Chapter 1

Introduction

Mean Field Particle Systems is about the study of particles which are represented by (stochastic) differential equations. This course in particular is concerned with the behaviour of the system as the size grows to infinity:

Definition 1.0.1 (Toy Mean Field Particle System)**.** Let $N \in \mathbb{N}$ then a Mean Field Particle System of first order is given by :

$$x_1(t), \ldots, x_n(t) \in \mathcal{C}^1([0,T];\mathbb{R}^d) \qquad x_i(0) = c_i.$$

Where each particle satisfies

$$dx_i = \frac{1}{N}\sum_{j=1}^{N} K(x_i, x_j)dt + \sigma dB_i(t).$$

Where B_i is a Brownian motion; For $\sigma = 0$ the system is called deterministic.

Example. Example choices for K are :

$$K(x_i, x_j) = \nabla(|x_i - x_j|^2).$$

or :

$$o\gamma = \frac{x_i - x_j}{|x_i - x_j|^d}.$$

Goal is to study what happens at $N \to \infty$, to do so we consider how the measure of a system converges

Definition 1.0.2 ((Empirical) Measure of a System)**.** Consider the point measure for every x_i : $\delta_{x_i(t)}$, then the measure of the System of order N is

:

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}.$$

Assumption 1.0.1. For initial data the empirical measure of a system converges $\mu_N(0) \to \mu(0)$ where μ is absolutely continuous with respect to the Lebesgue Measure

Corollary. By Radon Nikodym

$$d\mu = \rho_0 dx \quad \rho_0 \in L^1(\mathbb{R}^d).$$

It can be shown that μ solves a PDE, to do so we compute the derivative of μ using test functions

$$\forall \varphi \in C_0^\infty(\mathbb{R}^d).$$

$$\begin{aligned}
\frac{d}{dt}\langle \mu_N(t), \varphi\rangle = \frac{d}{dt}\int_{\mathbb{R}^d} \varphi(x) d\mu_N(t)(x) &= \frac{d}{dt}\int \frac{1}{N}\sum_{i=1}^{N} \varphi(x) d\delta_{x_i(t)} \\
&= \frac{1}{N}\sum_{i=1}^{N} \frac{d}{dt}\varphi(x_i(t)) \\
&\overset{\text{Chain.}}{=} \frac{1}{N}\sum_{i=1}^{N} \nabla\varphi(x_i(t)) \frac{d}{dt} x_i(t) \\
&= \frac{1}{N}\sum_{i=1}^{N} \nabla_x \varphi(x_i(t)) \cdot \underbrace{\frac{1}{N}\sum_{j=1}^{N} K(x_i(t), x_j(t))}_{\text{Def.}} \\
&= \frac{1}{N}\sum_{i=1}^{N} \nabla_x \varphi(x_i(t)) \cdot \frac{1}{N}\sum_{j=1}^{N} \int_{\mathbb{R}^d} K(x_i(t), y) d\delta_{x_j(t)}(y) \\
&= \frac{1}{N}\sum_{i=1}^{N} \nabla_x \varphi(x_i(t)) \cdot \int_{\mathbb{R}^d} K(x_i(t), y) d\mu_N(t) \\
&= \int_{\mathbb{R}^d} \nabla\varphi(x) \int_{\mathbb{R}^d} K(x,y) d\mu_N(t,y) d\mu_N(t,x)
\end{aligned}$$

.

Where the last line can be rewritten by using Integration by Parts (Divergence Theorem) :

$$\int_{\mathbb{R}^d} \nabla\varphi(x) \int_{\mathbb{R}^d} K(x,y)d\mu_N(t,y)d\mu_N(t,x) \overset{\text{Part.}}{=} -\langle \nabla \cdot (\mu_N \int_{\mathbb{R}^d} K(\cdot,y)d\mu_N(y)), \varphi\rangle$$

.

This means μ satisfies :

$$\partial_t \mu_N + \langle \nabla \cdot (\mu_N \int_{\mathbb{R}^d} K(\cdot,y)d\mu_N(y)), \varphi\rangle = 0 \quad \xrightarrow{N\to\infty} \partial_t \mu + \langle \nabla \cdot (\mu \int_{\mathbb{R}^d} K(\cdot,y)d\mu(y)), \varphi\rangle = 0.$$

In practical applications (Theoretical Physics , Biology) systems that are considered are often of second order

Definition 1.0.3 (Toy Second Order System)**.** Given $N \in \mathbb{N}$ a Second Order System is given by

$$(x_i(t), v_i(t)), \ldots, (x_N(t), v_N(t)) \in \mathbb{R}^{2d}.$$

Such that :

$$\frac{d}{dt}x_i(t) = v_i(t)$$
$$\frac{d}{dt}v_i(t) = \frac{1}{N}\sum_{j=1}^{N} F(\underbrace{x_i(t), v_i(t)}_{\text{Position and Velocity of itself}}; x_j(t), v_j(t)) + \sigma\frac{dB_t}{dt}$$

.

Example (Gravitational Force)**.** An example of F could be :

$$F(x,v,y,u) = \frac{x-y}{|x-y|^d}.$$

Definition 1.0.4 (Second Order Measure)**.** The Measure of a second order System is given by :

$$\mu_N(x,v) = \frac{1}{N}\sum_{i=1}^{N} \delta_{(x_i(t),v_i(t))}.$$

Exercise 1.0.1. Show what PDE μ solves for $\sigma = 0$, *Hint :* Calculate $\frac{d}{dt} < \mu_N, \varphi >$ for some test function $\varphi \in C_0^\infty(\mathbb{R}^{2d})$

Chapter 2

Deterministic Mean Field Particle Systems

The goal for this chapter is to determine when a unique solution exists to the Mean-Field-Equation arising from our Particle Systems. Beginning by recapping standard ODE Theory on when a solution exists to an ODE IVP and continuing with the notion of Weak Solutions and Distributions which allows us the generalize the above ODE results.

Definition 2.0.1 (Deterministic Mean Field Particle System). For $N \in \mathbb{N}$ a deterministic mean field particle system is given by N particles :

$$x_1(t), \dots, x_n(t) \in \mathcal{C}^1([0,T]; \mathbb{R}^d) \qquad x_i(0) = c_i.$$

With initial points :

$$x_i(0) = x_{i,0} \in \mathbb{R}^d.$$

And the relation :

$$\frac{d}{dt} x_i = \frac{1}{N} \sum_{j=1}^{N} K(x_i, x_j).$$

The system is then given by :

$$X_N = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix} \in \mathbb{R}^{dN}.$$

2.1 ODE Theory

Definition 2.1.1 (Initial Value Problem (standard)). For $\forall\ T > 0$ let the standard IVP be given by :

$$x' = f(t, x)$$
$$x|_{t=0} = x_0 \in \mathbb{R}^n.$$

with $t \in [0, T],\ x(t) \in \mathbb{R}^n$ and $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$

As opposed to Picard-Lindelöf where only need locally Lipschitz continuity since we first construct a solution on a small local subset and then extend this solution, here we require global Lipschitz continuity since we do not want to extend our solution.

