Lemma (3.2 Itô s isometry simple version)**.** For $f \in \mathcal{H}_0^2$ we have

$$\|f\|_{\mathcal{H}^2} = \|I(f)\|_{L^2}.$$

Proof. We have

$$\begin{aligned}
\|I(f)\|_{L^2} &= \mathbb{E}[(\sum_{i=1}^{n} a_i(B_{t_i} - B_{t_{i-1}}))^2] \\
&= \mathbb{E}[\sum_{i=1}^{n} a_i^2(B_{t_i} - B_{t_{i-1}})^2 + \sum_{i\neq j}^{n} a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] \\
&= \mathbb{E}[\sum_{i=1}^{n} a_i^2(B_{t_i} - B_{t_{i-1}})^2] \\
&= \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[a_i^2(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2 \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2 \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2] \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2](t_i - t_{i-1})
\end{aligned}$$

.

Since w.l.o.g take $i < j$

$$\begin{aligned}
\mathbb{E}[a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] &= \mathbb{E}[\mathbb{E}[a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{i-1}}]] \\
&= \mathbb{E}[a_i \underbrace{\mathbb{E}[(B_{t_i} - B_{t_{i-1}})]}_{=0} \mathbb{E}[a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{i-1}}]]
\end{aligned}$$

.

But

$$\|f\|_{\mathcal{H}^2} = \mathbb{E}[\int_0^T f^2 ds] = \sum_{i=1}^{n} \mathbb{E}[a_i^2](t_i - t_{i-1}).$$

□

Proposition (3.5)**.** For every $f \in \mathcal{H}^2$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^2$

such that

$$\|f_n - f\|_{\mathcal{H}^2} \xrightarrow{n\to\infty} 0.$$

Proof. □

Theorem (3.17 Riemann sum approximation)**.** If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $t_i = \frac{i}{n}T$ then for $n \to \infty$ we have

$$\sum_{i=1}^{n} f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \xrightarrow{\mathbb{P}} \int_0^T f(B_s)dB_s.$$

Proof. By Remark 3.12. we know that for any continuous function $g : \mathbb{R} \to \mathbb{R}$

$$f(\omega, t) = g(B_t(\omega)) \in \mathcal{H}^2_{\text{loc}}.$$

This follows since for a.s. $\omega \in \Omega$ the map

$$\varphi(\omega) : [0, T] \to \mathbb{R} : t \mapsto B_t(\omega)$$

is bounded. This gives us that

$$\sup_{t\in[0,T]} |g(B_t(\omega))| = \sup_{x\leq|m|} |g(x)| \leq C.$$

Where the last inequality follows from the fact that g is continuous and attains a maximum on the compact set $[-m, m]$ then we can check that for

$$\omega \in \{\varphi \text{ is bounded }\}.$$

The integral

$$\begin{aligned}\int_0^T g^2(B_t(\omega))dt &\leq \int_0^T |g(B_t(\omega))||g(B_t(\omega))|dt \\ &\leq \underbrace{\sup_{t\in[0,T]} |g(B_t(\omega))|}_{\leq C} \int_0^T |g(B_t(\omega))|dt \\ &\leq C^2T.\end{aligned}$$

Since $\mathbb{P}(\{\varphi \text{ is bounded }\}) = 1$ we get immediately

$$\mathbb{P}(\int_0^T g^2(B_t(\omega)) < \infty) = 1.$$

This tells us that for any continuous f and Brownian motion B

$$f(B) \in \mathcal{H}^2_{\text{loc}}.$$

we can rewrite $\{\varphi$ is bounded $\}$ as a stopping time instead and get

$$\tau_m = \inf\{t \in [0,T] : |B_t| \geq m\}.$$

which is a localizing sequence for $f(B)$ since by similar argument to above we have

$$|f(B_{\cdot \wedge \tau_m})| \leq \sup_{|x| \leq m} |f(x)| < \infty.$$

and we get

$$f_m = f \cdot \mathbb{1}_{[-m,m]} = f|_{[-m,m]}.$$

Where

$$f_m(B) \in \mathcal{H}^2.$$

By definition of the Itô integral for $f \in \mathcal{H}^2$ we already get that

$$I(f_m^{(n)}) = \sum_{i=1}^{n} a_i (B_{t_i} - B_{t_{i-1}}) \xrightarrow{L^2} \int_0^T f_m(B_t) dt.$$

where L^2 convergence implies $\mathbb{P}$ convergence.

Thus our goal in Step 2 is to show that in fact

$$f_m^{(n)} = \sum_{i=1}^{n} f_m(B_{t_{i-1}})(\omega) \mathbb{1}_{(t_{i-1}, t_i]}(s).$$

