Lemma (3.2 Itô s isometry simple version)**.** For $f \in \mathcal{H}_0^2$ we have

$$\|f\|_{\mathcal{H}^2} = \|I(f)\|_{L^2}.$$

Proof. We have

$$\begin{aligned}
\|I(f)\|_{L^2} &= \mathbb{E}[(\sum_{i=1}^{n} a_i(B_{t_i} - B_{t_{i-1}}))^2] \\
&= \mathbb{E}[\sum_{i=1}^{n} a_i^2(B_{t_i} - B_{t_{i-1}})^2 + \sum_{i \neq j}^{n} a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] \\
&= \mathbb{E}[\sum_{i=1}^{n} a_i^2(B_{t_i} - B_{t_{i-1}})^2] \\
&= \sum_{i=1}^{n} \mathbb{E}[\mathbb{E}[a_i^2(B_{t_i} - B_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2]] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2]\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2] \\
&= \sum_{i=1}^{n} \mathbb{E}[a_i^2](t_i - t_{i-1})
\end{aligned}$$

.

Since w.l.o.g take $i < j$

$$\begin{aligned}
\mathbb{E}[a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] &= \mathbb{E}[\mathbb{E}[a_i a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_{t_{i-1}}]] \\
&= \mathbb{E}[a_i \underbrace{\mathbb{E}[(B_{t_i} - B_{t_{i-1}})]}_{=0} \mathbb{E}[a_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})|\mathcal{F}_{t_{i-1}}]]
\end{aligned}$$

.

But

$$\|f\|_{\mathcal{H}^2} = \mathbb{E}[\int_0^T f^2 ds] = \sum_{i=1}^{n} \mathbb{E}[a_i^2](t_i - t_{i-1}).$$

□

Proposition (3.5)**.** For every $f \in \mathcal{H}^2$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^2$

such that

$$\|f_n - f\|_{\mathcal{H}^2} \xrightarrow{n\to\infty} 0.$$

Proof. The rough outline of the proof can be split up as follows

1. Show that we can assume f is bounded
2. Show that we can assume f is bounded adapted and continuous
3. Construct simple sequence that approximates f

For step 1 we can simply consider that for any (potentially) unbounded function f

$$f_n = -n \vee (f \wedge n).$$

is bounded and apply DCT

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{H}^2} = \lim_{n\to\infty} \mathbb{E}[\int_0^T (f_n - f)^2 dt] = \mathbb{E}[\int_0^T \lim_{n\to\infty} (f_n - f)^2 dt] = 0.$$

Thus we may assume f bounded

For step 2 we can use that we can get the height of a rectangle by dividing by its base s.t.

$$f_n(\cdot, t) := \frac{1}{(t - (t - \frac{1}{n})_+)} \int_{(t-\frac{1}{n})_+}^t f(\cdot, s) ds = n \int_{(t-\frac{1}{n})_+}^t f(\cdot, s) ds.$$

Now we note that since we work with random variables we condition ω on the set where

$$\lim_{n\to\infty} f_n(\omega, t) = f(\omega, t).$$

We need this set to have measure 1, such that

$$A := \{(\omega, t) \in \Omega \times [0, T] : \lim_{n\to\infty} f_n(\omega, t) \neq f(\omega, t)\}.$$

Has measure zero with respect to $\mathbb{P} \otimes \lambda$, we have by the fundamental theorem of calculus that

$$\lambda(\{t \in [0, T] : (\omega, t) \in A\}) = 0.$$

Or rather Lebesgue Differentiation theorem, otherwise we'd need the condition that $f(\omega, \cdot)$ only has countable many discontinuities. Thus we get that we may assume f is bounded and continuous.

