Lemma (3.2 Itô s isometry simple version)**.** For $f \in \mathcal{H}_0^2$ we have

$$\|f\|_{\mathcal{H}^2} = \|I(f)\|_{L^2}.$$

Proof. We have

$$\begin{aligned}
\|I(f)\|_{L^2} &= \mathbb{E}[(\sum_{i=1}^{n} a_i(B_{t_i} - B_{t_{i-1}}))^2] \\
&= \mathbb{E}[\sum_{i=1}^{n} a_i^2(B_{t_i} - B_{t_{i-1}})^2 + \sum_{i\neq j}^{n} a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] \\
&= \mathbb{E}[\sum_{i=1}^{n} a_i^2(B_{t_i} - B_{t_{i-1}})^2] \\
&= \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[a_i^2(B_{t_i} - B_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2]\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2](t_i - t_{i-1})
\end{aligned}$$

.

Since w.l.o.g take $i < j$

$$\begin{aligned}
\mathbb{E}[a_i a_j(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] &= \mathbb{E}[\mathbb{E}[a_i a_j(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_{t_{i-1}}]] \\
&= \mathbb{E}[a_i \underbrace{\mathbb{E}[(B_{t_i} - B_{t_{i-1}})]}_{=0} \mathbb{E}[a_j(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_{t_{i-1}}]]
\end{aligned}$$

.

But

$$\|f\|_{\mathcal{H}^2} = \mathbb{E}[\int_0^T f^2 ds] = \sum_{i=1}^{n} \mathbb{E}[a_i^2](t_i - t_{i-1}).$$

□

Proposition (3.5)**.** For every $f \in \mathcal{H}^2$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^2$

such that

$$\|f_n - f\|_{\mathcal{H}^2} \xrightarrow{n\to\infty} 0.$$

Proof. The rough outline of the proof can be split up as follows

1. Show that we can assume f is bounded
2. Show that we can assume f is bounded adapted and continuous
3. Construct simple sequence that approximates f

For step 1 we can simply consider that for any (potentially) unbounded function f

$$f_n = -n \vee (f \wedge n).$$

is bounded and apply DCT

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{H}^2} = \lim_{n\to\infty} \mathbb{E}[\int_0^T (f_n - f)^2 dt] = \mathbb{E}[\int_0^T \lim_{n\to\infty} (f_n - f)^2 dt] = 0.$$

Thus we may assume f bounded

For step 2 we can use that we can get the height of a rectangle by dividing by its base s.t.

$$f_n(\cdot, t) := \frac{1}{(t - (t - \frac{1}{n})_+)} \int_{(t-\frac{1}{n})_+}^t f(\cdot, s) ds = n \int_{(t-\frac{1}{n})_+}^t f(\cdot, s) ds.$$

Now we note that since we work with random variables we condition ω on the set where

$$\lim_{n\to\infty} f_n(\omega, t) = f(\omega, t).$$

We need this set to have measure 1, such that

$$A := \{(\omega, t) \in \Omega \times [0, T] : \lim_{n\to\infty} f_n(\omega, t) \neq f(\omega, t)\}.$$

Has measure zero with respect to $\mathbb{P} \otimes \lambda$, we have by the fundamental theorem of calculus that

$$\lambda(\{t \in [0, T] : (\omega, t) \in A\}) = 0.$$

Or rather Lebesgue Differentiation theorem, otherwise we'd need the condition that $f(\omega, \cdot)$ only has countable many discontinuities. Thus we get that we may assume f is bounded and continuous.

