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# Sheet 11

## 33 The distribution of heat

Consider the fundamental solution of the heat equation $\Phi(x, t)$ given in Definition 4.5.

**Exercise (a).** Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$

**Proof.** We recall

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{for } x \in \mathbb{R}^n, t > 0 \\ 0 & \text{for } x \in \mathbb{R}^n, t \leq 0 \end{cases}.$$

Pick $x \in \mathbb{R}^n$ and handle the cases $t > 0$ , $t < 0$ and $t = 0$, for $t = 0$ we have that for $n \in \mathbb{N}$, $\Phi(x,t)^{(n)} \equiv 0$ a smooth function, for $t < 0$ we also see that $\Phi(x,t)^{(n)} \equiv 0$ and $t \to 0^-$ are also 0 and we get no problem here.

We handle $t > 0$ by considering that from the last sheet we know that the derivatives of $e^{-x^2}$ are all of the form

$$p(x)e^{-x^2}.$$

For some polynomial $p(x)$ in this case we get extra terms including $t$ i.e

$$\Phi^{(n)}(x,t) = q(x,t)e^{-\frac{|x|^2}{4t}}.$$

similar to last sheet, the exponential growth for $x \neq 0$ (to 0 since $t \to 0^+ \Rightarrow e^{-\frac{|x|^2}{4t}} \to 0$) will "outperform" the polynomial growth and force the expression towards 0. For $x = 0$ the polynomial is 0 and thus the whole expression.

We get that the fundamental solution extends to a smooth function. $\square$

**Exercise** (b). Verify that this obey the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$

**Solution.** Let us calculate

$$\begin{aligned}\partial_t \Phi &= \frac{1}{(4\pi)^{\frac{n}{2}}}\left(-\frac{n}{2}\frac{1}{t^{\frac{n}{2}+1}} \cdot e^{-\frac{|x|^2}{4t}} + \frac{1}{t^{\frac{n}{2}}}\frac{|x|^2}{4t^2}e^{-\frac{|x|^2}{4t}}\right) \\ &\Phi(-\frac{n}{2}\frac{1}{t} + \frac{|x|^2}{4t^2}).\end{aligned}$$

and

$$\begin{aligned}\partial_{x_i}\Phi &= -\frac{x_i}{2t} \cdot \Phi \\ \partial_{x_i^2}\Phi &= \frac{x_i^2}{4t^2}\Phi - \frac{1}{2t}\Phi \\ &= \Phi(\frac{x_i^2}{4t^2} - \frac{1}{2t}).\end{aligned}$$

Then

$$\begin{aligned}\partial_t - \Delta\Phi &= \Phi(-\frac{n}{2t} + \frac{|x|^2}{4t^2} - \frac{|x|^2}{4t^2} + \frac{n}{2t}) \\ &= \Phi \cdot 0.\end{aligned}$$

□

**Exercise** (c). Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)dxdt.$$

**Proof.** For $t < 0$ the Integral vanishes because $\Phi(x,t) = 0$, since $\varphi \in C_0^\infty(K)$

$$\int_{\mathbb{R}} \varphi(x,t)dt = \int_{-T}^{T} \varphi(x,t).$$

For some $T > 0$ such that we get

$$\begin{aligned}H(\varphi) &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)dxdt \\ &= \int_{-T}^{T}\int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)dxdt \\ &= \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)dxdt\end{aligned}$$

.

□

**Exercise** (d). Conclude with the help of Lemma 4.6 and Theorem 4.7 that

$$|H(\varphi)| \leq T\|\varphi\|_{K,0}.$$

Hence $H$ is a continuous linear functional

**Proof.**

$$\begin{aligned}
|H(\varphi)| &= \left|\int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)dxdt\right| \\
&\leq \int_0^T \int_{\mathbb{R}^n} |\Phi(x,t)\varphi(x,t)|dxdt \\
&\leq \int_0^T \int_{\mathbb{R}^n} |\Phi(x,t)|\|\varphi\|_{K,0}dxdt \\
&\leq \int_0^T \|\varphi\|_{K,0} \int_{\mathbb{R}^n} |\Phi(x,t)|dxdt
\end{aligned}$$

.

