Exercise Sheet "Numerical Methods for ODEs"

The classroom exercises will be discussed on **September 11, 2023**.

Classroom Exercise 2.1 [Explicit Euler method]
Consider the IVP

$$y''(t) + ty'(t) + (1+t)y(t) = t^2, \quad y(0) = 0,\ y'(0) = 1.$$

Compute two steps using the explicit Euler method with step size $\tau = \frac{1}{2}$. Approximate the solutions of $y,\ y'$ and y'' at $t_1 = \frac{1}{2}$ and $t_2 = 1$.

Solution:
(a) Transfer to a first order system.

$$x_1 = y, x_2 = y'$$

$$\begin{aligned}
\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' &= \begin{pmatrix} x_2 \\ t^2 - tx_2 - (1+t)x_1 \end{pmatrix}; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\Rightarrow f(t,x) &= \begin{pmatrix} x_2 \\ t^2 - tx_2 - (1+t)x_1 \end{pmatrix} \\
u_n(0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = u_0
\end{aligned}$$

(b) Step 1

$$\begin{aligned}
u_1 := u_n(\frac{1}{2}) =& u_0 + \tau f(t_0, u_0) \\
=& \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0^2 - 0 \cdot 1 - (1+0) \cdot 0 \end{pmatrix} \\
=& \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}
\end{aligned}$$

(c) Step 2

$$\begin{aligned}
u_2 := u_n(1) =& u_1 + \tau f(t_1, u_1) \\
=& \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ (\frac{1}{2})^2 - \frac{1}{2} \cdot 1 - (1\frac{1}{2}0) \cdot \frac{1}{2} \end{pmatrix} \\
=& \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}
\end{aligned}$$

(d) Overview

t	0	1/2	1
y	0	1/2	1
y'	1	1	1/2
y"	0	-1	-3/2

Classroom Exercise 2.2 [Beyond Euler]
Euler's Method can be derived in many ways. Most of the ideas can be generalized.

1. If we integrate $x'(t) = f(t, x(t))$ from t_i to t_{i+1}, we get
$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(s, x(s))ds.$$
Calculate the integral with the rectangular and the trapezoidal rule. What happens?

2. What happens if the solution $x(t_{i+1})$ is approximated by the second order Taylor polynomial around t_i?

Solution:

We have
$$x(t_{i+1}) = x(t_i) + I$$
with
$$I = \int_{t_i}^{t_{i+1}} f(s, x(s))ds$$
Evaluation of I:

1. Rectangular rule
$$\begin{aligned} I =& (t_{i+1} - t_i) f(t_i, x(t_i)) \rightsquigarrow \text{ explicit Euler} \\ \text{or} \quad I =& (t_{i+1} - t_i) f(t_{i+1}, x(t_{i+1})) \rightsquigarrow \text{ implicit Euler} \end{aligned}$$

2. Trapezoidal rule
$$\begin{aligned} & I = (t_{i+1} - t_i) \frac{f(t_i, x(t_i) + f(t_{i+1}, x(t_{i+1})}{2} \\ \Rightarrow \quad & x(t_{i+1}) = x(t_i) + \frac{t_{i+1} - t_i}{2}(f(t_i, x(t_i)) + f(t_{i+1}, x(t_{i+1})) \\ & \rightsquigarrow \text{implicit Scheme} \end{aligned}$$

3. Taylor expansion
$$\begin{aligned} x(t_{i+1}) =& x(t_i) + (t_{i+1} - t_i)x'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} x''(t_i) \\ =& x(t_i) + (t_{i+1} - t_i) f(t_i, x(t_i)) + \frac{(t_{i+1} - t_i)^2}{2} (f_t(t_i, x(t_i)) + f_x(t_i, x(t_i)) f(t_i, x(t_i))) \end{aligned}$$
The number of evaluations of f increases, since f_t, f_x have to be calculated or approximated.

