# MEAN FIELD PARTICLE SYSTEMS AND THEIR LIMITS TO NONLOCAL PD'S

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# Contents

# Chapter 1

# PDE Approach To Solving the Makean-Vlasov Equation

## 1.1 Motivation

Above we saw an SDE approach to solving the Makean-Vlasov Equation, in this section we instead focus on a PDE based approach. From now on we assume $\sigma(Y(t), \mu(t)) = \sqrt{2}$ is a constant, then the (MVE) can be rewritten as

$$(\text{MVE*}) \begin{cases} Y(t) & = b(Y(t), \mu(t))dt + \sqrt{2}dW_t \\ Y(0) & = \xi \in L^2(\Omega) \\ \mu_0 & = \mathcal{L}(\xi) \end{cases}.$$

by applying Itôs formula for $\forall \varphi \in \mathcal{C}_0^\infty([0, T) \times \mathbb{R}^d)$

$$\begin{aligned} \varphi(Y(t), t) - \varphi(Y(0), 0) = & \int_0^t \frac{\partial \varphi}{\partial t}(Y(s), s) + \nabla\varphi(Y(s), s) \cdot b(Y(s), \mu(s)) \\ & + \frac{1}{2} \underbrace{\sqrt{2} \cdot \sqrt{2})}_{tr(\sigma \cdot \sigma^T)} \cdot \Delta\varphi(Y(s), s)ds \\ & + \int_0^t \nabla\varphi(Y(s), s)\sqrt{2}dW_s. \end{aligned}$$

and taking the expectation on both sides, such that the last term disappears

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(x, t)d\mu(t) - \int_{\mathbb{R}^d} \varphi(x, 0)d\mu_0 \\ & = \int_0^t \int_{\mathbb{R}^d} \frac{\partial \varphi}{\partial t}(x, s) + \nabla\varphi(x, s) \cdot b(x, \mu(s)) \cdot \Delta\varphi(x, s)d\mu(s)ds. \end{aligned}$$

This leads us to formulating the following weak PDE, if $\mu$ is regular enough i.e it has density and the density has enough regularity, then $\mu$ should satisfy

$$\begin{cases} \partial_t\mu - \Delta\mu + \nabla \cdot (b(x, \mu) \cdot \mu) = 0 \\ \mu(0) = \mu_0 \end{cases}.$$

**Remark.** Compare this weak PDE to the one we got in the discrete case, what do you notice ?

**Exercise.** Show that the integral equation and the weak formulation are equal.

**Remark.** Now suppose we find $\mu$ with density $u$ satisfying the weak PDE, then we can plug it in to the (MVE) equation to get

$$\begin{cases} dY_t = b(Y_t, u)dt + \sqrt{2}dW_t \\ Y(t) = \xi \in L^2(\Omega) \quad \mathcal{L}(\xi) = u \end{cases}.$$

Now if $b$ is bounded and Lipschitz continuous, then we get a solution $Y_t$. Now if $\overline{u}$ is the Law of $Y_t$. Then by Itô formula we have for $\forall \varphi \in \mathcal{C}_0^\infty$

$$\int_{\mathbb{R}^d} \varphi(x,t)d\overline{\mu(t)} - \int_{\mathbb{R}^d} \varphi(x,0)u_0(x)dx$$
$$= \int_0^t \int_{\mathbb{R}^d} \left(\frac{\partial \varphi}{\partial t}(x,s) + \nabla\varphi(x,s) \cdot b(x,u) - \Delta\varphi(x,s)\right) \overline{u}(x,t)dxds.$$

Which means $\overline{\mu}$ satisfies

$$\begin{cases} \partial_t \overline{\mu} - \Delta\overline{\mu} + \nabla \cdot (b(x,u) \cdot \overline{\mu}) = 0 \\ \overline{\mu}|_{t=0} = u_0 \end{cases}.$$

If we can prove $\overline{u} = u$, then we get a solution to the **??**.

