Leon Fiethen 1728330
Janik Sperling 1728567.

Sheet 9

27. It's not easy being green

Exercise. Suppose that Ω is a bounded domain. Prove that there is at most one Green's function on Ω

Proof. Since Ω is bounded this clearly hints at using the weak maximums principle, using property (i) of greens function we get :
Assume two Green's functions exists and label them $G, \tilde{G}$ then

$$u(y) = G(x,y) - \tilde{G}(x,y) = G(x,y) - \tilde{G}(x,y) + \underbrace{(\Phi(x-y) - \Phi(x-y))}_{=0}$$
$$= G(x,y) - \Phi(x-y) - (\tilde{G}(x,y) - \Phi(x-y)).$$

Then $u(y)$ is harmonic by 3.18 (i) and by the weak maximum principle $u(y) = 0$ $\square$

Exercise. On the other hand, suppose that Ω has Green's function G_Ω and that there exists a non-trivial solution to the Dirichlet problem

$$\Delta u = 0 \quad u|_{\partial\Omega} = 0.$$

We search for a function that satisfies (i) and (ii) from 3.18, since u is non-trivial

$$\tilde{G}(x,y) = G(x,y) + u(y) \neq G(x,y).$$

and we check (i)

$$y \mapsto \tilde{G}(x,y) - \Phi(x-y) = (G(x,y) - \Phi(x-y)) + u(y).$$

Both parts extend to a harmonic function for $x \in \Omega$, first half by virtue of being a Green's function , second part is harmonic by properties of being a result to the Dirichlet Problem.

For (ii) we check again for $y \in \partial\Omega$

$$y \mapsto \tilde{G}(x,y) - \Phi(x-y) = G(x,y) + u(y).$$

is 0 since G is a Greens function and we have $u|_{\partial\Omega} = 0$. It follows that $\tilde{G}$ is a second Greens Function.

28. Do nothing by halves

Let $H_1^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n | x_1 > 0\}$ be the upper half space and $H_1^0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n | x_1 = 0\}$ be the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \dots, x_n)$ reflection in the plane H^0

Exercise (a). Let $u \in \mathcal{C}^2(\overline{H_1^+})$ be a harmonic function that vanishes on H_1^0. Show that the function

$$v : \mathbb{R}^n \to \mathbb{R} \; x \mapsto \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}.$$

is harmonic

Proof. We want to check that

$$\Delta v = 0.$$

We know (by past exercise sheet) that if u is harmonic then $u(R_1(x))$ is harmonic also. We have a possible singularity at $x_1 = 0$ we have for $x_1 = 0$ that

$$x \in H_1^0.$$

But for $x \in H_1^0$ we have

$$R_1(x) = \mathbb{1}_{H_1^0}(x) = x.$$

And $u(x) = 0$, since u is continuous we also have for $x \in B(0, \varepsilon)$ that $v(x) \equiv 0$. Such that $x_1 = 0$ is not a singularity and

$$v \in \mathcal{C}^2.$$

We check the partial derivatives of v for $x \in B(0, \varepsilon)$

$$\partial_{x_i} v(x) = \lim_{h \to 0} \frac{v((0, \dots, x_i + h, \dots)) - v(x)}{h} = .$$

□

Exercise (b). Show that Green's function for H_1^+ is

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

Proof. We check (i) and (ii), For (i) we first note

$$G(x, y) - \Phi(x - y) = \Phi(R_1(x) - y).$$

we check for singularity at $R_1(x) = y$, since $x \in H_1^+$ then $R_1(x)$ is in $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \; : \; x_1 < 0\}$ but since $y \in H_1^+$ the case $R_1(x) = y$ is impossible, and since Φ is harmonic (without singularity) we get that

$G(x, y) - \Phi(x - y)$ extends to a harmonic function. For (ii) we check for $x \in \Omega$ and $y \in \partial\Omega$

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

The fundamental solution only depends on the Length $\|R_1(x) - y\|$ which is symmetric i.e.

$$\|R_1(x) - y\| = \|x - R_1(y)\|.$$

Since $y \in \partial\Omega := H_1^0$ we get $R_1(y) = y$ and

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y) = \Phi(x - y) - \Phi(x - R_1(y)) = \Phi(x - y) - \Phi(x - y) = 0.$$

