University of St.Gallen

EFMD EQUIS ACCREDITED

Methods: Statistics (4,120)

3. Probability Theory

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Contents

Learning Objectives

After that lecture, you know how …

- … the basic concepts of probability theory are applied in examples.
- … the basic concepts of probability theory are represented by **Venn diagrams or tree diagrams.**

Literature

Levine, D.M., K. A. Szabat, and D.F. Stephan. (2016). *Business Statistics: A First Course*, 7th ed. United States: Pearson, **Chapter 4**.*

Stinerock, R. (2018). *Statistics with R*. United Kingdom: Sage. **Chapter 4**.*

Shira, Joseph (2012). *Statistische Methoden der VWL und BWL*, 4th ed., Munich et al.: Pearson Studium, **Chapter 8**.

Weiers, R. M. (2011). *Introductory Business Statistics*,7th ed., Canada: Thomson South-Western, **Chapter 5**.

*Mandatory literature

I. Basic Terms

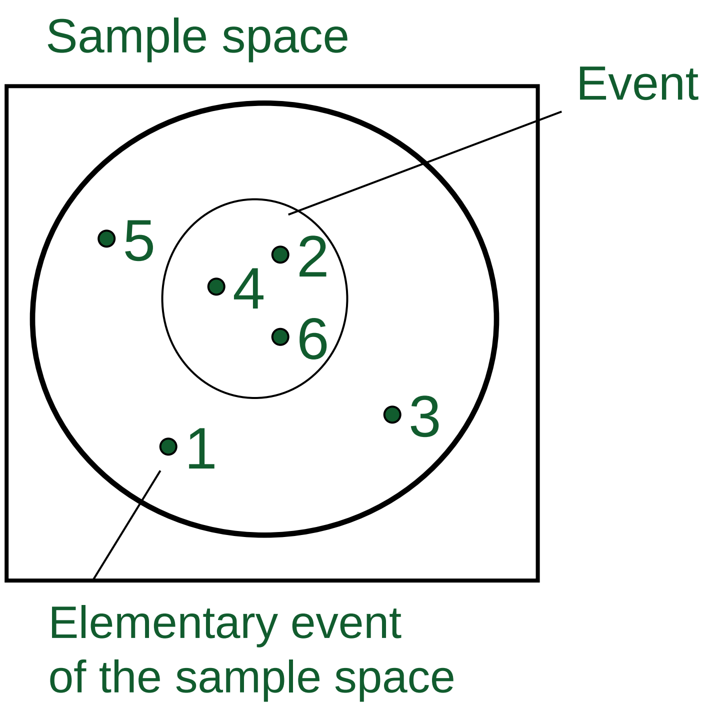
Random experiment: A process that is performed according to certain rules, can be repeated an arbitrary number of times, and whose result depends on chance (e.g., the number of dots (pips) after throwing a dice).

Elementary events: Each random experiment has several possible elementary results. When the experiment is performed, exactly one of them occurs.

Sample space: set Ω of all possible elementary events of the experiment (e.g., throwing of a cube has the sample space $\Omega = \{1,2,3,4,5,6\}$).

I. Basic Terms

(Composed) events: Any possible subset of the sample space (including Ω). An event occurs when an included elementary event occurs.

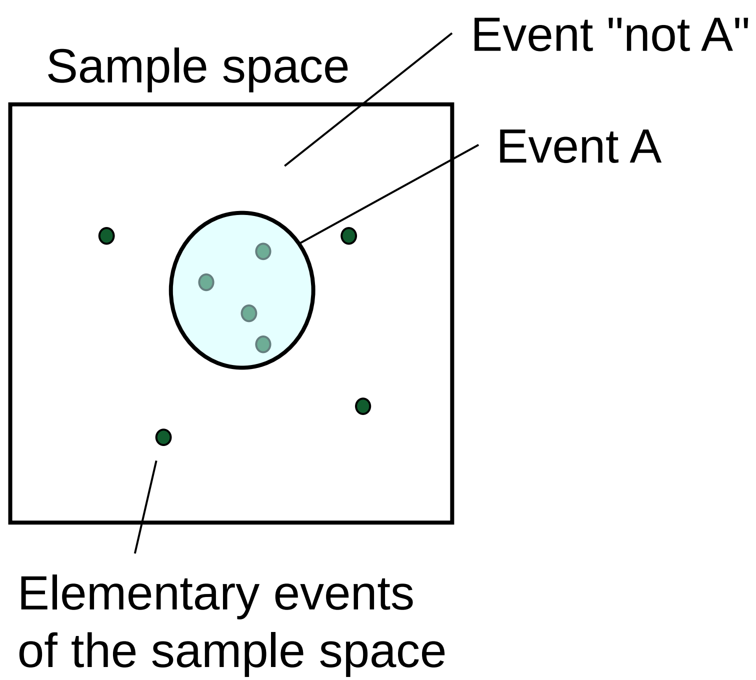


Example: ‚Even number‘ when throwing a dice is a (composed) event consisting of the elementary events ‚2‘, ‚4‘ and ‚6‘.

I. Basic Terms

Complementary event: The complementary event of an event *A* is the event „not *A*“. It is defined as the set of all elementary events of the sample space which are not included in *A*. We write:

$$\bar{A} = \Omega \setminus A$$



Probability: A number between 0 and 1 assigned to an event and representing the probability of the event occurring.

II. Probability Approaches

- ➡ **Classical** Approach
- ➡ **Relative Frequency** Approach
- ➡ **Subjective** Approach
- ➡ **Axiomatic** Approach

II. Classical Approach

The classical approach goes back to P.S. Laplace (1749-1827) and defines probability as the proportion of times that an event can be expected to occur in a random experiment.

