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Sheet 11

33 The distribution of heat

Consider the fundamental solution of the heat equation $\Phi(x, t)$ given in Definition 4.5.

Exercise (a). Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$

Proof. We recall

$$\Phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & \text{for } x \in \mathbb{R}^n, t > 0 \\ 0 & \text{for } x \in \mathbb{R}^n, t \leq 0 \end{cases}.$$

Pick $x \in \mathbb{R}^n$ and handle the cases $t > 0$, $t < 0$ and $t = 0$, for $t = 0$ we have that for $n \in \mathbb{N}$, $\Phi(x, t)^{(n)} \equiv 0$ a smooth function, for $t < 0$ we also see that $\Phi(x, t)^{(n)} \equiv 0$ and $t \rightarrow 0^-$ are also 0 and we get no problem here.

We handle $t > 0$ by considering that from the last sheet we know that the derivatives of e^{-x^2} are all of the form

$$p(x)e^{-x^2}.$$

For some polynomial $p(x)$ in this case we get extra terms including t i.e

$$\Phi^{(n)}(x, t) = q(x, t)e^{-\frac{|x|^2}{4t}}.$$

similar to last sheet, the exponential growth for $x \neq 0$ (to 0 since $t \rightarrow 0^+ \Rightarrow e^{-\frac{|x|^2}{4t}} \rightarrow 0$) will "outperform" the polynomial growth and force the expression towards 0. For $x = 0$ the polynomial is 0 and thus the whole expression.

We get that the fundamental solution extends to a smooth function. \square

Exercise (b). Verify that this obey the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$

Solution. We check

$$\partial_t \Phi(x, t) = -\frac{n}{2(4\pi)^{\frac{n}{2}}} \frac{1}{(t)^{\frac{n}{2}+1}} e^{-\frac{|x|^2}{4t}} + \frac{|x|^2}{4t^2} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

and

$$\begin{aligned} \partial_{x_i} \Phi(x, t) &= -\frac{x_i}{2t} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \\ \partial_{x_i^2} \Phi(x, t) &= \frac{x_i^2}{4t^2} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} - \frac{x_i}{2t^2} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}. \end{aligned}$$

Then

$$\Delta \Phi = \frac{|x|^2}{4t^2}.$$

Kann der Leon aufschreiben

□

Exercise (c). Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x, t) \varphi(x, t) dx dt.$$

Proof. For $t < 0$ the Integral vanishes because $\Phi(x, t) = 0$, since $\varphi \in C_0^\infty(K)$

$$\int_{\mathbb{R}} \varphi(x, t) dt = \int_{-T}^T \varphi(x, t) dt.$$

For some $T > 0$ such that we get

$$\begin{aligned} H(\varphi) &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \Phi(x, t) \varphi(x, t) dx dt \\ &= \int_{-T}^T \int_{\mathbb{R}^n} \Phi(x, t) \varphi(x, t) dx dt \\ &= \int_0^T \int_{\mathbb{R}^n} \Phi(x, t) \varphi(x, t) dx dt \end{aligned}$$

□

Exercise (d). Conclude with the help of Lemma 4.6 and Theorem 4.7 that

$$|H(\varphi)| \leq T \|\varphi\|_{K,0}.$$

Hence H is a continuous linear functional

Proof.

$$\begin{aligned}
 |H(\varphi)| &= \left| \int_0^T \int_{\mathbb{R}^n} \Phi(x, t) \varphi(x, t) dx dt \right| \\
 &\leq \int_0^T \int_{\mathbb{R}^n} |\Phi(x, t) \varphi(x, t)| dx dt \\
 &\leq \int_0^T \int_{\mathbb{R}^n} |\Phi(x, t)| \|\varphi\|_{K,0} dx dt \\
 &\leq \int_0^T \|\varphi\|_{K,0} \int_{\mathbb{R}^n} |\Phi(x, t)| dx dt
 \end{aligned}$$

Now for $t > 0$ we have

$$\int_{\mathbb{R}^n} |\Phi(x, t)| dx = \int_{\mathbb{R}^n} \Phi(x, t) dx = 1.$$

For $t \rightarrow 0$ we have by 4.7. that also

$$\int_{\mathbb{R}^n} |\Phi(x, t)| dx = \underbrace{\int_{\mathbb{R}^n} \Phi(x, t) \cdot 1 dx}_{u(x,t)} \rightarrow 1.$$

Thus

$$\int_0^T \|\varphi\|_{K,0} \int_{\mathbb{R}^n} |\Phi(x, t)| dx dt \leq T \cdot \|\varphi\|_{K,0}.$$

