Leon Fiethen 1728330
Janik Sperling 1728567.

Sheet 9

27. It's not easy being green

Exercise. Suppose that Ω is a bounded domain. Prove that there is at most one Green's function on Ω

Proof. Since Ω is bounded by using the weak maximums principle and using property (i) of Greens function.

Assume two Green's functions for Ω $G, \tilde{G}$ exist then

$$u(y) = G(x,y) - \tilde{G}(x,y) = G(x,y) - \tilde{G}(x,y) + \underbrace{(\Phi(x-y) - \Phi(x-y))}_{=0}$$
$$= G(x,y) - \Phi(x-y) - (\tilde{G}(x,y) - \Phi(x-y)).$$

Then $u(y)$ is harmonic by 3.18 (i) and by the weak maximum principle we must have $u(y) = 0$ $\square$

Exercise. On the other hand, suppose that Ω has Green's function G_Ω and that there exists a non-trivial solution to the Dirichlet problem

$$\Delta u = 0 \quad u|_{\partial\Omega} = 0.$$

Proof. We search for a function that satisfies properties (i) and (ii) from 3.18, since u is non-trivial

$$\tilde{G}(x,y) = G(x,y) + u(y) \neq G(x,y).$$

and we check (i)

$$y \mapsto \tilde{G}(x,y) - \Phi(x-y) = (G(x,y) - \Phi(x-y)) + u(y).$$

Both parts extend to a harmonic function for $x \in \Omega$, first half by virtue of being a Green's function , second part is harmonic by properties of being a result to the Dirichlet Problem.

For (ii) we check for $y \in \partial\Omega$

$$y \mapsto \tilde{G}(x,y) - \Phi(x-y) = G(x,y) + u(y).$$

is 0 since G is a Greens function and we have $u|_{\partial\Omega} = 0$. It follows that $\tilde{G}$ is a second Greens Function. $\square$

28. Do nothing by halves

Let $H_1^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n | x_1 > 0\}$ be the upper half space and $H_1^0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n | x_1 = 0\}$ be the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \dots, x_n)$ reflection in the plane H^0

Exercise (a). Let $u \in \mathcal{C}^2(\overline{H_1^+})$ be a harmonic function that vanishes on H_1^0. Show that the function

$$v : \mathbb{R}^n \to \mathbb{R}\ x \mapsto \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}.$$

is harmonic

Proof. We want to check that

$$\Delta v = 0.$$

We know (by past exercise sheet) that if u is harmonic then $u(R_1(x))$ is harmonic also, such that $v|_{x_1>0}$ and $v|_{x_1<0}$ are both harmonic.

We check the partial derivatives, at $x_1 > 0$ we have

$$\frac{\partial v}{\partial x_i} = \frac{\partial u}{\partial x_i}$$
$$\frac{\partial^2 v}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2}.$$

At $x_1 < 0$

$$\frac{\partial v}{\partial x_i} = -\frac{\partial u(R_1(x))}{\partial x_i}$$
$$\frac{\partial v}{\partial x_1} = \frac{\partial u(R_1(x))}{\partial x_1}.$$

and

$$\frac{\partial^2 v}{\partial x_i^2} = -\frac{\partial^2 u(R_1(x))}{\partial x_i^2}$$
$$\frac{\partial^2 v}{\partial x_1^2} = -\frac{\partial^2 u(R_1(x))}{\partial x_1^2}.$$

i.e for $i \neq 1$ in the first partial derivative we should get a problem

$$\lim_{x_1 \to 0^+} \frac{\partial v}{\partial x_i} \neq \lim_{x_1 \to 0^-} \frac{\partial v}{\partial x_i} \Leftrightarrow \lim_{x_1 \to 0^+} \frac{\partial u}{\partial x_i} \neq \lim_{x_1 \to 0^-} -\frac{\partial u}{\partial x_i}.$$

But since u is continuous and vanishes on H_1^0 we have that both sides of the limit are zero for $i \neq 1$, i.e we get

$$\lim_{x_1 \to 0^+} \frac{\partial v}{\partial x_i} = \lim_{x_1 \to 0^-} \frac{\partial v}{\partial x_i}.$$

We get that the partial derivatives of v extend continuous to $x_1 = 0$ and that v is harmonic. □

