MEAN FIELD PARTICLE SYSTEMS AND THEIR LIMITS TO NONLOCAL PD'S

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October 20, 2023

Contents

Abstract

This lecture aims to give an introduction on the mean field derivation of a family of non-local partial differential equations with and without diffusion

Chapter 1

Model description and Introduction

The following chapter will outline how the relevant particle models are defined, we differentiate between first and second order systems focusing here on first order systems while leaving the second order setting as exercises

1.1 1st Order Particle Systems

Definition 1.1.1 (1st Order Particle System)**.** We consider a system of N particles and denote by $(x_1(t), x_2(t), \ldots, x_N(t)) \in \mathcal{C}^1([0,T];\mathbb{R}^d),\ i = 1, \ldots, N$ the trajectories of the particles.

Our first order system is then governed by the system of ordinary differential equations

$$\begin{cases} dx_i(t) & = \frac{1}{N}\sum_{j=1}^N K(x_i, x_j)dt + \sigma dW_i(t), \quad 1 \le i \le N \\ x_i(t)|_{t=0} & = x_i(0) \end{cases}.$$

where $K : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a given function.

For $\sigma = 0$ we say the system is deterministic

We consider the following examples for K

Example. A common example for a well-behaved K is

$$K(x, y) = \nabla(|x - y|^2).$$

which is a locally Lipschitz continuous function.

Another typical interaction force which is not continuous is the potential field given by Coulomb potential, namely

$$K(x, y) = \nabla \frac{1}{|x - y|^{d-2}} = \frac{x - y}{|x - y|^d}.$$

Definition 1.1.2 (Empirical Measure)**.** For a set of particles $(x_1(t), x_2(t), \ldots, x_N(t)) \in \mathcal{C}^1([0,T];\mathbb{R}^d),\ i = 1, \ldots, N$ we define the empirical measure by

$$\mu^N(t) \triangleq \frac{1}{N}\sum_{j=1}^N \delta_{x_i(t)}.$$

Our goal is the study of the limit of this system as $N \to \infty$. An appropriate quantity is to consider the empirical measure 1.1.2. If the initial empirical measure converges in some sense to a measure $\mu(0)$ i.e.

$$\mu^N(0) \to \mu(0).$$

would $\mu^N(t)$ also converge to some measure $\mu(t)$?

$$\mu^N(t) \overset{?}{\to} \mu(t).$$

Furthermore, can we find an equation which $\mu(t)$ satisfies and in which sense does it satisfy this equation?

> **Note.** Consider the following case when the limit measure $\mu(t)$ is absolutely continuous with respect the Lebesgue measure, this means that
>
> $$d\mu(0,x) = \rho_0(x)dx \quad \rho_0 \in L^1(\mathbb{R}^d).$$
>
> would the limit function have the same property ?

1.2 Motivation For Partial Differential Equation

Let the following Proposition serve as a motivation on which partial differential equation $\mu(t)$ should satisfy and consider only the deterministic case for now.

Proposition 1.2.1. We say $\mu(t)$ solves the following partial differential equation (in the sense of distribution)

$$\partial_t \mu(t,x) + \nabla \cdot \left(\mu(t,x) \int_{\mathbb{R}^d} K(\cdot, y) d\mu(t,y) \right) = 0.$$

Proof. Take $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and calculate

$$\begin{aligned}
\frac{d}{dt} \langle \mu^N(t), \varphi \rangle &\triangleq \frac{d}{dt} \int_{\mathbb{R}^d} \varphi(x) d\mu^N(t,x) \\
&\overset{\text{Def.}}{=} \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{N} \sum_{j=1}^N \varphi(x) d\delta_{x_i(t)} \\
&\overset{\text{Lin.}}{=} \frac{1}{N} \sum_{j=1}^N \frac{d}{dt} \varphi(x_i(t)) \\
&= \frac{1}{N} \sum_{j=1}^N \nabla \varphi(x_i(t)) \cdot \frac{d}{dt} x_i(t) \\
&= \frac{1}{N} \sum_{j=1}^N \nabla \varphi(x_i(t)) \cdot \frac{1}{N} \sum_{j=1}^N K(x_i, x_j) \\
&= \frac{1}{N} \sum_{j=1}^N \nabla \varphi(x_i(t)) \cdot \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{R}^d} K(x_i, y) d\delta_{x_i(t)}(y) \\
&\overset{\text{Emp.}}{=} \frac{1}{N} \sum_{j=1}^N \nabla \varphi(x_i(t)) \cdot \int_{\mathbb{R}^d} K(x_i, y) d\mu^N(t,y) \\
&= \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{R}^d} \nabla \varphi(x) \cdot \int_{\mathbb{R}^d} K(x,y) d\mu^N(t,y) d\delta_{x_i(t)}(x) \\
&= \int_{\mathbb{R}^d} \nabla \varphi(x) \cdot \int_{\mathbb{R}^d} K(x,y) d\mu^N(t,y) d\mu^N(t,x) \\
&= -\left\langle \nabla \cdot \left(\mu^N(t,\cdot) \int_{\mathbb{R}^d} K(\cdot,y) d\mu^N(t,y) \right), \varphi \right\rangle.
\end{aligned}$$

i.e μ^N is a solution to

$$\partial_t \mu^N(t,x) + \nabla \cdot \left(\mu^N(t,x) \int_{\mathbb{R}^d} K(\cdot, y) d\mu^N(t,y) \right) = 0.$$

If we can now take the limit $N \to \infty$ we obtain that μ should satisfy the proposed PDE ☐

Corollary. If $\sigma > 0$ i.e our system is stochastic then we expect the limit partial differential equation to share a similar structure

$$\partial_t \mu(t,x) + \nabla \cdot \left(\mu(t,x) \int_{\mathbb{R}^d} K(\cdot, y) d\mu(t,y) \right) = \Delta \mu(t,x).$$

We define the stochastic case in detail later

1.3 2nd Order Particle Systems

We define a second order particle system as follows

Definition 1.3.1. Given the N particles

$$((x_1(t), v_1(t)), \ldots, (x_N(t), v_N(t))) \in \mathcal{C}^1([0,T]; \mathbb{R}^{2d}).$$

with initial values $x_i(0)$ for $i = 1, \ldots, N$

Then our second order system is then governed by

$$\text{(MPS)} \begin{cases} \frac{d}{dt} x_i(t) & = v_i(t) \\ \frac{d}{dt} v_i(t) & = \frac{1}{N} \sum_{j=1}^{N} F(x_i(t), v_i(t); x_j(t), v_j(t)) \quad 1 \leq i \leq N \end{cases}.$$

In this setting $(x_i(t), v_i(t))$ mean the position and velocity of the i-th particle respectively. An example for F would be

$$F(x, v; y, u) = \frac{x - y}{|x - y|^d}.$$

The empirical measure from Definition 1.1.2 can be rewritten to include the velocity as well

$$\mu^N \triangleq \frac{1}{N} \sum_{j=1}^{N} \delta_{x_i(t), v_i(t)}.$$

Exercise. Calculate for $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ the following in the second order case

$$\frac{d}{dt} \langle \mu^N(t), \varphi \rangle.$$

1.4 Lecture Structure

In Chapter 1, we are going to discuss the deterministic case for "Good" interaction forces (2-3 weeks) while giving a brief review of the well-posedness theory of ordinary differential equation. And prove the mean field limit in the framework of 1-Wasserstein distance.

The stochastic case will be studied in Chapter 2. Where we first review the mandatory concepts of probability theory, the definition of the Itô integral, and the well-posedness of stochastic differential equations. Then the propagation of chaos result of the interacting SDE system is studied, where the well-posedness of Mckean-Vlasov equation plays an important role. If time allows, we will study non-smooth interaction forces in chapter 3.

The first result is the convergence in probability, which implies the weak convergence of propagation of chaos. The second topic is to introduce the relative entropy method to get the convengence in L^1 space.

Chapter 2

MEAN-FIELD LIMIT IN THE DETERMINISTIC SETTING

In this chapter we focus on the deterministic version of the mean-field limit. Namely, we start from a system of deterministic interacting particle system with mean-field structure and prove that the corresponding empirical measure converges weakly to the measure valued solution of the corresponding partial differential equation. We are going to work only with the first order system, recall

Definition (1st Order Particle System)**.** We consider a system of N particles and denote by $(x_1(t), x_2(t), \dots, x_N(t)) \in \mathcal{C}^1([0,T];\mathbb{R}^d),\ i = 1,\dots,N$ the trajectories of the particles.

Our first order system is then governed by the system of ordinary differential equations

$$\begin{cases} dx_i(t) & = \frac{1}{N}\sum_{j=1}^{N} K(x_i, x_j)dt \quad 1 \le i \le N \\ x_i(t)|_{t=0} & = x_i(0) \in \mathbb{R}^d \end{cases}.$$

where $K : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a given function.