We now construct our simple function f_n as

$$f_{n,s}(\omega, t) := f(\omega, (s + \varphi_n(t - s)_+)).$$

where

$$\varphi_n = \sum_{i=1}^{2^n} \frac{j-1}{2^n} \mathbb{1}_{(\frac{j-1}{2^n}, \frac{j}{2^n}]}.$$

makes our time interval discrete (standard argument really), then we wanna show

$$\|f_n - f\|_{\mathcal{H}^2} = \mathbb{E}[\int_0^T |f_{n,s} - f|^2 dt] \to 0.$$

We have

$$\begin{aligned}
\mathbb{E}[\int_0^T |f_{n,s} - f|^2 dt] &\leq \mathbb{E}[\int_0^T \int_0^1 |f_{n,s} - f|^2 ds dt] \\
&= \mathbb{E}[\int_0^T \int_0^1 |f(\cdot, (s + \varphi_n(t-s))_+) - f|^2 ds dt] \\
&= \sum_{j \in \mathbb{Z}} \mathbb{E}[\int_0^T \int_{[t - \frac{j}{2^n}, t - \frac{j-1}{2^n}] \cap [0,1]} |f(\cdot, (s + \frac{j-1}{2^n}) - f|^2 ds dt] \\
&\leq (2^n + 1) 2^{-n} \int_{(0,1]} \mathbb{E}[\int_0^T |f(\cdot, t - 2^{-n}h) - f(\cdot, t)|^2 dt] dh.
\end{aligned}$$

Where we can show that the Expectation term goes to 0 □

Theorem (3.7.). For any $f \in \mathcal{H}^2$ there is a continuous martingale $X = (X_t)_{t \in [0,T]}$ with respect to $\mathcal{F}_t$ such that for all $t \in [0, T]$

$$X_t = I(f \mathbb{1}_{[0,t]}).$$

Proof. We first consider the simple case with

$$f_n = \sum_{i=0}^{m_n - 1} a_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}.$$

Then

$$\begin{aligned}
\mathbb{E}[X_t^n - X_s^n | \mathcal{F}_s] &= \mathbb{E}[\sum_{t_i > s} a_i (B_{t_{i+1}} - B_{t_i})] \\
&= 0.
\end{aligned}$$

Our end goal is to use a triangular inequality to use the simple case to bound the normal one i.e.

$$\begin{aligned}|\mathbb{E}[X_t - X_s|\mathcal{F}_s]| &= |\mathbb{E}[X_t - X_t^n + X_t^n - X_s + X_s^n - X_s^n|\mathcal{F}_s]| \\ &\leq |\mathbb{E}[X_t - X_t^n|\mathcal{F}_s]| + \underbrace{|\mathbb{E}[X_t^n - X_s^n|\mathcal{F}_s]|}_{=0} + |\mathbb{E}[X_s^n - X_s|\mathcal{F}_s]| \\ &= |\mathbb{E}[X_t - X_t^n|\mathcal{F}_s]| + |\mathbb{E}[X_s^n - X_s|\mathcal{F}_s]| \\ &\overset{\text{Jen}}{\leq} \mathbb{E}[|X_t - X_t^n||\mathcal{F}_s] + \mathbb{E}[|X_s^n - X_s||\mathcal{F}_s] \\ &.\end{aligned}$$

We consider $A \in \mathcal{F}_s$

$$\begin{aligned}\mathbb{E}[|X_t - X_t^n|\mathbb{1}_A] + \mathbb{E}[|X_s^n - X_s|\mathbb{1}_A] &\leq \mathbb{E}[|X_t - X_t^n|] + \mathbb{E}[|X_s^n - X_s|] \\ &\leq 2 \cdot \|f - f_n\|_{\mathcal{H}^2}.\end{aligned}$$

So we have $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$

For $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^2$ we have

$$\lim_{n\to\infty} f_n = f \in \mathcal{H}^2.$$

And

$$\lim_{n\to\infty} I(f_n) = I(f) \in L^2.$$

Which should be equivalent to for fixed $t \in [0, T]$

$$\begin{aligned}X_t^n &\xrightarrow{L^2} X_t \\ I(f_n\mathbb{1}_{[0,t]}) &\xrightarrow{L^2} I(f\mathbb{1}_{[0,t]}).\end{aligned}$$

We need to make the argument uniform in $t \in [0, T]$, which is Step 2 in the script i guess.

