Worksheet 03 Group 2

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```r
library(tidyverse)
#library(rjags)
library(coda)
library(bayesmeta)
library(pCalibrate)
```

```
## Warning in .recacheSubclasses(def@className, def, env): undefined subclass
## "ndiMatrix" of class "replValueSp"; definition not updated
```

```r
library(glue)
library(ggplot2)
```

Exercise 3:

a)

$y_1, \ldots, y_n \sim \mathcal{N}(m, k^{-1})$ $\qquad m \sim \mathcal{N}(\mu, \lambda^{-1})$

Prior: $f(m) = \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda}{2}(m-\mu)^2\right)$

Likelihood: $f(y|m) = \left(\frac{k}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{k}{2}(y-m)^2\right)$

$$f(y) = \overbrace{\int_{\mathbb{R}} \underbrace{f(y|m)}_{\text{likelihood}} \underbrace{f(m)}_{\text{prior}} \, dm}^{\text{marginal likelihood}}$$

$$= \int_{\mathbb{R}} \left(\frac{k}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{k}{2}(y-m)^2\right) \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{\lambda}{2}(m-\mu)^2\right) dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left(-\frac{k}{2}(y-m)^2 - \frac{\lambda}{2}(m-\mu)^2\right) dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left(k(y-m)^2 + \lambda(m-\mu)^2\right)\right) dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left(k(y^2+m^2-2ym) + \lambda(m^2+\mu^2-2m\mu)\right)\right) dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left(ky^2 + km^2 - 2kym + \lambda m^2 + \lambda\mu^2 - 2\lambda m\mu\right)\right) dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left(ky^2 + \lambda\mu^2 + km^2 + \lambda m^2 - 2m(ky + \lambda\mu)\right)\right) dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left(-\frac{1}{2}\left(ky^2 + \lambda\mu^2 + km^2 + \lambda m^2 - 2m(ky + \lambda\mu)\right)\right) dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\lambda+k)\left(\frac{km^2}{\lambda+k} + \frac{\lambda m^2}{\lambda+k} - 2m\frac{ky+\lambda\mu}{\lambda+k}\right) + ky^2 + \lambda\mu^2\right]\right\} dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\lambda+k)\left(m^2 - 2m\frac{ky+\lambda\mu}{\lambda+k}\right) + ky^2 + \lambda\mu^2\right]\right\} dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\lambda+k)\left(m^2 - 2m\frac{ky+\lambda\mu}{\lambda+k}\right) + ky^2 + \lambda\mu^2\right]\right\} dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\lambda+k)\left(m^2 - 2m\frac{ky+\lambda\mu}{\lambda+k} + \left(\frac{ky+\lambda\mu}{\lambda+k}\right)^2\right) - \frac{(ky+\lambda\mu)^2}{\lambda+k} + ky^2 + \lambda\mu^2\right]\right\} dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\lambda+k)\left(m - \frac{ky+\lambda\mu}{\lambda+k}\right)^2 - \frac{(ky+\lambda\mu)^2}{\lambda+k} + ky^2 + \lambda\mu^2\right]\right\} dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\lambda+k)\left(m - \frac{ky+\lambda\mu}{\lambda+k}\right)^2\right]\right\} \exp\left\{-\frac{1}{2}\left[-\frac{(ky+\lambda\mu)^2}{\lambda+k} + ky^2 + \lambda\mu^2\right]\right\} dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \exp\left\{-\frac{1}{2}\left[-\frac{(ky+\lambda\mu)^2}{\lambda+k} + ky^2 + \lambda\mu^2\right]\right\} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\lambda+k)\left(m - \frac{ky+\lambda\mu}{\lambda+k}\right)^2\right]\right\} dm$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \exp\left\{-\frac{1}{2}\left[-\frac{(ky+\lambda\mu)^2}{\lambda+k} + ky^2 + \lambda\mu^2\right]\right\} \int_{\mathbb{R}} \exp\left\{-\overbrace{\frac{\lambda+k}{2}}^{\text{precision}}\left(m - \overbrace{\frac{ky+\lambda\mu}{\lambda+k}}^{\text{mean}}\right)^2\right\} dm$$

in general,

for $x \sim \mathcal{N}(\mu, \frac{1}{k^2})$: $\quad f(x) = \sqrt{\frac{k}{2\pi}} \exp\left\{-\frac{k}{2}(x-\mu)^2\right\}$

$$= \frac{\sqrt{k\lambda}}{2\pi} \exp\left\{-\frac{1}{2}\left[-\frac{(ky+\lambda\mu)^2}{\lambda+k} + ky^2 + \lambda\mu^2\right]\right\} \sqrt{\frac{2\pi}{\lambda+k}} \underbrace{\int_{\mathbb{R}} \sqrt{\frac{\lambda+k}{2\pi}} \exp\left\{-\frac{\lambda+k}{2}\left(m - \frac{ky+\lambda\mu}{\lambda+k}\right)^2\right\} dm}_{=1}$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \exp\left\{-\frac{1}{2}\left[-\frac{(ky+\lambda\mu)^2}{\lambda+k} + ky^2 + \lambda\mu^2\right]\right\} \sqrt{\frac{2\pi}{\lambda+k}}$$

