Homework set #3

Based on lectures 3 – 4.

1. Is it possible for X, Y, Z to have the same distribution and satisfy $X = U(Y + Z)$, where $U \sim U(0, 1)$, and Y, Z are independent of U and each other?

2. Let $X \sim \mathrm{Bin}(n, p)$. Find the MGF of X. Show that for $\epsilon > p$

$$P(X/n \geq \epsilon) \leq \exp\left\{-n(1-\epsilon)\log\left(\frac{1-\epsilon}{1-p}\right) - n\epsilon\log\left(\frac{\epsilon}{p}\right)\right\}$$

3. Prove that for all $x > 0$

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) < 1 - \Phi(x) < \frac{1}{x}\phi(x).$$

4. Let $h(u)$ be the function appearing in the version of Bennett's inequality given in class. Show, as claimed, that $h(u) \geq u^2(2 + 2u/3)^{-1}$.

5. Let $Z_1, \ldots, Z_n$ be i.i.d. $N(0, \sigma^2)$ and write $U_i = \max(\min(Z_i, 1), -1)$. In what follows $t > 0$.

   (a) Prove $P(\sum_{i=1}^n Z_i > nt) \leq \frac{\sigma \exp(-\frac{nt^2}{2\sigma^2})}{\sqrt{2\pi n}\ t}$.

   (b) Prove $P(\sum_{i=1}^n U_i > nt) \leq \exp(-\frac{nt^2}{2})$.

   (c) Find EU_i and prove $\mathrm{Var}\, U_i \leq \sigma^2$.

   (d) Prove $P(\sum_{i=1}^n U_i > nt) \leq \exp(-\frac{nt^2}{2\sigma^2 + 4t/3})$.