Chapter 1

Introduction

Mean Field Particle Systems is about the study of particles which are represented by (stochastic) differential equations. This course in particular is concerned with the behaviour of the system as the size grows to infinity:

Definition 1.0.1 (Toy Mean Field Particle System)**.** Let $N \in \mathbb{N}$ then a Mean Field Particle System of first order is given by :

$$x_1(t), \dots, x_n(t) \in \mathcal{C}^1([0,T];\mathbb{R}^d) \qquad x_i(0) = c_i.$$

Where each particle satisfies

$$dx_i = \frac{1}{N}\sum_{j=1}^{N} K(x_i, x_j)dt + \sigma dB_i(t).$$

Where B_i is a Brownian motion; For $\sigma = 0$ the system is called deterministic.

Example. Example choices for K are :

$$K(x_i, x_j) = \nabla(|x_i - x_j|^2).$$

or :

$$o\gamma = \frac{x_i - x_j}{|x_i - x_j|^d}.$$

Goal is to study what happens at $N \to \infty$, to do so we consider how the measure of a system converges

Definition 1.0.2 ((Empirical) Measure of a System)**.** Consider the point measure for every x_i : $\delta_{x_i(t)}$, then the measure of the System of order N is

:

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}.$$

Assumption 1.0.1. For initial data the empirical measure of a system converges $\mu_N(0) \to \mu(0)$ where μ is absolutely continuous with respect to the Lebesgue Measure

Corollary. By Radon Nikodym

$$d\mu = \rho_0 dx \quad \rho_0 \in L^1(\mathbb{R}^d).$$

It can be shown that μ solves a PDE, to do so we compute the derivative of μ using test functions

$$\forall \phi \in C_0^\infty(\mathbb{R}^d).$$

$$\begin{aligned}
\frac{d}{dt} < \mu_N(t), \phi >= \frac{d}{dt}\int_{\mathbb{R}^d} \phi(x) d\mu_N(t)(x) &= \frac{d}{dt}\int \frac{1}{N}\sum_{i=1}^{N} \phi(x) d\delta_{x_i(t)} \\
&= \frac{1}{N}\sum_{i=1}^{N} \frac{d}{dt}\phi(x_i(t)) \\
&\overset{\text{Chain.}}{=} \frac{1}{N}\sum_{i=1}^{N} \nabla\phi(x_i(t)) \frac{d}{dt} x_i(t) \\
&= \frac{1}{N}\sum_{i=1}^{N} \nabla_x\phi(x_i(t)) \cdot \underbrace{\frac{1}{N}\sum_{j=1}^{N} K(x_i(t), x_j(t))}_{\text{Def.}} \\
&= \frac{1}{N}\sum_{i=1}^{N} \nabla_x\phi(x_i(t)) \cdot \frac{1}{N}\sum_{j=1}^{N} \int_{\mathbb{R}^d} K_(x_i(t), y) d\delta_{x_j(t)}(y) \\
&= \frac{1}{N}\sum_{i=1}^{N} \nabla_x\phi(x_i(t)) \cdot \int_{\mathbb{R}^d} K_(x_i(t), y) d\mu_N(t) \\
&= \int_{\mathbb{R}^d} \nabla\phi(x) \int_{\mathbb{R}^d} K(x,y) d\mu_N(t,y) d\mu_N(t,x)
\end{aligned}$$

.

Where the last line can be rewritten by using Integration by Parts (Divergence Theorem) :

$$\int_{\mathbb{R}^d} \nabla\phi(x) \int_{\mathbb{R}^d} K(x,y) d\mu_N(t,y) d\mu_N(t,x) \overset{\text{Part.}}{=} - < \nabla \cdot (\mu_N \int_{\mathbb{R}^d K(\cdot,y) d\mu_N(y)}), \phi >$$

.