We have

$$\mathbb{P}(\sup_{0\leq t\leq T} |X_t^n - X_t^m| \geq \varepsilon) \leq \varepsilon^{-2}\mathbb{E}[|X_T^n - X_T^m|^2] = \varepsilon^{-2}\|f_n - f_m\|_{\mathcal{H}^2}^2.$$

since for all $p > 1$

$$\mathbb{P}(\sup_{0\leq t\leq T} |X_t| \geq \varepsilon)\frac{1}{\varepsilon^p}\mathbb{E}[|X|_T^p].$$

By choosing a subequence we can get

$$\mathbb{P}(\sup_{0\leq t\leq T} |X_t^n - X_t^m| \geq 2^{-k}) \leq 2^{2k}\mathbb{E}[|X_T^n - X_T^m|^2] = \varepsilon^{-2}\|f_n - f_m\|_{\mathcal{H}^2}^2 \leq 2^{-k}.$$

Then we can apply Borel-Cantelli since

$$\sum_{k=0}^{\infty} \mathbb{P}(\sup_{0\le t\le T} |X_t^n - X_t^m| \ge 2^{-k}) < \infty.$$

and get $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and X^{n_k} is a pathwise cauchy sequence. $\square$

Proposition (3.10)**.** Let $f \in \mathcal{H}^2$ and ν be a stopping time satisfying

$$f\mathbb{1}_{[0,\nu]} = 0.$$

The integral process $X = (X_t)_{t\in[0,T]}$ with $X_t = \int_0^t f(\cdot, s)dB_s$, the fulfills

$$X\mathbb{1}_{[0,\nu]} = 0.$$

In particular for two functions $f, g \in \mathcal{H}^2$ with $f\mathbb{1}_{[0,\nu]} = g\mathbb{1}_{[0,\nu]}$ the integral processes coincide on $[0, \nu]$

Remark. This proposition is mostly used to prove that the same holds for $\mathcal{H}^2_{\text{loc}}$ functions as well since this allows us to use localizing sequences τ_m and use the fact that on

$$\{\tau_m = T\}.$$

The processes must coincide.

Proof. The proof follows similarly to before where we first prove the simple case, for that suppose

$$\begin{aligned} X_t &= I(f\mathbb{1}_{[0,t]}) \\ Y_t &= I(f\mathbb{1}_{[0,\nu]}\mathbb{1}_{[0,t]}). \end{aligned}$$

Then it suffices to consider the simplification $f = a\mathbb{1}_{[}(r,s]]$ for $0 \le r < s \le T$.
The important thing to note now is that

$$Y_t = I(f\mathbb{1}_{[0,\nu]}\mathbb{1}_{[0,t]}) = \int_0^t \underbrace{(f\mathbb{1}_{[0,\nu]}}_{\notin \mathcal{H}_0^2}.$$

For a function $h = f\mathbb{1}_{[0,\nu]}$ to be in $\mathcal{H}_0^2$ there needs to be a representation

$$h = \sum_{i=0}^{n-1} a_i\mathbb{1}_{[t_{i-1},t_i]}.$$

But ν is continuous, such that there cannot exist such a representation, which means we first need to consider the discrete stopping time ν^n as follows

$$s_{i,n} = r + (s-r)\frac{i}{2^n}$$
$$\nu^n = \sum_{i=0}^{2^n-1} s_{i+1,n}\mathbb{1}_{(s_{i,n},s_{i+1,n}]}(\nu).$$

We show

$$\begin{aligned} f\mathbb{1}_{[0,\nu^n]} &= f - f\mathbb{1}_{\nu^n,T} \\ &= f - f\sum_{i=0}^{2^n-1} \mathbb{1}_{(s_{i,n},s_{i+1,n}]}(\nu)\mathbb{1}_{(s_{i+1,n},T]} \in \mathcal{H}_0^2. \end{aligned}$$

then by definition of the integral for simple functions it follows

$$Y_t^N = \int_0^t f(\cdot,s)\mathbb{1}_{[0,\nu^n]}(u)dB_u = a(B_{s\wedge\nu^n\wedge t} - B_{r\wedge\nu^n\wedge t}).$$

And by continuity of B we can do

$$Y_t = \lim_{n\to\infty} Y_t^n = a(B_{s\wedge\nu\wedge t} - B_{r\wedge\nu\wedge t}).$$

But this is clearly $X_t\mathbb{1}_{[0,\nu]}$, it follows

$$X\mathbb{1}_{[0,\nu]} = Y\mathbb{1}_{[0,\nu]}.$$

Note that for $X\mathbb{1}_{[0,\nu]}$ there is no difficulty since we can first apply the definition of the simple integral, and then consider the stopping time

