)\} \neq \emptyset$.
3. For all ξ the cdf $F_S(s, \xi)$ is continuous.

Then one has **"unique" fiducial distribution** and **exact coverage** of one-sided confidence intervals, i.e.,

$$P(Q_s(\mathbf{U}^*) \leq \xi) = 1 - F_S(s, \xi).$$

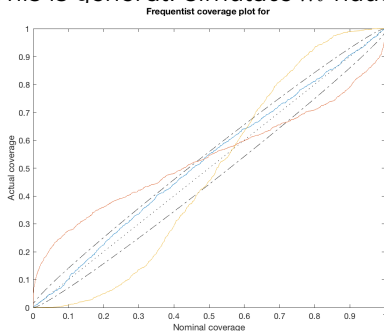
- If $S \sim \xi_0$ then $1 - F_S(S, \xi_0) \sim U(0, 1)$ – fiducial p-value.

Exact frequentist coverage

- Set $P(Q_s(\mathbf{U}^*) \leq C_\alpha(s)) = 1 - \alpha$.
- Coverage of upper confidence limit:

$$\begin{aligned} P_\xi(\xi \leq C_\alpha(S)) &= P_\xi(P(Q_S(\mathbf{U}^*) \leq \xi | S) \leq 1 - \alpha) \\ &= P(U(0, 1) \leq 1 - \alpha) = 1 - \alpha \end{aligned}$$

- This is general: simulate m fiducial p-values



exact
conservative
liberal

Exact frequentist coverage ($p > 1$)

- ▶ In the $p = 1$ setup before $(-\infty, C_\alpha(S)) = Q_S((0, 1 - \alpha))$
 - ▶ the CIs map to the same interval in the U space!
- ▶ Invert and generalize this:
 - ▶ select a $P(\mathcal{U}) = \alpha$, the set $Q_S(\mathcal{U})$ has both fiducial probability and confidence $1 - \alpha$
- ▶ Comments:
 - ▶ Links sets of $1 - \alpha$ fiducial probability for different S .
 - ▶ Similar to pivotal method of deriving CIs
 - ▶ Basis of Inferential Models (Liu & Martin, 2015)

Example -- Fieller's Problem

- ▶ $X \sim N(\mu_x, 1)$, $Y \sim N(\mu_y, 1)$, parameter of interest $\eta = \frac{\mu_x}{\mu_y}$
- ▶ DGE 1: $X = \mu_x + Z_x$, $Y = \mu_y + Z_y$
- ▶ Marginal Fiducial RV $Q_{x,y}(Z_x^*, Z_y^*) = \frac{x - Z_x^*}{y - Z_y^*}$
- ▶ Consider equal tailed regions of fiducial probability 95%:
 - ▶ When $|\mu_y| \gg 0$ good frequentist performance, when $\mu_y \approx 0$ poor performance.
 - ▶ Visualize $\mathcal{Z} = \{(z_x, z_y) \text{ in the interval}\}$

Example -- Fieller's Problem

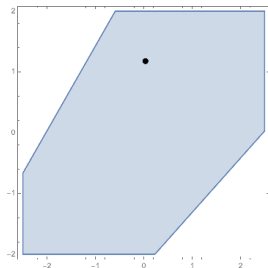
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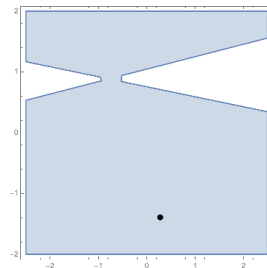
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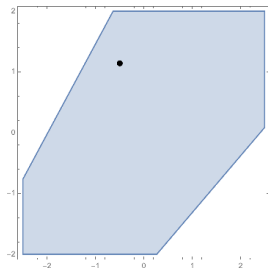
$$\mu_{x0} = 10, \mu_{y0} = 10$$