In the case of higher dimensional vectors we sometimes use the following notation

$$X_N(t) = \big(x_1(t), x_2(t), \dots, x_N(t)\big)^T \in \mathbb{R}^{dN}.$$

2.1 Review Of ODE Theory

Definition 2.1.1 (Intial Value Problem)**.** For $\forall T > 0$ and $f\ \ [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ we consider the initial value problem given by

$$(\text{IVP}) \begin{cases} \frac{d}{dt}x(t) & = f(t,x) \quad t \in [0,T] \\ x|_{t=0} & = x_0 \in \mathbb{R}^d \end{cases}.$$

Assumption A. $f \in \mathcal{C}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ and f is Lipschitz continuous in x, which means there $\exists L > 0$ such that $\forall (t,x),(t,y) \in [0,T]\times\mathbb{R}^d$

$$|f(t,x) - f(t,y)| \le L|x - y|.$$

Theorem 2.1.1 (Existence and uniqueness of IVP)**.** If Assumption A holds then the (IVP) has a unique solution $x \in \mathcal{C}^1([0,T];\mathbb{R}^d)$

Proof. We use Picard iteration to prove the existence, we can define the equivalent way of solving the (IVP) by considering the integral equation

$$x(t) - x_0 = \int_0^t f(s, x(s))ds \quad \forall t \in [0, T].$$

Then our Picard iteration is given by the following

$$\begin{aligned} x_1(t) &= x_0 + \int_0^t f(s, x_0)ds \\ x_2(t) &= x_0 + \int_0^t f(s, x_1(s))ds \\ &\vdots \\ x_m(t) &= x_0 + \int_0^t f(s, x_{m-1}(s))ds. \end{aligned}$$

By Assumption A and properties of integration we have $x_m(t) \in \mathcal{C}^1([0, T]; \mathbb{R}^d)$.
Due to completeness of $\mathcal{C}^1([0, T]; \mathbb{R}^d)$ we only need to show that $(x_m(t))_{m\in\mathbb{N}}$ is a Cauchy sequence to get the existence. We first prove by induction that for $m \geq 2$ it holds for some constant M that

$$|x_m(t) - x_{m-1}(t)| \leq \frac{ML^{m-1}|t|^m}{m!}.$$

IA For $m = 1$ it holds

$$\begin{aligned} |x_2(t) - x_1(t)| &\overset{\text{Tri.}}{\leq} \int_0^t |f(s, x_1(s)) - f(s, x_0)|ds \\ &\leq L \int_0^t |x_1(s_0) - x_0|ds_0 \\ &\leq L \int_0^t \int_0^{s_0} |f(s_1, x_0)|ds_1 ds_0 \\ &\leq ML \int_0^t (s_0 - 0)ds_0 \\ &= \frac{MLt^2}{2}. \end{aligned}$$

where we chose $M \geq \max_{s\in[0,T]}|f(s, x_0)|$
IV Suppose for $m \in \mathbb{N}$ it holds

$$|x_m(t) - x_{m-1}(t)| \leq \frac{ML^{m-1}|t|^m}{m!}.$$

IS $m \to m+1$

$$
\begin{aligned}
|x_{m+1}(t) - x_m(t)| &= \left| \int_0^t f(s, x_m(s)) - f(s, x_{m-1}(s)) ds \right| \\
&\overset{\text{Tri.}}{\leq} \int_0^t |f(s, x_m(s)) - f(s, x_{m-1})(s)| ds \\
&\leq L \int_0^t |x_m(s) - x_{m-1}(s)| ds \\
&\overset{\text{IV}}{\leq} L \int_0^t \frac{ML^{m-1}|s|^m}{m!} ds \\
&= \frac{ML^m |t|^{m+1}}{(m+1)!}.
\end{aligned}
$$

Now take arbitrary $p, m \in \mathbb{N}$ then by triangle inequality we obtain for $\forall t \in [0, T]$ that

$$
\begin{aligned}
|x_{m+p} - x_m(t)| &\leq \sum_{k=m+1}^{m+p} |x_k(t) - x_{k-1}(t)| \\
&\leq \sum_{k=m+1}^{m+p} M \frac{L^{k-1} T^k}{k!} \\
&= \frac{M}{L} \sum_{k=m+1}^{m+p} \frac{(LT)^k}{k!}
\end{aligned}
$$

.

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$$
\begin{aligned}
\frac{M}{L} \sum_{k=m+1}^{m+p} \frac{(LT)^k}{k!} &\leq \frac{M}{L} \frac{(LT)^{m+1}}{(m+1)!} \sum_{k=0}^{p-1} \frac{(LT)^k}{k!} \\
&\leq \frac{M}{L} \frac{(LT)^{m+1}}{(m+1)!} \sum_{k=0}^{\infty} \frac{(LT)^k}{k!} \\
&= \frac{M}{L} \frac{(LT)^{m+1}}{(m+1)!} e^{LT} \xrightarrow{m \to \infty} 0 \text{ uniformly in } t \in [0, T].
\end{aligned}
$$

Therefore $x_m(t)$ has a limit $x(t) \in \mathcal{C}^1([0, T]; \mathbb{R}^d)$ with

$$
\max_{t \in [0,T]} |x_m(t) - x(t)| \xrightarrow{m \to \infty} 0.
$$

Then by taking $m \to \infty$ in

$$
x_m(t) = x_0 + \int_{t_0}^t f(s, x_{m-1}(s)) ds.
$$

we get

$$
x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds.
$$

which means $x(t) \in \mathcal{C}^1([0, T]; \mathbb{R}^d)$ is a solution of the equivalent integral equation.

To prove the uniqueness suppose we have two solutions $x(t), \tilde{x}(t) \in \mathcal{C}^1([0, T]; \mathbb{R}^d)$ then they satisfy

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$
$$\tilde{x}(t) = x_0 + \int_{t_0}^{t} f(s, \tilde{x}(s))ds.$$

By taking the difference of these two solutions and using the Lipschitz continuity of f in x we obtain

$$\begin{aligned}|x(t) - \tilde{x}(t)| &\leq \int_0^t |f(s, x(s)) - f(s, \tilde{x(s)})|ds + |x_0 - x_0| \\ &\leq L \int_)^t |x(s) - \tilde{x}(s)|ds \\ &\leq L \int_0^t e^{-\alpha s}|x(s) - \tilde{x}(s)|e^{\alpha s}ds.\end{aligned}$$

For any $\alpha > 0$. By considering the quantity $P(t) = e^{-\alpha t}|x(t) - \tilde{x}(t)|$, we obtain

$$\begin{aligned}|x(t) - \tilde{x}(t)| &\leq L \int_0^t \max_{0\leq s\leq t}\{e^{-\alpha s}|x(s) - \tilde{x}(s)|\}e^{\alpha s}ds \\ &\leq L \max_{0\leq s\leq t}\{e^{-\alpha s}|x(s) - \tilde{x}(s)|\} \int_0^t e^{\alpha s}ds.\end{aligned}$$

We obtain

$$P(t) = e^{-\alpha t}|x(t) - \tilde{x}(t)| \leq \max_{t\in[0,T]} P(t) \leq \frac{L}{\alpha} \max_{t\in[0,T]} P(t) \quad \forall t \in [0, T].$$

By choosing $\alpha = 2L$ we have

$$\max_{t\in[0,T]} e^{-2Lt}|x(t) - \tilde{x}(t)| = 0.$$

i.e

$$x(t) = \tilde{x}(t) \quad \forall t \in [0, T].$$

This concludes the uniqueness proof □

Remark. An alternative proof for uniqueness uses Gronwall's inequality which we give in the following. Furthermore similar to the uniqueness proof, one can obtain that the solution $x(t; t_0, x_0)$ is continuously dependent on initial data

Lemma 2.1.1 (Gronwall's inequality)**.** Let $\alpha, \beta, \varphi \in \mathcal{C}([a, b]; \mathbb{R}^d)$ and $\beta(t) \geq 0$ for $\forall t \in [a, b]$ such that

$$0 \leq \varphi(t) \leq \alpha(t) + \int_a^t b(s)\varphi(s)ds \quad \forall t \in [a, b].$$

then

$$\varphi(t) \leq \alpha(t) + \int_a^t \beta(s)e^{\int_s^t \beta(\tau)d\tau}\alpha(s)ds \quad \forall t \in [a, b].$$

Specially if $\alpha(t) \equiv M$ then we have

$$\varphi(t) \leq Me^{\int_a^b \beta(\tau)d\tau} \quad \forall t \in [a, b].$$

Proof. Define

$$\psi(t) = \int_a^t \beta(\tau)\varphi(\tau)d\tau \quad \forall t \in [a, b].$$

because of the continuity of β and φ we get that ψ is differentiable on $[a, b]$ and

$$\psi'(t) = \beta(t)\varphi(t).$$

Since $\beta(t) \geq$ we have

$$\psi'(t) = \beta(t)\varphi(t) \leq \beta(t)(\alpha(t) + \psi(t)) \quad \forall t \in [a, b].$$

Then by multiplying both sides with $e^{-\int_a^t \beta(\tau)d\tau}$ we obtain

$$\begin{aligned}\frac{d}{dt}(e^{-\int_a^t \beta(\tau)d\tau}\psi(t)) &= e^{-\int_a^t \beta(\tau)d\tau}(\psi'(t) - \beta(t)\psi(t)) \\ &\leq \beta(t)\alpha(t)e^{-\int_a^t \beta(\tau)d\tau}.\end{aligned}$$

Integrate the above inequality from a to t to get

$$e^{-\int_a^t \beta(\tau)d\tau}\psi(t) - e^{-\int_a^t \beta(\tau)d\tau}\psi(a) \leq \int_a^t \beta(s)\alpha(s)e^{-\int_a^s \beta(\tau)d\tau}ds.$$

Which implies

$$\psi(t) \leq \int_a^t \beta(s)\alpha(s)e^{\int_s^t \beta(\tau)d\tau}ds.$$

and

$$\varphi(t) \leq \alpha(t) + \psi(t) \leq \alpha(t) + \int_a^t \beta(s)\alpha(s)e^{\int_s^t \beta(\tau)d\tau}ds.$$