Now for $t > 0$ we have

$$\int_{\mathbb{R}^n} |\Phi(x,t)|dx = \int_{\mathbb{R}^n} \Phi(x,t)dx = 1.$$

For $t \to 0$ we have by 4.7. that also

$$\int_{\mathbb{R}^n} |\Phi(x,t)|dx = \underbrace{\int_{\mathbb{R}^n} \Phi(x,t) \cdot 1dx}_{u(x,t)} \to 1.$$

Thus

$$\int_0^T \|\varphi\|_{K,0} \int_{\mathbb{R}^n} |\Phi(x,t)|dxdt \leq T \cdot \|\varphi\|_{K,0}.$$

□

**Exercise** (e). Extend Theorem 4.7 to show that

$$\int_{\mathbb{R}^n} \Phi(x-y,t)h(y,s)dy \to h(x,s).$$

as $t \to 0$ uniformly in $s$

**Proof.** In the last step of the proof of (iii) we bound

$$|h(y) - h(x)| \leq 2\sup\{|h(y)|, y \in \mathbb{R}^n\}.$$

By making the necessary assumption that $h \in \mathcal{C}_b(\mathbb{R}^n \times \mathbb{R})$ we instead bound

(for our $h$ now)

$$|h(y,s)-h(x,t)| \leq 2\sup\{|h(y,x)|, y,x \in \mathbb{R}^n \times \mathbb{R}\}.$$

everything else stays the same ☐

**Exercise (f).** Hence show that

$$\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)dydt \to \varphi(0,0).$$

as $\varepsilon \to 0$

**Proof.** Assuming we mean

$$\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi(y,t)(-\partial_t\varphi(y,t) - \Delta\varphi(y,t))dydt \to \varphi(0,0).$$

Then

$$\begin{aligned}\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi(y,t)(-\partial_t\varphi(y,t) - \Delta\varphi(y,t))dydt &= \int_{\mathbb{R}^n} -\Phi(y,\varepsilon)\varphi(y,\varepsilon)dy \\ &\quad - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \partial_t\Phi(y,t)(-\varphi - \Delta\varphi(y,t))dy \\ &= \int_{\mathbb{R}^n} -\Phi(y,\varepsilon)\varphi(y,\varepsilon)dy \\ &\quad - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \underbrace{\partial_t\Phi(y,t) - \Delta}_{=0}(-\varphi)dydt \\ &= \int_{\mathbb{R}^n} -\Phi(y,\varepsilon)\varphi(y,\varepsilon)dy\end{aligned}$$

.

Some sign is wrong but if we ignore that then in the end we have, since $\Phi$ only depends on the length of the $x$ variable

$$u(0,\varepsilon) = \int_{\mathbb{R}^n} \Phi(-y,\varepsilon)\varphi(y,\varepsilon)dy$$

.

which by e) tends to

$$\varphi(0,0).$$

as $\varepsilon \to 0$ ☐

**Exercise (g).** Prove that as $\varepsilon \to 0$

$$\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)dydt \to 0.$$

**Proof.** We have

$$\begin{aligned}
\left|\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)dydt - 0\right| &\leq \left|\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)dydt\right| \\
&\leq \int_0^\varepsilon \int_{\mathbb{R}^n} |\Phi(y,t)h(y,t)|dydt \\
&\leq \int_0^\varepsilon \sup_{x\in\mathbb{R}^n}|h(x,t)| \int_{\mathbb{R}^n} \Phi(y,t)dydt \\
&\leq \int_0^\varepsilon \sup_{(x,s)\in\mathbb{R}^n\times\mathbb{R}^+}|h(x,s)| \int_{\mathbb{R}^n} \Phi(y,t)dydt \\
&\leq \sup_{(x,s)\in\mathbb{R}^n\times\mathbb{R}^+}|h(x,s)| \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,t)\cdot 1dydt \\
&\leq \varepsilon \sup_{(x,s)\in\mathbb{R}^n\times\mathbb{R}^+}|h(x,s)| \\
&\to 0.
\end{aligned}$$