Thus G is Green's function for H_1^+ $\square$

Exercise (c). Compute the Green's function for B^+

Proof. By 3.20 we know that

$$G_{B(0,1)}(x, y) = \Phi(x - y) - \Phi(|x|(\tilde{x} - y)).$$

where $\tilde{x} = \frac{x}{|x|^2}$
We know that the greens function $B(0, 1)$ must be unique, lets say we have a greens function on B^+ call it G^+, we expect

$$G_{B(0,1)(x,y)}|_{B^+} \equiv G^+.$$

We consider

$$G(x, y) = \Phi(x - y) - \Phi(|x|(R_1(\tilde{x}) - y)).$$

We check (i)

$$G(x, y) - \Phi(x - y) = \Phi(|x|(R_1(x) - y)).$$

abs By similar argument to (b) we know the singularity is not a problem and (i) is satisfied by properties of the fundamental solution
For (ii) we check $x \in B^+$ and $y \in \partial B^+$, clearly the boundary ∂B^+ consists of two parts,

$$\partial B^+ = B^0 \cup \partial B^+ \cap H^+.$$

We consider the cases separately,for $x \in B^+$ and $y \in B^0$

$$G(x, y) = \Phi(x - y) - \Phi(|x|(R_1(\tilde{x}) - y)) = \Phi(x - y) - \Phi(|x|(\tilde{x} - y)).$$

And notice it doesn't work out lol , we choose new Green's function such that the above is 0 ,

$\tilde{G}(x,y) = \Phi(x-y) - \Phi(R_1(x)-y) - (\Phi(|x|(\tilde{x}-y)) - \Phi(|x|(R_1(\tilde{x})-y)))$.

then for $x \in B^+$ and $y \in B^0$

$$\begin{aligned}
\tilde{G}(x,y) &= \Phi(x-y) - \Phi(R_1(x)-y) - (\Phi(|x|(\tilde{x}-y)) - \Phi(|x|(R_1(\tilde{x})-y))) \\
&= \Phi(x-y) - \Phi(x-R_1(y)) - (\Phi(|x|(\tilde{x}-y)) - \Phi(|x|(\tilde{x}-y))) \\
&= \Phi(x-y) - \Phi(x-y) - (\Phi(|x|(\tilde{x}-y)) - \Phi(|x|(\tilde{x}-y))) \\
&= 0.
\end{aligned}$$

similar argument to (b), for $y \in \partial B^+ \cap H^+ \subset \partial B(0,1)$ we have by lecture

$$\||x|(\tilde{x}-y)\| = |x-y|.$$

and

$$\||x|(R_1(\tilde{x}-y))\| = |R_1(x)-y|.$$

then

$$\begin{aligned}
\tilde{G}(x,y) &= \Phi(x-y) - \Phi(R_1(x)-y) - (\Phi(|x|(\tilde{x}-y)) - \Phi(|x|(R_1(\tilde{x})-y))) \\
&= \Phi(x-y) - \Phi(|x|(\tilde{x}-y) + \Phi(|x|(R_1(\tilde{x})-y)) - \Phi(R_1(x)-y) \\
&= 0.
\end{aligned}$$

□

29. Teach a man to fish

Exercise (a). Using the Green's function of H_1^+ from the previous question, derive the following formal integral representation for a solution of the Dirichlet problem

$$\Delta u = 0 \qquad u|_{H_1^0} = g.$$

$$u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x-z|^n} d\sigma(z).$$

Proof. We assume g has sufficient regularity, by Greens representation we know

$$u(x) := \int_{H_1^+} G_{H_1^+}(x,y) f(y) d^n y - \int_{H_1^0} g(z) \nabla_z G_{H_1^+} \cdot N d\sigma(z).$$

solves the Dirichlet problem in fact as $f \equiv 0$

$$u(x) := -\int_{H_1^0} g(z) \nabla_z G_{H_1^+}(x,z) \cdot N d\sigma(z).$$

We calculate for $n > 2$

$$\nabla G(x,z) = \nabla_z(\Phi(x-z) - \Phi(R_1(x)-z)) = \nabla_z(\frac{1}{n(n-2)\omega_n|x-z|^{n-2}} - \frac{1}{n(n-2)\omega_n|x-z|^{n-2}}).$$

ja ka rechnen halt, □

Exercise (b). Show that if g is periodic that is, there is some vector $L \in \mathbb{R}^{n-1})$ with

$$g(x+L) = x.$$

for all $x \in \mathbb{R}^{n-1}$, then so is the solution

Proof. We have our solution by

$$u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x-z|^n} d\sigma(z).$$

We naively pick $L \in \mathbb{R}^{n-1}$ such that $g(x+L) = x$ for all $x \in \mathbb{R}^{n-1}$ and check

$$u(x+L) = \frac{2(x_1+L_1)}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x+L-z|^n}.$$

Consider

$$y = z - L.$$

then (we techincally have the jacobian but we may ignore that since this preserves volume and sign stays the same)

$$\begin{aligned} u(x+L) &= \frac{2(x_1+L_1)}{n\omega_n} \int_{H_1^0} \frac{g(y+L)}{|x-y|^n} d\sigma(y) \\ &= \frac{2(x_1+L_1)}{n\omega_n} \int_{H_1^0} \frac{y}{|x-y|^n} d\sigma(y) \end{aligned}$$

.

□