Exercise (b). Show that Green's function for H_1^+ is

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

Proof. We check (i) and (ii), For (i) we first note

$$G(x, y) - \Phi(x - y) = -\Phi(R_1(x) - y).$$

we check for singularity at $R_1(x) = y$, since $x \in H_1^+$ then

$$R_1(x) \in \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_1 < 0\}$$

but since $y \in H_1^+$, then $R_1(x) = y$ is impossible and since Φ is harmonic we get that $G(x, y) - \Phi(x - y)$ extends to a harmonic function (since we checked for the singularity).
For (ii) we check for $x \in \Omega$ and $y \in \partial\Omega$

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

The fundamental solution only depends on the Length $\|R_1(x) - y\|$ is symmetric i.e.

$$\|R_1(x) - y\| = \|x - R_1(y)\|.$$

Since $y \in \partial\Omega := H_1^0$ we get $R_1(y) = y$ and

$$\begin{aligned} G(x, y) &= \Phi(x - y) - \Phi(R_1(x) - y) \\ &= \Phi(x - y) - \Phi(x - R_1(y)) \\ &= \Phi(x - y) - \Phi(x - y) \\ &= 0. \end{aligned}$$

Thus G is Green's function for H_1^+ □

Exercise (c). Compute the Green's function for B^+

Proof. By 3.20 we know that

$$G_{B(0,1)}(x, y) = \Phi(x - y) - \Phi(|x|(\tilde{x} - y)).$$

where $\tilde{x} = \frac{x}{|x|^2}$

We know that the greens function $B(0,1)$ must be unique,then lets say we have a greens function on B^+ call it G^+, we expect

$$G_{B(0,1)(x,y)}|_{B^+} \equiv G^+.$$

We consider

$$G(x,y) = \Phi(x-y) - \Phi(|x|(R_1(\tilde{x}) - y)).$$

We check (i)

$$G(x,y) - \Phi(x-y) = -\Phi(|x|(R_1(\tilde{x}) - y)).$$

By similar argument to (b) we know the singularity is not a problem and (i) is satisfied by properties of the fundamental solution
For (ii) we check $x \in B^+$ and $y \in \partial B^+$, clearly the boundary ∂B^+ consists of two parts,

$$\partial B^+ = B^0 \cup (\partial B \cap H^+).$$

We consider the cases separately,for $x \in B^+$ and $y \in B^0$

$$\begin{aligned} G(x,y) = \Phi(x-y) - \Phi(|x|(R_1(\tilde{x}) - y)) &= \Phi(x-y) - \Phi(|x|(\tilde{x} - R_1(y))) \\ &= \Phi(x-y) - \Phi(|x|(\tilde{x} - y)). \end{aligned}$$

For $x \in B^+$ and $y \notin B^0$ it holds

$$\Phi(R_1(x) - y) = \Phi(x - R_1(y)) \neq \Phi(x-y).$$

And notice it doesn't work out lol , we choose new Green's function such that the above is 0 ,

$$\tilde{G}(x,y) = \Phi(x-y) - \Phi(R_1(x) - y) - (\Phi(|x|(\tilde{x} - y)) - \Phi(|x|(R_1(\tilde{x}) - y))).$$

then for $x \in B^+$ and $y \in B^0$

$$\begin{aligned} \tilde{G}(x,y) &= \Phi(x-y) - \Phi(R_1(x) - y) - (\Phi(|x|(\tilde{x} - y)) - \Phi(|x|(R_1(\tilde{x}) - y))) \\ &= \Phi(x-y) - \Phi(x - R_1(y)) - (\Phi(|x|(\tilde{x} - y)) - \Phi(|x|(\tilde{x} - R_1(y)))) \\ &= \Phi(x-y) - \Phi(x-y) - (\Phi(|x|(\tilde{x} - y)) - \Phi(|x|(\tilde{x} - y))) \\ &= 0. \end{aligned}$$

similar argument to (b), for $y \in \partial B \cap H^+ \subset \partial B(0,1)$ we have by lecture

$$\||x|(\tilde{x} - y)\| = |x-y|.$$

and

$$\||x|((R_1(\tilde{x}) - y))\| = |R_1(x) - y|.$$

Note that swapping what $R_1(\cdot)$ acts on doesn't give us anything here since $R_1(y) \neq y$.

$$
\begin{aligned}
\tilde{G}(x,y) &= \Phi(x-y) - \Phi(R_1(x)-y) - (\Phi(|x|(\tilde{x}-y)) - \Phi(|x|(R_1(\tilde{x})-y))) \\
&= \Phi(x-y) - \Phi(|x|(\tilde{x}-y) + \Phi(|x|(R_1(\tilde{x})-y)) - \Phi(R_1(x)-y) \\
&= 0.
\end{aligned}
$$