$$= \frac{\sqrt{k\lambda}}{2\pi} \exp\left\{\frac{(ky+\lambda\mu)^2}{2\lambda+k} - \frac{ky^2}{2} - \frac{\lambda\mu^2}{2}\right\} \sqrt{\frac{2\pi}{\lambda+k}}$$

Because this is an integral over a normal distribution's PDF with $m \sim \mathcal{N}\left(\frac{ky+\lambda\mu}{\lambda+k}, \lambda+k\right)$ we know that

$$= \frac{\sqrt{\kappa\lambda}}{2\pi} \exp\left\{ \frac{1}{2(\lambda+\kappa)} \left[2\lambda\mu y\kappa - \lambda\kappa y^2 - \kappa\lambda\mu^2 \right] \right\} \sqrt{\frac{2\pi}{\lambda+\kappa}}$$

$$= \frac{\sqrt{\kappa\lambda}}{2\pi} \exp\left\{ -\frac{1}{2(\lambda+\kappa)} \left[-2\lambda\mu y\kappa + \lambda\kappa y^2 + \kappa\lambda\mu^2 \right] \right\} \sqrt{\frac{2\pi}{\lambda+\kappa}}$$

$$= \frac{\sqrt{\kappa\lambda}}{2\pi} \exp\left\{ -\frac{\kappa\lambda}{2(\lambda+\kappa)} \left[-2\mu y + y^2 + \mu^2 \right] \right\} \sqrt{\frac{2\pi}{\lambda+\kappa}}$$

$$= \frac{\sqrt{\kappa\lambda}}{2\pi} \exp\left\{ -\frac{\kappa\lambda}{2(\lambda+\kappa)} (y-\mu)^2 \right\} \sqrt{\frac{2\pi}{\lambda+\kappa}}$$

$$= \sqrt{\frac{\kappa\lambda}{4\pi^2}} \sqrt{\frac{2\pi}{\lambda+\kappa}} \exp\left\{ -\frac{\kappa\lambda}{2(\lambda+\kappa)} (y-\mu)^2 \right\}$$

$$= \underbrace{\sqrt{\frac{\kappa\lambda}{2\pi(\lambda+\kappa)}} \exp\left\{ -\frac{\kappa\lambda}{2(\lambda+\kappa)} (y-\mu)^2 \right\}}_{\text{This is the form of a normal distribution's PDF!}}$$

precision: $\frac{\kappa\lambda}{\kappa+\lambda} = \frac{1}{\kappa} + \frac{1}{\lambda} = \kappa^{-1} + \lambda^{-1}$

$\Rightarrow$ var $= \frac{\kappa+\lambda}{\kappa\lambda}$

mean: μ

$$\Rightarrow y \sim \mathcal{N}\left(\mu, \frac{\kappa+\lambda}{\kappa\lambda}\right)$$

b)

$$f(y_{n+1}|y_{1:n}) = \int_{\mathbb{R}} f(y_{n+1}, m | y_{1:n}) \, dm$$

$$= \int_{\mathbb{R}} f(y_{n+1}| m, y_{1:n})\, f(m|y_{1:n})\,dm = \int_{\mathbb{R}} f(y_{n+1}|m)\, f(m|y_{1:n})\,dm$$

conditionally independent

Likelihood: $y_{n+1}|m \sim \mathcal{N}(m, \kappa^{-1}) \Rightarrow f(y_{n+1}|m) = \sqrt{\frac{\kappa}{2\pi}} \exp\left\{-\frac{\kappa}{2}(y_{n+1}-m)^2\right\}$

Posterior: $m|y_{1:n} \sim \mathcal{N}\left(\frac{\kappa n \bar{y} + \lambda\mu}{n\kappa+\lambda}, (n\kappa+\lambda)^{-1}\right)$ per last exercise sheet

$$\mu' := \frac{\kappa n \bar{y} + \lambda\mu}{n\kappa+\lambda} \quad \text{read "}\mu\text{ posterior"}$$

$$\kappa' := n\kappa + \lambda \quad \text{read "}\kappa\text{ posterior"}$$

$$\Rightarrow f(m|y_{1:n}) = \sqrt{\frac{\kappa'}{2\pi}} \exp\left\{-\frac{\kappa'}{2}(m-\mu')^2\right\}$$

hence

$$f(y_{n+1}|y_{1:n}) = \int_{\mathbb{R}} \sqrt{\frac{\kappa}{2\pi}} \exp\left\{-\frac{\kappa}{2}(y_{n+1}-m)^2\right\} \sqrt{\frac{\kappa'}{2\pi}} \exp\left\{-\frac{\kappa'}{2}(m-\mu')^2\right\} dm$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[\kappa(y_{n+1}-m)^2 + \kappa'(m-\mu')^2\right]\right\} dm$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[\kappa(y_{n+1}-m)^2 + \kappa'(m-\mu')^2\right]\right\} dm$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[\kappa y_{n+1}^2 - 2\kappa m y_{n+1} + \kappa m^2 + \kappa' m^2 - 2\kappa' m\mu' + \kappa'\mu'^2\right]\right\} dm$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\kappa+\kappa')\left(m^2 - 2m\frac{\kappa y_{n+1} + \kappa'\mu'}{\kappa+\kappa'}\right) + \kappa y_{n+1}^2 + \kappa'\mu'^2\right]\right\} dm$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\kappa+\kappa')\left(m^2 - 2m\frac{\kappa y_{n+1} + \kappa'\mu'}{\kappa+\kappa'} + \left(\frac{\kappa y_{n+1} + \kappa'\mu'}{\kappa+\kappa'}\right)^2\right) - \frac{(\kappa y_{n+1} + \kappa'\mu')^2}{\kappa+\kappa'} + \kappa y_{n+1}^2 + \kappa'\mu'^2\right]\right\} dm$$

completing the square

