MEAN FIELD PARTICLE SYSTEMS AND THEIR LIMITS TO NONLOCAL PD'S

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Contents

Chapter 1

PDE Approach To Solving the Makean-Vlasov Equation

1.1 Motivation

Above we saw an SDE approach to solving the Makean-Vlasov Equation, in this section we instead focus on a PDE based approach. From now on we assume $\sigma(Y(t), \mu(t)) = \sqrt{2}$ is a constant, then the (MVE) can be rewritten as

$$\text{(MVE*)} \begin{cases} Y(t) & = b(Y(t), \mu(t))dt + \sqrt{2}dW_t \\ Y(0) & = \xi \in L^2(\Omega) \\ \mu_0 & = \mathcal{L}(\xi) \end{cases} .$$

by applying Itôs formula for $\forall \varphi \in \mathcal{C}_0^\infty([0,T) \times \mathbb{R}^d)$

$$\begin{aligned} \varphi(Y(t), t) - \varphi(Y(0), 0) = & \int_0^t \frac{\partial \varphi}{\partial t}(Y(s), s) + \nabla \varphi(Y(s), s) \cdot b(Y(s), \mu(s)) \\ & + \frac{1}{2} \underbrace{\sqrt{2} \cdot \sqrt{2})}_{tr(\sigma \cdot \sigma^T)} \cdot \Delta \varphi(Y(s), s) ds \\ & + \int_0^t \nabla \varphi(Y(s), s) \sqrt{2} dW_s. \end{aligned}$$

and taking the expectation on both sides, such that the last term disappears

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(x, t) d\mu(t) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0 \\ & = \int_0^t \int_{\mathbb{R}^d} \frac{\partial \varphi}{\partial t}(x, s) + \nabla \varphi(x, s) \cdot b(x, \mu(s)) \cdot \Delta \varphi(x, s) d\mu(s) ds. \end{aligned}$$

This leads us to formulating the following weak PDE, if μ is regular enough i.e it has density and the density has enough regularity, then μ should satisfy

$$\begin{cases} \partial_t \mu - \Delta \mu + \nabla \cdot (b(x, \mu) \cdot \mu) = 0 \\ \mu(0) = \mu_0 \end{cases} .$$

Remark. Compare this weak PDE to the one we got in the discrete case, what do you notice ?

Exercise. Show that the integral equation and the weak formulation are equal.

Remark. Now suppose we find μ with density u satisfying the weak PDE, then we can plug it in to the (MVE) equation to get

$$\begin{cases} dY_t = b(Y_t, u)dt + \sqrt{2}dW_t \\ Y(t) = \xi \in L^2(\Omega) \quad \mathcal{L}(\xi) = u \end{cases}.$$

Now if b is bounded and Lipschitz continuous, then we get a solution Y_t. Now if $\overline{u}$ is the Law of Y_t. Then by Itô formula we have for $\forall \varphi \in \mathcal{C}_0^\infty$

$$\begin{aligned} &\int_{\mathbb{R}^d} \varphi(x,t) d\overline{\mu(t)} - \int_{\mathbb{R}^d} \varphi(x,0) u_0(x) dx \\ &= \int_0^t \int_{\mathbb{R}^d} \left(\frac{\partial \varphi}{\partial t}(x,s) + \nabla\varphi(x,s) \cdot b(x,u) - \Delta\varphi(x,s) \right) \overline{u}(x,t) dx ds. \end{aligned}$$

Which means $\overline{\mu}$ satisfies

$$\begin{cases} \partial_t \overline{\mu} - \Delta\overline{\mu} + \nabla \cdot (b(x,u) \cdot \overline{\mu}) = 0 \\ \overline{\mu}|_{t=0} = u_0 \end{cases}.$$

If we can prove $\overline{u} = u$, then we get a solution to the **??**.

