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Continuous Price Processes in Frictionless Markets Have Infinite Variation

The "continuous time model" in finance makes assumptions both about the characteristics of the market (absence of frictions, continuous trading opportunities, and so forth) and about the stochastic process that governs prices. Usually, these assumptions are introduced as if they were independent, that is, as if the characteristics of the market did not constrain the choice of stochastic process. In this paper we ask the following question: If trading is continuous and frictionless, what classes of stochastic process may be chosen that also admit equilibrium? We restrict our attention to processes whose sample paths are continuous. Our conclusion is given by the title of the paper.

I. Introduction

The "continuous time model" is central to much of financial economics, forming the basis of option-pricing theory (Black and Scholes 1973; Merton 1973*b*) and a number of intertemporal valuation models (Merton 1973*a*; Breeden 1979). The continuous time model involves two sets of assumptions. The first concerns the characteristics of the market (absence of frictions, continuous trading opportunities, and so forth); the second concerns the stochastic process governing prices. These assumptions are usually introduced as if they were independent; that is, as if the characteristics of the market did not constrain the nature of the stochastic process. In this paper we ask the following question: If trading is continuous and frictionless, what classes of stochastic processes may be chosen to describe prices which also admit equilibrium? Attention will be restricted to processes whose sample paths are continuous.

The most frequently employed continuous processes are diffusions, all intimately related to Brownian motion. Among the characteristics of such processes is that their sample paths display unbounded variation. (Variation, to be defined rigorously in Sec. III, represents the cumulative upward and downward displacement of the function in question.) This property makes the as-

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sumption of zero transaction costs quite critical in some cases. Consider the hedging strategy dictated by the usual derivation of the Black-Scholes option-pricing formula. If proportional transaction costs were positive rather than zero—no matter how small—then the total transaction costs would be infinite over any finite interval.

This observation motivates the following related question: Is it possible to represent prices with a process that has sample paths with bounded variation? We show that a continuous stochastic price process whose sample paths have bounded variation must necessarily admit arbitrage opportunities. The intuition behind this result is that continuous, bounded variation price processes are ''too smooth,'' resulting in significant positive correlation between successive small increments. An investor who can trade continuously and without cost would be able to spot the direction in which prices are moving and exploit this in trading. Thus, if arbitrage opportunities are not present, prices must have unbounded variation.

The effect of our analysis is to rule out certain stochastic processes as equilibrium price models, because they are too smooth. Is this result obvious? To argue that it is not, we identify three levels of increasing smoothness for continuous functions: (*a*) finite variation, (*b*) absolute continuity, and (*c*) absolute continuity with a continuous density. If a price process has paths that are smooth in the sense of *b*, this means that the rate of price change is well defined for each security and each time point, and *c* means that furthermore that rate varies continuously with time. It really is obvious that *c* implies arbitrage opportunities unless all securities are identical, and most economists familiar with continuous time models would say that *b* is obviously too smooth as well. So the contribution of this paper is to rule out processes whose paths fall in category *a* but not *b*. In Section VII we give an example which illustrates this distinction.

The paper is organized as follows. In Section II we give an heuristic account of the portfolio policies that lead to arbitrage when prices have bounded variation. Sections III–VI provide a rigorous version of this argument. Finally, Section VII contains the example and our conclusions.

II. The Heuristic Argument

Consider a market in which K securities are traded. For each security k we denote by X_k the associated price process, $X_k(t)$, representing the price of one share of security k and time t. Units are chosen so that $X_k(0) = 1$, and it is assumed throughout that X_k is continuous, strictly positive, and has finite variation with certainty ($k = 1, \ldots, K$). When discussing trading strategies, we denote by $S_k(t)$ the number of shares of security k held at time t and by $V(t)$ the total dollar value of the portfolio held at time t. That is,

$$V(t) = \sum_{k=1}^{K} X_k(t)S_k(t), \quad t \geq 0. \tag{1}$$

As indicated above, one might attempt arbitrage profits by buying the "hot" stocks and selling the cold ones. As it happens, one need not be so sophisticated; it is enough to buy those securities that have performed better *since time zero* and sell those that have performed worse since time zero. Roughly, the argument goes as follows.