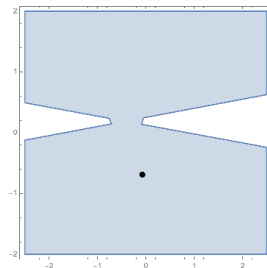
$$\mu_{x0} = 0.5, \mu_{y0} = 0.5$$

Example -- Fieller's Problem

- ▶ $X \sim N(\mu_x, 1)$, $Y \sim N(\mu_y, 1)$, parameter of interest $\eta = \frac{\mu_x}{\mu_y}$
- ▶ DGE 1: $X = \mu_x + Z_x$, $Y = \mu_y + Z_y$
- ▶ Marginal Fiducial RV $Q_{x,y}(Z_x^*, Z_y^*) = \frac{x - Z_x^*}{y - Z_y^*}$
- ▶ Consider equal tailed regions of fiducial probability 95%:
 - ▶ When $|\mu_y| \gg 0$ good frequentist performance, when $\mu_y \approx 0$ poor performance.
 - ▶ Visualize $\mathcal{Z} = \{(z_x, z_y) \text{ in the interval}\}$



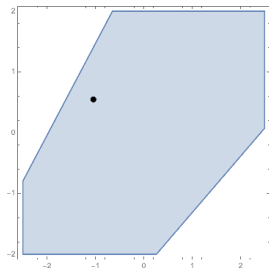
$$\mu_{x0} = 10, \mu_{y0} = 10$$



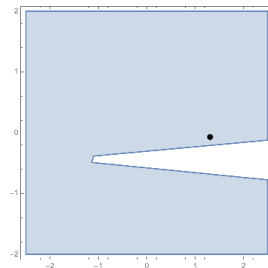
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$$\mu_{x0} = 10, \mu_{y0} = 10$$



$$\mu_{x0} = 0.5, \mu_{y0} = 0.5$$

Example -- Fieller's Solution

- ▶ $X \sim N(\mu_x, 1)$, $Y \sim N(\mu_y, 1)$, parameter of interest $\eta = \frac{\mu_x}{\mu_y}$
- ▶ DGE 2: $X = \mu_x + \frac{-\eta Z_1 + Z_2}{\sqrt{1+\eta^2}}$, $Y = \mu_y + \frac{Z_1 + \eta Z_2}{\sqrt{1+\eta^2}}$
- ▶ Fiducial density $r(\eta|x, y) \propto \frac{e^{-\frac{(x-\eta y)^2}{2(\eta^2+1)}} |y+x\eta|}{(\eta^2+1)^{3/2}}$
- ▶ Fieller picked the set of fiducial probability 95% corresponding to $\mathcal{Z} = \{|z_1| < 1.96\}$
 - ▶ Pros: Fiducial sets are linked so coverage is guaranteed
 - ▶ Cons: The shape of the sets is strange (interval, complement of interval, whole real line)

Ancillary Representation ($n > 1, p = 1$)

- (4) Let $(S(\mathbf{X}), \mathbf{A}(\mathbf{X}))$ be a smooth 1-1 transformation of $\mathbf{X} = \mathbf{G}(\mathbf{U}, \xi)$.
- ▶ $S(\mathbf{X})$ is **one dimensional** satisfying 1, 2, 3.
 - ▶ $\mathbf{A}(\mathbf{X})$ is a vector of **functional ancillary** statistics $(\frac{\partial}{\partial \xi} \mathbf{A} \circ \mathbf{G}(\mathbf{U}, \xi) = \mathbf{0})$.

Theorem (Majumder, 2015)

If (4) is satisfied GFI derived from (S, \mathbf{A}) is exact.

- ▶ Idea: The GFD is the same as FD based on $S = G_{S|\mathbf{a}}(\mathbf{U}_{\mathbf{a}}, \xi)$
 - ▶ $\mathbf{U}_{\mathbf{a}} \sim \mathbf{U} \mid \mathbf{a} = \mathbf{A}(\mathbf{G}(\mathbf{U}, \xi))$
 - ▶ $G_{S|\mathbf{a}}$ is the restriction of G to the domain of $\mathbf{U}_{\mathbf{a}}$.
- ▶ Same argument works for $p > 1$.

Various Asymptotic Results

$$r(\xi|\mathbf{x}) \propto f_{\mathbf{X}}(\mathbf{x}|\xi)J(\mathbf{x}, \xi) \text{ where } J(\mathbf{x}, \xi) = D \left(\left. \frac{d}{d\xi} \mathbf{G}(\mathbf{u}, \xi) \right|_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x}, \xi)} \right)$$

- ▶ Most start with $C_n^{-1}J(\mathbf{x}, \xi) \rightarrow J(\xi_0, \xi)$
- ▶ Bernstein-von Mises theorem for fiducial distributions provides asymptotic correctness of fiducial CIs H (2009, 2013), Sonderegger & H (2013) .
- ▶ Consistency of model selection H & Lee (2009), Lai, H & Lee (2015), H, Iyer, Lai & Lee (2016).
- ▶ Regular higher order asymptotics in Pal Majumdar & H (2016+).

