Chapter 1

General Concepts

1.1 Classification of Second order PDEs

1.1.1 General Problem

For PDEs of order above one no general methods exist to solve them and methods for solving differ quite a bit from each other, thus PDEs are classified by methods that solve them and once a new method to solve is found all pdes that can be solved by it are classified under it.

A general second order linear PDE has the following form :

Definition 1.1.1 (Second Order Linear PDE). A general second order linear PDE is given by :

$$Lu(x) = \sum_{i,j=1}^{n} a_{i,j}(x)\partial_i\partial_j u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u(x) = 0.$$

i.e, second order terms , first order terms and 0th order terms.

Where $a_{i,j}$ is a matrix of coefficients and pde's can be classified by the shape they take. The matrix $a_{i,j}$ is symmetric and diagonalizable as the partial derivatives are symetric **Schwarz's Theorem** $(a_{i,j} \equiv \frac{1}{2}(a_{i,j} + a_{j,i}))$

Elliptic PDEs

Definition 1.1.2 (Elliptic PDEs). If the matrix $a_{i,j}$ is the unity matrix and $b = c = 0$ then they are called elliptic pdes

Example (Laplace Equation)**.** Laplace Equation is given by :

$$\triangle u := \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Solutions are called harmonic functions. Important tool : a priori estimates i.e lower order derivatives can be estimated in terms of second order derivatives.

Major example whose investigation played a role in the development of elliptic theory is :

Example (Minimal surface equation)**.** Note $\nabla \cdot u = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

$$\nabla \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}.$$

The graphs of such solutions describe minimal surfaces. The area of such hypersurfaces in $\mathbb{R}^{n+1}$ does not change with infinitesimal variation.

Example : Soap bubbles are one example.

Boundary value problem is called Plateaus problem, first proof of existence received Field's Medal (Jesse Douglas)

1.1.2 Parabolic PDEs

Parabolic PDEs are linear PDEs where the matrix $a_{i,j}$ is considered as a symmetric bilinear form which is only semi-definite and belong to the boundary of the class of elliptic PDEs. semi-definite (all eigenvalues are non-negative / non positive).

Example (Heat equation)**.** The heat equation is given by :

$$\dot{u} - \triangle u = 0.$$

And describes diffusion processes, named after the prominent example of temperature.
Many stochastic processes have this property.

These are processes which level inhomogeneities of some quantity by some flow along the negative gradient of the quantity.
Interpretation : the rate $\dot{u}$ at which the material at a point will heat up (or cool down) is proportional to how much hotter (or cooler) the surrounding is.

Example (Ricci Flow).

$$\dot{g}_{i,j} = -2R_{i,j}.$$

This PDE describes a diffusion-like process on Riemannian manifolds, it levels the inhomogeneities of the metric (g).

Definition 1.1.3. Riemannian Manifold A Manifold is a locally euclidean space but not globally, common example are maps of an atlas, i.e we can locally embed the maps into $\mathbb{R}^n$ but globally thats impossible, a Riemanniann manifold is a n dimensional manifold with a function g that assigns every point $p \in M$ a scalar product.

1.1.3 Hyperbolic PDE

Hyperbolic PDEs are the second most important class of linear PDEs. The matrix $a_{i,j}$ has one eigenvalue of opposite sign than all other eigenvalues. An example is :

Example (Wave equation).

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = 0.$$

The wave equation describes the behavior of waves with constant finite speed. The investigation of these PDEs depend on understanding all trajectories which propagate by given speed.

