Chapter 1

Stochastic Mean Field Particle Systems

From now on let the underlying probability space be given by $(\Omega, \mathcal{F}, \mathbb{P})$.

1.1 Basics of probability

Definition 1.1.1 (Brownian Motion). Real valued stochastic process $W(\cdot)$ is called a Brownian motion (Wiener process) if

1. $W(0) = 0$a.s.
2. $W(t) - W(s) \sim \mathcal{N}(0, t-s)$, for all $t, s \geq 0$
3. $\forall 0 < t_1 < t_2 < \ldots < t_n$, $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent
4. $W(t)$ is continuous a.s (sample paths)

Remark (Properties).

1. $\mathbb{E}[W(t)] = 0$, $\mathbb{E}[W(t)^2] = t$, for all $t > 0$
2. $\mathbb{E}[W(t)W(s)] = t \wedge s$ a.s
3. $W(t) \in \mathcal{C}^\gamma[0, T]$, $\forall 0 < \gamma < \frac{1}{2}$.
4. $W(t)$ is nowhere differentiable a.s

additionally Brownian motions are martingales and satisfy the Markov property

Definition 1.1.2 (Progressively measurable). In addition to adaptation of a Stochastic process X_t we say it is progressively measurable w.r.t $\mathcal{F}_t$ if $X(s, \omega) : [0, t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}[0, t] \times \mathcal{F}_t$ measurable, i.e the t is included

Definition 1.1.3 (Simple functions). Instead of $\mathcal{H}^2$ she uses $\mathbb{L}^2(0,T)$ is the space of all real-valued progressively measurable proceses $G(\cdot)$ s.t

$$\mathbb{E}[\int_0^T G^2 dt] < \infty.$$

define $\mathbb{L}$ analog

Definition 1.1.4 (Step Process). $G \in \mathbb{L}^2(0,T)$ is called a step process when Partition of $[0,T]$ exists s.t $G(t) = G_k$ for all $t_k \leq t \leq t_{k+1}$, $k = 0, \ldots, m-1$ note G_k is $\mathcal{F}_{t_k}$ measurable R.V.

For step process we define the ito integral as a simple sum

Definition 1.1.5 (Ito integral for step process). Let $G \in \mathbb{L}^2(0,T)$ be a step process is given by

$$\int_0^T G(t) dW_t = \sum_{k=0}^{m-1} G_k(W(t_{k+1} - W(t_k))).$$

We take the left value of the process such that we converge against the right integral later

Remark. For two step processes $G, H \in \mathbb{L}^2(0,T)$ for all $a, b \in \mathbb{R}$, we have linearity (note they may have two different partitions, so we need to make a bigger (finer) one to include both,)

1. $\int_0^T (aG + bH) dW_t = a \int G + b \int H$
2. $\mathbb{E}[\int_0^T G dW_t] = 0$, because the Brownian motion has EV of 0
3. $\mathbb{E}[(\int_0^T G dW_t)^2] = \mathbb{E}[\int_0^T G^2 dt]$ Ito isometry

Proof. First property is just defining a new partition that includes both process. Second property, the Idea of the proof is that

$$\begin{aligned}\mathbb{E}[\int_0^t G dW_t] &= \mathbb{E}[\sum_{k=0}^{m-1} G_k(W_{t_{k+1}} - W_{t_k})] \\ &= \sum_{k=0}^{m-1} \mathbb{E}[G_k(W(t_{k+1}) - W(t_k))]\end{aligned}$$

.

Remember $G_k \sim \mathcal{F}_{t_k}$ m.b. and $W(t_{k+1}) - W(t_k)$ is mb. wrt to $W^t(t_k)$ future sigma algebra and it is independent of $\mathcal{F}_{t_k}$ s.t the expectation decomposes

$$\sum_{k=0}^{m-1} \mathbb{E}[G_k(W(t_{k+1}) - W(t_k))] = \sum_{k=0}^{m-1} \mathbb{E}[G_k]\mathbb{E}[W(t_{k+1} - W(t_k))] = C \cdot 0 = 0.$$

For the variance decompose into square and non square terms , the non square terms dissapear by property 2 the rest follows by the variance of Brownian motion, be careful of which terms are actually independent , at leas one will always be independent of the other 3 □