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

Example 1: for a 6-sided dice, the probability that 3 occurs:

$$P(A=3) = \frac{1}{1+1+1+1+1+1} = \frac{1}{6}$$

Example 2: Tossing a coin and it lands on ‚heads':

$$P(A=\text{head}) = \frac{1}{1+1} = \frac{1}{2}$$

II. Relative Frequency Approach

The relative frequency approach is empirically based on the **law of large numbers** and corresponds to the frequency with which an event occurs with a correspondingly large number of trials. The propability (*P*) is then equal to the limit value of the relative frequency of occurrence of *A*.

$$P(A) = \frac{\text{number of trials in which } A \text{ occurs}}{\text{total number of trials}}$$

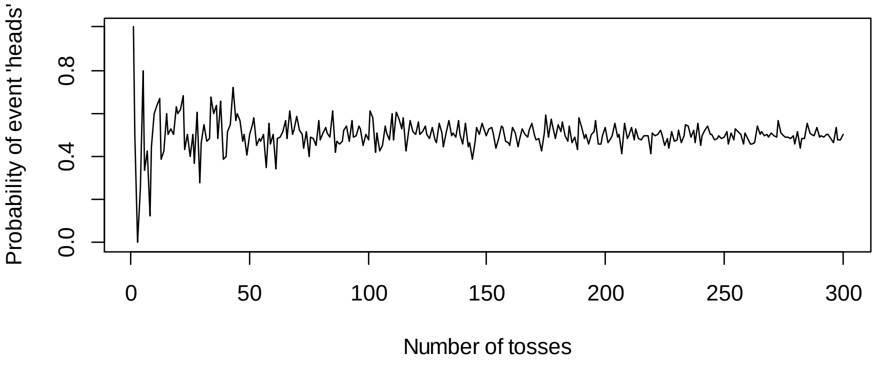
resp.: $P(A) = \lim_{n \to \infty} f_n(A)$

with $f_n(A) = \frac{1}{n}$ · (number of trials in which A occurred)

II. Relative Frequency Approach

Example «Coin Tossing»: With a growing number of trials (*N* tosses) the probability to get ‚heads' will approach 0.5 .

$$P(\text{head}) = \frac{\#\text{ of heads}}{N}$$



Example: The life insurance industry uses relative frequencies to calculate acceptability and premium for applicants. With a death rate of 221.77 (per 1000 male persons aged 90), the probability of death within the next year for a 90-year-old policyholder is 0.22.

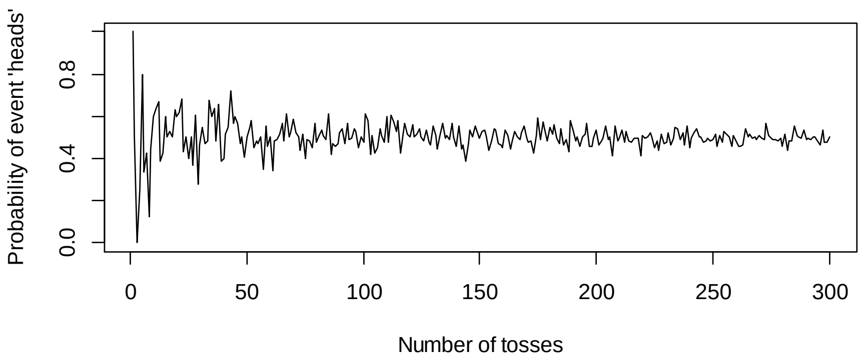
II. Relative Frequency Approach (R-Example 1)

Open the file "L3-Example_1.R" in R-Studio and reproduce the R-Code.

```
# simulate a sequence of coin tosses (1000 tosses in total)
c<-1000
n.trial<-rep(0,1000)
n.heads<-rep(0,1000)
p.heads<-rep(0,1000)
y<-data.frame(n.trial,p.heads,n.heads)

# for-Loop to execute consecutive coin tosses
for (i in 1:c){
x<-rbinom(i,1,0.5)
A<-sum(x)/length(x)
y$n.trial[i]<-length(x)
y$n.heads[i]<-sum(x)
y$p.heads[i]<-A
}

# representation of the probability over the number of attempts
# the par command is used to format the graphical representation.
op <- par(mfrow = c(2, 1), mgp = c(2.1, 1.1, 0), mar = .1+c(3,3,2,1))
plot(y$n.trial,y$p.heads,type="l",xlab="Number of Attempts",ylab="Probability of Event 'Head'")
par(op)
```



II. Subjective Approach

The approach is judgemental and **reflects the degreee to which one believes that an event will happen or not**. Subjective probabilities can be quantified by expressing odds (likelihood that something will happen).

1. If the quota for a certain event is *A* : *B*, then the probability reads:

$$P(A) = \frac{A}{A+B}$$

2. If the probability for a certain event is *x* (with $0 \leq x \leq 1$), the corresponding quota reads:

$$\frac{x}{1-x}$$

II. Subjective Approach (Example)

1. It is "twice as likely" that the price of a particular stock will rise instead of fall.

$$P(A) = \frac{2}{2+1} = \frac{2}{3} = 0.66$$

2. In 80% of the cases the forecasts of a certain financial analyst also occur.

$$P(A) = 0.8; \quad P(\bar{A}) = 0.2$$

$$odds(A): \frac{0.8}{0.2}:1 = 4:1$$

II. Axiomatic Approach

Generalization:

- The **axiomatic** approach goes back to Kolmogorov.
- The axiomatic approach does not explain the character of probability, but it defines **its mathematical properties**.
- According to Kolmogorov, **sets can be interpreted as events to** which a probability p can be assigned.

