Sheet 1

1.1

Mostly just calculating and showing the three properties

1. $W_0 = 0$
2. For any partition $(t_i)_{i\in\mathbb{N}}$ it holds that $W_0, W_{t_1} - W_{t_0}, \dots W_{t_n} - W_{t_{n-1}}$ are independent random variables
3. The increment's are normally distributed i.e $W_t - W_s \sim \mathcal{N}(0, |t-s|)$

For (iv) pick $\pm B_t$ as a counterexample , then the variance doesn't match for the increments

1.2

Exercise. Let $(X_t)_{t\in[0,\infty)}$ be a right-continuous real-valued, stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ and let $A \subset \mathbb{R}$. Prove that the hitting time

$$\tau_A \coloneqq \inf\{t \geq 0 : X_t \in A\}.$$

is a stopping time if

1. A is open and $(\mathcal{F}_t)_{t\in[0,\infty)}$ is right-continuous
2. A is closed and $(X_t)_{t\in[0,\infty)}$ is continuous

Proof. First we note that if A is open then $\tau_A = t$ does not imply $X_t \in A$, and that since X_t is right-continuous we have for any $\omega \in \{\tau_A = t\}$ that

$$t \mapsto X_t(\omega).$$

is right-continuous i.e for any $\varepsilon > 0$ there $\exists\delta > 0$ such that

$$s \in [t, t+\delta] \Rightarrow |X_s - X_t| < \varepsilon.$$

i.e if $X_t \in A$ then a small Ball (to the right) around t is also in A. This lets us do

$$\{\tau_A \leq t\} = \{\tau_A < t\} \cup \{\tau_A = t\}.$$

Where

$$\{\tau_A < t\} = \bigcup_{s<t}\{X_s \in A\} \overset{\text{Cont.}}{=} \bigcup_{s<t, s\in\mathbb{Q}} \{X_s \in A\}.$$

where the last union is over finite set each in $\mathcal{F}_s$ (X is adapted) such that

$$\{\tau_A < t\} \in \mathcal{F}_t.$$

For $\{\tau_A = t\}$ we consider

$$\{\tau_A \leq t\} = \bigcap \{\tau_A < t + \frac{1}{n}\}.$$

Which by right right-continuity and again a continuous argument lie in $\mathcal{F}_t^+ = \mathcal{F}_t$

I am unsure why X continuous is necessary since since X_t at any ω is already uniquely determined by its paths. We consider

$$d(x, A) = \inf_{y \in A} |x - y|.$$

Then

$$\{\tau_A = t\} = \{d(X_t, A) = 0\}.$$

we show that

$$A_n = \{y \in \mathbb{R} : d(y, A) < \frac{1}{n}\}.$$

Then

$$\bigcap A_n = A.$$

Since A is closed, then we want to show

$$\{\tau_A \leq t\} = \bigcap_{n \in \mathbb{N}} \{\tau_{A_n} \leq t\}.$$

And first note that $\tau_{A_n} \leq \tau_{A_{n+1}} \leq \tau_A$
We show that for $T = \sup_n \tau_{A_n}$

$$\tau_A \leq T.$$

Then we get the convergence, we do so by showing that $X_T \in A$ then by definition $\tau_A \leq T$.

$$d(X_T, A) = \inf_{y \in A} |X_T - y| \leq |X_T - X_{t_n}| + |X_{t_n} - y| \leq |X_T - X_{\tau_n}| + \frac{1}{n}.$$

And since X is continuous we get that there $\exists N \in \mathbb{N}$ such that for $n \geq N$

$$|X_T - X_{t_n}| < \frac{1}{n}.$$

in fact left continuous would have been enough (for this argument) we still need right continuous such that we can apply (i) to

$$\{\tau_A \leq t\} = \bigcap_{n \in \mathbb{N}} \{\tau_{A_n} < t\}.$$

$\square$

Excercise 1.3

Exercise. Let X and $X_n, n \in \mathbb{N}$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove the following statements

1. If $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable and $X_n \to X$ $\mathbb{P}$-a.s. then $X_n \to X$ in L^1
2. If X is integrable, then the family $\{\mathbb{E}[X|\mathcal{G}] \ : \ G \subseteq \mathcal{F}\}$ is uniformly integrable

Proof. For (i) alternative statement is if $\|X_n\| \to \|X\|$ and $X_n \to X$ a.s. then

$$\lim_{n\to\infty} \mathbb{E}[|X - X_n|] = 0.$$

We consider

$$|X - X_n| =\leq |X| + |X_n|.$$

I.e

$$|X| + |X_n| - |X - X_n| \geq 0.$$

Such that by fatou

$$0 \leq \mathbb{E}[\lim_{n\to\infty} |X| + |X_n| - |X - X_n|] = \mathbb{E}[2|X|] \leq \liminf \mathbb{E}[|f| + |f_n| - |f - f_n|] = \liminf(\mathbb{E}[|f|] + \mathbb{E}[f_n]) - \text{li}$$

Then we get by rearanging

$$\limsup \mathbb{E}[|f - f_n|] \leq \liminf(\mathbb{E}[|f|] + \mathbb{E}[f_n]) - \mathbb{E}[2|f|] = 0.$$

Now consider

$$\mathbb{E}[|f - f_n|] = \mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|\geq c}] + \mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|<c}].$$

The last is bounded by

$$\mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|\geq c}] + \mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|<c}] < \mathbb{E}[|f - f_n| \cdot \mathbb{1}_{|f-f_n|\geq c}] + c \cdot \mathbb{P}(\Omega).$$

By convergence in measure there exists $n \in \mathbb{N}$ such that $\mathbb{P}(|f - f_n| \geq c) < \delta$ where c is choses n such that $c \cdot \mathbb{P}(\Omega) < \frac{\varepsilon}{2}$ Now we prove $|f - f_n|$ is uniformly integrable, we have

$$\int_A |f| \leq \liminf \int_A |f_n| \leq \varepsilon.$$

for $\mathbb{P}(A) < \delta$ then

$$|f - f_n| < |f| + |f_n|.$$

i.e

$$\int_A |f - f_n| \leq \int_A |f| + |f_n| \leq \varepsilon.$$

Such that $|f - f_n|$ is uniformly integrable. $\square$

Let us summarize, in the hitting time exercise we know finite unions of open sets are in the σ-algebra such that we always want to rewrite it as that case, the right continuity of X allows us to show that any infinite union (over time) can be written as a finite one. the right continuity is useful be cause

$$[1, 2 + \frac{1}{n}) \to [1, 2].$$

And we need that $\{\tau < t + \frac{1}{n}\}$ are contained in $\mathcal{F}_t$.

In the closed case we argue that first

$$A = \bigcap A_n := \{y \in \mathbb{R} : d(y, A) < \frac{1}{n}\}.$$

this follows since A is closed, then we want the following convergence

$$\{\tau_A \leq t\} = \bigcap \{\tau_{A_n} < t\}.$$

We use the left continuity from X to prove that

$$\sup \tau_{A_n} \leq \tau_A \text{ and } \tau_A \leq \sup_{\tau_{A_n}}.$$

then since clearly $\tau_{A_n} \leq \tau_{A_n+1}$ the following holds

$$\lim_{n\to\infty} \tau_{A_n} = \tau_A.$$

the first direction holds immediately since for any n we must have that

$$\tau_{A_n} \leq \tau_A.$$

And

$$d(X_T, A) = \inf_{y \in A} |X_T - y| \leq |X_T - X_{t_n}| + |X_{t_n} - y| \leq |X_T - X_{\tau_n}| + \frac{1}{n}.$$