**Example.** A common choice of $b$ is the following for some kernel $K$

$$b(Y_t, u) = \int K(Y_t - y)u(y)dy = \int K(y)u(Y_t - y)dy.$$

then the regularity of $b$ by convolution depends on either $K$ or $u$

## 1.2 Problem Definition

**Definition 1.2.1** (Weak PDE)**.** Let $\mu$ have density $u$, then we write

$$(\text{PDE}) \begin{cases} \partial_t u - \Delta u + \nabla \cdot (b(x,u) \cdot u) = 0 \\ u(0) = u_0 \end{cases}.$$

Formalize by adding the relevant spaces

**Definition 1.2.2** (Sobolev Spaces)**.** We define roughly

$$H^1(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d)\}$$
$$\|u\|_{H_1} = \|u\|_2 + \|\nabla u\|_2.$$

where the gradient is defined for $\forall \varphi \in \mathcal{C}_0^\infty$

$$\nabla u = \langle \nabla u, \varphi \rangle = -\langle u, \nabla \varphi \rangle.$$

And the dual space

$$H^{-1}(\mathbb{R}^d) = (H^1(\mathbb{R}))' = \{l : l \text{ is bounded linear functional of } H^1(\mathbb{R}^d)\}.$$

Then

$$L^2([0,T]; H^1(\mathbb{R}^d)) = \{u : \int_0^T \|u(t)\|_{H^1} dt < \infty\}.$$

**Remark.** The Sobolev space $H^1$ is a separable Hilbert space

**Definition 1.2.3** (Weak Solution)**.** We say that a function

$$u \in L^2([0,T]; H^1(\mathbb{R}^d) \cap L^\infty([0,T]; L^2(\mathbb{R}^d))).$$

with $\partial_t u \in L^2([0,T]; H^{-1}(\mathbb{R}^d))$ is a weak solution of the (PDE) if for $\forall \varphi \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)$ it holds

$$\begin{aligned}\int_0^T \langle \partial_t u, \varphi \rangle_{(H^{-1},H^1)} dt = & \int_0^T \int_{\mathbb{R}^d} \nabla\varphi \cdot (b(x,u) \cdot u) dxdt \\ & - \int_0^T \int_{\mathbb{R}^d} \nabla u \cdot \nabla \varphi dxdt.\end{aligned}$$

## 1.3 Heat Equation and the Heat Kernel

**Motivation**

**Definition 1.3.1** (Heat equation)**.** The following PDE is called the inhomogenes Heat equation with source term $f$

$$(\text{HE}) \begin{cases} \partial_t u(x,t) - \Delta u(x,t) & = f(x,t) \\ u|_{t=0} & = u_0 \end{cases}.$$

**Remark.** Compare this to our PDE which looks similar, but is in fact non-linear

$$\partial_t u - \Delta u + \nabla \cdot (b(x,u) \cdot u) = 0.$$

**Remark.** Let us suppose $K(x,t)$ is a heat kernel, then

$$\begin{aligned}u(x,t) &= \int_{\mathbb{R}^d} K(x-y,t) u_0(y) dy - \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s) \nabla \cdot (b(y,u(y,s)u(y,s)) dyds \\ &= u_1(x,t) + u_2(x,t).\end{aligned}$$

is a solution to the inhomogenous Heat-Equation, this is called Duhamel's principle i.e. we can "add" up solutions to homogeneous problems and get the solution to the inhomogenous.

**Remark.** We say the heat kernel is the density of the Brownian Motion.

### 1.3.1 Derivation by Fourier Transform

**Definition 1.3.2** (Fourier Transform)**.** For $x \in \mathbb{R}^d$ the Fourier transform is defined as

$$\mathcal{F} : L^2 \to L^2 \; u \mapsto \hat{u}.$$

where

$$\hat{u}(k) = \int_{\mathbb{R}^d} u(x) e^{ix \cdot k} dx.$$

**Exercise.** Proof

$$-\widehat{\Delta u} = |k|^2 \hat{u}(k).$$

*Hint*

$$\widehat{\nabla u} = \frac{k}{i}\hat{u}(k).$$

**Remark.** Using the Fourier transformation we can transform our PDE into an ODE

$$\begin{cases} \partial_t \hat{u} - \widehat{\Delta u} & = \hat{f} \\ \hat{u}|_{t=0} & = \hat{u}_0 \end{cases}.$$

that is

$$\begin{cases} \partial_t \hat{u}(k) + |k|^2 \hat{u}(k) = \hat{f}(k) \\ \hat{u}_0(k) = \hat{u}_0 \end{cases}.$$

where

$$\hat{u}(k,t) = e^{-|k|^2 t}\hat{u}_0(k) + \int_0^t e^{-|k|^2(t-\tau)}\hat{f}(k,\tau)d\tau.$$

