Chapter 1

Brownian Motion and Martingales

These are merely my thoughts and notes to the stochastic calc script and will only include the entire statement if i had any special thoughts to it.

Definition 1.0.1 (usual conditions)**.** The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is said to satisfy the usual conditions if :

1. $\mathcal{F}_0$ contains all Pr-null sets $\mathcal{N}$ ("completeness")
2. $\mathcal{F}_t = \mathcal{F}_{t+} \coloneqq \bigcap_{s>t} F_s$ for $t \in [0,T)$ ("right-continuity")

Remark. Completeness assures us that any modifications to an adapted stochastic process is again adapted (same null sets).
Right-continuity can be thought of as giving us the ability to slightly peak into the future, consider the following hitting time :

$$\tau_A = \inf\{t \in [0,T] | X_t \in A\}.$$

For some open set $A \subset \mathbb{R}$ then for any $s \in [0,T]$ the event :

$$\{\tau_A = s\} = \{X_s \in \overline{A}\}.$$

Meaning that at time s we do not know if X_s is already in A or just right on the boundary of entering, such that we need the ability to peak slightly into the future.

Proposition 1.0.1. Let $(B_t)_{t\in[0,T]}$ be a Brownian motion. The completed natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ of a Brownian motion $(B_t)_{t\in[0,T]}$ is defined by

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^B, \mathcal{N}).$$

is right-continuous

Proof. Idea is to show $\mathcal{F}_{t+} \subseteq \mathcal{F}_t$ by taking any continuous and bounded $f : \mathbb{R}^d \to \mathbb{R}$ and showing that for $d \in \mathbb{N}$, $0 \le t_1 < t_2 < \ldots < t_d$

$$\mathbb{E}[f(B_{t_1}, \ldots, B_{t_d}) \mid \mathcal{F}_{t+}] \text{ is } \mathcal{F}_t\text{-measurable.}$$

We know that by the properties of Brownian motions any increment $B_t - B_s$ is independent of $\mathcal{F}_s^B$, take $k \in \{1, \ldots, d-1\}$ such that $t_k \le t \le t_{k+1}$. For $n \in \mathbb{N}$ large :

$$t + \frac{1}{n} < t_{k+1}.$$

and :

$$\lim_{n\to\infty} t + \frac{1}{n} = t.$$

Idea is to first show that $\mathbb{E}[f(B_{t_1}, \ldots, B_{t_d}) \mid \mathcal{F}_{t+\frac{1}{n}}]$ converges against a $\mathcal{F}_t$ measurable limit and converges against $\mathbb{E}[. \mid \mathcal{F}_{t+}]$ which concludes the proof.

$$\begin{aligned}
&\mathbb{E}[f(B_{t_1}, \ldots, B_{t_d}) \mid \mathcal{F}_{t+\frac{1}{n}}] \\
&\quad = \mathbb{E}[f(B_{t_1}, \ldots, B_{t_k}, \underbrace{B_{t+\frac{1}{n}} + (B_{t_{k+1}} - B_{t+\frac{1}{n}})}_{=0}, \ldots, \underbrace{B_{t+\frac{1}{n}} + (B_{t_d} - B_{t+\frac{1}{n}})}_{=0}) \mid \mathcal{F}_{t+\frac{1}{n}}] \\
&\quad = \int_{\mathbb{R}^{d-k}} f(B_{t_1}, \ldots, B_{t_k}, B_{t+\frac{1}{n}} + x_1, \ldots, B_{t+\frac{1}{n}} + x_{d-k})\rho_n(x)dx.
\end{aligned}$$

Question? why do we ignore the first t_k elements
Convergence of integral is shown by DCT against :

$$\int_{\mathbb{R}^{d-k}} f(B_{t_1}, \ldots, B_{t_k}, B_t + x_1, \ldots, B_t + x_{d-k})\rho(x)dx.$$

Which is clearly $\mathcal{F}_t$ mb.
Convergence of left hand side is shown by backward martingale theorem.

