

Lemma (3.2 Itô's isometry simple version). For $f \in \mathcal{H}_0^2$ we have

$$\|f\|_{\mathcal{H}^2} = \|I(f)\|_{L^2}.$$

Proof. We have

$$\begin{aligned} \|I(f)\|_{L^2}^2 &= \mathbb{E}[(\sum_{i=1}^n a_i(B_{t_i} - B_{t_{i-1}}))^2] \\ &= \mathbb{E}[\sum_{i=1}^n a_i^2(B_{t_i} - B_{t_{i-1}})^2 + \sum_{i \neq j} a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] \\ &= \mathbb{E}[\sum_{i=1}^n a_i^2(B_{t_i} - B_{t_{i-1}})^2] \\ &= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[a_i^2(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\ &= \sum_{i=1}^n \mathbb{E}[a_i^2 \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\ &= \sum_{i=1}^n \mathbb{E}[a_i^2 \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2]] \\ &= \sum_{i=1}^n \mathbb{E}[a_i^2] \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2] \\ &= \sum_{i=1}^n \mathbb{E}[a_i^2](t_i - t_{i-1}) \end{aligned}$$

Since w.l.o.g take $i < j$

$$\begin{aligned} \mathbb{E}[a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] &= \mathbb{E}[\mathbb{E}[a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{i-1}}]] \\ &= \mathbb{E}[a_i \underbrace{\mathbb{E}[(B_{t_i} - B_{t_{i-1}})]}_{=0} \mathbb{E}[a_j (B_{t_j} - B_{t_{j-1}})(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{i-1}}]] \end{aligned}$$

But

$$\|f\|_{\mathcal{H}^2}^2 = \mathbb{E}[\int_0^T f^2 ds] = \sum_{i=1}^n \mathbb{E}[a_i^2](t_i - t_{i-1}).$$

□

Proposition (3.5). For every $f \in \mathcal{H}^2$ there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{H}_0^2$

such that

$$\|f_n - f\|_{\mathcal{H}^2} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. The rough outline of the proof can be split up as follows

1. Show that we can assume f is bounded
2. Show that we can assume f is bounded adapted and continuous
3. Construct simple sequence that approximates f

For step 1 we can simply consider that for any (potentially) unbounded function f

$$f_n = -n \vee (f \wedge n).$$

is bounded and apply DCT

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{\mathcal{H}^2} = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T (f_n - f)^2 dt \right] = \mathbb{E} \left[\int_0^T \lim_{n \rightarrow \infty} (f_n - f)^2 dt \right] = 0.$$

Thus we may assume f bounded

For step 2 we can use that we can get the height of a rectangle by dividing by its base s.t.

$$f_n(\cdot, t) := \frac{1}{(t - (t - \frac{1}{n})_+)} \int_{(t - \frac{1}{n})_+}^t f(\cdot, s) ds = n \int_{(t - \frac{1}{n})_+}^t f(\cdot, s) ds.$$

Now we note that since we work with random variables we condition ω on the set where

$$\lim_{n \rightarrow \infty} f_n(\omega, t) = f(\omega, t).$$

We need this set to have measure 1, such that

$$A := \{(\omega, t) \in \Omega \times [0, T] : \lim_{n \rightarrow \infty} f_n(\omega, t) \neq f(\omega, t)\}.$$

Has measure zero with respect to $\mathbb{P} \otimes \lambda$, we have by the fundamental theorem of calculus that

$$\lambda(\{t \in [0, T] : (\omega, t) \in A\}) = 0.$$

Or rather Lebesgue Differentiation theorem, otherwise we'd need the condition that $f(\omega, \cdot)$ only has countable many discontinuities. Thus we get that we may assume f is bounded and continuous.

We now construct our simple function f_n as

$$f_{n,s}(\omega, t) := f(\omega, (s + \varphi_n(t - s)_+)).$$

where

$$\varphi_n = \sum_{j=1}^{2^n} \frac{j-1}{2^n} \mathbb{1}_{(\frac{j-1}{2^n}, \frac{j}{2^n}]}$$

makes our time interval discrete (standard argument really), then we wanna show

$$\|f_n - f\|_{\mathcal{H}^2} = \mathbb{E}[\int_0^T |f_{n,s} - f|^2 dt] \rightarrow 0.$$

We have

$$\begin{aligned} \mathbb{E}[\int_0^T |f_{n,s} - f|^2 dt] &\leq \mathbb{E}[\int_0^T \int_0^1 |f_{n,s} - f|^2 ds dt] \\ &= \mathbb{E}[\int_0^T \int_0^1 |f(\cdot, (s + \varphi_n(t-s))_+) - f|^2 ds dt] \\ &= \sum_{j \in \mathbb{Z}} \mathbb{E}[\int_0^T \int_{[t-\frac{j}{2^n}, t-\frac{j-1}{2^n}] \cap [0,1]} |f(\cdot, (s + \frac{j-1}{2^n}) - f|^2 ds dt] \\ &\leq (2^n + 1) 2^{-n} \int_{(0,1]} \mathbb{E}[\int_0^T |f(\cdot, t - 2^{-n}h) - f(\cdot, t)|^2 dt] dh. \end{aligned}$$

Where we can show that the Expectation term goes to 0 \square

Theorem (3.7.). For any $f \in \mathcal{H}^2$ there is a continuous martingale $X = (X_t)_{t \in [0, T]}$ with respect to \mathcal{F}_t such that for all $t \in [0, T]$

$$X_t = I(f \mathbb{1}_{[0, t]}).$$

Proof. We first consider the simple case with

$$f_n = \sum_{i=0}^{m_n-1} a_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}.$$