This means μ satisfies :

$$\partial_t \mu_N + \nabla \cdot (\mu_N \cdot \int K(\cdot, y) d\mu_N(y)) = 0 \xrightarrow{N \to \infty} \partial_t \mu + \nabla \cdot (\mu \cdot \int K(\cdot, y) d\mu(y)) = 0.$$

In practical applications (Theoretical Physics , Biology) systems that are considered are often of second order

Definition 1.0.3 (Toy Second Order System)**.** Given $N \in \mathbb{N}$ a Second Order System is given by

$$(x_i(t), v_i(t)), \ldots, (x_N(t), v_N(t)) \in \mathbb{R}^{2d}.$$

Such that :

$$\begin{aligned} &\frac{d}{dt} x_i(t) = v_i(t) \\ &\frac{d}{dt} v_i(t) = \frac{1}{N} \sum_{j=1}^{N} F(\underbrace{x_i(t), v_i(t)}_{\text{Position and Velocity of itself}} ; x_j(t), v_j(t)) + \sigma \frac{dB_t}{dt} \end{aligned}$$

.

Example (Gravitational Force)**.** An example of F could be :

$$F(x, v, y, u) = \frac{x - y}{|x - y|^d}.$$

Definition 1.0.4 (Second Order Measure)**.** The Measure of a second order System is given by :

$$\mu_N(x, v) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i(t), v_i(t))}.$$

Exercise 1. Show what PDE μ solves for $\sigma = 0$, *Hint :* Calculate $\frac{d}{dt} < \mu_N, \phi >$ for some test function $\phi \in C_0^\infty(\mathbb{R}^{2d})$

Chapter 2

Deterministic Mean Field Particle Systems

2.1 Goal

Definition 2.1.1 (Deterministic Mean Field Particle System)**.** For $N \in \mathbb{N}$ a deterministic mean field particle system is given by N particles :

$$x_1(t), \ldots, x_n(t) \in \mathcal{C}^1([0,T]; \mathbb{R}^d) \qquad x_i(0) = c_i.$$

With initial points :

$$x_i(0) = x_{i,0} \in \mathbb{R}^d.$$

And the relation :

$$\frac{d}{dt} x_i = \frac{1}{N} \sum_{j=1}^{N} K(x_i, x_j).$$

The system is then given by :

$$X_N = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix} \in \mathbb{R}^{dN}.$$

The goal is to solve the above $d \cdot N$ dimensional system under assumptions on K.

2.2 ODE Theory

Definition 2.2.1 (Initial Value Problem (standard))**.** For $\forall T > 0$ the standard ode system is given by :

$$x' = f(t, x)$$
$$x|_{t=0} = x_0 \in \mathbb{R}^n.$$

with $t \in [0, T],\ x(t) \in \mathbb{R}^n$

Assumption 2.2.1. Condition for global existence : f is continuous in $(t, x) \in [0, T] \times \mathbb{R}^n$
Condition for uniqueness : f is Lipschitz continuous in x

Theorem 2.2.1 (Picard Iteration)**.** Whenever Assumption 2.2.1 holds the standard IVP has a unique solution $x \in \mathcal{C}^1([0, T]; \mathbb{R}^n)$

Proof. Rewriting the IVP using integration :

$$x(t) - x(0) = \int_0^t f(s, x(s))ds \quad \forall t \in [0, T].$$

We construct the following sequence by :

$$\begin{aligned} x_1(t) &= x_0 + \int_0^t f(s, x_0)ds \\ x_2(t) &= x_0 + \int_0^t f(s, x(s))ds \qquad\qquad \vdots \\ x_m(t) &= x_0 + \int_0^t f(s, x_{m-1}(s))ds. \end{aligned}$$

Step 1: First part of the proof consists of proving the above sequence is converging
Step 2: The second part is then showing the limit is a solution to the IVP

Under Assumption 2.2.1 we know that f is continuous such that $(x_n)_{n \in N} \subset \mathcal{C}^1([0, T]; \mathbb{R}^n)$, As $\mathcal{C}^1$ is complete any sequence that is cauchy must also converge against a limit in the space.

$$\begin{aligned}|x_2 - x_1| &= |\int_0^t f(s, x_1(s))ds - \int_0^t f(s, x_0(s))ds| = |\int_0^t f(s, x_{m-1}(s)) - f(s, x_{n-1}(s))ds| \\ &\leq \int_0^t |f(s, x_1(s)) - f(s, x_0(s))|ds \\ &\overset{\text{Lip.}}{\leq} L \int_0^t |x_1(s) - x_0(s)|ds \\ &= L \int_0^t |\int_0^{s_0} f(s, x_0)ds|ds_0 \\ &\leq L \cdot \int_0^t \int_0^{s_0} |f(s, x_0)|dsds_0 \\ &\leq L \underbrace{M}_{=\max_{s\in[0,T]}|f(s,x_0)|} \frac{t^2}{2}.\end{aligned}$$