Plugging in $\mathbb{1}_A$ $\square$

Chapter 2

Ito-Integration

Lemma 2.0.1 (Itos isometry)**.** For $f \in \mathcal{H}_0^2$ (f is of form $f(\omega, s) = \sum_{i=0}^{n-1} a_i(\omega)\mathbb{1}_{(t_i]}$, a_i is any random variable, its just a discrete case)

Proof. Idea is similar to law of large numbers , i.e split the quadratic and non quadratic terms are 0 as $\mathbb{E}[(B_{t_{i+1}} - B_{t_i})] = 0$ is 0 and for a brownian motion the increments are Normal with mean $t_{i+1} - t_i$ □

Remark. The idea is to define a space of simple functions that lies dense in the space of functions (Stochastic processes) this allows us to prove properties of simple functions and transfer that to more complex one , think Stochastic 1

The following Proposition does just that and tells us that this sequence exists

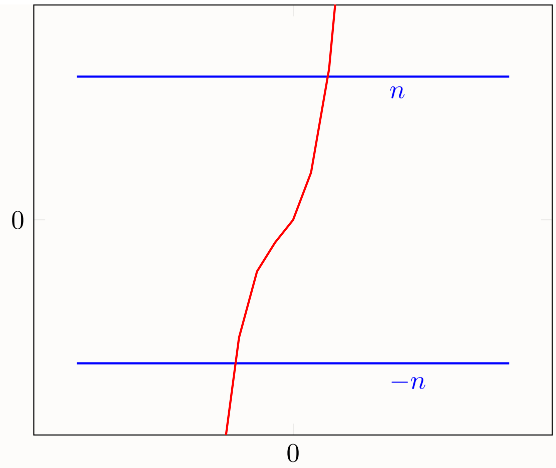
Proposition 2.0.1. For every f $f \in \mathcal{H}^2$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}_0^2$ such that

$$\|f_n - f\|_{\mathcal{H}^2} \to 0 \quad \text{as } n \to \infty.$$

Proof. In essence the proof first shows that we can define us f_n as

$$f_n = \max(-n, \min(f, n)).$$

This is just this



Then as f_n is bounded by f clearly we can use *DCT* to get the convergence, (swap integral and limit)

Second step ☐

Definition 2.0.1 (3.11)**.** For a fixed $T \in (0, \infty)$ we introduce

$$\mathcal{H}^2_{\text{loc}} := \{f : \Omega \times [0,T] \to \mathbb{R} : f \text{ is measurable, adapted and } \int_0^T f^2(\cdot, s)ds < \infty \mathbb{P}\text{-a.s.}\}.$$

clearly $\mathbb{E}[I(f^2)] < \infty$ implies $f \in \mathcal{H}^2_{\text{loc}}$ find example for f that is in $\mathcal{H}^2_{\text{loc}}$ but is not in $\mathcal{H}^2$,

Remark. Localizing sequence is an increasing sequence ν_n of $[0,T]$ stopping times such that for $f \in \mathcal{H}^2_{\text{loc}}$ if $f\mathbb{1}_{[0,\nu_n]} \in \mathcal{H}^2$ for all $n \in \mathbb{N}$ and

$$\mathbb{P}(\bigcup_{n=1}^{\infty} \{\nu_n = T\}) = 1.$$

Question , what does the above allow us to do

Example. Consider any $f \in \mathcal{H}^2_{\text{loc}}$ and ν_n then

$$\begin{aligned}
\mathbb{E}[\int_0^T f^2(\cdot, s)]ds &= \mathbb{E}[\int_0^T f^2(\cdot, s)ds \cdot \mathbb{1}_{\bigcup_{n=1}^{\infty}\{v_n = T\}}] \\
&\overset{?}{=} \int_0^T \int_\Omega f^2(\cdot, s) \cdot \mathbb{1}_{\bigcup_{n=1}^{\infty}\{v_n = T\}} d\mathbb{P}ds \\
&= \int_0^T \int_\Omega \sum_{n=1}^{\infty} f^2(\cdot, s) \cdot \mathbb{1}_{\{v_n = T\}} d\mathbb{P}ds \\
&\overset{?}{=} \int_0^T \sum_{n=1}^{\infty} \int_\Omega f^2(\cdot, s) \cdot \mathbb{1}_{\{v_n = T\}} d\mathbb{P}ds \\
&\overset{?}{=} \sum_{n=1}^{\infty} \int_\Omega \int_0^T f^2(\cdot, s) \cdot \mathbb{1}_{\{v_n = T\}} d\mathbb{P}ds.
\end{aligned}$$

