Chapter 1

Stochastic Mean Field Particle Systems

From now on let the underlying probability space be given by $(\Omega, \mathcal{F}, \mathbb{P})$.

1.1 Basics of probability

Definition 1.1.1 (Brownian Motion). Real valued stochastic process $W(\cdot)$ is called a Brownian motion (Wiener process) if

1. $W(0) = 0$a.s.
2. $W(t) - W(s) \sim \mathcal{N}(0, t-s)$, for all $t, s \geq 0$
3. $\forall 0 < t_1 < t_2 < \ldots < t_n$, $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent
4. $W(t)$ is continuous a.s (sample paths)

Remark (Properties).

1. $\mathbb{E}[W(t)] = 0$, $\mathbb{E}[W(t)^2] = t$, for all $t > 0$
2. $\mathbb{E}[W(t)W(s)] = t \wedge s$ a.s
3. $W(t) \in \mathcal{C}^\gamma[0, T]$, $\forall 0 < \gamma < \frac{1}{2}$.
4. $W(t)$ is nowhere differentiable a.s

additionally Brownian motions are martingales and satisfy the Markov property

Definition 1.1.2 (Progressively measurable). In addition to adaptation of a Stochastic process X_t we say it is progressively measurable w.r.t $\mathcal{F}_t$ if $X(s, \omega) : [0, t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}[0, t] \times \mathcal{F}_t$ measurable, i.e the t is included

Definition 1.1.3 (Simple functions). Instead of $\mathcal{H}^2$ she uses $\mathbb{L}^2(0,T)$ is the space of all real-valued progressively measurable proceses $G(\cdot)$ s.t

$$\mathbb{E}[\int_0^T G^2 dt] < \infty.$$

define $\mathbb{L}$ analog

Definition 1.1.4 (Step Process). $G \in \mathbb{L}^2(0,T)$ is called a step process when Partition of $[0,T]$ exists s.t $G(t) = G_k$ for all $t_k \leq t \leq t_{k+1}$, $k = 0, \ldots, m-1$ note G_k is $\mathcal{F}_{t_k}$ measurable R.V.

For step process we define the ito integral as a simple sum

Definition 1.1.5 (Ito integral for step process). Let $G \in \mathbb{L}^2(0,T)$ be a step process is given by

$$\int_0^T G(t)dW_t = \sum_{k=0}^{m-1} G_k(W(t_{k+1} - W(t_k))).$$

We take the left value of the process such that we converge against the right integral later

Remark. For two step processes $G, H \in \mathbb{L}^2(0,T)$ for all $a, b \in \mathbb{R}$, we have linearity (note they may have two different partitions, so we need to make a bigger (finer) one to include both,)

1. $\int_0^T (aG + bH)dW_t = a \int G + b \int H$
2. $\mathbb{E}[\int_0^T GdW_t] = 0$, because the Brownian motion has EV of 0
3. $\mathbb{E}[(\int_0^T GdW_t)^2] = \mathbb{E}[\int_0^T G^2 dt]$ Ito isometry

Proof. First property is just defining a new partition that includes both process. Second property, the Idea of the proof is that

$$\begin{aligned}\mathbb{E}[\int_0^t GdW_t] &= \mathbb{E}[\sum_{k=0}^{m-1} G_k(W_{t_{k+1}} - W_{t_k})] \\ &= \sum_{k=0}^{m-1} \mathbb{E}[G_k(W(t_{k+1}) - W(t_k))]\end{aligned}$$

.

Remember $G_k \sim \mathcal{F}_{t_k}$ m.b. and $W(t_{k+1}) - W(t_k)$ is mb. wrt to $W^t(t_k)$ future sigma algebra and it is independent of $\mathcal{F}_{t_k}$ s.t the expectation decomposes

$$\sum_{k=0}^{m-1} \mathbb{E}[G_k(W(t_{k+1}) - W(t_k))] = \sum_{k=0}^{m-1} \mathbb{E}[G_k]\mathbb{E}[W(t_{k+1} - W(t_k))] = C \cdot 0 = 0.$$

For the variance decompose into square and non square terms , the non square terms dissapear by property 2 the rest follows by the variance of Brownian motion, be careful of which terms are actually independent , at leas one will always be independent of the other 3 □