By repeatedly using the Lipschitz continuity of f the following induction assumption is motivated :

$$|x_m(t) - x_{m-1}(t)| \leq ML^{m-1}\frac{t^m}{m!}. \tag{IA}$$

(IS) : $m \to m + 1$

$$\begin{aligned}|x_{m+1}(t) - x_m(t)| &\overset{\text{Lip.}}{\leq} L \int_0^t |x_m(s) - x_{m-1}(s)|ds \\ &\overset{\text{IA.}}{\leq} L \int_0^t \frac{ML^{m-1}s^m}{m!}ds = ML^m \frac{t^{m+1}}{(m+1)!}.\end{aligned}$$

For any $n, m \in \mathbb{N}$ and assuming without loss of generality that $n > m$ such that $n = m + p$ for $p \in N$:

$$\begin{aligned}|x_n(t) - x_m(t)| = |x_{m+p}(t) - x_m(t)| &\leq \sum_{k=m+1}^{m+p} |x_k(t) - x_{k-1}(t)| \overset{\text{Ind.}}{\leq} M \sum_{k=m+1}^{m+p} \frac{L^{k-1}T^k}{k!} \\ &= \frac{M}{L} \sum_{k=m+1}^{m+p} \frac{(LT)^k}{k!} = \frac{M}{L}\frac{(LT)^{m+1}}{(m+1)!}\sum_{k=0}^{p-1}\frac{(LT)^k}{k!} \\ &\leq \frac{M}{L}\frac{(LT)^{m+1}}{(m+1)!}e^{LT} \xrightarrow{m\to\infty} 0 \text{ uniformly in } t \in [0, T].\end{aligned}$$

This shows that $(x_m)_{m\in\mathbb{N}}$ is Cauchy and has a limit $x \in \mathcal{C}([0, T]; \mathbb{R}^n)$ with :

$$\max_{t\in[0,T]} |x_m(t) - x(t)| \to 0.$$

It remains to show that $x(t)$ is a solution to the IVP i.e :

$$x(t) = \lim_{m\to\infty} x_0 + \int_0^t f(s, x_{m-1}(s))ds \leftrightarrow x_0 + \int_0^t f(s, x(s))ds.$$

Which can be shown by :

$$\begin{aligned}
|\lim_{m\to\infty} \int_0^t f(s, x_{m-1}(s)) - f(s, x(s))ds| &\leq \lim_{m\to\infty} \int_0^t |f(s, x_{m-1}(s)) - f(s, x(s))|ds \\
&\leq \lim_{m\to\infty} L \int_0^t |x_{m-1}(s) - x(s)|ds \\
&\leq \lim_{m\to\infty} Lt \cdot \max_{s\in[0,t]} |x_{m-1}(s)x(s)| \\
&\leq \lim_{m\to\infty} Lt \cdot \max_{s\in[0,T]} |x_{m-1}(s)x(s)| \\
&= 0.
\end{aligned}$$

It remains to show that the solution is unique, for that assume $x, \hat{x} \in \mathcal{C}([0,T]; \mathbb{R}^n)$ are both solutions to the IVP. Meaning that :

$$\begin{aligned}
x(t) &= x_0 + \int_0^t f(s, x(s))ds \\
\hat{x}(t) &= x_0 + \int_0^t f(s, \hat{x}(s))ds.
\end{aligned}$$

Then :

$$\begin{aligned}
|x - \hat{x}| &\leq \int_0^t |f(s, x(s)) - f(s, \hat{x}(s))|ds \leq L \cdot \int_0^t |x(s) - \hat{x}(s)|ds \\
&= L \int_0^t \underbrace{e^{-\alpha s}|x(s) - \hat{x}(s)|}_{=\rho(s)} e^{\alpha s} ds \\
&\leq L \max_{t\in[0,T]} \rho(t) \cdot \frac{1}{\alpha}(e^{\alpha t} - 1) \\
&\leq L \max_{t\in[0,T]} \rho(t) \cdot \frac{1}{\alpha} \cdot e^{\alpha t}.
\end{aligned}$$