We now construct our simple function f_n as

$$f_{n,s}(\omega, t) := f(\omega, (s + \varphi_n(t - s)_+)).$$

where

$$\varphi_n = \sum_{i=1}^{2^n} \frac{j-1}{2^n} \mathbb{1}_{(\frac{j-1}{2^n},\frac{j}{2^n}]}.$$

makes our time interval discrete (standard argument really), then we wanna show

$$\|f_n - f\|_{\mathcal{H}^2} = \mathbb{E}[\int_0^T |f_{n,s} - f|^2 dt] \to 0.$$

We have

$$\begin{aligned}
\mathbb{E}[\int_0^T |f_{n,s} - f|^2 dt] &\leq \mathbb{E}[\int_0^T \int_0^1 |f_{n,s} - f|^2 ds dt] \\
&= \mathbb{E}[\int_0^T \int_0^1 |f(\cdot,(s+\varphi_n(t-s))_+) - f|^2 ds dt] \\
&= \sum_{j\in\mathbb{Z}} \mathbb{E}[\int_0^T \int_{[t-\frac{j}{2^n},t-\frac{j-1}{2^n}]\cap[0,1]} |f(\cdot,(s+\frac{j-1}{2^n}) - f|^2 ds dt] \\
&\leq (2^n+1)2^{-n} \int_{(0,1]} \mathbb{E}[\int_0^T |f(\cdot,t-2^{-n}h) - f(\cdot,t)|^2 dt] dh.
\end{aligned}$$

Where we can show that the Expectation term goes to 0 $\square$

$$\int_0^T |f_n(\cdot,(t-h)_+) - f(\cdot,(t-h)_+)|^2 dt = \int_0^h |f_n(\cdot,0) - f(\cdot,0)|^2 dt + \int_h^t \ldots \leq h|f_n(\cdot,0) - f(\cdot,0)|^2 + \|f - f_n\|_{\mathcal{H}^2}$$

Theorem (3.7.). For any $f \in \mathcal{H}^2$ there is a continuous martingale $X = (X_t)_{t\in[0,T]}$ with respect to $\mathcal{F}_t$ such that for all $t \in [0,T]$

$$X_t = I(f\mathbb{1}_{[0,t]}).$$

Proof. We first consider the simple case with

$$f_n = \sum_{i=0}^{m_n-1} a_i^n \mathbb{1}_{(t_i^n,t_{i+1}^n]}.$$

Then

$$\begin{aligned}
\mathbb{E}[X_t^n - X_s^n|\mathcal{F}_s] &= \mathbb{E}[\sum_{t_i>s} a_i(B_{t_{i+1}} - B_{t_i})] \\
&= 0.
\end{aligned}$$

Our end goal is to use a triangular inequality to use the simple case to bound the normal one i.e.

$$
\begin{aligned}
|\mathbb{E}[X_t - X_s|\mathcal{F}_s]| &= |\mathbb{E}[X_t - X_t^n + X_t^n - X_s + X_s^n - X_s^n|\mathcal{F}_s]| \\
&\leq |\mathbb{E}[X_t - X_t^n|\mathcal{F}_s]| + \underbrace{|\mathbb{E}[X_t^n - X_s^n|\mathcal{F}_s]|}_{=0} + |\mathbb{E}[X_s^n - X_s|\mathcal{F}_s]| \quad = |\mathbb{E}[X_t - X_t^n|\mathcal{F}_s]| + |\mathbb{E}[X_s^n - X_s| \\
&\overset{\text{Jen}}{\leq} \mathbb{E}[|X_t - X_t^n||\mathcal{F}_s] + \mathbb{E}[|X_s^n - X_s||\mathcal{F}_s]
\end{aligned}
$$

.

We consider $A \in \mathcal{F}_s$

$$
\begin{aligned}
\mathbb{E}[|X_t - X_t^n|\mathbb{1}_A] + \mathbb{E}[|X_s^n - X_s|\mathbb{1}_A] &\leq \mathbb{E}[|X_t - X_t^n|] + \mathbb{E}[|X_s^n - X_s|] \\
&\leq 2 \cdot \|f - f_n\|_{\mathcal{H}^2}.
\end{aligned}
$$

So we have $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$

For $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^2$ we have

$$\lim_{n\to\infty} f_n = f \in \mathcal{H}^2.$$

And

$$\lim_{n\to\infty} I(f_n) = I(f) \in L^2.$$

Which should be equivalent to for fixed $t \in [0, T]$

$$
\begin{gathered}
X_t^n \xrightarrow{L^2} X_t \\
I(f_n \mathbb{1}_{[0,t]}) \xrightarrow{L^2} I(f \mathbb{1}_{[0,t]}).
\end{gathered}
$$

We need to make the argument uniform in $t \in [0, T]$, which is Step 2 in the script i guess.