Then

$$\begin{aligned} \mathbb{E}[X_t^n - X_s^n | \mathcal{F}_s] &= \mathbb{E}[\sum_{t_i > s} a_i (B_{t_{i+1}} - B_{t_i})] \\ &= 0. \end{aligned}$$

Our end goal is to use a triangular inequality to use the simple case to bound

the normal one i.e.

$$\begin{aligned}
|\mathbb{E}[X_t - X_s | \mathcal{F}_s]| &= |\mathbb{E}[X_t - X_t^n + X_t^n - X_s + X_s^n - X_s^n | \mathcal{F}_s]| \\
&\leq |\mathbb{E}[X_t - X_t^n | \mathcal{F}_s]| + \underbrace{|\mathbb{E}[X_t^n - X_s^n | \mathcal{F}_s]|}_{=0} + |\mathbb{E}[X_s^n - X_s | \mathcal{F}_s]| \\
&= |\mathbb{E}[X_t - X_t^n | \mathcal{F}_s]| + |\mathbb{E}[X_s^n - X_s | \mathcal{F}_s]| \\
&\stackrel{\text{Jen}}{\leq} \mathbb{E}[|X_t - X_t^n| | \mathcal{F}_s] + \mathbb{E}[|X_s^n - X_s| | \mathcal{F}_s]
\end{aligned}$$

We consider $A \in \mathcal{F}_s$

$$\begin{aligned}
\mathbb{E}[|X_t - X_t^n| \mathbb{1}_A] + \mathbb{E}[|X_s^n - X_s| \mathbb{1}_A] &\leq \mathbb{E}[|X_t - X_t^n|] + \mathbb{E}[|X_s^n - X_s|] \\
&\leq 2 \cdot \|f - f_n\|_{\mathcal{H}^2}.
\end{aligned}$$

So we have $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$

For $(f_n)_{n \in \mathbb{N}} \subset \mathcal{H}_0^2$ we have

$$\lim_{n \rightarrow \infty} f_n = f \in \mathcal{H}^2.$$

And

$$\lim_{n \rightarrow \infty} I(f_n) = I(f) \in L^2.$$

Which should be equivalent to for fixed $t \in [0, T]$

$$\begin{aligned}
X_t^n &\xrightarrow{L^2} X_t \\
I(f_n \mathbb{1}_{[0, t]}) &\xrightarrow{L^2} I(f \mathbb{1}_{[0, t]}).
\end{aligned}$$

We need to make the argument uniform in $t \in [0, T]$, which is Step 2 in the script i guess.

We have

$$\mathbb{P}(\sup_{0 \leq t \leq T} |X_t^n - X_t^m| \geq \varepsilon) \leq \varepsilon^{-2} \mathbb{E}[|X_T^n - X_T^m|^2] = \varepsilon^{-2} \|f_n - f_m\|_{\mathcal{H}^2}^2.$$

since for all $p > 1$

$$\mathbb{P}(\sup_{0 \leq t \leq T} |X_t| \geq \varepsilon) \leq \frac{1}{\varepsilon^p} \mathbb{E}[|X_T|^p].$$

By choosing a subsequence we can get

$$\mathbb{P}(\sup_{0 \leq t \leq T} |X_t^n - X_t^m| \geq 2^{-k}) \leq 2^{2k} \mathbb{E}[|X_T^n - X_T^m|^2] = \varepsilon^{-2} \|f_n - f_m\|_{\mathcal{H}^2}^2 \leq 2^{-k}.$$

Then we can apply Borel-Cantelli since

$$\sum_{k=0}^{\infty} \mathbb{P}(\sup_{0 \leq t \leq T} |X_t^n - X_t^m| \geq 2^{-k}) < \infty.$$

and get $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and X^{n_k} is a pathwise cauchy sequence. \square

Proposition (3.10). Let $f \in \mathcal{H}^2$ and ν be a stopping time satisfying

$$f \mathbb{1}_{[0, \nu]} = 0.$$

The integral process $X = (X_t)_{t \in [0, T]}$ with $X_t = \int_0^t f(\cdot, s) dB_s$, the fulfills

$$X \mathbb{1}_{[0, \nu]} = 0.$$

In particular for two functions $f, g \in \mathcal{H}^2$ with $f \mathbb{1}_{[0, \nu]} = g \mathbb{1}_{[0, \nu]}$ the integral processes coincide on $[0, \nu]$

Remark. This proposition is mostly used to prove that the same holds for $\mathcal{H}_{\text{loc}}^2$ functions as well since this allows us to use localizing sequences τ_m and use the fact that on

$$\{\tau_m = T\}.$$

The processes must coincide.

Proof. The proof follows similarly to before where we first prove the simple case, for that suppose

$$\begin{aligned} X_t &= I(f \mathbb{1}_{[0, t]}) \\ Y_t &= I(f \mathbb{1}_{[0, \nu]} \mathbb{1}_{[0, t]}). \end{aligned}$$

Then it suffices to consider the simplification $f = a \mathbb{1}_{[r, s]}$ for $0 \leq r < s \leq T$.