1.2 Existence of Solutions

There exists PDEs with smooth coefficients without solutions, an example to this is :

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = f(x, y).$$

Sucht that :

$f(-x, y) = f(x, y)$

there exists a sequence of positive numbers $\rho_n \downarrow 0$,such that f vanishes on a neighbourhood of the circles bou

The idea to proof there exists no solution in a neighbourhood of $(0, 0) \in \mathbb{R}^2$ is to :

1. If the function $u(x, y)$ is a solution then due to (i) $-u(-x, y)$ is also a solution. Such that $u \equiv frac12(u(x, y) - u(-x, y))$ and assume $u(-x, y) = -u(x, y)$

2. We claim that every solution u vanishes on the circles boundary $\partial B(0,\rho_n)$.

This leads to a contradiction by the Divergence Theorem :

$$\int_{B(0,\rho_n)} f dxdy = \int_{B(0,\rho_n)} (\frac{\partial u}{\partial x} + ix\frac{\partial u}{\partial y})dxdy = \int_{B(0,\rho_n)\nabla\cdot} \begin{pmatrix} u \\ ixu \end{pmatrix} dxdy$$

$$\overset{\text{Div Th.}}{=} \int_{\partial B(0,\rho_n)} \begin{pmatrix} u \\ ixu \end{pmatrix} \cdot N(x,y)d\sigma(x,y) = 0$$

.

1.3 Regularity of Solutions

Regularity of a differential equation refers to the local properties of the corresponding functions. The most general functions we consider are distributions, which have the lowest regularity.

Distributions contain measurable functions with the next highest regularity. The highest regularity are smooth functions and analytic functions.

1.4 Boundary Value Problems

In general partial differential equations have an infinite dimensional space of solutions. Similar to how solutions in the ODE case can be uniquely determined (by fixing the values of the derivatives), in PDEs solutions are functions on higher dimensional domains $\Omega \subset \mathbb{R}^n$ such that a natural condition is the specification of the values of the solution and some of its derivatives on the boundary of the domain.

1.5 Divergence Theorem

The divergence theorem is a generalization of the fundamental theorem of calculus to higher dimensions. It states that the surface integral of a vector field over a closed surface, which is called the "flux" through the surface, is equal to the volume integral of the divergence over the region inside the surface. The Idea behind it can be classified into two definitions

Definition 1.5.1. A continuously differentiable homeomorphism $\Phi : \mathbb{R}^k \supset U \to A \subset \mathbb{R}^n$ is called a $k-$dimensional parameterization of A. It is called regular if the Jacobian φ' has full rank k at every point of U.

Definition 1.5.2. Let $A \subset \mathbb{R}^n$ be a subset with a regular parameterization Φ and f a continuous function on A. We define :

$$\int_A f d\sigma := \int_U f \circ \Phi \sqrt{det((\Phi')^T \Phi')} d\mu_{\mathbb{R}^k}.$$

Φ' is the Jacobian.

Some subsets cannot be regularly parameterised, usually this is because the cannot be covered by a single parameterisation an example of this is a sphere, as a sphere is compact there cannot exists a homeomorphism between the sphere and any open set $U \subset \mathbb{R}^k$ this can be solved by using more than one parameterisation, thus the following definition

Definition 1.5.3 (Submanifold)**.** A subster $A \subset \mathbb{R}^n$ is called a $k-$dimensional submanifold if there exists subsets A_i such that each A_i has a regular $k-$dimensional parameterization and $A = \cup A_i$

Issue : subsets can overlap ,which leads to double counts when integrating over the parameterisations. An answer to this are partitions of unity (not practically useful)

Definition 1.5.4. Let $\Omega \subset \mathbb{R}^n$ be covered by a countable family $U_{i\in\mathbb{N}}$ of open subsets. A smooth partition of unity is a countable family $(h_i)_{i\in\mathbb{N}}$ of smooth functions $h_i : \Omega \to [0,1]$ such that :

1. Each $x \in \Omega$ has a neighbourhood on which all but finite many h_i vanish identically
2. For all $x \in \Omega$ we have $\sum_{i=1}^{\infty} h_i(x) = 1$
3. Each h_i vanishes outside of U_i

For every family of open subsets of $\mathbb{R}^n$ there exists a smooth partition of unity.