For general $f \in \mathcal{H}^2$ we choose $(f_n) \subset \mathcal{H}_0^2$ such that

$$\|f - f_n\|_{\mathcal{H}^2} \to 0.$$

then we already know that $X^n = Y^n$ such that

$$X\mathbb{1}_{[0,\nu]} = \lim_{n\to\infty} X^n\mathbb{1}_{[0,\nu]} = \lim_{n\to\infty} Y^n\mathbb{1}_{[0,\nu]} = Y\mathbb{1}_{[0,\nu]}.$$

□

Proposition (3.13)**.** For every $f \in \mathcal{H}^2_{\text{loc}}$ there is a localizing sequence $(\nu_n)_{n\in\mathbb{N}}$

Proof. We want ν_n such that for $f \in \mathcal{H}^2_{\text{loc}}$ it holds

$$f\mathbb{1}_{[0,\nu_n]} \in \mathcal{H}^2.$$

and

$$\mathbb{P}(\bigcup_{n=1}^{\infty}\{\nu_n = T\}) = 1.$$

We already have

$$\mathbb{P}(\int_0^T f^2(\cdot,s)ds < \infty) = 1.$$

a natural choice of localizing sequence is then

$$\nu_n = \inf\{t \in [0,T] : \int_0^t f^2 ds \geq n\}.$$

we have

$$\|f\mathbb{1}_{[0,\nu_n]}\|_{\mathcal{H}^2} < \infty.$$

since ν_n conditions on a set such that we only have ω where we are bounded. Since f is adapted the hitting time is a stopping time, and we consider

$$\bigcup_{n=1}^{\infty}\{\nu_n = T\} = \{\int_0^T f^2 < \infty\} = 1.$$

□

Definition (3.14)**.** Let $f \in \mathcal{H}^2_{\mathrm{loc}}$ and ν_n be a localizing sequence for f. The Itô integral process $(\int_0^t f(\cdot,s)dB_s)$ is defined as the continuous process $X = (X_t)_{t\in[0,T]}$ such that

$$\int_0^t f(\cdot,s)dB_s = X_t = \lim_{n\to\infty}\int_0^t f(\cdot,s)\mathbb{1}_{[0,\nu_n]}(s)dB_s \quad \mathbb{P}\text{-a.s.}.$$

Theorem (3.15)**.** For $f \in \mathcal{H}^2_{\mathrm{loc}}$ there exists a continous local martingale $(X_t)_{t\in[0,T]}$ such that for any localizing sequence $(\nu_n)_{n\in\mathbb{N}}$ of f it holds

$$\int_0^t f(\cdot,s)\mathbb{1}_{[0,\nu_n]}(s)dB_s \xrightarrow{n\to\infty} X_t.$$

for $t \in [0,T]$. In particular this is well defined and (X_t) does not depend on the choice of localizing sequence.

Proof. Remember that any proof involving local Martingales or $\mathcal{H}^2_{\mathrm{loc}}$ processes we need to work through a localizing sequence. Let $f \in \mathcal{H}^2_{\mathrm{loc}}$ and (ν_n) be a corresponding localizing sequence (it exists), define the localized integral process

$$X_t^n = \int_0^t f(\cdot,s)\mathbb{1}_{[0,\nu_n]}(s)dB_s.$$

we first show that X_t^n has a continuous limit (X_t) i.e

$$
\begin{aligned}
X_t &= \lim_{n\to\infty} X_t^n \\
X_t &= \lim_{n\to\infty} \int_0^t f(\cdot, s)\mathbb{1}_{[0,\nu_n]}(s)dB_s.
\end{aligned}
$$

The expression above is only useful if we view it on the set such that

$$
\begin{aligned}
&t \mapsto X_t^n(\omega) \text{ is continous} \\
&\min\{n \in \mathbb{N} : \nu_n(\omega) = T\} < \infty.
\end{aligned}
$$