□

Exercise (e). Extend Theorem 4.7 to show that

$$\int_{\mathbb{R}^n} \Phi(x - y, t) h(y, s) dy \rightarrow h(x, s).$$

as $t \rightarrow 0$ uniformly in s

Proof. In the last step of the proof of (iii) we bound

$$|h(y) - h(x)| \leq 2 \sup\{|h(y)|, y \in \mathbb{R}^n\}.$$

By making the necessary assumption that $h \in \mathcal{C}_b(\mathbb{R}^n \times \mathbb{R})$ we instead bound (for our h now)

$$|h(y, s) - h(x, t)| \leq 2 \sup\{|h(y, x)|, y, x \in \mathbb{R}^n \times \mathbb{R}\}.$$

everything else stays the same

□

Exercise (f). Hence show that

$$\int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \Phi(-\partial_t \varphi - \Delta \varphi) dy dt \rightarrow \varphi(0, 0).$$

as $\varepsilon \rightarrow 0$

Proof. Assuming we mean

$$\int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \Phi(y, t)(-\partial_t \varphi(y, t) - \Delta \varphi(y, t)) dy dt \rightarrow \varphi(0, 0).$$

Then

$$\begin{aligned} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \Phi(y, t)(-\partial_t \varphi(y, t) - \Delta \varphi(y, t)) dy dt &= \int_{\mathbb{R}^n} -\Phi(y, \varepsilon) \varphi(y, \varepsilon) dy \\ &\quad - \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \partial_t \Phi(y, t)(-\varphi - \Delta \varphi(y, t)) dy \\ &= \int_{\mathbb{R}^n} -\Phi(y, \varepsilon) \varphi(y, \varepsilon) dy \\ &\quad - \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \underbrace{\partial_t \Phi(y, t) - \Delta(-\varphi)}_{=0} dy dt \\ &= \int_{\mathbb{R}^n} -\Phi(y, \varepsilon) \varphi(y, \varepsilon) dy \end{aligned}$$

Some sign is wrong but if we ignore that then in the end we have, since Φ only depends on the length of the x variable

$$u(0, \varepsilon) = \int_{\mathbb{R}^n} \Phi(-y, \varepsilon) \varphi(y, \varepsilon) dy$$

which by e) tends to

$$\varphi(0, 0).$$

as $\varepsilon \rightarrow 0$

□

Exercise (g). Prove that as $\varepsilon \rightarrow 0$

$$\int_0^{\varepsilon} \int_{\mathbb{R}^n} \Phi(y, t) h(y, t) dy dt \rightarrow 0.$$

Proof. We have

$$\begin{aligned}
\left| \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y, t) h(y, t) dy dt - 0 \right| &\leq \left| \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y, t) h(y, t) dy dt \right| \\
&\leq \int_0^\varepsilon \int_{\mathbb{R}^n} |\Phi(y, t) h(y, t)| dy dt \\
&\leq \int_0^\varepsilon \sup_{x \in \mathbb{R}^n} |h(x, t)| \int_{\mathbb{R}^n} \Phi(y, t) dy dt \\
&\leq \int_0^\varepsilon \sup_{(x, s) \in \mathbb{R}^n \times \mathbb{R}^+} |h(x, s)| \int_{\mathbb{R}^n} \Phi(y, t) dy dt \\
&\leq \sup_{(x, s) \in \mathbb{R}^n \times \mathbb{R}^+} |h(x, s)| \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y, t) \cdot 1 dy dt \\
&\leq \varepsilon \sup_{(x, s) \in \mathbb{R}^n \times \mathbb{R}^+} |h(x, s)| \\
&\rightarrow 0.
\end{aligned}$$

Where we used similar argument to part d for handling the Φ integral \square

34. Heat death of the universe

Exercise (a). Suppose that $h \in C_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and u is defined as in Theorem 4.7. Show

$$\sup_{x \in \mathbb{R}^n} |u(x, t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \|h\|_{L^1}.$$

Proof. Take $t > 0$

$$\begin{aligned}
\sup_{x \in \mathbb{R}^n} |u(x, t)| &\leq \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(x - y, t) h(y)| dy \\
&\leq \|h\|_{L^1} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(x - y, t)| dy
\end{aligned}$$

and use the transformation $z = \frac{1}{\sqrt{4t}}(x - y)$ which has determinant $-(4t)^{\frac{n}{2}}$

$$\frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dy = \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-z^2} dz = 1.$$

So i don't quite see where the $\frac{1}{(4\pi t)^{\frac{n}{2}}}$ term comes from and get

$$\sup_{x \in \mathbb{R}^n} |u(x, t)| \leq \|h\|_{L^1}.$$

\square