Let p be a fixed real number. Consider a portfolio with an initial investment of one dollar and no net investment thereafter, which is continuously adjusted so that the dollar value of holdings of security k at time t is proportional to $[X_k(t)]^p$. In symbols, this last property is expressed by

$$\frac{X_k(t)S_k(t)}{V(t)} = \frac{[X_k(t)]^p}{\sum_{j=1}^{K}[X_j(t)]^p}. \tag{2}$$

For high values of p, (2) says that relatively larger amounts of the higher priced securities are held, and vice versa for low values of p. If $p = 1$, then (2) describes the buy-and-hold strategy where $S_k(t) = 1/K$ for all k and t. If $p = 0$, (2) describes the continuous-dollar-equalization strategy, where one equalizes the dollar value of holdings in each security at each point in time.

Now, just exactly how does one adjust one's portfolio over time to achieve (2)? Obviously, (2) gives

$$S_k(t) = \frac{V(t)\,[X_k(t)]^{p-1}}{\Sigma[X_j(t)]^p}, \tag{3}$$

but it remains to express V in terms not involving $S_1, \ldots, S_K$. In differential terms, the zero net new investment requirement is expressed as

$$dV(t) = \Sigma\, S_k(t)dX_k(t), \tag{4}$$

this saying that all changes in portfolio value are due to capital gains. Now (3) and (4) together give, for the case $p \neq 0$,

$$\begin{aligned} dV(t) &= \frac{V(t)\Sigma[X_k(t)]^{p-1}dX_k(t)}{\Sigma[X_k(t)]^p} \\ &= \frac{V(t)\Sigma\,\dfrac{1}{p}\,d[X_k(t)]^P}{\Sigma[X_k(t)]^p}. \end{aligned} \tag{5}$$

That is,

$$\frac{dV(t)}{V(t)} = \frac{1}{p}\,\frac{d\,\Sigma[X_k(t)]^p}{\Sigma[X_k(t)]^p}. \tag{6}$$

or

$$d \log V(t) = \frac{1}{p} d \log \Sigma[X_k(t)]^p. \tag{7}$$

Using the fact that $V(0) = X_k\,(0) = 1$, we deduce from (7) that

$$V(t) = \left\{\frac{1}{K}\sum_{k=1}^{K}\,[X_k(t)]^p\right\}^{1/p}. \tag{8}$$

Thus $V(t)$ is the pth order *power mean* of the prices $X_1(t), \ldots, X_K(t)$, provided that the heuristics have not led us astray (as they would have in the case of diffusions). Hereafter this pth order power mean will be denoted by $V^p(t)$ rather than $V(t)$, and trading strategy (3) will be called the p-strategy. A similar set of calculations suggest that in the case $p = 0$,

$$V^0(t) = \left\{\prod_{k=1}^{K} X_k(t)\right\}^{1/K},$$

the geometric mean. Note that execution of these strategies requires only knowledge of past and present prices.

Together, (3) and (8) spell out in explicit terms a trading strategy that yields at time t a portfolio with market value $V^p(t)$. How does this strategy constitute an arbitrage opportunity? The following property of power means is proved by Hardy, Littlewood, and Polya (1959); it is an easy consequence of Jensen's inequality.

PROPOSITION 1: If $-\infty < p < q < \infty$, then $V^p(t) \leqslant V^q(t)$, with equality only when $X_1(t) = \ldots = X_K(t)$.

To make arbitrage profits, buy $1/K$ shares of each security at time zero and manage this portfolio according to the q-strategy, and sell $1/K$ shares of each security at time zero and manage these short positions according to the p-strategy. This yields a profit of $V^q(t) - V^p(t)$ at time t with no investment.

In what sense is the preceding argument heuristic? Expressed in mathematical terms, the answer is that we have not justified the progression from (5) to (8) rigorously. To put this in economic terms, we have not really shown that the p-strategy requires zero net new investment after time zero. In the language of Harrison and Kreps (1979), we have not shown that the p-strategy is "self-financing." The usual condition given to justify manipulations like (5)–(8) is differentiability of the price process. In the sections to follow, it will be shown that this justification extends to the more general case under consideration and hence that the p-strategy really *is* self-financing.