The first gives us that

$$
t \mapsto \int_0^t f(\cdot, s)\mathbb{1}_{[0,\nu_n]}(s)dB_s.
$$

Show X is continuous
For $\varepsilon > 0$ and $t \in [0, T]$ find $\delta > 0$ such that

$$
s \in B_\delta(t) \Rightarrow |X_t(\omega) - X_s(\omega)| < \varepsilon.
$$

suppose w.l.o.g $s < t$

$$
\begin{aligned}
|X_t(\omega) - X_s(\omega)| &= \left| \lim_{n\to\infty} \left(\int_0^t f\mathbb{1}_{[0,\nu_n]} - \int_0^s f\mathbb{1}_{[0,\nu_n]} \right) \right| \\
&= \left| \lim_{n\to\infty} \int_s^t f\mathbb{1}_{[0,\nu_n]} dB_s \right| \\
&\leq \lim_{n\to\infty} \int_s^t |f\mathbb{1}_{[0,\nu_n]}| dB_s.
\end{aligned}
$$

Since X_t^n is continuous the integral can be made arbitrarily small and $f \cdot \mathbb{1}_{[0,\nu_n]} \in \mathcal{H}^2$. Thus the limit exists and is continuous, for the independence of localizing sequence we get immediately by prop 3.10 (identity) that for $\tau_n := \nu_n \wedge \tilde{\nu}_n$

$$
X^n \mathbb{1}_{[0,\tau_m]} = \tilde{X}^n \mathbb{1}_{[0,\tau_m]}.
$$

where

$$
\tilde{X}^n = \int_0^\cdot f(\cdot, s)\mathbb{1}_{[0,\tilde{\nu}_n]}(s).
$$

then

$$
\lim_{n\to\infty} X^n = \lim_{n\to\infty} \tilde{X}^n.
$$

on $[0, \tau_m]$ and $\tau_m \uparrow T$.

It remains to show that (X_t) is a local martingale, this is simply to show that $f \in \mathcal{H}^2$ then we already know that $\int f$ is a martingale,

$$\sigma_n = \inf\{t \in [0,T] : \int_0^t f^2(\cdot,s) \geq n\} \wedge T.$$

clearly this is a localizing sequence for f then

$$X_{t\wedge\sigma_n} = \int_0^t \underbrace{f(\cdot,s)\mathbb{1}_{[0,\sigma_n]}}_{\in\mathcal{H}^2} dB_s.$$

and we conclude with Theorem 3.7 $\square$

Theorem (3.17)**.** Let $f, g \in \mathcal{H}^2_{\text{loc}}$ and ν be a stopping time such that

$$f\mathbb{1}_{[0,\nu]} = g\mathbb{1}_{[0,\nu]}.$$

then

$$\int_0^t f(\cdot s) dB_s \mathbb{1}_{[0,\nu]} \stackrel{\mathbb{P}}{=} \int_0^t g(\cdot s) dB_s \mathbb{1}_{[0,\nu]}.$$

Proof. You know the drill, localizing sequence and then apply the result for the normal, we have that

$$\tau_n = \inf\{t \in [0,T] : \int_0^t f(\cdot,s)^2 ds \geq n \vee \int_0^t g(\cdot,s)^2 ds \geq n\} \wedge T.$$

Clearly $f\mathbb{1}_{[0,\tau_n]} \in \mathcal{H}^2$ since we stop the moment one of the integral processes exceeds n then we have

$$X^n = \int_0^\cdot \underbrace{f(\cdot,s)\mathbb{1}_{[0,\tau_n]}}_{\in\mathcal{H}^2}$$

$$Y^n = \int_0^\cdot \underbrace{g(\cdot,s)\mathbb{1}_{[0,\tau_n]}}_{\in\mathcal{H}^2}.$$

then prop 3.10 gives

$$X^n\mathbb{1}_{[0,\nu]} = Y^n\mathbb{1}_{[0,\nu]}.$$

Theorem 3.15 gives us the existence of the limit

$$\lim_{n\to\infty} X^n.$$

$\square$

Theorem (3.17 Riemann sum approximation)**.** If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $t_i = \frac{i}{n}T$ then for $n \to \infty$ we have

$$\sum_{i=1}^{n} f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \xrightarrow{\mathbb{P}} \int_0^T f(B_s)dB_s.$$