We have

$$\mathbb{P}(\sup_{0\leq t\leq T} |X_t^n - X_t^m| \geq \varepsilon) \leq \varepsilon^{-2}\mathbb{E}[|X_T^n - X_T^m|^2] = \varepsilon^{-2}\|f_n - f_m\|_{\mathcal{H}^2}^2.$$

since for all $p > 1$

$$\mathbb{P}(\sup_{0\leq t\leq T} |X_t| \geq \varepsilon)\frac{1}{\varepsilon^p}\mathbb{E}[|X|_T^p].$$

By choosing a subequence we can get

$$\mathbb{P}(\sup_{0\leq t\leq T} |X_t^n - X_t^m| \geq 2^{-k}) \leq 2^{2k}\mathbb{E}[|X_T^n - X_T^m|^2] = \varepsilon^{-2}\|f_n - f_m\|_{\mathcal{H}^2}^2 \leq 2^{-k}.$$

Then we can apply Borel-Cantelli since

$$\sum_{k=0}^{\infty} \mathbb{P}(\sup_{0 \le t \le T} |X_t^n - X_t^m| \ge 2^{-k}) < \infty.$$

and get $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and X^{n_k} is a pathwise cauchy sequence. $\square$

Theorem (3.17 Riemann sum approximation)**.** If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $t_i = \frac{i}{n} T$ then for $n \to \infty$ we have

$$\sum_{i=1}^{n} f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \xrightarrow{\mathbb{P}} \int_0^T f(B_s) dB_s.$$

Proof. By Remark 3.12. we know that for any continuous function $g : \mathbb{R} \to \mathbb{R}$

$$f(\omega, t) = g(B_t(\omega)) \in \mathcal{H}^2_{\text{loc}}.$$

This follows since for a.s. $\omega \in \Omega$ the map

$$\varphi(\omega) : [0, T] \to \mathbb{R} : t \mapsto B_t(\omega)$$

is bounded. This gives us that

$$\sup_{t \in [0,T]} |g(B_t(\omega))| = \sup_{x \le |m|} |g(x)| \le C.$$

Where the last inequality follows from the fact that g is continuous and attains a maximum on the compact set $[-m, m]$ then we can check that for

$$\omega \in \{\varphi \text{ is bounded } \}.$$

The integral

$$\begin{aligned} \int_0^T g^2(B_t(\omega)) dt &\le \int_0^T |g(B_t(\omega))||g(B_t(\omega))| dt \\ &\le \underbrace{\sup_{t \in [0,T]} |g(B_t(\omega))|}_{\le C} \int_0^T |g(B_t(\omega))| dt \\ &\le C^2 T. \end{aligned}$$

Since $\mathbb{P}(\{\varphi \text{ is bounded } \}) = 1$ we get immediately

$$\mathbb{P}(\int_0^T g^2(B_t(\omega)) < \infty) = 1.$$

This tells us that for any continuous f and Brownian motion B

$$f(B) \in \mathcal{H}^2_{\text{loc}}.$$

we can rewrite $\{\varphi$ is bounded $\}$ as a stopping time instead and get

$$\tau_m = \inf\{t \in [0, T] : |B_t| \geq m\}.$$

which is a localizing sequence for $f(B)$ since by similar argument to above we have

$$|f(B_{\cdot \wedge \tau_m})| \leq \sup_{|x| \leq m} |f(x)| < \infty.$$

and we get

$$f_m = f \cdot \mathbb{1}_{[-m,m]} = f|_{[-m,m]}.$$

Where

$$f_m(B) \in \mathcal{H}^2.$$

By definition of the Itô integral for $f \in \mathcal{H}^2$ we already get that

$$I(f_m^{(n)}) = \sum_{i=1}^{n} a_i(B_{t_i} - B_{t_{i-1}}) \xrightarrow{L^2} \int_0^T f_m(B_t)dt.$$

where L^2 convergence implies $\mathbb{P}$ convergence.