Of course, it makes sense to take q very large and p very negative. Examining the limiting values, we see that

$$\lim_{q\to\infty}\left(\frac{1}{K}\,\Sigma\,X_k(t)^q\right)^{1/q} = \max_{k=1,\ldots,K}\{X_k(t)\}$$

and

$$\lim_{p\to-\infty}\left(\frac{1}{K}\,\Sigma\,X_k(t)^p\right)^{1/p} = \min_{k=1,\ldots,K}\{X_k(t)\},$$

(the convergence is not uniform). Furthermore, as $q \to \infty$, the associated strategy comes closer to holding only the highest-priced security (equal division when there is more than one security at the maximum). The fact that $V^q(t) \to \max_k X_k(t)$ as $q \to \infty$ lends credibility to such a strategy of "going with the leader," provided the price process is smooth enough to justify the heuristics. (In the case where the X_k are diffusions, the described strategy has such a pervasive set of discontinuities that it is not readily approximated by simple strategies. In trying to go with the leader in a realistic way, one cannot keep up when the lead changes hands.)

Our analysis is an extension of earlier work by Hodges and Schaefer (1974), who used the strategies described above to argue that geometric stock price indices (e.g., the Valueline Index) were misleading because the conditions under which a geometric price index represents a feasible portfolio strategy also lead to arbitrage opportunities. Hodges and Schaefer were able to demonstrate the connection between continuous dollar equalization and the geometric mean only in the case where price paths are differentiable.

III. Notation and Definitions

Let f be a real-valued function on $[0, \infty)$. Consider an interval $[a, b]$, $0 \leqslant a \leqslant b \leqslant \infty$ and a finite partition $P = (x_1, x_2, \ldots, x_n)$ of $[a, b]$, where $a \leqslant x_1 < x_2 < \ldots < x_n \leqslant b$. Now define $v_f(P) = \Sigma_{i=2}^n |f(x_i) - f(x_{i-1})|$. Further define $V_f[a, b] = \sup\{v_f(P){:}P$ is a finite partition of $[a, b]\}$. If $V_f[a, b] < \infty$ we say that f has bounded variation over $[a, b]$, or that $f \in BV[a, b]$, and $V_f[a, b]$ is called the total variation of f over $[a, b]$. If $V_f[a, b] < \infty$ for all such choices of $[a, b]$, then we say that $f \in BV[0, \infty)$. The proof of the following proposition may be found in Royden (1963, p. 100).

Proposition 2: $f \in BV[a, b]$ if and only if f is the difference of two increasing functions on $[a, b]$.

The decomposition given in Royden also shows that if $f \in BV[a, b]$ is continuous, it is the difference between two continuous increasing functions whose sum is $V_f[a, t]$, $t \in (a, b)$. Thus we have the following:

PROPOSITION 3: If $f \in BV[a, b]$ is continuous, then $V_f[a, t]$ is continuous in t on $[a, b]$. It is also easy to see that if $f \in BV[a, c]$ and $a < b < c$, then $V_f[a, b] + V_f[b, c] = V_f[a, c]$.

IV. Riemann-Stieltjes Integrals

We claimed earlier that continuous price processes whose sample paths have bounded variation admit arbitrage opportunities. The proof hinges on the valuation of certain Riemann-Stieltjes integrals. These integrals have the form $\int_0^T f(t)dg(t)$, $0 \leqslant T < \infty$, where f and g are real-valued functions on $[0, \infty)$. In this section we record some properties of Riemann-Stieltjes integration that will be needed later. For a definition of the Riemann-Stieltjes integral, see Bartle (1976, pp. 212–14).

PROPOSITION 4: If f is continuous and $g \in BV[0, T]$, the f is integrable with respect to g on $[0, T]$.

Remark. This is proved in Bartle (1976, theorem 30.2) for the case where g is monotone. But if proposition 2 above and the bilinearity property are used, the extension follows easily. (The bilinearity property, theorem 29.5 of Bartle [1976], says that the integral is linear in both f and g).

The next two results are not found in standard introductions to Riemann-Stieltjes integration. Nevertheless, we do not claim originality.

