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Sheet 10

30

Exercise (a). Solutions of PDEs that are constant in the time variable are called "steady-state" solutions. Describe steady-state solutions of the inhomogenous heat equation

Proof. Suppose we have a steady-state solution u, then by assumption

$$\dot{u} = 0.$$

such that

$$\dot{u} - \Delta u = f \leftrightarrow -\Delta u = f.$$

Which are all the solutions to the Poisson equation. □

Exercise (b). Consider the heat equation $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$ with smooth initial condition $u(x, 0) = h(x)$. Suppose that the Laplacian an of h is a constant. Show that there is a solution whose time derivative is constant

Proof. If the time derivative is a constant then u is linear in time i.e

$$u(x, t) = u_1 + t \cdot c.$$

where by initial condition

$$u(x, 0) = h(x) = u_1.$$

Then we have

$$\dot{u} - \Delta u = 0 \leftrightarrow c - c_2 = 0.$$

So $c = c_2 = \Delta h$ is a solution. □

Exercise (c). Consider "translational solutions" to the heat equation on $\mathbb{R} \times \mathbb{R}^+$(i.e. n=1). These are solutions of the form $u(x, t) = F(x - bt)$. Find all such solutions.

Proof. We have $u(x,t) = F(z(x,t)) = F(x-bt)$

$$\begin{aligned} \dot{u} &= -bF' \\ \frac{du}{dx} &= F' \\ \frac{d^2u}{dx^2} &= F'' \\ . \end{aligned}$$

where $F' = \frac{d}{dz}F$, then

$$\dot{u} - \Delta F = 0 \leftrightarrow -bF' - F^{''} = 0.$$

By integration we must have

$$-bF - F' = c.$$

For some constant c, then

$$F' = -(c + bF).$$

Which is a first order linear ode with solution ... $\square$

Exercise (d). If u is a solution to the heat equation, show for every $\lambda \in \mathbb{R}$ that

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t).$$

is also a solution to the heat equation

Proof. We check

$$\begin{aligned} \dot{u}_\lambda &= \lambda^2 \dot{u} \\ \Delta u_\lambda &= \lambda^2 \Delta u. \end{aligned}$$

Then

$$\dot{u}_\lambda - \Delta u_\lambda = \lambda^2(\dot{u} - \Delta u) = 0.$$

$\square$

31. The Fourier transform

In this question we expand on some details from Section 4.1. Recall that the Fourier transform of a function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is defined to be function $\hat{h}(k) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} h(x) dx.$$

Lemma 4.3 shows that it is well-defined for Schwartz functions.

Exercise (a). Argue that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \exp(-x^2)$ is a Schwartz function

Proof. We want to check that for any $l \in \mathbb{N}$ and $k \in \mathbb{N}$

$$\sup_{x \in \mathbb{R}} |x|^{2l} |f^{(k)}(x)|.$$

is bounded, first we check a couple derivatives of f

$$\begin{aligned} f' &= -2xe^{-x^2} = -2x \cdot f \\ f'' &= -2f - 2xf' = -2f + 4x^2 f \\ f^{(3)} &= -2f' + 8xf + 4x^2 f' = 4xf + 8xf + 4x^2(-2f + 4x^2 f) = 12xf - 8x^2 f + 16x^4 f. \end{aligned}$$

Or in other words

$$f^{(k)} = p(x) \cdot f = p(x) e^{-x^2}.$$

for some $n = k + 1$ order polynomial we get

$$p(x) e^{-x^2} = \frac{p(x)}{e^{x^2}}.$$

Where for every $d \in \mathbb{N}$ we have by cutting off the series

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{|x^2|^k}{k!} \geq 1 + \frac{x^d}{d!}.$$

Now suppose $p(x)$ is a polynomial of order $n \in \mathbb{N}$ then we have

$$|p(x)| \leq C|x|^n.$$

For some $C > 0$ and $|x| \geq 1$, Proof

$$|p(x)| = |\sum_{i=0}^{n} x^i \cdot a_i| \leq \sum_{i=0}^{n} |x^i a_i| \leq a_0 + \sum_{i=1}^{n} |x^i a_i|.$$

then

$$\frac{|p(x)|}{|x|^n} \leq a_0 + a_n + \sum_{i=1}^{n-1} \frac{1}{|x|^{n-i} a_i} \leq C.$$