The case with $\alpha(t) \equiv M$ is handled by using the main theorem of Differential and Integral calculus

$$\begin{aligned}\varphi(t) &\leq M\left(1 + \int_a^t \beta(s)e^{\int_s^t \beta(\tau)d\tau}ds\right) \\ &= M(1 - e^{\int_s^t \beta(\tau)d\tau}|_a^t) \\ &= Me^{\int_a^t \beta(\tau)d\tau}.\end{aligned}$$

$\square$

2.2 Mean-field particle system, well-posedness and problem setting

Let us again give the model and problem setting

Definition 2.2.1 (1st Order Particle System)**.** We consider a system of N particles and denote by $(x_1(t), x_2(t), \dots, x_N(t)) \in \mathcal{C}^1([0,T];\mathbb{R}^d),\ i = 1, \dots, N$ the trajectories of the particles. Our first order system is then governed by the system of ordinary differential equations

$$\text{(MPS)} \begin{cases} dx_i(t) & = \frac{1}{N}\sum_{j=1}^{N} K(x_i, x_j)dt \quad 1 \le i \le N \\ x_i(t)|_{t=0} & = x_i(0) \in \mathbb{R}^d \end{cases}.$$

where $K : \mathbb{R}^{2d} \to \mathbb{R}^d$ is anti-symmetric i.e

$$K(x,y) = -K(y,x) \quad K(x,x) = 0.$$

Assumption B. $K \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R}^d)$ and there exists some $L > 0$ such that $\forall x, y \in \mathbb{R}^d$ it holds

$$\sup_y |\nabla_x K(x,y)| + \sup_x |\nabla_y K(x,y)| \le L.$$

Lemma 2.2.1. When Assumption B holds for K then for $\forall T > 0$ the (MPS) has a unique solution

$$X_N(t) = (x_1(t), x_2(t), \dots, x_N(t)) \in \mathcal{C}^1([0,T];\mathbb{R}^{dN}).$$

and for any fixed $t \in [0,T]$ the map

$$X_N(t,\cdot) : \mathbb{R}^{dN} \to \mathbb{R}^{dN} \ :\ x \mapsto X_N(t,x)$$

is a bijection

In the introduction we saw that the empirical measure satisfies a partial differential equation

Definition 2.2.2 (PDE Problem). Let $\mu^N(t)$ be the empirical measure

$$\mu^N(t) \triangleq \frac{1}{N}\sum_{j=1}^{N} \delta_{x_i(t)}.$$

Then from the introduction we know that for $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ the empirical measure satisfies

$$\frac{d}{dt}\langle \mu^N(t), \varphi\rangle = \langle \mu^N(t), \nabla\varphi \cdot \mathcal{K}\mu^N(t)\rangle.$$

where

$$\mathcal{K}\mu^N(\cdot) = \int_{\mathbb{R}^d} K(\cdot, y) d\mu^N(y).$$

Idea. If $\mu^N \to \mu$ in some sense, then the limiting measure μ should also satisfy

$$\begin{cases} \partial_t \mu + \nabla \cdot (\mu \mathcal{K}\mu) = 0 \\ \mu^N(0) \to \mu_0 \end{cases}.$$

in the sense weak sense i.e. the "sense of distributions" which we define in the following section

2.3 A short introduction for Distributions

Definition 2.3.1. Let $\Omega \subset \mathbb{R}^d$ be an open subset then the space of test functions $\mathcal{D}(\Omega)$ consists of all the functions in $\mathcal{C}_0^\infty(\Omega)$ supplemented by the following convergence

We say $\varphi_m \to \varphi \in \mathcal{C}_0^\infty(\Omega)$ iff

1. There exists a compact set $\exists K \subset \Omega$ such that $\operatorname{supp} \varphi_m \subset K$ for $\forall m$
2. For all multi indices α it holds

$$\sup_K |\partial^\alpha \varphi_m - \partial^\alpha \varphi| \xrightarrow{m\to\infty} 0.$$

Remark. $\mathcal{D}(\Omega)$ is a linear space

Definition 2.3.2 (Multi-Index)**.** A multi-index $\alpha \in \mathbb{N}_0^n$ of length $|\alpha| = \sum_i \alpha_i$ for example $\alpha = (0,2,1) \in \mathbb{N}_0^3$ can be used to denote partial derivatives of higher order as such :

$$\partial^\alpha = \prod_i (\frac{\partial}{\partial x_i})^{\alpha_i}.$$

Definition 2.3.3 (Distribution)**.** The space of Distributions is denoted by $\mathcal{D}'(\Omega)$ and is the dual space of $\mathcal{D}(\Omega)$ i.e. it is the linear space of all continuous linear functions on $\mathcal{D}(\Omega)$

We say a functional $T : \mathcal{D}(\Omega) \to \mathbb{C}$ is continuous linear iff

1. $\langle T, \alpha\varphi_1 + \beta\varphi_2 \rangle = \alpha\langle T, \varphi_1\rangle + \beta\langle T, \varphi_2\rangle$
2. If $\varphi_m \to \varphi$ in $\mathcal{D}(\Omega)$ then $\langle T, \varphi_m\rangle \to \langle T, \varphi\rangle$

We can define several operations on the space of distributions but since most of them are not used in this Lecture we only define the multiplication with a smooth function

Definition 2.3.4. For a smooth function $f \in \mathcal{C}^\infty$ and a distribution $T \in \mathcal{D}'$ the product is defined as follows

$$\langle Tf, \varphi\rangle = \langle T, f\varphi\rangle \quad \forall \varphi \in \mathcal{D}.$$

Remark. Multiplication between two Distributions $T, F \in \mathcal{D}'$ is not well defined, instead the convolution of two Distributions is defined

Example. For functions $f \in L^1_{\text{loc}}(\Omega)$ we can define the associated distribution $T_f \in \mathcal{D}'(\Omega)$ is defined by

$$\langle T_f, \varphi\rangle = \int_\Omega f(x)\varphi(x)dx \quad \forall \varphi \in \mathcal{D}(\Omega).$$

and say $L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$

Similarly $L^p_{\text{loc}} \subset \mathcal{D}'(\Omega)$, using Hölder's inequality one obtains $L^p_{\text{loc}}(\Omega) \subset L^q_{\text{loc}}(\Omega)$ for $1 < q < p < \infty$

Remark. The support of a distribution is also well-defined

Theorem 2.3.1. L^1_{loc} functions are uniquely deterimed by distributions. More precisely for two functions $f, g \in L^1_{\text{loc}}(\Omega)$ if

$$\int_\Omega f\varphi dx = \int_\Omega g\varphi dx \quad \forall \varphi \in \mathcal{D}(\Omega).$$

then $f = g$ a.e. in Ω

Proof. This proof is left as an exercise □

Example. The set of probability density functions on $\mathbb{R}$ is a subset of $\mathcal{D}'(\mathbb{R})$. For any probability density function $P(x)$ the associated distribution $T_P \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle T_P, \varphi \rangle = \int_{\mathbb{R}} \varphi(x) P(x) dx \quad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Example. The set of measures $\mathcal{M}(\Omega)$ is a subset of $\mathcal{D}'(\Omega)$. For any $\mu \in \mathcal{M}(\Omega)$ the associated distribution T_μ is defined by

$$\langle T_\mu, \varphi \rangle = \int_\Omega \varphi(x) d\mu \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Example. An important example of a distribution which is not defined in the above way is the Delta distribution $\delta_y(x)$ (concentrated on $y \in \mathbb{R}^d$)

$$\langle \delta_y, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(x) d\delta_y(x) = \varphi(y) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

where

$$\delta_y(E) = \begin{cases} 1, & y \in E \\ 0, & y \notin E \end{cases}.$$

The empirical measure μ^N is actually given by using the Delta distribution

$$\mu^N(t) \triangleq \frac{1}{N} \sum_{j=1}^{N} \delta_{x_i(t)} \quad \langle \mu^N, \varphi \rangle = \frac{1}{N} \sum_{j=1}^{N} \varphi(x_i(t)).$$

We define the convergence for a sequence of distributions as follows

Definition 2.3.5. For a sequence of distributions $(T_m)_{m\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ we was it converges against a limit $T \in \mathcal{D}'(\Omega)$ iff

$$\langle T_m, \varphi \rangle \to \langle T, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Based on this convergence we give some examples in the approximation of $\delta_0(x)$

Example (Heat Kernel)**.** The heat kernel for $x \in \mathbb{R}$ and $t > 0$ is given by

$$f_t(x) = \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{|x|^2}{4t}}.$$

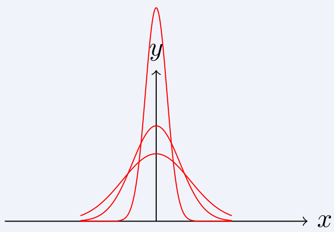


Figure 2.1: Heat Kernel for different t

Lemma 2.3.1. The sequence of distributions associated to the heat kernel converge to the Delta distribution

Proof. We consider the limit $t \to 0^+$ and obtain $\forall \varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\begin{aligned} \lim_{t\to 0^+} \int_{\mathbb{R}} f_t(x)\varphi(x) &= \lim_{t\to 0^+} \int_{\mathbb{R}} \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{|x|^2}{4t}} \varphi(x) \\ &= \lim_{t\to 0^+} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \varphi(2\sqrt{t}y) dy \\ &= \varphi(0) = \langle \delta_0, \varphi \rangle. \end{aligned}$$

where we used $x = 2\sqrt{t}y$ $\square$

Example. For the rectangular functions

$$Q_n(x) = \begin{cases} \frac{n}{2}, & |x| \leq \frac{1}{n} \\ 0, & |x| > \frac{1}{n} \end{cases}.$$