**Lemma 1.3.1** (Inverse transformation of the Fourier transformation)**.**

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-\tau))^{\frac{d}{2}}} e^{\frac{-|x-y|^2}{4(t-\tau)}} f(y,\tau)dyd\tau.$$

**Definition 1.3.3** (Heat Kernel)**.** The following is called Heat Kernel

$$K(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}.$$

and for $\forall t > 0$ it is a solution to the homogeneous heat equation

$$\partial_t K - \Delta K = 0.$$

And

$$K \xrightarrow{t \to 0^+} \delta.$$

In the sense of distributions.

**Theorem 1.3.1** (Solution To Heat Equation)**.** Let $K$ be the Heat kernel and initial data $u_0 \in \mathcal{C}_b(\mathbb{R}^d)$ and $f \in \mathcal{C}^{2,1}(\mathbb{R}^d \times [0,T])$ with compact support (schwarz function would work as well, since they lie dense in compact)

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy - \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s)\nabla \cdot (b(y,u(y,s)u(y,s))dyds \\ &= u_1(x,t) + u_2(x,t). \end{aligned}$$

is a solution to the heat equation, in fact $u_1$ and $u_2$ are solutions to

$$(\mathrm{P1})\begin{cases}\partial_t u_1 - \Delta u_1 = 0\\ u_1(0) = u_0\end{cases} \qquad (\mathrm{P2})\begin{cases}\partial_t u_2 - \Delta u_1 = f\\ u_2(0) = u_0\end{cases}.$$

respectively

**Proof.** We begin by showing that $u_1$ is a solution to (P1) by showing

$$\lim_{t\to 0^+} u_1(x,t) = \lim_{t\to 0^+}\int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy \overset{!}{=} u_0.$$

$$\begin{aligned}\lim_{t\to 0^+}\int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy &= \lim_{t\to 0^+}\int_{\mathbb{R}^d}\frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-\frac{|x-y|^2}{4t}}u_0(y)dy\\ &= \lim_{t\to 0^+}\int_{\mathbb{R}^d}\frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-|z|^2}u_0(x+2\sqrt{t}z)dz\\ &= \int_{\mathbb{R}^d}\lim_{t\to 0^+}\frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-|z|^2}u_0(x+2\sqrt{t}z)dz\\ &= u_0(x).\end{aligned}$$

where we used the change of variables

$$\frac{x-y}{2\sqrt{t}} = -z.$$

Then for $\forall t > 0$

$$\begin{aligned}\partial_t u_1 - \Delta u_1 &= (\partial_t - \Delta)\int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy\\ &= \int_{\mathbb{R}^d}(\partial_t - \Delta)K(x-y,t)u_0(y)dy\\ &= 0.\end{aligned}$$

by properties of the Heat-Kernel.

For $u_2(x,t)$

$$u_2(x,t) = \int_0^t\int_{\mathbb{R}^d} K(y,s)f(x-y,t-s)dyds.$$

First note that

$$\lim_{t\to 0^+} u_2(x,t) = 0.$$

Then by applying

$$\begin{aligned}(\partial_t - \Delta)u_2 &= \int_0^t\int_{\mathbb{R}^d} K(y,s)(\partial_t - \Delta_x)f(x-y,t-s)dyds\\ &\quad+\int_{\mathbb{R}^d} K(y,t)f(x-y,0)dy\\ &= \int_0^{\varepsilon}\int_{\mathbb{R}^d} K(y,s)\left(-\partial_s - \Delta_y\right)f(x-y,t-s)dyds + \int_{\varepsilon}^t\int_{\mathbb{R}^d}\dots\\ &\quad+\int_{\mathbb{R}^d} K(y,t)f(x-y,0)dy\\ &= I_\varepsilon + J_\varepsilon + L.\end{aligned}$$