Definition 1.1.6 (Ito Formula)**.** If $u \in \mathcal{C}^{2,1}(\mathbb{R} \times [0,T]; R)$ then

$$\begin{aligned} du(x(t),t) &= \frac{\partial u}{\partial t}(x(t),t)dt + \frac{\partial u}{\partial x}(x(t),t)dx + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2 dt \\ &= \frac{\partial u}{\partial x}(x(t),t)GdW_t + (\frac{\partial u}{\partial t}(x(t),t)) + \frac{\partial u}{\partial x}(x(t),t)F + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2 dt. \end{aligned}$$

For $dX = Fdt + GdW_t$ for $F \in L^1([0,T])$, $G \in L^2([0,T])$

Proof. The proof is split into the steps

1.
$$\begin{aligned} d(W_t^2) &= 2W_t dW_t + dt \\ d(tW_t) &= W_t dt + tdW_t. \end{aligned}$$

2.
$$\begin{aligned} dX_i &= F_i dt + G_i dW_t \\ d(X_1, X_2) &= X_2 dX_1 + X_1 dX_2 + G_1 G_2 dt \\ &. \end{aligned}$$

3.
$$u(x) = x^m \quad m \geq 2.$$

4. Itos formula for $u(x,t) = f(x)g(t)$ where f is a polynomial

I.e we prove the Ito formula for functions of the form $u(x) = x^m$ and then Step 1 :

1. $d(W_t^2) = 2W_t dW_t + dt$ which is equivalent to $W^2(t) = W_0^2 + \int_0^t 2W_s dW_t + \int_0^t ds$

2. $d(tW_t) = W_t dt + t dW_t$ which is equivalent to $tW(t) - sW(0) = \int_0^t W_s ds + \int_0^t s dW_s$

Actually $\forall$ a.e $\omega \in \Omega$:

$$2\int_0^t W_s dW_s = 2 \lim_{n\to\infty} .$$

Now we prove (2) $tW_t - 0W_0 = \int_0^t W_s ds + \int_0^t s dW_s$

$$\int_0^t s dW_s + \int_0^t W_s ds = \lim_{n\to\infty} \sum_{k=0}^{n-1} t_k^n (W(t_{k+1}^n) - W(t_k^n)) + \sum_{k=0}^{n-1} W(t_{k+1}^n (t_{k+1}^n - t_k^n)).$$

We choose the right value for the second integral

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} (-t_k^n W(t_k)^n + t_{k+1}^n W(t_{k+1}^n)) = W(t)t - W(0)\cdot 0.$$

Its product rule

$$\begin{aligned} dX_1 &= F_1 dt + G_1 dW_t \\ dX_2 &= F_2 dt + G_2 dW_t. \end{aligned}$$

This can be written as

$$d(X_1, X_2) = X_2 dX_1 + X_1 dX_2.$$

this shorthand notation actually means

$$\begin{aligned} X_1(t)X_2(t) - X_1(0)X_2(0) = &\int_0^t X_2 F_1 ds + \int_0^t X_2 G_1 dW_s \\ &+ \int_0^t X_1 F_2 ds + \int_0^t X_1 G_2 dW_s \\ &+ \int_0^t G_1 G_2. \end{aligned}$$

We prove for F_1, F_2, G_1, G_2 are time independent

$$
\begin{aligned}
&\int_0^t (X_2 dX_1 + X_1 dX_2 + G_1 G_2 ds) \\
&= \int_0^t (X_2 F_1 + X_1 F_2 + G_1 G_2) ds + \int_0^t (X_2 G_1 + X_1 G_2) dW_s \\
&= \int_0^t (\underbrace{F_2 F_1 s + F_1 G_2 W_s}_{=X_2} + \underbrace{F_1 F_2 s + F_2 G_1 W_s}_{=X_1} + G_1 G_2) ds \\
&+ \int_0^t (F_2 G_1 s + G_2 G_1 W_s + F_1 G_2 s + G_1 G_2 W_s) dW_s \\
&= G_1 G_2 t + F_1 F_2 t^2 + (F_1 G_2 + F_2 G_1) \underbrace{\left(\int_0^t W_s ds + \int_0^t s dW_s \right)}_{tW_t} + 2 G_1 G_2 \underbrace{\int_0^t W_s dW_s}_{W_t^2 - t} \\
&= G_1 G_2 t + F_1 F_2 t^2 + (F_1 G_2 + F_2 G_1) t W_t + G_1 G_2 W_t^2 - G_1 G_2 t \\
&= X_1(t) \cdot X_2(t).
\end{aligned}
$$