Theorem 1.5.1 (Divergence Theorem)**.** Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)-$ dimensional sub-manifold of $\mathbb{R}^n$. Let $F : \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously to $\partial\Omega$. Then we have :

$$\int_\Omega \nabla \cdot F d\mu = \int_{\partial\Omega} F \cdot N d\sigma.$$

where N is the outward pointing normal. (last component is positive)

Proof. First Note that $\nabla \cdot F = \partial_1 F_1 + \ldots + \partial_n F_n$ Idea is to do the proof component wise and proof a statement about each component and put them back together

In the special case of $F, F' = 0$ on the boundary then extend F by 0 to $\mathbb{R}^n$ to show that the both sides vanish. We know Ω is bounded and as such is contained by a Box and by continous extension we can integrate over that box

$$\int_{\Omega} \nabla \cdot F = \int_{\text{Box}} \nabla \cdot F = \sum \int_{\text{box}} \partial_i F_i.$$

Where

$$\begin{aligned}\int_{\text{box}} \partial_i F_i &= \int_{-R}^{R} \iint_{-R}^{R} \partial_i F_i dx_1 dx_2 \dots dx_n \\ &= \iint [\int_{-R}^{R} \partial_i F_i]_{-R}^{R} = 0\end{aligned}$$

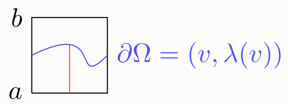
.

Such that the sum is 0 as well

In the general case we do it similar , where the existence of the box is guaranteed by the fact that every submanifold is a graph over some box. We are able to cover $\overline{\Omega}$ by Ω and all the boxes $V' \times (a,b) \subset \mathbb{R}^n$, by compactness we only need finite many sets to cover $\overline{\Omega}$, by choosing a partition of unity we avoid including the same point multiple times :

$$\int_{\partial\Omega} F \cdot N d\sigma = \sum \int_{\partial\Omega \cap U_i} (h_i \cdot F) N d\sigma.$$

Remember that each h_i vanishes outside U_i, in the case $U_i = \Omega$ then $h_i F$ is 0 on the boundary and is covered by the first case. Normal case is $U_i = V' \times (a,b)$ for some a, b



Where the area below the curve is Ω in essence we integrate over lines :

$$x \mapsto \int_{a}^{\lambda(x)} F_i(x,z) dz.$$

Note that $\lambda(x)$ is the height at point x such that the integral of the ∂_i term :

$$\int_{\Omega \cap U_i} \partial_i F_i = \int_{V'} \int_{a}^{\lambda(x)} \partial_{x_i} F_i(x,z) dz d^{n-1}x = \int_{V'} \lambda(x) \cdot F_i(x, \lambda(x)) d^{n-1}x.$$

Last equality follows from the fact that F is zero on $V' \times \{a\}$ (by property of the partition of unity h_i) $\square$

Definition 1.5.5 (Formula outward pointing Normal). The formula for the outward pointing normal is given by

$$N = \pm \frac{1}{\sqrt{1 + |\nabla \lambda|}^2} \begin{pmatrix} -\nabla \lambda \\ 1 \end{pmatrix}.$$

Where λ is a height function,this says in essence that every sub manifold is a graph over a coordinate plane. The explicit formula of N comes from the fact that we can determine every tangent and then just determine N such that the dualproduct (dot product) is 0

Definition 1.5.6 (Projection). Let P be a projection otno the jth coordinate

$$P(x) = (x_1, \dots, \underbrace{c}_{jth}, \dots x_n).$$

To get the derivative is just the identity except at the jth coordinate where it is 0

Now if there exists a regular parameterization of the plane $\Phi : \mathbb{R}^k \to \mathbb{R}^n$ to get a regular parameterization we can project onto the coordinate which would not be linear independent which gives us full rank

1.6 Distributions

Main trick is to use integration by parts to "transfer" the integration from one function to the other, see :

$$F_f(\Phi) = \int_\Omega f \Phi d\mu.$$

$$F_{f'}(\Phi) = \int_\Omega f' \Phi d\mu = -\int_\Omega f \Phi' d\mu.$$

Where the boundary terms vanish as Φ is a test function, that vanish on outside of a compact set ?.