Definition 1.1.6 (Ito Formula)**.** If $u \in \mathcal{C}^{2,1}(\mathbb{R} \times [0,T]; R)$ then

$$\begin{aligned} du(x(t),t) &= \frac{\partial u}{\partial t}(x(t),t)dt + \frac{\partial u}{\partial x}(x(t),t)dx + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2 dt \\ &= \frac{\partial u}{\partial x}(x(t),t)GdW_t + (\frac{\partial u}{\partial t}(x(t),t)) + \frac{\partial u}{\partial x}(x(t),t)F + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2 dt. \end{aligned}$$

For $dX = Fdt + GdW_t$ for $F \in L^1([0,T])$, $G \in L^2([0,T])$

Proof. The proof is split into the steps

1. $$\begin{aligned} d(W_t^2) &= 2W_t dW_t + dt \\ d(tW_t) &= W_t dt + tdW_t. \end{aligned}$$

2. $$\begin{aligned} dX_i &= F_i dt + G_i dW_t \\ d(X_1, X_2) &= X_2 dX_1 + X_1 dX_2 + G_1 G_2 dt \\ &. \end{aligned}$$

3. $$u(x) = x^m \quad m \geq 2.$$

4. Itos formula for $u(x,t) = f(x)g(t)$ where f is a polynomial

I.e we prove the Ito formula for functions of the form $u(x) = x^m$ and then Step 1 :

1. $d(W_t^2) = 2W_t dW_t + dt$ which is equivalent to $W^2(t) = W_0^2 + \int_0^t 2W_s dW_t + \int_0^t ds$
2. $d(tW_t) = W_t dt + tdW_t$ which is equivalent to $tW(t) - sW(0) = \int_0^t W_s ds + \int_0^t s dW_s$

Actually $\forall$ a.e $\omega \in \Omega$:

$$2\int_0^t W_s dW_s = 2\lim_{n\to\infty} .$$

Now we prove (2) $tW_t - 0W_0 = \int_0^t W_s ds + \int_0^t s dW_s$

$$\int_0^t s dW_s + \int_0^t W_s ds = \lim_{n\to\infty} \sum_{k=0}^{n-1} t_k^n (W(t_{k+1}^n) - W(t_k^n)) + \sum_{k=0}^{n-1} W(t_{k+1}^n (t_{k+1}^n - t_k^n)).$$

We choose the right value for the second integral

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} (-t_k^n W(t_k)^n + t_{k+1}^n W(t_{k+1}^n)) = W(t)t - W(0)\cdot 0.$$

Its product rule

$$\begin{aligned} dX_1 &= F_1 dt + G_1 dW_t \\ dX_2 &= F_2 dt + G_2 dW_t. \end{aligned}$$

This can be written as

$$d(X_1, X_2) = X_2 dX_1 + X_1 dX_2.$$

this shorthand notation actually means

$$\begin{aligned} X_1(t)X_2(t) - X_1(0)X_2(0) = &\int_0^t X_2 F_1 ds + \int_0^t X_2 G_1 dW_s \\ &+ \int_0^t X_1 F_2 ds + \int_0^t X_1 G_2 dW_s \\ &+ \int_0^t G_1 G_2. \end{aligned}$$

We prove for F_1, F_2, G_1, G_2 are time independent

$$
\begin{aligned}
&\int_0^t (X_2 dX_1 + X_1 dX_2 + G_1 G_2 ds)\\
&= \int_0^t (X_2 F_1 + X_1 F_2 + G_1 G_2) ds + \int_0^t (X_2 G_1 + X_1 G_2) dW_s\\
&= \int_0^t (\underbrace{F_2 F_1 s + F_1 G_2 W_s}_{=X_2} + \underbrace{F_1 F_2 s + F_2 G_1 W_s}_{=X_1} + G_1 G_2) ds\\
&+ \int_0^t (F_2 G_1 s + G_2 G_1 W_s + F_1 G_2 s + G_1 G_2 W_s) dW_s\\
&= G_1 G_2 t + F_1 F_2 t^2 + (F_1 G_2 + F_2 G_1) \underbrace{\left(\int_0^t W_s ds + \int_0^t s dW_s \right)}_{tW_t} + 2 G_1 G_2 \underbrace{\int_0^t W_s dW_s}_{W_t^2 - t}\\
&= G_1 G_2 t + F_1 F_2 t^2 + (F_1 G_2 + F_2 G_1) t W_t + G_1 G_2 W_t^2 - G_1 G_2 t\\
&= X_1(t) \cdot X_2(t).
\end{aligned}
$$

