Chapter 1

Stochastic Mean Field Particle Systems

From now on let the underlying probability space be given by $(\Omega, \mathcal{F}, \mathbb{P})$.

1.1 Basics of probability

Definition 1.1.1 (Brownian Motion). Real valued stochastic process $W(\cdot)$ is called a Brownian motion (Wiener process) if

1. $W(0) = 0$a.s.
2. $W(t) - W(s) \sim \mathcal{N}(0, t-s)$, for all $t, s \geq 0$
3. $\forall 0 < t_1 < t_2 < \ldots < t_n$, $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent
4. $W(t)$ is continuous a.s (sample paths)

Remark (Properties).

1. $\mathbb{E}[W(t)] = 0$, $\mathbb{E}[W(t)^2] = t$, for all $t > 0$
2. $\mathbb{E}[W(t)W(s)] = t \wedge s$ a.s
3. $W(t) \in \mathcal{C}^\gamma[0,T]$, $\forall 0 < \gamma < \frac{1}{2}$.
4. $W(t)$ is nowhere differentiable a.s

additionally Brownian motions are martingales and satisfy the Markov property

Definition 1.1.2 (Progressively measurable). In addition to adaptation of a Stochastic process X_t we say it is progressively measurable w.r.t $\mathcal{F}_t$ if $X(s,\omega) : [0,t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}[0,t] \times \mathcal{F}_t$ measurable, i.e the t is included

Definition 1.1.3 (Simple functions). Instead of $\mathcal{H}^2$ she uses $\mathbb{L}^2(0,T)$ is the space of all real-valued progressively measurable proceses $G(\cdot)$ s.t

$$\mathbb{E}[\int_0^T G^2 dt] < \infty.$$

define $\mathbb{L}$ analog

Definition 1.1.4 (Step Process). $G \in \mathbb{L}^2(0,T)$ is called a step process when Partition of $[0,T]$ exists s.t $G(t) = G_k$ for all $t_k \leq t \leq t_{k+1}$, $k = 0, \dots, m-1$ note G_k is $\mathcal{F}_{t_k}$ measurable R.V.

For step process we define the ito integral as a simple sum

Definition 1.1.5 (Ito integral for step process). Let $G \in \mathbb{L}^2(0,T)$ be a step process is given by

$$\int_0^T G(t) dW_t = \sum_{k=0}^{m-1} G_k(W(t_{k+1} - W(t_k))).$$

We take the left value of the process such that we converge against the right integral later

Remark. For two step processes $G, H \in \mathbb{L}^2(0,T)$ for all $a, b \in \mathbb{R}$, we have linearity (note they may have two different partitions, so we need to make a bigger (finer) one to include both,)

1. $\int_0^T (aG + bH) dW_t = a \int G + b \int H$
2. $\mathbb{E}[\int_0^T G dW_t] = 0$, because the Brownian motion has EV of 0
3. $\mathbb{E}[(\int_0^T G dW_t)^2] = \mathbb{E}[\int_0^T G^2 dt]$ Ito isometry

Proof. First property is just defining a new partition that includes both process. Second property, the Idea of the proof is that

$$\begin{aligned}\mathbb{E}[\int_0^t G dW_t] &= \mathbb{E}[\sum_{k=0}^{m-1} G_k(W_{t_{k+1}} - W_{t_k})] \\ &= \sum_{k=0}^{m-1} \mathbb{E}[G_k(W(t_{k+1}) - W(t_k))]\end{aligned}$$

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Remember $G_k \sim \mathcal{F}_{t_k}$ m.b. and $W(t_{k+1}) - W(t_k)$ is mb. wrt to $W^t(t_k)$ future sigma algebra and it is independent of $\mathcal{F}_{t_k}$ s.t the expectation decomposes

$$\sum_{k=0}^{m-1} \mathbb{E}[G_k(W(t_{k+1}) - W(t_k))] = \sum_{k=0}^{m-1} \mathbb{E}[G_k]\mathbb{E}[W(t_{k+1} - W(t_k))] = C \cdot 0 = 0.$$

For the variance decompose into square and non square terms , the non square terms dissapear by property 2 the rest follows by the variance of Brownian motion, be careful of which terms are actually independent , at leas one will always be independent of the other 3 □

Definition 1.1.6 (Ito Formula).