Example. A common choice of b is the following for some kernel K

$$b(Y_t, u) = \int K(Y_t - y) u(y) dy = \int K(y) u(Y_t - y) dy.$$

then the regularity of b by convolution depends on either K or u

1.2 Problem Definition

Definition 1.2.1 (Weak PDE). Let μ have density u, then we write

$$\text{(PDE)} \begin{cases} \partial_t u - \Delta u + \nabla \cdot (b(x,u) \cdot u) = 0 \\ u(0) = u_0 \end{cases}.$$

Formalize by adding the relevant spaces

Definition 1.2.2 (Sobolev Spaces). We define roughly

$$\begin{aligned} H^1(\mathbb{R}^d) &= \{u \in L^2(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d)\} \\ \|u\|_{H_1} &= \|u\|_2 + \|\nabla u\|_2. \end{aligned}$$

where the gradient is defined for $\forall \varphi \in \mathcal{C}_0^\infty$

$$\nabla u = \langle \nabla u, \varphi \rangle = -\langle u, \nabla\varphi \rangle.$$

And the dual space

$$H^{-1}(\mathbb{R}^d) = (H^1(\mathbb{R}))' = \{l : l \text{ is bounded linear functional of } H^1(\mathbb{R}^d)\}.$$

Then

$$L^2([0,T]; H^1(\mathbb{R}^d)) = \{u : \int_0^T \|u(t)\|_{H^1} dt < \infty\}.$$

Remark. The Sobolev space H^1 is a separable Hilbert space

Definition 1.2.3 (Weak Solution)**.** We say that a function

$$u \in L^2([0,T]; H^1(\mathbb{R}^d) \cap L^\infty([0,T]; L^2(\mathbb{R}^d))).$$

with $\partial_t u \in L^2([0,T]; H^{-1}(\mathbb{R}^d))$ is a weak solution of the (PDE) if for $\forall \varphi \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)$ it holds

$$\begin{aligned} \int_0^T \langle \partial_t u, \varphi \rangle_{(H^{-1}, H^1)} dt = \int_0^T \int_{\mathbb{R}^d} \nabla \varphi \cdot (b(x,u) \cdot u) dx dt \\ - \int_0^T \int_{\mathbb{R}^d} \nabla u \cdot \nabla \varphi dx dt. \end{aligned}$$

1.3 Heat Equation and the Heat Kernel

Motivation

Definition 1.3.1 (Heat equation)**.** The following PDE is called the inhomogenes Heat equation with source term f

$$\text{(HE)} \begin{cases} \partial_t u(x,t) - \Delta u(x,t) & = f(x,t) \\ u|_{t=0} & = u_0 \end{cases}.$$

Remark. Compare this to our PDE which looks similar, but is in fact non-linear

$$\partial_t u - \Delta u + \nabla \cdot (b(x,u) \cdot u) = 0.$$

Remark. Let us suppose $K(x,t)$ is a heat kernel, then

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^d} K(x-y,t) u_0(y) dy - \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s) \nabla \cdot (b(y,u(y,s)u(y,s)) dy ds \\ &= u_1(x,t) + u_2(x,t). \end{aligned}$$

is a solution to the inhomogenous Heat-Equation, this is called Duhamel's principle i.e. we can "add" up solutions to homogeneous problems and get the solution to the inhomogenous.

Remark. We say the heat kernel is the density of the Brownian Motion.

1.3.1 Derivation by Fourier Transform

Definition 1.3.2 (Fourier Transform)**.** For $x \in \mathbb{R}^d$ the Fourier transform is defined as

$$\mathcal{F} : L^2 \to L^2 \ u \mapsto \hat{u}.$$

where

$$\hat{u}(k) = \int_{\mathbb{R}^d} u(x) e^{ix \cdot k} dx.$$

Exercise. Proof

$$-\widehat{\Delta u} = |k|^2 \hat{u}(k).$$

Hint

$$\widehat{\nabla u} = \frac{k}{i}\hat{u}(k).$$

Remark. Using the Fourier transformation we can transform our PDE into an ODE

$$\begin{cases} \partial_t \hat{u} - \widehat{\Delta u} & = \hat{f} \\ \hat{u}|_{t=0} & = \hat{u}_0 \end{cases}.$$

that is

$$\begin{cases} \partial_t \hat{u}(k) + |k|^2 \hat{u}(k) = \hat{f}(k) \\ \hat{u}_0(k) = \hat{u}_0 \end{cases}.$$

where

$$\hat{u}(k,t) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t e^{-|k|^2(t-\tau)}\hat{f}(k,\tau)d\tau.$$

