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Sheet 7

Unclear if $u \in \mathcal{C}(\mathbb{R}^n)$ is $u : \mathbb{R}^n \to \mathbb{R}$ or $u : \mathbb{R}^n \to \mathbb{R}^n$, we consider the first case here but it doesn't change any of the following proofs

19. Twirling towards freedom

Let $u \in \mathcal{C}^2(\mathbb{R}^n)$ be a harmonic function. Show that the following functions are also harmonic.

Note Whenever we write $\frac{\partial v}{\partial x_i}$ we are really referring to $\tilde{x}(x)_i$

Exercise (a). $v(x) = u(x + b)$ for $b \in \mathbb{R}^n$

Solution. Using chain rule gives

$$\frac{\partial v}{\partial x_i} = \frac{\partial u}{\partial x_i} \cdot \frac{\partial (x + b)}{\partial x_i} = \frac{\partial u}{\partial x_i} \cdot 1.$$

Second derivative gets another $\cdot \, 1$.

Such that summing gives

$$\triangle v = 1 \cdot \triangle u = 0.$$

$\square$

Exercise (b). $v(x) = u(ax)$ for $a \in \mathbb{R}^n$

Solution. Again using chain rule

$$\frac{\partial v}{\partial x_i} = \frac{\partial v}{\partial x_i} \cdot \frac{\partial (ax)}{\partial x_i} = \frac{\partial v}{\partial x_i} \cdot a.$$

For the second derivative :

$$\frac{\partial v}{\partial x_i} = \frac{\partial^2 u}{\partial x_i^2} \cdot \frac{\partial (ax)}{\partial x_i} \cdot a = \frac{\partial^2 u}{\partial x_i} \cdot a^2.$$

$$\triangle v = a^2 \triangle u = a^2 \cdot 0 = 0.$$

$\square$

Exercise (c). $v(x) = u(Rx)$ for $R(x_1, \ldots, x_n) = (-x_1, \ldots, x_n)$

Solution. Using chain rule

$$\frac{\partial v}{\partial x_i} = \frac{\partial v}{\partial x_i} \cdot \frac{\partial (Rx)}{\partial x_i} = \begin{cases} -1 & \frac{\partial v}{\partial x_1} \text{ if } i = 1 \\ & \frac{\partial v}{\partial x_i} \end{cases}.$$

Second derivative if $i = 1$ we get another -1 and they cancel out s.t.

$$\frac{\partial^2 v}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2}.$$

Summing gives

$$\triangle v = \triangle u = 0.$$

$\square$

Exercise (d). $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$

Solution. Note $(Ax)_i = \sum_{j=1}^n A_{ji} x_j$ such that using the chain rule

$$\frac{\partial v}{\partial x_i} = \frac{\partial u(Ax)}{\partial (Ax)_i} \frac{\partial Ax}{\partial x_i} = \nabla u \cdot \underbrace{A_i}_{i\text{-th col}}.$$

We get the gradient of u since we take the derivative of u with respect to $(Ax)_i$

Second derivative

$$\frac{\partial}{\partial x_i} \frac{\partial v}{\partial x_i} = \frac{\partial}{\partial (Ax)_i} \nabla u(Ax) \cdot A_i \cdot \frac{\partial Ax}{\partial x_i} = \nabla \cdot (\nabla u \cdot A_i) A_i.$$

Note since A is orthogonal we get $AA^T = \mathbb{1}$, writing out the terms as sum for clarity

$$\sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} A_{ji} A_{ki} = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0.$$

$\square$

20. Harmonic Polynomials in Two Variables

Exercise (a). Let $u \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic

Solution. First note that $u^{(k)}(x_0) \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$
and $x_0 \in \mathbb{R}^n$
IA: $k = 1$

$$\triangle(u^{(1)}) = \left(\frac{\partial^2}{\partial x_i^2}, \dots, \frac{\partial^2}{\partial x_i^2} \right) \begin{pmatrix} \frac{\partial u(x_0)}{\partial x_1} \\ \vdots \\ \frac{\partial u(x_0)}{\partial x_n} \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \cdot \frac{\partial u}{\partial x_i}.$$

The order of differentiation does not matter such that we get

$$\triangle(u^{(1)}) = \frac{d}{dx}(\triangle u^1) = 0.$$

IV: Suppose the statement holds for $k \in \mathbb{N}$
IS: $k \to k + 1$
We know that $u^{(k)} \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ is harmonic by IV. such that by relabeling $u^{(k)} := v \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ we can again apply the argument from the IA. which concludes the proof $\qquad \square$

Exercise (b). Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x,y) = (x + iy)^n$ and let $u_n(x,y)$ and $v_n(x,y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic.

