Exercise 8 Let $y \sim N(0, \sigma^2)$

(a) The normal PDF is: $(2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{(y-0)^2}{2\sigma^2}}$

κ^{-1} is the variance. It is the reciprocal of the precision, κ. Analoguously for λ^{-1} and λ.

μ is our prior. $f(y,\mu)$ is the likelihood.

Therefore, $f(y|H1) = \int f(y,\mu) f(\mu) d\mu$

$$f(y,\mu) f(\mu) = (2\pi\kappa^{-1})^{-\frac{1}{2}} exp^{-\frac{\kappa}{2}(y-\mu)^2} \cdot (2\pi\lambda^{-1})^{-\frac{1}{2}} exp^{-\frac{\lambda}{2}(\mu-v)^2}$$

Focusing solely on the exponent terms:

$$exp^{-\frac{\kappa}{2}(y-\mu)^2 - \frac{\lambda}{2}(\mu-v)^2} = exp^{-\frac{\kappa}{2}(y^2-2\mu y+\mu^2) - \frac{\lambda}{2}(\mu^2-2v\mu+\cancel{\mu^2})}$$

Ignoring the exponent: $-\frac{\kappa}{2}y^2 - \kappa\mu y - \frac{\kappa}{2}\mu^2 - \frac{\lambda}{2}\mu^2 - \lambda v\mu - \frac{\lambda}{2}v^2$

Re-arranging: $= -\frac{1}{2}(\kappa y^2 - 2\kappa\mu y - \kappa\mu^2 - \lambda\mu^2 - 2\lambda v\mu - \lambda v^2)$

Factorising: $= -\frac{1}{2}(\mu^2(-\kappa-\lambda) - \mu(2\kappa y + 2\lambda v) + \kappa y^2 - \lambda v^2)$

$$= \underbrace{\frac{1}{2}((\kappa+\lambda)\mu^2) + \frac{1}{2}((2\kappa y + 2\lambda v)\mu)} + \frac{1}{2}(\lambda v^2 - \kappa y^2) \quad (1)$$

We are after a perfect square, and want to re-write the part of (1) in a squiggly bracket as: $\frac{1}{2}(\kappa+\lambda)(\mu-a)^2$

We note that the "coefficient" on μ^2 is $\kappa+\lambda$, whilst the coefficient on μ is $(2\kappa y + 2\lambda v) = 2(\kappa y + \lambda v)$

To find a, we solve:

$$2a(\kappa+\lambda) = 2(\kappa y + \lambda v)$$

$$a = \frac{\kappa y + \lambda v}{\kappa + \lambda} \quad (2)$$

We now plug this into the normal PDF as the mean.