Theorem 2.1.1 (Picard Iteration). Whenever $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$is globally Lipschitz continuous in the second component the standard IVP has a unique solution $x \in \mathcal{C}^1([0, T]; \mathbb{R}^n)$

Proof. We begin by defining our Picard-Iteration by

$$\begin{aligned} x_1(t) &= x_0 + \int_0^t f(s, x_0) ds \\ x_2(t) &= x_0 + \int_0^t f(s, x(s)) ds \\ &\vdots \\ x_m(t) &= x_0 + \int_0^t f(s, x_{m-1}(s)) ds. \end{aligned}$$

The proof will be split into two steps

- **Step 1:** The first part of the proof consists of showing the above defined sequence is Cauchy and thus converges
- **Step 2:** The second part is then showing the limit is a solution to the IVP

We know that f is continuous such that $(x_n)_{n\in N} \subset \mathcal{C}^1([0,T];\mathbb{R}^n)$, by completeness we know that any sequence that is Cauchy must also converge against a limit in the space.

As such we show our sequence is Cauchy by first considering the distance between any two points

$$\begin{aligned}
|x_2 - x_1| &= \left|\int_0^t f(s,x_1(s))ds - \int_0^t f(s,x_0(s))ds\right| = \left|\int_0^t f(s,x_{m-1}(s)) - f(s,x_{n-1}(s))ds\right| \\
&\leq \int_0^t |f(s,x_1(s)) - f(s,x_0(s))|ds \\
&\overset{\text{Lip.}}{\leq} L\int_0^t |x_1(s) - x_0(s)|ds \\
&= L\int_0^t \left|\int_0^{s_0} f(s,x_0)ds\right| ds_0 \\
&\leq L\cdot \int_0^t \int_0^{s_0} |f(s,x_0)|ds ds_0 \\
&\leq L \underbrace{M}_{=\max_{s\in[0,T]}|f(s,x_0)|} \frac{t^2}{2}.
\end{aligned}$$

We can extend to arbitrary $m \in \mathbb{N}$ by using induction

$$|x_m(t) - x_{m-1}(t)| \leq ML^{m-1}\frac{t^m}{m!}. \qquad \text{(IA)}$$

(IS) : $m \to m+1$

$$\begin{aligned}
|x_{m+1}(t) - x_m(t)| &\overset{\text{Lip.}}{\leq} L\int_0^t |x_m(s) - x_{m-1}(s)|ds \\
&\overset{\text{IA.}}{\leq} L\int_0^t \frac{ML^{m-1}s^m}{m!}ds = ML^m\frac{t^{m+1}}{(m+1)!}.
\end{aligned}$$

Now for any $n, m \in \mathbb{N}$ and assuming without loss of generality that $n > m$ we can write $n = m + p$ for $p \in \mathbb{N}$:

$$\begin{aligned}
|x_n(t) - x_m(t)| = |x_{m+p}(t) - x_m(t)| &\leq \sum_{k=m+1}^{m+p} |x_k(t) - x_{k-1}(t)| \overset{\text{Ind.}}{\leq} M \sum_{k=m+1}^{m+p} \frac{L^{k-1}T^k}{k!} \\
&= \frac{M}{L}\sum_{k=m+1}^{m+p}\frac{(LT)^k}{k!} = \frac{M}{L}\frac{(LT)^{m+1}}{(m+1)!}\sum_{k=0}^{p-1}\frac{(LT)^k}{k!} \\
&\leq \frac{M}{L}\frac{(LT)^{m+1}}{(m+1)!}e^{LT} \xrightarrow{m\to\infty} 0 \text{ uniformly in } t \in [0,T].
\end{aligned}$$

This shows that $(x_m)_{m\in\mathbb{N}}$ is Cauchy and has a limit $x \in \mathcal{C}([0,T];\mathbb{R}^n)$ with

$$\max_{t\in[0,T]}|x_m(t) - x(t)| \to 0.$$

It remains to show that $x(t)$ is a solution to the IVP i.e :

$$x(t) = \lim_{m\to\infty} x_0 + \int_0^t f(s, x_{m-1}(s))ds \leftrightarrow x_0 + \int_0^t f(s, x(s))ds.$$

Consider the difference between both sides

$$\begin{aligned}
|\lim_{m\to\infty}\int_0^t f(s,x_{m-1}(s)) - f(s,x(s))ds| &\leq \lim_{m\to\infty}\int_0^t |f(s,x_{m-1}(s)) - f(s,x(s))|ds \\
&\leq \lim_{m\to\infty} L\int_0^t |x_{m-1}(s) - x(s)|ds \\
&\leq \lim_{m\to\infty} Lt\cdot \max_{s\in[0,t]}|x_{m-1}(s)x(s)| \\
&\leq \lim_{m\to\infty} Lt\cdot \max_{s\in[0,T]}|x_{m-1}(s)x(s)| \\
&= 0.
\end{aligned}$$

It remains to show that the solution is unique, for that assume $x, \hat{x} \in \mathcal{C}([0,T];\mathbb{R}^n)$ are both solutions to the IVP. Meaning that :

$$\begin{aligned}
x(t) &= x_0 + \int_0^t f(s,x(s))ds \\
\hat{x}(t) &= x_0 + \int_0^t f(s,\hat{x}(s))ds.
\end{aligned}$$

Then :

$$\begin{aligned}
|x - \hat{x}| &\leq \int_0^t |f(s,x(s)) - f(s,\hat{x}(s))|ds \leq L\cdot\int_0^t |x(s) - \hat{x}(s)|ds \\
&= L\int_0^t \underbrace{e^{-\alpha s}|x(s) - \hat{x}(s)|}_{=\rho(s)} e^{\alpha s}ds \\
&\leq L\max_{t\in[0,T]}\rho(t)\cdot\frac{1}{\alpha}(e^{\alpha t} - 1) \\
&\leq L\max_{t\in[0,T]}\rho(t)\cdot\frac{1}{\alpha}\cdot e^{\alpha t}.
\end{aligned}$$

By rearranging with the initial term :

$$\begin{aligned}
\rho(t) = e^{-\alpha t}|x(t) - \hat{x}(t)| &\leq \frac{L}{\alpha}\max_{t\in[0,T]}\rho(t) \\
\max_{t\in[0,T]}\rho(t) &\leq \frac{L}{\alpha}\max t\in[0,T]\rho(t).
\end{aligned}$$

by choosing $\alpha = 2L$:

$$\max_{t\in[0,T]} e^{-2Lt}|x(t) - \hat{x}(t)| = 0.$$

And the solutions must be equal for $\forall t \in [0, T]$. $\square$

The reason this proof deviates from the standard Picard-Lindelöf theorem, is that for our systems we require Global existence, doing so by requiring f to be globally Lipschitz continuous.

