Chapter 1

Brownian Motion and Martingales

These are merely my thoughts and notes to the stochastic calc script and will only include the entire statement if i had any special thoughts to it.

Definition 1.0.1 (usual conditions)**.** The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is said to satisfy the usual conditions if :

1. $\mathcal{F}_0$ contains all Pr-null sets $\mathcal{N}$ ("completeness")
2. $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} F_s$ for $t \in [0, T)$ ("right-continuity")

Remark. Completeness assures us that any modifications to an adapted stochastic process is again adapted (same null sets).
Right-continuity can be thought of as giving us the ability to slightly peak into the future, consider the following hitting time :

$$\tau_A = \inf\{t \in [0, T] | X_t \in A\}.$$

For some open set $A \subset \mathbb{R}$ then for any $s \in [0, T]$ the event :

$$\{\tau_A = s\} = \{X_s \in \overline{A}\}.$$

Meaning that at time s we do not know if X_s is already in A or just right on the boundary of entering, such that we need the ability to peak slightly into the future.

Proposition 1.0.1. Let $(B_t)_{t\in[0,T]}$ be a Brownian motion. The completed natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ of a Brownian motion $(B_t)_{t\in[0,T]}$ is defined by

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^B, \mathcal{N}).$$

is right-continuous

Proof. Idea is to show $\mathcal{F}_{t+} \subseteq \mathcal{F}_t$ by taking any continuous and bounded $f : \mathbb{R}^d \to \mathbb{R}$ and showing that for $d \in \mathbb{N}$, $0 \leq t_1 < t_2 < \ldots < t_d$

$$\mathbb{E}[f(B_{t_1}, \ldots, B_{t_d}) \mid \mathcal{F}_{t+}] \text{ is } \mathcal{F}_t\text{-measurable.}$$

□