A function P defined on a system of events is called **probability**, if it fulfills the **Kolmogorovian axioms.**

III. Axioms of Kolmogorov

- The probability P(A) of an event A is a definite, non-negative real number between the values 0 and 1.

$$\mathbf{0 \leq \boldsymbol{P(A)} \leq 1}$$

- The safe event has the probability 1.

$$\boldsymbol{P(\Omega) = 1 \;\; \text{with } \Omega = \{A_1, A_2, \ldots, A_n\}}$$

- For two disjoint events A_1, A_2 with ($A_1 \cap A_2 = \varnothing$) it holds:

$$\boldsymbol{P(A_1 \cup A_2) = P(A_1) + P(A_2)}$$

III. Axioms of Kolmogorov

- For each event A it holds

$$\mathbf{P}(\overline{A}) = 1 - \mathbf{P}(A)$$

- Probability of the impossible event $\varnothing$:

$$P(\varnothing) = 1 - P(\Omega) = 0$$

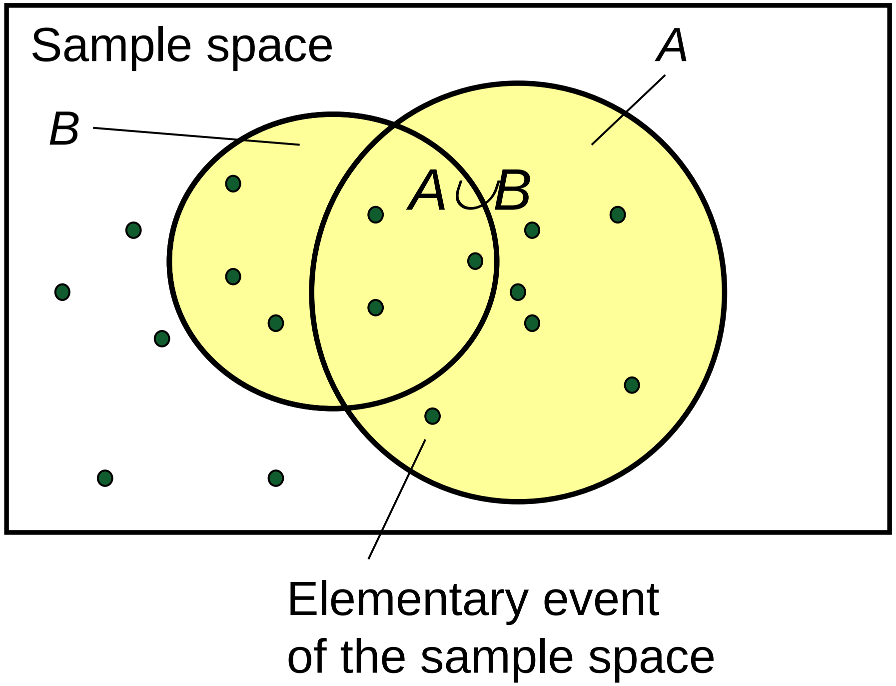
- Generalization on n mutually exclusive events $A_1, A_2, \ldots, A_n$ yields:

$$\Rightarrow P(A_1 \cup A_2 \cup \ldots. \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

IV. Venn-Diagrams

Union of events:

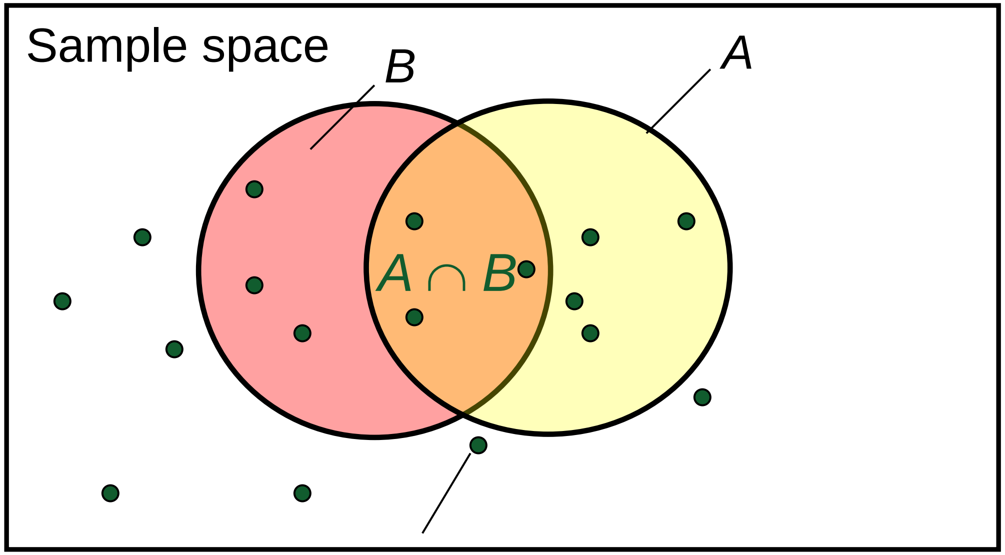
The union $A \cup B$ of two events A and B is defined as the set of all elementary events that belong to ***A* or *B* or both**.



IV. Venn-Diagrams

Intersection of events:

The intersection $A \cap B$ of two events A and B is defined as the set of all elementary events that belong to ***A* and *B***.