Solution. Treating the Laplacian in $\mathbb{C}$ as we do in $\mathbb{R}$

$$\partial_x f_n = n(x + iy)^{n-1}$$
$$\partial_y f_n = n \cdot i(x + iy)^{n-1}.$$

Then

$$\triangle f_n = n(n-1)(x + iy)^{n-2} + (i^2 n(n-1)(x + iy))$$
$$= n(n-1) \cdot \left((x + iy)^{n-2} - (x + iy)^{n-2} \right)$$
$$= 0.$$

Since u_n, v_n are real and imaginary part , their Laplacian must be 0 as well. $\qquad \square$

Exercise (c). A homogeneous polynomial of degree n in two variables is a

polynomial of the form

$$p = \sum a_k x^k y^{n-k}.$$

Show that $\partial_x p$ and $\partial_y p$ are homogeneous of degree $n - 1$

Solution. Calculating

$$\partial_x p = \frac{\partial}{\partial x} \sum_{k=0}^{n} a_k x^k y^{n-k} = \sum_{k=0}^{n} k \cdot a_k x^{k-1} y^{n-k}.$$

For $k = 0$ we have $k \cdot a_k = 0$ such that the first term vanishes

$$\sum_{k=0}^{n} k a_k x^{k-1} y^{n-k} = \sum_{k=1}^{n} k a_k x^{k-1} y^{n-k} = \sum_{i=0}^{n-1} (i+1) a_{i+1} x^i y^{(n-1)-i}.$$

For the case ∂_y we get the term $(n - k) \cdot a_k$ which is 0 for $k = n$ such that the last term of our sum vanishes and the degree is reduced by 1

□

Exercise (d). Show that such a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n

Solution. "⇐": Let p be a linear combination of u_n and v_n then by (b) we know u_n and v_n are harmonic, since the Laplacian is a linear operator, we conclude p must also be harmonic.

Other direction is missing

□

Louisville's Theorem

Liouville's theorem (3.9 in the script) says that if u is bounded and harmonic, then u is constant. In this question we give a geometric proof in $\mathbb{R}^2$ using globular means defined when $B(x, r) \subset \Omega$ through

$$\mathcal{M}[v](x, r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y) dy.$$

Recall from Theorem 3.5 that if u is harmonic then it obeys $u(x) = \mathcal{M}[u](x, r)$.

Exercise (a). Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centered on the two points respectively. Show that the area of the intersection is

$$\text{area} B(a, r) \cap B(b, r) = 2r^2 a \cos(dr^{-1}) - 2d\sqrt{r^2 - d^2}.$$

Solution. W.l.o.g we may assume $a = (0,0)$ and $b = (2d, 0)$, otherwise we just shift the plane to be centered at a first and rotate (both volume preserving)

In that case we can solve for the points of intersection by

$$x^2 + y^2 = r^2 = (x - 2d)^2 + y^2.$$

Such that

$$x^2 - (x - 2d)^2 = 0$$
$$x^2 - (x^2 - 4dx + 4d^2) = 0$$
$$4dx - 4d^2 = 0$$
$$x = d.$$

And y

$$d^2 + y^2 = r^2 \Rightarrow y = \pm\sqrt{r^2 - d^2}.$$

Now either by integrating we get

$$x^2 + y^2 = r^2 \Rightarrow x = \sqrt{r^2 - y^2}.$$

$$2 \cdot \left(\int_{-\sqrt{r^2-d^2}}^{\sqrt{r^2-d^2}} \sqrt{r^2 - y^2} dy - 2d \cdot \sqrt{r^2 - d^2} \right).$$

Or the prettier way

We can get the light-green area by calculating the area of the triangle and subtracting it from the area of the sector and multiplying by 4, this gives the following
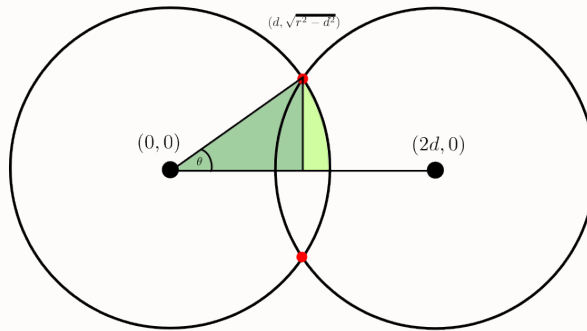
Triangle area

$$\frac{1}{2} \cdot d \cdot \sqrt{r^2 - d^2}.$$

Sector area , since $\theta = \cos^{-1}(\frac{d}{r})$ and the area of the sector is proportional to the angle where $\theta = 2\pi$ gives the full circle we get

$$\frac{r^2 \theta}{2}.$$

Putting it together

$$4 \cdot \left(\frac{r^2 \cos^{-1}(\frac{d}{r})}{2} - \frac{d}{2}\sqrt{r^2 - d^2} \right)$$

$$= 2r^2 \cos^{-1}(dr^{-1}) - 2d\sqrt{r^2 - d^2}.$$

Exercise (b). Suppose that v is a bounded function on the plane $-C \leq v(x) \leq C$ for all x and some constant C. Show that

$$|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)| \leq \frac{2C}{\omega_2}(\pi - 2a\cos(\frac{d}{r})) - \frac{2d}{r}\sqrt{1-d^2r^2}.$$