Theorem 2.1.2. The solution $x(t, t_0, x_0) \in \mathcal{C}$ is continuously dependent on (t_0, x_0)

Theorem 2.1.3 (Gronwalls inequality)**.** For $\alpha, \beta, \varphi \in \mathcal{C}([a, b]; \mathbb{R})$ $\beta \geq 0$ and

$$0 \leq \varphi(t) \leq \alpha(t) + \int_a^t \beta(s)\varphi(s)ds, \ \forall t \in [a, b].$$

then :

$$\varphi(t) \leq \alpha(t) + \int_a^t \beta(s) \exp\left(\int_s^t \beta(\tau)d\tau\right) \alpha(s)ds.$$

Proof. Denote $\psi(t) = \int_a^t \beta(s)\varphi(s)ds$ then

$$\begin{aligned}\psi'(t) &= \beta(t)\varphi(t) \leq \beta(t)\alpha(t) + \beta(t)\psi(t)\\ &= \beta(t) \cdot (\alpha(t) + \psi(t))\end{aligned}$$

.

Recall $\dot{x} + a(t)x + b(t) = 0$

$$\begin{aligned}(\dot{\psi}(t) - \beta(t)\psi(t))e^{-\int_a^t \beta(s)ds} &\leq \beta(t)\alpha(t) \cdot e^{-\int_a^t \beta(s)ds}\\ (e^{-\int_a^t \beta(s)ds}\psi(t))' &\leq \beta(t)\alpha(t) \cdot e^{-\int_a^t \beta(s)ds}.\end{aligned}$$

Integrating gives :

$$(e^{-\int_a^t \beta(s)ds}\psi(t)) \overset{\psi(a)=0}{\leq} \int_a^t \beta(s)\alpha(s)e^{-\int_a^s \beta(r)dr}ds.$$

$\square$

Definition 2.1.2 (Regularity)**.** A function $K : \mathbb{R}^{2d} \to \mathbb{R}^d$ is called regular if :

1. $K \in \mathcal{C}^1(\mathbb{R}^{2d}; \mathbb{R}^d)$ (gives local lipschitz)
2. And $\exists L > 0$ s.t. :

$$\sup_y |\nabla_x K(x,y)| + \sup_x |\nabla_y K(x,y)| \leq L.$$

Remark. We further assume K has the following properties :

$$K(x,y) = -K(y,x) \qquad \text{(antisymmetric)}$$
$$K(x,x) = 0.$$

Theorem 2.1.4. For regular K the MPS has a solution for all $T > 0$

$$\begin{cases} \frac{d}{dt} x_i &= \frac{1}{N} \sum_{j=1}^N K(x_i, x_j), 1 \leq i \leq N \\ x_i(0) &= x_{i,0} \in \mathbb{R}^d \end{cases}.$$

has a unique solution by Picard-Iteration :

$$X_N(t) = \big(x_1(t), x_2(t), \ldots, x_N(t)\big) \in \mathcal{C}^1([0,T]; \mathbb{R}^{dN}).$$

Definition 2.1.3 (Empirical Measure of a System)**.** Consider the point measure for every x_i : $\delta_{x_i(t)}$, then the measure of the System of order N is given by

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)}.$$

As shown in the introduction μ_N is a (weak-) solution to the following PDE

$$\partial_t \mu_N + \nabla \cdot (\mu_N \cdot \int K(\cdot, y) d\mu_N(y)) = 0.$$

Intuition. Now for $N \to \infty$ if we have $\mu_N \xrightarrow{\text{in some sense}} \mu$ then μ is a (weak) solution to

$$\partial_t \mu + \nabla \cdot (\mu \cdot \int K(\cdot, y) d\mu(y)) = 0.$$

with

$$\mu_0 \leftarrow \mu_N(0).$$

2.2 Weak Solutions and Distributions

Distributions are a more general class of functions and can be seen as the dual space of the space of test functions

Definition 2.2.1 (Multi-Index)**.** A multi-index $\gamma \in \mathbb{N}_0^n$ of length $|\gamma| = \sum_i \gamma_i$ for example $\gamma = (0, 2, 1) \in \mathbb{N}_0^3$ can be used to denote partial derivatives of higher order as such :

$$\partial^\gamma = \prod_i (\frac{\partial}{\partial x_i})^{\gamma_i}.$$

Remark. Only sensible cause partial derivatives commute as otherwise the index would be ambiguous.

Definition 2.2.2 (Test Functions)**.** For $\Omega \subset \mathbb{R}^d$ the space of test functions $\mathcal{D}(\Omega) \supset \mathcal{C}_0^\infty(\Omega)$. We say a sequence of test functions $(\varphi_m)_{m\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\Omega)$ converges against some limit $\varphi \in \mathcal{C}_0^\infty(\Omega)$ iff.

1. $\exists$ a compact set $K \subset \Omega$ s.t. $\operatorname{supp} \varphi_m \subset K$ for all $m \in \mathbb{N}$
2. $\forall$ multi-indexes $\alpha \in \mathbb{N}_0^n$:

$$\sup_K |\partial^\alpha \varphi_m - \partial^\alpha \varphi| \xrightarrow{m\to\infty} 0.$$

Remark. $\mathcal{D}(\Omega)$ is a linear space

Definition 2.2.3 (Distribution)**.** The space of distributions $\mathcal{D}(\Omega)'$ is the dual space of $\mathcal{D}(\Omega)$ i.e. $\mathcal{D}(\Omega)'$ contains all the continuous linear functionals T

$$T : \mathcal{D}(\Omega) \to \mathbb{K}.$$

Remark. Continuity refers to the notion that for a sequence $(\varphi_m)_{m\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ with limit φ then :

$$\varphi_m \to \varphi \;\Rightarrow\; T(\varphi_m) \to T(\varphi).$$

linearity :

$$T(\alpha\varphi_1 + \beta\varphi_2) = \alpha T(\varphi_1) + \beta T(\varphi_2).$$

We sometimes write $\langle T, \varphi\rangle$ instead of $T(\varphi)$

Definition 2.2.4 (Convergence)**.** For a sequence of distributions $(T_m)_{m\in\mathbb{N}} \subset \mathcal{D}(\Omega)'$ we say it converges against a limit $T \in \mathcal{D}(\Omega)$ iff

$$\langle T_m, \varphi\rangle \to \langle T, \varphi\rangle, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Example. Every locally integrable function $f \in L^1_{\text{loc}}(\Omega) := \{f \mid \forall K \subset \Omega, \int_K f(x)dx < \infty\}$ defines a Distribution by :

$$T_f(\varphi) = \langle T_f, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx. \quad \forall \varphi \in \mathcal{D}(\Omega).$$

i.e. $L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$

(Probability -) Measures $\mu \in \mathcal{M}(\Omega)$ define a distributions , by :

$$\langle T_\mu, \varphi\rangle = \int_{\mathbb{R}^d} \varphi(x)d\mu(x) < \infty \quad \forall \varphi \in \mathcal{D}(\Omega).$$

A prominent example is the δ distribution defined by :

$$\langle \delta, \varphi\rangle = \int_{\mathbb{R}^d} \varphi(x)d\delta = \varphi(0).$$

Remember for a measurable set E

$$\delta_x(E) = \begin{cases} 1, & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}.$$

We further introduce a class of operators called Mollifiers, which take non-smooth functions (like step functions) and smooth them out. In our case they prove useful as we can define test functions with desired properties easily, since the mollifier-kernel as defined below is required to be $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the resulting function is a test function.