Then

$$Q_n \xrightarrow{n\to\infty} \delta_0(x).$$

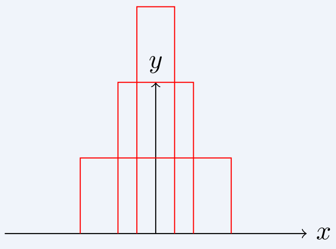


Figure 2.2: Rectangular functions for different n

Example. The Dirichlet kernel

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}} = 1 + 2\sum_{k=1}^{n}\cos(kx).$$

Then

$$D_n \xrightarrow{n\to\infty} 2\pi\delta_0(x).$$

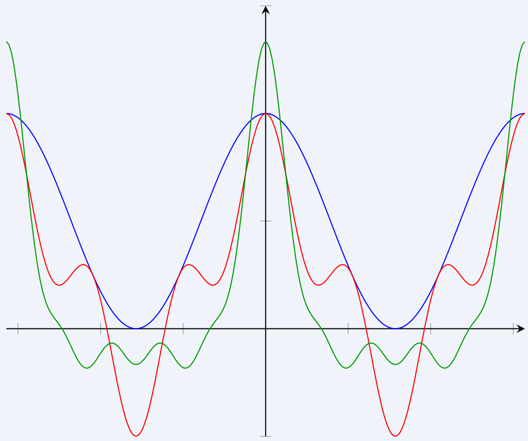


Figure 2.3: Dirichlet kernel for different n

2.4 Weak Derivative Of Distributions

Definition 2.4.1. For all distributions $\forall T \in \mathcal{D}'(\Omega)$ we define the derivative $\partial_i T$ by

$$\langle \partial_i T, \varphi\rangle \coloneqq -\langle T, \partial_i \varphi\rangle \quad \forall \varphi \in \mathcal{D}(\Omega)\langle \partial_i^\alpha T, \varphi\rangle \qquad \coloneqq (-1)^{|\alpha|}\langle T, \partial_i^\alpha \varphi\rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

For multi index α

Exercise. Prove the function $-\langle T, \partial_i \varphi\rangle$ is a continuous and linear function

Hint: Consider the case where $T \coloneqq T_f$ for $f \in L^1_{\text{loc}}$

We give a couple examples

Example. For $\forall \varphi \in \mathcal{D}(\Omega)$ the weak derivative of the Dirac Delta distribution is given by

$$\langle \delta_0', \varphi \rangle = -\langle \delta_0, \varphi' \rangle = -\varphi(0)$$
$$\langle \delta_0^{(k)}, \varphi \rangle = (-1)^k \varphi^{(k)}(0).$$

Lemma 2.4.1. The weak derivative of the 1-D Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}.$$

is the Dirac Delta distribution

Proof. For $\forall \varphi \in \mathcal{D}(\Omega)$ it holds

$$\begin{aligned} \langle H', \varphi \rangle &\overset{\text{Def.}}{=} -\langle H, \varphi' \rangle \\ &= -\int_{-\infty}^{\infty} H(x) \varphi'(x) dx \\ &= -\int_{0}^{\infty} \varphi'(x) dx \\ &= \varphi(0) \\ &= \langle \delta_0, \varphi \rangle. \end{aligned}$$

Therefore

$$H' = \delta_0.$$

□

We can now go on to properly formulate the mean field partial differential equation in a weak sense

2.5 Weak Formulation Of The Mean Field Partial Differential Equation

Using the notation of the empirical measure we can rewrite our earlier definition of the (MPS) as follows

$$\begin{cases} \frac{d}{dt} x_i(t) & = \langle K(x_i, \cdot), \mu^N(t, \cdot) \rangle = \int_{\mathbb{R}^d} K(x_i, y) d\mu^N(t, y) \\ x_i(0) & = x_{i,0} \in \mathbb{R}^d, t \in [0, T] \end{cases}.$$

As has been discussed before, the empirical measure satisfies for $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^D)$

$$\frac{d}{dt} \langle \mu^N, \varphi \rangle = \langle \mu^N \mathcal{K} \mu^N, \nabla \varphi \rangle = \langle -\operatorname{div}(\mu^N \mathcal{K} \mu^N), \varphi \rangle.$$

where

$$\mathcal{K} \mu^N(x) = \int_{\mathbb{R}^d} K(x, y) d\mu^N(y),$$

which means that the empirical measure μ^N satisfies the following equation in the sense of distribution

$$\text{(MPDE)} \quad \partial_t \mu^N + \operatorname{div}(\mu^N \mathcal{K} \mu^N) = 0.$$

Exercise. Show $\mu^N \mathcal{K} \mu^N$ is a distribution for smooth $K(x, y)$

Next we concentrate on the following PDE

Definition 2.5.1 (Mean Field Equation (MFE)). Define the mean field equation as

$$\text{(MFE)} \begin{cases} \partial_t + \operatorname{div}(\mu \mathcal{K} \mu) & = 0 \\ \mu|_{t=0} & = \mu_0 \end{cases}.$$

where

$$\mathcal{K}\mu^N(x) = \int_{\mathbb{R}^d} K(x, y) d\mu^N(y),$$

We give the definition of the weak solution of (MFE)

Definition 2.5.2 (Weak Solution of MFE). For all $t \in [0, T]$, $\mu(t) \in \mathcal{M}(\mathbb{R}^d)$ is called a weak solution of (MFE), where $\mathcal{M}(\mathbb{R}^d)$ denotes the space of measures on $\mathbb{R}^d$
For $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ it holds

$$\langle \mu(t), \varphi \rangle - \langle \mu_0, \varphi \rangle = \int_0^t \langle \mu(s) \mathcal{K} \mu(s), \nabla \varphi \rangle.$$

Remark. If $\mu_0 = \mu^N(0)$ i.e. the initial data is given by an empirical measure, then $\mu^N(t, \cdot)$ is a weak solution of the (MFE)

We define the following initial value problem, the so called characteristics equation

Definition 2.5.3 (Push Forwad Measure)**.** For a measurable function X and a measure $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ denote the push forward measure for any Borel set $B \subset \mathbb{R}^d$ by

$$X\#\mu_0 \coloneqq \mu_0(X^{-1}(B)).$$

Definition 2.5.4 (Characteristics equation)**.**

$$\begin{cases} \frac{d}{dt}x(t,x_0,\mu_0) & = \int_{\mathbb{R}^d} K(x(t,x_0,\mu_0),y)d\mu(y,t) \\ x(0,x_0,\mu_0) & = x_0 \quad \forall x_0 \in \mathbb{R}^d \\ \mu(\cdot,t) & = x(t,\cdot,\mu_0)\#\mu_0 \end{cases}.$$

The solution flow $x(t,\cdot,\mu_0)$ gives for any time $t > 0$ a map

$$x(t,\cdot,\mu_0) \ : \ \mathbb{R}^d \to \mathbb{R}^d.$$

Remark. It can be easily checked that the push forward measure $\mu(t)$ obtained in the Characteristics equation is a weak solution of the (MFE)

Remark. The solution space of the Characteristics equation is given by

$$\mathcal{P}_1(\mathbb{R}^d) = \{\mu \in \mathcal{P}(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |x| d\mu(x) < \infty\}.$$

where $\mathcal{P}(\mathbb{R}^d)$ is the space of all probability measures

Assumption C (Regularity)**.** We say an interaction force K is regular if $K \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^d)$ and there exists an $L > 0$ such that

$$\sup_y |\nabla_x K(x,y)| + \sup_x |\nabla_y K(x,y)| \leq L.$$

Actually this assumption has already been used in order to show the well-posedness of the particle system

Theorem 2.5.1 (Existence and Uniqueness of Characteristics Equation)**.** Let Assumption C hold for K and $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$ then the Characteristics equation has a unique solution $x(t, x_0, \mu_0) \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^d)$ and $x(t, \cdot, \mu_0)\#\mu_0 \in \mathcal{P}_1$ for $\forall t > 0$

Proof. The proof is based on Picard iteration.