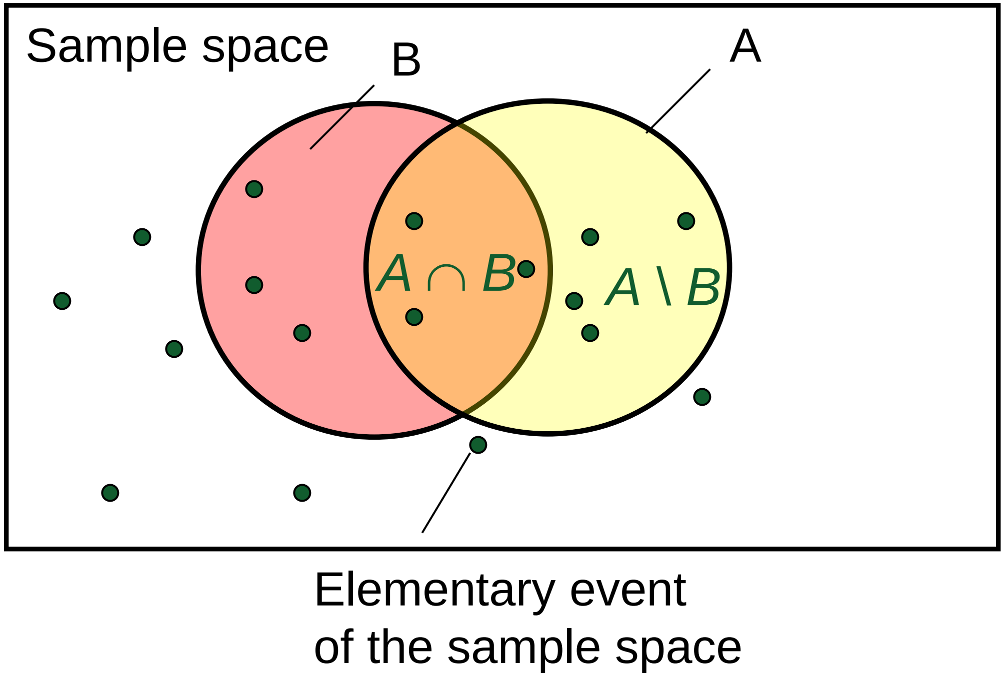
Elementary event
of the sample space

IV. Venn-Diagrams

Relative complement:

Event A consists of the two disjoint events A\B (A without B) and the intersection A∩B of the two events A and B. From this it follows:

$$P\left(A \setminus B\right) = P\left(A\right) - P(A \cap B)$$



IV. Contingency Table

The **dependence** or **independence**, respectively, of two events and their corresponding complementary events can be presented in a contingency table.

	A	$\bar{A}$	**Total**
B	$P(A \cap B)$	$P(\bar{A} \cap B)$	$P(B)$
$\bar{B}$	$P(A \cap \bar{B})$	$P(\bar{A} \cap \bar{B})$	$P(\bar{B})$
Total	$P(A)$	$P(\bar{A})$	**Σ=1**

IV. Contingency Table (Example)

There are 20 red and 30 green cubes in an urn. 5 red and 10 green cubes are marked with a *. The events 'red cube', 'green cube' and 'cube with *' are called *A*, *B* and *C*, respectively.

	C	$\bar{C}$	**total**
A	**5**	**15**	**20**
$B=\bar{A}$	**10**	**20**	**30**
total	**15**	**35**	**50**

1. P(*A*) = 20 / 50 = 0.4

2. P($A \cap \bar{C}$) = 15 / 50 = 0.3

3. P($A \cup \bar{C}$) = 0.8 ⇨ 3 different ways to calculate: :

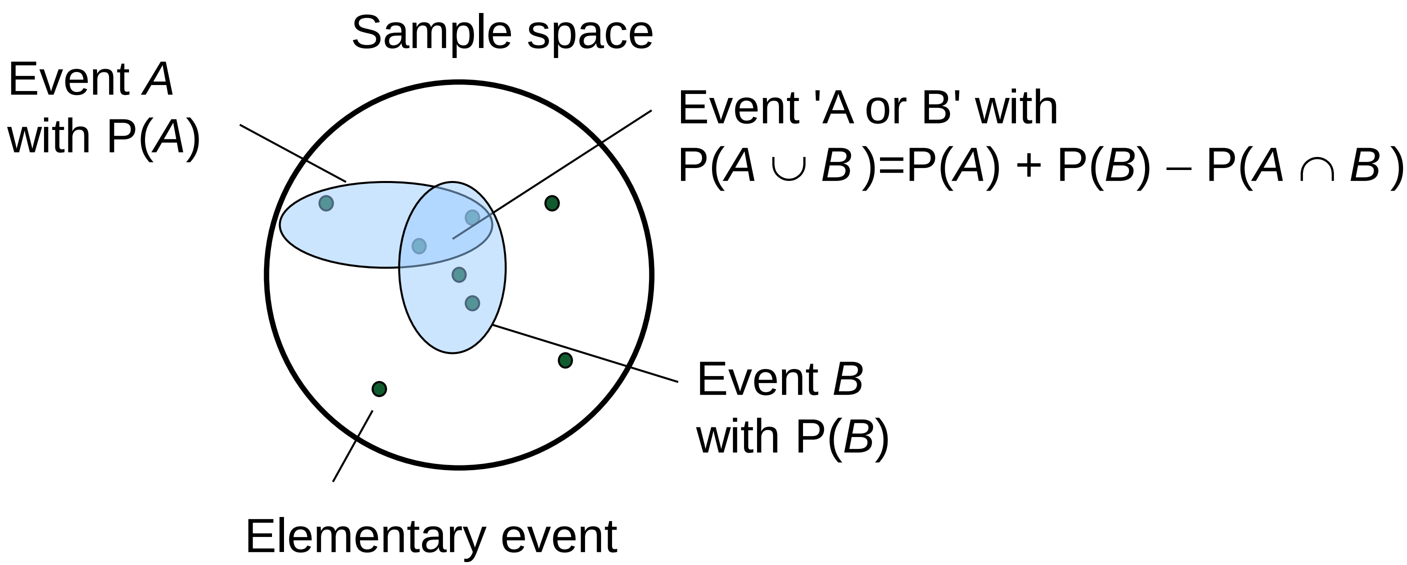
- (5 + 15 + 20) / 50 = 0.8 $P(A\cap C)+P(A\cap\bar{C})+P(\bar{A}\cap\bar{C})$
- (35 + 20 – 15) / 50 = 0.8 $P(\bar{C})+P(A)-P(\bar{C}\cap A)$
- (35 + 5) / 50 = 0.8 $P(\bar{C})+P(A\cap C)$

V. Addition Rule

The events are mutually exclusive: Let A and B be arbitrary events of a random experiment. The probability for the event $A \cup B$ can be calculated by summing up P(A) and P(B).