Another look

- ▶ Do fiducial probabilities correspond to (asymptotic) coverage?
- ▶ What is needed?
 - ▶ Convergence of the posteriors to something nice
 - ▶ Linkage of credible sets across all potential data.

LARGE

Data \mathbf{X}_n generated using $\xi_{n,0}$ and GFD measures R_{n,\mathbf{X}_n} .

1. $t_n(\mathbf{X}_n) \xrightarrow{\mathcal{D}} \mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2)$
2. $\mathbf{T}_1 = \mathbf{H}_1(\mathbf{V}_1, \zeta), \mathbf{T}_2 = \mathbf{H}_2(\mathbf{V}_2)$
 - ▶ \mathbf{H}_1 and \mathbf{H}_2 are one-to-one
 $Q_{t_1}(\mathbf{v}_1) = \zeta$ solves $t_1 = \mathbf{H}_1(\mathbf{v}_1, \zeta)$
GFD $R_t \sim Q_{t_1}(\mathbf{V}_1^*) \mid \mathbf{H}_2(\mathbf{V}_2^*) = t_2$.
 - ▶ $R_t(\partial C) = 0$ for open C
3. Homeomorphic injective Ξ_n
 - ▶ $\Xi_n(\zeta_{n,0}) = 0$
 - ▶ $t_n(\mathbf{x}_n) \rightarrow t$
implies $R_{n,\mathbf{y}_n} \Xi_n^{-1} \xrightarrow{\mathcal{W}} R_t$.

Bernstein-von Mises Thm:

centered and scaled MLE

$$\mathbf{T} = \zeta + \mathbf{V}, \mathbf{V} \sim N(0, I_{\xi_0}^{-1})$$

$$N(t, I_{\xi_0}^{-1})$$

$$\Xi_n(\xi) = \sqrt{n}(\xi - \xi_0)$$

LARGE theorem

Theorem (H., Lai & Lee, 2016)

Assume LARGE, fix $0 < \alpha < 1$, select open $C_n(\mathbf{x}_n)$

1. $R_{n,\mathbf{x}_n}(C_n(\mathbf{x}_n)) = \alpha$
2. $t_n(\mathbf{x}_n) \rightarrow \mathbf{t}$ implies $\Xi_n(C_n(\mathbf{x}_n)) \rightarrow C(\mathbf{t})$
3. the set $\mathcal{V}_{\mathbf{t}_2} = \{\mathbf{v} : \mathbf{t} = \mathbf{H}(\mathbf{v}, \xi) \text{ for } \xi \in C(\mathbf{t})\}$ depends on $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ only through ancillary \mathbf{t}_2 .

Then the sets $C_n(\mathbf{x}_n)$ are α asymptotic confidence sets.

- If $\mathbf{T}_1 = \mathbf{H}_1(\mathbf{V}_1, \xi)$ has group structure – 3) means $C(\mathbf{t})$ is invariant in \mathbf{t}_1 .

Example - Uniform(θ, θ^2)

The conditions are satisfied

► Fiducial Density $r(\theta|x) \propto \frac{\bar{x}(2\theta-1)-\theta^2}{(\theta^2-\theta)^{n+1}} I_{(\sqrt{x_{(n)}}, x_{(1)})}(\theta)$

► Transformations:

$$t_n(\mathbf{y}) = n \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} y_{(1)} - \theta_0 \\ \frac{\theta_0^2 - y_{(n)}}{2\theta_0} \end{pmatrix}, \quad \Xi_n(\theta) = n(\theta - \theta_0)$$

► Limit $T_1 = \xi + V_1$, $T_2 = V_2$, where V_1, V_2 are dependent.

► Limiting fiducial distribution

$$r(\zeta|\mathbf{t}) \propto \exp\left(-\zeta \frac{2\theta_0-1}{\theta_0^2-\theta_0}\right) I_{(T_1-T_2, T_1+T_2)}(\zeta).$$

► One sided credible sets have correct asymptotic coverage!