The important thing to note now is that

$$Y_t = I(f \mathbb{1}_{[0, \nu]} \mathbb{1}_{[0, t]}) = \int_0^t \underbrace{(f \mathbb{1}_{[0, \nu]})}_{\notin \mathcal{H}_0^2}.$$

For a function $h = f \mathbb{1}_{[0, \nu]}$ to be in \mathcal{H}_0^2 there needs to be a representation

$$h = \sum_{i=0}^{n-1} a_i \mathbb{1}_{[t_{i-1}, t_i]}.$$

But ν is continuous, such that there cannot exist such a representation, which

means we first need to consider the discrete stopping time ν^n as follows

$$s_{i,n} = r + (s - r) \frac{i}{2^n}$$

$$\nu^n = \sum_{i=0}^{2^n-1} s_{i+1,n} \mathbb{1}_{(s_{i,n}, s_{i+1,n}]}(\nu).$$

We show

$$\begin{aligned} f \mathbb{1}_{[0, \nu^n]} &= f - f \mathbb{1}_{\nu^n, T} \\ &= f - f \sum_{i=0}^{2^n-1} \mathbb{1}_{(s_{i,n}, s_{i+1,n}]}(\nu) \mathbb{1}_{(s_{i+1,n}, T]} \in \mathcal{H}_0^2. \end{aligned}$$

then by definition of the integral for simple functions it follows

$$Y_t^N = \int_0^t f(\cdot, s) \mathbb{1}_{[0, \nu^n]}(u) dB_u = a(B_{s \wedge \nu^n \wedge t} - B_{r \wedge \nu^n \wedge t}).$$

And by continuity of B we can do

$$Y_t = \lim_{n \rightarrow \infty} Y_t^n = a(B_{s \wedge \nu \wedge t} - B_{r \wedge \nu \wedge t}).$$

But this is clearly $X_t \mathbb{1}_{[0, \nu]}$, it follows

$$X \mathbb{1}_{[0, \nu]} = Y \mathbb{1}_{[0, \nu]}.$$

Note that for $X \mathbb{1}_{[0, \nu]}$ there is no difficulty since we can first apply the definition of the simple integral, and then consider the stopping time

For general $f \in \mathcal{H}^2$ we choose $(f_n) \subset \mathcal{H}_0^2$ such that

$$\|f - f_n\|_{\mathcal{H}^2} \rightarrow 0.$$

then we already know that $X^n = Y^n$ such that

$$X \mathbb{1}_{[0, \nu]} = \lim_{n \rightarrow \infty} X^n \mathbb{1}_{[0, \nu]} = \lim_{n \rightarrow \infty} Y^n \mathbb{1}_{[0, \nu]} = Y \mathbb{1}_{[0, \nu]}.$$

□

Proposition (3.13). For every $f \in \mathcal{H}_{\text{loc}}^2$ there is a localizing sequence $(\nu_n)_{n \in \mathbb{N}}$

Proof. We want ν_n such that for $f \in \mathcal{H}_{\text{loc}}^2$ it holds

$$f \mathbb{1}_{[0, \nu_n]} \in \mathcal{H}^2.$$

and

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} \{\nu_n = T\}\right) = 1.$$

We already have

$$\mathbb{P}\left(\int_0^T f^2(\cdot, s)ds < \infty\right) = 1.$$

a natural choice of localizing sequence is then

$$\nu_n = \inf\{t \in [0, T] : \int_0^t f^2 ds \geq n\}.$$

we have

$$\|f\mathbb{1}_{[0, \nu_n]}\|_{\mathcal{H}^2} < \infty.$$

since ν_n conditions on a set such that we only have ω where we are bounded. Since f is adapted the hitting time is a stopping time, and we consider

$$\bigcup_{n=1}^{\infty} \{\nu_n = T\} = \left\{\int_0^T f^2 < \infty\right\} = 1.$$

□

Definition (3.14). Let $f \in \mathcal{H}_{\text{loc}}^2$ and ν_n be a localizing sequence for f . The Itô integral process $(\int_0^t f(\cdot, s)dB_s)$ is defined as the continuous process $X = (X_t)_{t \in [0, T]}$ such that

$$\int_0^t f(\cdot, s)dB_s = X_t = \lim_{n \rightarrow \infty} \int_0^t f(\cdot, s)\mathbb{1}_{[0, \nu_n]}(s)dB_s \quad \mathbb{P}\text{-a.s..}$$

Theorem (3.15). For $f \in \mathcal{H}_{\text{loc}}^2$ there exists a continuous local martingale $(X_t)_{t \in [0, T]}$ such that for any localizing sequence $(\nu_n)_{n \in \mathbb{N}}$ of f it holds

$$\int_0^t f(\cdot, s)\mathbb{1}_{[0, \nu_n]}(s)dB_s \xrightarrow{n \rightarrow \infty} X_t.$$

for $t \in [0, T]$. In particular this is well defined and (X_t) does not depend on the choice of localizing sequence.

Proof. Remember that any proof involving local Martingales or $\mathcal{H}_{\text{loc}}^2$ processes we need to work through a localizing sequence. Let $f \in \mathcal{H}_{\text{loc}}^2$ and (ν_n) be a corresponding localizing sequence (it exists), define the localized integral process

$$X_t^n = \int_0^t f(\cdot, s)\mathbb{1}_{[0, \nu_n]}(s)dB_s.$$

we first show that X_t^n has a continuous limit (X_t) i.e

$$X_t = \lim_{n \rightarrow \infty} X_t^n$$

$$X_t = \lim_{n \rightarrow \infty} \int_0^t f(\cdot, s) \mathbb{1}_{[0, \nu_n]}(s) dB_s.$$

The expression above is only useful if we view it on the set such that

$$t \mapsto X_t^n(\omega) \text{ is continuous}$$

$$\min\{n \in \mathbb{N} : \nu_n(\omega) = T\} < \infty.$$

The first gives us that

$$t \mapsto \int_0^t f(\cdot, s) \mathbb{1}_{[0, \nu_n]}(s) dB_s.$$

Show X is continuous

For $\varepsilon > 0$ and $t \in [0, T]$ find $\delta > 0$ such that

$$s \in B_\delta(t) \Rightarrow |X_t(\omega) - X_s(\omega)| < \varepsilon.$$

suppose w.l.o.g $s < t$

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &= \left| \lim_{n \rightarrow \infty} \left(\int_0^t f \mathbb{1}_{[0, \nu_n]} - \int_0^s f \mathbb{1}_{[0, \nu_n]} \right) \right| \\ &= \left| \lim_{n \rightarrow \infty} \int_s^t f \mathbb{1}_{[0, \nu_n]} dB_s \right| \\ &\leq \lim_{n \rightarrow \infty} \int_s^t |f \mathbb{1}_{[0, \nu_n]}| dB_s. \end{aligned}$$

