Exercise Sheet
"Numerical Methods for ODEs"

The classroom exercises will be discussed on **September 25, 2023**.

Classroom Exercise 4.1 [Runge method]
Consider the IVP

$$x'(t) = y(t), \quad y'(t) = 4x(t) - 2ty(t), \quad x(0) = \frac{1}{5}, \, y(0) = \frac{1}{2}.$$

Use one step of the Runge (modified Euler) method to obtain approximations for $x(0.1)$ and $y(0.1)$.

Classroom Exercise 4.2 [Runge-Kutta]
Consider the following Butcher array of an explicit Runge-Kutta scheme:

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & \frac{1}{4} & \frac{3}{4} & \\ \hline & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{array}$$

a) Consider the IVP

$$x'(t) = \lambda x(t), \qquad x(0) = 1$$

and derive an explicit formula for $\mathbf{\Psi}^{\tau,0}\mathbf{x}$ using one step of the Runge-Kutta scheme given above.

b) Consider the error $\mathbf{\Psi}^{\tau,0}\mathbf{x} - x(\tau)$. How does this error behaves for $\tau \to 0$? What does this mean for the consistency order of the Runge-Kutta scheme?

Classroom Exercise 4.3 [Error estimation]
Consider the convergence theorem (Theorem 2.11) from the lecture for the IVP $x'(t) = -\lambda x(t)$, $x(0) = x_0 \in \mathbb{R}$ on $[0, T]$. Given a truncation error tolerance $\epsilon > 0$ how should τ_Δ be chosen? Determine the (smallest) constants p, C, L for the Euler scheme of the above ODE with $T = 1$, $\epsilon = 0.1$, $x_0 = 1$, $\lambda = 1$ and an equidistant grid.

Programming Exercise 4.4 [Runge-Kutta methods](4 Points)
Implement an explicit Runge-Kutta solver in MATLAB. Try to implement a function that has an input/output structure similar to the one used by MATLAB's ODE solvers and which contains the Butcher array as an additional input, i.e.,

```
[T,Y] = explicit_Runge_Kutta(odefun,tspan,y0,A,b,c).
```

Your code should also work for systems of ODEs. Apply your function to the following IVPs

1. $y' = -5y;\ y(0) = 1; t \in [0, 3]$,
2. $y'' = -y - y';\ y(0) = 1,\ y'(0) = 0, t \in [0, 10]$,

and use it with the Butcher arrays for the Euler, Runge and standard Runge-Kutta method.

Compare for each method the different approximate solutions for the equidistant step sizes $\tau \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\}$ and the exact solutions in one plot.

To be handed in: Executable Matlab code and figures of the programming exercises via ILIAS.