For $|x| \geq 1$ and some constant $C > 0$

Then

$$\begin{aligned}
\sup_{|x|\geq 1} |x|^{2l} |p(x) e^{-x^2}| &= \sup_{|x|\geq 1} \frac{|x|^{2l}|p(x)|}{e^{-x^2}} \\
&\leq \sup_{|x|\geq 1} \frac{|x|^{2l} C |x^n|}{1 + \frac{x^d}{d!}} \\
&= C \sup_{|x|\geq 1} \frac{|x|^d}{1 + \frac{x^d}{d!}} \\
&\leq C \sup_{|x|\geq 1} \frac{|x|^d}{\frac{x^d}{d!}} \\
&= C d!.
\end{aligned}$$

where $d = 2l + n$
For $x \in [-1, 1]$ we get that $p(x)$ is bounded by some constant $\tilde{C}$, as it is continuous and $[-1, 1]$ is compact. Such that

$$\sup_{x\in\mathbb{R}} |x|^{2l} |p(x) e^{-x^2}| \leq \max\{\tilde{C}, C d!\}.$$

for any polynomial $p(x)$, and thus e^{-x^2} is a Schwartz function. □

Exercise (b)**.** Consider

$$I^2 = \left(\int_{\mathbb{R}} e^{-x^2} dx \right)^2 = \int_{\mathbb{R}^2} e^{-x^2 - y^2} dx dy.$$

By changing to polar coordinates, compute this integral

Proof. Using $\Phi(r, \varphi) = (r \cos\varphi, r \sin\varphi) := (x, y)$ then

$$J\Phi = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -r \sin(\varphi) & r \cos(\varphi) \end{pmatrix}.$$

and

$$\det J\Phi = r \cos^2 + r \sin^2 = r.$$

Where $r \in [0, \infty)$ and $\theta \in [0, 2\pi]$

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy &= \int_0^\infty \int_0^{2\pi} re^{-r^2(\cos^2(\varphi)+\sin^2(\varphi))} d\varphi dr \\ &= \int_0^\infty \int_0^{2\pi} re^{-r^2} d\varphi dr \\ &= 2\pi \int_0^\infty re^{-r^2} dr \\ &= \pi.\end{aligned}$$

□

So $I = \sqrt{\pi}$ like we saw in the lecture ($n = 1$).

Exercise (c). Prove the rescaling law for Fourier transforms: if $h(x) = g(ax)$ then

$$\hat{h}(k) = |a|^{-n} \hat{g}(a^{-1}k).$$

Proof. We compute

$$\begin{aligned}\hat{h}(k) &= \int_{\mathbb{R}^n} e^{-2\pi ik\cdot x} h(x) dx \\ &= \int_{\mathbb{R}^n} e^{-2\pi ik\cdot x} g(ax) dx\end{aligned}$$

.

By using the transformation

$$z = ax \leftrightarrow \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = a \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Then we get the determinant

$$\frac{1}{a^n}.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-2\pi ik\cdot x} g(ax) dx &= \int_{\mathbb{R}^n} |a^{-n}| e^{-2\pi ik\cdot(z\cdot\frac{1}{a})} g(z) dz \\ &= \int_{\mathbb{R}^n} |a^{-n}| e^{-2\pi \frac{ik}{a}\cdot z} g(z) dz \\ &= |a|^{-n} \hat{g}(a^{-1}k).\end{aligned}$$

□

Exercise (d). Prove the shift law for Fourier transforms: if $h(x) = g(x-a)$, then

$$\hat{h}(k) = e^{-2\pi i a \cdot k}\hat{g}(k).$$

Proof. We compute

$$\begin{aligned}\hat{h}(k) &= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} h(x)dx \\ &= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} g(x-a)dx\end{aligned}$$

.

Using the transform

$$z = x - a \leftrightarrow \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix}.$$

Which has determinant 1 then

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} g(x-a)dx &= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot(z+a)} g(z)dx \\ &= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot z + k\cdot a} g(z)dx \\ &= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot z} \cdot e^{-2\pi i k\cdot a)} g(z)dx \\ &= e^{-2\pi i k\cdot a}\int_{\mathbb{R}^n} e^{-2\pi i k\cdot z} g(z)dx \\ &= e^{-2\pi i k\cdot a}\hat{g}(k).\end{aligned}$$

□

Exercise (e). Show that δ is a tempered distribution

Proof. We recall

Definition (Tempered). Suppose that φ_m is a sequence of test functions that converges to zero in $\mathcal{S}$, i.e. $\lim_{m\to\infty} \rho_{l,\alpha}(\varphi_m) = 0$ for all $l \in \mathbb{N}, \alpha \in \mathbb{N}_0^n$. We say that F is a tempered distribution $F \in \mathcal{S}'$ if $\lim_{m\to\infty} F(\varphi_m) = 0$. Where

$$\rho_{l,\alpha}(\varphi_m) = \sup|x|^{2l}|\partial^\alpha \varphi_m|.$$

Since $\alpha \in \mathbb{N}_0$ in our case we have

$$\rho_{l,0}(\varphi_m) = \sup|x|^{2l}|\varphi_m(x)| \to 0.$$

and for all $m \in \mathbb{N}$ we have by properties of sup

$$\sup|x|^{2l}|\varphi_m(x)| \geq |\varphi_m(0)| \geq 0.$$

So we have for $\forall m \in \mathbb{N}$

$$0 \leq |\varphi_m(0)| = |\delta(\varphi_m)| \leq |x|^{2l}|\varphi_m(x)| \xrightarrow{m\to\infty} 0.$$