Since X_t^n is continuous the integral can be made arbitrarily small and $f \cdot \mathbb{1}_{[0, \nu_n]} \in \mathcal{H}^2$. Thus the limit exists and is continuous, for the independence of localizing sequence we get immediately by prop 3.10 (identity) that for $\tau_n := \nu_n \wedge \tilde{\nu}_n$

$$X^n \mathbb{1}_{[0, \tau_m]} = \tilde{X}^n \mathbb{1}_{[0, \tau_m]}.$$

where

$$\tilde{X}^n = \int_0^\cdot f(\cdot, s) \mathbb{1}_{[0, \tilde{\nu}_n]}(s).$$

then

$$\lim_{n \rightarrow \infty} X^n = \lim_{n \rightarrow \infty} \tilde{X}^n.$$

on $[0, \tau_m]$ and $\tau_m \uparrow T$.

It remains to show that (X_t) is a local martingale, this is simply to show that $f \in \mathcal{H}^2$ then we already know that $\int f$ is a martingale,

$$\sigma_n = \inf\{t \in [0, T] : \int_0^t f^2(\cdot, s) ds \geq n\} \wedge T.$$

clearly this is a localizing sequence for f then

$$X_{t \wedge \sigma_n} = \int_0^t \underbrace{f(\cdot, s) \mathbb{1}_{[0, \sigma_n]}}_{\in \mathcal{H}^2} dB_s.$$

and we conclude with Theorem 3.7 □

Theorem (3.17). Let $f, g \in \mathcal{H}_{\text{loc}}^2$ and ν be a stopping time such that

$$f \mathbb{1}_{[0, \nu]} = g \mathbb{1}_{[0, \nu]}.$$

then

$$\int_0^t f(\cdot, s) dB_s \mathbb{1}_{[0, \nu]} \stackrel{\mathbb{P}}{=} \int_0^t g(\cdot, s) dB_s \mathbb{1}_{[0, \nu]}.$$

Proof. You know the drill, localizing sequence and then apply the result for the normal, we have that

$$\tau_n = \inf\{t \in [0, T] : \int_0^t f(\cdot, s)^2 ds \geq n \vee \int_0^t g(\cdot, s)^2 ds \geq n\} \wedge T.$$

Clearly $f \mathbb{1}_{[0, \tau_n]} \in \mathcal{H}^2$ since we stop the moment one of the integral processes exceeds n then we have

$$\begin{aligned} X^n &= \int_0^\cdot \underbrace{f(\cdot, s) \mathbb{1}_{[0, \tau_n]}}_{\in \mathcal{H}^2} \\ Y^n &= \int_0^\cdot \underbrace{g(\cdot, s) \mathbb{1}_{[0, \tau_n]}}_{\in \mathcal{H}^2}. \end{aligned}$$

then prop 3.10 gives

$$X^n \mathbb{1}_{[0, \nu]} = Y^n \mathbb{1}_{[0, \nu]}.$$

Theorem 3.15 gives us the existence of the limit

$$\lim_{n \rightarrow \infty} X^n.$$

□

Theorem (3.17 Riemann sum approximation). If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous

function and $t_i = \frac{i}{n}T$ then for $n \rightarrow \infty$ we have

$$\sum_{i=1}^n f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \xrightarrow{\mathbb{P}} \int_0^T f(B_s)dB_s.$$

Proof. By Remark 3.12. we know that for any continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$

$$f(\omega, t) = g(B_t(\omega)) \in \mathcal{H}_{\text{loc}}^2.$$

This follows since for a.s. $\omega \in \Omega$ the map

$$\varphi(\omega) : [0, T] \rightarrow \mathbb{R} : t \mapsto B_t(\omega)$$

is bounded. This gives us that

$$\sup_{t \in [0, T]} |g(B_t(\omega))| = \sup_{x \leq |m|} |g(x)| \leq C.$$

Where the last inequality follows from the fact that g is continuous and attains a maximum on the compact set $[-m, m]$ then we can check that for

$$\omega \in \{\varphi \text{ is bounded}\}.$$

The integral

$$\begin{aligned} \int_0^T g^2(B_t(\omega))dt &\leq \int_0^T |g(B_t(\omega))||g(B_t(\omega))|dt \\ &\leq \underbrace{\sup_{t \in [0, T]} |g(B_t(\omega))|}_{\leq C} \int_0^T |g(B_t(\omega))|dt \\ &\leq C^2 T. \end{aligned}$$

Since $\mathbb{P}(\{\varphi \text{ is bounded}\}) = 1$ we get immediately

$$\mathbb{P}\left(\int_0^T g^2(B_t(\omega))dt < \infty\right) = 1.$$

This tells us that for any continuous f and Brownian motion B

$$f(B) \in \mathcal{H}_{\text{loc}}^2.$$

we can rewrite $\{\varphi \text{ is bounded}\}$ as a stopping time instead and get

$$\tau_m = \inf\{t \in [0, T] : |B_t| \geq m\}.$$

which is a localizing sequence for $f(B)$ since by similar argument to above we have

$$|f(B_{\cdot \wedge \tau_m})| \leq \sup_{|x| \leq m} |f(x)| < \infty.$$

and we get

$$f_m = f \cdot \mathbb{1}_{[-m, m]} = f|_{[-m, m]}.$$

Where

$$f_m(B) \in \mathcal{H}^2.$$

By definition of the Itô integral for $f \in \mathcal{H}^2$ we already get that

$$I(f_m^{(n)}) = \sum_{i=1}^n a_i(B_{t_i} - B_{t_{i-1}}) \xrightarrow{L^2} \int_0^T f_m(B_t) dt.$$

where L^2 convergence implies \mathbb{P} convergence.