Lemma 1.3.1 (Inverse transformation of the Fourier transformation)**.**

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-\tau))^{\frac{d}{2}}} e^{\frac{-|x-y|^2}{4(t-\tau)}} f(y,\tau)dyd\tau.$$

Definition 1.3.3 (Heat Kernel)**.** The following is called Heat Kernel

$$K(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}.$$

and for $\forall t > 0$ it is a solution to the homogeneous heat equation

$$\partial_t K - \Delta K = 0.$$

And

$$K \xrightarrow{t \to 0^+} \delta.$$

In the sense of distributions.

Theorem 1.3.1 (Solution To Heat Equation)**.** Let K be the Heat kernel and initial data $u_0 \in \mathcal{C}_b(\mathbb{R}^d)$ and $f \in \mathcal{C}^{2,1}(\mathbb{R}^d \times [0,T])$ with compact support (schwarz function would work as well, since they lie dense in compact)

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy - \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s)\nabla \cdot (b(y,u(y,s)u(y,s))dyds \\ &= u_1(x,t) + u_2(x,t). \end{aligned}$$

is a solution to the heat equation, in fact u_1 and u_2 are solutions to

$$\text{(P1)}\begin{cases} \partial_t u_1 - \Delta u_1 = 0 \\ u_1(0) = u_0 \end{cases} \qquad \text{(P2)}\begin{cases} \partial_t u_2 - \Delta u_1 = f \\ u_2(0) = u_0 \end{cases}.$$

respectively

Proof. We begin by showing that u_1 is a solution to (P1) by showing

$$\lim_{t\to 0^+} u_1(x,t) = \lim_{t\to 0^+} \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy \overset{!}{=} u_0.$$

$$\begin{aligned}
\lim_{t\to 0^+} \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy &= \lim_{t\to 0^+} \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy \\
&= \lim_{t\to 0^+} \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-|z|^2} u_0(x+2\sqrt{t}z)dz \\
&= \int_{\mathbb{R}^d} \lim_{t\to 0^+} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-|z|^2} u_0(x+2\sqrt{t}z)dz \\
&= u_0(x).
\end{aligned}$$

where we used the change of variables

$$\frac{x-y}{2\sqrt{t}} = -z.$$

Then for $\forall t > 0$

$$\begin{aligned}
\partial_t u_1 - \Delta u_1 &= (\partial_t - \Delta) \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy \\
&= \int_{\mathbb{R}^d} (\partial_t - \Delta)K(x-y,t)u_0(y)dy \\
&= 0.
\end{aligned}$$

by properties of the Heat-Kernel.
For $u_2(x,t)$

$$u_2(x,t) = \int_0^t \int_{\mathbb{R}^d} K(y,s)f(x-y,t-s)dyds.$$

First note that

$$\lim_{t\to 0^+} u_2(x,t) = 0.$$

Then by applying

$$\begin{aligned}
(\partial_t - \Delta)u_2 &= \int_0^t \int_{\mathbb{R}^d} K(y,s)(\partial_t - \Delta_x)f(x-y,t-s)dyds \\
&+ \int_{\mathbb{R}^d} K(y,t)f(x-y,0)dy \\
&= \int_0^\varepsilon \int_{\mathbb{R}^d} K(y,s)\left(-\partial_s - \Delta_y\right) f(x-y,t-s)dyds + \int_\varepsilon^t \int_{\mathbb{R}^d} \dots \\
&+ \int_{\mathbb{R}^d} K(y,t)f(x-y,0)dy \\
&= I_\varepsilon + J_\varepsilon + L.
\end{aligned}$$