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\kappa+\kappa')\left(m^2 - \frac{\kappa y_{n+1} + \kappa'\mu'}{\kappa+\kappa'}\right)^2 - \frac{(\kappa y_{n+1} + \kappa'\mu')^2}{\kappa+\kappa'} + \kappa y_{n+1}^2 + \kappa'\mu'^2\right]\right\} dm$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\kappa+\kappa')\left(m^2 - \frac{\kappa y_{n+1} + \kappa'\mu'}{\kappa+\kappa'}\right)^2\right]\right\} \exp\left\{-\frac{1}{2}\left[-\frac{(\kappa y_{n+1} + \kappa'\mu')^2}{\kappa+\kappa'} + \kappa y_{n+1}^2 + \kappa'\mu'^2\right]\right\} dm$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \exp\left\{-\frac{1}{2}\left[-\frac{(\kappa y_{n+1} + \kappa'\mu')^2}{\kappa+\kappa'} + \kappa y_{n+1}^2 + \kappa'\mu'^2\right]\right\} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\left[(\kappa+\kappa')\left(m^2 - \frac{\kappa y_{n+1} + \kappa'\mu'}{\kappa+\kappa'}\right)^2\right]\right\} dm$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \exp\left\{-\frac{1}{2}\left[-\frac{(\kappa y_{n+1} + \kappa'\mu')^2}{\kappa+\kappa'} + \kappa y_{n+1}^2 + \kappa'\mu'^2\right]\right\} \int_{\mathbb{R}} \exp\left\{-\underbrace{\frac{\kappa+\kappa'}{2}}_{\text{precision}}\left(m^2 - \underbrace{\frac{\kappa y_{n+1} + \kappa'\mu'}{\kappa+\kappa'}}_{\text{mean}}\right)^2\right\} dm$$

this looks like a normal distribution's PDF, so let's add a normalizing constant

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi} \exp\left\{-\frac{1}{2}\left[-\frac{(\kappa y_{n+1} + \kappa'\mu')^2}{\kappa+\kappa'} + \kappa y_{n+1}^2 + \kappa'\mu'^2\right]\right\}$$

$$\sqrt{\frac{2\pi}{\kappa+\kappa'}} \underbrace{\int_{\mathbb{R}} \sqrt{\frac{\kappa+\kappa'}{2\pi}} \exp\left\{-\frac{\kappa+\kappa'}{2}\left(m^2 - \frac{\kappa y_{n+1} + \kappa'\mu'}{\kappa+\kappa'}\right)^2\right\} dm}_{=1}$$

$$= \frac{\sqrt{\kappa\kappa'}}{2\pi}\sqrt{\frac{2\pi}{\kappa+\kappa'}}\exp\left\{-\frac{1}{2}\left[-\frac{(\kappa y_{n+1}+\kappa'\mu')^2}{\kappa+\kappa'}+\kappa y_{n+1}^2+\kappa'\mu'^2\right]\right\}$$

$$= \sqrt{\frac{\kappa\kappa'}{2\pi(\kappa+\kappa')}}\exp\left\{-\frac{1}{2}\left[-\frac{(\kappa y_{n+1}+\kappa'\mu')^2}{\kappa+\kappa'}+\kappa y_{n+1}^2+\kappa'\mu'^2\right]\right\}$$

$$= \sqrt{\frac{\kappa\kappa'}{2\pi(\kappa+\kappa')}}\exp\left\{\frac{1}{2}\left[\frac{(\kappa y_{n+1}+\kappa'\mu')^2}{\kappa+\kappa'}-\kappa y_{n+1}^2-\kappa'\mu'^2\right]\right\}$$

$$= \sqrt{\frac{\kappa\kappa'}{2\pi(\kappa+\kappa')}}\exp\left\{\frac{1}{2}\left[\frac{(\kappa y_{n+1}+\kappa'\mu')^2}{\kappa+\kappa'}-\kappa y_{n+1}^2-\kappa'\mu'^2\right]\right\}$$

$$= \sqrt{\frac{\kappa\kappa'}{2\pi(\kappa+\kappa')}}\exp\left\{\frac{1}{2(\kappa+\kappa')}\left[\kappa^2 y_{n+1}^2+2\kappa y_{n+1}\kappa'\mu'+\kappa'^2\mu'^2-\kappa^2 y_{n+1}^2-\kappa\kappa' y_{n+1}^2-\kappa\kappa'\mu'^2-\kappa'^2\mu'^2\right]\right\}$$

$$= \sqrt{\frac{\kappa\kappa'}{2\pi(\kappa+\kappa')}}\exp\left\{\frac{1}{2(\kappa+\kappa')}\left[2\kappa y_{n+1}\kappa'\mu'-\kappa\kappa' y_{n+1}^2-\kappa\kappa'\mu'^2\right]\right\}$$

$$= \sqrt{\frac{\kappa\kappa'}{2\pi(\kappa+\kappa')}}\exp\left\{-\frac{1}{2(\kappa+\kappa')}\left[-2\kappa y_{n+1}\kappa'\mu'+\kappa\kappa' y_{n+1}^2+\kappa\kappa'\mu'^2\right]\right\}$$

$$= \sqrt{\frac{\kappa\kappa'}{2\pi(\kappa+\kappa')}}\exp\left\{-\frac{\kappa\kappa'}{2(\kappa+\kappa')}\left[-2 y_{n+1}\mu'+y_{n+1}^2+\mu'^2\right]\right\}$$

$$= \sqrt{\frac{\kappa\kappa'}{2\pi(\kappa+\kappa')}}\exp\left\{-\underbrace{\frac{\kappa\kappa'}{2(\kappa+\kappa')}}_{\text{precision}}(y_{n+1}-\underbrace{\mu'}_{\text{mean}})^2\right\}$$

$$\text{Variance} = \left(\frac{\kappa\kappa'}{\kappa+\kappa'}\right)^{-1} = \frac{\kappa+\kappa'}{\kappa\kappa'} = \kappa^{-1}+\kappa'^{-1}$$

$$\Rightarrow y_{n+1} \mid y_{1:n} \sim \mathcal{N}(\mu', \kappa^{-1}+\kappa'^{-1})$$

where

$$\boxed{\begin{aligned}\mu' &:= \frac{\kappa n\bar{y}+\lambda\mu}{n\kappa+\lambda}\\ \kappa' &:= n\kappa+\lambda\end{aligned}}$$

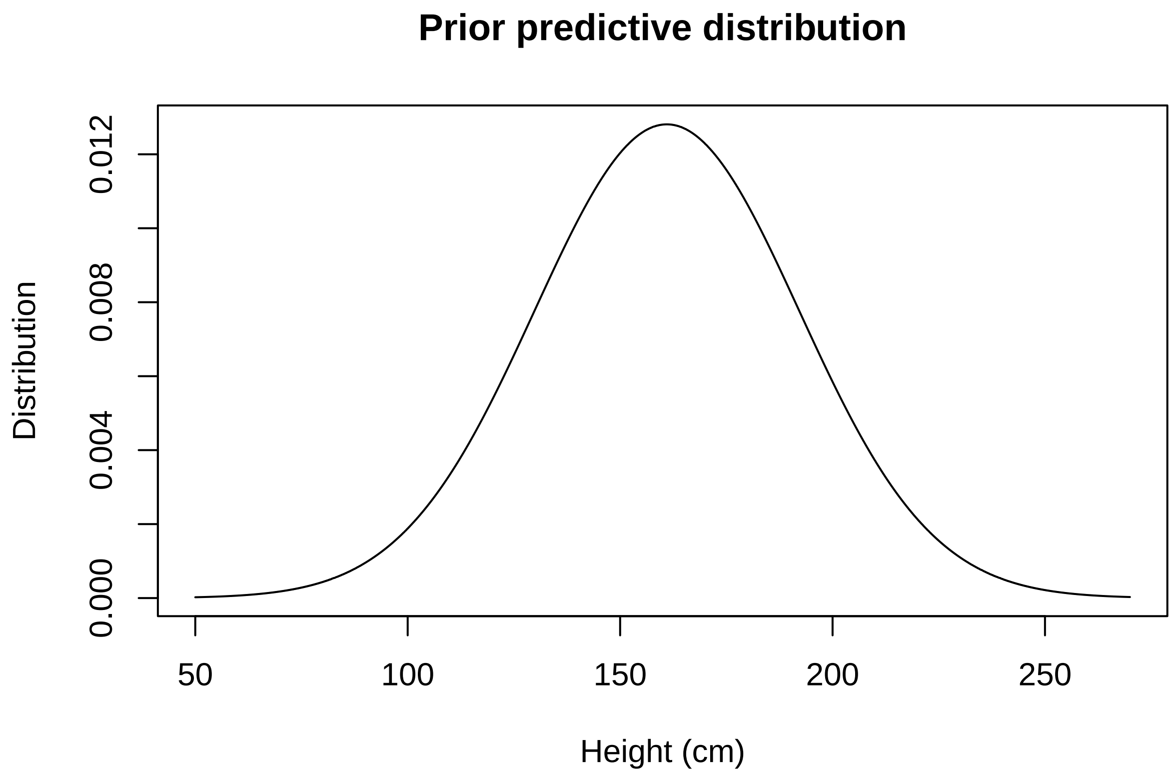
Exercise 4:

Apply analytical formulas derived in Exercise 3 above to the vector of height (cm) measurements 166, 168, 168, 177, 160, 170, 172, 159, 175, 164, 175, 167, 164 of 13 Swiss females. Assume that $y_1, ..., y_n$ are observations generated by N(m, κ^{-1}) distribution with $\kappa = 1/900$. Moreover, assume a N(μ, λ^{-1}) prior for m with $\mu = 161$ and $\lambda = 1/70$.

a) Plot the prior predictive distribution for one observation y and compute its expectation and standard deviation. Estimate P[y > 200] for one future observation of Height.

The prior predictive distribution of one future observation y is N(μ, λ^{-1}+κ^{-1}).