Thus our goal in Step 2 is to show that in fact

$$f_m^{(n)} = \sum_{i=1}^n f_m(B_{t_{i-1}})(\omega) \mathbb{1}_{(t_{i-1}, t_i]}(s).$$

we clearly have

$$f_m^{(n)} \in \mathcal{H}_0^2.$$

Then it remains to show $f_m^{(n)} \xrightarrow{\mathcal{H}^2} f_m$

$$\begin{aligned}
\mathbb{E} \left[\int_0^T (f_m^{(n)} - f_m)^2 ds \right] &= \mathbb{E} \left[\int_0^T \left(\sum_{i=1}^n f_m(B_{t_{i-1}}) \mathbb{1}_{(t_{i-1}, t_i]}(s) - f_m(B_s) \right)^2 ds \right] \\
&= \mathbb{E} \left[\int_0^T \left(\sum_{i=1}^n f_m(B_{t_{i-1}}) \mathbb{1}_{(t_{i-1}, t_i]}(s) - \sum_{i=1}^n f_m(B_s) \mathbb{1}_{t_{i-1}, t_i} \right)^2 ds \right] \\
&= \mathbb{E} \left[\int_0^T \sum_{i=1}^n (f_m(B_{t_{i-1}}) - f_m(B_s) \mathbb{1}_{(t_{i-1}, t_i]}(s))^2 ds \right. \\
&\quad \left. + \underbrace{\sum_{i,j=1}^n \underbrace{(f_m(B_{t_{i-1}}) - f_m(B_s) \mathbb{1}_{(t_{i-1}, t_i]}(s))(f_m(B_{t_{j-1}}) - f_m(B_s) \mathbb{1}_{(t_{j-1}, t_j]}(s))}_{[t_{i-1}, t_i] \cap [t_{j-1}, t_j] = \emptyset} ds}_{=0} \right] \\
&= \mathbb{E} \left[\int_0^T \sum_{i=1}^n (f_m(B_{t_{i-1}}) - f_m(B_s) \mathbb{1}_{(t_{i-1}, t_i]}(s))^2 ds \right] \\
&\leq \mathbb{E} \left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f_m(B_{t_{i-1}}) - f_m(B_s))^2 ds \right] \\
&\leq \mathbb{E} \left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sup_{r \in [t_{i-1}, t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 ds \right] \\
&\leq \sum_{i=1}^n \mathbb{E} \left[\sup_{r \in [t_{i-1}, t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \int_{t_{i-1}}^{t_i} ds \right] \\
&\leq \frac{T}{n} \sum_{i=1}^n \mathbb{E} \left[\sup_{r \in [t_{i-1}, t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \right]
\end{aligned}$$

Where we can bound

$$\sup_{r \in [t_{i-1}, t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2.$$

further by considering that f is continuous and thus for

$$\mu_{f_m}(h) := \sup\{|f_m(x) - f_m(y)| : x, y \in \mathbb{R} \text{ with } |x - y| \leq h\}.$$

we get that

$$\sup_{r \in [t_{i-1}, t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \leq \mu_{f_m} \left(\sup_{r \in [t_{i-1}, t_i]} |B_{t_{i-1}} - B_r| \right).$$

putting it together

$$\begin{aligned}
\mathbb{E}\left[\int_0^T (f_m^{(n)} - f_m)^2 ds\right] &\leq \frac{T}{n} \sum_{i=1}^n \mathbb{E}\left[\sup_{r \in [t_{i-1}, t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2\right] \\
&\leq \frac{T}{n} \sum_{i=1}^n \mathbb{E}[\mu_{f_m}(\sup_{r \in [t_{i-1}, t_i]} |B_{t_{i-1}} - B_r|)^2] \\
&\leq \frac{T}{n} \mathbb{E}[n \cdot \mu_{f_m}(\sup_{r \in [t_{i-1}, t_i], i \leq n} |B_{t_{i-1}} - B_r|)^2] \\
&\leq T \mathbb{E}[\mu_{f_m}(\sup_{r \in [t_{i-1}, t_i], i \leq n} |B_{t_{i-1}} - B_r|)^2]
\end{aligned}$$

Since f_m is continuous the modulus of continuity must tend to 0 as $n \rightarrow \infty$.

Thus we have shown that $f_m^{(n)} \xrightarrow{\mathcal{H}^2} f_m \Rightarrow I(f_m^{(n)}) \xrightarrow{L^2} I(f_m)$

Now on the set $\{\tau_m = T\}$ we have

$$f(B) = f_m(B).$$

and by persistence of identity also

$$\int_0^T f(B_s) dB_s = \int_0^T f_m(B_s) dB_s.$$

For

$$A_{n,\varepsilon} = \left\{ \left| \sum_{i=1}^n f(B_{t_{i-1}}) \cdot (B_{t_i} - B_{t_{i-1}}) - \int_0^T f(B_s) dB_s \right| \geq \varepsilon \right\}.$$

Then we get

$$\sum_{i=1}^n f(B_{t_{i-1}}) \cdot (B_{t_i} - B_{t_{i-1}}) \xrightarrow{\mathbb{P}} \int_0^T f(B_s) dB_s.$$

if $\mathbb{P}(A_{n,\varepsilon}) \rightarrow 0$

$$\begin{aligned}
\mathbb{P}(A_{n,\varepsilon}) &= \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m < T\}) + \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m = T\}) \\
&\leq \mathbb{P}(\{\tau_m < T\}) + \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m = T\}) \\
&\xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