Where we used similar argument to part d for handling the $\Phi$ integral □

## 34. Heat death of the universe

**Exercise (a).** Suppose that $h \in \mathcal{C}_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and u is defined as in Theorem 4.7. Show

$$\sup_{x\in\mathbb{R}^n}|u(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}}\|h\|_{L^1}.$$

**Proof.** Take $t > 0$ and check with 4.7.

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n}|u(x,t| &\leq \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|\Phi(x-y,t)h(y)|dy \\
&\leq \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|\Phi(x-y,t)||h(y)|dy \\
&= \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4t}}|h(y)|dy \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}}\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\underbrace{e^{-\frac{|x-y|^2}{4t}}}_{\leq 1}|h(y)|dy \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}}\sup_{x\in\mathbb{R}^n}\underbrace{\int_{\mathbb{R}^n}|h(y)|\,dy}_{\|h\|_{L^1}} \\
&\leq \frac{1}{(4\pi t)^{\frac{n}{2}}}\|h\|_{L^1}.
\end{aligned}$$

□

**Exercise** (b). Let $I_m$ be the function from Theorem 4.7. that solves the heat equation on $\mathbb{R}^n$ with $I_m(x,0) = mk(x)$ for m a constant and $k : \mathbb{R}^n \to [0,1]$ a smooth function of compact support such that

$$k|_\Omega \equiv 1.$$

why must k exist ? Why does $I_m \to 0$ as $t \to \infty$? What boundary conditions on $\Omega$ does it obey ?

**Proof.** Let us first assume that we find $I_m$ such that

$$I_m(x,0) = mk(x).$$

with $k$ smooth and compact support, then $k \in \mathcal{C}_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and by part a) we get that

$$\sup_{x\in\mathbb{R}^n}|u(x,t)| \leq \frac{m}{(4\pi t)^{\frac{n}{2}}}\|k\|_{L^1} \xrightarrow{t\to\infty} 0.$$

For the boundary conditions

$$\partial\Omega_T = (\partial\Omega \times (0,T]) \cup (\overline{\Omega} \times 0).$$

we check individually $t = 0$ and $x \in \Omega$ then we have by (iii) (or rather assumption)

$$I_m(x,0) = mk(x) = m. \qquad k|_\Omega \equiv 1$$

For $x \in \partial\Omega$ and $t \in (0, T]$

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) m k(y) dy = \int_K \Phi(x-y,t) m k(y) dy$$

.

We have $k(y) \geq 0$ and for $\Phi(x-y,t) > 0$ , $(t > 0)$, so the sign of $u$ depends on the sign of $m$

For the existence of $k$ we consider that since $\Omega$ is bounded, then it can be contained in a compact set $K$ such that we set $\operatorname{supp} k = K$ and $k|_{\Omega} \equiv 1$ which is smooth, for $\Omega \setminus K$ let $k \to 0$ in a smooth way. $\square$

**Exercise** (c). Use the monotonicity property to show that u tends to zero.

**Proof.** We learned that if we have two problems with

$$h_1 \geq h_2$$
$$g_1 \geq g_2.$$

then

$$u_1 \geq u_2.$$

Let $u$ be a solution to the homogeneous heat equation such that, for some $h \in \mathcal{C}_b(\mathbb{R}^n) \cap L^1$

$$u(x,0) = h(x).$$

and on $\partial\Omega \times \mathbb{R}^+$

$$u(x,t) = 0.$$

we can then construct two solutions by considering

$$a := \sup_{x\in\Omega} |u(x,0)| = \sup_{x\in\Omega} |h(x)|.$$

Then

$$I_{-a}(x,0) \leq u(x,0) \leq I_a(x,0).$$

so we must also have

$$I_{-a}(x,t) \leq u(x,t) \leq I_a(x,t).$$

but as shown above we have

$$\lim_{t\to\infty} I_{-a}(x,t) = 0 = \lim_{t\to\infty} I_a(x,t).$$

thus $u(x,t) \to 0$ as well. $\square$