The events are not mutually exclusive: The probability of the event $A \cup B$ can be calculated by summing up P(A) and P(B). However, in doing so the area $A \cap B$ is counted twice. Thus, it has to be subtracted again:

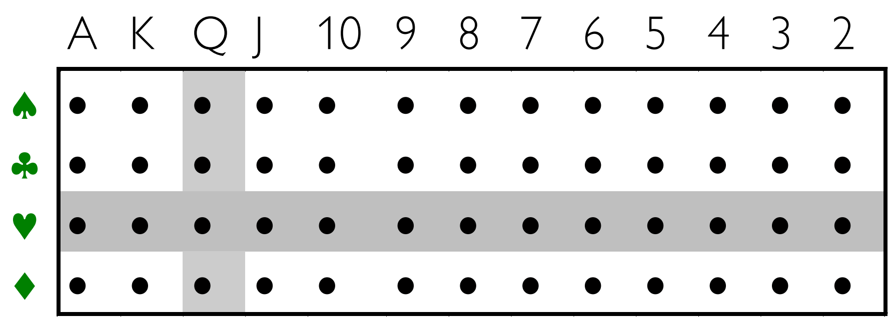
$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



V. Addition theorem (example)

Drawing a card. What is the probability of getting a queen (Q) or hearts (♥)?

P(Q or ♥) = ?



Number of elements in the state space: 52

An elementary event (a particular card) has the probability of 1/52.

$\Rightarrow$ P(Q) = 4/52.

$\Rightarrow$ P(♥) = 13/52.

$\Rightarrow$ P(♥ $\cap$ Q) = 1/52. It follows: P(Q $\cup$ ♥) = 4/52 + 13/52 - 1/52 = 16/52

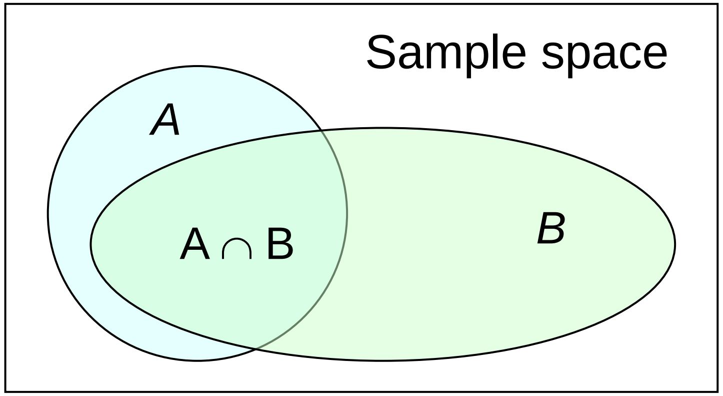
VI. Conditional probability

Occasionally, one is interested in the occurence of certain events conditional on other events. P(*B* | *A*) is the conditional probability that event *B* occurs given that the event *A* has already occurred.

P(*B* | *A*) **Conditional probability** of event *B* given *A*

P(*A*∩*B*) **Joint probability** of events *A* and *B*

P(*A*) **Marginal probability** of event *A* (with P(*A*) > 0):



P(*B* | *A*) = P(*A*∩B) / P(*A*)

VII. Multiplication Rule

Sometimes the probabilities $P(B \mid A)$ and $P(A)$ are known, while the probability $P(A \cap B)$ is to be calculated. For this, the multiplication rule can be used. The equation follows from the conditional probability formula.

$$\mathbf{P(A \cap B) = P(B \mid A) \cdot P(A)}$$
$$\mathbf{P(A \cap B) = P(A \mid B) \cdot P(B)}$$

Dependent events: The occurrence of one event affects the probability of the occurrence of another event.

Independent events: The occurrence of one event has no influence on the probability of occurrence of the other event.

For *independent* events: $\mathbf{P(A \cap B) = P(A) \cdot P(B)}$

Events A and B are *stochastically independent* if:
$\mathbf{P(B \mid A) = P(B \mid \overline{A}) = P(B)}$ **oder** $\mathbf{P(A \mid B) = P(A \mid \overline{B}) = P(A)}$

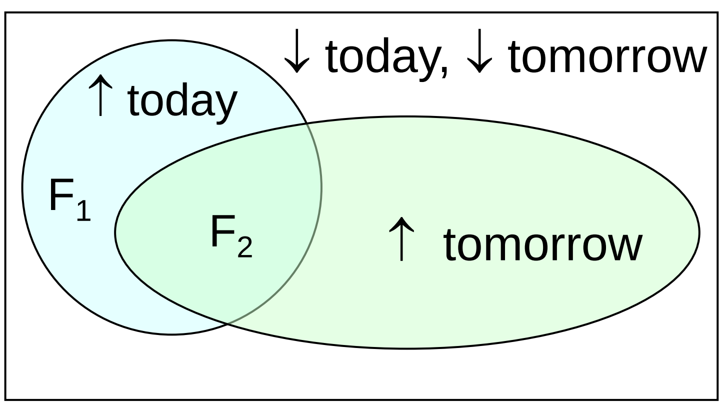
VII. Tree Diagrams (Example)

What is the probability of the price of a security to rise today and tomorrow?

$$P(\uparrow \text{today}) = P(\downarrow \text{today}) = 0.5$$

Probability "↑ tomorrow" is 0.6 if the stock has risen today. Thus, the **conditional probability** is given by**:**

$$P(\uparrow \text{tomorrow} \mid \uparrow \text{today}) = 0.6$$



$F_1 \equiv P(\uparrow \text{today})$
$F_2 \equiv P(\uparrow \text{today and} \uparrow \text{tomorrow})$