Solution. We get

$$|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)| = \frac{1}{\omega_2 r^2}\left|\left(\int_{B(a,r)} v(y)dy - \int_{B(b,r)} v(y)dy\right)\right|$$

.

leaving the constant

$$= \left|\left(\int_{B(a,r)\setminus B(b,r)} v(y)dy - \int_{B(b,r)\setminus B(a,r)} v(y)dy\right)\right|$$

$$\leq \int_{B(a,r)\setminus B(b,r)} |v(y)|dy + \int_{B(b,r)\setminus B(a,r)} |v(y)|dy$$

$$\overset{\text{def.}}{\leq} \int_{B(a,r)\setminus B(b,r)} Cdy - \int_{B(b,r)\setminus B(a,r)} Cdy$$

.

Then by drawing a picture and looking at it we know

$$\text{area}(B(a,r)\setminus B(b,r)) + \text{area}(B(a,r)\setminus B(b,r)) = 2\pi r^2 - 2\cdot\text{area}(B(a,r)\cap B(b,r)).$$

Using (a)

$$|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)| \leq \frac{2C}{\omega_2}\cdot\left(\pi - 2\cos^{-1}(\frac{d}{r}) + \frac{2d}{r^2}\sqrt{r^2-d^2}\right).$$

$\square$

Exercise (c). Let $u \in \mathcal{C}^2(\mathbb{R}^2)$ be a bounded harmonic function. Complete the proof that u is constant

Solution. By Theorem 3.5 we know that

$$u(x) = \mathcal{M}[u](x,r).$$

Picking any points $a, b \in \mathbb{R}^2$ we check the distance

$$|u(a) - u(b)| = |\mathcal{M}[u](a,r) - \mathcal{M}[u](b,r)| \leq C(r,d).$$

Where $r > d := \|a - b\|$, i.e

$$\lim_{h \to 0} \frac{|u(a+h) - u(a)|}{h} = \lim_{h \to 0} \frac{|\mathcal{M}[u](a+h, r(h)) - \mathcal{M}[u](a, r(h))|}{h}$$

$$\leq \lim_{h \to 0} \frac{1}{h} \cdot C(r(h), h)$$

$$= 0.$$

Where $C(r(h), h) \to 0$ as $h \to 0$, thus u must be constant if it is bounded and harmonic $\qquad \square$

Weak Tea

In this question we try to generalise the idea of spherical means to distribution in the way suggested in the lectures. Let $\lambda_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the standard mollifier and define a sphere mollifier

$$\Lambda_{x,r,\varepsilon}(y) = \frac{\lambda_\varepsilon(|y - x| - r)}{n\omega_n |y - x|^{n-1}}.$$

Exercise (a). Describe the support of $\Lambda_{x,r,\varepsilon}$

Solution. By definition it is all y such that $\Lambda_{x,r,\varepsilon}(y) \neq 0$ i.e all y such that $|y - x| - r \in \operatorname{supp} \lambda_\varepsilon := \overline{B(0, \varepsilon)}$, Which gives

$$|y - x| - r \leq \varepsilon \Rightarrow |y - x| \leq \varepsilon + r.$$

$$y \in B(x, \varepsilon + r).$$

$\qquad \square$

Exercise (b). We may try to define the spherical mean of a distribution F as $\lim\limits_{\varepsilon \to 0} F(\Lambda_{x,r,\varepsilon})$. However this does not always exist. Let G be the distribution in Exercise 17(d), submanifold inte- gration on the unit circle. Show that $G(\Lambda_{0,1,\varepsilon}) = \lambda_\varepsilon(1)$ and therefore the limit does not exist.