□

This inequality is just $\mathbb{P}(A) \leq \mathbb{P}(B)$ if $A \subset B$

Remark (3.12). For any continuous $g : \mathbb{R} \rightarrow \mathbb{R}$ we have $f(\omega, t) = g(B_t(\omega)) \in \mathcal{H}_{\text{loc}}^2$ since B is a.s. pathwise bounded on $[0, T]$

Proof. Consider $\omega \in \Omega$ a.s., then

$$\sup_{t \in [0, T]} |g(B_t(\omega))| \leq C.$$

for some $C \geq 0$, then we have

$$\begin{aligned} \int_0^T g^2(B_t(\omega)) dt &= \int_0^T g(B_t(\omega)) g(B_t(\omega)) dt \\ &\leq \int_0^T \sup_{t \in [0, T]} |g(B_t(\omega))| \cdot |g(B_t(\omega))| dt \\ &\leq \sup_{t \in [0, T]} |g(B_t(\omega))| \int_0^T |g(B_t(\omega))| dt \\ &\leq C^2 \cdot T. \end{aligned}$$

□

Theorem (3.18). For any twice continuous differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ we have

$$f(B_t) = f(0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds \quad \mathbb{P}\text{-a.s..}$$

Proof. The main tools used are a second order Taylor expansion then using our Riemann Sum convergence for the first integral and for the second we have a normal (pointwise) Riemann sum.

We write

$$\begin{aligned} f(B_t) - f(0) &= \sum_{i=0}^{n-1} (f(B_{t_i}) - f(B_{t_{i-1}})) \\ &= \sum_{i=0}^{n-1} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) + \frac{1}{2} \sum_{i=0}^{n-1} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 + \sum_{i=0}^n r(B_{t_i}, B_{t_{i-1}}). \end{aligned}$$

The first sum converges by 3.17 to

$$\sum_{i=0}^{n-1} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \rightarrow \int f'(B_s) dB_s.$$

The second one we write as

$$\frac{1}{2} \sum_{i=0}^{n-1} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 = \frac{1}{2} \sum_{i=0}^{n-1} f''(B_{t_{i-1}})((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})) + \frac{1}{2} \sum_{i=0}^{n-1} f''(B_{t_{i-1}})(t_i - t_{i-1}).$$

Then the second term is the integral we want i.e.

$$\frac{1}{2} \sum_{i=0}^{n-1} f''(B_{t_{i-1}})(t_i - t_{i-1}) \rightarrow \frac{1}{2} \int f''(B_s) ds.$$

Such that we need to show the first part converges against 0 \mathbb{P} -a.s., we do so by showing it converges to 0 in L^2 instead where we again make use of the independence of Brownian increments, and that they have mean 0, plus the fact that

$$\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2] = t_i - t_{i-1}.$$

the remainder term is a little more complicated, we rewrite

$$\begin{aligned} r(x, y) &= \int_x^y (y - u)(f''(u) - f''(x)) du \\ &= (y - x)^2 \int_0^1 (1 - t)(f''(x + t(y - x)) - f''(x)) dt. \end{aligned}$$

then if f has compact support it holds

$$|r(x, y)| \leq |y - x|^2 |h(x, y)|.$$

for bounded h with $h(x, x) = 0$ and compact support (that is $\text{supp } f$), we show that the error term converges to 0 in L^1 by bounding h , that consists of splitting up Ω as follows

$$\Omega = \{|x - y| < \delta\} \cup \{|x - y| \geq \delta\}.$$

on the first set we know by continuity $h(x, y) < \varepsilon$ on the second we bound by using the $\|h\|_\infty$ which exists since h continuous and of compact support, bounding the probability of the second set by Markov inequality ($f(x) = x^2$). Since we can choose δ as we want we may take $\varepsilon = 0$

Now we need to argue that we are allowed to assume f compact support, this follows by similar argument to Riemann approximation. \square

Theorem (3.21). For any $f \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ we have

$$f(t, B_t) = f(0, 0) + \int_0^t f_x(s, B_s) dB_s + \int_0^t f_t(s, B_s) + \frac{1}{2} f_{xx}(s, B_s) ds.$$

for $t \in [0, T]$, \mathbb{P} -a.s.

Proof. We do a second order Taylor expansion in space and 1 in time \square

$$\begin{aligned}
f(t, B_t) - f(0, 0) &= \sum_{i=1}^n f(t_i, B_{t_i}) - f(t_{i-1}, B_{t_{i-1}}) \\
&= \sum_{i=1}^n f(t_{i-1}, B_{t_i}) - f(t_{i-1}, B_{t_i}) + \sum_{i=1}^n f(t_i, B_{t_{i-1}}) - f(t_i, B_{t_{i-1}}) \\
&= \sum_{i=1}^n f_t(t_{i-1}, B_{t_i})(t_i - t_{i-1}) \\
&\quad + \sum_{i=1}^n f_x(t_i, B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \\
&\quad + \sum_{i=1}^n \frac{1}{2} f_{xx}(t_i, B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 \\
&\quad + \sum_{i=1}^n r(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i}).
\end{aligned}$$

where

$$r(s, t, x, y) = \int_s^t (f_t(r, y) - f_t(s, y)) dr + \int_x^y (y - u)(f_{xx}(s, u) - f_{xx}(s, x)) du.$$

Now if f has compact we get a bound again and use the same arguments

Lemma. Proof that if $X_n \xrightarrow{L^p} X$ then $X_n \xrightarrow{\mathbb{P}} X$