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Computational Issues

- ▶ When lucky: Direct Monte Carlo inversion of DGE
- ▶ Typically need a normalizing constant to $J(\mathbf{x}, \xi) f(\mathbf{x}|\xi)$
 - ▶ Ride the Bayesian Wave:
MCMC (Gibbs, MH), SMC (trick is to resample right)
 - ▶ For p small – numerical integration
- ▶ Fiducial Quirk – Interested in Frequentist Properties
 - ▶ m -number of MC samples vs k -number of replications:
 $k \sim Cm^2$

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Distributed Data

- ▶ Motivation
 - ▶ n is so big that the \mathbf{X} 's cannot be loaded to one computer
 - ▶ the data are at different sites
 - ▶ data cannot be released off site for privacy concerns
- ▶ partition \mathbf{x} into K subsets, where each subsets can be analyzed
 - ▶ e.g., use a computer cluster for parallel processing, where K is the number of nodes (or workers)
 - ▶ merge results from different nodes

Importance sampling for Massive Data

- ▶ $\mathbf{x} = \mathbf{x}_1 \cup \mathbf{x}_2 \cup \dots \cup \mathbf{x}_K$
- ▶ $r(\boldsymbol{\theta}|\mathbf{x})$ – the generalized fiducial density of \mathbf{x}
- ▶ $r(\boldsymbol{\theta}|\mathbf{x}_k)$ – the generalized fiducial density of \mathbf{x}_k
- ▶ On each worker sample from $q_k(\boldsymbol{\theta})$

$$r(\boldsymbol{\theta}|\mathbf{x}) \propto \sum_k \text{'importance weight'} \times q_k(\boldsymbol{\theta})$$

- ▶ depending on the problem, $r(\boldsymbol{\theta}|\mathbf{x})$ and $r_k(\boldsymbol{\theta}|\mathbf{x}_k)$ may be known only up to normalizing constant.

Naive scheme

- ▶ generate a fiducial sample for data on each node
- ▶ compute the weight $w_k(\boldsymbol{\theta}) = \frac{r(\boldsymbol{\theta}|\mathbf{x})}{r_k(\boldsymbol{\theta}|\mathbf{x}_k)}$
- ▶ Not feasible and very inefficient!
 - ▶ Target of order $n^{-1/2}$
 - ▶ Fiducial sample on each worker of order $n_k^{-1/2}$.
 - ▶ Most realizations get extremely small weights.

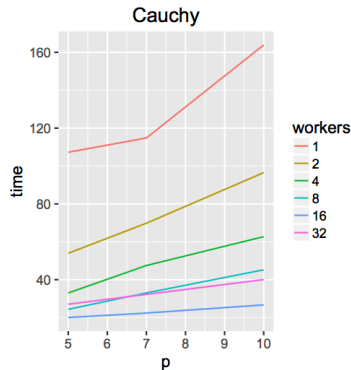
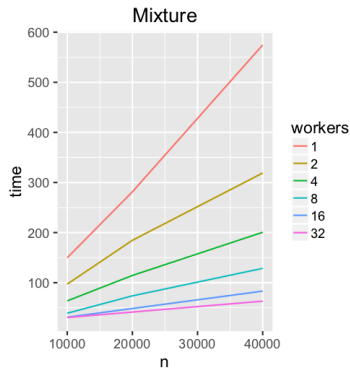
Improved scheme

- ▶ Each worker computes MLE $\hat{\theta}_k$ and empirical Fisher Information \hat{I}_k and passes it to other workers
- ▶ Each worker simulates a sample from
$$q(\mathbf{x}_k) \propto r_k(\boldsymbol{\theta}|\mathbf{x}_k) \times \prod_{j \neq k} g(\boldsymbol{\theta}|\hat{\theta}_j, \hat{I}_j)$$
 - ▶ Practical choice $g \sim \text{Normal}(\hat{\theta}_j, \gamma \hat{I}_k^{-1})$.
- ▶ Weight $w_k(\boldsymbol{\theta}) \approx \prod_{j \neq k} \frac{f(\mathbf{x}_k, \boldsymbol{\theta})}{g(\boldsymbol{\theta}|\hat{\theta}_j, \hat{I}_j)}$
(Computed in parallel.)
- ▶ We proved consistency and asymptotic normality of the approximation error.