Thus our goal in Step 2 is to show that in fact

$$f_m^{(n)} = \sum_{i=1}^{n} f_m(B_{t_{i-1}})(\omega)\mathbb{1}_{(t_{i-1}, t_i]}(s).$$

we clearly have

$$f_m^{(n)} \in \mathcal{H}^2_0.$$

Then it remains to show $f_m^{(n)} \xrightarrow{\mathcal{H}^2} f_m$

$$
\begin{aligned}
\mathbb{E}[\int_0^T (f_m^{(n)} - f_m)^2 ds] &= \mathbb{E}[\int_0^T (\sum_{i=1}^n f_m(B_{t_{i-1}}) \mathbb{1}_{(t_{i-1},t_i]}(s) - f_m(B_s))^2] \\
&= \mathbb{E}[\int_0^T (\sum_{i=1}^n f_m(B_{t_{i-1}}) \mathbb{1}_{(t_{i-1},t_i]}(s) - \sum_{i=1}^n f_m(B_s) \mathbb{1}_{t_{i-1},t_i})^2] \\
&= \mathbb{E}\Big[\int_0^T \sum_{i=1}^n (f_m(B_{t_{i-1}}) - f_m(B_s) \mathbb{1}_{(t_{i-1},t_i]}(s))^2 \\
&+ \underbrace{\sum_{i,j=1}^n \underbrace{(f_m(B_{t_{i-1}}) - f_m(B_s)\mathbb{1}_{(t_{i-1},t_i]}(s))(f_m(B_{t_{j-1}}) - f_m(B_s)\mathbb{1}_{(t_{j-1},t_j]}(s))}_{[t_{i-1},t_i]\cap[t_{j-1},t_j]=\emptyset}}_{=0} ds\Big] \\
&= \mathbb{E}\Big[\int_0^T \sum_{i=1}^n (f_m(B_{t_{i-1}}) - f_m(B_s) \mathbb{1}_{(t_{i-1},t_i]}(s))^2 ds] \\
&\leq \mathbb{E}[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f_m(B_{t_{i-1}}) - f_m(B_s))^2] \\
&\leq \mathbb{E}[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 ds] \\
&\leq \sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \int_{t_{i-1}}^{t_i} ds] \\
&\leq \frac{T}{n} \sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2]
\end{aligned}
$$

.

Where we can bound

$$\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2.$$

further by considering that f is continuous and thus for

$$\mu_{f_m}(h) := \sup\{|f_m(x) - f_m(y)| : x, y \in \mathbb{R} \text{ with } |x-y| \leq h\}.$$

we get that

$$\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \leq \mu_{f_m}(\sup_{r\in[t_{i-1},t_i]} |B_{t_{i-1}} - B_r|).$$

putting it together

$$\begin{aligned}
\mathbb{E}[\int_0^T (f_m^{(n)} - f_m)^2 ds] &\le \frac{T}{n}\sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2] \\
&\le \frac{T}{n}\sum_{i=1}^n \mathbb{E}[\mu_{f_m}(\sup_{r\in[t_{i-1},t_i]} |B_{t_{i-1}} - B_r|)^2] \\
&\le \frac{T}{n}\mathbb{E}[n\cdot\mu_{f_m}(\sup_{r\in[t_{i-1},t_i], i\le n\}} |B_{t_{i-1}} - B_r|)^2] \\
&\le T\mathbb{E}[\mu_{f_m}(\sup_{r\in[t_{i-1},t_i], i\le n\}} |B_{t_{i-1}} - B_r|)^2]
\end{aligned}$$

.