In general distributions are a way to define solutions for partial differential equations that may not posses a regular solution that is continuously differentiable. Thus distributions are a special case of functions that act as weak solutions for linear differential equations (solutions in the sense of distributions)

1.6.1 Test Functions

Test functions are infinitely differentiable functions that vanish outside of their compact support. We say for an open set $\Omega \subset \mathbb{R}^n$ the set of test functions $\mathcal{D}(\Omega)$ are such functions with the following notion of convergence : We say test functions converge $f_n \to f$ if there is a compact subset $K \subset \Omega$ such that $\forall n \in \mathbb{N}\ :\ supp f_n \in K$ and $\partial t^\alpha f_n \to \partial^\alpha f$ in the supremum norm on K for every multi-index α.

Mollifier

Mollifier or approximate identities is a subset of test functions $(\lambda_\varepsilon)_{\varepsilon>0}$ with $supp\lambda_\varepsilon = \overline{B(0,\varepsilon)}$ and $\int \lambda_\varepsilon d\mu = 1$, the standard molifier is defined as :

$$\lambda(x) := \begin{cases} C\exp(\frac{1}{|x|^2-1}), & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}.$$

Then the standard mollifier is given by :

$$\lambda_\varepsilon(x) = \varepsilon^{-n}\lambda(\frac{x}{\varepsilon}).$$

They have the property that for any continuous function f on Ω and suppose $0 \in \Omega$ then :

$$\int_\Omega f\lambda_\varepsilon d\mu \approx \int_{B(0,\varepsilon)} f(0)\lambda_\varepsilon d\mu = f(0).$$

This is in fact an equality as $\varepsilon \downarrow 0$, the proof can be summarized as choosing a compact subset of Ω then taking an ε ball around any point x such that the Ball $B(x,\varepsilon) \subset \Omega$:

$$|f_\varepsilon(x) - f(x)| = |\int_\Omega \lambda_\varepsilon(x-y)(f(y)-f(x))d^n y| \leq \sup_{y\in B(x,\varepsilon)} |f(y)-f(x)|.$$

when $\varepsilon \downarrow 0$ the sup goes to 0 uniformly.

Usecase : Mollifiers are used to prove that properties valid for smooth functions are also valid in nonsmooth situations and in our case to give notion to product of distributions.

1.6.2 Formal Definition

Definition 1.6.1. For any function $f \in L^1_{loc}(\Omega)$ a distribution is given by :

$$F_f : \mathcal{D}(\Omega) \to \mathbb{R}, \qquad \Phi \mapsto \int_\Omega f\Phi d\mu.$$

We define the space of distributions as

Definition 1.6.2. On an open subset $\Omega \subset \mathbb{R}^n$ the space of distributions $\mathcal{D}'(\Omega)$ is defined as the vector space of all linear maps $F : \mathcal{D}(\Omega) \to \mathbb{R}$ which are continuous with respect to the seminorms

$$\|\cdot\|_{K,\alpha} : \mathcal{C}_0^\infty(\Omega) \to \mathbb{R} \qquad \Phi \mapsto \|\Phi\|_{K,\alpha} := \sup_{x\in K}|\partial^\alpha\Phi(x)|.$$

meaning for each compact $K \subset \Omega$ there exist finite many multiindicies α_i and constants $C_i > 0$ such that the following holds for all testfunctions $\Phi \in \mathcal{D}(\Omega)$:

$$|F(\Phi)| \leq C_1\|\Phi\|_{K,\alpha_1} + \ldots + C_M\|\Phi\|_{K,\alpha_M}.$$

The space of distributions $\mathcal{D}'$ can be regarded as the dual space of $\mathcal{D}$

Corollary. We get the following convergence property :

If $\Phi_n \to \Phi$ in $\mathcal{D}(\Omega)$ then the values $F(\Phi_n) \to F(\Phi)$ and we say a sequence of distributions F_n converges to F if $F_n(\Phi) \to F(\Phi)$ for all test functions Φ