We are allowed to exchange the order because of the Heat-Kernel since it decays very fast this gives uniform integrability the further away we get from the origin
We have, since $f \in \mathcal{C}_b^{2,1}$ we get that its term is bounded and the integral for the Heat kernel is just 1

$$\begin{aligned}
|I_\varepsilon| &\leq C \cdot \varepsilon \\
J_\varepsilon &= \int_\varepsilon^t \int_{\mathbb{R}^d} K(y,s)(-\partial_t - \Delta_y) f(x-y,t-s)dyds \\
&= \int_\varepsilon^t \int_{\mathbb{R}^d} \underbrace{(-\partial_t - \Delta_y)K(y,s)}_{=0} f(x-y,t-s)dyds \\
&+ \int_{\mathbb{R}^d} K(y,\varepsilon) - f(x-y,t-\varepsilon)dy \\
&\underbrace{- \int_{\mathbb{R}^d} K(y,t)f(x-y,0)dy}_{=Li}.
\end{aligned}$$

□

Together we have

$$\begin{aligned}
\partial_t u_2 - \Delta u_2 &= \lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}^d} \underbrace{K(y,\varepsilon)}_{\to \delta} f(x-y,t-\varepsilon)dy + \underbrace{C\varepsilon}_{\to 0} \right) \\
&= f(x,t).
\end{aligned}$$

This shows that

$$u(x,t) = \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy - \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s)\nabla \cdot (b(y,u(y,s)u(y,s))dyds$$

.

is a solution to the inhomogenous heat equation.

**Remark.** When changing order or variable of derivative one has to be aware of the boundary terms appearing. For example when changing the order for the $\partial_t$ derivative one gets two boundary terms , where one is not well behaved $t = 0$

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (b(x,u) \cdot u) = 0 \\ u|_{t=0} = u_0 \in L^1 \;\; \int (1+|x|^2)u_0 < \infty \end{cases}.$$

Formally

$$u(x,t) = \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy - \int_0^t \int_{\mathbb{R}^d} K(x-y,t-\tau)\nabla \cdot (b(y,u(y,\tau)) \cdot u(y,\tau))dyd\tau.$$

Now we start with bounded drift term for a linear equation.

**Definition 1.3.4** (LDE). For bounded drift term $\overline{b} \in L^\infty$ we define

$$(\text{LDE}) \begin{cases} \partial_t - \Delta u + \nabla \cdot (\overline{b}(x,t)u) = 0 \\ u|_{t=0} = u_0 \end{cases}.$$

**Remark.** By first proving the existence of a solution to this simpler equation we can then construct an iteration that will yield a solution to the more complex non-linear , non-local one.

**Theorem 1.3.2** (Uniqueness and Existence of LDE Solution ). If $b \in L^\infty([0,T] \times \mathbb{R}^d)$ and $u_0 \in L^1(\mathbb{R}^d)$, then the $LDE$ has a unique solution $u \in L^\infty([0,T]; L^1(\mathbb{R}^d))$

$$u(x,t) = \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \nabla K(x-y,t-\tau) \cdot (\bar{b}(y,\tau)u(y,\tau))dyd\tau.$$

**Proof.** We prove again by Iteration, and fix point argument, consider a map

$$\begin{aligned} &\mathcal{T} : L^\infty([0,T]; L^1(\mathbb{R}^d)) \to L^\infty([0,T]; L^1(\mathbb{R}^d)) \\ &u \mapsto \mathcal{T}(u) = \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \nabla K(x-y,t-\tau) \cdot (\bar{b}(y,\tau)u(y,\tau))dyd\tau. \end{aligned}$$

We need to check $\mathcal{T}(u) \in L^\infty([0,T]; L^1(\mathbb{R}^d))$, for $\forall t > 0$

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{T}(u)(x,t)|dx \leq& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x-y,t)|u_0(y)|dydx \\ &+ \int_0^t d\tau \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy|\nabla K(x-y,t-\tau)\bar{b}(y,\tau)u(y,\tau)| \\ &= I + II. \end{aligned}$$