Since f_m is continuous the modulus of continuity must tend to 0 as $n \to \infty$. Thus we have shown that $f_m^{(n)} \xrightarrow{\mathcal{H}^2} f_m \Rightarrow I(f_m^{(n)}) \xrightarrow{L^2} I(f_m)$
Now on the set $\{\tau_m = T\}$ we have

$$f(B) = f_m(B).$$

and by persistence of identity also

$$\int_0^T f(B_s)dB_s = \int_0^T f_m(B_s)dB_s.$$

For

$$A_{n,\varepsilon} = \{|\sum_{i=1}^n f(B_{t_{i-1}})\cdot(B_{t_i} - B_{t_{i-1}})) - \int_0^T f(B_s)dB_s| \ge \varepsilon\}.$$

Then we get

$$\sum_{i=1}^n f(B_{t_{i-1}})\cdot(B_{t_i} - B_{t_{i-1}})) \xrightarrow{\mathbb{P}} \int_0^T f(B_s)dB_s.$$

if $\mathbb{P}(A_{n,\varepsilon}) \to 0$

$$\begin{aligned}
\mathbb{P}(A_{n,\varepsilon}) &= \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m < T\}) + \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m = T\}) \\
&\le \mathbb{P}(\{\tau_m < T\}) + \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m = T\}) \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

This inequality is just $\mathbb{P}(A) \le \mathbb{P}(B)$ if $A \subset B$

□

Remark (3.12)**.** For any continuous $g : \mathbb{R} \to \mathbb{R}$ we have $f(\omega, t) = g(B_t(\omega)) \in \mathcal{H}^2_{\mathrm{loc}}$ since B is a.s. pathwise bounded on $[0, T]$

Proof. Consider $\omega \in \Omega$ a.s., then

$$\sup_{t\in[0,T]} |g(B_t(\omega))| \leq C.$$

for some $C \geq 0$, then we have

$$\begin{aligned}
\int_0^T g^2(B_t(\omega))dt &= \int_0^T g(B_t(\omega))g(B_t(\omega))dt \\
&\leq \int_0^T \sup_{t\in[0,T]} |g(B_t(\omega))| \cdot |g(B_t(\omega))| dt \\
&\leq \sup_{t\in[0,T]} |g(B_t(\omega))| \int_0^T |g(B_t(\omega))| dt \\
&\leq C^2 \cdot T.
\end{aligned}$$

□

Example (5.1)**.** Consider the SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t.$$

then we can solve this SDE by making the ansatz $X_t = f(B_t, t)$ and using Itôs formula

$$\begin{aligned}
df(B_t, t) &= f(0,0) + f_x(B_t)dB_t + (\frac{1}{2} f_{xx} + f_t)dt \\
&\triangleq \mu f(B_t)dt + \sigma f(B_t)dB_t.
\end{aligned}$$

This implies

$$f_x(B_t) = \sigma f.$$

Such that

$$f = \exp(\sigma \cdot x + g(t)).$$

then

$$\begin{aligned}
g'(t) \cdot f + \frac{1}{2}\sigma^2 f &= \mu f \\
g'(t) &= \mu - \frac{\sigma^2}{2} \\
g(t) &= (\mu - \frac{\sigma^2}{2})t + g_0.
\end{aligned}$$

Which gives the solution

$$X_t = \exp(\sigma B_t + (\mu - \frac{\sigma^2}{2})t + g_0).$$

Definition. In general a linear SDE has the form

$$X_t = (\alpha(t)X_t + \beta(t))dt + (\varphi(t)X_t + \psi(t))dB_t.$$

and a solution is given by

$$X_t = x_0 \exp(Y_t) + \int_0^t \exp(Y_t - Y_s)(\beta(s) - \psi(s)\varphi(s))ds + \int_0^t \exp(Y_t - Y_s)\psi(s)dB_s.$$

Where

$$Y_t = \int_0^t \varphi(s)dB_s + \int_0^t (\alpha(s) - \frac{1}{2}\varphi^2(s))ds.$$