Chapter 3

Problem Sheet 4

3.1 4.1

Question 1. Let $(B_t)_{t\in[0,T]}$ be a brownian motion. Without using Ito formula show that

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t).$$

Solution. Without using Ito formula we only know how to evaluate Simple Functions $f \in \mathcal{H}_0^2$ this is done by

$$\int_0^t f dB_s = I(f\mathbb{1}_{[0,t]}) = \sum_{i=0}^{n-1} f_i(B_{t_{i+1}} - B_{t_i}).$$

In our case we have $f(\cdot, s) = B_s \in \mathcal{H}^2$ we approximate by

$$f_n(s) = \sum_{i=1}^{n-1} \mathbb{1}_{(t_i,t_{i+1}]}(s) B_{t_i}.$$

Where we decompose the interval $[0, T]$ into $t_i = \frac{i}{n}T$ First show that

$f_n \to f$, consider

$$\begin{aligned}
\|f_n - f\|^2_{\mathcal{H}^2} &= \mathbb{E}[\int_0^T \sum_{i=0}^{n-1} (\mathbb{1}_{(t_i,t_{i+1}]}(t) B_{t_i} - B_t)^2 dt] \\
&\overset{\text{Fub}}{=} \int_0^T \sum_{i=0}^{n-1} \mathbb{E}[(\mathbb{1}_{(t_i,t_{i+1}]}(t) B_{t_i} - B_t)^2] dt \\
&= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}[(B_{t_i} - B_t)^2] dt \\
&\overset{\text{Def.}}{=} \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} |t - t_i| dt \\
&= \sum_{i=0}^{n-1} \frac{1}{2}(t_{i+1} - t_i)^2 \\
&= \sum_{i=0}^{n-1} \frac{T^2}{2n^2} \\
&= \frac{T}{2n} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Such that we get

$$I(f\mathbb{1}_{[0,t]}) = \lim_{n\to\infty} I(f_n \mathbb{1}_{[0,t]}) = \lim_{n\to\infty} \sum_{t_i \le t} .$$

□

Chapter 4

Uebung

Definition 4.0.1. For $X = (X_t)_{t\in[0,T]}$ continuous local martingale then, The process $\langle X\rangle = (\langle X\rangle_t)_{t\in[0,T]}$ is given by

$$\langle X\rangle_t = \lim_{n\to\infty} \sum_{J\in\Pi_n} (\triangle_{J\cap[0,T]} X)^2.$$

The limit is needed such that the non decreasing property is satisfied , i.e while the summands are always non negative , the increments might be different in size such that the process is not non-decreasing , note that the difference between $tandt + \varepsilon$ is that the intersection $J \cap [0,t]$ is taken over a bigger intervall
Dont think this is true (he said it) look below , the granularity does not change if we fix an n, the limit is just needed for the uniqueness tbh.

Remark. The Process is well defined as it is independent of the sequence of partitions Π_n

1. $\langle X\rangle_0 = 0$, and non decreasing
2. And $(X_t^2 - \langle X\rangle_{t\in[0,T]})$ is again a continuous local martingale

Why is it unique, as it is non decreasing and 0 at time 0, we get a unique process by (ii) as the only way for it to be a martingale is to be the same at every time t as the Martingale property allows us to trace back at any time t to time 0

$$\langle X\rangle_{t+1} - \langle X\rangle_t = \sum_{J\in\Pi_n} (\triangle_{J\cap[0,t+1]} X)^2 - (\triangle_{J\cap[0,t]} X)^2 = \sum_{J\in\Pi_n} (\triangle_{J\cap[t,t+1]} X)^2 > 0.$$