Where $X_2(t) = \int_0^t F_2 ds + \int_0^t G_2 dW_s \overset{\text{Cons.}}{=} F_2 t + G_2 W_t$

Extend the above idea by considering step processes (F_1, F_2, G_1, G_2)instead of time independent. Step processes are constant (related to time) and we can use the above prove for every time step t and just consider a summation over it.

For general $F_1, F_2 \in L^1(0,T), G_1, G_2 \in L^2(0,T)$ then we take step processes to approximate them

$$
\begin{aligned}
&\mathbb{E}[\int_0^T |F_i^n - F_i| dt] \to 0 \\
&\mathbb{E}[\int_0^T |G_i^n - G_i|^2 dt] \to 0
\end{aligned}
$$

.

$$
X_i(t)^n = X_i(0) + \int_0^t F_i^n ds + \int_0^t G_i^n dW_s.
$$

It holds

$$
\begin{aligned}
X_1^n(t) X_2(t)^n - X_1(0) X_2(0) &= \int_0^t X_2(s)^n F_1^n(s) ds + \int_0^t X_2(s) G_1(s)^n dW_s \\
&+ \int_0^t X_1^n(s) F_2^n(s) ds + \int_0^t X_1(s)^n G_2^n(s) dW_s + \int_0^t G_1(s)^n G_2^n(s) ds.
\end{aligned}
$$

Only thing left is a convergence result (i.e DCT) sinc the processes are bounded or smth like that.

Step 3 if $u(x) = x^m,\ \forall m = 0, \dots$ to prove

$$d(X^m) = mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}G^2dt.$$

For $m = 2$ the result is obtained by the product rule, By induction we prove for arbitrary m

(IV) Suppose the statement hold for $m-1$
(IS) $m-1 \to m$

$$\begin{aligned}
d(X^m) = d(X \cdot X^{m-1}) &= XdX^{m-1} + X^{m-1}dx + (m-1)X^{m-2}G^2dt \\
&\overset{\text{IS}}{=} X(m-1)X^{m-2}dx + X \cdot \frac{1}{2}(m-1)(m-2)X^{m-3}G^2dt + X^{m-1}dx + (m-1)X^{m-2}G^2dt \\
&= mX^{m-1}dx + (m-1)(\frac{m}{2} - 1 + 1)X^{m-2}G^2dt \\
&= \underbrace{mX^{m-1}}_{\partial_x u}dx + \frac{1}{2}\underbrace{m(m-1)X^{m-2}}_{\partial_x^2 u}G^2dt.
\end{aligned}$$

Now $u(x) = x^m$

$$dX = Fdt + GdW_t.$$

Step 4 If $u(x,t) = f(x)g(t)$ where f is a polynomial

$$\begin{aligned}
d(u(x,t)) = d(f(x)g(t)) &= f(x)dg + gdf(x) + G \cdot 0dt \\
&\overset{\text{S3}}{=} f(x)g'(t)dt + gf'(x)dx + \frac{1}{2}gf''(x)G^2dt.
\end{aligned}$$

Itos formula is true for $f(x)g(t)$, it should thus also be true for functions $u(x,t) = \sum_{i=1}^m g^i(t)f^i(x)$

Step 5: if $u \in \mathcal{C}^{2,1}$ then we know there exists a sequence of polynomials $f^i(x)$ s.t

$$u_n(x,t) = \sum_{i=1}^n f^i(x)g^i(t).$$

Then $u_n \to u$ uniformly for any compact set $K \subset \mathbb{R} \times [0,T]$, we can thus apply Itos formula for each of the u_n and take the limit term wise □

Remark. One can get the existence of the polynomial sequence by using Hermetian polynomials

