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Sheet 8

23 Back in the saddle

Question 1. Suppose that $u \in \mathcal{C}^2(\mathbb{R}^2)$ is harmonic with critical point at x_0, Assume the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle

By Ana I we know that u has a saddle point if the eigenvalues of the hessian at x_0 are of opposing sign i.e. the determinant of $\det(H(u)) \leq 0$

Solution.

$$\det(H(u)) = \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x \partial y}^{2}.$$

Since u is harmonic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}.$$

then we get

$$\det(H(u)) = \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x \partial y}^{2} = -(\frac{\partial^2 u}{\partial x^2})^2 - (\frac{\partial u}{\partial x \partial y})^2 \leq 0.$$

$\square$

24 Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called subharmonic if for all $x \in \Omega$ and $r > 0$ with $B(x, r) \subset \Omega$ it lies below its spherical mean

$$v(x) \leq \mathcal{S}[v](x, r).$$

Question 2 (a). Prove that every subharmonic function obeys the maximum-principle i.e. if the maximum of v can be found inside Ω then v is constant

Solution. Suppose $x_0 \in \Omega$ is the maximum of v then on a ball of radius $r > 0$ around x_0 we have

$$0 \geq v(x_0) - S[v](x_0, r) = \frac{1}{C} \int_{\partial B(x_0, r)} \underbrace{v(x_0) - v(y)}_{v(x_0) \text{ is max}} d\sigma(y) \geq 0.$$

We conclude that for all $y \in \partial B(x_0, r)$

$$v(x_0) = v(y).$$

Now we want to extend this to the entire Ball $\forall y \in B(x_0, r)$ and then argue that we can cover Ω by Balls where this holds.

We know that

$$\int_{B(x_0, \tilde{r})} v(x_0) - v(y) d\mu(y) = \int_0^{\tilde{r}} \int_{\partial B(x_0, \tilde{r})} v(x_0) - v(y) d\sigma(y) dr = 0.$$

We conclude v is constant on the entire Ball $\forall y \in B(x_0, r)$ (and note our original argument didn't depend on the value of r besides the ball being contained)

Now we know that if v attains a maxima x_0 it must be constant on a small ball centered at x_0 with $r > 0$. By compactness we can cover $\overline{\Omega}$ by finite many balls of radius $\frac{r}{2} > 0$

$$B(\gamma_1, \frac{r}{2}), \ldots, B(\gamma_n, \frac{r}{2}).$$

Pick $\gamma_1 = x_0$ then v is constant on the first ball and the next center γ_2 is contained in the ball $B(x_0, r)$ (otherwise relabel) such that v is also constant on this ball, by repeating this argument we get that v must be constant on all balls. $\square$

Question 3 (b). Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \leq 0$ in Ω

Solution. Assume first that $-\Delta v \leq 0$ in Ω and define

$$\tilde{v}(r) = S[v(x) - v](x, r).$$

for $x \in \mathbb{R}$ then by the divergence theorem we get that

$$\frac{d}{dr}\tilde{v}(r) = -\frac{1}{C}\int_{B(x,r)} \Delta v d\mu.$$

By assumption

$$\frac{d}{dr}\tilde{v}(r) \leq 0.$$

Such that we must have for all $\tilde{r} \leq r$

$$\tilde{v}(r) - \tilde{v}(\tilde{r}) \leq 0.$$

But

$$\begin{aligned}\tilde{v}(r) - \tilde{v}(\tilde{r}) &= v(x) - S[v](x, r) - (v(x) - S[v](x, \tilde{r})) \\ &= S[v](x, \tilde{r}) - S[v](x, r) \\ &\leq 0.\end{aligned}$$

i.e.

$$S[v](x, \tilde{r}) \leq S[v](x, r).$$

by continuity of v for $\tilde{r} \to 0$ we have

$$v(x) \leq S[v](x, r).$$

which is the sub-harmonic property

Now for the case v sub-harmonic implies $-\Delta v \leq 0$ we do this by proving that $-\Delta v > 0$ it follows v not sub harmonic, but this just means we reverse all the inequalities above and make them strict such that we get

$$v(x) > S[v](x, r).$$

i.e. v is not sub-harmonic $\square$

Question 4 (c). Let $u : \overline{\Omega} \to \mathbb{R}$ be a harmonic function. Show that $\|\nabla u\|^2$ is subharmonic