Let $C_1 = \int_{\mathbb{R}^d} |x| d\mu_0(x)$ and define the following Banach space

$$X \coloneqq \{v \in \mathcal{C}(\mathbb{R}^d) \ \ \|v\|_X < \infty\}.$$

Where

$$\|v\|_X \coloneqq \sup_{x \in \mathbb{R}^d} \frac{|v(x)|}{1 + |x|}.$$

As preparations we need the following estimates for the nonlocal term, by using Assumption C for K we have for $\forall v, w \in X$

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} K(v(x), v(y)) d\mu_0(y) - \int_{\mathbb{R}^d} K(w(x), w(y)) d\mu_0(y) \right| \\
&\leq L \int_{\mathbb{R}^d} |v(x) - w(x)| + |v(y) - w(y)| d\mu_0(y) \\
&\leq L\|v - w\|_X (1 + |x|) + L\|v - w\|_X \int_{\mathbb{R}^d} (1 + |y|) d\mu_0(y) \\
&\leq L(2 + C_1)\|v - w\|_X (1 + |x|).
\end{aligned}$$

Now define the Picard iteration for $\forall y \in \mathbb{R}^d$

$$\begin{aligned}
x_0(t, y) &= y \\
x_1(t, y) &= y + \int_0^t \int_{\mathbb{R}^d} K(x_0(s, y), x_0(s, z)) d\mu_0(z) ds \\
&\vdots \\
x_m(t, y) &= y + \int_0^t \int_{\mathbb{R}^d} K(x_{m-1}(s, y), x_{m-1}(s, z)) d\mu_0(z) ds \\
&.
\end{aligned}$$

Then we can bound the difference between x_1 and x_0 by

$$\begin{aligned}
|x_1(t, y) - x_0(t, y)| &= \left| \int_0^t \int_{\mathbb{R}^d} K(x_0(s, y), x_0(s, z)) d\mu_0(z) ds \right| \\
&= \left| \int_0^t \int_{\mathbb{R}^d} K(y, z) d\mu_0(z) ds \right| \\
&\leq \int_0^{|t|} \int_{\mathbb{R}^d} L(|y| + |z|) d\mu_0(z) ds \\
&= \int_0^{|t|} L(|y| + C_1) ds \\
&\leq L(1 + C_1)(1 + |y|)|t|.
\end{aligned}$$

Furthermore for $\forall m \geq 1$ we have

$$\begin{aligned}
&|x_m(t,y) - x_{m-1}(t,y)| \\
&= \left| \int_0^t \int_{\mathbb{R}^d} \left(K(x_{m-1}(s,y), x_{m-1}(s,z)) - K(x_{m-2}(s,y), x_{m-2}(s,z))\right) d\mu_0(z) ds \right| \\
&\leq L(2+C_1) \int_0^{|t|} \|x_{m-1}(s,\cdot) - x_{m-2}(s,\cdot)\|_X (1+|y|) ds.
\end{aligned}$$

hence by dividing both sides by $1+|y|$ we have

$$\begin{aligned}
\|x_m(t,\cdot) - x_{m-1}(t,\cdot)\|_X &\leq L(2+C_1) \int_0^{|t|} \|x_{m-1}(s,\cdot) - x_{m-2}(s,\cdot)\|_X ds \\
&\leq \frac{((2+C_1)L|t|)^d}{(m-1)!}.
\end{aligned}$$

which implies for $\forall m > n \to \infty$

$$\|x_m(t,\cdot) - x_n(t,\cdot)\|_X \leq \sum_{i=n}^{m-1} \|x_{i+1}(t,\cdot) - x_i(t,\cdot)\|_X \to 0.$$

Therefore for $T > 0$

$$x_m(t,\cdot) \to x(t,\cdot) \text{ in } X \text{ uniformly in } [-T,T].$$

and $x \in \mathcal{C}(\mathbb{R};\mathbb{R}^d)$ satisfies that, after taking the limit in Picard iteration $\forall y \in \mathbb{R}^d$

$$x(t,y) = y + \int_0^t \int_{\mathbb{R}^d} K(x(s,y), x(s,z)) d\mu_0(z) ds.$$

By the fundamental theorem of calculus and Assumption C we know that for $y \in \mathbb{R}^d$ and $x(t,y) \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^d)$

$$\frac{d}{dt} x(t,y) = \int_{\mathbb{R}^d} K(x(t,y), x(t,z)) d\mu_0(z) = \int_{\mathbb{R}^d} K(x(t,y), z') d\mu(z',t).$$

where $\mu(\cdot,t)$ is the push forward measure of μ_0 along $x(t,\cdot)$
For uniqueness consider two solutions $x, \tilde{x}$ then by taking the difference we have

$$x(t,y) - \tilde{x}(t,y) = \int_0^t \int_{\mathbb{R}^d} \left(K(x(s,y), x(s,z)) - K(\tilde{x}(s,y), \tilde{x}(s,z))\right) d\mu_0(z) ds.$$

Using estimates similarly to before we obtain

$$\|x(t,\cdot) - \tilde{x}(t,\cdot)\|_X \leq L(2+C_1) \int_0^{|t|} \|x(s,\cdot) - \tilde{x}(s,\cdot)\|_X ds.$$

By applying Gronwall's inequality we get

$$\|x(t,\cdot) - \tilde{x}(t,\cdot)\|_X = 0.$$

where clearly $\|x(0,\cdot) - \tilde{x}(0,\cdot)\|_X = 0$ $\square$

2.5.1 Stability

Let's remind us of the N-particle system (MPS), the Mean field equation (MFE) and its weak solution as defined in Definition 2.5.2. We have thus far done the following things

1. If $\mu_0 = \mu_N(0)$ then $\mu_N(t)$ is a weak solution of (MFE)
2. If $\mu_0 = \mathcal{P}_1(\mathbb{R}^d)$ and the assumption on Regularity hold for K,then

$x(t, \cdot, \mu_0)\#\mu_0 \in \mathcal{P}_1$ is the solution of (MFE)

We will prove the stability of the mean field PDE, which means directly that

$$\mu_N(0) \to \mu(0) \Rightarrow \mu_N(t) \to \mu(t).$$

by using the so called Monge-Kantorovich distance (or Wasserstein distance)

Definition 2.5.5 (Monge-Kantorovich Distance). For two measures $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ $p \geq 1$ with

$$\mathcal{P}_p(\mathbb{R}^d) = \{\mu \in \mathcal{P}(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |x|^p d\mu(x) < \infty\}.$$

the Monge-Kantorovich distance $\mathrm{dist}_{\mathrm{MK},p}(\mu, \nu)$ or $W^p(\mu, \nu)$ is defined by

$$\operatorname*{dist}_{\mathrm{MK},p}(\mu, \nu) = W^p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p d\pi(x,y) \right)^{\frac{1}{p}}.$$

where

$$\Pi(\mu, \nu) = \{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} \pi(\cdot, dy) = \mu(\cdot) \text{ and } \int_{\mathbb{R}^d} \pi(dx, \cdot) = \nu(\cdot)\}.$$

Remark. For $\forall \varphi, \psi \in \mathcal{C}(\mathbb{R}^d)$ such that $\varphi(x) \sim O(|x|^p)$ for $|x| \gg 1$ and $\psi(y) \sim O(|y|^p)$ for $|y| \gg 1$, for $\pi \in \Pi(\mu, \nu)$ it holds

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\varphi(x) + \psi(y)) d\pi(x,y) = \int_{\mathbb{R}^d} \varphi(x) + d\mu(x) + \int_{\mathbb{R}^d} \psi(y) d\nu(y).$$

Remark (Kantorovich-Rubinstein duality). It can be shown that the W^1 distance can be computed by

$$\operatorname*{dist}_{\mathrm{MK},1}(\mu, \nu) = W^1(\mu, \nu) = \sup_{\varphi \in \mathrm{Lip}(\mathbb{R}^d), \mathrm{Lip}(\varphi) \leq 1} \left| \int_{\mathbb{R}^d} \varphi(x) d\mu(x) - \int_{\mathbb{R}^d} \varphi(x) d\nu(x) \right|.$$

Theorem 2.5.2 (Dobrushin's stability). Let $\mu_0, \overline{\mu}_0 \in \mathcal{P}_1(\mathbb{R}^d)$ and $(x(t, \cdot, \mu_0), \mu(\cdot, t))$, $(x(t, \cdot, \overline{\mu}_0), \overline{\mu}_0(\cdot, t))$ be solutions of Theorem 2.5.1. Then $\forall t > 0$ it hold

$$\operatorname*{dist}_{\mathrm{MK},1}(\mu(\cdot, t), \overline{\mu}(\cdot, t)) \leq e^{2|t|L} \operatorname*{dist}_{\mathrm{MK},1}(\mu_0, \overline{\mu}_0).$$

Proof. Let (x_0, μ_0) and $(\overline{x}_0, \overline{\mu}_0)$ be two initial data pairs of problem Theorem 2.5.1 and $\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)$ taking the difference of these two problems, we have

$$\begin{aligned} & x(t, x_0, \mu_0) - x(t, \overline{x}_0, \overline{\mu}_0) \\ & = x_0 - \overline{x}_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s, x_0, \mu_0), y) d\mu(s, y) ds \\ & \qquad - \int_0^t \int_{\mathbb{R}^d} K(x(s, \overline{x}_0, \overline{\mu}_0), y) d\overline{\mu}(s, y) ds. \end{aligned}$$

where $\mu(\cdot,t) = x(t,\cdot,\mu_0)\#\mu_0$ and $\overline{\mu}(\cdot,t) = x(t,\cdot,\overline{\mu}_0)\#\overline{\mu}_0$. Now we compute further and get

$$\begin{aligned}
&x(t,x_0,\mu_0) - x(t,\overline{x}_0,\overline{\mu}_0)\\
&= x_0 - \overline{x}_0 + \int_0^t \int_{\mathbb{R}^d} K(x(s,x_0,\mu_0), x(s,z,\mu_0)) d\mu_0(z) ds\\
&\qquad - \int_0^t \int_{\mathbb{R}^d} K(x(s,\overline{x}_0,\overline{\mu}_0), x(s,\overline{z},\overline{\mu}_0)) d\overline{\mu}_0(\overline{z}) ds\\
&= x_0 - \overline{x_0} + \int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} (K(x(s,x_0,\mu_0), x(s,z,\mu_0))\\
&\qquad - K(x(s,\overline{x}_0,\overline{\mu}_0), x(s,\overline{z},\overline{\mu}_0)))\, d\pi_0(z,\overline{z}) ds.
\end{aligned}$$