Definition 2.2.5 (Mollifier-Kernel)**.** A mollifier is given by a function $j \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ with the following properties

1. $j \geq 0$
2. $\operatorname{supp} j \subset \overline{B}_1(0)$
3. $\int_{\mathbb{R}^d} j(x)dx = 1$

Example.

$$j(x) = \begin{cases} k\exp(-\frac{1}{1-|x|^2}) & \text{if } |x| < 1 \\ 0 & \text{if otherwise} \end{cases}.$$

where k is given s.t the integral is 1

Using the above example we can define a class of mollifiers called Standard Mollifier as follows

Definition 2.2.6 (Standard Mollifier)**.** For $\varepsilon > 0$ define the standard mollifier by

$$j_\varepsilon(x) = \frac{1}{\varepsilon^d} j(\frac{x}{\varepsilon}).$$

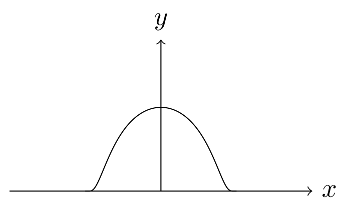


Figure 2.1: Example of a Standard Mollifier

Exercise 2.2.1. Proof that the standard mollifier converges to the delta distribution

$$j_\varepsilon \xrightarrow{\varepsilon \to 0} \delta_0.$$

Mollification is then defined as the convolution of a locally integrable function and our standard mollifier

Definition 2.2.7 (Mollification Operator)**.** $\forall u \in L^1_{\text{loc}}(\mathbb{R}^d)$ define

$$J_\varepsilon(u)(x) = j_\varepsilon \star u(x) = \int_{\mathbb{R}^d} j_\varepsilon(x-y)u(y)dy < \infty.$$

Lemma 2.2.1. If $u \in L^1(\mathbb{R}^d)$ and $\operatorname{supp} u$ is compact in $\mathbb{R}^d$ then for all fixed $\varepsilon > 0$

$$J_\varepsilon(u) = j_\varepsilon \star u \in \mathcal{C}_0^\infty(\mathbb{R}^d).$$

Furthermore if $u \in \mathcal{C}_0(\mathbb{R}^d)$ then :

$$J_\varepsilon(u) = j_\varepsilon \xrightarrow{\varepsilon \to 0} u \quad \text{uniformly in } \operatorname{supp} u.$$

Proof. Let $K = \operatorname{supp} u \subset \mathbb{R}^d$ compact then

$$\operatorname{supp} j_\varepsilon \star u = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, K) \leq \varepsilon\}.$$

$\forall i \in \{1, \dots d\}$ consider

$$\frac{\partial}{\partial x_i} \int_{\mathbb{R}^d} j_\varepsilon(x-y) u(y) dy.$$

Wether we can switch the derivative and the integral depends on wether we can bound the following

$$\frac{\partial}{\partial x_i} j_\varepsilon(x-y)|_K \leq \frac{C(j')}{\varepsilon^{d+1}} < \infty.$$

We only need to consider K, since the support is limited to K by property of u and j, then by DCT we can switch derivative and integral infinite many times

For $u \in \mathcal{C}_0(\mathbb{R}^d)$ we want to proof that

$$\|J_\varepsilon - u\|_{L^\infty(\operatorname{supp} u)} \xrightarrow{\varepsilon \to 0} 0.$$

For any $x \in \operatorname{supp} u$ it suffices to show that the convergence does not depend on x

$$\begin{aligned}
|j_\varepsilon \star u(x) - u(x)| &= \left| \int_{\mathbb{R}} j_\varepsilon(x-y)(u(y) - u(x)) dy \right| \\
&\leq \max_{\substack{x,y \in \operatorname{supp} u \\ |x-y| < \varepsilon}} |u(y) - u(x)| \cdot \int_{\mathbb{R}^d} j_\varepsilon(x-y) dy \\
&= \max_{\substack{x,y \in \operatorname{supp} u \\ |x-y| < \varepsilon}} |u(y) - u(x)| \xrightarrow{\varepsilon \to 0} 0.
\end{aligned}$$

Note the first equality follows since the integral of any mollifier is equal to 1 and it acts as a "smart 1"

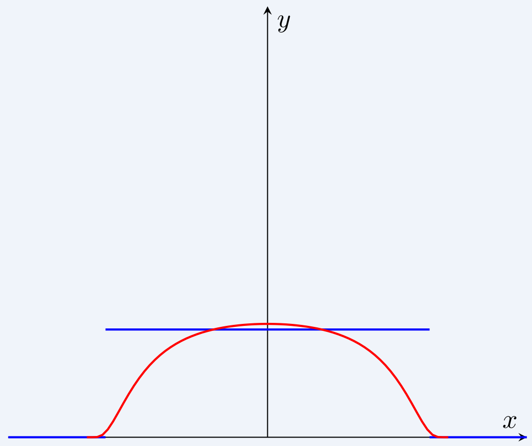
where we have used in the last step that $u \in C(\operatorname{supp} u)$ is uniformly continuous i.e $\forall \eta > 0$, $\exists \delta > 0$ s.t $\forall$ $x, y \in \operatorname{supp} u$ and $|x-y| < \delta$ we have

$$|u(x) - u(y)| < \eta.$$

$\square$

An example of how Mollification can be useful consider the following example :

Example. Consider the following step function $\mathbb{1}_{[-1,1]}(\cdot)$, then the mollification $J_\varepsilon(u)$ will look like this



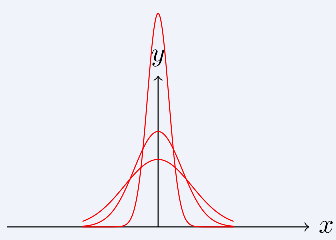
Exercise 2.2.2. Proof that the Mollification of the step function $\mathbb{1}_{[-1,1]}(\cdot)$ is a smooth function

Coming back to our empirical measure from 2.1.3, we can see the corresponding distribution is defined by :

$$\langle \mu_n, \varphi \rangle = \frac{1}{N} \sum_{i=1}^{N} \varphi(x_i).$$

Some examples in approximation of δ distribution

Example (Heat Kernel). The heat kernel $f_t(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$ approximates the δ distribution



Proof.