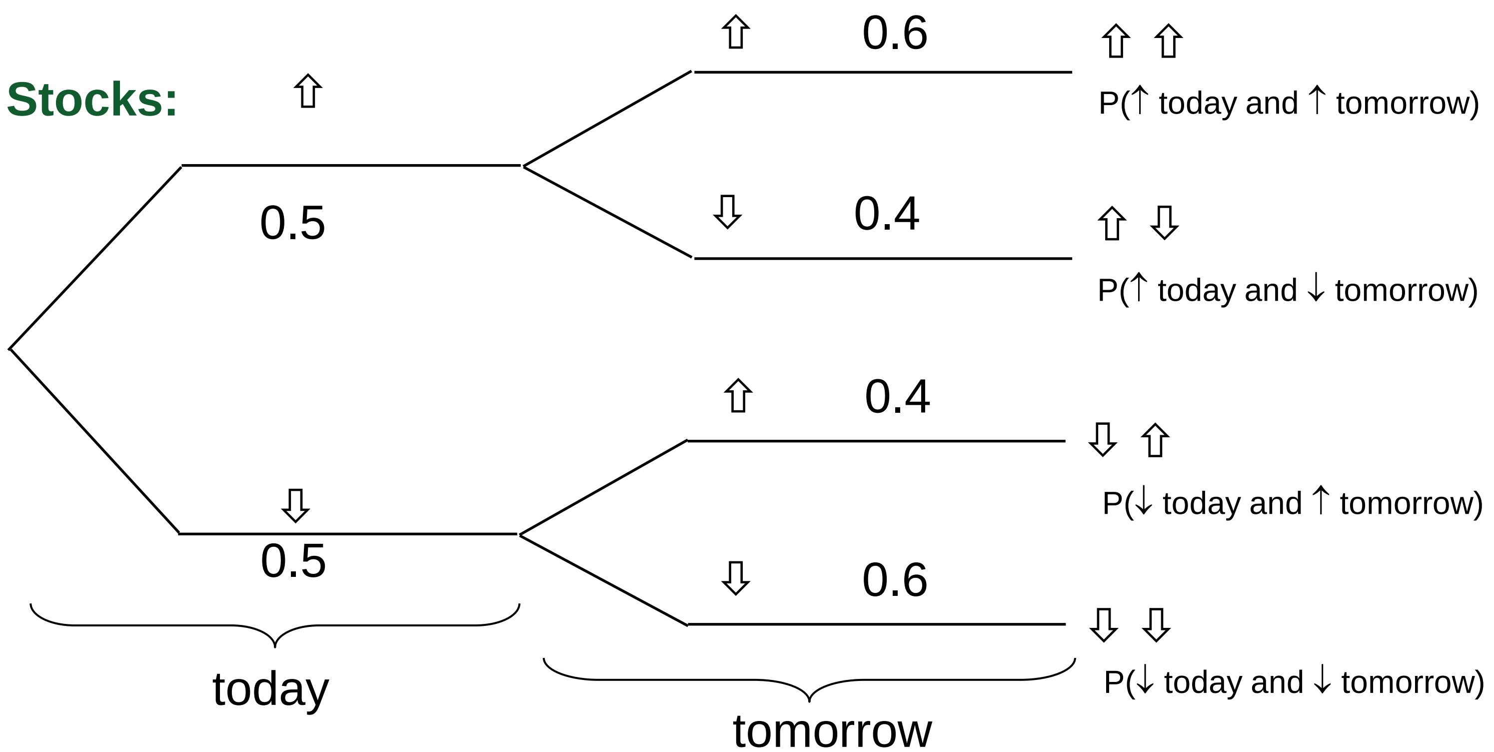
$$P(\uparrow \text{tomorrow} \mid \uparrow \text{today}) = F_2 \mid F_1 = 0.6$$

$$P(\uparrow \text{today and} \uparrow \text{tomorrow})$$
$$= P(\uparrow \text{tomorrow} \mid \uparrow \text{today}) \cdot P(\uparrow \text{today})$$
$$= 0.6 \text{ x } 0.5 = 0.3$$

VII. Tree Diagram

Tree diagrams visually summarize the occurrence or non-occurrence of two or more events. They are especially useful for events that occur in sequence. The probabilities of the possible events on each step are plotted at the branches.

Example Stocks:



VII. Tree Diagram

P(↑ today, ↓ tomorrow) = P(↑ today) · P(↓ tomorrow | ↑ today)

P(↓ tomorrow | today ↑) = 1 - P(↑ tomorrow | ↑ today) = 0.4

P(↑ today, ↓ tomorrow) = 0.5 · 0.4 = 0.2

Analogous:

P(↓ today, ↑ tomorrow) = P(↓ today) · P(↑ tomorrow | ↓ today)

P(↑ tomorrow | ↓ today) = 1 - P(↓ tomorrow | ↓ today) = 0.4

P(↓ today, ↑ tomorrow) = 0.5 · 0.4 = 0.2

VII. tree diagram

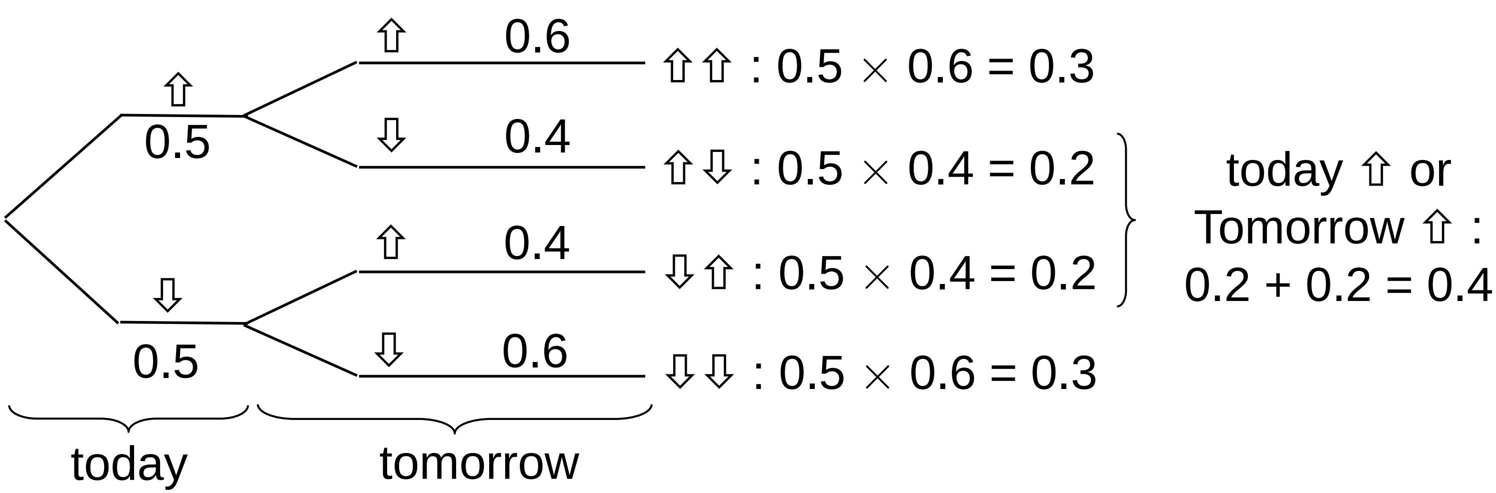
= P(↑ today, ↓ tomorrow) + P(↓ today, ↑ tomorrow)