Proof. Step 1 :

1. $d(W_t^2) = 2W_t dW_t + dt$ which is equivalent to $W^2(t) = W_0^2 + \int_0^t 2W_s dW_t + \int_0^t ds$
2. $d(tW_t) = W_t dt + t dW_t$ which is equivalent to $tW(t) - sW(0) = \int_0^t W_s ds + \int_0^t s dW_s$

Actually $\forall$ a.e $\omega \in \Omega$:

$$2\int_0^t W_s dW_s = 2\lim_{n\to\infty}.$$

Now we prove (2) $tW_t - 0W_0 = \int_0^t W_s ds + \int_0^t s dW_s$

$$\int_0^t s dW_s + \int_0^t W_s ds = \lim_{n\to\infty} \sum_{k=0}^{n-1} t_k^n (W(t_{k+1}^n) - W(t_k^n)) + \sum_{k=0}^{n-1} W(t_{k+1}^n (t_{k+1}^n - t_k^n)).$$

We choose the right value for the second integral

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} (-t_k^n W(t_k)^n + t_{k+1}^n W(t_{k+1}^n)) = W(t)t - W(0) \cdot 0.$$

Its product rule

$$dX_1 = F_1 dt + G_1 dW_t$$
$$dX_2 = F_2 dt + G_2 dW_t.$$

This can be written as

$$d(X_1, X_2) = X_2 dX_1 + X_1 dX_2.$$

this shorthand notation actually means

$$\begin{aligned}
X_1(t)X_2(t) - X_1(0)X_2(0) = &\int_0^t X_2 F_1 ds + \int_0^t X_2 G_1 dW_s \\
&+ \int_0^t X_1 F_2 ds + \int_0^t X_1 G_2 dW_s \\
&+ \int_0^t G_1 G_2.
\end{aligned}$$

We prove for F_1, F_2, G_1, G_2 are time independent

$$\begin{aligned}
&\int_0^t (X_2 dX_1 + X_1 dX_2 + G_1 G_2 ds) \\
&= \int_0^t (X_2 F_1 + X_1 F_2 + G_1 G_2) ds + \int_0^t (X_2 G_1 + X_1 G_2) dW_s \\
&= \int_0^t (\underbrace{F_2 F_1 s + F_1 G_2 W_s}_{=X_2} + \underbrace{F_1 F_2 s + F_2 G_1 W_s}_{=X_1} + G_1 G_2) ds \\
&+ \int_0^t (F_2 G_1 s + G_2 G_1 W_s + F_1 G_2 s + G_1 G_2 W_s) dW_s \\
&= G_1 G_2 t + F_1 F_2 t^2 + (F_1 G_2 + F_2 G_1) \underbrace{\left(\int_0^t W_s ds + \int_0^t s dW_s \right)}_{tW_t} + 2 G_1 G_2 \underbrace{\int_0^t W_s dW_s}_{W_t^2 - t} \\
&= G_1 G_2 t + F_1 F_2 t^2 + (F_1 G_2 + F_2 G_1) t W_t + G_1 G_2 W_t^2 - G_1 G_2 t \\
&= X_1(t) \cdot X_2(t).
\end{aligned}$$

Where $X_2(t) = \int_0^t F_2 ds + \int_0^t G_2 dW_s \overset{\text{Cons.}}{=} F_2 t + G_2 W_t$

Extend the above idea by considering step processes (F_1, F_2, G_1, G_2)instead of time independent. Step processes are constant (related to time) and we can use the above prove for every time step t and just consider a summation over it.