There for by assumption on Regularity for K, we have

$$\begin{aligned}
&|x(t,x_0,\mu_0) - x(t,\overline{x}_0,\overline{\mu}_0)|\\
&\leq |x_0 - \overline{x}_0| + L\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,x_0,\mu_0) - x(s,\overline{x}_0,\overline{\mu}_0)|\\
&\qquad\qquad + |x(s,z,\mu_0) - x(s,\overline{z},\overline{\mu}_0)| d\pi_0(z,\overline{z}) ds\\
&\\
&\leq |x_0 - \overline{x}_0| + L\int_0^t |x(s,x_0,\mu_0) - x(s,\overline{x}_0,\overline{\mu}_0)| ds\\
&\qquad + L\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,z,\mu_0) - x(s,\overline{z},\overline{\mu}_0)| d\pi_0(z,\overline{z}) ds.
\end{aligned}$$

Next we integrate both sides in $x_0, \overline{x}_0$ with respect to the measure π_0

$$\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(t,x_0,\mu_0) - x(t,\overline{x}_0,\overline{\mu}_0)| d\pi_0(x_0,\overline{x}_0)\\
&\leq \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x_0 - \overline{x}_0| d\pi_0(x_0,\overline{x}_0)\\
&+ L\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,x_0,\mu_0) - x(s,\overline{x}_0,\overline{\mu}_0)| d\pi_0(x_0,\overline{x}_0) ds\\
&+ L\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,z,\mu_0) - x(s,\overline{z},\overline{\mu}_0)| d\pi_0(z,\overline{z}) ds\\
&.
\end{aligned}$$

By denoting

$$D[\pi_0](t) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x(s,z,\mu_0) - x(s,\overline{z},\overline{\mu}_0)| d\pi_0(z,\overline{z}).$$

we have obtained the estimate

$$D[\pi_0](t) \leq D[\pi_0](0) + 2L\int_0^t D[\pi_0](s) ds.$$

which implies by Gronwall's inequality that

$$D[\pi_0](t) \leq D[\pi_0](0) e^{2Lt}.$$

Now let $\varphi_t : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d$ be the map such that

$$\varphi_t(x_0,\overline{x}_0) = (x(t,x_0,\mu_0), x(t,\overline{x}_0,\overline{\mu}_0)).$$

and for arbitrary $\pi_0 \in \Pi(\mu_0, \nu_0)$, $\pi_t := \varphi_t \# \pi_0$ be the push forward measure of π_0 by φ_t. It is obvious that

$$\pi_t = \varphi_t \# \pi_0 \in \Pi(\mu(\cdot,t), \overline{\mu}(\cdot,t)).$$

Therefore

$$\begin{aligned}
\operatorname*{dist}_{\mathrm{MK},1}(\mu(\cdot,t), \overline{\mu}(\cdot,t)) &= \inf_{\pi \in \Pi(\mu(\cdot,t),\overline{\mu}(\cdot,t))} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |z - \overline{z}| d\pi(z, \overline{z}) \\
&\le \inf_{\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x(t, z, \mu_0) - x(t, \overline{z}, \overline{\mu}_0)| d\pi(z, \overline{z}) \\
&= \inf_{\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)} D[\pi_0](t) \\
&\le \inf_{\pi_0 \in \Pi(\mu_0, \overline{\mu}_0)} D[\pi_0](0) e^{2Lt} \\
&= e^{2Lt} \operatorname*{dist}_{\mathrm{MK},1}(\mu_0, \overline{\mu}_0).
\end{aligned}$$

□

2.6 Mollification Operator

Definition 2.6.1 (Mollification-Kernel)**.** A function $j(x) \in \mathcal{C}_0^\infty$ is called a mollification kernel if it satisfies the following properties

1. $j(x) \ge 0$
2. $\operatorname{supp} j \subset \overline{B_1(0)}$
3. $\int_{\mathbb{R}^d} j(x) dx = 1$

A typical example of a smooth kernel is given by

Example.

$$j(x) = \begin{cases} k \exp(-\frac{1}{1-|x|^2}) & \text{if } |x| < 1 \\ 0 & \text{if otherwise} \end{cases}.$$

where k is given s.t the integral is 1

Remark. Based on the given function j it is easy to prove that its rescaled sequence converges to the Dirac Delta distribution in the weak sense

$$j_\varepsilon(x) = \frac{1}{\varepsilon^d} j(\frac{x}{\varepsilon}) \xrightarrow{\varepsilon \to 0} \delta_0.$$

Exercise. Prove that for $\varphi(x) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ it holds that $\forall x \in \mathbb{R}^d$

$$\lim_{\varepsilon \to 0} j_\varepsilon \star \varphi(x) = \varphi(x).$$

Definition 2.6.2 (Mollification Operator)**.** For $\forall u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we define the following function as its mollification

$$J_\varepsilon(u)(x) \triangleq j_\varepsilon(x) \star u(x) = \int_{\mathbb{R}^d} j_\varepsilon(x - y) u(y) dy.$$

where J_ε is called the mollification operator

Remark. Notice that $\operatorname{supp} j_\varepsilon(x) \subset \overline{B_\varepsilon(0)}$ we obtain

$$J_\varepsilon(u)(x) = \int_{B_\varepsilon(0)} j_\varepsilon(x-y)u(y)dy < \infty.$$

Lemma 2.6.1.

1. If $u(x) \in L^1(\mathbb{R}^d)$ and $\operatorname{supp} u(x)$ is compact in $\mathbb{R}^d$ then

$$J_\varepsilon(u) = j_\varepsilon \star u \in \mathcal{C}_0^\infty \quad \forall \varepsilon > 0.$$

2. if $u \in C_0(\mathbb{R}^d)$ then

$$J_\varepsilon(u) \xrightarrow{\varepsilon \to 0} u \text{ uniformly on } \operatorname{supp} u.$$

Proof. 1. Let $K = \operatorname{supp} u \subset \mathbb{R}^d$ be compact , then we have

$$\operatorname{supp} j_\varepsilon \star u = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, K) \leq \varepsilon\}.$$

is also compact. For the differentiability it is enough to show the first order partial differentiability at any given point, as the argument for higher order differentiability is analog Now for $\forall x \in \operatorname{supp} j_\varepsilon \star u$ we have that $\forall i = 1, 2, \dots, d$

$$\frac{\partial}{\partial x_i} \int_{\mathbb{R}^d} j_\varepsilon(x-y)u(y)dy = \int_K \frac{\partial}{\partial x_i} j_\varepsilon(x-y)u(y)dy.$$

where we have used the fact that

$$|\frac{\partial}{\partial x_i} j_\varepsilon(x-y)u(y)| \leq |\frac{\partial}{\partial x_i} j_\varepsilon(x-y)| \|u\|_{L^1} \leq \frac{Cj'}{\varepsilon^d}.$$

to show the uniform integrability of $\frac{\partial}{\partial x_i} j_\varepsilon(x-y)u(y)$

For (2) we need to prove that for $u \in \mathcal{C}_0(\mathbb{R}^d)$ it holds

$$\|J_\varepsilon(u) - u\|_{L^\infty(\operatorname{supp} u)} \xrightarrow{\varepsilon \to 0} 0.$$

Actually $\forall x \in \operatorname{supp} u$ we have the following estimate

$$\begin{aligned} |j_\varepsilon \star u(x) - u(x)| &= \left| \int_{\mathbb{R}^d} j_\varepsilon(x-y)(u(y)-u(x))dy \right| \\ &= \left| \int_{\operatorname{supp} u} j_\varepsilon(x-y)(u(y)-u(x))dy \right| \\ &\leq \max_{\substack{x,y \in \operatorname{supp} u \\ |x-y|<\varepsilon}} |u(y)-u(x)| \int_{\mathbb{R}^d} j_\varepsilon(x-y)dy \xrightarrow{\varepsilon \to 0} 0. \end{aligned}$$

where we have used the fact that $u \in \mathcal{C}(\operatorname{supp} u)$ which means u is uniformly continuous to obtain the limit in the last step above □

2.6.1 Conservation of Mass

Let $u(t)$ be the push forward measure obtained from Definition 2.5.4, one can check that is is a weak solution of (MFE) by using test functions. Furthermore we obtain that if the initial measure has a probability density, then the solution is also integrable for any fixed time t

Corollary. Let f_0 be a probability density of μ_0 on $\mathbb{R}^d$ with

$$\int_{\mathbb{R}^d} |x| f_0(x) dx < \infty.$$

Then the Cauchy problem

$$\begin{cases} \partial_t f + \nabla \cdot (f \mathcal{K} f) & = 0 \\ f|_{t=0} & = f_0 \end{cases}.$$

has a unique weak solution $f(t, \cdot) \in L^1(\mathbb{R}^d)$ and $\|f(t, \cdot)\|_{L^1(\mathbb{R}^d)} = 1$. The weak solution in the sense of distribution means that $\forall \varphi \in \mathcal{C}_0^\infty$ it holds for all $0 \leq \tilde{t} < t < \infty$

$$\int_{\mathbb{R}^d} \varphi(x) f(t, x) dx - \int_{\mathbb{R}^d} \varphi(x) f(\tilde{t}, x) dx = \int_{\tilde{t}}^t \int_{\mathbb{R}^d} f(s, x) \mathcal{K} f(s, x) dx ds.$$

Proof. We need to prove $\forall t \in \mathbb{R}$ and $\mu_t \in \mathcal{P}_1(\mathbb{R}^d)$ absolutely continuous with respect to the Lebesgue measure i.e. $\forall B \in \mathcal{B}$ and $\int_B d\lambda = 0$ it holds $\mu_t(B) = 0$. The mass conservation property, $\|f(t, \cdot)\|_{L^1(\mathbb{R}^d)} = 1$ comes from the definition of probability measures $\square$

Exercise. Let $\mu_t \in \mathcal{P}_1(\mathbb{R}^d)$ be an absolutely continuous measure with respect to the Lebesgue measure, then proof that for $\forall B \in \mathcal{B}$ such that

$$\int_B d\lambda = 0.$$

it holds that $\mu_t(B) = 0$

In the next we give an alternative proof of the conservation of mass without using the characteristics presentation and instead only use the definition of a weak solution

Proof. In the weak solution formulation it holds

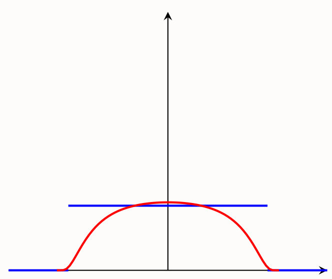
$$\int_{\mathbb{R}^d} \varphi(x) f(t, x) dx = \int_{\mathbb{R}^d} \varphi(x) f(\tilde{t}, x) dx + \int_{\tilde{t}}^t \iint_{\mathbb{R}^{2d}} f(s, x) K(x, y) f(s, y) \nabla \varphi(x) dx dy ds.$$

where the test function $\varphi \in \mathcal{C}_0^\infty$ is chosen arbitrarily. Now we take a sequence of test functions defined as follows.