→ The two events "↑ today, ↓ tomorrow" and "↓ today, ↑ tomorrow" are mutually exclusive. Therefore, the probabilities can be added together.

That's why:

= P(↑ today, ↓ tomorrow) + P(↓ today, ↑ tomorrow)

= 0.2 + 0.2 = 0.4



VII. Tree Diagram (R-Example 2)

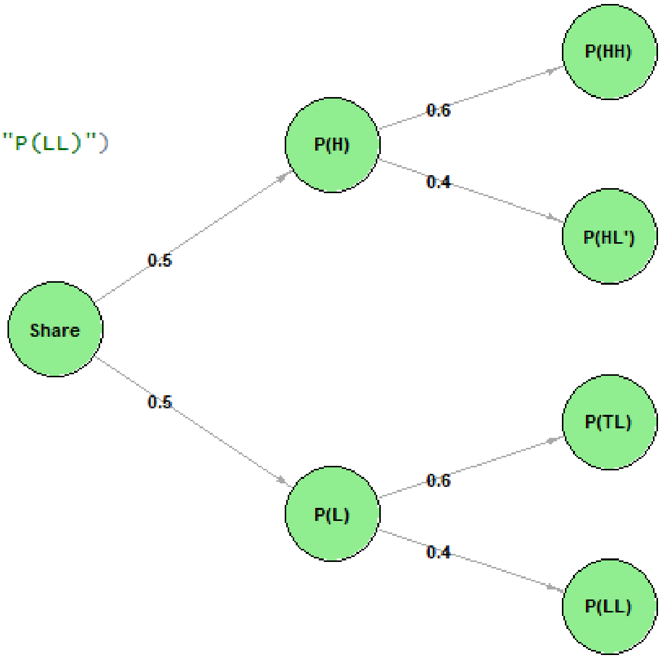
Open the file "L3-Example_2.R" in R-Studio and reproduce the R-Code.

```
# the "igraph" package must be installed for this example
# install.packages('igraph')
library(igraph)

# determine the size of the tree:
g <- graph.tree(n = 2 ^ 3 - 1, children = 2)
# labeling of the nodes:
node_labels <- c("Share", "P(H)", "P(L)", "P(HH)", "P(HL')", "P(TL)", "P(LL)")
# labeling of the paths:
edge_labels <- c("0.5", "0.5", "0.6", "0.4", "0.6", "0.4")

# definition des Layouts
layout <- layout.reingold.tilford(g)
# rotate the graph (we want a horizontal tree diagram)
layout <- -layout[,2:1]

# drawing the tree diagram
plot(g,
     layout = layout,                    # draw graphs as tree
     vertex.size = 35,                   # size of the nodes
     vertex.color = 'Lightgreen',        # color of the nodes
     vertex.label = node_labels,         # labelling nodes
     vertex.label.cex = .7,              # size of node labels
     vertex.label.family = "Helvetica",  # font of node labels
     vertex.label.font = 2,              # highlighting node labels
     vertex.label.color = 'Black',       # color node labels
     edge.label = edge_labels,           # path labeling
     edge.label.cex = .7,                # size path labels
     edge.label.family = "Helvetica",    # fonth path labels
     edge.label.font = 2,                # highlighting path labels
     edge.label.color = 'Black',         # color path labels
     edge.arrow.size = .5,               # arrow size
     edge.arrow.width = .5               # arrow thickness
)
```



Appendix: Combinatorics

Combinatorics analyzes in how many different ways a given number of elements can be arranged and combined. The topic of combinatorics is closely related to the concept of probability. Mostly we are interested in different combinations of possible events and the determination of the corresponding probability of randomly choosing a particular combination. It is important that the postulation of equal probabilities is fulfilled.

Permutations refer to the number of different ways in which objects can be arranged **in order**. Each item can appear only once and each order of arrangement is a separate permutation. **Combinations** consider the possible set of objects **independent of the order** in which the objects are arranged.

Appendix: Combinatorics

The symbol ***n!*** is called a factorial and is defined as

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

Special cases:

$$1! = 1$$

$$0! := 1$$

Principle of multiplication: if there are *m* ways a first event can happen and *n* ways in which a second event can happen, the total number of possibilities is $m \cdot n$

- Following an event that can happen in n_1 ways, the second event can happen n_2 ways, the total number of both ways can happen is $n_1 n_2$
- If each of *k* independent events can occur in *n* different ways, the total number of possibilities is n_k

Appendix: Combinatorics (Examples)

License plates: The car license plate contains 6 different combinations of numbers or letters. All 26 letters and 10 numbers are possible. How many different possibilities are there?

Solution: $36^6 = 2{,}176{,}782{,}336$ different possibilities

Government formation: The newly elected president must nominate his cabinet. From 3 possible candidates he can appoint the new Minister of Finance. He must appoint the new Foreign Minister from 5 other suitable candidates. How many different possibilities are there to fill the ministerial offices?