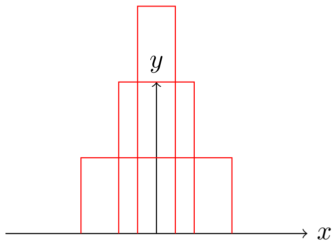
$$\begin{aligned}\lim_{t\to 0+}\langle f_t,\varphi\rangle &= \lim_{t\to 0+}\int_{\mathbb{R}} f_t(x)\varphi(x)dx = \lim_{t\to 0+}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}\varphi(x)dx\\ &= \lim_{t\to 0+}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-y^2}\varphi(2ty)dy\\ &\overset{*}{=} \varphi(0) = \langle\delta,\varphi\rangle.\end{aligned}$$

with the transformation $\frac{x}{\sqrt{2t}} = y$ □

Exercise 2.2.3. Show the change of integration and limit is valid in *

Further examples include :

$$Q_n(x) = \begin{cases}\frac{n}{2}, & \text{if } |x| \le \frac{1}{n}\\ 0 & \text{if } |x| > \frac{1}{n}\end{cases}.$$



And the dirichlet kernel

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} = 1 + 2\sum_{k=1}^{n}\cos(kx) \;\to 2\pi\delta.$$

To define the notion of a distribution solving a PDE we need to first define the way we take the derivatives of distributions

Definition 2.2.8 (Weak derivative of Distributions)**.** $\forall T \in \mathcal{D}(\Omega)'$. $\partial_i T$ is given by

$$\langle\partial_i T,\varphi\rangle := -\langle T,\partial_i\varphi\rangle \quad \forall\varphi\in\mathcal{D}(\Omega).$$

We first show this for all distributions that are defined by a $f \in L^1_{\text{loc}}$, every other distribution T also has to satisfy this property.

Exercise 2.2.4. Proof the above equality for distributions $f \in L^1_{\text{loc}}$ and show for arbitrary distribution T that :

$$-\langle T, \partial_i \varphi \rangle.$$

is continuous and linear.

Hint : Integration by parts; Why does it vanish on the boundary ?

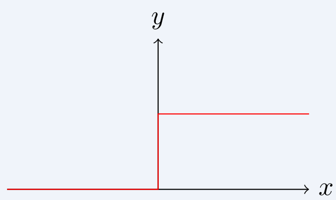
Example. For the δ distribution :

$$\langle \delta^{(k)}, \varphi \rangle = (-1)^k \varphi^{(k)}(0).$$

Example. Heaviside The Heaviside step function is defined by :

$$H = \begin{cases} 1, & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \in L^1_{\text{loc}}.$$

$$\begin{aligned} \langle H', \varphi \rangle &:= -\langle H, \varphi' \rangle = -\int_{-\infty}^{\infty} H(x)\varphi'(x)dx \\ &= -\int_0^{\infty} \varphi'(x)dx = \varphi(0) = \langle \delta, \varphi \rangle. \end{aligned}$$



Using all the above we can rewrite our many particle system (MPS) by using the empiric measure and distributions

$$\begin{cases} \frac{d}{dt} x_i & = \langle K(x_i, \cdot), \mu_N \rangle = \int K(x_i, y) d\mu_N(y) \\ x_i(0) & = x_{i,0} \end{cases}.$$

Definition 2.2.9 (Weak Solution of MFE). We say μ is a weak solution of the Mean-Field-Equation (MFE) iff for $\forall t \in [0, T]$, $\mu_t \in \mathcal{M}(\mathbb{R}^d)$ satisfies for all test functions $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ the following equation

$$\langle \mu_t, \varphi \rangle - \langle \mu_0, \varphi \rangle = \int_0^t \langle \mu_s K \mu_s, \varphi \rangle ds.$$

Remark. If $\mu_0 = \mu_{N,0}$ then $\mu_{N,t}$ is a weak solution of the MFE

Theorem 2.2.1. Let the empirical measure 2.1.3 be denoted by μ_N then for "good" (regular ?) $K(x,y)$ we have

$$\frac{d}{dt}\langle \mu_N, \varphi \rangle = \langle \mu_N \underbrace{\int K(x,y) d\mu_N(y)}_{=K\mu_N} \rangle = -\langle \nabla \cdot \nabla(\mu_N K \mu_N), \varphi \rangle.$$

Such that μ_N is a weak solution of the MPDE

$$\partial_t \mu_N + \nabla \cdot \nabla(\mu_N K \mu_N) = 0.$$

Remark. Note when we talk about weak solution, it means the PDE is solved in the sense of distributions.

Exercise 2.2.5. Show $\mu_N K \mu_N$ as defined above is a distribution for regular/ good $K(x,y)$

Proof. □

Definition 2.2.10 (characteristic problem for MFE)**.** The corresponding characteristic is given by :

$$\begin{cases} \frac{d}{dt} x(t, x_0, \mu_0) & = \int_{\mathbb{R}^d} K(x(t, x_0, \mu_0), y) d\mu_t(y) \\ x(0, x_0, \mu_0) & = x_0 \in \mathbb{R}^d \\ \mu_t & = x(t, \cdot, \mu_0) \# \mu_0 \end{cases}.$$

Notation (Push Forward)**.** For a measurable map $X : (\mathbb{R}^d, \mathcal{B}) \xrightarrow{X} (\mathbb{R}^d, \mathcal{B})$ and a measure $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ we have :

$$\forall B \in \mathcal{B} \ , \ X \# \mu_0 = \mu_0(X^{-1}(B)).$$

Exercise 2.2.6. Show that if $x(t, x_0, \mu_0) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^d)$ exists then $\mu_t =$

$x(t,\cdot,\mu_0)\#\mu_0$ is a weak solution of MFE

Definition 2.2.11. Space of probability measures with bounded first moment

$$\mathcal{P}_1(\mathbb{R}) := \{\mu_0 \in \mathcal{M}_+(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x| d\mu_0(x) < \infty\}.$$

Theorem 2.2.2 (Uniqueness of Solution). For regular K (2.1.2) and $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$ then the characteristic problem 2.2.10 has a unique solution $x(t,x_0,\mu_0) \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^d)$ and $\mu_T \in \mathcal{P}_1(\mathbb{R}^d)$, for all $t > 0$

Proof. Existence

We consider

$$\begin{aligned} x(t,x_0,\mu_0) &= x_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s,x_0,\mu_0),y) d\mu_s(y) ds \\ &\overset{\text{psh frwd.}}{=} x_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s,x_0,\mu_0), x(s,\zeta,\mu_0)) d\mu_0(\zeta) ds. \end{aligned}$$

We define the following iteration for all $y \in \mathbb{R}^d$

$$\begin{aligned} x_0(t,y) &= y \\ x_1(t,y) &= y + \int_0^t \int_{\mathbb{R}^d} k(x_0(s,y), x_0(s,\zeta)) d\mu_0(\zeta) ds \\ &\vdots \\ x_n(t,y) &= y + \int_0^t \int_{\mathbb{R}^d} k(x_{n-1}(s,y), x_{n-1}(s,\zeta)) d\mu_0(\zeta) ds \\ &. \end{aligned}$$

Similar to our proof in we show the sequence is cauchy :

$$\begin{aligned} |x_n(t,y) - x_{n-1}(t,y)| &\le \int_0^t \int_{\mathbb{R}^d} |K(x_{n-1}(s,y), x_{n-1}(s,\zeta)) - K(x_{n-2}(s,y), x_{n-2}(s,\zeta))| d\mu_0(\zeta) ds \\ &\le L \int_0^t \int_{\mathbb{R}^d} |x_{n-1}(s,y) - x_{n-2}(s,y)| + |x_{n-1}(s,\zeta) - x_{n-2}(s,\zeta)| d\mu_0(\zeta) ds \\ &. \end{aligned}$$