Where $X_2(t) = \int_0^t F_2 ds + \int_0^t G_2 dW_s \overset{\text{Cons.}}{=} F_2 t + G_2 W_t$

Extend the above idea by considering step processes (F_1, F_2, G_1, G_2)instead of time independent. Step processes are constant (related to time) and we can use the above prove for every time step t and just consider a summation over it.

For general $F_1, F_2 \in L^1(0, T), G_1, G_2 \in L^2(0, T)$ then we take step processes to approximate them

$$
\begin{aligned}
\mathbb{E}[\int_0^T |F_i^n - F_i| dt] \to 0\\
\mathbb{E}[\int_0^T |G_i^n - G_i|^2 dt] \to 0
\end{aligned}
$$

.

$$
X_i(t)^n = X_i(0) + \int_0^t F_i^n ds + \int_0^t G_i^n dW_s.
$$

It holds

$$
\begin{aligned}
X_1^n(t) X_2(t)^n - X_1(0) X_2(0) = &\int_0^t X_2(s)^n F_1^n(s) ds + \int_0^t X_2(s) G_1(s)^n dW_s\\
&+ \int_0^t X_1^n(s) F_2^n(s) ds + \int_0^t X_1(s)^n G_2^n(s) dW_s + \int_0^t G_1(s)^n G_2^n(s) ds.
\end{aligned}
$$

Only thing left is a convergence result (i.e DCT) sinc the processes are bounded or smth like that.

Step 3 if $u(x) = x^m,\ \forall m = 0, \ldots$ to prove

$$d(X^m) = mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}G^2dt.$$

For $m = 2$ the result is obtained by the product rule, By induction we prove for arbitrary m

(IV) Suppose the statement hold for $m-1$
(IS) $m-1 \to m$

$$\begin{aligned}
d(X^m) &= d(X \cdot X^{m-1}) = XdX^{m-1} + X^{m-1}dx + (m-1)X^{m-2}G^2dt \\
&\stackrel{\text{IS}}{=} X(m-1)X^{m-2}dx + X \cdot \frac{1}{2}(m-1)(m-2)X^{m-3}G^2dt + X^{m-1}dx + (m-1)X^{m-2}G^2dt \\
&= mX^{m-1}dx + (m-1)(\frac{m}{2} - 1 + 1)X^{m-2}G^2dt \\
&= \underbrace{mX^{m-1}}_{\partial_x u}dx + \frac{1}{2}\underbrace{m(m-1)X^{m-2}}_{\partial_x^2 u}G^2dt.
\end{aligned}$$

Now $u(x) = x^m$

$$dX = Fdt + GdW_t.$$

Step 4 If $u(x,t) = f(x)g(t)$ where f is a polynomial

$$\begin{aligned}
d(u(x,t)) = d(f(x)g(t)) &= f(x)dg + gdf(x) + G \cdot 0dt \\
&\stackrel{\text{S3}}{=} f(x)g'(t)dt + gf'(x)dx + \frac{1}{2}gf''(x)G^2dt.
\end{aligned}$$

Itos formula is true for $f(x)g(t)$, it should thus also be true for functions $u(x,t) = \sum_{i=1}^m g^i(t)f^i(x)$

Step 5: if $u \in \mathcal{C}^{2,1}$ then we know there exists a sequence of polynomials $f^i(x)$ s.t

$$u_n(x,t) = \sum_{i=1}^n f^i(x)g^i(t).$$

Then $u_n \to u$ uniformly for any compact set $K \subset \mathbb{R} \times [0,T]$, we can thus apply Itos formula for each of the u_n and take the limit term wise $\square$

Remark. One can get the existence of the polynomial sequence by using Hermetian polynomials

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

Exercise. If $u \in \mathcal{C}^\infty$, $\frac{\partial u}{\partial x} \in \mathcal{C}_b$ then prove Step 4 $\Rightarrow$ Step 5

Use Taylor expansion and use the uniform convergence of the Taylor series on any compact support

Remark (Multi Dimensional Brownian Motion)**.** Multi dimensional Brownian motion

$$W(t) = (W^1(t), \dots, W^m(t)) \in \mathbb{R}^m$$

.