We are allowed to exchange the order because of the Heat-Kernel since it decays very fast this gives uniform integrability the further away we get from the origin
We have, since $f \in \mathcal{C}_b^{2,1}$ we get that its term is bounded and the integral for the Heat kernel

is just 1

$$\begin{aligned}
|I_\varepsilon| &\le C \cdot \varepsilon \\
J_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^d} K(y,s)(-\partial_t - \Delta_y) f(x-y, t-s) dy ds \\
&= \int_\varepsilon^t \int_{\mathbb{R}^d} \underbrace{(-\partial_t - \Delta_y) K(y,s)}_{=0} f(x-y,t-s) dy ds \\
&+ \int_{\mathbb{R}^d} K(y,\varepsilon) - f(x-y, t-\varepsilon) dy \\
&\underbrace{- \int_{\mathbb{R}^d} K(y,t) f(x-y,0) dy}_{=Li} .
\end{aligned}$$

$\square$

Together we have

$$\begin{aligned}
\partial_t u_2 - \Delta u_2 &= \lim_{\varepsilon \to 0} \left(\int_{\mathbb{R}^d} \underbrace{K(y,\varepsilon)}_{\to \delta} f(x-y, t-\varepsilon) dy + \underbrace{C\varepsilon}_{\to 0} \right) \\
&= f(x,t).
\end{aligned}$$

This shows that

$$u(x,t) = \int_{\mathbb{R}^d} K(x-y,t) u_0(y) dy - \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s) \nabla \cdot (b(y, u(y,s) u(y,s)) dy ds$$

.

is a solution to the inhomogenous heat equation.

Remark. When changing order or variable of derivative one has to be aware of the boundary terms appearing. For example when changing the order for the ∂_t derivative one gets two boundary terms , where one is not well behaved $t = 0$

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (b(x,u) \cdot u) = 0 \\ u|_{t=0} = u_0 \in L^1 \ \ \int (1 + |x|^2) u_0 < \infty \end{cases} .$$

Formally

$$u(x,t) = \int_{\mathbb{R}^d} K(x-y,t) u_0(y) dy - \int_0^t \int_{\mathbb{R}^d} K(x-y,t-\tau) \nabla \cdot (b(y, u(y,\tau)) \cdot u(y,\tau)) dy d\tau.$$

Now we start with bounded drift term for a linear equation.

Definition 1.3.4 (LDE). For bounded drift term $\bar{b} \in L^\infty$ we define

$$(\text{LDE}) \begin{cases} \partial_t - \Delta u + \nabla \cdot (\bar{b}(x,t) u) = 0 \\ u|_{t=0} = u_0 \end{cases} .$$

Remark. By first proving the existence of a solution to this simpler equation we can then construct an iteration that will yield a solution to the more complex non-linear , non-local one.

Theorem 1.3.2 (Uniqueness and Existence of LDE Solution). If $b \in L^\infty([0,T] \times \mathbb{R}^d)$ and

$u_0 \in L^1(\mathbb{R}^d)$, then the LDE has a unique solution $u \in L^\infty([0,T]; L^1(\mathbb{R}^d))$

$$u(x,t) = \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \nabla K(x-y,t-\tau) \cdot (\bar{b}(y,\tau)u(y,\tau))dyd\tau.$$

Proof. We prove again by Iteration, and fix point argument, consider a map

$$\begin{aligned}
&\mathcal{T} : L^\infty([0,T]; L^1(\mathbb{R}^d)) \to L^\infty([0,T]; L^1(\mathbb{R}^d)) \\
&u \mapsto \mathcal{T}(u) = \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \nabla K(x-y,t-\tau) \cdot (\bar{b}(y,\tau)u(y,\tau))dyd\tau.
\end{aligned}$$

We need to check $\mathcal{T}(u) \in L^\infty([0,T]; L^1(\mathbb{R}^d))$, for $\forall t > 0$

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathcal{T}(u)(x,t)|dx &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x-y,t)|u_0(y)|dydx \\
&+ \int_0^t d\tau \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy |\nabla K(x-y,t-\tau)\bar{b}(y,\tau)u(y,\tau)| \\
&= I + II.
\end{aligned}$$

Since we have fixed $t > 0$ we use Fubini

$$\begin{aligned}
I &\le \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} K(x-y,t)dx}_{=1} |u_0(y)|dy \\
&\le \|u_0\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