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

Exercise 1.1.1. If $u \in \mathcal{C}^\infty$, $\frac{\partial u}{\partial x} \in \mathcal{C}_b$ then prove Step 4 $\Rightarrow$ Step 5

Use Taylor expansion and use the uniform convergence of the Taylor series on any compact support

Remark (Multi Dimensional Brownian Motion)**.** Multi dimensional Brownian motion

$$W(t) = (W^1(t), \ldots, W^m(t)) \in \mathbb{R}^m$$

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In each direction we should have a 1 dimensional Brownian motion and any two directions should be independent. We use the natural filtration $\mathcal{F}_t = \sigma(W(s); 0 \leq s \leq t)$

Definition 1.1.7 (Multi-Dimensional Ito's Integral)**.** We the define the n dimensional integral for $G \in L^2_{n \cdot m}([0,T])$, $G_{ij} \in L^2([0,T])$ $1 \leq i \leq n$, $1 \leq j \leq m$

$$\int_0^T G dW_t = \begin{pmatrix} \vdots \\ \int_0^T G_{ij} dW_t^j \\ \vdots \end{pmatrix}_{n \times 1} .$$

With the Properties

$$\mathbb{E}[\int_0^T G dW_t] = 0$$

$$\mathbb{E}[(\int_0^T G dW_t)^2] = \mathbb{E}[\int_0^T |G|^2 dt].$$

Where $|G|^2 = \sum_{i,j}^{n,m} |G_{ij}|^2$

Definition 1.1.8 (Multi-Dimensional Ito process)**.** We define the n dimensional Ito process as

$$X(t) = X(s) + \int_s^t F_{n \times 1}(r) dr + \int_0^t G_{n \times m}(r) dW_{m \times 1}(r)$$

$$dX^i = F^i dt + \sum_{j=1}^m G^{ij} dW_t^i \qquad 1 \leq i \leq n.$$

Theorem 1.1.1 (Multi Dimensional Ito's formula)**.** We define the n dimensional Ito's formula as $u \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0,T], \mathbb{R})$

$$du(x(t),t) = \frac{\partial u}{\partial t}(x(t),t)dt + \nabla u(x(t),t) \cdot dx(t) \\ + \frac{1}{2}\sum \frac{\partial^2 u}{\partial x_i \partial x_j}(x(t),t)\sum_{l=1}^{m} G^{il}G^{il}dt.$$

Proposition 1.1.1. For real valued processes X_1, X_2

$$\begin{cases} dX_1 &= F_1 dt + G_1 dW_1 \\ dX_2 &= F_2 dt + G_2 dW_2 \end{cases} \Rightarrow d(X_1, X_2) = XdX_2 + X_2 dX_1 + \sum_{k=1}^{m} G_1^k G_2^k dt.$$

Working with SDEs relies on a lot of notational rules as seen in the differential notation is just shorthand for the Integral form

Definition 1.1.9. Formal multiplication rules for SDEs

$$(dt)^2 = 0 \ , \ dtdW^k = 0 \ , \ dW^k dW^l = \delta_{kl}dt.$$

Using this notation we can simply itos formula as follows

$$\begin{aligned} du(X,t) &= \frac{\partial u}{\partial t}dt + \nabla_x u \cdot dX + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} dX^i dX^j \\ &= \frac{\partial u}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial u}{\partial X^i}F^i dt + \sum_{i=1}^{n} \frac{\partial u}{\partial X_i}\sum_{i=1}^{m} G^{ik}dW_k \\ &+ \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\left(F^i dt + \sum_{k=1}^{m} G^{ik}dW_k\right)\left(F^j dt + \sum_{l=1}^{m} G^{i;}dW_l\right) \\ &= (\frac{\partial u}{\partial t} + F \cdot \nabla u + \frac{1}{2}H \cdot D^2 u)dt + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\sum_{k=1}^{m} G^{ik}dW_k. \end{aligned}$$

Where

$$dX^i = F^i dt + \sum_{k=1}^{m} G^{ik}dW_k$$

$$H_{ij} = \sum_{k=1}^{m} G^{ik}G^{jk} \ , \ A \cdot B = \sum_{i,j=1}^{m} A_{ij}B_{ij}.$$

