Lemma 0.0.1. Let $(X_t)_{t\in[0,T]}$ be an Itô process with representations

$$X_t = X_0 + \int_0^t a(\cdot,s)ds + \int_0^t b(\cdot,s)dB_s = \tilde{X}_0 + \int_0^t \tilde{a}(\cdot,s)ds + \int_0^t \tilde{b}(\cdot,s)dB_s.$$

$X_0 = \tilde{X}_0$, then $a = \tilde{a}$ and $b = \tilde{b}$

Proof. We have

$$0 = \int_0^t a(\cdot,s) - \tilde{a}(\cdot,s) + \int_0^t b(\cdot,s) - \tilde{b}(\cdot,s)dB_s.$$

Which follows by taking the difference, i.e

$$\int_0^t a(\cdot,s) - \tilde{a}(\cdot,s) = -\int_0^t b(\cdot,s) - \tilde{b}(\cdot,s)dB_s.$$

This is a local martingale that is continuous and of finite variation □

Let us prove that

$$(\int_0^t a(\cdot,s)ds).$$

is of finite variation

Proof. Define

$$A_t = \int_0^t a(\cdot,s)ds.$$

We consider

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^n A_{t_i} - A_{t_{i-1}} &= \lim_{n\to\infty}\sum_{i=1}^n \int_0^{t_i} a(\cdot,s)ds - \int_0^{t_{i-1}} a(\cdot,s)ds \\
&= \lim_{n\to\infty}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} a(\cdot,s)ds \\
&\le \lim_{n\to\infty}\sum_{i=1}^n \sup_{t\in[t_{i-1},t_i]} |a(\cdot,s)|(t_i - t_{i-1}) \\
&\le \lim_{n\to\infty}\sum_{i=1}^n \sup_{t\in[0,T]} |a(\cdot,s)|(t_i - t_{i-1}) \\
&\le \lim_{n\to\infty} C\sum_{i=1}^n (t_i - t_{i-1}) \\
&< \infty.
\end{aligned}$$

$$\begin{aligned}\langle A\rangle_t = \lim_{n\to\infty}\sum_{i=1}^{n}(A_t - A_{t-1})^2 &= \lim_{n\to\infty}\sum_{i=1}^{n}(\int_0^{t_i} a(\cdot,s)ds - \int_0^{t_{i-1}} a(\cdot,s)ds)^2\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i} a(\cdot,s)ds)^2\\ &\leq \lim_{n\to\infty}\sum_{i=1}^{n}(t_i - t_{i-1})\int_{t_{i-1}}^{t_i} a(\cdot,s)^2 ds\\ &= (T)\lim_{n\to\infty}\sum_{i=1}^{n}.\end{aligned}$$

$$(\int_{t_{i-1}}^{t_i} |a(\cdot,s)\cdot 1|ds)^2 \leq (\int_{t_{i-1}}^{t_i} |a(\cdot,s)|^2 ds)ds \cdot \int_{t_{i-1}}^{t_i} 1^2 = (t_i - t_{i-1})\int_{t_{i-1}}^{t_i} |a(\cdot,s)|^2 dsds.$$

□

Lemma 0.0.2. Show

$$|g|_t = \sup_{\Pi}\sum_{J\in\Pi}|\Delta_{J\cap[0,t]}g| = \lim_{n\to\infty}\sum_{J\in\Pi_n}|\Delta_{J\cap[0,t]}g|.$$

For a zero sequence of partitions

Proof. We proof this by 3 steps first, w.l.o.g we consider Partitions as follows

$$0 = t_0 < t_1 < \ldots < t_n = t.$$

And define

$$|g|_t^n = \sum_{i=1}^{n}|g(t_i) - g(t_{i-1})|.$$

Then we show that

$$|g|_t^n \leq |g|_t^{n+1}.$$

We have

$$|g|_t^n - |g|_t^{n+1} = \sum_{i=1}^{n}|g(t_i) - g(t_{i-1})| - \sum_{j=1}^{n+1}|g(\tilde{t}_j) - g(\tilde{t}_{j-1})|.$$

W.l.o.g we consider $t_i = \frac{i}{n} \cdot t$ then

$$\begin{aligned} t_i^n - t_i^{n+1} &= t \cdot (\frac{i}{n} - \frac{i}{n+1}) \\ &= t \cdot (\frac{i(n+1)}{n(n+1)} - \frac{i \cdot n}{n(n+1)}) \\ &= t \cdot (\frac{i}{n(n+1)}). \end{aligned}$$

I.e

$$|g|_t^n - |g|_t^{n+1} = \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| - \sum_{j=1}^{n+1} |g(\tilde{t}_j) - g(\tilde{t}_{j-1})|$$

.

□