```
my_mu <- 161
my_lambda <- 1/70
my_kappa <- 1/900
my_seq <- seq(50, 270, by = 0.01)
plot(my_seq, dnorm(my_seq, mean = my_mu, sd = sqrt(1/my_lambda+1/my_kappa)), type = "l", main = "Prior 
```



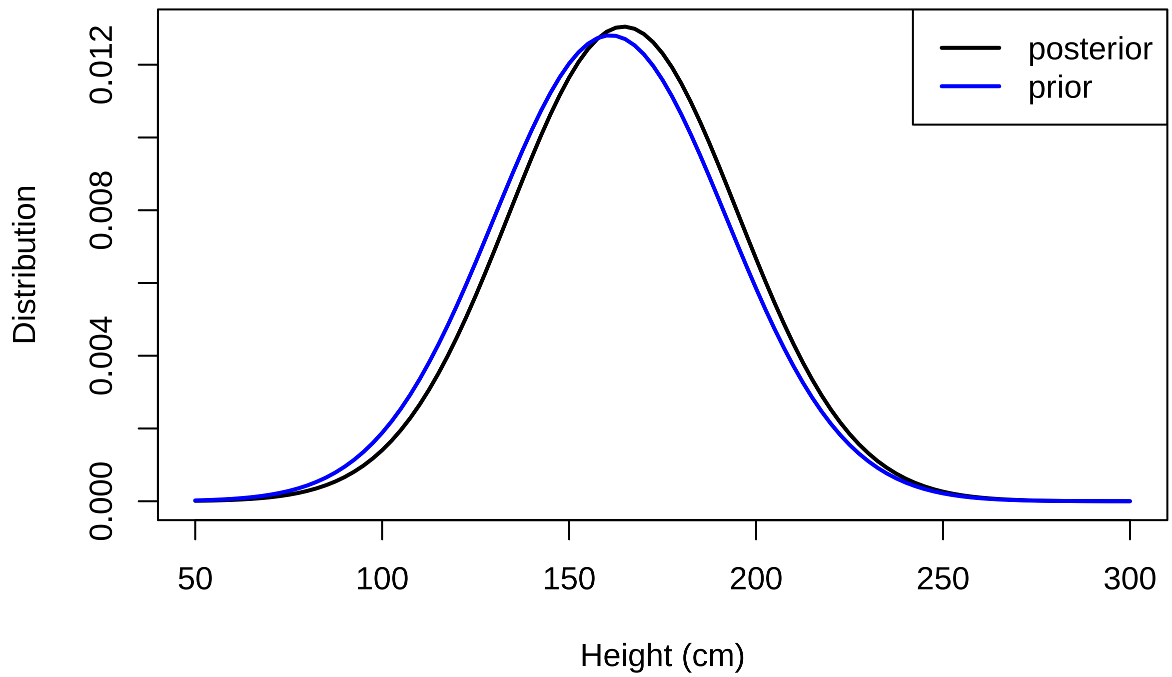

