Lehrstuhl für Mathematik IV
Prof. boshi Li Chen
Paul Nikolaev

Universität Mannheim
Fakultät für Wirtschaftsinformatik
und Wirtschaftsmathematik

11.09.2023

Mean-field particle systems (Exercise sheet 1)

Exercise 1: Consider the following seconder order system

$$\begin{aligned} \mathrm{d}X_t^i &= \mathrm{d}V_t^i \\ \mathrm{d}V_t^i &= \frac{1}{N}\sum_{j=1}^N F(t, X_t^i, V_t^i, X_t^j, V_t^j)\mathrm{d}t \end{aligned}$$

on $[0,T]$ for some smooth interaction force $F\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \mapsto \mathbb{R}^d$. Following the lecture, we assume that the empirical measure

$$\mu_t^N(\mathrm{d}x, \mathrm{d}v) := \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^i, V_t^i)}$$

converges (in some sense) to the measure μ_t, which has a density ρ_t for each $t \in [0,T]$. Derive an equation for $(\rho_t, t \geq 0)$ similar to the lecture. How would you classify such equation?

Exercise 2:
Let K be a smooth function and consider the following Vlasov system.

$$\begin{cases} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) - E_t(x,v)\nabla_v f(t,x,v) = 0 \\ E(t,x) = \int_{\mathbb{R}^d} K(x-y)\int_{\mathbb{R}^d} f_t(y,v)\mathrm{d}v\mathrm{d}y. \end{cases}$$

Write down the associated particle system, i.e. a system of equations such that after taking formally the limit $N\to\infty$, the measure is a solution to the above kinetic equation.

Exercise 3: This goal of this exercise is to recall the Cauchy-Lipschitz Theorem of ODE's and some facts about dynamical systems.
Let $T > 0$, $F\colon [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ be a continuously differentiable in the space variable and continuous in the time variable. Furthermore, we have the following linear growth condition

$$|F(t,x)| \leq C(1+|x|). \tag{1}$$

for all $t \in [0,T]$.

(i) Consider the following Cauchy problem on $[s, T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = F(t, Y(t))$$
$$Y(s) = x.$$

Show that the ODE has a unique solution $Y(s, t, x)$. What is the regularity of the map

$$(s, t, y) \mapsto Y(s, t, y)?$$

(ii) Compute

$$\partial_s Y(s, t, y) + F(t, y) \cdot \nabla_y Y(s, t, y).$$

Hint: Use the flow property of the $Y(s, t, x)$

(iii) Let $f_0 \in C^1(\mathbb{R}^d)$. Prove that the following transport equation

$$\begin{cases} \partial_t f(t, y) + F(t, y) \cdot \nabla_y f(t, y) & = 0 \\ f(0, x) = f_0(x) \end{cases}$$

has a unique solution $f \in C^1([0, T] \times \mathbb{R}^d)$ given by

$$f(s, x) = f_0(Y(s, 0, y)).$$