For $\forall R > 0$

$$\varphi_R(x) = \begin{cases} 1, & |x| \leq R \\ \text{smooth}, & R < |x| < 2R, \\ 0, & |x| \geq R \end{cases}.$$

An example of this is the mollification of a step function i.e. $\varphi_R = j_{\frac{R}{2}} \cdot \mathbb{1}_{B_{\frac{3R}{2}}}$



One obtains directly for the gradient estimate $|\nabla\varphi_R(x)| \leq \frac{C}{R}$. Therefore with this test function, we obtain from the weak solution formula that

$$\begin{aligned}\left|\int_{\mathbb{R}^d} f(t,x)\varphi_R(x)dx - \int_{\mathbb{R}^d} f(\tilde{t},x)\varphi_R(x)dx\right| &= \left|\int_{\tilde{t}}^t \iint_{\mathbb{R}^{2d}} f(s,x)K(x,y)f(s,y)\nabla\varphi_R(x)dxdyds\right| \\ &\leq \frac{CL}{R}\int_{\tilde{t}}^t \iint_{\mathbb{R}^{2d}} (1+|x|+|y|)f(s,x)f(s,y)|\nabla\varphi_R(x)|dxdyds \\ &\leq \frac{C}{R}|t-\tilde{t}|.\end{aligned}$$

Where C depends on $\|(1+|\cdot|)f(t,\cdot)\|_{L^1(\mathbb{R}^d)}$. Since

$$|f(t,x)\varphi_R(x)| \leq |f(t,x)| \quad \forall x \in \mathbb{R}^d.$$

we can use the dominant convergence theorem to obtain

$$\int_{\mathbb{R}^d} f(t,x)\varphi_R(x)dx \xrightarrow{R\to\infty} \int_{\mathbb{R}^d} f(t,x)dx > 0.$$

Therefore passing to the limit $R \to \infty$ we have

$$\int_{\mathbb{R}^d} f(t,x)dx = \int_{\mathbb{R}^d} f_0(x)dx.$$

□

2.7 Mean Field Limit

Theorem 2.7.1 (Mean Field Limit). For $f_0 \in L^1(\mathbb{R}^d)$, let $\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_{i,0}}$ such that

$$\operatorname*{dist}_{\mathrm{MK},1}(\mu_0^N, f_0) \xrightarrow{N\to\infty} 0.$$

Let $X_N(t)$ be the solution of the N particle system (MPS) with its empirical measure

$$\mu^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t,X_{N,0})}.$$

Then

$$\operatorname*{dist}_{\mathrm{MK},1}(\mu^N(t), f(t,\cdot)) \leq e^{2Lt} \operatorname*{dist}_{\mathrm{MK},1}(\mu_0^N, f_0) \xrightarrow{N\to\infty} 0.$$

And $\mu^N(t) \rightharpoonup f(t,\cdot)$ weakly in measures , i.e for $\forall\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ it holds

$$\int_{\mathbb{R}^d} \varphi(x)d\mu^N(t,x) \xrightarrow{N\to\infty} \int_{\mathbb{R}^d} \varphi(x)f(t,x)dx.$$

Proof. The stability result from Theorem 2.5.2 gives us already the convergence rate estimate. We are left to prove the weak convergence in measure. Note $\forall\varphi \in \mathrm{Lip}(\mathbb{R}^d)$ we have

$$\begin{aligned}\left|\int_{\mathbb{R}^d} \varphi(x)d\mu^N(t,x) - \int_{\mathbb{R}^d} \varphi(x)f(t,x)dx\right| &= |\iint_{\mathbb{R}^d\times\mathbb{R}^d} (\varphi(x)-\varphi(y))d\pi_t(x,y)| \\ &\leq \mathrm{Lip}(\varphi)\iint_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|d\pi_t(x,y) \\ &\to 0.\end{aligned}$$

where $\pi_t \in \Pi(\mu^N(t), f(t,\cdot))$

Since $\mathrm{Lip}(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$ and because the total mass is 1, the above also holds for test functions in $\mathcal{C}_b(\mathbb{R}^d)$. Hence the weak convergence in measure is true. The fact that $\mathrm{Lip}(\mathbb{R}^d)$ is dense in $\mathcal{C}_0$ can be obtained by using the mollification operator introduced in Definition 2.6.2. More precisely we have to show that $\forall \varphi \in C_b^\infty$ it holds

$$\int_{\mathbb{R}^d} \varphi(x) d\mu^N(t,x) \xrightarrow{N\to\infty} \int_{\mathbb{R}^d} \varphi(X) f(t,x) dx.$$

Notice we have shown that the above convergence holds for all $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$.

For $\forall \varphi \in \mathcal{C}_b^\infty$ and $\forall \varepsilon > 0$ we choose $R > 1$ s.t.

$$\frac{2\|\varphi\|_{L^\infty(\mathbb{R}^d)} M_1}{R} \leq \frac{\varepsilon}{2}.$$

where $M_1 = \int_{\mathbb{R}^d} |x| d\mu^N(t,x)$. Let $\varphi_m \in \mathcal{C}_0^\infty(B_{2R})$ be the approximation of φ on $B_{\frac{3R}{2}}$. This means that $\exists M >)$ such that for $\forall m > M$ it holds

$$\|\varphi_m - \varphi\|_{L^\infty(B_R)} < \frac{\varepsilon}{4}.$$

Now we take $\varphi_{M+1} \in \mathcal{C}_0^\infty(B_{2R})$ which is obviously Lipschitz continuous. Therefore the convergence holds. Then $\exists N_1 > 0$ such that $\forall N > N_1$ we have

$$\left| \int_{\mathbb{R}^d} \varphi_{M+1}(x)(d\mu^N(t,x) - f(t,x))dx \right| < \frac{\varepsilon}{4}.$$

To summarize we obtain that

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} \varphi(x) d\mu^N(t,x) - \int_{\mathbb{R}^d} \varphi(x) f(t,x) dx \right| &\leq \left| \int_{B_R} \varphi(x)(d\mu^N(t,x) - f(t,x)dx) \right| \\
&+ \left| \int_{B_R^c} \varphi(x)(d\mu^N(t,x) - f(t,x)dx) \right| \\
&\leq \left| \int_{B_R} \varphi_{M+1}(x)(d\mu^N(t,x) - f(t,x)dx) \right| \\
&+ \left| \int_{B_R} (\varphi_{M+1}(x) - \varphi(x))(d\mu^N(t,x) - f(t,x)dx) \right| \\
&+ \left| \int_{B_R^c} |\varphi(x)| \frac{|x|}{R} (d\mu^N(t,x) + f(t,x)dx) \right| \\
&< \frac{\varepsilon}{4} + \|\varphi_{M+1} - \varphi\|_{L^\infty(B_R)} + \frac{2}{R} \|\varphi\|_{L^\infty(\mathbb{R}^d)} M_1 \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

□

This concludes the chapter on the Mean-field Limit in the deterministic setting, we have thus far reviewed the basics of relevant ODE Theory, introduced the Mean-Field particle system (MPS) and the associated Mean-Field equation (MFE) and finished by proving a convergence result for the Mean-Field Limit

Chapter 3

MEAN FIELD LIMIT FOR SDE SYSTEM

3.1 Basic On Probability Theory

This section is dedicated to a small review of basic concepts in probability theory in preparations of SDE's

Definition 3.1.1 (σ-Algebra)**.** Let Ω be a given set, then a $\sigma-$algebra $\mathcal{F}$ on Ω is a family of subsets of Ω s.t.

1. $\emptyset \in \mathcal{F}$
2. $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$
3. If $A_1, A_2, \ldots \in \mathcal{F}$ countable, then

$$A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}.$$

Definition 3.1.2 (Measure Space)**.** A tuple $(\Omega, \mathcal{F})$ is called a measurable space. The elements of $\mathcal{F}$ are called measurable sets

Definition 3.1.3 (Probability Measure)**.** A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function

$$\mathbb{P} \ : \ \mathcal{F} \to [0, 1].$$

s.t.