Proof. Let us first remember what the two convergences mean, for the first we have that

$$\|X_n - X\|_{L^p}^p = \int_{\Omega} |X_n - X|^p \rightarrow 0.$$

The second means that for $\forall \varepsilon > 0$:

$$\mathbb{P}(|X_n - X| \geq \varepsilon) \rightarrow 0.$$

Using Markov

$$\mathbb{P}(|X_n - X| \geq \varepsilon) \leq \frac{\mathbb{E}[|X_n - X|^p]}{\varepsilon^p} \rightarrow 0.$$

□

Lemma. If $X_n \xrightarrow{L^p} X$ then $X_n \xrightarrow{L^q} X$ for $p > q$

Proof. We recall Hölder for $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$ it holds

$$\mathbb{E}[XY] \leq \mathbb{E}[X^p]^{\frac{1}{p}} \mathbb{E}[Y^q]^{\frac{1}{q}}.$$

We want to get :

$$\mathbb{E}[|X_n - X|^q] = \int_{\Omega} |X_n - X|^q d\mathbb{P} \leq \int_{\Omega} |X_n - X|^p d\mathbb{P} \cdot 1 \rightarrow 0.$$

i.e we need $r, s \geq 1$ such that $\frac{1}{r} + \frac{1}{s} = 1$ and

$$q \cdot r = p \Rightarrow r = \frac{p}{q}.$$

and

$$1 = \frac{1}{s} + \frac{q}{p}$$

$$\frac{p-q}{q} = \frac{1}{s}.$$

Note that any finite measure space suffices. \square

Definition (4.1). A stochastic process $X = (X_t)_{t \in [0, T]}$ is called Itô process if there is a a.s. representation

$$X_t = X_0 + \int_0^t a(\cdot, s) ds + \int_0^t b(\cdot, s) dB_s.$$

where $X_0 \in \mathbb{R}$ and $a, b : \Omega \times [0, T] \rightarrow \mathbb{R}$ are adapted, measurable processes satisfying

$$\mathbb{P}\left(\int_0^t |a(\omega, s)| ds < \infty\right) = 1 \text{ and } \mathbb{P}\left(\int_0^t |b(\omega, s)|^2 ds < \infty\right) = 1.$$

Remark. The stochastic integral process $A = (\int_0^t a(\cdot, s) ds)_{t \in [0, T]}$ is continuous, adapted process of finite variation.

Proof. We show that it has finite variation, let $\Pi_n \subset \Pi_{[0, T]}$ such that $|\Pi_n| \rightarrow 0$, and $\omega \in \{\int_0^t |a(\omega, s)| ds < \infty\}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{J \in \Pi_n} \Delta_{J \cap [0, T]} A &= \lim_{n \rightarrow \infty} \sum_{J \in \Pi_n: J=(s, t]} \left| \int_s^t a(\omega, r) dr \right| \\ &\leq \lim_{n \rightarrow \infty} \sum_{J \in \Pi_n: J=(s, t]} \int_s^t |a(\omega, r)| dr \\ &= \int_0^t |a(\omega, s)| ds < \infty. \end{aligned}$$

We can show this for any process X_s such that X_s is FV since

$$X = X_1 - X_2.$$

By definition of FV where X_1, X_2 are monotone increasing then

$$\begin{aligned}\int_0^t a(\cdot, s) dX_s &= \int_0^t a^+(\cdot, s) - a^-(\cdot, s) dX_s \\ &= \int_0^t a^+ dX_s^{(1)} + \int_0^t a^-(\cdot, s) dX_s^{(2)} - \left(\int_0^t a^+ dX_s^{(2)} + \int_0^t a^-(\cdot, s) dX_s^{(1)} \right).\end{aligned}$$

Which is the difference of two monotone increasing functions for $X_s = s$ we could just pick $s = 0$ \square

Lemma (4.2). Let $X = (X_t)_{t \in [0, T]}$ be an Itô process with representation

$$X_t = X_0 + \int_0^t a(\cdot, s) ds + \int_0^t b(\cdot, s) dB_s = \tilde{X}_0 + \int_0^t \tilde{a}(\cdot, s) ds + \int_0^t \tilde{b}(\cdot, s) dB_s.$$

for $t \in [0, T]$ and $X_0 = \tilde{X}_0$. Then we have $a = \tilde{a}$ and $b = \tilde{b} \mathbb{P} \otimes \lambda$ -a.s.

Proof. We have by linearity

$$0 = \int_0^t a(\cdot, s) - \tilde{a} ds + \int_0^t b(\cdot, s) - \tilde{b} dB_s.$$

this implies

$$-\int_0^t a(\cdot, s) - \tilde{a} ds = \underbrace{\int_0^t b(\cdot, s) - \tilde{b} dB_s}_{B_t}.$$

Right is a local martingale, and left is a process of finite variation. Then we know that

$$B_t = 0 \quad \mathbb{P}\text{-a.s.}$$

Then we have by Ito-isometry that

$$0 = \|B\|_{\mathcal{H}^2}^2 = \mathbb{E} \left[\int_0^t (b(\cdot, s) - \tilde{b}(\cdot, s))^2 ds \right] = \int_0^t \mathbb{E} [(b(\cdot, s) - \tilde{b}(\cdot, s))^2] ds.$$

Which gives the implication, since we can split up the integral into two parts, 1 where they are equal i.e 0 and the other one where they are not equal to each other, for a we do the same \square

Proposition (4.3). For any Itô process $X = (X_t)$ with rep

$$X_t = X_0 + \int_0^t a(\cdot, s) ds + \int_0^t b(\cdot, s) dB_s.$$

the quadratic variation of X is given by

$$\langle X \rangle_t = \lim_{n \rightarrow \infty} \sum_{J \in \Pi_n} (\Delta_{J \cap [0, t]} X)^2 = \int_0^t b^2(\cdot, s) ds.$$