This shows δ is a tempered distribution. $\square$

Exercise (f). Compute the Fourier transform of δ

Proof. For $\varphi \in \mathcal{S} \subset \mathcal{C}_0^\infty$, and since δ is a tempered distribution by the above $\hat{\delta}(\varphi) = \delta(\hat{\varphi})$

$$\begin{aligned}
\hat{\delta}(\varphi) &= \delta(\hat{\varphi}(k)) \\
&= \delta\left(\int_{\mathbb{R}^n} e^{-2\pi i(\star)\cdot x}\varphi(x)dx\right) \\
&= \int_{\mathbb{R}^n} e^{-2\pi i 0\cdot x}\varphi(x)dx \\
&= \int_{\mathbb{R}^n} \varphi(x)dx
\end{aligned}$$

.

$\square$

Exercise (g). Try to compute the Fourier transform of 1 using Definition 4.8. What is the difficulty?

Proof. If we want to use Definition we identify 1 with the distribution

$$F_1(\varphi) = \int_{\mathbb{R}^n} 1 \cdot \varphi.$$

We recognize this as

$$F_1(\varphi) = \hat{\delta}(\varphi) = \delta(\hat{\varphi}).$$

Since δ is tempered so is F_1 and we use 4.8.

$$\begin{aligned}\hat{F}_1(\varphi) = F_1(\hat{\varphi}) &= \int_{\mathbb{R}^n} 1 \cdot \hat{\varphi} dk \\ &= \int_{\mathbb{R}^n} e^{2\pi i k \cdot 0} \hat{\varphi} dk \\ &= \mathcal{F}^{-1}(\hat{\varphi})(0) \\ &= \varphi(0) \\ &= \delta(\varphi).\end{aligned}$$

□

32 One step at a time

Exercise. Prove the following identity for the fundamental solution in one dimension ($n = 1$)

$$\Phi(x, s+t) = \int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s) dy.$$

Interpret this equation in the context of Theorem 4.7.

Proof. We calculate for $t, s > 0$

$$\int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s) dy = \int_{\mathbb{R}} \frac{1}{\sqrt{16\pi^2 ts}} e^{-\frac{|x-y|^2}{4t} - \frac{|y|^2}{4s}} dy.$$

and for simplicity consider first we want to get $(-A + By - Cy^2)$

$$\begin{aligned}-\frac{|x-y|^2}{4t} - \frac{|y|^2}{4s} &= -\frac{x^2 - 2xy + y^2}{4t} - \frac{y^2}{4s} \\ &= -\frac{x^2}{4t} + \frac{x}{2t} \cdot y - \frac{y^2}{4t} - \frac{1}{4s} y^2 \\ &= -\underbrace{\frac{x^2}{4t}}_{A} + \underbrace{\frac{x}{2t}}_{B} \cdot y - \underbrace{\left(\frac{1}{4t} + \frac{1}{4s}\right)}_{C} y^2.\end{aligned}$$

Then by the hint we get for the integral $\sqrt{\frac{\pi}{C}} \exp(\frac{B^2}{4C} - A)$

$$\begin{aligned}\int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s) dy &= \int_{\mathbb{R}} \frac{1}{\sqrt{16\pi^2 ts}} e^{-\frac{|x-y|^2}{4t} - \frac{|y|^2}{4s}} dy \\ &= \frac{1}{\sqrt{16\pi^2 ts}} \sqrt{\frac{\pi}{(\frac{1}{4t} + \frac{1}{4s})}} \exp\left(\frac{x^2}{16t^2 \cdot (\frac{1}{4t} + \frac{1}{4s})} - \frac{x^2}{t}\right) \\ &\vdots \\ &= \Phi(x, s+t).\end{aligned}$$

Where we pinky promise we did the intermediate transformations.

Theorem 4.7 says that, for $h \in \mathcal{C}_b(\mathbb{R}^n, \mathbb{R})$

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)h(y)d^n y.$$

has the properties

1. $u \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^+)$
2. $\dot{u} - \Delta u = 0$
3. u extend continuously to $\mathbb{R}^n \times [0,\infty)$ with $\lim_{t\to 0} u(x,t) = h(x)$

Now lets say we have u as given by the representation of 4.7., and take $n=1$ since we've only shown the identity for that, then

$$\begin{aligned} u(x,t+s) &= \int_{\mathbb{R}} \Phi(x-y,t+s)h(y)dy \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Phi(x-y-z,t)\Phi(z,s)dz \right) h(y)dy \\ &= \int_{\mathbb{R}} \tilde{u}(x-z,t)\Phi(z,s)dz. \end{aligned}$$

Which by Theorem 4.7 is a solution to the Cauchy problem with initial condition

$$h(y) := \tilde{u}(y,t).$$

So we can always construct a new heat equation by taking the previous state as our new initial state i.e. the initial condition only matters till the immediately following state. $\square$