For general $F_1, F_2 \in L^1(0,T), G_1, G_2 \in L^2(0,T)$ then we take step processes to approximate them

$$\begin{aligned}
\mathbb{E}[\int_0^T |F_i^n - F_i| dt] \to 0 \\
\mathbb{E}[\int_0^T |G_i^n - G_i|^2 dt] \to 0
\end{aligned}$$

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$$X_i(t)^n = X_i(0) + \int_0^t F_i^n ds + \int_0^t G_i^n dW_s.$$

It holds

$$\begin{aligned}X_1^n(t)X_2(t)^n - X_1(0)X_2(0) &= \int_0^t X_2(s)^n F_1^n(s)ds + \int_0^t X_2(s)G_1(s)^n dW_s \\ &+ \int_0^t X_1^n(s)F_2^n(s)ds + \int_0^t X_1(s)^n G_2^n(s)dW_s + \int_0^t G_1(s)^n G_2^n(s)ds.\end{aligned}$$

Only thing left is a convergence result (i.e DCT) sinc the processes are bounded or smth like that.

Step 3 if $u(x) = x^m,\ \forall m = 0, \ldots$ to prove

$$d(X^m) = mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}G^2dt.$$

For $m = 2$ the result is obtained by the product rule, By induction we prove for arbitrary m

(IV) Suppose the statement hold for $m - 1$
(IS) $m - 1 \to m$

$$\begin{aligned}d(X^m) &= d(X \cdot X^{m-1}) = XdX^{m-1} + X^{m-1}dx + (m-1)X^{m-2}G^2dt \\ &\overset{\text{IS}}{=} X(m-1)X^{m-2}dx + X \cdot \frac{1}{2}(m-1)(m-2)X^{m-3}G^2dt + X^{m-1}dx + (m-1)X^{m-2}G^2dt \\ &= mX^{m-1}dx + (m-1)(\frac{m}{2} - 1 + 1)X^{m-2}G^2dt \\ &= \underbrace{mX^{m-1}}_{\partial_x u}dx + \frac{1}{2}\underbrace{m(m-1)X^{m-2}}_{\partial_x^2 u}G^2dt.\end{aligned}$$

Now $u(x) = x^m$

$$dX = Fdt + GdW_t.$$

□

1.2 Bad K

1.3 Convergence

Chapter 2

Excercise Sheets

2.1 Sheet 1 (11.09.2023)

2.1.1 Excercise 1

Question 1. Consider the second order system :

$$dX_t^i = V_t^i$$
$$dV_t^i = \frac{1}{N}\sum_{i=1}^{N} F(t, X_t^i, V_t^i, X_t^j, V_t^j)dt.$$

on $[0,T]$ for some smooth interaction force $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ following the lecture we assume the empirical measure :

$$\mu_t^N(dx, dv) = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_t^i, V_t^i)}.$$

converges in some sense to the measure μ_t with density ρ_t for each t. Derive an equation for $\rho, t \geq 0$ similar to the lecture.

Solution. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^?)$ and calculate :

$$\begin{aligned}
\frac{d}{dt}\langle \mu_N, \varphi\rangle = \frac{d}{dt}\int_{\mathbb{R}^{2d}} \varphi(x,v) d\mu_N(t)(dx,dv) = \frac{d}{dt}\int \frac{1}{N}\sum_{i=1}^N \varphi(x,v) d\delta_{(x_i^t, v_i^t)} \\
&\overset{*}{=} \frac{1}{N}\sum_{i=1}^N \frac{d}{dt}\varphi(x_i(t), v_i(t)) \\
&\overset{\text{Chain.}}{=} \frac{1}{N}\sum_{i=1}^N \partial_x\varphi \cdot \dot{x}_i + \partial_v \cdot \dot{v}_i \\
&= \frac{1}{N}\sum_{i=1}^N \partial_x\varphi \cdot v_i(t) + \partial_v\varphi \cdot \sum_{j=1}^N F(t, x_i(t), v_i(t), x_j(t), v_j(t))
\end{aligned}$$

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□

Chapter 3

Appendix

Theorem 3.0.1 (Divergence Theorem). Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)-$ dimensional sub-manifold of $\mathbb{R}^n$. Let $F : \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously to $\partial\Omega$. Then we have :

$$\int_{\Omega} \nabla \cdot F d\mu = \int_{\partial\Omega} F \cdot N d\sigma.$$

where N is the outward pointing normal. (last component is positive)