Solution. First we can check that indeed $\Lambda \in \mathcal{C}_0^\infty$ such that $G(\Lambda_{0,1,\varepsilon})$ is valid

$$\Lambda_{0,1,\varepsilon} = \frac{\lambda_\varepsilon(|y| - 1)}{2\omega_2 |y|}.$$

We have $C := \{x^2 + y^2 = 1\}$ i.e $C := \partial B(0,1)$ then

$$G(\Lambda_{0,1,\varepsilon}) = \int_{\{x^2+y^2=1\}} \frac{\lambda_\varepsilon(|y|-1)}{2\omega_2|y|} d\sigma$$

$$= \int_{\partial B(0,1)} \frac{\lambda_\varepsilon(0)}{2\omega_2} d\sigma = \lambda_\varepsilon(0).$$

If we didn't shift by $r = 1$ it would be 1. For $\varepsilon \to 0$ we have $\lambda_\varepsilon(0) \to \infty$ $\quad \square$

Exercise (c)**.** Show that the limit $\lim_{\varepsilon \to 0} F(\Lambda_{x,r,\varepsilon})$

Solution. Let F be a harmonic distribution i.e $\triangle F = 0$ then F has the weak mean value property by Theorem 3.6 and we know that F vanishes on any test function with $\int \psi d\mu = 0$ where $\psi \in \mathcal{C}_0^\infty((0,r))$
We know that necessarily we have

$$\int_\Omega \Lambda_{x,r,\varepsilon} d\mu = 1.$$

Choose any test function φ such that its integral is 1 then $\tilde{\varphi} = \varphi - \Lambda_{x,r,\varepsilon}$

$$\int_\Omega \tilde{\varphi} = 0.$$

Since distribution are linear operators

$$0 = F(\tilde{\varphi}) = F(\varphi) - F(\Lambda_{x,r,\varepsilon}).$$

Thus the limit exists. $\quad \square$

0.1   Tutorial

19.

WE can generalise the argument for all linear transformations

$$y = Ax + b \Rightarrow y_j = \sum_k A_{jk}x_k + b_j.$$

Then define

$$v(x) = u(Ax + b).$$

then by chain rule

$$\frac{\partial v}{\partial x_i} = \sum \frac{\partial u}{\partial y_i}\frac{\partial y}{\partial x_i}$$

$$= \sum_l \frac{\partial u}{\partial y_l} A_{li}.$$

And

$$\frac{\partial^2 v}{\partial x_j \partial x_i} = \sum \frac{\partial}{\partial x_j} \frac{\partial u}{\partial y_i} A_{li}$$
$$= \sum_{l,m} \frac{\partial u}{\partial y_m \partial y_l} A_{mj} A_{li}.$$

Then we get for the laplacian

$$\triangle v = \sum_i \frac{\partial^2 v}{\partial x_i^2}$$
$$= \sum_{l,m} \frac{\partial u}{\partial y_m \partial y_l} (\sum A_{mi} A_{li}) \qquad\qquad = \triangle u.$$

Last equality for $A \in O(M)$, writing it out as a matrix

$$H(v)_{ij} = \sum_{l,m} H(u)_{m,l} A_{mj} A_{li}.$$

This is the same as

$$\sum_{m,l} A_{jm}^T H(u)_{ml} A_{li}.$$

Then

$$\triangle v = \operatorname{trace}(H(v))$$
$$= \operatorname{trace}(A^T H(u) A)$$
$$= \operatorname{trace}(A A^T H(u)).$$

20

All Harmonic polynomials have a saddle point at $(0,0)$

P^d homoegnous degree d

$$P^D = \binom{n+d-1}{n-1}.$$

Then

$$\triangle : P^d \to P^{d-2}.$$

For $p = x_1^2 x_2 x_3^2$ we get

$$\triangle p = 2x_2 x_3^2 + 0 + 2x_1^2 x_2.$$

$\triangle$ is surjective since

$$\dim \ker \triangle = \binom{n+d-1}{n-1} - \binom{n+d-3}{n-1}.$$

n choose k for the amount of harmonic polynomials, harmonic functions are smooth and indeed also harmonic i.e they are equal to their taylor series i.e theyre polynomials

If u has the MVP for spheres S it also has it for the globular mean

$$M[u](x,r) = \frac{n}{r^n} \int_0^r u(x)s^{n-1}ds$$
$$= \frac{n}{r^n}u(x)\frac{1}{n}[s_n]_0^r$$
$$= u(x).$$

Other direction if u has MVP for M then

$$u(x) = M[u](x,r).$$

And

$$\frac{1}{n}r^n u(x) = \int_0^r s^{n-1}S[u](x,s)ds$$
$$\frac{1}{n}nr^{n-1}u(x) = r^{n-1}S[u](x,r).$$

Proof of 21, is to lower bound the are of the Lens created by the intersection of two circles by using the circle that fits inside it, when $r \to \infty$ the shared values of both circles become very similar and thus their difference should tend to 0

Small note in

$$\Lambda = \frac{\lambda_\varepsilon(|y-x|)}{n\dots}.$$

The $|\cdot|$ refers to the norm in the relevant dimension, i.e $y \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$