By rearranging with the initial term :

$$\begin{aligned}
\rho(t) = e^{-\alpha t}|x(t) - \hat{x}(t)| &\leq \frac{L}{\alpha} \max_{t\in[0,T]} \rho(t) \\
\max_{t\in[0,T]} \rho(t) &\leq \frac{L}{\alpha} \max t \in [0,T] \rho(t).
\end{aligned}$$

by choosing $\alpha = 2L$:

$$\max_{t\in[0,T]} e^{-2Lt}|x(t) - \hat{x}(t)| = 0.$$

And the solutions must be equal for $\forall t \in [0, T]$. $\square$

The reason this proof deviates from the standard Picard-Lindelöf theorem, is that for our systems we require Global existence, doing so by requiring f to be globally Lipschitz continuous.

Theorem 2.2.2. The solution $x(t, t_0, x_0) \in \mathcal{C}$ is continuously dependent on (t_0, x_0)

Theorem 2.2.3 (Gronwalls inequality)**.** For $\alpha, \beta, \phi \in \mathcal{C}([a,b]; \mathbb{R})$ $\beta \geq 0$ and

$$0 \leq \phi(t) \leq \alpha(t) + \int_a^t \beta(s)\phi(s)ds, \ \forall t \in [a,b].$$

then :

$$\phi(t) \leq \alpha(t) + \int_a^t \beta(s) \exp\left(\int_s^t \beta(\tau)d\tau\right) \alpha(s)ds.$$

Proof. Denote $\psi(t) = \int_a^t \beta(s)\phi(s)ds$ then

$$\begin{aligned}\psi'(t) &= \beta(t)\phi(t) \leq \beta(t)\alpha(t) + \beta(t)\psi(t) \\ &= \beta(t) \cdot (\alpha(t) + \psi(t))\end{aligned}$$

.

Recall $\dot{x} + a(t)x + b(t) = 0$

$$\begin{aligned}(\dot{\psi}(t) - \beta(t)\psi(t))e^{-\int_a^t \beta(s)ds} &\leq \beta(t)\alpha(t) \cdot e^{-\int_a^t \beta(s)ds} \\ (e^{-\int_a^t \beta(s)ds}\psi(t))' &\leq \beta(t)\alpha(t) \cdot e^{-\int_a^t \beta(s)ds}.\end{aligned}$$

Integrating gives :

$$(e^{-\int_a^t \beta(s)ds}\psi(t)) \overset{\psi(a)=0}{\leq} \int_a^t \beta(s)\alpha(s)e^{-\int_a^s \beta(r)dr}ds.$$

□

Definition 2.2.2 (Regularity)**.** A function $K : \mathbb{R}^{2d} \to \mathbb{R}^d$ is called regular if :

1. $K \in \mathcal{C}^1(\mathbb{R}^{2d}; \mathbb{R}^d)$ (gives local lipschitz)
2. And $\exists L > 0$ s.t. :

$$\sup_y |\nabla_x K(x,y)| + \sup_x |\nabla_y K(x,y)| \leq L.$$

Remark. We further assume K has the following properties :

$$K(x,y) = -K(y,x) \qquad \text{(antisymmetric)}$$
$$K(x,x) = 0.$$

Theorem 2.2.4. For regular K the MPS has a solution for all $T > 0$

$$\begin{cases} \frac{d}{dt}x_i & = \frac{1}{N}\sum_{j=1}^{N} K(x_i, x_j), 1 \le i \le N \\ x_i(0) & = x_{i,0} \in \mathbb{R}^d \end{cases}.$$

has a unique solution by Picard-Iteration :

$$X_N(t) = \big(x_1(t), x_2(t), \ldots, x_N(t)\big) \in \mathcal{C}^1([0,T]; \mathbb{R}^{dN}).$$

Definition 2.2.3 (Empirical Measure of a System)**.** Consider the point measure for every x_i : $\delta_{x_i(t)}$, then the measure of the System of order N is given by

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}.$$

As shown in the introduction μ_N is a (weak-) solution to the following PDE

$$\partial_t \mu_N + \nabla \cdot (\mu_N \cdot \int K(\cdot, y) d\mu_N(y)) = 0.$$

Intuition. Now for $N \to \infty$ if we have $\mu_N \xrightarrow{\text{in some sense}} \mu$ then μ is a (weak) solution to

$$\partial_t \mu + \nabla \cdot (\mu \cdot \int K(\cdot, y) d\mu(y)) = 0.$$

with

$$\mu_0 \leftarrow \mu_N(0).$$

2.3 Weak Solutions and Distributions

Distributions are a more general class of functions and can be seen as the dual space of the space of test functions

