Calculus and Linear Algebra Workshop Notes and Problems - Basics of Integration

October 5, 2015

1 Definition of Definite Integral

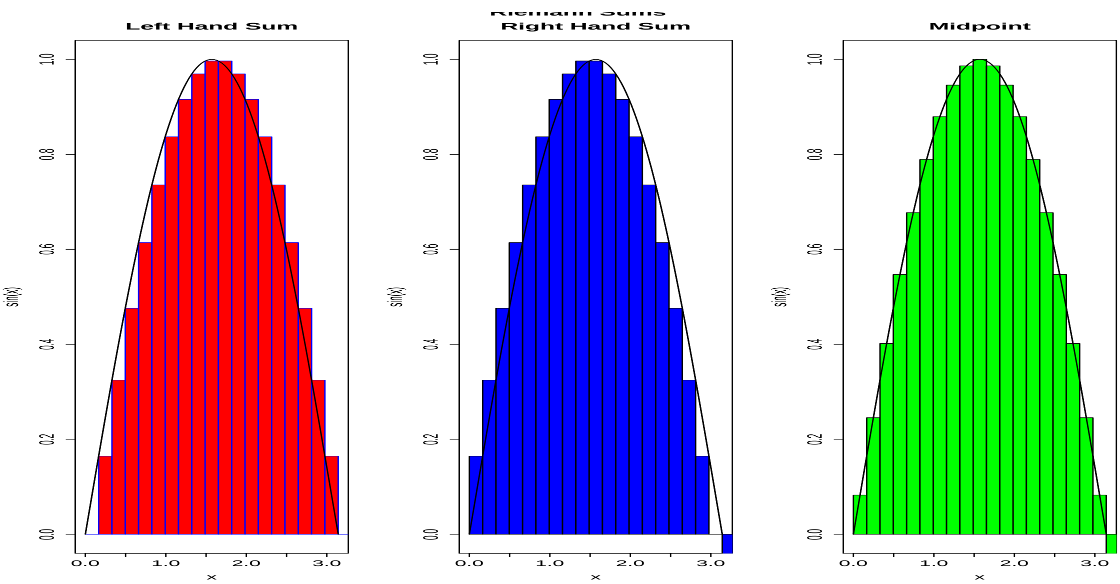
Like the derivative, the definite integral is defined as a limit.

Definition 1. *Let f be a function defined on the interval $[a, b]$. We define the definite integral of f on $[a, b]$ as:*

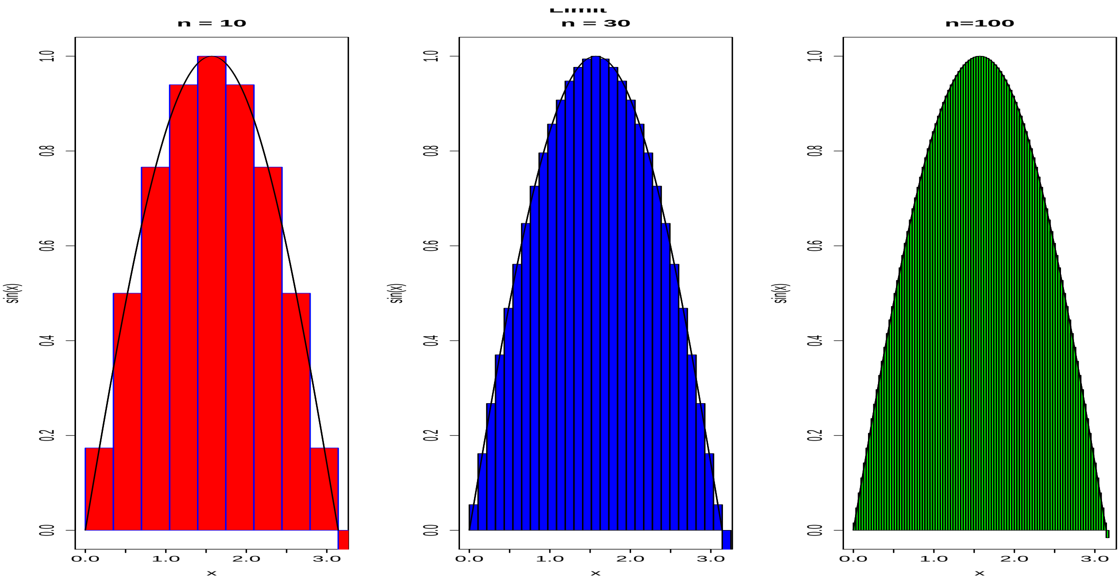
$$\int_a^b f(x)dx = \lim_{\max(\Delta_x)\to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

where $a = x_1 \leq x_2 \leq ... \leq x_n = b$ is a partition of the interval $[a, b]$ and $\Delta x_k = x_{k+1}x_k$ for $k = 1, ..., n1$ and $x_k \leq x_k \leq x_{k+1}$.