we clearly have

$$f_m^{(n)} \in \mathcal{H}_0^2.$$

Then it remains to show $f_m^{(n)} \xrightarrow{\mathcal{H}^2} f_m$

$$\begin{aligned}
\mathbb{E}[\int_0^T (f_m^{(n)} - f_m)^2 ds] &= \mathbb{E}[\int_0^T (\sum_{i=1}^n f_m(B_{t_{i-1}})\mathbb{1}_{(t_{i-1},t_i]}(s) - f_m(B_s))^2] \\
&= \mathbb{E}[\int_0^T (\sum_{i=1}^n f_m(B_{t_{i-1}})\mathbb{1}_{(t_{i-1},t_i]}(s) - \sum_{i=1}^n f_m(B_s)\mathbb{1}_{t_{i-1},t_i})^2] \\
&= \mathbb{E}\Big[\int_0^T \sum_{i=1}^n (f_m(B_{t_{i-1}}) - f_m(B_s)\mathbb{1}_{(t_{i-1},t_i]}(s))^2 \\
&+ \underbrace{\sum_{i,j=1}^n \underbrace{(f_m(B_{t_{i-1}}) - f_m(B_s)\mathbb{1}_{(t_{i-1},t_i]}(s))(f_m(B_{t_{j-1}}) - f_m(B_s)\mathbb{1}_{(t_{j-1},t_j]}(s))}_{[t_{i-1},t_i]\cap[t_{j-1},t_j]=\emptyset}}_{=0} ds\Big] \\
&= \mathbb{E}\Big[\int_0^T \sum_{i=1}^n (f_m(B_{t_{i-1}}) - f_m(B_s)\mathbb{1}_{(t_{i-1},t_i]}(s))^2 ds] \\
&\leq \mathbb{E}[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f_m(B_{t_{i-1}}) - f_m(B_s))^2] \\
&\leq \mathbb{E}[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 ds] \\
&\leq \sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \int_{t_{i-1}}^{t_i} ds] \\
&\leq \frac{T}{n} \sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2]
\end{aligned}$$

.

Where we can bound

$$\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2.$$

further by considering that f is continuous and thus for

$$\mu_{f_m}(h) := \sup\{|f_m(x) - f_m(y)| : x, y \in \mathbb{R} \text{ with } |x-y| \leq h\}.$$

we get that

$$\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \leq \mu_{f_m}(\sup_{r\in[t_{i-1},t_i]} |B_{t_{i-1}} - B_r|).$$

putting it together

$$\begin{aligned}
\mathbb{E}[\int_0^T (f_m^{(n)} - f_m)^2 ds] &\leq \frac{T}{n}\sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2] \\
&\leq \frac{T}{n}\sum_{i=1}^n \mathbb{E}[\mu_{f_m}(\sup_{r\in[t_{i-1},t_i]} |B_{t_{i-1}} - B_r|)^2] \\
&\leq \frac{T}{n}\mathbb{E}[n \cdot \mu_{f_m}(\sup_{r\in[t_{i-1},t_i],i\leq n\}} |B_{t_{i-1}} - B_r|)^2] \\
&\leq T\mathbb{E}[\mu_{f_m}(\sup_{r\in[t_{i-1},t_i],i\leq n\}} |B_{t_{i-1}} - B_r|)^2]
\end{aligned}$$

.

Since f_m is continuous the modulus of continuity must tend to 0 as $n \to \infty$.
Thus we have shown that $f_m^{(n)} \xrightarrow{\mathcal{H}^2} f_m \Rightarrow I(f_m^{(n)}) \xrightarrow{L^2} I(f_m)$
Now on the set $\{\tau_m = T\}$ we have

$$f(B) = f_m(B).$$

and by persistence of identity also

$$\int_0^T f(B_s)dB_s = \int_0^T f_m(B_s)dB_s.$$

For

$$A_{n,\varepsilon} = \{|\sum_{i=1}^n f(B_{t_{i-1}}) \cdot (B_{t_i} - B_{t_{i-1}})) - \int_0^T f(B_s)dB_s| \geq \varepsilon\}.$$

Then we get

$$\sum_{i=1}^n f(B_{t_{i-1}}) \cdot (B_{t_i} - B_{t_{i-1}})) \xrightarrow{\mathbb{P}} \int_0^T f(B_s)dB_s.$$

if $\mathbb{P}(A_{n,\varepsilon}) \to 0$

$$\begin{aligned}
\mathbb{P}(A_{n,\varepsilon}) &= \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m < T\}) + \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m = T\}) \\
&\leq \mathbb{P}(\{\tau_m < T\}) + \mathbb{P}(A_{n,\varepsilon} \cap \{\tau_m = T\}) \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

This inequality is just $\mathbb{P}(A) \leq \mathbb{P}(B)$ if $A \subset B$

□

Remark (3.12)**.** For any continuous $g : \mathbb{R} \to \mathbb{R}$ we have $f(\omega, t) = g(B_t(\omega)) \in \mathcal{H}^2_{\text{loc}}$ since B is a.s. pathwise bounded on $[0, T]$

Proof. Consider $\omega \in \Omega$ a.s., then

$$\sup_{t\in[0,T]} |g(B_t(\omega))| \le C.$$

for some $C \ge 0$, then we have

$$\begin{aligned}
\int_0^T g^2(B_t(\omega))dt &= \int_0^T g(B_t(\omega))g(B_t(\omega))dt \\
&\le \int_0^T \sup_{t\in[0,T]} |g(B_t(\omega))| \cdot |g(B_t(\omega))|dt \\
&\le \sup_{t\in[0,T]} |g(B_t(\omega))| \int_0^T |g(B_t(\omega))|dt \\
&\le C^2 \cdot T.
\end{aligned}$$

□