Definition 1.6.3. The delta distortion is a special case of distortion such that it does not correspond to an element of $L^1_{loc}(\mathbb{R}^n)$:

$$\delta : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R} \qquad \Phi \mapsto \Phi(0).$$

It is the limit of the sequence of distributions corresponding to the mollifier λ_ε

1.6.3 Opereations on Distributions

The goal is to define as many operations on distributions as possible. The first operation we define is convolution, by first defining it on $\mathcal{C}_0^\infty(\mathbb{R}^n)$

$$(g \star f)(x) := \int_{\mathbb{R}^n} g(x-y)f(y)d^ny = \int_{\mathbb{R}^n} g(z)f(x-z)d^nz.$$

by using integration by parts we get :

$$\partial^\alpha(g \star f) = (\partial^\alpha g) \star f = g \star (\partial^\alpha f).$$

convolution is well behaved in respect to integration by using volume preserving transformation $z = y - x, y = y$, and preservers symmetry of functions, also proven by coordinate transformation $y = Oz + b$

Chapter 2

Task Sheets

2.1 Dont cross the streams

Consider the PDE $\partial_x u + 2x\partial_y u = 0$ on the domain $y > 0$ with the boundary condition $u(x, 0) = g(x)$

Question 1. (a) Show that the boundary hyperplane $\{y = 0\}$ is non-characteristic at $(x, 0)$, except for $x = 0$.

Solution. Consider the PDE as a function $F(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u(x, y), x, y) = 0$ then we need to show that :

$$\partial_{p_1} F(p_0, p_1, z, x, y) \neq 0.$$

In our case :

$$\partial_{p_1} F(p_0, p_1, z, x, y) = \partial_{p_1} (p_0 + 2xp_1) = 2x.$$

Which as required is non zero on $(x, 0)$ except for $x = 0$ □

Question 2. (b) What condition does the PDE impose on the boundary data g at the point $(0, 0)$

Solution. At the point $(0, 0)$ we know $F(d\frac{g(0)}{dx}, p_1, g(0), 0, 0) = 0$ but

$$F(d\frac{dg(0)}{dx}, p_1, g(0), 0, 0) = \frac{dg}{dx}(0) + 2 \cdot 0 \cdot p_1 = \frac{dg}{dx}(0) = 0.$$

The PDE requires g to be constant at $(0, 0)$ □

Question 3. (c) Determine the characteristic curves of this PDE.

Solution. Consider $z(s) = u(x(s), y(s))$ then :

$$z' = \partial_x u x' + \partial_y y'.$$

Choosing $x' = 1$ and $y' = 2x$ aligns with the PDE such that

$$x(s) = s + x_0$$
$$y(s) = 2xs + y_0.$$

Considering the initial condition

$$z(0) = u(x(0), y(0)) = u(x_0, 0).$$

Meaning $y(s) = 2xs$ this leads to :

$$z(s) = u(x_0 + s, 2xs) = g(x_0).$$

For (x, y) we get :

$$x_0 = x - s$$
$$s = \frac{y}{2x}.$$

Such that :

$$u(x, y) = g(x - \frac{y}{2x}).$$

□

Question 4. By considering the y-derivative of u on the boundary hyperplane, show that there is no $\mathcal{C}^1$ solution with the initial data $g(x) = x$

Solution. By (b) we know that the PDE requires $g' = 0$ at $(0, 0)$ but for initial data $g(x) = x$ we get $\frac{dg}{dx}(0) = 1 \neq 0$, such that no $\mathcal{C}^1$ solution may exist, other way is using that $H = \{y = 0\}$ is non-characteristic at $(0, 0)$ □

2.2 Its just a jump to the left

We consider the IVP from Example 1.10., as we saw for small t the method of characteristics gives a unique solution

$$u_{t<1}(x,t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x \end{cases}.$$