Solution. By the previous sheet we know that any partial derivative of a harmonic function is again harmonic such that

$$\frac{\partial u}{\partial x_i}.$$

is harmonic,by (d) we have that for any convex function $f\circ(\frac{\partial u}{\partial x_i})$ is subharmonic, since $f(x) = x^2$ is convex, and the sum of sub-harmonic is trivially also sub-harmonic. □

Question 5 (d). Show that $f \circ u$ is sub-harmonic for any smooth convex function $f : \mathbb{R} \to \mathbb{R}$

Solution. We calculate

$$\begin{aligned}
\triangle f(u(x)) &= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \\
&\overset{\text{Chn.}}{=} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f'(u) \cdot \frac{\partial u}{\partial x_i}) \\
&= \sum_{i=1}^{n} f''(u)(\frac{\partial u}{\partial x_i})^2 + \sum_{i=1}^{n} f'(u)\frac{\partial^2 u}{\partial x_i^2} \\
&= \sum_{i=1}^{n} C''(\frac{\partial u}{\partial x_i})^2 + \sum_{i=1}^{n} C'\frac{\partial^2 u}{\partial x_i^2} \\
&= C''\|\nabla u\|^2 + C'\Delta u \geq 0.
\end{aligned}$$

it follows by convexity of f that $-\Delta f \leq 0$ □

Question 6 (e). Let v_1, v_2 be two sub harmonic functions. Show that $v = \max(v_1, v_2)$ is sub harmonic

Solution. We show this for v_1 for v_2 the process is analog

$$v_1(x) \overset{\text{Ass.}}{\leq} S[v_1](x,r) = \frac{1}{C}\int_{\partial B(x,r)} v_1(y)d\sigma(y) \leq \frac{1}{C}\int_{\partial B(x,r)} \max(v_1,v_2)d\sigma(y).$$

this implies

$$\max(v_1, v_2)(x) \leq S[v](x,r).$$

□

25. Never judge a book by its cover

Let $\Omega \subset \mathbb{R}^n$ be an open ,connected and bounded subset and let $f : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous functions. Consider then the two Dirichlet problems

$$-\Delta u = f \quad u|_{\partial\Omega} = g_k.$$

for $k = 1, 2$. Let u_1, u_2 be respective solutions such that they are twice continuously differentiable on Ω and continuous on $\overline{\Omega}$. Show that if $g_1 \leq g_2$ on $\partial\Omega$ then $u_1 \leq u_2$ on Omega

Solution. As always we take the difference of two solutions of the inhomogeneous equation and get a solution to the homogeneous problem i.e.

$$\tilde{u} = u_1 - u_2.$$

is harmonic, such that we consider

$$\int_{\partial\Omega} u_1 - u_2 d\sigma = \int_{\partial\Omega} g_1 - g_2 d\sigma \leq 0.$$

By the Weak Maximum Principle (3.11) a harmonic function takes its maximum on the boundary this means that any $y \in \Omega$ such that

$$u_1(y) - u_2(y) \geq 0.$$

presents a contradiction as y would be the maxima. Thus we have on the entirety of Ω $u_1 \leq u_2$ □

To be or not to be

Question 7. Consider the Dirichlet problem for the Laplace equation Δu on Ω with $u = g$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and g is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed. Consider $\Omega = B(0,1) \setminus \{0\}$, so that the boundary $\partial\Omega = \partial B(0,1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0,1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.

Solution. We suppose a solution u exists on $\Omega = B(0,1) \setminus \{0\}$ with the hint we use Lemma 3.23 and get that u extends as a harmonic function on $\tilde{\Omega} = B(0,1)$, now we are in a similar case described below 3.14 where by the Initial condition we have $u \equiv 0$ on $\partial\tilde{\Omega} := \partial B(0,1)$ and know by the weak maximum principle that both minimum and maximum ($-u$ and u) on $\tilde{\Omega}$ is 0 i.e $u \equiv 0$ on all of $\tilde{\Omega}$, this is a direct contradiction to $g_2 = 1$ and no solution can exist. □