Definition 2.3.1 (Multi-Index)**.** A multi-index $\gamma \in \mathbb{N}_0^n$ of length $|\gamma| = \sum_i \gamma_i$ for example $\gamma = (0,2,1) \in \mathbb{N}_0^3$ can be used to denote partial derivatives of higher order as such :

$$\partial^\gamma = \prod_i (\frac{\partial}{\partial x_i})^{\gamma_i}.$$

Remark. Only sensible cause partial derivatives commute as otherwise the index would be ambiguous.

Definition 2.3.2 (Test Functions)**.** For $\Omega \subset \mathbb{R}^d$ the space of test functions $\mathcal{D}(\Omega) \supset \mathcal{C}_0^\infty(\Omega)$. We say a sequence of test functions $(\phi_m)_{m\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\Omega)$ converges against some limit $\phi \in \mathcal{C}_0^\infty(\Omega)$ iff.

1. $\exists$ a compact set $K \subset \Omega$ s.t. $\operatorname{supp}\phi_m \subset K$ for all $m \in \mathbb{N}$
2. $\forall$ multi-indexes $\alpha \in \mathbb{N}_0^n$:

$$\sup_K |\partial^\alpha \phi_m - \partial^\alpha \phi| \xrightarrow{m\to\infty} 0.$$

Remark. $\mathcal{D}(\Omega)$ is a linear space

Definition 2.3.3 (Distribution)**.** The space of distributions $\mathcal{D}(\Omega)'$ is the dual space of $\mathcal{D}(\Omega)$ i.e. $\mathcal{D}(\Omega)'$ contains all the continuous linear functionals T

$$T : \mathcal{D}(\Omega) \to \mathbb{K}.$$

Remark. Continuity refers to the notion that for a sequence $(\phi_m)_{m\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ with limit ϕ then :

$$\phi_m \to \phi \;\Rightarrow\; T(\phi_m) \to T(\phi).$$

linearity :

$$T(\alpha\phi_1 + \beta\phi_2) = \alpha T(\phi_1) + \beta T(\phi_2).$$

We sometimes write $\langle T, \phi\rangle$ instead of $T(\phi)$

Definition 2.3.4 (Convergence)**.** For a sequence of distributions $(T_m)_{m\in\mathbb{N}} \subset \mathcal{D}(\Omega)'$ we say it converges against a limit $T \in \mathcal{D}(\Omega)$ iff

$$\langle T_m, \phi\rangle \to \langle T, \phi\rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

Example. Every locally integrable function $f \in L^1_{\text{loc}}(\Omega) := \{f \mid \forall K \subset$

$\Omega, \int_K f(x)dx < \infty\}$ defines a Distribution by :

$$T_f(\phi) = \langle T_f, \phi\rangle = \int_\Omega f(x)\phi(x)dx. \quad \forall \phi \in \mathcal{D}(\Omega).$$

i.e. $L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$

Example. Probability densities are distributions in the same sense as for L^1_{loc} functions

Example. (Probability -) Measures $\mu \in \mathcal{M}(\Omega)$ define a distributions , by :

$$\langle T_\mu, \phi\rangle = \int_{\mathbb{R}^d} \phi(x)d\mu(x) < \infty \quad \forall \phi \in \mathcal{D}(\Omega).$$

A prominent example is the δ distribution defined by :

$$\langle \delta, \phi\rangle = \int_{\mathbb{R}^d} \phi(x)d\delta = \phi(0).$$

Remember for a measurable set E

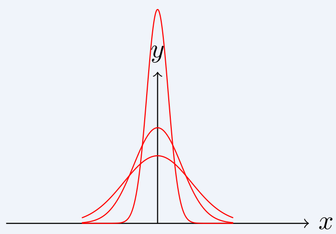
$$\delta_x(E) = \begin{cases} 1, & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}.$$

Coming back to our empirical measure from 2.2.3, we can see the corresponding distribution is defined by :

$$\langle \mu_n, \phi\rangle = \frac{1}{N}\sum_{i=1}^{N} \phi(x_i).$$

Some examples in approximation of δ distribution

Example (Heat Kernel). The heat kernel $f_t(x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$ approximates the δ distribution



Proof.

