Sheet 1

1.1

Mostly just calculating and showing the three properties

1. $W_0 = 0$
2. For any partition $(t_i)_{i\in\mathbb{N}}$ it holds that $W_0, W_{t_1} - W_{t_0}, \dots W_{t_n} - W_{t_{n-1}}$ are independent random variables
3. The increment's are normally distributed i.e $W_t - W_s \sim \mathcal{N}(0, |t-s|)$

For (iv) pick $\pm B_t$ as a counterexample , then the variance doesn't match for the increments

1.2

Exercise. Let $(X_t)_{t\in[0,\infty)}$ be a right-continuous real-valued, stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and let $A \subset \mathbb{R}$. Prove that the hitting time

$$\tau_A := \inf\{t \geq 0 : X_t \in A\}.$$

is a stopping time if

1. A is open and $(\mathcal{F}_t)_{t\in[0,\infty)}$ is right-continuous
2. A is closed and $(X_t)_{t\in[0,\infty)}$ is continuous

Proof. First we note that if A is open then $\tau_A = t$ does not imply $X_t \in A$, and that since X_t is right-continuous we have for any $\omega \in \{\tau_A = t\}$ that

$$t \mapsto X_t(\omega).$$

is right-continuous i.e for any $\varepsilon > 0$ there $\exists \delta > 0$ such that

$$s \in [t, t+\delta] \Rightarrow |X_s - X_t| < \varepsilon.$$

i.e if $X_t \in A$ then a small Ball (to the right) around t is also in A. This lets us do

$$\{\tau_A \leq t\} = \{\tau_A < t\} \cup \{\tau_A = t\}.$$

Where

$$\{\tau_A < t\} = \bigcup_{s<t} \{X_s \in A\} \overset{\text{Cont.}}{=} \bigcup_{s<t, s\in\mathbb{Q}} \{X_s \in A\}.$$

where the last union is over finite set each in $\mathcal{F}_s$ (X is adapted) such that

$$\{\tau_A < t\} \in \mathcal{F}_t.$$

For $\{\tau_A = t\}$ we consider

$$\{\tau_A \leq t\} = \bigcap \{\tau_A < t + \frac{1}{n}\}.$$

Which by right right-continuity and again a continuous argument lie in $\mathcal{F}_t^+ = \mathcal{F}_t$

I am unsure why X continuous is necessary since since X_t at any ω is already uniquely determined by its paths. We consider

$$d(x, A) = \inf_{y \in A} |x - y|.$$

Then

$$\{\tau_A = t\} = \{d(X_t, A) = 0\}.$$

we show that

$$A_n = \{y \in \mathbb{R} : d(y, A) < \frac{1}{n}\}.$$

Then

$$\bigcap A_n = A.$$

Since A is closed, then we want to show

$$\{\tau_A \leq t\} = \bigcap_{n \in \mathbb{N}} \{\tau_{A_n} \leq t\}.$$

And first note that $\tau_{A_n} \leq \tau_{A_{n+1}} \leq \tau_A$
We show that for $T = \sup_n \tau_{A_n}$

$$\tau_A \leq T.$$

Then we get the convergence, we do so by showing that $X_T \in A$ then by definition $\tau_A \leq T$.

$$d(X_T, A) = \inf_{y \in A} |X_T - y| \leq |X_T - X_{t_n}| + |X_{t_n} - y| \leq |X_T - X_{\tau_n}| + \frac{1}{n}.$$

And since X is continuous we get that there $\exists N \in \mathbb{N}$ such that for $n \geq N$

$$|X_T - X_{t_n}| < \frac{1}{n}.$$

in fact left continuous would have been enough (for this argument) we still need right continuous such that we can apply (i) to

$$\{\tau_A \leq t\} = \bigcap_{n \in \mathbb{N}} \{\tau_{A_n} < t\}.$$

□

Excercise 1.3

Exercise. Let X and $X_n, n \in \mathbb{N}$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove the following statements

1. If $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable and $X_n \to X$ $\mathbb{P}$-a.s. then $X_n \to X$ in L^1
2. If X is integrable, then the family $\{\mathbb{E}[X|\mathcal{G}] \ : \ G \subseteq \mathcal{F}\}$ is uniformly integrable

Proof. For (i) alternative statement is if $\|X_n\| \to \|X\|$ and $X_n \to X$ a.s. then

$$\lim_{n\to\infty} \mathbb{E}[|X - X_n|] = 0.$$

We consider

$$|X - X_n| =\leq |X| + |X_n|.$$

I.e

$$|X| + |X_n| - |X - X_n| \geq 0.$$

Such that by Fatou

$$\begin{aligned} 0 \leq \mathbb{E}[\lim_{n\to\infty} |X| + |X_n| - |X - X_n|] = \mathbb{E}[2|X|] &\leq \liminf \mathbb{E}[|f| + |f_n| - |f - f_n|] \\ &= \liminf(\mathbb{E}[|f|] + \mathbb{E}[f_n]) - \limsup \mathbb{E}[|f - f_n|]. \end{aligned}$$