Question 5. **(a)** Derive this solution for yourself for extra practice

Solution. We consider Burgers equation $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ with the following continuous initial values $u(x, 0) = g(x)$

and

$$g(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 1-x & \text{for } 0 \leq x < 1 \\ 0 & \text{for } 1 \leq x \end{cases} .$$

Deriving the characteristic equations by setting $z(s) = u(x(s), s)$

$$z' = \partial_x u x' + \partial_t u.$$

such that by choosing $x' = u$

$$z' = 0.$$

i.e z is constant such that

$$x(s) = x_0 + z_0 \cdot s.$$

With $z_0 = u(x_0, 0) = g(x_0)$ such that for arbitrary $(x,t) \in \mathbb{R} \times \mathbb{R}^{>0}$ we want to evaluate which characteristic they lie on i.e the value of x_0, we do this by first considering the shape of our characteristics :

$$x + g(x) \cdot t = \begin{cases} x + 1 \cdot t & \text{for } x \leq 0 \\ x - t(1-x) & \text{for } 0 \leq x < 1 \\ x & \text{for } 1 \leq x \end{cases} .$$

We know that to get a unique solution the characteristics cannot cross, for $t = 1$ we get a crossing such that we need to restrict our solution to $t < 1$, to get our desired x_0 we take the inverse of the above mapping (invertible cause stricly monotone for $t < 1$) with

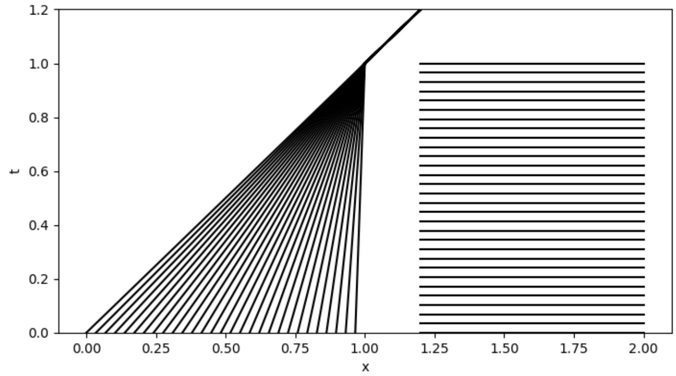
$$y \mapsto \begin{cases} y - t & \text{for } y \leq t \\ \frac{y-t}{1-t} & \text{for } t < x < 1 \\ y & \text{for } 1 \leq x \end{cases} .$$

Therefore we get the desired solution :

$$u_{t<1}(x,t) = \begin{cases} x - t & \text{for } y \leq t \\ \frac{x-1}{t-1} & \text{for } t < x < 1 \\ 0 & \text{for } 1 \leq x \end{cases} .$$

□

Question 6. (b) Draw the corresponding characteristics diagram in the (x,t)- plane



Solution. □

Question 7. **(c)** Describe the graph of disontinuities y(t). Compute the Rankine-Hugonoit condition for v.

Solution. The disontinuity occurs when our solution jumps from 0 to 1, this occurs when $x \uparrow t$ or $x \downarrow t$, i.e along the line $y(t) = t$, we compute the condition by :

$$\dot{y} = \frac{f(u^r) - f(u^l)}{u^r - u^l}.$$

In our case $f = \frac{1}{2}u^2$ and $\dot{y} = 1$

$$1 \neq \frac{0 - \frac{1}{2}}{-1} = \frac{1}{2}.$$

Such that the condition is not fulfilled □

Question 8. **(d)** How much mass (i.e the integral of v over x) is being lost in the system described by v for $t > 1$

Solution. Integrating for fixed t

$$F_{[a,b]}(t) = \int_a^b v(x,t)dx = \int_a^t v(x,t)dx + \int_t^b v(x,t)dx$$
$$= \int_a^t 1dx = t - a.$$

i.e $F' = 1$ the mass over $[a, b]$ changes by 1 for every increase in t. □

2.3 You're not in traffic, you are traffic

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper- to-bumper they can't move, call this density u_m.