In each direction we should have a 1 dimensional Brownian motion and any two directions should be independent. We use the natural filtration $\mathcal{F}_t = \sigma(W(s); 0 \leq s \leq t)$

Definition 1.1.7 (Multi-Dimensional Ito's Integral)**.** We the define the n dimensional integral for $G \in L^2_{n \cdot m}([0,T])$, $G_{ij} \in L^2([0,T])$ $1 \leq i \leq n$, $1 \leq j \leq m$

$$\int_0^T G dW_t = \begin{pmatrix} \vdots \\ \int_0^T G_{ij} dW_t^j \\ \vdots \end{pmatrix}_{n \times 1} .$$

With the Properties

$$\mathbb{E}[\int_0^T G dW_t] = 0$$

$$\mathbb{E}[(\int_0^T G dW_t)^2] = \mathbb{E}[\int_0^T |G|^2 dt].$$

Where $|G|^2 = \sum_{i,j}^{n,m} |G_{ij}|^2$

Definition 1.1.8 (Multi-Dimensional Ito process)**.** We define the n dimensional Ito process as

$$X(t) = X(s) + \int_s^t F_{n \times 1}(r) dr + \int_0^t G_{n \times m}(r) dW_{m \times 1}(r)$$

$$dX^i = F^i dt + \sum_{j=1}^m G^{ij} dW_t^i \qquad 1 \leq i \leq n.$$

Theorem 1.1.1 (Multi Dimensional Ito's formula)**.** We define the n dimensional Ito's formula as $u \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0,T], \mathbb{R})$

$$\begin{aligned} du(x(t),t) &= \frac{\partial u}{\partial t}(x(t),t)dt + \nabla u(x(t),t) \cdot dx(t) \\ &+ \frac{1}{2} \sum \frac{\partial^2 u}{\partial x_i \partial x_j}(x(t),t) \sum_{l=1}^m G^{il} G^{il} dt. \end{aligned}$$

Proposition 1.1.1. For real valued processes X_1, X_2

$$\begin{cases} dX_1 &= F_1 dt + G_1 dW_1 \\ dX_2 &= F_2 dt + G_2 dW_2 \end{cases} \Rightarrow d(X_1, X_2) = XdX_2 + X_2 dX_1 + \sum_{k=1}^{m} G_1^k G_2^k dt.$$

Working with SDEs relies on a lot of notational rules as seen in the differential notation is just shorthand for the Integral form

Definition 1.1.9. Formal multiplication rules for SDEs

$$(dt)^2 = 0\ ,\ dtdW^k = 0\ ,\ dW^k dW^l = \delta_{kl} dt.$$

Using this notation we can simply itos formula as follows

$$\begin{aligned} du(X,t) &= \frac{\partial u}{\partial t} dt + \nabla_x u \cdot dX + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} dX^i dX^j \\ &= \frac{\partial u}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial u}{\partial X^i} F^i dt + \sum_{i=1}^{n} \frac{\partial u}{\partial X_i} \sum_{i=1}^{m} G^{ik} dW_k \\ &+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \left(F^i dt + \sum_{k=1}^{m} G^{ik} dW_k \right) \left(F^j dt + \sum_{l=1}^{m} G^{i;} dW_l \right) \\ &= (\frac{\partial u}{\partial t} + F \cdot \nabla u + \frac{1}{2} H \cdot D^2 u) dt + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \sum_{k=1}^{m} G^{ik} dW_k. \end{aligned}$$

Where

$$\begin{aligned} dX^i &= F^i dt + \sum_{k=1}^{m} G^{ik} dW_k \\ H_{ij} &= \sum_{k=1}^{m} G^{ik} G^{jk}\ ,\ A \cdot B = \sum_{i,j=1}^{m} A_{ij} B_{ij}. \end{aligned}$$