Experiments

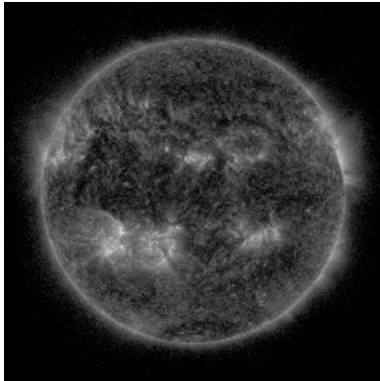
- ▶ All good performance ($K = 1, 2, 4, 8, 16, 32$)
 - ▶ Gaussian mixture: $0.6N(-1, 1) + 0.4N(1, 1)$ ($n = 4 \times 10^5$)
 - ▶ Linear regression with Cauchy errors ($n = 10^5$,
 $p_T = 4, p = 6, 8, 11$)

Computational time

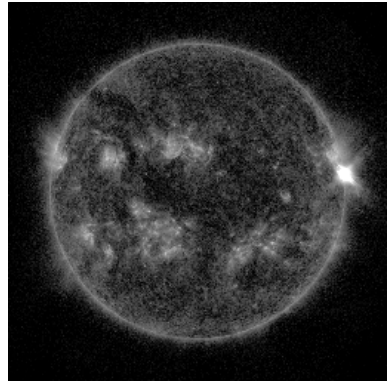


- For Cauchy: speed improves until $K = 16$ then deteriorates (Cheng & Shang, 2015)

Sun Spots



low activity



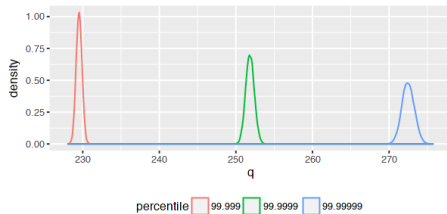
high activity

- The bright flare on the right has value 253. Is this high?

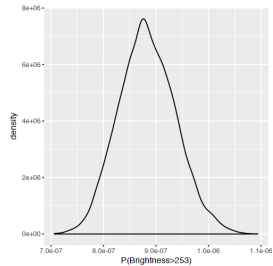
Data

- ▶ Solar Dynamics Observatory (SDO), launched on 2010
- ▶ one instrument is Atmospheric Imaging Assembly (AIA) (Schuh et al. 2013)
 - ▶ photographs the sun in 8 wavelengths every 12s
 - ▶ image size: 4096×4096
 - ▶ 1.5 TB compressed data per day
 - ▶ same as 3 TB raw (i.e., uncompressed) data per day
- ▶ ultimate goal: detect and predict solar flares
- ▶ Tool: GFD for Generalized Pareto (Wandler & H, 2012)

GFD for extreme quantiles



Large Quantiles



Fiducial Excedance
Probability

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Right Censored Data

- Data generating equation:

$$T_i = F^{-1}(U_i), C_i = G_i^{-1}(W_i | F^{-1}(U_i)), \quad U_i, W_i \text{ i.i.d. Uniform}(0,1)$$

observe only $X_i = \min(T_i, C_i), \delta_i = I_{\{T_i \leq C_i\}}$.

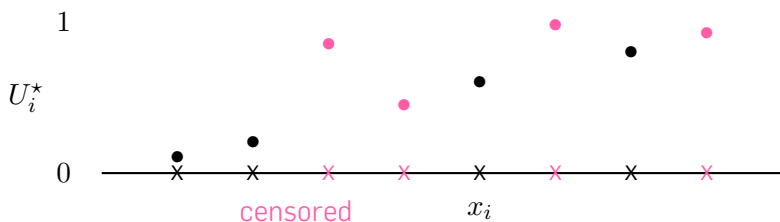
- Solve for F and G :

$$\begin{aligned} \delta_i = 1 & \quad F^*(x_i - \epsilon) < U_i^* \leq F^*(x_i) \\ \delta_i = 0 & \quad F^*(x_i) < U_i^* \end{aligned}$$

$$G_i^*(x_i - \epsilon | x_i) \leq W_i^* \quad G_i^*(x_i - \epsilon | F^{-1}(U_i^*)) < W_i^* \leq G_i^*(x_i | F^{-1}(U_i^*))$$

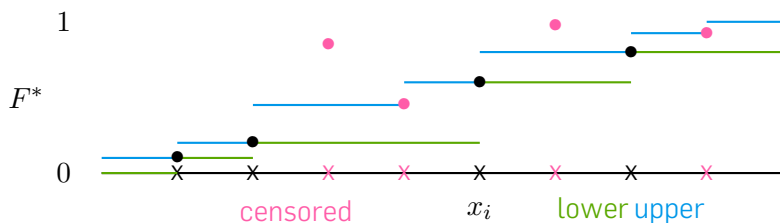
- Generate U^* conditional on existence of F^* solving the equations.