PROPOSITION 5 (chain rule): Let f and g be continuous on $[0, \infty)$ with $g \in BV[0, \infty)$. If $\phi: \mathbf{R} \rightarrow \mathbf{R}$ is continuously differentiable, then

$$\int_0^T f(t)d\phi[g(t)] = \int_0^T f(t)\phi'[g(t)]dg(t), \quad 0 < T < \infty.$$

(Implicit here is the statement that both integrals exist.)

PROOF: See Appendix A.

PROPOSITION 6 (change of variables): Let h be continuous on $[0, \infty)$ and let $g \in BV\ [0, \infty)$ be continuous. Then

$$\int_0^t h[g(u)]dg(u) = \int_{g(0)}^{g(t)} h(v)dv.$$

Remark. This result is usually stated for the case where g is monotone, but the direct method of proof used for that case does not extend to our case.

PROOF: See Appendix B.

V. Formal Verification

We now formally verify that V^p satisfies (4). This amounts to a rigorous justification of the progression from (5) to (8) in Section II. As in Section II, assume $p > 0$ and define

$$S_k(t) = V^p(t) \frac{[X_k(t)]^{p-1}}{\Sigma_j [X_j(t)]^p},$$

where $V^p(t) = \{(1/K) \Sigma_k [X_k(t)]^p\}^{1/p}$ and $X_1, \ldots, X_K$ are stochastic processes whose sample paths are continuous and strictly positive and have bounded variation.

PROPOSITION 7: $V^p(t) = V^p(0) + \Sigma_k \int_0^t S_k(u) dX_k(u)$.

COROLLARY: $\Sigma_k \int_0^t X_k(u) dS_k(u) = 0$.

PROOF: The corollary follows from the proposition by integration by parts (see Bartle 1976, p. 219). If $p \neq 0$, then

$$\begin{aligned}
\sum_{k=1}^{K} \int_0^t S_k(u) dX_k(u) &= \sum_{k=1}^{K} \int_0^t \left(\frac{1}{K}\right)^{1/p} \{\Sigma_j [X_j(u)]^p\}^{1/p-1} [X_k(u)]^{p-1} dX_k(u) \\
&= \Sigma_k \int_0^t \left(\frac{1}{K}\right)^{1/p} \{\Sigma_j [X_j(u)]^p\}^{1/p-1} d\left\{\frac{1}{p} [X_k(u)]^p\right\} \\
&= \left(\frac{1}{K}\right)^{1/p} \int_0^t \frac{1}{p} \{\Sigma_j [X_j(u)]^p\}^{1/p-1} d\{\Sigma_k [X_k(u)]^p\} \\
&= \left(\frac{1}{K}\right)^{1/p} \int_{\Sigma_k [X_k(0)]^p}^{\Sigma_k [X_k(t)]^p} \frac{1}{p} v^{1/p-1} dv \\
&= \left(\frac{1}{K}\right)^{1/p} \left(\{\Sigma_k [X_k(t)]^p\}^{1/p} - \{\Sigma_k [X_k(0)]^p\}^{1/p}\right) \\
&= V^p(t) - V^p(0).
\end{aligned}$$

The first equality is by substitution, the second from proposition 5, the third from the bilinearity property, and the fourth from proposition 6. The case $p = 0$ is similar and is left to the reader. Q.E.D.

Proposition 7 and its corollary provide two articulations of the self-financing character of the trading strategy $S_1, \ldots, S_K$. The second term on the right-hand side of the proposition represents the investor's total earnings (or capital gains) up to time t, so the proposition states that all changes in the market value of the portfolio are due to capital gains or losses, as opposed to cash withdrawal (through liquidation) or the infusion of new funds. The corollary states the same thing in simpler terms. Its left-hand side represents the net cost of transactions (cumulative cost of purchases minus cumulative revenue from sales) between times zero and t, and this is found to be zero. To paraphrase, the increment of revenue generated by the sale of securities over any period is exactly balanced by the cost of security purchases over that same period.

The preceding statements are valid only if the definition of the Riemann-Stieltjes integral is such that $\Sigma_k \int_0^t S_k(u)dX_k(u)$ is the correct expression for capital gains over $[0, T]$, and that $\Sigma_k \int_0^t X_k(u)dS_k(u)$ is the correct expression for the net cost of transactions over $[0, T]$. This is the subject of the next section.