Typical example

$$G^T G = \sigma I_{n \times n}.$$

Example. If F and G are deterministic

$$dX_{n\times 1}F(t)_{n\times 1}dt + G_{n\times m}dW_t m \times 1.$$

Then for arbitrary test function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ then by Ito's formula

$$\begin{aligned} u(x(t)) - u(x(0)) = &\int_0^t \nabla u(x(s)) \cdot F(s)ds + \int_0^t \frac{1}{2}(G^T G) : D^2 u(x(s))ds \\ &+ \int_0^t \nabla u(x(s)) \cdot G(s)dW_s. \end{aligned}$$

Let $\mu(s,\cdot)$ be the law of $X(s)$ then we take the expectation of the above integral

$$\begin{aligned} &\int_{\mathbb{R}^n} u(x)d\mu(s,x) - \int_{\mathbb{R}^n} u(x)d\mu_0(x) = \int_0^t \int_{\mathbb{R}^n} \nabla u(x) \cdot F(s)d\mu(s,x) \\ &+ \int_0^t \int_{\mathbb{R}^n} \frac{1}{2}(G^T(s)G(s)) : D^2 u(x) \cdot d\mu(s,x) + 0. \end{aligned}$$

Definition 1.1.10 (Parabolic Operator)**.**

$$\partial_t u - \frac{1}{2}\sum_{i,j=1}^n D_{ij}(\sum_{k=1}^m G^{ik}G^{kj})\mu + \nabla \cdot (F\mu) = 0.$$

Example. If $F = 0$ $m = n$ and $G = \sqrt{2}I_{n\times n}$ then

$$dX = \sqrt{2}dW_t.$$

And the law of X , μ fulfills the heat equation

$$\mu_t = \triangle \mu = 0.$$

How does this all translate to our Mean field Limit, consider a particle system given by

$$\begin{cases} dX_N &= F(X_N)dt + \sqrt{2}dW_{dN\times 1} \\ dx_i &= \frac{1}{N}\sum K(x_i,x_j)dt + \sqrt{2}dW_t^1 \qquad 1 \le i \le N \;\; N \to \infty \\ x_i(0) &= x_{0,i} \\ \mu_N(t) &= \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)} \end{cases}.$$

At time $t = 0$ the x_i are independent random variables at any time $t > 0$ they are dependent and the particles have joint law

$$(x_1(t),\ldots,x_N(t)) \sim u(x_1,\ldots,x_n).$$

Where $u \in \mu(\mathbb{R}^{dN}$ by Ito's formula we get for arbitrary test function $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{dN})$

$$\varphi(X_N) = \varphi(X_N(0)) + \int_0^t \nabla_{dN}\varphi \cdot \begin{pmatrix} \vdots \\ \frac{1}{n}\sum_{j=1}^N K(x_i, x_j) \\ \vdots \end{pmatrix} X_N$$
$$+ \int_0^t \triangle_{X_N}\varphi dt + \int_0^t \sqrt{2}\nabla\varphi dW_t^i.$$

Taking the expectation on both sides, then the last term disappears by definition of Ito processes

$$\partial_t - \sum_{i=1}^N \triangle_i u + \sum_{i=1}^N \nabla_{x_i}\left(\frac{1}{N}\sum_{j=1}^N K(x_i, x_j)u\right) = 0.$$

Now consider the Mean-Field-Limit, if the joint particle law can be rewritten as the tensor product of a single $\overline{u}$

$$u(x_1, \dots, x_N) = \overline{u}^{\otimes N}.$$

the equation simplifies

$$\partial_t - \sum_{i=1}^N \triangle_i u + \sum_{i=1}^N \nabla_{x_i}\left(\overline{u}^{\otimes N} k \star \overline{u}(x_i)\right) = 0.$$

1.2 Bad K

1.3 Convergence

Chapter 2

Appendix

Theorem 2.0.1 (Divergence Theorem). Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)-$ dimensional sub-manifold of $\mathbb{R}^n$. Let $F : \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that $\triangledown F$ continuously to $\partial\Omega$. Then we have :

$$\int_\Omega \triangledown \cdot F d\mu = \int_{\partial\Omega} F \cdot N d\sigma.$$

where N is the outward pointing normal. (last component is positive)