Typical example

$$G^T G = \sigma I_{n \times n}.$$

Example. If F and G are deterministic

$$dX_{n\times 1} F(t)_{n\times 1} dt + G_{n\times m} dW_t m \times 1.$$

Then for arbitrary test function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ then by Ito's formula

$$\begin{aligned} u(x(t)) - u(x(0)) = \int_0^t \nabla u(x(s)) \cdot F(s) ds + \int_0^t \frac{1}{2} (G^T G) : D^2 u(x(s)) ds \\ + \int_0^t \nabla u(x(s)) \cdot G(s) dW_s. \end{aligned}$$

Let $\mu(s, \cdot)$ be the law of $X(s)$ then we take the expectation of the above integral

$$\int_{\mathbb{R}^n} u(x)d\mu(s,x) - \int_{\mathbb{R}^n} u(x)d\mu_0(x) = \int_0^t \int_{\mathbb{R}^n} \nabla u(x) \cdot F(s)d\mu(s,x)$$
$$+ \int_0^t \int_{\mathbb{R}^n} \frac{1}{2}(G^T(s)G(s)) : D^2u(x) \cdot d\mu(s,x) + 0.$$

Definition 1.1.10 (Parabolic Operator)**.**

$$\partial_t u - \frac{1}{2}\sum_{i,j=1}^{n} D_{ij}(\sum_{k=1}^{m} G^{ik}G^{kj})\mu + \nabla \cdot (F\mu) = 0.$$

Example. If $F = 0$ $m = n$ and $G = \sqrt{2}I_{n\times n}$ then

$$dX = \sqrt{2}dW_t.$$

And the law of X , μ fulfills the heat equation

$$\mu_t = \triangle\mu = 0.$$

How does this all translate to our Mean field Limit, consider a particle system given by

$$\begin{cases} dX_N &= F(X_N)dt + \sqrt{2}dW_{dN\times 1} \\ dx_i &= \frac{1}{N}\sum K(x_i,x_j)dt + \sqrt{2}dW_t^1 \qquad 1 \le i \le N \;\; N \to \infty \\ x_i(0) &= x_{0,i} \\ \mu_N(t) &= \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)} \end{cases}.$$

At time $t = 0$ the x_i are independent random variables at any time $t > 0$ they are dependent and the particles have joint law

$$(x_1(t), \dots, x_N(t)) \sim u(x_1, \dots, x_n).$$

Where $u \in \mu(\mathbb{R}^{dN}$ by Ito's formula we get for arbitrary test function $\forall\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{dN})$

$$\varphi(X_N) = \varphi(X_N(0)) + \int_0^t \nabla_{dN}\varphi \cdot \begin{pmatrix} \vdots \\ \frac{1}{n}\sum_{j=1}^{N} K(x_i,x_j) \\ \vdots \end{pmatrix} X_N$$
$$+ \int_0^t \triangle_{X_N}\varphi dt + \int_0^t \sqrt{2}\nabla\varphi dW_t^i.$$

Taking the expectation on both sides, then the last term disappears by definition of Ito processes

$$\partial_t - \sum_{i=1}^{N}\triangle_i u + \sum_{i=1}^{N}\nabla_{x_i}\left(\frac{1}{N}\sum_{j=1}^{N}K(x_i,x_j)u\right) = 0.$$

Now consider the Mean-Field-Limit, if the joint particle law can be rewritten as the tensor product of a single $\overline{u}$

$$u(x_1, \ldots, x_N) = \overline{u}^{\otimes N}.$$

the equation simplifies

$$\partial_t - \sum_{i=1}^{N} \triangle_i u + \sum_{i=1}^{N} \nabla_{x_i} \left(\overline{u}^{\otimes N} k \star \overline{u}(x_i) \right) = 0.$$

1.2 Solving Stochastic Differential Equations

The setup of the following section will be the following

Definition 1.2.1 (Basic Setup). We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, With a $m - D$ dimensional Brownian motion $W(\cdot)$. Let X_0 be an $n - D$ dimensional random variable independent of $W(0)$, then our Filtration is given by

$$\mathcal{F}_t = \sigma(X_0) \cup \sigma(W(s), 0 \leq s \leq t).$$

Note for better understanding the dimensions will be included in the following definition, but we generally leave them out.