VI. Approximating Simple Strategies

There are two ways to justify the use of Riemann-Stieltjes integrals. The first approach will be sketched. The second approach is technically more difficult and thus will be examined rigorously.

Fix a time $T > 0$, let N be a large fixed integer and let $t_n = nT/N$ for $n = 0, 1, \ldots, N$. Consider a trading strategy which at time $t_0 = 0$ buys $1/K$ units of each security and holds them until time t_1, at which time the portfolio is liquidated and $S_k(t_1)$ units of each security are purchased. Similarly, at each time t_i, the portfolio is liquidated and $S_k(t_i)$ units of each security are purchased. Holdings in this strategy are the same as $S_k(t)$ at each time t_i, and are close to $S_k(t)$ in each interval $[t_i, t_{i+1})$ by virtue of the continuity of S_k. The described strategy is not self-financing, but the net cost of transactions over $[0, t]$ for such a strategy is seen to be an approximating sum for $\Sigma_k \int_0^t X_k(u)dS_k(u)$, and the capital gains from such a strategy is seen to be an approximating sum for $\Sigma_k \int_0^t S_k(u)dX_k(u)$. The details of this argument are straightforward and are left to the reader.

Thus the use of Riemann-Stieltjes theory is justified by showing that simple strategies "close" to S_k are "nearly" self-financing. Alternatively, one might approximate S_k by a simple strategy which is self-financing and of which the value at time T is close to $V^P(T)$.

Fix a time $T > 0$. Let N be a large fixed integer and let $t_n = nT/N$ for $n = 0, 1, \ldots, N$. Consider a trading strategy which initially buys $1/K$ shares of each security $k = 1, 2, \ldots, K$ and thereafter trades only at times t_n. At those times the portfolio is adjusted so that the dollar value of holdings of each security k is proportional to $[X_k(t_n)]^p$, much like the continually adjusted strategy S^p. No net infusion of funds takes place at these times. Thus it is seen that (employing the obvious notation)

$$S_{Nk}(t_n) = V_N^p(t_n) \frac{[X_k(t_n)]^{p-1}}{\Sigma_j [X_j(t_n)]^p},$$

where $V_N^p(t_n) = \Sigma_k S_{Nk}(t_{n-1}) X_k(t_n)$.

The following fact is elementary and the proof is left to the reader.

PROPOSITION 8: Let f be integrable with respect to g. Then

$$\left| \int_0^t f(u)dg(u) \right| \leqslant \sup_{0 \leqslant u \leqslant t} |f(u)| V_g[0, t].$$

PROPOSITION 9: For each sample path (for which $X_1, \ldots, X_K$ are continuous and have bounded variation), $\lim_{N \to \infty} V_N^p(T) = V^p(T)$.

PROOF: Let $\delta = \inf_{0<t<T} V^p(t)'' > 0$. Given $\epsilon > 0$, let N be large enough so that $|S_k(t_n) - S_k(\xi)| < \epsilon$ for all $\xi \in [t_n, t_{n+1}]$ and $V_{X_k}[t_n, t_{n+1}] < \delta/K$, $k = 1, 2, \ldots, K$ and $n = 0, 1, \ldots, N$. This is possible by the uniform continuity of the functions S_k and of $V_{X_k}[0, \cdot]$ (see proposition 3). Note that