Question 9. (a) What properties do you think that f should have? Does $f(u) = s_m u(1 - \frac{u}{u_m})$ have these properties

Solution. The flux f should describe the flow of cars given the density of cars on any given stretch, to properly describe this f needs to consider the case of 0 cars i.e density $u = 0$ and the case $u = u_m$. The flux of traffic f should be a function of the number of cars u and the speed each car moves at i.e :

$$f(u) = u \cdot s(u).$$

Where we know that $s'(u)|_{u=0} = s_m$ and $s'(u)|_{u=u_m} = 0$ i.e

$$s(u) = s_m \cdot (1 - \frac{u}{u_m}).$$

plugging into f gives the described function □

Question 10. (b) Find a function f that meets your condition or use the f from the previous part, and write down a PDE to describe the traffic flow

Solution. We take $f(u) = u \cdot s(u)$ as previously defined and model using a conservation law

$$\dot{u} = -\frac{\partial f(u)}{\partial x}.$$

i.e the change of density on any given stretch in t should be proportional to the fluxes change in x □

Question 11. **(c)** Find all solutions that are constant in time

Solution. Solutions constant in time implies

$$\dot{u} = 0.$$

such that

$$0 = -\partial_x f(u) \Rightarrow 0 = -\partial_u f \cdot \partial_x u.$$

Where

$$\partial_u f = -u \cdot \frac{s_m}{u_m} + s_m(1 - \frac{u}{u_m}) = s_m \cdot (1 - 2\frac{u}{u_m}).$$

Which gives

$$0 = \partial_x u \cdot s_m \cdot (1 - 2\frac{u}{u_m}).$$

Such that any constant solution must have

$$\partial_x u = 0.$$

or

$$1 - 2\frac{u}{u_m} = 0 \Rightarrow 1 = 2\frac{u}{u_m} \Rightarrow \frac{u_m}{2} = u.$$

Meaning all constant functions u solve the PDE. □

Question 12. **(d)** Consider the situation of a traffic light at $x = 0$: to the left of the traffic light , the cars are queued up at maximum density. To the right of the trafic light, the road is empty. Now, at time $t = 0$ the traffic light turns green. Give a discontinuous solution that obeys the Rankine-Hugonoit condition as well as a continuous solution

Solution. We have $u(-\varepsilon, t = 0) = u_m$ and $u(\varepsilon, t = 0) = 0$, The Rankine-Hugonoit condition is given by

$$\dot{y} = \frac{f(u^r(y,t)) - f(u^l(y,t))}{u^r(y,t) - u^l(y,t)}.$$

Where $\lim_{x \uparrow y(t)} u(x,t) = u^l(y,t)$, u^r analog, the jump occurs at $t = 0$ when the light switches to green and cars go from full stop to full speed, i.e $u^r(y,t) = 0$ and $u^l(y,t) = u_m$

$$\dot{y} = \frac{f(u^r) - f(u^l)}{u^r - u^l} = \frac{f(0) - f(u_m)}{0 - u_m} = \frac{0 - 0}{-u_m} = 0.$$

discontinuous solution

$$u(x,t) = \begin{cases} 0 & \text{if } x \geq t \\ u_m & \text{if } x < t \end{cases}.$$

By (c) on both sides of $y(t) = 0$ u is a solution and fulfills the Rankine-Hugonoit condition (hätte eigentlich gesagt es sollte $x > t$ und $x < t$ aber was is bei $x = t$)

For a continuous solution consider the characteristic curves by

$$x(t) = x_0 + tf'(g(x_0)) \Rightarrow x(t) = x_0 + t \cdot (s_m(1 - 2\frac{g(x_0)}{u_m})).$$

Consider for $s_m = 1, u_m = 2$ (doenst matter really)

$$x(t) = x_0 + t \cdot (1 - g(x_0)).$$

Initial condition $u(x_0, 0) = g(x_0)$ where we know that $g(-\varepsilon) = 2$ and $g(\varepsilon) = 1$ furthermore we know

$\square$