Then we get by rearanging

$$\limsup \mathbb{E}[|f - f_n|] \leq \liminf(\mathbb{E}[|f|] + \mathbb{E}[f_n]) - \mathbb{E}[2|f|] = 0.$$

Now consider

$$\mathbb{E}[|f - f_n|] = \mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|\geq c}] + \mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|<c}].$$

The last is bounded by

$$\mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|\geq c}] + \mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|<c}] < \mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|\geq c}] + c \cdot \mathbb{P}(\Omega).$$

By convergence in measure there exists $n \in \mathbb{N}$ such that $\mathbb{P}(|f - f_n| \geq c) < \delta$ where c is choses n such that $c \cdot \mathbb{P}(\Omega) < \frac{\varepsilon}{2}$ Now we prove $|f - f_n|$ is uniformly integrable, we have

$$\int_A |f| \leq \liminf \int_A |f_n| \leq \varepsilon.$$

for $\mathbb{P}(A) < \delta$ then

$$|f - f_n| < |f| + |f_n|.$$

i.e

$$\int_A |f - f_n| \leq \int_A |f| + |f_n| \leq \varepsilon.$$

Such that $|f - f_n|$ is uniformly integrable. $\square$

Let us summarize, in the hitting time exercise we know finite unions of open sets are in the σ-algebra such that we always want to rewrite it as that case, the right continuity of X allows us to show that any infinite union (over time) can be written as a finite one. the right continuity is useful be cause

$$[1, 2 + \frac{1}{n}) \to [1, 2].$$

And we need that $\{\tau < t + \frac{1}{n}\}$ are contained in $\mathcal{F}_t$.

In the closed case we argue that first

$$A = \bigcap A_n := \{y \in \mathbb{R} : d(y, A) < \frac{1}{n}\}.$$

this follows since A is closed, then we want the following convergence

$$\{\tau_A \leq t\} = \bigcap \{\tau_{A_n} < t\}.$$

We use the left continuity from X to prove that

$$\sup \tau_{A_n} \leq \tau_A \text{ and } \tau_A \leq \sup_{\tau_{A_n}}.$$

then since clearly $\tau_{A_n} \leq \tau_{A_n+1}$ the following holds

$$\lim_{n\to\infty} \tau_{A_n} = \tau_A.$$

the first direction holds immediately since for any n we must have that

$$\tau_{A_n} \leq \tau_A.$$

And for the second we consider

$$d(X_T, A) = \inf_{y \in A} |X_T - y| \leq |X_T - X_{t_n}| + |X_{t_n} - y| \leq |X_T - X_{\tau_n}| + \frac{1}{n}.$$

which goes to 0 for $n \to \infty$

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2.1

Do not forget to show the integrability of the processes, besides that its just using smart 0 to get the result one wants, at (iii) one has to recognize that the mean of functions of equal distribution are the same and then

$$B_s = B_s - B_0 \sim \mathcal{N}(0, s).$$

2.2

For (i) the direction indistinguishable $\Rightarrow$ modification is trivial, for the other way we recognize that

$$A = \{X_t = Y_t \ , \ \forall t \in [0,T] \cap \mathbb{Q}\} = \bigcap_{t \in [0,T] \cap \mathbb{Q}} \{X_t = Y_t\}.$$

Then A^c is a null set by property of being a union of null sets.
I.e for $\omega \in A$ we already have for rational times t

$$X_t(\omega) = Y_t(\omega).$$

For real times t we argue by

$$X_t(\omega) = \lim_{k \to \infty} X_{q_k}(\omega) = \lim_{k \to \infty} Y_{q_k}(\omega) = Y_t(\omega).$$

BIG ISSUE WITH THE ABOVE
When talking in the language of probability one always needs to consider that everything is only defined up to null sets, i.e the $\omega \in A$ is not guaranteed to also be in $\omega \overset{?}{\in} \{X \text{right continuous}\}$. which is why we need to consider $\omega \in A \cap B \cap C$ where B, C guarantee that we can perform the operations.

2.3

(i) is an ok assumption to make since we otherwise consider the shifted process $M_t = M_t - \mathbb{E}[M_t]$, (ii) The telescoping argument here is fairly important, since its a common tool, and then we can proceed from there by letting go $n \to \infty$ see

$$M_t - M_0 = \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}}).$$

As $n \to \infty$

$$M_t - M_0 = \int_0^t dM_t.$$

same argument is used later in proving Itos formula.