To get rid of the ζ we define the following banach space $\mathcal{X} = \{v \in$

$\mathcal{C}(\mathbb{R}^d; \mathbb{R}^d) : \sup_x \frac{|v(x)|}{1+|x|} < \infty\}$ with norm

$$\|v\| = \sup_{x\in\mathbb{R}^d} \frac{|v(x)|}{1+|x|}.$$

We can then further approximate by :

$$\begin{aligned} &L \int_0^t \int_{\mathbb{R}^d} |x_{n-1}(s,y) - x_{n-2}(s,y)| + |x_{n-1}(s,\zeta) - x_{n-2}(s,\zeta)| d\mu_0(\zeta) ds \\ &\le L \int_0^t |x_{n-1}(s,y) - x_{n-2}(s,y)| + \|x_{n-1}(s,\cdot) - x_{n-2}(s,\cdot)\|_{\mathcal{X}}(1+C_1) ds. \end{aligned}$$

Where $C_1 = \int |x| d\mu_0(x_0)$ is the first moment of our initial measure. Now we divide both sides of the inequality by $1+|y|$, and take the supremum in y

$$\|x_n(t,\cdot) - x_{n-1}(t,\cdot)\|_{\mathcal{X}} \le L(2+C_1) \int_0^{|t|} \|x_{n-1}(s,\cdot) - x_{n-2}(s,\cdot)\|_{\mathcal{X}} ds.$$

Then for $\forall n > m \gg 1$ we have

$$\|x_n(t,\cdot) - x_m(t,\cdot)\|_{\mathcal{X}} \le \sum_{i=m}^{n-1} \|x_{i+1}(t,\cdot) - x_i(t,\cdot)\|_{\mathcal{X}} \xrightarrow{m\to\infty} 0.$$

Therefore $(x_n(t,\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$.

Now suppose $x_n(t,\cdot) \to x(t,\cdot)$:

$$\begin{aligned} x(t,y) &= y + \int_0^t \int_{\mathbb{R}^d} K(x(s,y), x(s,\zeta)) d\mu_0(\zeta) ds \\ &= y + \int_0^t \int_{\mathbb{R}^d} K(x(s,y), z) d\mu_0(z) ds \end{aligned}$$

.

This concludes the **Existence** proof

Uniqueness :

This proof closely mimics the one presented in by using the space $\mathcal{X}$ □

Remark. Showing the convergence of our Picard Iteration here is slightly more complicated, forcing us to use a different norm to get a simpler estimate to work with, remember similar trick as in functional analysis with

$$\|f\|_L = \sup_{t\in[0,1]} e^{-Lt}|f(t)|.$$

Exercise 2.2.7. Show that $\mathcal{X} = \{v \in \mathcal{C}(\mathbb{R}^d; \mathbb{R}^d) : \sup_x \frac{|v(x)|}{1+|x|} < \infty\}$ with norm

$$\|v\| = \sup_{x \in \mathbb{R}^d} \frac{|v(x)|}{1+|x|}.$$

is a banach space

Hint : Compare to supremums norm

2.3 Wasserstein Distance

2.3.1 Goal

The goal of this section is to consider as $N \to \infty$ how the empirical measure $\mu_{N,\cdot}$ converges

$$\mu_{N,0} \xrightarrow{?} \mu_0$$
$$\mu_{N,t} \xrightarrow{?} \mu_t.$$

we have already shown that for arbitrary given measure μ_0 (on both sides of the arrows) the PDE problem is uniquely solved, the idea of the Mean Field Limit problem is to prove a stability result for the above convergence.

2.3.2 Weak Convergence of Measure (Wasserstein Distance)

Definition 2.3.1 (Weak Setting of PDE problem)**.** For all test functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \varphi(x) d\mu(t,x) - \int_{\mathbb{R}^d} \varphi(x) d\mu_0 = \int_0^t \int_{\mathbb{R}^d} K\mu(s,x) \nabla\varphi(x) d\mu(s,x).$$

Where

$$K\mu(x) = \int_{\mathbb{R}^d} K(x,y) d\mu(y).$$

To give a small recap of what we have done so far :

1. If $\mu_0 = \mu_{N,0}$ then $\mu_{N,t}$ is a weak solution of the above PDE
2. Solve the PDE for given $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$ for regular K then

   $$\mu_t = x_t \# \mu_0.$$

   is a weak solution of the PDE

The next goal is to consider the problem

$$\text{if } \mu_{N,0} \to \mu_0 \quad \text{then } \mu_{N,t} \to \mu_t.$$

$\Leftrightarrow$ stability of PDE

Definition 2.3.2 (Wasserstein Distance)**.** For all $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, $(p \geq 1)$ the Wasserstein Distance of μ and ν is given by

$$W^p(\mu,\nu) = \operatorname*{dist}_{MK,p}(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int \int_{\mathbb{R}^{2d}} |x-y|^p \pi(dxdy) \right)^{\frac{1}{p}}.$$

Where

$$\Pi(\mu,\nu) = \{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \int_{\mathbb{R}^d \times E} \pi(dx,dy) = \nu(E)$$
$$\int_{E \times \mathbb{R}^d} \pi(dx,dy) = \mu(E)\}.$$

Exercise 2.3.1. For two deterministic measures δ_x, δ_y prove

$$W^1(\delta_x,\delta_y) = |x-y|.$$

Remark. If $\varphi, \psi \in \mathcal{C}(\mathbb{R}^d)$ s.t :

$$\varphi(x) \sim |x|^p \quad \forall\ |x| \gg 1$$
$$\psi(x) \sim |y|^p \quad \forall\ |y| gg1.$$

Then

$$\int \int_{\mathbb{R}^{2d}} (\varphi(x) + \psi(y)) \pi(dx,dy) = \int_{\mathbb{R}^d} \varphi(x) d\mu(x) + \int_{\mathbb{R}^d} \psi(y) d\nu(y).$$

Corollary (Kontonovich-Rubinstein duality)**.**

$$\operatorname*{dist}_{MK,1}(\mu,\nu) = W^1(\mu,\nu)$$
$$= \sup_{\varphi \in \mathrm{Lip}(\mathbb{R}^d)} \left| \int_{\mathbb{R}^d} \varphi(x) d\mu(x) - \int_{\mathbb{R}^d} \varphi(x) d\nu(x) \right|.$$

$\mathrm{Lip}(\varphi) = 1$

The proof of the above can be found in the book xyz

Theorem 2.3.1 (Dobrushin's stability)**.** Let $\mu_0, \overline{\mu}_0 \in \mathcal{P}_1(\mathbb{R}^d)$
Then let $(x(t,\cdot,\mu_0), \mu_t(\cdot)), (x(t,\cdot,\overline{\mu}_0), \overline{\mu}_t(\cdot))$ be solutions of the corresponding PDE problem. For arbitrary $\forall\ t > 0$ it holds that the distance

$$\operatorname*{dist}_{MK,1}(\mu_t,\overline{\mu}_t) \leq e^{2|t|L} \operatorname*{dist}_{MK,1}(\mu_0,\overline{\mu}_0).$$