Definition 1.2.2 (SDE). Given the above basic setup we are trying to solve equations of the type

$$\begin{cases} d\underbrace{X_t}_{n\times 1} &= \underbrace{b}_{n\times 1}(X_t, t)dt + \underbrace{B}_{n\times m}(X_t, t)d\underbrace{W_t}_{m\times 1} \quad 0 \leq t \leq T \\ X_t|_{t=0} &= X_0 \quad X \ : \ (t, \omega) \to \mathbb{R}^n \end{cases}.$$

Where

$$\begin{aligned} b &: (x, t) \in \mathbb{R}^n \times [0, T] \to \mathbb{R}^n \\ B &: (x, t) \in \mathbb{R}^n \times [0, T] \to M^{nxm}. \end{aligned}$$

Remark. The differential equation should always be understood as the Integral equation

$$X_t - X_0 = \int_0^t b(X_s, s)ds + \int_0^t B(X_s, s)dW_s.$$

Definition 1.2.3 (Solution). We say an $\mathbb{R}^n$-valued stochastic process $X(\cdot)$ is a solution of the SDE if

1. X_t is progressively measurable w.r.t $\mathcal{F}_t$
2. (drift) $F := b(X_t, t) \in L^1_n([0, T]) \ \Leftrightarrow \ \int_0^t \mathbb{E}[F_s]ds < \infty$

3. (diffusion) $G := B(X_t, t) \in L^2_{n\times m}([0,T]) \Leftrightarrow \int_0^t \mathbb{E}[|G_s|^2]ds < \infty$

Reminder that (1) implies that for any given $t \in [0,T]$ X_t is random variable measurable with respect to $\mathcal{F}_t$

The goal from now on is to prove the existence and uniqueness of such solutions, we formulate the following theorem, one should remember that if the diffusion term $B(X_t, t)$ is 0 then we get a unique solution iff $b(X_t, t)$ is Lipschitz

Theorem 1.2.1 (Existence and Solution)**.** Suppose $b : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ and $B : \mathbb{R}^n \times [0,T] \to M^{n\times m}$, then we get the necessary condition that they are continuous and (globally) Lipschitz continuous with respect to x i.e $\exists\, L > 0$ such that for arbitrary $\forall\, x, \tilde{x} \in \mathbb{R}^n$ and $t \in [0,T]$ it holds

$$|b(x,t) - b(\tilde{x},t)| + |B(x,t) - B(\tilde{x},t)| \leq L|x - \tilde{x}|.$$

and the linear growth condition

$$|b(x,t)| + |B(x,t)| \leq L(1 + |x|).$$

The initial data X_0 should be square integrable $x_0 \in L^2_n(\Omega)$ and that X_0 is independent of $W^t(0)$

Whenever the above conditions hold then there exists a unique solution $X \in L^2_n([0,T])$ of the SDE.

Proof. We begin by proving the uniqueness of solution.

Suppose we have two solutions X and $\tilde{X}$ to the SDE then we need to show that they are indistinguishable, then by using the definition of a solution

$$X_t - \tilde{X}_t = \int_0^t (b(X_s, s) - b(\tilde{X}_s, s))ds + \int_0^t B(X_s, s) - B(\tilde{X}(s), s)dW_s.$$

If the diffusion term were to be 0 we could use a Grönwall type inequality and get the uniqueness. To work with the diffusion term we consider the square of the above and apply Itos isometry. Note that generally $|a+b|^2 \nleq (a^2 + b^2)$ which is why we need the extra 2.

$$|X_t - \tilde{X}_t|^2 \leq 2|\int_0^t (b(X_s, s) - b(\tilde{X}_s, s))ds|^2 + |\int_0^t B(X_s, s) - B(\tilde{X}(s), s)dW_s|^2.$$

Now consider the following

$$
\begin{aligned}
\mathbb{E}[|X_t - \tilde{X}_t|^2] \leq 2\mathbb{E}[| \int_0^t |b(X_s, s) - b(\tilde{X}_s, x)| ds|^2] \\
+ 2\mathbb{E}[| \int_0^t B(X_s, s) - B(\tilde{X}_s, s) dW_s|^2] \\
&\overset{\text{Hold.}}{\leq} 2t\mathbb{E}[\int_0^t |b(X_s, s) - b(\tilde{X})s, s)|^2 ds] + 2\mathbb{E}[\int_0^t |B(X_s, s) - B(\tilde{X}_s, s)|^2 ds] \\
&\overset{\text{Lip.}}{\leq} 2(t+1)L^2 E[\int_0^t |X_s - \tilde{X}_s|^2 ds] \\
&= 2(t+1)L^2 \int_0^t E[|X_s - \tilde{X}_s|^2] ds
\end{aligned}
$$

.