```
expect <- my_mu
stand_dev <- sqrt(1/my_lambda + 1/my_kappa)
prob <- 1-pnorm(200, mean = my_mu, sd = sqrt(1/my_lambda + 1/my_kappa))
```

The expectation is 161 and the standard deviation is 31.145.

P[y >200] for one future observation of Height is 0.105.

b) Plot the posterior predictive distribution for one future observation y_{n+1} given that $y_1, ..., y_n$ have been observed and compute its expectation and standard deviation. Estimate P[$y_{n+1} >$ 200$y_1, ..., y_n$] for one future observation y_{n+1} of Height.

```
obs <- c(166, 168, 168, 177, 160, 170, 172, 159, 175, 164, 175, 167, 164)
n <- length(obs)
mean_obs <- mean(obs)

mu_post <- (my_kappa*n*mean_obs+my_lambda*my_mu)/(n*my_kappa+my_lambda)
lambda_post <- n*my_kappa+my_lambda

curve(dnorm(x, mean = mu_post, sd = sqrt(1/lambda_post+1/my_kappa)), xlab = "Height (cm)", ylab = "Dist
curve(dnorm(x, mean = my_mu, sd = sqrt(1/my_lambda+1/my_kappa)), from = 50, to = 300, lwd = 2, add = TRU
legend("topright", legend = c("posterior", "prior"), col = c("black", "blue"), lwd = 2)
```