Proof. By Remark 3.12. we know that for any continuous function $g : \mathbb{R} \to \mathbb{R}$

$$f(\omega, t) = g(B_t(\omega)) \in \mathcal{H}^2_{\text{loc}}.$$

This follows since for a.s. $\omega \in \Omega$ the map

$$\varphi(\omega) : [0, T] \to \mathbb{R} : t \mapsto B_t(\omega)$$

is bounded. This gives us that

$$\sup_{t\in[0,T]} |g(B_t(\omega))| = \sup_{x\leq|m|} |g(x)| \leq C.$$

Where the last inequality follows from the fact that g is continuous and attains a maximum on the compact set $[-m, m]$ then we can check that for

$$\omega \in \{\varphi \text{ is bounded }\}.$$

The integral

$$\begin{aligned}\int_0^T g^2(B_t(\omega))dt &\leq \int_0^T |g(B_t(\omega))||g(B_t(\omega))|dt \\ &\leq \underbrace{\sup_{t\in[0,T]} |g(B_t(\omega))|}_{\leq C} \int_0^T |g(B_t(\omega))|dt \\ &\leq C^2 T.\end{aligned}$$

Since $\mathbb{P}(\{\varphi \text{ is bounded }\}) = 1$ we get immediately

$$\mathbb{P}(\int_0^T g^2(B_t(\omega)) < \infty) = 1.$$

This tells us that for any continuous f and Brownian motion B

$$f(B) \in \mathcal{H}^2_{\text{loc}}.$$

we can rewrite $\{\varphi \text{ is bounded }\}$ as a stopping time instead and get

$$\tau_m = \inf\{t \in [0, T] : |B_t| \geq m\}.$$

which is a localizing sequence for $f(B)$ since by similar argument to above we have

$$|f(B_{\cdot \wedge \tau_m})| \leq \sup_{|x| \leq m} |f(x)| < \infty.$$

and we get

$$f_m = f \cdot \mathbb{1}_{[-m,m]} = f|_{[-m,m]}.$$

Where

$$f_m(B) \in \mathcal{H}^2.$$

By definition of the Itô integral for $f \in \mathcal{H}^2$ we already get that

$$I(f_m^{(n)}) = \sum_{i=1}^{n} a_i(B_{t_i} - B_{t_{i-1}}) \xrightarrow{L^2} \int_0^T f_m(B_t)dt.$$

where L^2 convergence implies $\mathbb{P}$ convergence.

Thus our goal in Step 2 is to show that in fact

$$f_m^{(n)} = \sum_{i=1}^{n} f_m(B_{t_{i-1}})(\omega)\mathbb{1}_{(t_{i-1},t_i]}(s).$$

we clearly have

$$f_m^{(n)} \in \mathcal{H}_0^2.$$

Then it remains to show $f_m^{(n)} \xrightarrow{\mathcal{H}^2} f_m$

$$
\begin{aligned}
\mathbb{E}[\int_0^T (f_m^{(n)} - f_m)^2 ds] &= \mathbb{E}[\int_0^T (\sum_{i=1}^n f_m(B_{t_{i-1}})\mathbb{1}_{(t_{i-1},t_i]}(s) - f_m(B_s))^2] \\
&= \mathbb{E}[\int_0^T (\sum_{i=1}^n f_m(B_{t_{i-1}})\mathbb{1}_{(t_{i-1},t_i]}(s) - \sum_{i=1}^n f_m(B_s)\mathbb{1}_{t_{i-1},t_i})^2] \\
&= \mathbb{E}\Big[\int_0^T \sum_{i=1}^n (f_m(B_{t_{i-1}}) - f_m(B_s)\mathbb{1}_{(t_{i-1},t_i]}(s))^2 \\
&+ \underbrace{\sum_{i,j=1}^n \underbrace{(f_m(B_{t_{i-1}}) - f_m(B_s)\mathbb{1}_{(t_{i-1},t_i]}(s))(f_m(B_{t_{j-1}}) - f_m(B_s)\mathbb{1}_{(t_{j-1},t_j]}(s))}_{[t_{i-1},t_i]\cap[t_{j-1},t_j]=\emptyset}}_{=0} ds\Big] \\
&= \mathbb{E}\Big[\int_0^T \sum_{i=1}^n (f_m(B_{t_{i-1}}) - f_m(B_s)\mathbb{1}_{(t_{i-1},t_i]}(s))^2 ds] \\
&\leq \mathbb{E}[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f_m(B_{t_{i-1}}) - f_m(B_s))^2] \\
&\leq \mathbb{E}[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 ds] \\
&\leq \sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \int_{t_{i-1}}^{t_i} ds] \\
&\leq \frac{T}{n} \sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2]
\end{aligned}
$$

.