$$\frac{S_k(t_n)}{V^p(t_n)} = \frac{S_{Nk}(t_n)}{V_N^p(t_n)}$$

for $n = 0, 1, \ldots, N$, so that

$$\begin{aligned}\sum_{k=1}^{K} S_k(t_n)X_k(t_{n+1}) &= \frac{V^p(t_n)}{V_N^p(t_n)} \sum_{k=1}^{K} S_{Nk}(t_n)X_k(t_{n+1}) \\ &= V^p(t_n)\,\frac{V_N^p(t_{n+1})}{V_N^p(t_n)}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\left| V^p(t_{n+1}) - V^p(t_n)\,\frac{V_N^p(t_{n+1})}{V_N^p(t_n)} \right| \\ &= \left| V^p(t_{n+1}) - \sum_{k=1}^{K} S_k(t_n)X_k(t_{n+1}) \right| \\ &= \left| V^p(t_{n+1}) - V^p(t_n) - \sum_{k=1}^{K} S_k(t_n)\,[X_k(t_{n+1}) - X_k(t_n)] \right| \\ &= \left| V^p(t_{n+1}) - V^p(t_n) - \sum_{k=1}^{K} \int_{t_n}^{t_{n+1}} S_k(t_n)dX_k(u) \right| \\ &= \left| \sum_{k=1}^{K} \int_{t_n}^{t_{n+1}} [S_k(u) - S_k(t_n)]dX_k(u) \right| \\ &\leqslant \epsilon \sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}].\end{aligned}$$

The last inequality is by proposition 8. Dividing the left-hand side of the inequality above by $V^p(t_{n+1})$ and the right-hand side by δ $[\leqslant V^p(t_{n+1})]$, we have

$$\left| 1 - \frac{V^p(t_n)}{V^p(t_{n+1})}\,\frac{V_N^p(t_{n+1})}{V_N^p(t_n)} \right| \leqslant \frac{\epsilon}{\delta} \sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}],$$

or

$$1 - \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}] \leqslant \frac{V^p(t_n)}{V^p(t_{n+1})}\,\frac{V_N^p(t_{n+1})}{V_N^p(t_n)}$$
$$\leqslant 1 + \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}].$$

Since $V_{X_k}(t_n, t_{n+1}) < \delta/K$, we may take the logarithm of each term:

$$\log\left[1 - \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]\right] \leqslant \log\left[\frac{V_N^p(t_{n+1})}{V^p(t_{n+1})}\right] - \log\left[\frac{V_N^p(t_n)}{V^p(t_n)}\right]$$
$$\leqslant \log\left[1 + \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]\right],$$

Summing up,

$$\sum_{n=0}^{N-1}\log\left[1 - \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]\right] \leqslant \log\left[\frac{V_N^p(T)}{V^p(T)}\right]$$
$$\leqslant \sum_{n=0}^{N-1}\log\left[1 + \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]\right].$$

Now by the mean value theorem we know that for some $\xi \in \{-\epsilon/\delta \sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}], 0\}$,

$$\log\left(1 - \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]\right) = \frac{-1}{1+\xi}\,\frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}].$$

Also, for some $\eta \in [0, \epsilon/\delta \sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]]$

$$\log\left(1 + \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]\right) = \frac{1}{1+\eta}\,\frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}].$$

Now $\epsilon/\delta \sum_{k=1}^{K} V_{X_k}(t_n, t_{n+1}) \leqslant \epsilon$, so that $\xi \geqslant -\epsilon$, and

$$\frac{-1}{1-\epsilon}\,\frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[0, T] = \frac{-1}{1-\epsilon}\,\frac{\epsilon}{\delta}\sum_{n=0}^{N-1}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]$$
$$\leqslant \log V_N^p(T) - \log V^p(T) \leqslant \frac{\epsilon}{\delta}\sum_{n=0}^{N-1}\sum_{k=1}^{K} V_{X_k}[t_n, t_{n+1}]$$
$$\leqslant \frac{\epsilon}{\delta}\sum_{k=1}^{K} V_{X_k}[0, T].$$

Since the first and last terms are arbitrarily small by taking ϵ small, the same holds true for the middle term. Q.E.D.

VII. Concluding Remarks

We set out to show that in a frictionless market with continuous trading, if equilibrium prices are continuous, they must have unbounded variation with positive probability. The proof involved showing that without unbounded variation, arbitrage opportunities are present. Thus we have provided a link between the characteristics of the market (absence of frictions, continuous trading, etc.) and the characteristics of admissible price processes.

Section I raised the question whether our result is obvious. We argued that while absolute continuity in prices obviously gives rise to arbitrage opportunities unless all securities are identical, the same is not true for price processes that have finite variation but are not absolutely continuous. We conclude with an example of such a process.