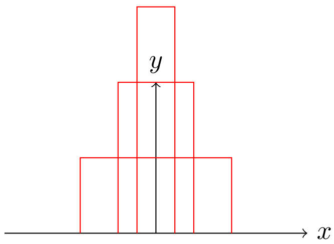
$$\begin{aligned}\lim_{t\to 0+}\langle f_t,\phi\rangle &= \lim_{t\to 0+}\int_{\mathbb{R}} f_t(x)\phi(x)dx = \lim_{t\to 0+}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}\phi(x)dx\\ &= \lim_{t\to 0+}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-y^2}\phi(2ty)dy\\ &\overset{*}{=}\phi(0)=\langle\delta,\phi\rangle.\end{aligned}$$

with the transformation $\frac{x}{\sqrt{2t}} = y$ ☐

Exercise 2. Show the change of integration and limit is valid in *

Further examples include :

$$Q_n(x) = \begin{cases}\frac{n}{2}, & \text{if } |x| \le \frac{1}{n}\\ 0 & \text{if } |x| > \frac{1}{n}\end{cases}.$$



And the dirichlet kernel

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} = 1 + 2\sum_{k=1}^{n}\cos(kx) \ \to 2\pi\delta.$$

To define the notion of a distribution solving a PDE we need to first define the way we take the derivatives of distributions

Definition 2.3.5 (Weak derivative of Distributions)**.** $\forall T \in \mathcal{D}(\Omega)'$. $\partial_i T$ is given by

$$\langle\partial_i T,\phi\rangle := -\langle T,\partial_i\phi\rangle \quad \forall\phi\in\mathcal{D}(\Omega).$$

We first show this for all distributions that are defined by a $f \in L^1_{\text{loc}}$, every other distribution T also has to satisfy this property.

Exercise 3. Proof the above equality for distributions $f \in L^1_{\text{loc}}$ and show for arbitrary distribution T that :

$$-\langle T,\partial_i\phi\rangle.$$

is continuous and linear.

Hint : Integration by parts; Why does it vanish on the boundary ?

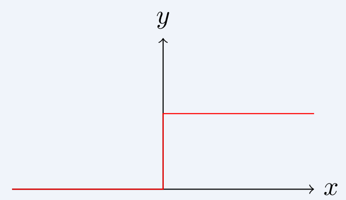
Example. For the δ distribution :

$$\langle \delta^{(k)}, \phi \rangle = (-1)^k \phi^{(k)}(0).$$

Example. Heaviside The Heaviside step function is defined by :

$$H = \begin{cases} 1, & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \in L^1_{\text{loc}}.$$

$$\begin{aligned} \langle H', \phi \rangle &\coloneqq -\langle H, \phi' \rangle = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx \\ &= -\int_{0}^{\infty} \phi'(x)dx = \phi(0) = \langle \delta, \phi \rangle. \end{aligned}$$



Using all the above we can rewrite our many particle system (MPS) by using the empiric measure and distributions

$$\begin{cases} \frac{d}{dt}x_i & = \langle K(x_i, \cdot), \mu_N \rangle = \int K(x_i, y) d\mu_N(y) \\ x_i(0) & = x_{i,0} \end{cases}.$$

Definition 2.3.6 (Weak Solution of MFE)**.** We say μ is a weak solution of the Mean-Field-Equation (MFE) iff for $\forall t \in [0, T]$, $\mu_t \in \mathcal{M}(\mathbb{R}^d)$ satisfies for all test functions $\forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ the following equation

$$\langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle = \int_0^t \langle \mu_s K \mu_s, \phi \rangle ds.$$

Remark. If $\mu_0 = \mu_{N,0}$ then $\mu_{N,t}$ is a weak solution of the MFE

Theorem 2.3.1. Let the empirical measure 2.2.3 be denoted by μ_N then for "good" (regular ?) $K(x,y)$ we have

$$\frac{d}{dt}\langle\mu_N,\phi\rangle = \langle\mu_N \underbrace{\int K(x,y)d\mu_N(y)\rangle}_{=K\mu_N} = -\langle\nabla\cdot\nabla(\mu_N K\mu_N),\phi\rangle.$$

Such that μ_N is a weak solution of the MPDE

$$\partial_t\mu_N + \nabla\cdot\nabla(\mu_N K\mu_N) = 0.$$

Remark. Note when we talk about weak solution, it means the PDE is solved in the sense of distributions.