Solution: The President has a total of 15 different options.

Appendix: Combinatorics (Example)

Computer setup: A computer can be combined in the following options to meet individual customer needs. These options include:

- 2 different processors,
- 3 different operating systems,
- 4 levels of memory,
- 4 different hard drives,
- 10 different monitors.

How many different types of the computer must the company be able to build?

Solution: $960 = 2 \cdot 3 \cdot 4 \cdot 4 \cdot 10$ possible combinations

Appendix: Combinatorics

n different elements can be arranged differently as follows (**significant sequence**):

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$$

Each order of *n* different elements is called **permutation**. Note that each element can appear **only once**!

Generalization: n elements, arranged **in order**, and the elements n_i belong to group *i* (with *m* different groups, i.e., $n=n_1+n_2+\ldots+n_m$)

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_m!}$$

Example: Number of permutations of the $n = 10$ letters of the word *STATISTICS*:

$$\frac{10!}{3!3!1!2!1!} = 50,400$$

Appendix: Combinatorics

Choose *r* from *n* elements...

… with consideration of the order (permutation):

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$$

… without consideration of the order (combination):

$$\frac{n!}{r!(n-r)!} = \frac{n \cdot (n-1) \ldots (n-r+1) \cdot (n-r)!}{r!(n-r)!}$$

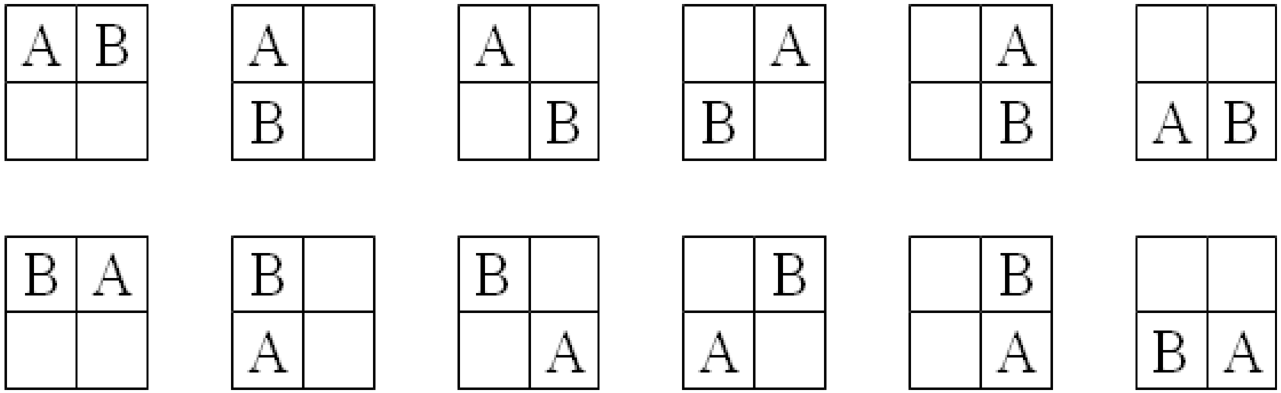
$$= \frac{n(n-1) \ldots (n-r+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (r-1) \cdot r} = \binom{n}{r}$$

Appendix: Combinatorics (Example)

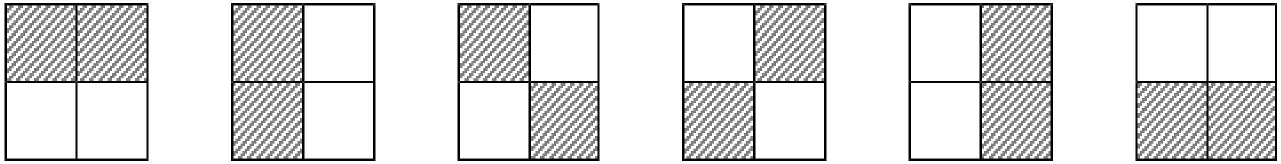
Place $r = 2$ figures A & B on a mini chessboard with

$n = 2 \cdot 2 = 4$ fields ...

- ... **with** consideration of the order:



- ... **without** consideration of the order:



Appendix: Combinatorics (Examples)

Envelopes: 5 letters are randomly put into 5 different envelopes; how many permutations are possible?

$$5! = 120$$

Numbers: How many 6-digit numbers with different digits (i.e., with consideration of the order) can be formed from the numbers 1,2,3,…,9?

$$\frac{9!}{(9-6)!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 60,480$$

Card game: 3 cards are drawn from a card deck (36 cards). How many different results are possible?

$$\binom{36}{3} = \frac{36!}{3! \cdot 33!} = \frac{36 \cdot 35 \cdot 34}{3 \cdot 2 \cdot 1} = 7,140$$

Appendix: Combinatorics

The number of ways to select an ordered sample of *k* subjects from a population that has *n* distinguishable elements is given by:

- n^k if sampling is done **with** replacement
- $n(n-1)(n-2)...(n-k+1)$ if sampling is done **without** replacement

the number of ways to select an unordered sample of *k* subjects from a population that has *n* distinguishable members is given by:

- $\frac{(n-1+k)!}{(n-1)!k!}$ if sampling is done **with** replacement
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ if sampling is done **without** replacement

Appendix: Combinatorics (R-Example)

Open the file "L3-Example_3.R" in R-Studio and reproduce the R-Code.

```
# example: balls
# there are 10 balls numbered from 1 to 10 in a jar.
# two balls are drawn without putting them back. how many possible combinations are there?
choose(10,2)
factorial(10)/(factorial(2)*factorial(10-2))
# both calculation types yield the same result.

# example: cards
# how many different 5 card hands can be drawn from a 52 card deck?
choose(52,5)
factorial(52)/(factorial(5)*factorial(52-5))
# both calculation types yield the same result.
```