Let X and Y be two independent Brownian motions with $X(0) = Y(0) = 0$, and define $Z(t) = \sup_{0 \leq s \leq t} X(s) - \sup_{0 \leq s \leq t} Y(s)$, $t \geq 0$. Each sample path of Z is the difference of two continuous increasing functions, so each is continuous and has finite variation on finite intervals. It is well known, however, that the maximum process of a Brownian motion increases only on a set of time points having zero Lebesque measure; increases in the maximum process are very jerky. Thus Z is flat except on a set of measure zero, and it is meaningless to talk about the rate of increase or rate of decrease for Z. The derivative of Z exists and equals zero at almost all time points, and yet Z is not constant.

Consider a market with one stock and one bond. Assume that the bond price process is constant (the riskless interest rate is zero) and the stock price at time t is $\exp[Z(t)]$. It is important that investors cannot see X and Y; they have knowledge of the past and present stock prices only. Do arbitrage opportunities exist in this market? We have shown in this paper that they do, but it is not a priori obvious. In particular, one cannot earn arbitrage profits by holding stock only when the stock price is rising, because the stock price is almost always constant.

Appendix A

Proof of Proposition 5

PROOF: Existence of the second integral is by proposition 4. Let M bound $|f|$ on $[0, T]$, and let $\epsilon > 0$ be given. Let P_ϵ be a partition such that if $P = (0, x_1, \ldots, x_n)$ is any refinement of P_ϵ, and if $\xi_i \in [x_{i-1}, x_i]$ and $\nu_i \in [x_{i-1}, x_i]$, then

$$\left| \sum_{i=1}^{n} f(\xi_i)\phi'[g(\xi_i)][g(x_i) - g(x_{i-1})] - \int_0^T f(t)\phi'[g(t)]dg(t) \right| < \epsilon,$$

and

$$|\phi'[g(\nu_i)] - \phi'[g(\xi_i)]| < \epsilon \quad i = 1, 2, \ldots, n.$$

The first inequality is possible from the definition of the integral, and the second arises from the (uniform) continuity of $\phi'[g(\cdot)]$ on $[0, T]$.

Now for any such partition P there exists $\eta_i \in (x_{i-1}, x_i)$ such that $\phi[g(x_i)] - \phi[g(x_{i-1})] = \phi'[g(\eta_i)][g(x_i) - g(x_{i-1})]$. This follows from the mean value and intermediate value theorems. Thus,

$$\begin{aligned}
&\left|\sum_{i=1}^{n} f(\xi_i)\{\phi[g(x_i)] - \phi[g(x_{i-1})]\} - \int_0^T f(t)\phi'[g(t)]dg(t)\right| \\
&= \left|\sum_{i=1}^{n} f(\xi_i)\ \phi'(\eta_i)[g(x_i) - g(x_{i-1})] - \int_0^T f(t)\phi'[g(t)]dg(t)\right| \\
&\leqslant \left|\sum_{i=1}^{n} f(\xi_i)(\phi'[g(\eta_i)] - \phi'[g(\xi_i)][g(x_i) - g(x_{i-1})]\right| \\
&+ \left|\sum_{i=1}^{n} f(\xi_i)\phi'[g(\xi_i)][g(x_i) - g(x_{i-1})] - \int_0^T f(t)\phi'[g(t)]dg(t)\right|.
\end{aligned}$$

Now the first term of the expression above is less than or equal to $M\epsilon\ \Sigma_{i=1}^{n} |g(x_i) - g(i=1)| \leqslant M\epsilon V_g[0, T]$, so that

$$\left|\sum_{i=1}^{n} f(\xi_i)\{\phi[g(x_i)] - \phi[g(x_{i-1})]\} - \int_0^T f(t)\phi'[g(t)]dg(t)\right| \leqslant (M \cdot V_g[0, T] + 1)\epsilon.$$

Thus the first integral exists and equals the second. Q.E.D.

Appendix B

Proof of Proposition 6

PROOF: Let $H(y) = \int_{g(0)}^{y} h(v)dv$. Then

$$\begin{aligned}
\int_0^t h[g(u)]dg(u) &= \int_0^t H'[g(u)]dg(u) = \int_0^t dH[g(u)] \\
&= H[g(t)] - H[g(0)] = \int_{g(0)}^{g(t)} h(v)dv.
\end{aligned}$$

The first equality is by the fundamental theorem of calculus, and the second equality follows from proposition 5. Q.E.D.

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