Proof. For initial data $\mu_0, \overline{\mu}_0$ we want to compare the trajectories

$$\begin{aligned} &x(t,x_0,\mu_0) - x(t,\overline{x}_0,\overline{\mu}_0) \\ &= x_0 - \overline{x}_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s,x_0,\mu 0), x(s,z,\mu_0)) d\mu_0(z) ds \\ &\qquad - \int_0^t \int_{\mathbb{R}^d} K(x(s,\overline{x}_0,\overline{\mu}_0), x(s,\overline{z},\overline{\mu}_0)) d\overline{\mu}_0(\overline{z}) ds. \end{aligned}$$

We need to combine the above two integrals together, while the time integrals are the same, but the space integral has to be converted into 2d dimensions by inserting $1 = \int_{\mathbb{R}^d} d\mu_0(z)$

$$\begin{aligned} x(t,x_0,\mu_0) - x(t,\overline{x}_0,\overline{\mu}_0) + \int_0^t \iint K(x(s,x_0,\mu 0), x(s,z,\mu_0)) \\ - K(x(s,\overline{x}_0,\overline{\mu}_0), x(s,\overline{z},\overline{\mu}_0)) d(\mu_0 \times \overline{\mu}_0)(z,\overline{z}) ds. \end{aligned}$$

Let $\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)$ then we can estimate point-wise for fixed $x_0, \overline{x}_0$:

$$\begin{aligned} &\|x(t,x_0,\mu_0) - x(t,\overline{x}_0,\overline{\mu}_0)\| \\ &\leq \|x_0 - \overline{x}_0\| + L \int_0^t \iint_{\mathbb{R}^{2d}} \|x(s,x_0,\mu_0) - x(s,\overline{x}_0,\overline{\mu}_0)\| \\ &\qquad + \|x(s,z,\mu_0) - x(s,\overline{z},\overline{\mu}_0)\| d\pi_0(z,\overline{z}) ds. \end{aligned}$$

Now taking the integral on both sides :

$$\begin{aligned} &\iint_{\mathbb{R}^{2d}} \|x(t,x_0,\mu_0) - x(t,\overline{x}_0,\overline{\mu}_0)\| d\pi_0(x_0,\overline{x}_0) \\ &\leq \iint_{\mathbb{R}^{2d}} \|x_0 - \overline{x}_0\| d\pi_0(x_0,\overline{x}_0) \\ &+ L \int_0^t \iint_{\mathbb{R}^{2d}} \|x(s,x_0,\mu_0) - x(s,\overline{x}_0,\overline{\mu}_0)\| d\pi_0(z,\overline{z}) ds \\ &+ L \int_0^t \int \int_{\mathbb{R}^{2d}} \|x(s,z,\mu_0) - x(s,\overline{z},\overline{\mu}_0)\| d\pi_0(z,\overline{z}) d\pi_0(z,\overline{z}) ds. \end{aligned}$$

Now we define :

$$D[\pi_0](t) = \int \int_{\mathbb{R}^d} \|x(s,z,\mu_0) - x(s,\overline{z},\overline{\mu}_0)\| d\pi_0(z,\overline{z}).$$

We obtain that the distance based on the measure π_0 at time t :

$$D[\pi_0](t) \leq D[\pi_0](0) + 2L \int_0^t D[\pi_0](s) ds.$$

We can now use Gronwalls inequality

$$D[\pi_0](t) \leq D[\pi_0](0) \cdot e^{2L|t|}.$$

Where the above inequality holds for arbitrary $\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)$

$$\inf_{\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)} D[\pi_0](t) \leq \inf_{\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)} D[\pi_0](0) \cdot e^{2Lt} = \operatorname*{dist}_{MK,1}.$$

Now all we need to show is that

$$\operatorname*{dist}_{MK,1} = \inf_{\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)} D[\pi_0](t) = \inf_{\pi_t \in \Pi(\mu_t, \overline{\mu}_t)} \int \int |x - \overline{x}| d\pi_t(x, \overline{x}).$$

It remains to be prove that for

$$\varphi_t(x_0, \overline{x}_0) = (x(t, x_0, \mu_0), x(t, \overline{x}_0, \overline{\mu}_0)).$$

the push forward measure $\varphi_t \# \pi_0 \in \Pi(\mu_t, \overline{\mu}_t)$ for $\forall$ $\pi_0 \in \Pi(\mu, \overline{\mu}_0)$ □

Exercise 2.3.2. Prove that for arbitrary $\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)$ the push forward measure $\varphi_t \# \pi_0 \exists \Pi(\mu_t, \overline{\mu}_t)$ for

$$\varphi_t(x_0, \overline{x}_0) : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \; \varphi_t(x_0, \overline{x}_0) = (x(t, x_0, \mu_0), x(t, \overline{x}_0, \overline{\mu}_0)).$$

We are also interested in what happpens when the initial data is good

Corollary. If μ_0 has a density $f_0 \in L^1(\mathbb{R}^d)$a with finite first moment :

$$\int_{\mathbb{R}^d} |x| f_0(x) dx < \infty.$$

Then the Cauchy problem

$$\begin{aligned} \partial_t f + \nabla \cdot (fkf) &= 0 \\ f_0|_{t=0} &= f_0. \end{aligned}$$

has a unique weak solution $f \in \mathcal{C}(\mathbb{R}; L^1(\mathbb{R}^d))$ i.e $\forall$ $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ it holds :

$$\int_{\mathbb{R}^d} \varphi(x) f(x,t) dx - \int_{\mathbb{R}^d} \varphi(x) f_0(x) dx = \int_0^t \int_{\mathbb{R}^d} f(x,t) K f(x,t) \nabla \varphi(x) dx ds.$$

and for all $t \in \mathbb{R}$

$$\|f(\cdot, t)\|_{L^1(\mathbb{R}^d)} \in \mathcal{C}(\mathbb{R}).$$

Proof. $\forall t \in \mathbb{R}$ proving $f_t \in L^1(\mathbb{R}^d)$ is left as an exercise

The weak formulation of the problem is :

$$\int_{\mathbb{R}^d} \varphi(x) f(t,x) dx = \int_{\mathbb{R}^d} \varphi(x) f(t,x) dx + \int_0^t \int_{\mathbb{R}^d} f(s,x) K f(s,x) \nabla \varphi(x) dx ds.$$

The idea is to choose a special test function φ as follows

$$\varphi_R(x) = \begin{cases} 1 & |x| \leq R \\ \text{smooth} & R < |x| < 2R \; . \\ 0 & |x| \geq 2R \end{cases}$$

to get the above desired function it suffices to take the mollification of the indicator function $\mathbb{1}_{[-R,R]}$ with $\frac{R}{2}$ we then get