Where the following Hoelders inequality was used

$$
\begin{aligned}
\left(\int_0^t 1|f|ds\right)^2 &\leq \left(\int_0^t 1^2 ds\right)^{\frac{1}{2}\cdot 2} \cdot \left(\int_0^t |f|^2 ds\right)^{\frac{1}{2}\cdot 2} \\
&\leq t \int_0^t |f|^2 ds.
\end{aligned}
$$

Now by Gronwalls inequality we have

$$\mathbb{E}[|X_t - \tilde{X}_t|^2] = 0.$$

i.e X_t and $\tilde{X}_t$ are modifications of each other and it remains to show that they are actually indistinguishable.

Define

$$A_t = \{\omega \in \Omega \mid |X_t - \tilde{X}_t| > 0\} \qquad \mathbb{P}(A_t) = 0.$$

$$\mathbb{P}(\max_{t \in \mathbb{Q} \cap [0,T]} |X_t - \tilde{X}_t| > 0) = \mathbb{P}(\bigcup_{k=1}^{\infty} A_{t_k}) = 0.$$

Now since $X_t(\omega)$ is continuous in t we can extend the maximum over the entire interval $[0, T]$

$$\max_{t \in \mathbb{Q} \cap [0,T]} |X_t - \tilde{X}_t| = \max_{t \in [0,T]} |X_t - \tilde{X}_t|.$$

Then the probability over the entire interval must also be 0

$$\mathbb{P}(\max_{t \in [0,T]} |X_t - \tilde{X}_t| > 0) = 0 \quad \text{i.e. } X_t = \tilde{X}_t \ \forall t \text{ a.s..}$$

This concludes the uniqueness proof, for existence as in the deterministic case we use Picard iteration.

Define the Picard iteration by

$$\begin{aligned} X_t^0 &= X_0 \\ &\vdots \\ X_t^{n+1} &= X_0 + \int_0^t b(X_s^n, s)ds + \int_0^t B(X_s^n, s)dW_s. \end{aligned}$$

Let $d(t)^n = \mathbb{E}[|X_t^{n+1} - X_t^n|^2]$ we claim that by induction $d^n(t) \le \frac{(Mt)^{n+1}}{(n+1)!}$ for some $M > 0$.
IA: For $n = 0$ we have

$$\begin{aligned} d(t)^0 = \mathbb{E}[|X_t^1 - X_t^0|^2] &\le \mathbb{E}[2(\int_0^t b(X_0, s)ds)^2 + 2(\int_0^t B(X_0, s)dW_s)^2] \\ &\le 2t\mathbb{E}[\int_0^t L^2(1 + X_0^2)ds] + 2\mathbb{E}[\int_0^t L^2(1 + X_0)ds] \\ &\le tM \qquad \text{where } M \ge 2L^2(1 + \mathbb{E}[X_0^2]) + 2L^2(1 + T). \end{aligned}$$

IV: suppose the assumption holds for $n - 1 \in \mathbb{N}$
IS: Take $n - 1 \to n$ then

$$\begin{aligned} d^n(t) = \mathbb{E}[|X_t^{n+1} - X_t^n|^2] &\le 2L^2T\mathbb{E}[\int_0^t |X_s^n - X_s^{n-1}|^2 ds] + 2L^2\mathbb{E}[\int_0^t |X_s^n - X_s^{n-1}|^2 ds] \\ &\overset{\text{IV}}{\le} 2L^2(1 + T)\int_0^t \frac{(Ms)^n}{n!}ds \\ &= 2L^2(1 + t)\frac{M^n}{(n + 1)!}t^{n+1} \le \frac{M^{n+1}t^{n+1}}{(n + 1)!}. \end{aligned}$$

Issue now is that because of Ω we cannot use completeness to argue the convergence, instead we use a similar argument to the uniqueness proof.