Exercise 4. Show $\mu_N K\mu_N$ as defined above is a distribution for regular/ good $K(x,y)$

Proof. □

Definition 2.3.7 (characteristic problem for MFE)**.** The corresponding characteristic is given by :

$$\begin{cases} \frac{d}{dt}x(t,x_0,\mu_0) & = \int_{\mathbb{R}^d} K(x(t,x_0,\mu_0),y)d\mu_t(y) \\ x(0,x_0,\mu_0) & = x_0 \in \mathbb{R}^d \\ \mu_t & = x(t,\cdot,\mu_0)\#\mu_0 \end{cases}.$$

Notation (Push Forward)**.** For a measurable map $X : (\mathbb{R}^d,\mathcal{B}) \xrightarrow{X} (\mathbb{R}^d,\mathcal{B})$ and a measure $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ we have :

$$\forall B \in \mathcal{B}\ ,\ \ X\#\mu_0 = \mu_0(X^{-1}(B)).$$

Exercise 5. Show that if $x(t,x_0,\mu_0) \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^d)$ exists then $\mu_t = x(t,\cdot,\mu_0)\#\mu_0$ is a weak solution of MFE

Definition 2.3.8. Space of probability measures with bounded first moment

$$\mathcal{P}_1(\mathbb{R}) := \{\mu_0 \in \mathcal{M}_+(\mathbb{R}^d) : \int_{\mathbb{R}^d}|x|d\mu_0(x) < \infty\}.$$

Theorem 2.3.2 (Uniqueness of Solution)**.** For regular K (2.2.2) and $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$ then the characteristic problem 2.3.7 has a unique solution $x(t,x_0,\mu_0) \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^d)$ and $\mu_T \in \mathcal{P}_1(\mathbb{R}^d)$, for all $t > 0$

Proof. Existence

We consider

$$\begin{aligned} x(t,x_0,\mu_0) &= x_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s,x_0,\mu_0),y)d\mu_s(y)ds \\ &\overset{\text{psh frwd.}}{=} x_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s,x_0,\mu_0),x(s,\zeta,\mu_0))d\mu_0(\zeta)ds. \end{aligned}$$

We define the following iteration for all $y \in \mathbb{R}^d$

$$\begin{aligned} x_0(t,y) &= y \\ x_1(t,y) &= y + \int_0^t \int_{\mathbb{R}^d} k(x_0(s,y),x_0(s,\zeta))d\mu_0(\zeta)ds \\ &\vdots \\ x_n(t,y) &= y + \int_0^t \int_{\mathbb{R}^d} k(x_{n-1}(s,y),x_{n-1}(s,\zeta))d\mu_0(\zeta)ds \\ &. \end{aligned}$$

Similar to our proof in we show the sequence is cauchy :

$$\begin{aligned} |x_n(t,y)-x_{n-1}(t,y)| &\leq \int_0^t \int_{\mathbb{R}^d} |K(x_{n-1}(s,y),x_{n-1}(s,\zeta)) - K(x_{n-2}(s,y),x_{n-2}(s,\zeta))|d\mu_0(\zeta)ds \\ &\leq L\int_0^t \int_{\mathbb{R}^d} |x_{n-1}(s,y)-x_{n-2}(s,y)| + |x_{n-1}(s,\zeta)-x_{n-2}(s,\zeta)|d\mu_0(\zeta)ds \\ &. \end{aligned}$$

To get rid of the ζ we define the following banach space $\mathcal{X} = \{v \in \mathcal{C}(\mathbb{R}^d;\mathbb{R}^d) : \sup_x \frac{|v(x)|}{1+|x|} < \infty\}$ with norm

$$\|v\| = \sup_{x\in\mathbb{R}^d} \frac{|v(x)|}{1+|x|}.$$

We can then further approximate by :

$$\begin{aligned} &L\int_0^t \int_{\mathbb{R}^d} |x_{n-1}(s,y)-x_{n-2}(s,y)| + |x_{n-1}(s,\zeta)-x_{n-2}(s,\zeta)|d\mu_0(\zeta)ds \\ &\leq L\int_0^t |x_{n-1}(s,y)-x_{n-2}(s,y)| + \|x_{n-1}(s,\cdot)-x_{n-2}(s,\cdot)\|_{\mathcal{X}}(1+C_1)ds. \end{aligned}$$

