Leon Fiethen 1728330
Janik Sperling 1728567.

Sheet 7

23

Question 1. Suppose that $u \in \mathcal{C}^2(\mathbb{R}^2)$ is harmonic with critical point at x_0. Assume the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle

Solution. Since $u \in \mathcal{C}^2(\mathbb{R}^2)$ is harmonic and assume assume x_0 is a maximum, then by the maximum principle u must be constant and $u' = 0$, therefore $H(u)$ must have 0 determinant too, for minima we argue that x_0 minima of u implies a maxima of $-u$ and the same argument applies.

It follows if u has a Hessian with non zero determinant it must have a saddle point at x_0 $\square$

24

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called subharmonic if for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ it lies below its spherical mean

$$v(x) \leq \mathcal{S}[v](x,r).$$

Question 2 (a). Prove that every subharmonic function obeys the maximum-principle i.e. if the maximum of v can be found inside Ω then v is constant

Solution. Suppose $x_0 \in \Omega$ is the maximum of v then on a ball of radius $r > 0$ around x_0 we have

$$0 \geq v(x_0) - S[v](x_0,r) = \frac{1}{C}\int_{\partial B(x_0,r)} v(x_0) - v(y) d\sigma(y) \geq 0.$$

As from x_0 maxima it follows that for $\forall y \in B(x_0,r)$

$$v(x_0) - v(y) \geq 0.$$

We conclude that for all $y \in \partial B(x_0,r)$

$$v(x_0) = v(y).$$

Now suppose there exists $y_1 \in B(x_0, r)$ such that $|v(y_1) - v(x_0)| > 0$ then by continuity we have $\forall x \in B(y_1, r)$ also (where $r > 0$ is sufficiently small)$|v(x) - v(x_0)| > 0$ such that

$$\begin{aligned} 0 &< \frac{1}{C} \int_{B(y_1,r)} v(x_0) - v(y) \\ &\leq \frac{1}{C} \int_{B(y_1,r)} S[v](x_0, r) - v(y) d\mu(y) \\ &= S[v](x_0, r) - \frac{1}{C} \int_{B(y_1,r)} v(y) d\mu(y). \end{aligned}$$

for $r \to 0$ we must have

$$0 < S[v](x_0, r) - \frac{1}{C} \int_{B(y_1,r)} v(y) d\mu(y) \leq x_0 - y_1 \geq 0.$$

a contradiction. (idk about this fully tbh).

Now we know that if v attains a maxima x_0 it must be constant on a small ball centered at x_0 with $r > 0$. By compactness we can cover $\overline{\Omega}$ by finite many balls of radius $\frac{r}{2} > 0$

$$B(\gamma_1, \frac{r}{2}), \ldots, B(\gamma_n, \frac{r}{2}).$$

Pick $\gamma_1 = x_0$ then v is constant on the first ball and the next center γ_2 must necessarily be contained in the ball $B(x_0, r)$ such that v is also constant on this ball, by repeating this argument we get that v must be constant on all balls. $\square$

Question 3 (b). Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \leq 0$ in Ω

Solution. Assume first that $-\Delta v \leq 0$ in Ω and define

$$\tilde{v}(r) = S[v(x) - v](x, r).$$

for $x \in \mathbb{R}$ then by the divergence theorem we get that

$$\frac{d}{dr}\tilde{v}(r) = -\frac{1}{C} \int_{B(x,r)} \Delta v d\mu.$$

$\square$