Visual Demonstration



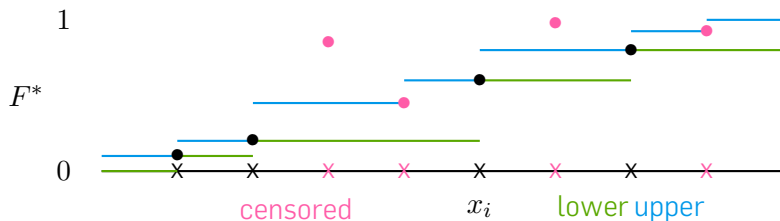
U_i^* generated so that there is a solution

Visual Demonstration



F^* is any cdf between bounds

Visual Demonstration



F^* is any cdf between bounds

Fact: For any failure time $E^*[F_L^*(s)] < \hat{F}(s) < E^*[F_U^*(s)]$.

Theoretical Results

- Known: KM estimator

$$\sqrt{n}[\hat{F}(t) - F(t)] \xrightarrow{\mathcal{D}} W_t \text{ in } \mathcal{D}[0, T]$$

(W – mean zero Gaussian process with known covariance depending on censoring)

- Theorem (Cui, Hannig): Under similar assumptions

$$\sqrt{n}[F^*(t) - \hat{F}(t)]|\mathbf{X} \xrightarrow{\mathcal{D}} W_t \text{ almost surely.}$$

Implementation

- ▶ Fast MC algorithm for generating samples from fiducial distribution.
- ▶ Quantiles of fiducial at given x – approximate pointwise CIs.
- ▶ Simultaneous CI
 - ▶ Center on mean (or Kaplan Mayer).
 - ▶ Find L_∞ ball containing $1 - \alpha$ fiducial sample curves.
- ▶ Fiducial p-value for testing $H_0 : F = F_0$
 - ▶ Find largest α so that F_0 is in the $1 - \alpha$ CI.
- ▶ For difference of two populations – use difference of fiducial distributions

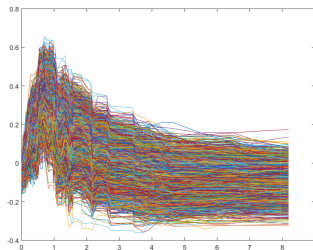
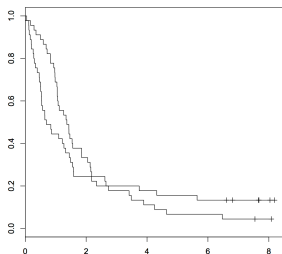
Lessons from Simulation

We simulated several one and two sample settings

- ▶ Pointwise simulations show good coverage
- ▶ Simultaneous CI also show good/conservative coverage
- ▶ Two sample testing
 - ▶ Good type 1 error
 - ▶ Compared to log rank tests and sup-log rank test: competitive.
- ▶ Room for improvement:
 - ▶ Use something else than L_∞ ball – half-region depth?

Gastrointestinal Tumor Study Group (1982)

- Clinical trial of chemotherapy vs. chemotherapy combined with radiotherapy.



- Different: Log-rank statistics p-value 0.63 (hazard crossing); Fiducial p-value 0.003.
- Simulation with estimated hazard shows fiducial more powerful than competitors.

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Model Selection

$$\blacktriangleright \mathbf{X} = \mathbf{G}(M, \boldsymbol{\xi}_M, \mathbf{U}), \quad M \in \mathcal{M}, \boldsymbol{\xi}_M \in \boldsymbol{\xi}_M$$

Theorem: (H, Iyer, Lai, Lee 2016) Under assumptions

$$r(M|\mathbf{y}) \propto q^{|M|} \int_{\boldsymbol{\xi}_M} f_M(\mathbf{y}, \boldsymbol{\xi}_M) J_M(\mathbf{y}, \boldsymbol{\xi}_M) d\boldsymbol{\xi}_M$$

- ▶ Need for penalty – in fiducial framework additional equations $0 = P_k, \quad k = 1, \dots, \min(|M|, n)$
 - ▶ Default value $q = n^{-1/2}$ (motivated by MDL)

Alternative to penalty - see poster!