Where $C_1 = \int |x|d\mu_0(x_0)$ is the first moment of our initial measure. Now we divide both sides of the inequality by $1+|y|$, and take the supremum in y

$$\|x_n(t,\cdot)-x_{n-1}(t,\cdot)\|_{\mathcal{X}} \leq L(2+C_1)\int_0^{|t|} \|x_{n-1}(s,\cdot)-x_{n-2}(s,\cdot)\|_{\mathcal{X}}ds.$$

Then for $\forall n > m \gg 1$ we have

$$\|x_n(t,\cdot) - x_m(t,\cdot)\|_{\mathcal{X}} \leq \sum_{i=m}^{n-1} \|x_{i+1}(t,\cdot) - x_i(t,\cdot)\|_{\mathcal{X}} \xrightarrow{m\to\infty} 0.$$

Therefore $(x_n(t,\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$.

Now suppose $x_n(t,\cdot) \to x(t,\cdot)$:

$$\begin{aligned} x(t,y) &= y + \int_0^t \int_{\mathbb{R}^d} K(x(s,y), x(s,\zeta)) d\mu_0(\zeta) ds \\ &= y + \int_0^t \int_{\mathbb{R}^d} K(x(s,y), z) d\mu_0(z) ds \end{aligned}$$

.

This concludes the **Existence** proof

Uniqueness :

This proof closely mimics the one presented in by using the space $\mathcal{X}$ □

Remark. Showing the convergence of our Picard Iteration here is slightly more complicated, forcing us to use a different norm to get a simpler estimate to work with, remember similar trick as in functional analysis with

$$\|f\|_L = \sup_{t\in[0,1]} e^{-Lt}|f(t)|.$$

Exercise 6. Show that $\mathcal{X} = \{v \in \mathcal{C}(\mathbb{R}^d;\mathbb{R}^d) : \sup_x \frac{|v(x)|}{1+|x|} < \infty\}$ with norm

$$\|v\| = \sup_{x\in\mathbb{R}^d} \frac{|v(x)|}{1+|x|}.$$

is a banach space

Hint : Compare to supremums norm

2.4 Wasserstein Distance

2.4.1 Goal

The goal of this section is to consider as $N \to \infty$ how the empirical measure $\mu_{N,\cdot}$ converges

$$\begin{aligned} \mu_{N,0} &\overset{?}{\to} \mu_0 \\ \mu_{N,t} &\overset{?}{\to} \mu_t. \end{aligned}$$

we have already shown that for arbitrary given measure μ_0 (on both sides of the arrows) the PDE problem is uniquely solved, the idea of the Mean Field Limit problem is to prove a stability result for the above convergence.

2.4.2 Weak Convergence of Measure (Wasserstein Distance)

Definition 2.4.1 (Wasserstein Distance). For all $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, $(p \geq 1)$ the Wasserstein Distance of μ and ν is given by

$$W^p(\mu, \nu) = \underset{MK,p}{\text{dist}}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int \int_{\mathbb{R}^{2d}} |x - y|^p \pi(dxdy) \right)^{\frac{1}{p}}.$$

Where

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \int_{\mathbb{R}^d} \pi(dx, \cdot) = \nu(\cdot)$$
$$\int_{\mathbb{R}^d} \pi(dy, \cdot) = \mu(\cdot)\}.$$

Exercise 7. For two deterministic measures δ_x, δ_y prove

$$W^1(\delta_x, \delta_y) = |x - y|.$$

Chapter 3

Stochastic Mean Field Particle Systems

5-6 weeks

3.1 Basics of probability

3.2 Bad K

3.3 Convergence

Chapter 4

New Results

4.1 Relative entropy Method

Goal to prove "strong" convergence in L^1

Chapter 5

Appendix

Theorem 5.0.1 (Divergence Theorem). Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)-$ dimensional sub-manifold of $\mathbb{R}^n$. Let $F : \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously to $\partial\Omega$. Then we have :

$$\int_{\Omega} \nabla \cdot F d\mu = \int_{\partial\Omega} F \cdot N d\sigma.$$

where N is the outward pointing normal. (last component is positive)