```
prob_post <- 1-pnorm(200, mean = mu_post, sd = sqrt(1/lambda_post+1/my_kappa))
```

The expectation is 164.558 and the standard deviation is 30.575.

P[y_{n+1} >200] for one future observation of Height is 0.123.

c) Compare the results obtained for predictive distribution with those obtained for the posterior in Exercise 4 of Worksheet 2. Discuss how much posterior, prior predictive, and posterior predictive distributions differ.

In Exercise 4 of Worksheet 2 the posterior distribution of m$|y_1, ..., y_n$ was derived. Whereas in this worksheet's exercise 4 the posterior distribution of $y_{n+1}|y_1, ..., y_n$ was derived. The values for the mean of the posterior predictive distribution are the same (164.558). However, the standard deviations differ with much larger values for the posterior distribution of y_{n+1}.

The mean of the prior predictive distribution is with 161 lower than the mean of the posterior predictive distribution (164.558). This shift is due to a sample mean larger than 161. The variance of the posterior predictive distribution is with 934.807 slightly smaller than the variance of the prior predictive distribution (970).

Exercise 5:

Change-of-variables formula. Derivation of Inverse Gamma (IG) and Square Root Inverse Gamma (SIG) with parameters $a = 1.6$ and $b = 0.4$.

Change of variables formula

r.v. $X \sim G(a, b)$

density $f(x) = \frac{b^{a}}{\Gamma(a)} x^{a-1} e^{-bx}$

→ density of $Y = \frac{1}{X}$ (IG prior for σ^2)

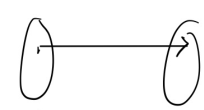
→ density of $Z = \sqrt{Y} = \sqrt{\frac{1}{X}}$ (SIG prior for σ)

Change of variable formula

$$\boxed{f_Y(y) = f_X\Big(\underbrace{g^{-1}(y)}_{\text{inverse}}\Big) \cdot \Bigg|\underbrace{\frac{d^{-1}(y)}{dy}}_{\text{derivative of inverse}}\Bigg|}$$