- ▶ Penalty is used to discourage models with many parameters
- ▶ Real issue: Not too many parameters but a smaller model can do almost the same job

$$r(M|\mathbf{y}) \propto \int_{\Xi_M} f_M(\mathbf{y}, \boldsymbol{\xi}_M) J_M(\mathbf{y}, \boldsymbol{\xi}_M) h_M(\boldsymbol{\xi}_M) d\boldsymbol{\xi}_M,$$

$$h_M(\boldsymbol{\xi}_M) = \begin{cases} 0 & \text{a smaller model predicts nearly as well} \\ 1 & \text{otherwise} \end{cases}$$

- ▶ Motivated by non-local priors of [Johnson & Rossell \(2009\)](#)

Regression

- ▶ $Y = X\beta + \sigma Z$
- ▶ First idea $h_M(\beta_M) = I_{\{|\beta_i| > \epsilon, i \in M\}}$ – issue: collinearity
- ▶ Better:

$$h_M(\beta_M) := I_{\{\frac{1}{2}\|X^T(X_M\beta_M - Xb_{min})\|_2^2 \geq \epsilon_M\}}$$

where b_{min} solves

$$\min_{b \in R^p} \frac{1}{2} \|X^T(X_M\beta_M - Xb)\|_2^2 \quad \text{subject to} \quad \|b\|_0 \leq |M| - 1.$$

- ▶ algorithm – Bertsimas et al (2016)
- ▶ similar to Dantzig selector Candes & Tao (2007)
different norm and target
- ▶ Call this: ϵ -admissible subset

GFD

$$r(M|\mathbf{y}) \propto \pi^{\frac{|M|}{2}} \Gamma\left(\frac{n - |M|}{2}\right) RSS_M^{-\left(\frac{n - |M| - 1}{2}\right)} E[h_M^\varepsilon(\beta_M^*)]$$

Observations:

- Expectation with respect to within model GFD (usual T)
- $r(M|y)$ negligibly small for large models because of h , e.g., $|M| > n$ implies $r(M|y) = 0$.
- Implemented using Grouped Independence Metropolis Hastings (Andrieu & Roberts, 2009).

Main Result

Theorem Williams & H (2017+)

Suppose the true model is given by M_T . Then under certain conditions, for a fixed positive constant $\alpha < 1$,

$$r(M_T|y) = \frac{r(M_T|y)}{\sum_{j=1}^{n^\alpha} \sum_{M:|M|=j} r(M|y)} \xrightarrow{P} 1 \text{ as } n, p \rightarrow \infty.$$

Some Conditions

- ▶ Number of Predictors: $\liminf_{\substack{n \rightarrow \infty \\ p \rightarrow \infty}} \frac{n^{1-\alpha}}{\log(p)} > 2$,
- ▶ For the true model/parameter $p_T < \log n^\gamma$

$$\varepsilon_{M_T} \leq \frac{1}{18} \|X^T(\mu_T - Xb_{min})\|_2^2$$

where b_{min} minimizes the norm subject to $\|b\|_0 \leq p_T - 1$.

- ▶ For a large model $|M| > p_T$ and large enough n or p ,

$$\frac{9}{2} \|X^T(H_M - H_{M(-1)})\mu_T\|_2^2 < \varepsilon_M,$$

where H_M and $H_{M(-1)}$ are the projection matrix for M and M with a covariate removed respectively.

Default ε

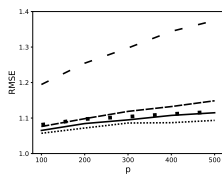
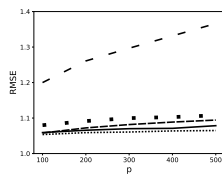
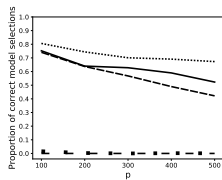
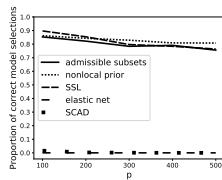
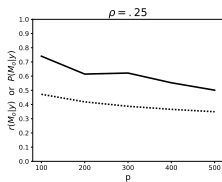
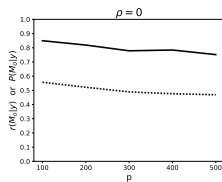
$$\varepsilon = \Lambda_M \hat{\sigma}_M^2 \left(\frac{n^{0.51}}{9} + |M| \frac{\log(p\pi)^{1.1}}{9} - p_T \right)_+,$$

- ▶ $\Lambda_M := \text{tr}((H_M X)' H_M X)$ with $H_M := X_M (X_M' X_M)^{-1} X_M'$
- ▶ $\hat{\sigma}_M^2 := \text{RSS}_M / (n - |M|)$
- ▶ Tuning parameter p_T represents belief about true $|M_T|$.