Proof. We have that

$$X_t = A_t + M_t.$$

For A_t continuous adapted and of Finite variation and M_t a local martingale, we first note that

$$(\Delta_{J \cap [0, T]} X)^2 = (\Delta_{J \cap [0, T]} A)^2 + 2(\Delta_{J \cap [0, T]} M)(\Delta_{J \cap [0, T]} A) + (\Delta_{J \cap [0, T]} M)^2.$$

We have

$$\lim_{n \rightarrow \infty} \sum_{J \in \Pi_n} (\Delta_{J \cap [0, T]} A)^2 = \lim_{n \rightarrow \infty} \sum_{J \in \Pi_n} (\Delta_{J \cap [0, T]} A)(\Delta_{J \cap [0, T]} A) \leq \lim_{n \rightarrow \infty} \sup_{J \in \Pi_n} (\Delta_{J \cap [0, T]} A) \sum_{J \in \Pi_n} (\Delta_{J \cap [0, T]} A) \rightarrow 0.$$

Since the first part is a bounded constant, and the second goes to 0 since A is of FV.

By the same logic we have that the middle term goes to 0 and all that remains is

$$\langle X \rangle_t = \lim_{n \rightarrow \infty} \sum_{J \in \Pi_n} (\Delta_{J \cap [0, T]} X)^2 = \lim_{n \rightarrow \infty} \sum_{J \in \Pi_n} (\Delta_{J \cap [0, T]} M)^2 = \langle M \rangle_t.$$

Thus we have to show

$$\langle M \rangle_t = \int_0^t b^2(\cdot, s) ds.$$

For that we know it suffices to show that

$$(M_t^2 - \int_0^t b^2(\cdot, s) ds)_{t \in [0, T]}.$$

is a Martingale, first take a localizing sequence τ for M_t (for notation sake we leave it out)

$$\begin{aligned} & \mathbb{E}[M_t^2 - \int_0^t b^2(\cdot, s) ds - M_s^2 + \int_0^s b^2(\cdot, s) ds] \\ &= \mathbb{E}[M_t^2 - M_s^2 - \int_s^t b^2 ds] \\ &= \mathbb{E}[(M_t - M_s)^2 - \int_s^t b^2 ds]. \end{aligned}$$

We have $M_t - M_s$ (localized) is a martingale such that we can use Ito isometry

$$(M_t^{T_n} - M_s^{T_n})\mathbb{1}_A = (M_t^{T_n} - M_s)\mathbb{1}_A\mathbb{1}_{\tau_n \geq s} \int_s^t \mathbb{1}_A\mathbb{1}_{\tau_n \geq s} b().$$

which is in \mathcal{H}^2 then we have directly

$$\begin{aligned} \mathbb{E}[(M_{t \wedge \tau_n} - M_{s \wedge \tau_n})^2 \mathbb{1}_A] &= \mathbb{E}[(\int_0^T \mathbb{1}_A \mathbb{1}_{\dots} b dB_r)^2] \\ &= \mathbb{E}[(\int_0^T \mathbb{1}_A \mathbb{1}_{\dots} b^2 dr) \\ &= \mathbb{E}[\mathbb{1}_A \int_{s \wedge \tau_n}^{t \wedge \tau_n} b^2 dr]. \end{aligned}$$

which with the before discussion concludes the proof \square

Lemma (4.5). Let $X = (X_t)_{t \in [0, T]}$ be an Ito process with rep

$$X_t = X_0 + \int_0^t b(\cdot, s) dB_s.$$

Then the integral process $(\int_0^t f(\cdot, s) dX_s)$ is a continuous local martingale and If $\mathbb{E}[\int_0^T f^2 b^2 ds] < \infty$ then Ito isometry holds true

Proof. For (i) we have by definition $f \in \mathcal{L}(X)$ that

$$\int_0^t f(\cdot, s) dX_s = \int_0^t \underbrace{f(\cdot, s) b(\cdot, s)}_{\in \mathcal{H}_{loc}^2} dB_s.$$

Thus its a continuous local martingale \square

Example (5.1). Consider the SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t.$$

then we can solve this SDE by making the ansatz $X_t = f(B_t, t)$ and using Itô's formula

$$\begin{aligned} df(B_t, t) &= f(0, 0) + f_x(B_t) dB_t + (\frac{1}{2} f_{xx} + f_t) dt \\ &\triangleq \mu f(B_t) dt + \sigma f(B_t) dB_t. \end{aligned}$$

This implies

$$f_x(B_t) = \sigma f.$$

Such that

$$f = \exp(\sigma \cdot x + g(t)).$$

then

$$g'(t) \cdot f + \frac{1}{2}\sigma^2 f = \mu f$$

$$g'(t) = \mu - \frac{\sigma^2}{2}$$

$$g(t) = (\mu - \frac{\sigma^2}{2})t + g_0.$$

Which gives the solution

$$X_t = \exp(\sigma B_t + (\mu - \frac{\sigma^2}{2})t + g_0).$$

Definition. In general a linear SDE has the form

$$dX_t = (\alpha(t)X_t + \beta(t))dt + (\varphi(t)X_t + \psi(t))dB_t.$$

and a solution is given by

$$X_t = x_0 \exp(Y_t) + \int_0^t \exp(Y_t - Y_s)(\beta(s) - \psi(s)\varphi(s))ds + \int_0^t \exp(Y_t - Y_s)\psi(s)dB_s.$$

Where

$$Y_t = \int_0^t \varphi(s)dB_s + \int_0^t (\alpha(s) - \frac{1}{2}\varphi^2(s))ds.$$