□

29. Teach a man to fish

Exercise (a). Using the Green's function of H_1^+ from the previous question, derive the following formal integral representation for a solution of the Dirichlet problem

$$\Delta u = 0 \qquad u|_{H_1^0} = g.$$

$$u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x-z|^n} d\sigma(z).$$

Proof. We assume g has sufficient regularity, by Greens representation we know

$$u(x) := \int_{H_1^+} G_{H_1^+}(x,y) f(y) d^n y - \int_{H_1^0} g(z) \nabla_z G_{H_1^+} \cdot N d\sigma(z).$$

solves the Dirichlet problem in fact as $f \equiv 0$

$$u(x) := -\int_{H_1^0} g(z) \nabla_z G_{H_1^+}(x,z) \cdot N d\sigma(z).$$

We calculate for $n > 2$ and

$$
\begin{aligned}
\nabla_z G(x,z) &= \nabla_z(\Phi(x-z) - \Phi(R_1(x)-z)) \\
&= \frac{-1}{n\omega_n} \cdot \left(\frac{x-z}{|x-z|^n} + \frac{x-z}{|R_1(x)-z|^n}\right)
\end{aligned}
$$

.

If $z \in H_1^0$ we know $|R_1(x)-z| = |x-z|$

$$
\begin{aligned}
\nabla_z G(x,z) &= \frac{-1}{n\omega_n} \cdot \left(\frac{x-z}{|x-z|^n} + \frac{x-z}{|R_1(x)-z|^n}\right) \\
&= \frac{-2}{n\omega_n} \frac{x-z}{|x-z|^n}.
\end{aligned}
$$

$$
\begin{aligned}
u(x) &= \frac{-2}{n\omega_n} \int_{H_1^0} g(z) \cdot \frac{x-z}{|z-x|^n} \cdot \underbrace{N}_{=-x_1 \cdot \frac{1}{|x-z|}} \sigma(z) \\
&= \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|z-x|^n} \sigma(z).
\end{aligned}
$$

□

Exercise (b). Fixed the task description
Show that if g is periodic that is, there is some vector $L \in \mathbb{R}^{n-1}$) with

$$
g(x+L) = g(x).
$$

for all $x \in \mathbb{R}^{n-1}$, then so is the solution

Proof. We have our solution by

$$
u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x-z|^n} d\sigma(z).
$$

We pick $\tilde{L} \in \mathbb{R}^n = (0, L)$ such that $g(x+L) = g(x)$ for all $x \in \mathbb{R}^{n-1}$ and check

$$
u(x+\tilde{L}) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x+\tilde{L}-z|^n}.
$$

Consider

$$
y = z - \tilde{L}.
$$

since the above transformation is volume preserving the determinant of the Jacobean is ± 1, and in-fact its 1

$$
\begin{aligned}
u(x+L) &= \frac{2x_1}{n\omega_n} \int_{H_1^0 - L} \frac{g(y+L)}{|x-y|^n} d\sigma(y) \\
&= \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(y)}{|x-y|^n} d\sigma(y) \\
&= u(x).
\end{aligned}
$$

□

Exercise (c). Now consider the plane $n = 2$ with g function with compact support. Approximate the value of $u(x)$ for large $|x|$. What interesting things can you say about the growth of u

Proof. All the arguments made in 3.21 in the script should carry over to this case, if we consider the inversion through the boundary of the unit circle
We write

$$\begin{aligned} u(x) &= \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x-z|^n} d\sigma(z) \\ &= \int_{H_1^0} K(x,z) g(z) d\sigma(z). \end{aligned}$$

Where

$$K(x,z) = \frac{2x_1}{n\omega_n} \frac{1}{|x-z|^n}.$$

One can check

$$\int_{H_1^0} K(x,z) d\sigma(z) = 1.$$

and g bounded as $g \in \mathcal{C}(K)$ for some compact set $K \subset H_1^0$

Then we can get a similar approximation like in the script, for some small $\delta > 0$ and $|x - x_0| < \delta$

$$|u(x) - g(x_0)| \leq 2\varepsilon.$$

i.e we get for all $x_0 \in H_1^0$

$$\lim_{x \to x_0} u(x) = g(x_0).$$

So if we consider the inversion of x where $|x|$ is large then $\tilde{x}$ should be very close to the boundary B_1^0 i.e we get

$$\lim_{|x| \to \infty} u(x) = \lim_{x \to x_0} u(\tilde{x}) = g(\tilde{x_0}).$$

For $x_0 \in B_1^0$

$\square$