Each term in the sum above is the area of a rectangle, with height $f(x_k)$ and width $x_{k+1}x_k$. In the limit, we make the width of the rectangles approach zero. The quantity we calculate is the area under the curve of the function $f(x)$ on the interval $[a, b]$.
Most calculus courses first define particular "Riemann Sums". The following figure illustrates 3 different sums - the left hand sum, the right hand sum and the mid-point. These are just different choices for x_k^*.



Clearly, in the limit that the intervals shrink to zero, it doesn't matter which x_k^* we choose:



2 Properties of Integrals

The following are useful properties of the integral:

- Linearity:

$$\int cf(x)dx = c\int f(x)dx \text{ and } \int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$$

- $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- $$\int_a^a f(x)dx = 0$$

- For any c such that $a \leq c \leq b$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

3 Computing Integrals and the Fundamental Theorem of Calculus

Now, we do not want to have to compute the limit of areas of rectangles and fortunately, we don't have to. The fundamental theorem tells us that:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is an *anti-derivative* of f, i.e.

$$F'(x) = f(x)$$

3.1 Common Integration Techniques

3.1.1 Power Rule

Therorem 1. *The indefinite integral of x^n, for $n \neq 1$ is*

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \textit{ for } n \neq 1$$

Note that n *need not be an integer.*

3.1.2 Indefinite Integrals of Commonly Used Functions

Note: indefinite integrals (anti-derivatives) are not unique. Because the derivative of a constant is zero, we can add any constant to an indefinite integral and it is still correct. In the following, we will ignore this and simply write the anti-derivative where the constant $C = 0$.

- $$\int \frac{1}{x} dx = \log(x)$$
- $$\int e^x dx = e^x$$
- $$\int \sin(x) dx = -\cos(x)$$
- $$\int \cos(x) = \sin(x)$$

3.1.3 Substitution

What do we do when we do not automatically see the derivative of a function - but we see that there may be a composite function we are dealing with? In other words, our integrand looks like something of the form:

$$f'(g(x))g'(x)$$

In other words, it looks like the result of differentiation using the chain rule. Well, we try to find $g(x)$ and substitute into the integral. The trick is that we must not only substitute our integrand - we also need to substitute the differential dx. This is best seen by example:

Example 1. *Compute the indefinite integral:*

$$\int x e^{-x^2} dx$$

First, we let $u = -x^2$. If we make that substitution, we have to substitute:

$$du = -2x dx$$

Now, it looks like our integral almost *has the form $e^u du$ - BUT - we are missing a factor of -2. That is easy to deal with - we just multiply by -2 inside the integral, and divide by -2 outside: Therefore:*

$$\int xe^{-x^2}dx = -\frac{1}{2}\int -2xe^{-x^2}dx = -\frac{1}{2}\int e^u du = -e^u + C$$

Now, we just back-substitute:

$$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + C$$

Here are a bunch for you to try:

- $$\int \frac{x}{x^2+1}$$
- $$\int \cos(\cos(x))\sin(x)dx$$
- $$x^2e^{-4x^3}dx$$

3.1.4 Integration by Parts

Integration by parts is kind of the Hail Mary play of integration. We don't know what else to do, and it looks like splitting the integrand into a product might work (integration by parts is sometimes called the 'inverse product rule). The formula is as follows:

$$\int udv = uv - \int vdu$$

Let's do an example that we saw in class:

$$\int xe^{-x}dx$$

Let $u = x$ and $dv = e^{-x}dx$. Then $du = dx$ (yay!) and $v = -e^{-x}$, so

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx$$

$$= -xe^{-x} - e^{-x} + C$$

Your turn!

- $$\int \log(x)dx$$
- $$\int \sqrt{x}\log(x)dx$$
- $$\int x\sin(x)dx$$