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} f(t,x)\varphi_R(x)dx - \int_{\mathbb{R}^d} f(\hat{t},x)\varphi_R(x)dx \right| \\
&\leq \left| \int_{\hat{t}}^{t} \iint_{\mathbb{R}^{2d}} K(x,y)f(s,y)f(s,x)\nabla\varphi_R(x)dxds \right| \\
&\leq \frac{C}{R} \int_{\hat{t}}^{t} \iint_{\mathbb{R}^{2d}} (1+|x|+|y|)f(s,y)f(s,x)dxdyds \\
&\leq \frac{\tilde{C}}{R}|t-\hat{t}| \leq \hat{C}|t-\hat{t}|.
\end{aligned}$$

i.e for $F_R(t) = \int_{\mathbb{R}^d} f(t,x)\varphi_R(x)dx$ we get

$$|F_R(t) - F_R(\hat{t})| \leq C \cdot |t-\hat{t}| \xrightarrow{t \to \hat{t}} 0.$$

on the other hand

$$\int_{\mathbb{R}^d} f(t,x)\varphi_R(x)dx \xrightarrow{R\to\infty} \int_{\mathbb{R}^d} f(t,x)dx.$$

This limit is a result of how we defined $\varphi_R(x)$, we choose $\varphi_R(x)$ such that it converges against 1 as $R \to \infty$ pointwise □

Exercise 2.3.3. We already know the weak-solution μ_t exists such that it remains to show that for $\forall\, t \in \mathbb{R}$, $\mu_t \in \mathcal{P}_1(\mathbb{R}^d)$ is absolute continuous with respect to the Lebesgue measure, prove this statement

Exercise 2.3.4. Show that $|\nabla\varphi_R(x)| \leq \frac{C}{R}$

Exercise 2.3.5. Proof that $\varphi_R(x) \to 1$ as $R \to \infty$

Theorem 2.3.2 (mean field limit)**.** For arbitrary initial data $\forall\, f_0 \in L^1(\mathbb{R}^d)$

, let $\mu_{N,0} = \frac{1}{n}\sum_{i=1}^{N}\delta_{x_{i,0}}$ s.t the distance

$$\underset{MK,1}{\text{dist}}(\mu_{N,0}, f_0) \xrightarrow{N\to\infty}).$$

and $x_N(t)$ be the solution of many particle system wiht ID $x_{i,0}$ then the corresponding empirical measure $\mu_{N,t} = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ it holds :

$$\underset{MK,1}{\text{dist}}(\mu_{N,t}, f_t) \leq e^{2Lt}\underset{MK,1}{\text{dist}}(\mu_{N,0}, f_0).$$

Where f_t is the corresponding density resulting from the previous corollary of the PDE weak solution.
and furthermore $\mu_{N,t} \xrightarrow{N\to\infty} f(\cdot, t)$ weakly in measure i.e

$$\forall \varphi \in \mathcal{C}_b(\mathbb{R}^d) \ : \ \int \varphi d\mu_{N,t} \to \int \varphi(x) f_t(x) dx.$$

Proof. We have to show that $\forall\ \varphi \in C_0(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi d\mu_{N,t} \to \int_{\mathbb{R}^d} \varphi f_t dx.$$

means $\forall\ \varphi \in Lip(\mathbb{R}^d)$ it holds

$$\int \varphi d\mu_{N,t} \to \int \varphi f_t dx.$$

The proof relies on using ε-δ language and combining that with our mollification results
$\forall\ \varepsilon > 0$ set $R \gg 1$ s.t $\frac{2C\|\varphi\|_\infty}{2R} \leq \frac{\varepsilon}{2}$, and let $\varphi_m \in \mathcal{C}_0^\infty(B_{2R})$ s.t $\exists\ M > 0\ \forall\ m > M$

$$\|\varphi_m - \varphi\|_{L^\infty(B_R)} < \frac{\varepsilon}{4}.$$

For $\varphi_{M+1} \in \mathcal{C}_0^\infty(B_{2R})$ there $\exists\ N_1$ s.t $\forall\ N > N_1$

$$\left|\int \varphi_{M+1}(d\mu_{N,t} - f_t dx)\right| < \frac{\varepsilon}{4}.$$

Using the above

$$
\begin{aligned}
&\left|\int \varphi d\mu_{N,t} - \int \varphi f_t dx\right| \\
&\leq \left|\int_{B_R} \varphi(x)(d\mu_{N,t} f_t dx)\right| + \left|\int_{B_R^C} \varphi(x)(d\mu_{N,t} - f_t d(x))\right| \\
&\overset{\text{Tri.}}{\leq} \left|\int_{B_R} \varphi(x)_{M+1}(d\mu_{N,t} - f_t dx)\right| + \left|\int_{B_R} (\varphi_{M+1} - \varphi)(d\mu_{N,t} - f_t dx)\right| \\
&+ \left|\int_{B_R^c} \varphi(x)\frac{|x|}{R}(d\mu_{N,t} - f_t dx)\right| \\
&\leq \frac{C}{R} < \frac{\varepsilon}{2}.
\end{aligned}
$$

□

Exercise 2.3.6. Find a sequence of empirical measures that converges against a density.

Chapter 3

Excercise Sheets

3.1 Sheet 1 (11.09.2023)

3.1.1 Excercise 1

Question 1. Consider the second order system :

$$
\begin{aligned}
dX_t^i &= V_t^i \\
dV_t^i &= \frac{1}{N}\sum_{i=1}^{N} F(t, X_t^i, V_t^i, X_t^j, V_t^j)dt.
\end{aligned}
$$

on $[0, T]$ for some smooth interaction force $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ following the lecture we assume the empirical measure :

$$
\mu_t^N(dx, dv) = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_t^i, V_t^i)}.
$$

converges in some sense to the measure μ_t with density ρ_t for each t. Derive an equation for $\rho, t \geq 0$ similar to the lecture.

Solution. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^?)$ and calculate :

$$\begin{aligned}
\frac{d}{dt}\langle \mu_N, \varphi\rangle = \frac{d}{dt}\int_{\mathbb{R}^{2d}} \varphi(x,v) d\mu_N(t)(dx,dv) = \frac{d}{dt}\int \frac{1}{N}\sum_{i=1}^{N} \varphi(x,v) d\delta_{(x_i^t, v_i^t)} \\
&\overset{*}{=} \frac{1}{N}\sum_{i=1}^{N} \frac{d}{dt}\varphi(x_i(t), v_i(t)) \\
&\overset{\text{Chain.}}{=} \frac{1}{N}\sum_{i=1}^{N} \partial_x\varphi \cdot \dot{x}_i + \partial_v \cdot \dot{v}_i \\
&= \frac{1}{N}\sum_{i=1}^{N} \partial_x\varphi \cdot v_i(t) + \partial_v\varphi \cdot \sum_{j=1}^{N} F(t, x_i(t), v_i(t), x_j(t), v_j(t))
\end{aligned}$$

.

$\square$

Chapter 4

Appendix

Theorem 4.0.1 (Divergence Theorem). Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)-$ dimensional sub-manifold of $\mathbb{R}^n$. Let $F : \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously to $\partial\Omega$. Then we have :

$$\int_{\Omega} \nabla \cdot F d\mu = \int_{\partial\Omega} F \cdot N d\sigma.$$

where N is the outward pointing normal. (last component is positive)