Classroom Exercise 2.3 [Multivariate Taylor formula]

Let $\Omega_0 \subset \mathbb{R}^d$, $f \in C^n(\Omega_0, \mathbb{R}^m)$ and $x \in \Omega_0$ be given. The n-dimensional derivative at the point x is defined by the **symmetric, multilinear** mapping
$$\begin{aligned} & f^{(n)}(x) \colon (\mathbb{R}^d)^n \to \mathbb{R}^m \\ & (h_1, \ldots, h_n) \mapsto \sum_{i_1, \ldots, i_n = 1}^{d} \frac{\partial^n f(x)}{\partial x_{i_1} \cdots \partial x_{i_n}} h_{1,i_1} \cdots h_{n,i_n}. \end{aligned}$$

If $f \in C^{p+1}(\Omega_0, \mathbb{R}^m)$ and $h \in \mathbb{R}^d$ such that $x + h \in \Omega_0$, then the Taylor formula

$$f(x+h) = \sum_{n=0}^{p} \frac{1}{n!} f^{(n)}(x)(h, \ldots, h) + O(||h||^{p+1})$$

holds for $||h|| \to 0$.

1. Show that
$$f(x+\tau h) = f(x) + \tau f^{(1)}(x)(h) + \frac{\tau^2}{2} f^{(2)}(x)(h,h) + O(\tau^3)$$
for $\tau \to 0$ holds.

2. Calculate the Taylor expansion of
$$f(x + \tau h + \tau^2 g)$$
around x for some $\tau > 0$, h, $g \in \mathbb{R}^d$ up to the order $p = 2$ and simplify the expression as far as possible.

Solution:

1. We have
$$\begin{aligned} f(x+\tau h) =& f(x) + f^{(1)}(x)(\tau h) + \frac{1}{2} f^{(2)}(x)(\tau h, \tau h) + \mathcal{O}(||\tau h||^3) \\ =& f(x) + \tau f^{(1)}(x)(h) + \frac{\tau}{2} f^{(2)}(x)(h, \tau h) + \mathcal{O}(\tau^3 ||h||^3) \\ =& f(x) + \tau f^{(1)}(x)(h) + \frac{\tau^2}{2} f^{(2)}(x)(h,h) + \mathcal{O}(\tau^3) \end{aligned}$$

2. We have
$$\begin{aligned} & f(x+\tau h + \tau^2 g) \\ =& f(x) + f^{(1)}(x)(\tau h + \tau^2 g) + \frac{1}{2} f^{(2)}(x)(\tau h + \tau^2 g, \tau h + \tau^2 g) + \mathcal{O}(||\tau h + \tau^2 g||^3) \\ =& f(x) + \tau f^{(1)}(x)(h + \tau g) + \frac{\tau^2}{2} f^{(2)}(x)(h + \tau g, h + \tau g) + \mathcal{O}(\tau^3 ||h + \tau g||^3) \\ =& f(x) + \tau f^{(1)}(x)h + \tau^2 f^{(1)}(x)(g) + \frac{\tau^2}{2}(f^{(2)}(x)(h,h) + 2\tau f^{(2)}(x)(h,g) + \tau^2 f^{(2)}(x)(g,g)) + \mathcal{O}(\tau^3) \\ =& f(x) + \tau f^{(1)}(x)h + \tau^2 (f^{(1)}(x)(g) + \frac{1}{2} f^{(2)}(x)(h,h)) + \mathcal{O}(\tau^3) \end{aligned}$$

Programming Exercise 2.4 [Explicit Euler](3 points)
Implement Euler's Method in MATLAB. Try to implement a function that has an input/output structure similar to the one used by MATLAB's ODE solvers (e.g. `ode23`). The optional arguments can be neglected. For more information about `ode23` see:

http://de.mathworks.com/help/matlab/ref/ode23.html

1. Solve $y'(t) = -2y(t),\ y(0) = 2$ for $t \in [0, 3]$ on an equidistant grid with different step sizes $\tau \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\}$. Plot the approximate solutions together with the analytic solution into one figure and label the axes properly.

2. Your code should also work for systems of ODEs. Therefore, use your Euler method to solve

$$y'' + y' + y = 0, \quad y(0) = 1, y'(0) = 0$$

for $t \in [0, 10]$ on an equidistant grid with different step sizes $\tau \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\}$. Compare the approximate solutions with the analytic solution in one figure.

3. Solve the following predator-prey model

$$\begin{aligned} x'(t) &= \frac{1}{2}x(t) - \frac{1}{3}x(t)y(t), && x(0) = 1, \\ y'(t) &= -y(t) + x(t)y(t), && y(0) = 1, \end{aligned}$$

for $t \in [0, 10]$ on an equidistant grid with different step sizes $\tau \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\}$. Compare the approximate solutions to the solution obtained by MATLAB's ODE solver `ode23`. Plot your results.

To be handed in: Executable Matlab code and figures of the programming exercises via ILIAS.