$$\begin{aligned} &\mathbb{E}[\max_{0\le t\le T} |X_t^{n+1} - X_t^n|^2] \\ &\le \mathbb{E}[\max_{0\le t\le T} 2\left|\int_0^t b(X_s^n, s) - b(X_s^{n-1}, s)ds\right|^2 + 2\left|\int_0^t B(X_s^n, s) - B(X_s^{n-1}, s)dW_s\right|^2] \\ &\le 2TL^2\mathbb{E}[\int_0^T |X_s^n - X_s^{n-1}|^2 ds] + 2\mathbb{E}[\max_{0\le t\le T}\left|\int_0^t B(X_s^n, s) - B(X_s^{n-1}, s)dsW_s\right|] \\ &\le 2TL^2\mathbb{E}[\int_0^T |X_s^n - X_s^{n-1}|^2 ds] + 8\mathbb{E}[\int_0^T |B(X_s^n, s) - B(X_s^{n-1}, s)|^2 ds] \\ &\le C \cdot \mathbb{E}[\int_0^T |X_s^n - X_s^{n-1}|^2 ds]. \end{aligned}$$

Where we used the following Doobs martingales Lp inequality

$$\mathbb{E}[\max_{0\le s\le t} |X(s)|^p] \le (\frac{p}{p - 1})^p\mathbb{E}[|X(t)|^p].$$

By Picard iteration we know the distance $d^n(t) = \mathbb{E}[|X_s^n - X_s^{n-1}|^2]$ is bounded by

$$
\begin{aligned}
C \cdot \mathbb{E}[\int_0^T |X_s^n - X_s^{n-1}|^2 ds] &= C \cdot \int_0^T \mathbb{E}[|X_s^n - X_s^{n-1}|^2] ds \\
&\leq \int_0^T \frac{(Mt)^n}{(n)!} \\
&= C \frac{M^n T^{n+1}}{(n+1)!}.
\end{aligned}
$$

Further more we get with a Markovs inequality

$$
\begin{aligned}
\underbrace{\mathbb{P}(\max_{0\leq t\leq T} |X_t^{n+1} - X_t^n|^2 > \frac{1}{2^n})}_{A_n} &\leq 2^{2n} \mathbb{E}[\max_{0\leq t\leq T} |X_t^{n+1} - X_t^n|^2] \\
&\leq 2^{2n} \frac{CM^n T^{n+1}}{(n+1)!}.
\end{aligned}
$$

Then by Borel-Cantelli we know

$$
\sum_{n=0}^{\infty} \mathbb{P}(A_n) \leq C \sum_{n=0}^{\infty} 2^{2n} \frac{(MT)^n}{(n+1)!} < \infty \Rightarrow \mathbb{P}(\bigcap_{n=0}^{\infty} \bigcup_{m=n}^{\infty} A_m) = 0.
$$

Define by a telescope argument

$$
X_t^n = X_t^0 + \sum_{j=1}^{n-1} (X_t^{j+1} - X_t^j).
$$

Then the above converges to

$$
X_t = X_0 + \int_0^t b(X_s, s) ds + \int_0^t B(X_s, s) dW_s.
$$

□

Remark. Uniqueness in a stochastic sense means that for two solution $X, \tilde{X}$ we have

$$
\mathbb{P}(X(t) = \tilde{X}(t),\ \forall t \in [0,T]) = 1 \Leftrightarrow \max_{0\leq t\leq T} |x(t) - \tilde{x}(t)| = 0 \text{ a.s.}.
$$

I.e they are indistinguishable

As a small side note we consider this example to distinguish modifications and indistinguishable.

Example. First note that for any $t \in [0,T]$ we have the following inclusion

$$
A := \{X(t) = \tilde{X}(t),\ \forall\ t \in [0,T]\} \subset \{X(t) = \tilde{X}(t)\} := A_t.
$$

i.e

$$\mathbb{P}(A) \leq P(A_t).$$

Such that indistinguishability implies modification where modification means

$$\forall\ t \in [0, T]\ :\ \mathbb{P}(A_t) = 1.$$

Chapter 2

Appendix

Theorem 2.0.1 (Divergence Theorem). Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)-$ dimensional sub-manifold of $\mathbb{R}^n$. Let $F : \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that $\triangledown F$ continuously to $\partial\Omega$. Then we have :

$$\int_{\Omega} \triangledown \cdot F d\mu = \int_{\partial\Omega} F \cdot N d\sigma.$$

where N is the outward pointing normal. (last component is positive)