History of the Monte Carlo method

Big picture contribution

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March 18, 2024

The History

- Revolutionary idea of Comte de Buffon (1707-1788):
  - Using randomness in a deterministic way
  - Buffon's Needle for calculating Pi (π)
- 1940s Los Alamos, World War II:
  - Simulations for nuclear weapons research
  - Named after Monte Carlo Casino in Monaco

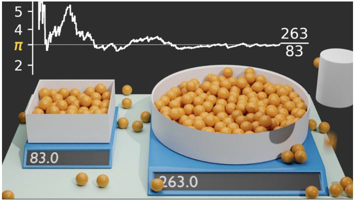
The Method

Many simulations follow this pattern:

- Model system with probability density functions (PDFs)
- Sample repeatedly from PDFs
- Compute statistics of interest

Requirement: **exact object is known**
$\rightarrow$ e.g. finance, physics and engineering

Marble Example: Estimating Pi (π)



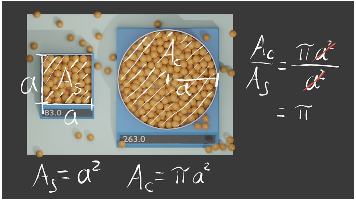


Figure: The number Pi is determined by randomly dropping marbles into two bowls. Proportion of marbles ending up in the two bowls approaches π as the simulation progresses. See YouTube video.

Law of Large Numbers

Let $(X_1, \ldots, X_n)$ be a sequence of independent and identically distributed (i.i.d.) random variables with finite expectation μ. Then, as $n \to \infty$,

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} \mu$$

Held and Sabanes Bove (2014).

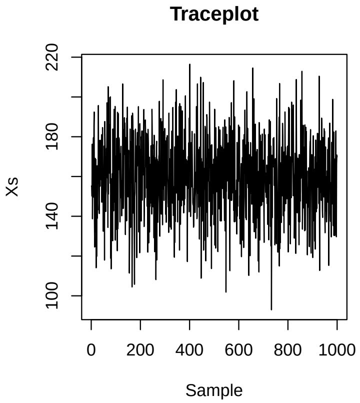
Inverse Transform Sampling: Generating Samples

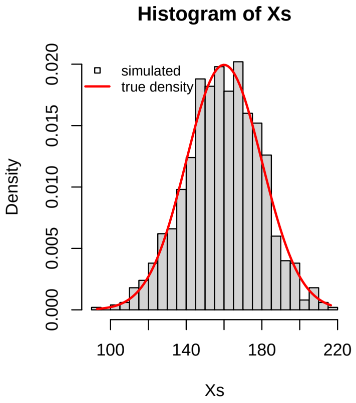
- Generate i.i.d. samples from uniform distribution [0, 1]
- Inverse transform sampling:
  - Apply inverse CDF to transform uniform samples to target distribution
  - Works because $P[F^{-1}(U) \leq x] = P[U \leq F(x)] = F(x)$
- Example: Exponential distribution
  - For $F(x) = 1 - \exp(-\lambda x), x \geq 0$,
  - Set $1 - \exp(-\lambda x) = u$,
  - Solve for x: $x = -\frac{1}{\lambda} \log(1 - u)$
- Enables sampling from any distribution with known inverse CDF

Worksheet 1: Random Sample vs True Distribution

Let the random variable $X \sim N(\mu, \sigma^2)$, with $\mu = 160$ and $\sigma = 20$. Generate a Monte Carlo sample *Xs* of size $M = 1000$.

```r
M <- 1000
mu <- 160
sigma <- 20
set.seed(44566)
Xs <- rnorm(n = M, mean = mu, sd = sigma)
```


Traceplot
Xs
100
140
180
220
0
200
400
600
800
1000
Sample


Histogram of Xs
simulated
true density
Density
0.000
0.005
0.010
0.015
0.020
100
140
180
220
Xs

Take Home Message

Why we use it:

- In Bayesian analysis posteriors aren't easy to work with
- Calculation of integrals is complex
- Monte Carlo simulation is used to numerically solve a complex problem through repeated random sampling

Bibliography

Harrison, R. L., Granja, C., and Leroy, C. (2010). Introduction to monte carlo simulation. http://dx.doi.org/10.1063/1.3295638.

Held, L. and Sabanes Bove, D. (2014). *Applied Statistical Inference: Likelihood and Bayes*. Springer. https://doi.org/10.1007/978-3-642-37887-4.

MarbleScience (2020). Monte carlo simulation [video]. https://www.youtube.com/watch?v=7ESK5SaP-bc [Accessed: 2024-03-16].