Chapter 1

First Oder PDEs

The main Method of solving first order PDE's is the method of characteristics

1.1 Homogeneous Transport Equation

Definition 1.1.1. For a function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with $b \in \mathbb{R}^n$ the transport equation is defined as

$$\dot{u} + b \cdot \nabla u = 0.$$

Theorem (1.2.). For a continuous differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ the transport equation

$$\dot{u} + b \cdot \nabla u = 0.$$

with

$$u(x, 0) = g(x).$$

has a solution

Proof. By method of characteristics we have for

$$z(s) = u(x(s), t(s)).$$

that

$$z'(s) = \nabla \frac{\partial x}{\partial s} + \dot{u} \frac{\partial t}{\partial s}.$$

Thus

$$\begin{aligned} x'(s) &= b \\ t'(s) &= 1. \end{aligned}$$

then

$$x(s) = b \cdot s + x_0.$$

and

$$z' = 0.$$

Since at $s = 0$ we have

$$z(0) = u(x_0, 0) = g(x_0).$$

we get a solution for any x, t by

$$x = b \cdot s + x_0.$$

thus

$$x_0 = x - b \cdot s.$$

and

$$u(x, t) = g(x - b \cdot t).$$

□

Corollary. The solution is unique if the characteristics do not cross, that means if for any x , $x_0 = x - b \cdot t$ is unique.

1.2 Inhomogeneous Transport Equation

Theorem. Given a vector $b \in \mathbb{R}^n$ a function $f : \mathbb{R}^n \times \mathbb{R}$ and an initial value $g : \mathbb{R}^n \to \mathbb{R}$ the Cauchy problem for the inhomogenous transport equation is given by

$$\dot{u} + b \cdot \nabla u = f \qquad u(x, 0) = g(x).$$

We could either , use the method of characteristics to arrive at

$$u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s) ds.$$

Again we chose $t(s) = s$ and $x(s) = x_0 + sb$ then integrating, and the initial condition tells us what z_0 is.
Alternatively we recognize that this is Duhamels Principle, since

$$f(x_0 + sb, s).$$

is a solution to the Homogeneous Cauchy problem with initial condition

$$u(x, 0) = f(x).$$

1.3 Scalar Conservation Laws

Definition 1.3.1. For a smooth function $f : \mathbb{R} \to \mathbb{R}$ the following is called scalar conservation law

$$\dot{u} + \frac{\partial f(u(x,t))}{\partial x} = \dot{u} + f'(u(x,t)) \cdot \frac{\partial u(x,t)}{\partial x} = 0.$$

Corollary. The name conservation law comes form the fact, that if $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a solution then

$$\frac{d}{dt}\int_a^b u(x,t) = \int_a^b \dot{u}(x,t)dx = -\int_a^b \frac{\partial f(u(x,t))}{\partial x}dx = f(u(a,t)) - f(u(b,t)).$$

Theorem (1.4). If $f \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$ and $g \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ with $f''(g(x), g'(x)) > -\alpha$ for all $x \in \mathbb{R}$ and some $\alpha \geq 0$ then there is a unique $\mathcal{C}^1$ solution of the initial value problem for the scalar conservation law

$$\dot{u} + f'\nabla u = 0 \qquad u(x,0) = g(x).$$

on $(x,t) \in \mathbb{R} \times [0, \alpha^{-1})$ for $\alpha > 0$ and on $(x,t) \in \mathbb{R} \times [0, \infty)$ for $\alpha = 0$

Proof. When looking at PDE's or IVP's we generally ask three questions

1. Existence of a solution
2. Uniqueness of a solution
3. Regularity of a solution

For the existence part we get by method of characteristics that

$$u(x + tf'(g(x)), t) = g(x).$$

so a solution exists, this solution is unique if the characteristics do not cross, we check that

$$\frac{d}{dx}x + tf'(g(x)) = 1 + tf''(g(x))g'(x).$$

which by assumption

$$1 + tf''(g(x))g'(x) \geq 1 - t\alpha > 0.$$

for all $t \in [0, \alpha^{-1})$ this means the characteristic curves are strictly monotone increasing, thus for two points $x \neq y$ $x_0 \neq y_0$ and the curves never cross.
For regularity, we have that $u \in \mathcal{C}^{1,1}$, since

$$u(y,t) = g(x).$$

where

$$x + tf'(g(x)) = y.$$

□

1.4 Non characteristic Hyper surfaces

The goal of this section is to generalize the method of characteristics to general first oder PDEs

$$F(\nabla u(x), u(x), x) = 0.$$

In the end the goal is to reduce the problem to some problem on a Hyper surface on which the solution is given by the initial value problem, then by studying how the solution behaves when leaving the hypersurface we attain a general solution. For that we first show that we can reduce every Cauchy problem to the form

$$u(y) = g(y) \text{ for all } y \in \Omega \cap H \text{ where } H = \{x \in \mathbb{R}^n | x \cdot e_n = x_0 \cdot e_n\}.$$

where $e_n = (0, \dots, 0, 1)$