First consider the gradient of K

$$\begin{aligned}
\int_{\mathbb{R}^d} \nabla K(x,s)dx &= \frac{1}{(4\pi s)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{s}} \left|\frac{4}{2\sqrt{s}}\right| e^{-\frac{|x|^2}{4s}} dx \\
&\le \frac{1}{\sqrt{s}} C.
\end{aligned}$$

Then for the second term we get

$$\begin{aligned}
II &\le \|\bar{b}\|_{L^\infty} \int_0^t d\tau \int_{\mathbb{R}^d} |\nabla K(x,t-\tau)|dx \int_{\mathbb{R}^d} u(y,\tau)dy \\
&\le \|\bar{b}\|_{L^\infty} \|u\|_{L^\infty(L^1)} C \cdot \int_0^t \frac{1}{\sqrt{s}} dy \\
&= C \cdot \sqrt{t-\tau}.
\end{aligned}$$

Note we we do not need to consider y in K since we can use a translation,$L^\infty(L^1) = L^\infty([0,T]; L^1(\mathbb{R}^d))$

This shows that our map $\mathcal{T}$ is indeed well defined, next we proof $\mathcal{T}(u)$ is a contraction, for

$\forall u_1, u_2 \in L^\infty(L^1)$, for $t^\star$ s.t. $C\|\bar{b}\|_\infty \sqrt{t^\star} < \frac{1}{2}$

$$
\begin{aligned}
&\|\sqcup^\star(u_1) - \mathcal{T}(u_2)\|_{L^\infty(L^1)} \\
&= \operatorname*{ess\,sup}_{0\le t\le t^\star} \int_{\mathbb{R}^d} |\mathcal{T}(u_1) - \mathcal{T}(u_2)|(x,t)dx \\
&\le \operatorname*{ess\,sup}_{0\le t\le t^\star} \int_0^t d\tau \int_{\mathbb{R}^d} dy |\nabla K(x-y,t-\tau)(\bar{b}(y,\tau)(u_1-u_2))(y,\tau)| \\
&\le \operatorname*{ess\,sup}_{0\le t\le t^\star} \|\bar{b}\|_{L^\infty} \|u_1-u_2\|_{L^\infty(L^1)} \int_0^t \frac{1}{\sqrt{t-\tau}} d\tau \\
&\le \operatorname*{ess\,sup}_{0\le t\le t^\star} C\|\bar{b}\|_{L^\infty} \sqrt{t} \|u_1-u_2\|_{L^\infty(L^1)} \\
&\le C\|\bar{b}\|_\infty \sqrt{t^\star} \|u_1-u_2\|_{L^\infty(L^1)}.
\end{aligned}
$$

Then $\mathcal{T}$ is a contraction. Since $t^\star$ only depends on $\|\bar{b}\|_\infty$ and dimension d. Then for any given $T > 0$ we can repeat the above argument finite many time and obtain

$$u \in L^\infty([0,T]; L^1(\mathbb{R}^d)).$$

□

Exercise. Think about wether you can proof it for b satisfying linear growth condition

$$|\bar{b}| \le C(1+|x|).$$

Let us discuss how a solution to the LDE leads back to a solution to the more complex

$$\partial_t u - \Delta u + \nabla \cdot (b(x,u)u) = 0.$$

where $u \in L^\infty(L^1)$ under the assumption on $b(x,u)$

$$|b(x,u) - b(\tilde{x},\tilde{u})| \le L(|x-\tilde{x}| + W_2(u,\tilde{u})).$$

By fixing $\tilde{x} = 0$

$$|b(x,u) - b(0,\delta_0)| \le L(|x| + W_2(u,\delta_0)).$$

where

$$W_2(u,\delta_0) \le \left(\int_{\mathbb{R}^d} |x|^2 u(x) dx\right)^{\frac{1}{2}}.$$

when b is unbounded we consider , the cutoff

$$\bar{b}(x, v(x,t)) = \min\{b(x,v(x,t)), M\}.$$