Since we have fixed $t > 0$ we use Fubini

$$\begin{aligned} I &\leq \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} K(x-y,t)dx}_{=1} |u_0(y)|dy \\ &\leq \|u_0\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

First consider the gradient of $K$

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla K(x,s)dx &= \frac{1}{(4\pi s)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{1}{\sqrt{s}} \Big|\frac{4}{2\sqrt{s}}\Big| e^{-\frac{|x|^2}{4s}} dx \\ &\leq \frac{1}{\sqrt{s}} C. \end{aligned}$$

Then for the second term we get

$$\begin{aligned} II &\leq \|\bar{b}\|_{L^\infty} \int_0^t d\tau \int_{\mathbb{R}^d} |\nabla K(x,t-\tau)|dx \int_{\mathbb{R}^d} u(y,\tau)dy \\ &\leq \|\bar{b}\|_{L^\infty} \|u\|_{L^\infty(L^1)} C \cdot \int_0^t \frac{1}{\sqrt{s}} dy \\ &= C \cdot \sqrt{t-\tau}. \end{aligned}$$

Note we we do not need to consider $y$ in $K$ since we can use a translation, $L^\infty(L^1) = L^\infty([0,T]; L^1(\mathbb{R}^d))$

This shows that our map $\mathcal{T}$ is indeed well defined, next we proof $\mathcal{T}(u)$ is a contraction, for

$\forall u_1, u_2 \in L^\infty(L^1)$, for $t^\star$ s.t. $C\|\bar{b}\|_\infty\sqrt{t^\star} < \frac{1}{2}$

$$\begin{aligned}
&\|\sqcup^\star(u_1) - \mathcal{T}(u_2)\|_{L^\infty(L^1)} \\
&= \operatorname*{ess\,sup}_{0\le t\le t^\star} \int_{\mathbb{R}^d} |\mathcal{T}(u_1) - \mathcal{T}(u_2)|(x,t)dx \\
&\le \operatorname*{ess\,sup}_{0\le t\le t^\star} \int_0^t d\tau \int_{\mathbb{R}^d} dy |\nabla K(x-y, t-\tau)(\bar{b}(y,\tau)(u_1-u_2))(y,\tau)| \\
&\le \operatorname*{ess\,sup}_{0\le t\le t^\star} \|\bar{b}\|_{L^\infty}\|u_1-u_2\|_{L^\infty(L^1)} \int_0^t \frac{1}{\sqrt{t-\tau}} d\tau \\
&\le \operatorname*{ess\,sup}_{0\le t\le t^\star} C\|\bar{b}\|_{L^\infty}\sqrt{t}\|u_1-u_2\|_{L^\infty(L^1)} \\
&\le C\|\bar{b}\|_\infty\sqrt{t^\star}\|u_1-u_2\|_{L^\infty(L^1)}.
\end{aligned}$$

Then $\mathcal{T}$ is a contraction. Since $t^\star$ only depends on $\|\bar{b}\|_\infty$ and dimension d. Then for any given $T > 0$ we can repeat the above argument finite many time and obtain

$$u \in L^\infty([0,T]; L^1(\mathbb{R}^d)).$$

□

**Exercise.** Think about wether you can proof it for $b$ satisfying linear growth condition

$$|\bar{b}| \le C(1+|x|).$$

Let us discuss how a solution to the LDE leads back to a solution to the more complex

$$\partial_t u - \Delta u + \nabla\cdot(b(x,u)u) = 0.$$

where $u \in L^\infty(L^1)$ under the assumption on $b(x,u)$

$$|b(x,u) - b(\tilde{x},\tilde{u})| \le L(|x-\tilde{x}| + W_2(u,\tilde{u})).$$

By fixing $\tilde{x} = 0$

$$|b(x,u) - b(0,\delta_0)| \le L(|x| + W_2(u,\delta_0)).$$

where

$$W_2(u,\delta_0) \le \left(\int_{\mathbb{R}^d} |x|^2 u(x)dx\right)^{\frac{1}{2}}.$$

when $b$ is unbounded we consider , the cutoff

$$\bar{b}(x, v(x,t)) = \min\{b(x, v(x,t)), M\}.$$