Where we can bound

$$\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2.$$

further by considering that f is continuous and thus for

$$\mu_{f_m}(h) := \sup\{|f_m(x) - f_m(y)| : x, y \in \mathbb{R} \text{ with } |x - y| \leq h\}.$$

we get that

$$\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2 \leq \mu_{f_m}(\sup_{r\in[t_{i-1},t_i]} |B_{t_{i-1}} - B_r|).$$

putting it together

$$
\begin{aligned}
\mathbb{E}[\int_0^T (f_m^{(n)} - f_m)^2 ds] &\leq \frac{T}{n}\sum_{i=1}^n \mathbb{E}[\sup_{r\in[t_{i-1},t_i]} (f_m(B_{t_{i-1}}) - f_m(B_r))^2] \\
&\leq \frac{T}{n}\sum_{i=1}^n \mathbb{E}[\mu_{f_m}(\sup_{r\in[t_{i-1},t_i]} |B_{t_{i-1}} - B_r|)^2] \\
&\leq \frac{T}{n}\mathbb{E}[n\cdot\mu_{f_m}(\sup_{r\in[t_{i-1},t_i], i\leq n\}} |B_{t_{i-1}} - B_r|)^2] \\
&\leq T\mathbb{E}[\mu_{f_m}(\sup_{r\in[t_{i-1},t_i], i\leq n\}} |B_{t_{i-1}} - B_r|)^2]
\end{aligned}
$$

.

Since f_m is continuous the modulus of continuity must tend to 0 as $n \to \infty$.
Thus we have shown that $f_m^{(n)} \xrightarrow{\mathcal{H}^2} f_m \Rightarrow I(f_m^{(n)}) \xrightarrow{L^2} I(f_m)$
Now on the set $\{\tau_m = T\}$ we have

$$f(B) = f_m(B).$$

and by persistence of identity also

$$\int_0^T f(B_s)dB_s = \int_0^T f_m(B_s)dB_s.$$

For

$$A_{n,\varepsilon} = \{|\sum_{i=1}^n f(B_{t_{i-1}})\cdot(B_{t_i} - B_{t_{i-1}})) - \int_0^T f(B_s)dB_s| \geq \varepsilon\}.$$

Then we get

$$\sum_{i=1}^n f(B_{t_{i-1}})\cdot(B_{t_i} - B_{t_{i-1}})) \xrightarrow{\mathbb{P}} \int_0^T f(B_s)dB_s.$$

if $\mathbb{P}(A_{n,\varepsilon}) \to 0$

$$
\begin{aligned}
\mathbb{P}(A_{n,\varepsilon}) &= \mathbb{P}(A_{n,\varepsilon}\cap\{\tau_m < T\}) + \mathbb{P}(A_{n,\varepsilon}\cap\{\tau_m = T\}) \\
&\leq \mathbb{P}(\{\tau_m < T\}) + \mathbb{P}(A_{n,\varepsilon}\cap\{\tau_m = T\}) \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}
$$

This inequality is just $\mathbb{P}(A) \leq \mathbb{P}(B)$ if $A \subset B$

□

Remark (3.12)**.** For any continuous $g : \mathbb{R} \to \mathbb{R}$ we have $f(\omega, t) = g(B_t(\omega)) \in \mathcal{H}^2_{\text{loc}}$ since B is a.s. pathwise bounded on $[0, T]$

Proof. Consider $\omega \in \Omega$ a.s., then

$$\sup_{t\in[0,T]} |g(B_t(\omega))| \leq C.$$

for some $C \geq 0$, then we have

$$\begin{aligned}
\int_0^T g^2(B_t(\omega))dt &= \int_0^T g(B_t(\omega))g(B_t(\omega))dt \\
&\leq \int_0^T \sup_{t\in[0,T]} |g(B_t(\omega))| \cdot |g(B_t(\omega))|dt \\
&\leq \sup_{t\in[0,T]} |g(B_t(\omega))| \int_0^T |g(B_t(\omega))|dt \\
&\leq C^2 \cdot T.
\end{aligned}$$

□

Theorem (3.18)**.** For any twice continuous differentiable function $f : \mathbb{R} \to \mathbb{R}$ we have

