Chapter 1

Stochastic Mean Field Particle Systems

1.1 Basics of probability

Definition 1.1.1 (Brownian Motion). Real valued stochastic process $W(\cdot)$ is called a Brownian motion (Wiener process) if

1. $W(0) = 0$a.s.
2. $W(t) - W(s) \sim \mathcal{N}(0, t-s)$, for all $t, s \geq 0$
3. $\forall 0 < t_1 < t_2 < \ldots < t_n$, $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent
4. $W(t)$ is continuous a.s (sample paths)

Remark (Properties).

1. $\mathbb{E}[W(t)] = 0$, $\mathbb{E}[W(t)^2] = t$, for all $t > 0$
2. $\mathbb{E}[W(t)W(s)] = t \wedge s$ a.s
3. $W(t) \in \mathcal{C}^{\gamma}[0, T]$, $\forall 0 < \gamma < \frac{1}{2}$.
4. $W(t)$ is nowhere differentiable a.s

additionally Brownian motions are martingales and satisfy the markov property

Definition 1.1.2 (Ito Integral).

1.2 Bad K

1.3 Convergence

Chapter 2

Excercise Sheets

2.1 Sheet 1 (11.09.2023)

2.1.1 Excercise 1

Question 1. Consider the second order system :

$$
\begin{aligned}
dX_t^i &= V_t^i \\
dV_t^i &= \frac{1}{N} \sum_{i=1}^{N} F(t, X_t^i, V_t^i, X_t^j, V_t^j) dt.
\end{aligned}
$$

on $[0, T]$ for some smooth interaction force $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ following the lecture we assume the empirical measure :

$$\mu_t^N(dx, dv) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_t^i, V_t^i)}.$$

converges in some sense to the measure μ_t with density ρ_t for each t. Derive an equation for $\rho, t \geq 0$ similar to the lecture.

Solution. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^?)$ and calculate :

$$\begin{aligned}
\frac{d}{dt}\langle \mu_N, \varphi\rangle = \frac{d}{dt}\int_{\mathbb{R}^{2d}} \varphi(x,v) d\mu_N(t)(dx,dv) = \frac{d}{dt}\int \frac{1}{N}\sum_{i=1}^N \varphi(x,v) d\delta_{(x_i^t, v_i^t)} \\
&\overset{*}{=} \frac{1}{N}\sum_{i=1}^N \frac{d}{dt}\varphi(x_i(t), v_i(t)) \\
&\overset{\text{Chain.}}{=} \frac{1}{N}\sum_{i=1}^N \partial_x\varphi \cdot \dot{x}_i + \partial_v \cdot \dot{v}_i \\
&= \frac{1}{N}\sum_{i=1}^N \partial_x\varphi \cdot v_i(t) + \partial_v\varphi \cdot \sum_{j=1}^N F(t, x_i(t), v_i(t), x_j(t), v_j(t)) \\
&.
\end{aligned}$$

□

Chapter 3

Appendix

Theorem 3.0.1 (Divergence Theorem). Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)-$ dimensional sub-manifold of $\mathbb{R}^n$. Let $F : \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously to $\partial\Omega$. Then we have :

$$\int_{\Omega} \nabla \cdot F d\mu = \int_{\partial\Omega} F \cdot N d\sigma.$$

where N is the outward pointing normal. (last component is positive)