For (iii) we just consider the limit to $n \to \infty$ and then argue by DCT (we can bound like the following)

$$(M_{t_i} - M_{t_{i-1}})^2 = (M_{t_i} - M_{t_{i-1}})(M_{t_i} - M_{t_{i-1}}) \leq \sup_i (M_{t_i} - M_{t_{i-1}}) \cdot |M_{t_i} - M_{t_{i-1}}|.$$

For (iv) we use fatou, also a natural bound

Lemma 0.0.1. Let $(X_t)_{t \in [0,T]}$ be an Itô process with representations

$$X_t = X_0 + \int_0^t a(\cdot, s) ds + \int_0^t b(\cdot, s) dB_s = \tilde{X}_0 + \int_0^t \tilde{a}(\cdot, s) ds + \int_0^t \tilde{b}(\cdot, s) dB_s.$$

$X_0 = \tilde{X}_0$, then $a = \tilde{a}$ and $b = \tilde{b}$

Proof. We have

$$0 = \int_0^t a(\cdot, s) - \tilde{a}(\cdot, s) + \int_0^t b(\cdot, s) - \tilde{b}(\cdot, s) dB_s.$$

Which follows by taking the difference, i.e

$$\int_0^t a(\cdot, s) - \tilde{a}(\cdot, s) = -\int_0^t b(\cdot, s) - \tilde{b}(\cdot, s) dB_s.$$

This is a local martingale that is continuous and of finite variation □

Let us prove that

$$(\int_0^t a(\cdot, s) ds).$$

is of finite variation

Proof. Define

$$A_t = \int_0^t a(\cdot, s) ds.$$

We consider

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^n A_{t_i} - A_{t_{i-1}} &= \lim_{n\to\infty} \sum_{i=1}^n \int_0^{t_i} a(\cdot, s) ds - \int_0^{t_{i-1}} a(\cdot, s) ds \\
&= \lim_{n\to\infty} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} a(\cdot, s) ds \\
&\leq \lim_{n\to\infty} \sum_{i=1}^n \sup_{t\in[t_{i-1}, t_i]} |a(\cdot, s)|(t_i - t_{i-1}) \\
&\leq \lim_{n\to\infty} \sum_{i=1}^n \sup_{t\in[0,T]} |a(\cdot, s)|(t_i - t_{i-1}) \\
&\leq \lim_{n\to\infty} C \sum_{i=1}^n (t_i - t_{i-1}) \\
&< \infty.
\end{aligned}$$

$$
\begin{aligned}
\langle A\rangle_t = \lim_{n\to\infty}\sum_{i=1}^{n}(A_t - A_{t-1})^2 &= \lim_{n\to\infty}\sum_{i=1}^{n}(\int_0^{t_i} a(\cdot,s)ds - \int_0^{t_{i-1}} a(\cdot,s)ds)^2 \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i} a(\cdot,s)ds)^2 \\
&\leq \lim_{n\to\infty}\sum_{i=1}^{n}(t_i - t_{i-1})\int_{t_{i-1}}^{t_i} a(\cdot,s)^2 ds \\
&= (T)\lim_{n\to\infty}\sum_{i=1}^{n}.
\end{aligned}
$$

$$
(\int_{t_{i-1}}^{t_i} |a(\cdot,s)\cdot 1|ds)^2 \leq (\int_{t_{i-1}}^{t_i} |a(\cdot,s)|^2 ds)ds \cdot \int_{t_{i-1}}^{t_i} 1^2 = (t_i - t_{i-1})\int_{t_{i-1}}^{t_i} |a(\cdot,s)|^2 dsds.
$$

□

Lemma 0.0.2. Show

$$
|g|_t = \sup_{\Pi}\sum_{J\in\Pi}|\Delta_{J\cap[0,t]}g| = \lim_{n\to\infty}\sum_{J\in\Pi_n}|\Delta_{J\cap[0,t]}g|.
$$

For a zero sequence of partitions

Proof. Let $(\Pi)_{n\in\mathbb{N}}$ be a zero-sequence of partitions and define

$$
|g|_t^n = \sum_{J\in\Pi^n}|\Delta_{J\cap[0,t]}g|.
$$

then showing that

$$
|g|_t^{n+1} \geq |g|_t^n.
$$

And

$$
\sup_{\Pi}\sum_{J\in\Pi}|\Delta_{J\cap[0,t]}g| \geq |g|_t^n.
$$

Gives

$$
\lim_{n\to\infty}|g|_t^n = \sup_{\Pi}\sum_{J\in\Pi}|\Delta_{J\cap[0,t]}g|.
$$

□