1. $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$
2. If $A_1, A_2, \ldots \in \mathcal{F}$ s.t. $A_i \cap A_j = \emptyset \ \forall i \neq j$ then

$$\mathbb{P}(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mathbb{P}(A_j).$$

Definition 3.1.4 (Probability Space)**.** The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. $F \in \mathcal{F}$ is called event. We say the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, if $\mathcal{F}$ contains all zero-measure sets i.e. if

$$\inf\{\mathbb{P}(F) \ : \ F \in \mathcal{F}, G \subset F\} = 0.$$

then $G \in \mathcal{F}$ and $\mathbb{P}(G) = 0$. Without loss of generality we use in this lecture $(\Omega, \mathcal{F}, \mathbb{P})$ as complete probability space

Definition 3.1.5 (Almost Surely). If for some $F \in \mathcal{F}$ it holds $\mathbb{P}(F) = 1$ the we say that F happens with probability 1 or almost surely (a.s.)

Remark. Let $\mathcal{H}$ be a family of subsets of Ω, then there exists a smallest $\sigma-$algebra of Ω called $\mathcal{U}_{\mathcal{H}}$ with

$$\mathcal{U}_{\mathcal{H}} = \bigcap_{\substack{\mathcal{H} \subset \mathcal{U} \\ \mathcal{H}\ \sigma-\text{alg.}}} \mathcal{H}.$$

Example. The $\sigma-$algebra generated by a topology τ of Ω , $\mathcal{U}_\tau \triangleq \mathcal{B}$ is called the Borel $\sigma-$algebra, the elements $B \in \mathcal{B}$ are called Borel sets.

Definition 3.1.6 (Measurable Functions). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, a function

$$Y \ : \ \Omega \to \mathbb{R}^d.$$

is called measurable if and only if

$$Y^{-1}(B) \in \mathcal{F}.$$

holds for all $B \in \mathcal{B}$ or equivalent for all $B \in \tau$

Example. Let $X : \Omega \to \mathbb{R}^d$ be a given function, then the $\sigma-$algebra $\mathcal{U}(X)$ generated by X is

$$\mathcal{U}(X) = \{X^{-1}(B) \ : \ B \in \mathcal{B}\}.$$

Lemma 3.1.1 (Doob-Dynkin). If $X, Y \ : \ \Omega \to \mathbb{R}^d$ are given then Y is $\mathcal{U}(X)$ measurable if and only if there exists a Boreal measurable function $g \ : \ \mathbb{R}^d \to \mathbb{R}^d$ such that

$$Y = g(x).$$

Exercise. Proof the above lemma

From now on we denote $(\Omega, \mathcal{F}, \mathbb{P})$ as a given probability space.

Definition 3.1.7 (Random Variable). A random variable $X \ : \ \Omega \to \mathbb{R}^d$ is a $\mathcal{F}-$measurable function. Every random variable induces a probability measure or $\mathbb{R}^d$

$$\mu_X(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathcal{B}.$$

This measure is called the distribution of X

Definition 3.1.8 (Expectation and Variance). Let X be a random variable, if

$$\int_\Omega |X(\omega)| d\mathbb{P}(\omega) < \infty.$$

then

$$\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}^d} x d\mu_X(x).$$

is called the expectation of X (w.r.t. $\mathbb{P}$)

$$\mathbb{V}[X] = \int_\Omega |X - \mathbb{E}[X]|^2 d\mathbb{P}(\omega).$$

is called variance and there exists the simple relation

$$\mathbb{V}[X] = \mathbb{E}[|X - \mathbb{E}[X]|^2] = \mathbb{E}[|X|^2] - \mathbb{E}[X]^2.$$

Remark. If $f : \mathbb{R}^d \to \mathbb{R}$ measurable and

$$\int_\Omega |f(X(\omega))| d\mathbb{P}(\Omega) < \infty.$$

then

$$\mathbb{E}[f(x)] = \int_\Omega f(X(\omega)) d\mathbb{P}(\omega) = \int_{\mathbb{R}^d} f(x) d\mu_X(x).$$

Definition 3.1.9 (L^p spaces)**.** Let $X : \Omega \to \mathbb{R}^d$ be a random variable and $p \in [1, \infty)$. With the norm

$$\|X\|_p = \|X\|_{L^p(\mathbb{P})} = \left(\int_\Omega |X(\omega)|^p d\mathbb{P}(\omega) \right)^{\frac{1}{p}}.$$

If $p = \infty$

$$\|X\|_\infty = \inf\{N \in \mathbb{R} : |X(\omega)| \leq N \text{ a.s.}\}.$$

the space $L^p(\mathbb{P}) = L^p(\Omega) = \{X \ : \ \Omega \to \mathbb{R}^d \mid \|X\|_p \leq \infty\}$ is a Banach space.

Remark. If $p = 2$ then $L^2(\mathbb{P})$ is a Hilbert space with inner product

$$\langle X, Y \rangle = \mathbb{E}[X(\omega) \cdot Y(\Omega)] = \int_\Omega X(\omega) \cdot Y(\omega) d\mathbb{P}(\omega).$$

Definition 3.1.10 (Distribution Functions)**.** Note for $x, y \in \mathbb{R}^d$ we write $x \leq y$ if $x_i \leq y_i$ for $\forall i$

1. $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^d$ is a random variable the ints distribution function $F_x : \mathbb{R}^d \to [0, 1]$ is defined by

$$F_X(x) = \mathbb{P}(X \leq x) \quad x \in \mathbb{R}^d.$$

2. If $X_1, \ldots, X_m : \Omega \to \mathbb{R}^d$ are random variables, their joint distribution function is

$$\begin{aligned} &F_{X_1,\ldots,X_m} : (\mathbb{R}^d)^m \to [0, 1] \\ &F_{X_1,\ldots,X_M} = \mathbb{P}(X_1 \leq x_1, \ldots, X_m \leq x_m) \quad \forall x_i \in \mathbb{R}^d. \end{aligned}$$

Definition 3.1.11 (Density Function Of X)**.** If there exists a non-negative function $f(x) \in$

$L^1(\mathbb{R}^d;\mathbb{R})$ such that

$$F(x) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} f(y)dy \quad y = (y_1, \dots, y_n).$$

then f is called density function of X and

$$\mathbb{P}(X^{-1}(B)) = \int_B f(x)dx \quad \forall B \in \mathcal{B}.$$

Example. Let X be random variable with density function $x \in \mathbb{R}$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2} e^{-\frac{|x-m|^2}{2\sigma^2}}}.$$

then we say that X has a Gaussian (or Normal) distribution with mean m and variance σ^2 and write

$$X \sim \mathcal{N}(m, \sigma^2).$$

Obviously

$$\int_{\mathbb{R}} xf(x)dx = m \qquad \int_{\mathbb{R}} |x-m|^2 f(x)dx = \sigma^2.$$

Definition 3.1.12 (Independent Events)**.** Events $A_1, \dots, A_n \in \mathcal{F}$ are called independet if $\forall 1 \le k_1 < \dots < k_m \le n$ it holds

$$\mathbb{P}(A_{k_1} \cap A_{k_2} \cap \dots \cap A_{k_m}) = \mathbb{P}(A_{k_1})\mathbb{P}(A_{k_2}) \dots \mathbb{P}(A_{k_m}).$$

Definition 3.1.13 (Independent $\sigma-$Algebra)**.** Let $\mathcal{F}_j \subset \mathcal{F}$ be $\sigma-$algebras for $j = 1, 2, \dots$. Then we say $\mathcal{F}_j$ are independet if for $\forall 1 \le k_1 < k_2 < \dots < k_m$ and $\forall A_{k_j} \in \mathcal{F}_{k_j}$ it holds

$$\mathbb{P}(A_{k_1} \cap A_{k_2} \cap \dots \cap A_{k_m}) = \mathbb{P}(A_{k_1})\mathbb{P}(A_{k_2}) \dots \mathbb{P}(A_{k_m}).$$

Definition 3.1.14 (Independent Random Variables)**.** We say random variables $X_1, \dots, X_m : \Omega \to \mathbb{R}^d$ are independet if for $\forall B_1, \dots, B_m \subset \mathcal{B}$ in $\mathbb{R}^d$ it holds

$$\mathbb{P}(X_{j_1} \in B_{j_1}, \dots, X_{j_k} \in B_{j_k}) = \mathbb{P}(X_{j_1} \in B_{j_1}) \dots \mathbb{P}(X_{j_k} \in B_{j_k}).$$

which is equivalent to proving that $\mathcal{U}(X_1), \dots, \mathcal{U}(X_k)$ are independet

Theorem 3.1.1. $X_1, \dots, X_m : \Omega \to \mathbb{R}^d$ are independet if and only if

$$F_{X_1,\dots,X_m}(x_1, \dots, x_m) = F_{X_1}(x_1) \dots F_{x_m}(x_m) \quad \forall x_i \in \mathbb{R}^d.$$

Exercise. Proof the above theorem

Theorem 3.1.2. If $X_1, \dots, X_m : \Omega \to \mathbb{R}$ are independet and $\mathbb{E}[|X_i|] < \infty$ then

$$\mathbb{E}[|X_1, \dots, X_m|] < \infty.$$

and

$$\mathbb{E}[X_1 \dots X_m] = \mathbb{E}[X_1] \dots \mathbb{E}[X_m].$$

Exercise. Proof the above theorem

Theorem 3.1.3. $X_1, \ldots, X_m : \Omega \to \mathbb{R}$ are independet and $\mathbb{V}[X_i] < \infty$ then

$$\mathbb{V}[X_1 + \ldots + X_m] = \mathbb{V}[X_1] + \ldots + \mathbb{V}[X_m].$$

Exercise. Proof the above theorem