Simulation setup 1

- ▶ Generate 1000 data vectors y from linear model with $\beta_{M_o}^0 = (-1.5, -1, -.8, -.6, .6, .8, 1, 1.5)'$, and $\sigma_{M_o}^0 = 1$.
- ▶ The $n \times p$ design matrix X is generated with rows from the $N_p(0, \Sigma)$ distribution, where the diagonal components $\Sigma_{ii} = 1$ and the off-diagonal components $\Sigma_{ij} = \rho$ for $i \neq j$.
- ▶ Implement 10-fold cross-validation scheme for choosing the tuning parameter p_o (prior to starting the algorithm).
- ▶ Set $n = 100$, and consider $p = 100, 200, 300, 400, 500$.

Simulation results 1



Simulation setup 2

To illustrate the difference from the nonlocal prior approach, for $n = 30$, generate data from the following model.

$$Y \sim N_n \left(1 \cdot x^{(1)} + 1 \cdot x^{(2)} + \cdots + 1 \cdot x^{(9)}, I_n \right),$$

where $x^{(1)}, x^{(2)}, x^{(3)} \stackrel{\text{iid}}{\sim} N_n(0, I_n)$, and

$$\begin{aligned} x^{(4)} &\sim N_n \left(\begin{array}{ccc} .25 \cdot x^{(1)} & & \end{array}, .1^2 I_n \right) \\ x^{(5)} &\sim N_n \left(\begin{array}{ccc} & .5 \cdot x^{(2)} & \end{array}, .1^2 I_n \right) \\ x^{(6)} &\sim N_n \left(\begin{array}{ccc} & - & .75 \cdot x^{(3)} \end{array}, .1^2 I_n \right) \\ x^{(7)} &\sim N_n \left(\begin{array}{ccc} x^{(1)} & + & x^{(3)} \end{array}, .1^2 I_n \right) \\ x^{(8)} &\sim N_n \left(\begin{array}{ccc} & x^{(2)} & - x^{(3)} \end{array}, .1^2 I_n \right) \\ x^{(9)} &\sim N_n \left(\begin{array}{ccc} x^{(1)} & + & x^{(2)} + x^{(3)} \end{array}, .1^2 I_n \right) \end{aligned}$$

Simulation results 2

| | MAP size | RMSE | $P(M_{\text{MAP}} y)$ |
|-----------------------------------|----------|-------|-----------------------|
| ε -admissible subsets | 3.476 | 1.138 | .365 |
| nonlocal prior | 8.997 | 1.197 | .333 |

- ▶ RMSE of an out-of-sample test set of 30 observations
- ▶ Averaged over 1000 synthetic data sets
- ▶ Nonlocal prior procedure typically includes all 9 covariates even though the y can be mostly explained by 3.

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- Conclusions

BFF

- ▶ Many great minds contributed to foundations of statistics in the past – Fisher, Neyman, de Finetti, Lindley, Savage, LeCam, Cox, Efron, Berger, Fraser, Reid, Dempster, Dawid, ...
 - ▶ Area was not known for harmonious relationships and respectful discourse

the “protracted battle” among leading statistics founding fathers “has left statistics without a philosophy that matches contemporary attitudes.” (Kass, 2011)

Can Bayesian, Fiducial and Frequentist
become Best Friends Forever?

Fiducial Future

- ▶ What is it that we provide? “Can we solve something others cannot?”
 - ▶ GFI: General purpose method that often works well
- ▶ Computational convenience and efficiency
 - ▶ Fiducial options in software
 - ▶ Deep learning?
- ▶ New kind of theoretical guarantees
- ▶ Applications
 - ▶ The proof is in the pudding

I have a dream ...

- ▶ One famous statistician said (I paraphrase)
"I use Bayes because there is no need to prove asymptotic theorem; it is correct."
- ▶ I have a dream that people will gain similar trust in fiducial inspired approaches.

Thank you!