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \quad \mathbb{P}\text{-a.s.}.$$

Proof. The main tools used are a second order Taylor expansion then using our Riemann Sum convergence for the first integral and for the second we have a normal (pointwise) Riemann sum.
We write

$$\begin{aligned}
f(B_t) - f(0) &= \sum_{i=0}^{n-1} f(B_{t_i}) - f(B_(t_{i-1}))) \\
&= \sum_{i=0}^{n-1} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) + \frac{1}{2}\sum_{i=0}^{n-1} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 + \sum_{i=0}^{n} r(B_{t_i}, B_{t_{i-1}}).
\end{aligned}$$

The first sum converges by 3.17 to

$$\sum_{i=0}^{n-1} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \to \int f'(B_s)dB_s.$$

The second one we write as

$$\frac{1}{2}\sum_{i=0}^{n-1} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2 = \frac{1}{2}\sum_{i=0}^{n-1} f''(B_{t_{i-1}})((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})) + \frac{1}{2}\sum_{i=0}^{n-1} f''(B_{t_{i-1}})(t_i - t_{i-1}).$$

Then the second term is the integral we want i.e.

$$\frac{1}{2}\sum_{i=0}^{n-1} f''(B_{t_{i-1}})(t_i - t_{i-1}) \to \frac{1}{2}\int f''(B_s)ds.$$

Such that we need to show the first part converges against 0 $\mathbb{P}$-a.s., we do so by showing it converges to 0 in L^2 instead where we again make use of the independence of Brownian increments , and that they have mean 0, plus the fact that

$$\mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2] = t_i - t_{i-1}.$$

the remainder term is a little more complicated, we rewrite

$$\begin{aligned} r(x, y) &= \int_x^y (y-u)(f''(u) - f''(x))du \\ &= (y-x)^2 \int_0^1 (1-t)(f''(x + t(y-x))f''(x))dt. \end{aligned}$$

then if f has compact support it holds

$$|r(x, y)| \leq |y - x|^2 |h(x, y)|.$$

for bounded h with $h(x, x) = 0$ and compact support (that is $\operatorname{supp} f$), we show that the error term converges to 0 in L^1 by bounding h, that consists of splitting up Ω as follows

$$\Omega = \{|x - y| < \delta\} \cup \{|x - y| \geq \delta\}.$$

on the first set we know by continuity $h(x, y) < \varepsilon$ on the second we bound by using the $\|h\|_\infty$ which exists since h continuous and of compact support,bounding the probability of the second set by Markov inequality ($f(x) = x^2$). Since we can choose δ as we want we may take $\varepsilon = 0$

Now we need to argue that we are allowed to assume f compact support, this follows by similar argument to Riemann approximation. □

Example (5.1). Consider the SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t.$$

then we can solve this SDE by making the ansatz $X_t = f(B_t, t)$ and using Itôs formula

$$\begin{aligned} df(B_t, t) &= f(0, 0) + f_x(B_t)dB_t + (\frac{1}{2}f_{xx} + f_t)dt \\ &\triangleq \mu f(B_t)dt + \sigma f(B_t)dB_t. \end{aligned}$$

This implies

$$f_x(B_t) = \sigma f.$$

Such that

$$f = \exp(\sigma \cdot x + g(t)).$$

then

$$\begin{aligned} g'(t) \cdot f + \frac{1}{2}\sigma^2 f &= \mu f \\ g'(t) &= \mu - \frac{\sigma^2}{2} \\ g(t) &= (\mu - \frac{\sigma^2}{2})t + g_0. \end{aligned}$$

Which gives the solution

$$X_t = \exp(\sigma B_t + (\mu - \frac{\sigma^2}{2})t + g_0).$$

Definition. In general a linear SDE has the form

$$X_t = (\alpha(t)X_t + \beta(t))dt + (\varphi(t)X_t + \psi(t))dB_t.$$

and a solution is given by

$$X_t = x_0 \exp(Y_t) + \int_0^t \exp(Y_t - Y_s)(\beta(s) - \psi(s)\varphi(s))ds + \int_0^t \exp(Y_t - Y_s)\psi(s)dB_s.$$

Where

$$Y_t = \int_0^t \varphi(s)dB_s + \int_0^t (\alpha(s) - \frac{1}{2}\varphi^2(s))ds.$$