· applied when transformation $g(\cdot)$ is one-to-one & differentiable

↳ bijective



$X \xrightarrow[g(x)]{} y$

$X \xleftarrow[g^{-1}(y)]{} y$

Figure 1: Change of variable formula

1) derive density of $Y = \frac{1}{X}$

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left| \frac{d^{-1}(y)}{dy} \right|$$

find inverse transformation

$$g(x) = y = \frac{1}{X}$$

$$x = \frac{1}{y} \rightarrow g^{-1}(y) = \frac{1}{y}$$

derive

$$\frac{d g^{-1}(y)}{dy} = y^{-1}$$

$$= -y^{-2} = -\frac{1}{y^2}$$

plug in

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$$

$$f(y) = \frac{b^a}{\Gamma(a)} \left(\frac{1}{y}\right)^{(a-1)} e^{-b/y} \left| -\frac{1}{y^2} \right|$$

$$= \frac{b^a}{\Gamma(a)} \left(y^{-1}\right)^{(a-1)} e^{-b/y} \; y^{-2}$$

$$= \frac{b^a}{\Gamma(a)} y^{-a+1-2} e^{-b/y}$$

$$= \frac{b^a}{\Gamma(a)} y^{-(a+1)} e^{-b/y}$$

→ density function of inverse gamma (IG distribution)

Figure 2: Inverse Gamma

2) derive density of $Z = \sqrt{Y} = \sqrt{\frac{1}{X}}$

inverse function:

$$g(x) = z = \sqrt{\frac{1}{x}}$$

$$z^2 = \frac{1}{x}$$

$$x = \frac{1}{z^2} \rightarrow g^{-1}(z) = \frac{1}{z^2}$$

$$\frac{d\, g^{-1}(z)}{dz} = -2\, z^{-3}$$

$$= \frac{-2}{z^3}$$

plug in

$$f(z) = \frac{b^a}{\Gamma(a)} \left(\frac{1}{z^2}\right)^{(a-1)} e^{-b/z^2} \left|\frac{-2}{z^3}\right|$$

$$= \frac{b^a}{\Gamma(a)} (z^{-2})^{(a-1)} \cdot 2\, z^{-3} \cdot e^{-b/z^2}$$

$$= \frac{2b^a}{\Gamma(a)} z^{-2a+2-3} e^{-b/z^2}$$

$$= \frac{2b^a}{\Gamma(a)} z^{-2a-1} e^{-b/z^2}$$

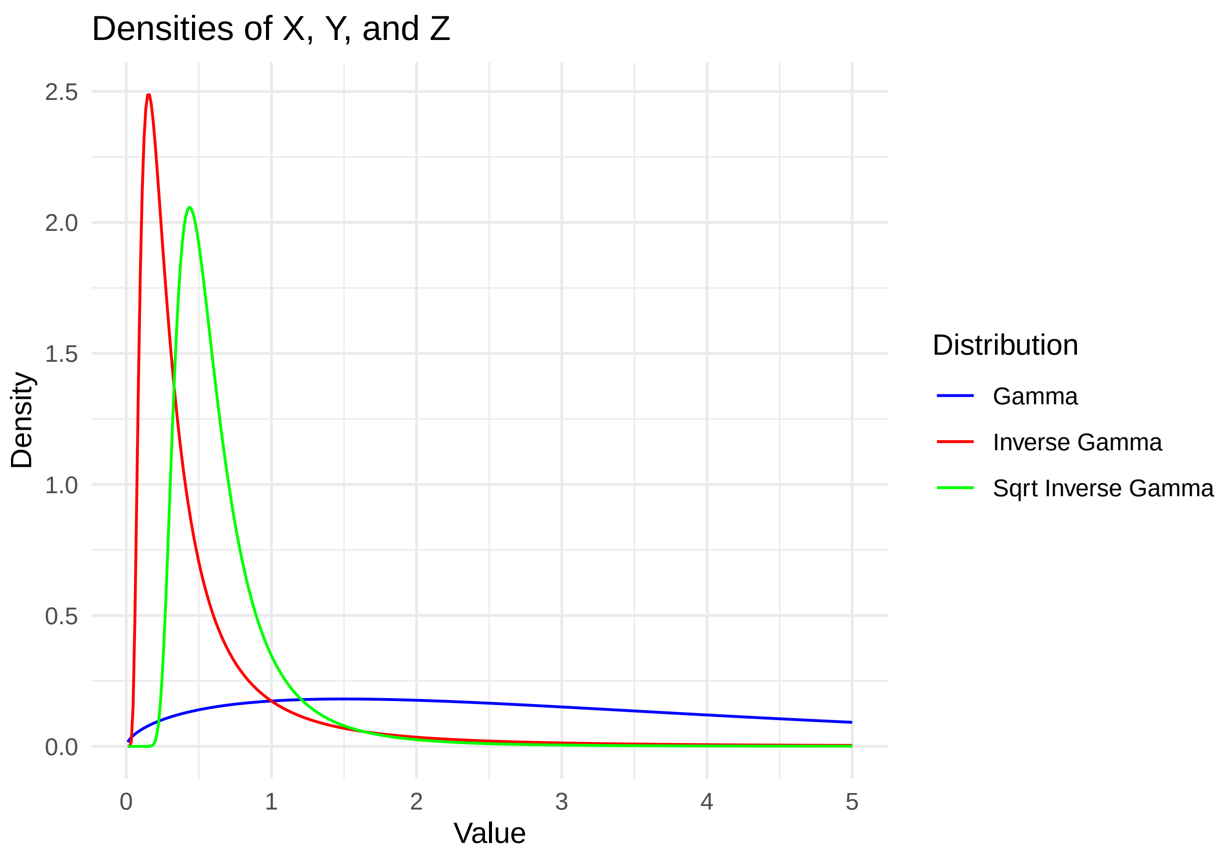
$$= \frac{2b^a}{\Gamma(a)} z^{-(2a+1)} \cdot e^{-b/z^2}$$

density function of Square root Inverse G.
(SIG)

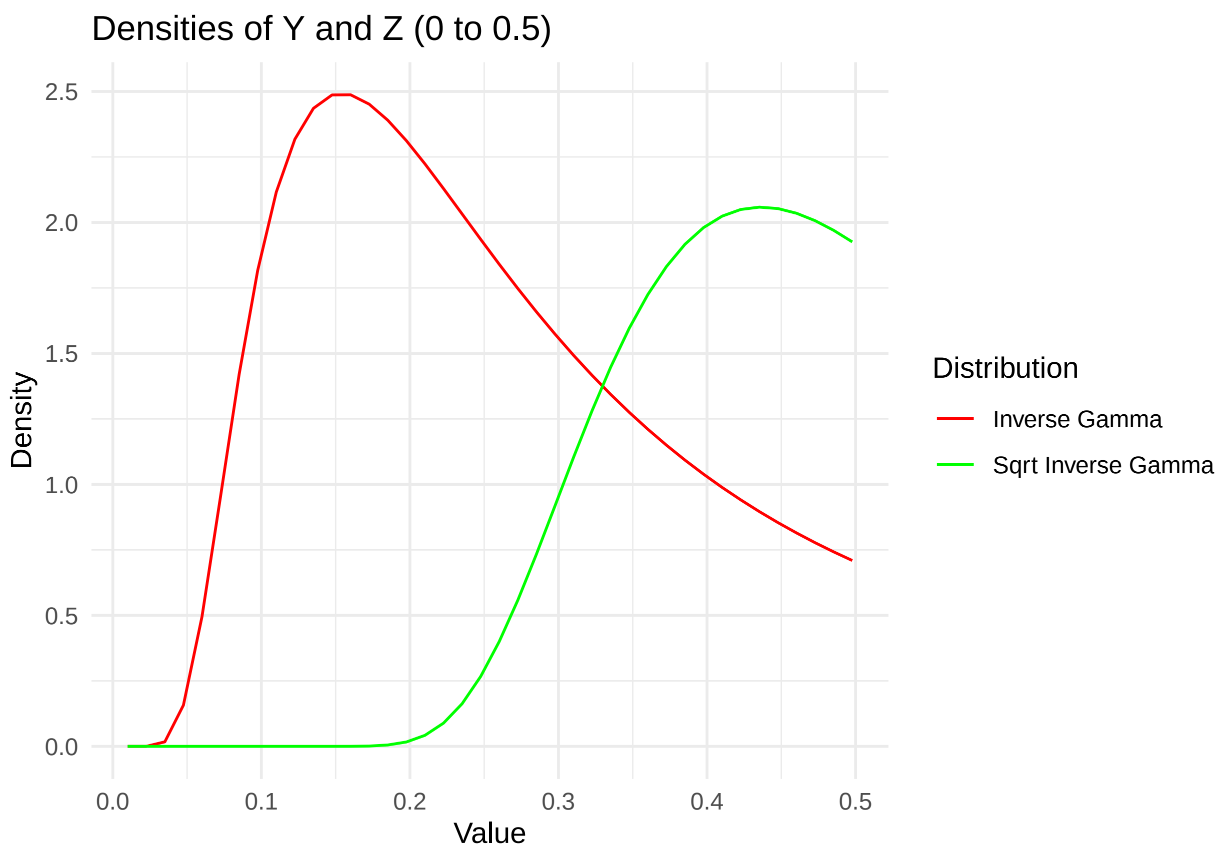
Figure 3: Square root Inverse Gamma

Plots:

```r
# shape parameters
a <- 1.6
b <- 0.4

# gamma
gamma_pdf <- function(x, a, b) {b^a / gamma(a) * x^(a-1) * exp(-b*x)
}

# inverse gamma
IG_pdf <- function(y, a, b) b^a / gamma(a) * y^(-(a + 1)) * exp(-b / y)


# square root gamma
SIG_pdf <- function(z, a, b) 2 * b^a / gamma(a) * z^(-(2 * a + 1)) * exp(-b / (z^2))

# generate values
x_values <- seq(0.01, 5, length = 400)

# data frames for plotting
df_gamma <- data.frame(x = x_values, y = sapply(x_values, gamma_pdf, a, b),
                       Distribution = 'Gamma')
df_IG <- data.frame(x = x_values, y = sapply(x_values, IG_pdf, a, b),
                    Distribution = 'Inverse Gamma')
df_SIG <- data.frame(x = x_values, y = sapply(x_values, SIG_pdf, a, b),
                     Distribution = 'Sqrt Inverse Gamma')

# combine data frames
df <- rbind(df_gamma, df_IG, df_SIG)

# plotting all
ggplot(df, aes(x = x, y = y, color = Distribution)) +
  geom_line() +
  theme_minimal() +
  labs(title = 'Densities of X, Y, and Z', x = 'Value', y = 'Density') +
  scale_color_manual(values = c('Gamma' = 'blue',
                                'Inverse Gamma' = 'red',
                                'Sqrt Inverse Gamma' = 'green'))
```




```
# filter data frame for domain range between 0 and 0.5
df_filtered <- df[df$x <= 0.5,]

# plotting Y and Z for the range 0 to 0.5
ggplot(df_filtered[df_filtered$Distribution != 'Gamma', ],
       aes(x = x, y = y, color = Distribution)) +
  geom_line() +
  theme_minimal() +
  labs(title = 'Densities of Y and Z (0 to 0.5)', x = 'Value', y = 'Density') +
  scale_color_manual(values = c('Inverse Gamma' = 'red',
                                'Sqrt Inverse Gamma' = 'green'))
```



Interpretation:

- The **IG(1.6, 0.4)** distribution spikes quickly in the assignment of probabilities, showing a sharp increase in probability for small values.
- The **SIG(1.6, 0.4)** distribution assigns probability 0 to values (very!) close to 0, indicating a rapid decrease in probability near 0. However, this increase in probability doesn't occur until around 0.2, indicating a slower rise in probability compared to the IG distribution.