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Sheet 7

Unclear if $u \in \mathcal{C}(\mathbb{R}^n)$ is $u : \mathbb{R}^n \to \mathbb{R}$ or $u : \mathbb{R}^n \to \mathbb{R}^n$, we consider the first case here but it doesn't change any of the following proofs

19. Twirling towards freedom

Let $u \in \mathcal{C}^2(\mathbb{R}^n)$ be a harmonic function. Show that the following functions are also harmonic.

Note Whenever we write $\frac{\partial v}{\partial x_i}$ we are really referring to $\tilde{x}(x)_i$

Exercise (a). $v(x) = u(x+b)$ for $b \in \mathbb{R}^n$

Solution. Using chain rule gives

$$\frac{\partial v}{\partial x_i} = \frac{\partial u}{\partial x_i} \cdot \frac{\partial (x+b)}{\partial x_i} = \frac{\partial u}{\partial x_i} \cdot 1.$$

Second derivative gets another $\cdot\ 1$.

Such that summing gives

$$\triangle v = 1 \cdot \triangle u = 0.$$

□

Exercise (b). $v(x) = u(ax)$ for $a \in \mathbb{R}^n$

Solution. Again using chain rule

$$\frac{\partial v}{\partial x_i} = \frac{\partial v}{\partial x_i} \cdot \frac{\partial (ax)}{\partial x_i} = \frac{\partial v}{\partial x_i} \cdot a.$$

For the second derivative :

$$\frac{\partial v}{\partial x_i} = \frac{\partial^2 u}{\partial x_i^2} \cdot \frac{\partial (ax)}{\partial x_i} \cdot a = \frac{\partial^2 u}{\partial x_i} \cdot a^2.$$

$$\triangle v = a^2 \triangle u = a^2 \cdot 0 = 0.$$

□

Exercise (c). $v(x) = u(Rx)$ for $R(x_1, \dots, x_n) = (-x_1, \dots, x_n)$

Solution. Using chain rule

$$\frac{\partial v}{\partial x_i} = \frac{\partial v}{\partial x_i} \cdot \frac{\partial (Rx)}{\partial x_i} = \begin{cases} -1 & \frac{\partial v}{\partial x_1} \text{ if } i = 1 \\ & \frac{\partial v}{\partial x_i} \end{cases}.$$

Second derivative if $i = 1$ we get another -1 and they cancel out s.t.

$$\frac{\partial^2 v}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2}.$$

Summing gives

$$\triangle v = \triangle u = 0.$$

□

Exercise (d). $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$

Solution. Note $(Ax)_i = \sum_{j=1}^n A_{ji} x_j$ such that using the chain rule

$$\frac{\partial v}{\partial x_i} = \frac{\partial u(Ax)}{\partial (Ax)_i} \frac{\partial Ax}{\partial x_i} = \nabla u \cdot \underbrace{A_i}_{i\text{-th col}}.$$

We get the gradient of u since we take the derivative of u with respect to $(Ax)_i$

Second derivative

$$\frac{\partial}{\partial x_i} \frac{\partial v}{\partial x_i} = \frac{\partial}{\partial (Ax)_i} \nabla u(Ax) \cdot A_i \cdot \frac{\partial Ax}{\partial x_i} = \nabla \cdot (\nabla u \cdot A_i) A_i.$$

Note since A is orthogonal we get $AA^T = \mathbb{1}$, writing out the terms as sum for clarity

$$\sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} A_{ji} A_{ki} = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0.$$

□

20. Harmonic Polynomials in Two Variables

Exercise (a). Let $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic

Solution. First note that $u^{(k)}(x_0) \in \mathcal{C}^\infty(\mathbb{R}^n)$ for any $k \in \mathbb{N}$ and $x_0 \in \mathbb{R}^n$
IA: $k = 1$

$$\triangle(u^{(1)}) = \left(\frac{\partial^2}{\partial x_i^2}, \dots, \frac{\partial^2}{\partial x_i^2}\right) \begin{pmatrix} \frac{\partial u(x_0)}{\partial x_1} \\ \vdots \\ \frac{\partial u(x_0)}{\partial x_n} \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \cdot \frac{\partial u}{\partial x_i}.$$

The order of differentiation does not matter such that we get

$$\triangle(u^{(1)}) = \frac{d}{dx}(\triangle u^1) = 0.$$

IV: Suppose the statement holds for $k \in \mathbb{N}$
IS: $k \to k+1$
We know that $u^{(k)} \in \mathcal{C}^\infty(\mathbb{R}^n)$ is harmonic by IV. such that by relabeling $u^{(k)} := v \in \mathcal{C}^\infty(\mathbb{R}^n)$ we can again apply the argument from the IA. which concludes the proof □

Exercise (b). Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x, y) = (x + iy)^n$ and let $u_n(x, y)$ and $v_n(x, y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic.

Solution. Treating the Laplacian in $\mathbb{C}$ as we do in $\mathbb{R}$

$$\partial_x f_n = n(x + iy)^{n-1}$$
$$\partial_y f_n = n \cdot i(x + iy)^{n-1}.$$

Then

$$\begin{aligned}\triangle f_n &= n(n-1)(x+iy)^{n-2} + (i^2 n(n-1)(x+iy)) \\ &= n(n-1) \cdot \left((x+iy)^{n-2} - (x+iy)^{n-2}\right) \\ &= 0.\end{aligned}$$

Since u_n, v_n are real and imaginary part , their Laplacian must be 0 as well. □

Exercise (c). A homogeneous polynomial of degree n in two variables is a polynomial of the form

$$p = \sum a_k x^k y^{n-k}.$$

Show that $\partial_x p$ and $\partial_y p$ are homogeneous of degree $n-1$

Solution. Calculating

$$\partial_x p = \frac{\partial}{\partial x} \sum_{k=0}^{n} a_k x^k y^{n-k} = \sum_{k=0}^{n} k \cdot a_k x^{k-1} y^{n-k}.$$

For $k = 0$ we have $k \cdot a_k = 0$ such that the first term vanishes

$$\sum_{k=0}^{n} k a_k x^{k-1} y^{n-k} = \sum_{k=1}^{n} k a_k x^{k-1} y^{n-k} = \sum_{i=0}^{n-1} (i+1) a_{i+1} x^i y^{(n-1)-i}.$$

For the case ∂_y we get the term $(n-k) \cdot a_k$ which is 0 for $k = n$ such that the last term of our sum vanishes and the degree is reduced by 1

□

Exercise (d). Show that such a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n

Solution. "$\Leftarrow$": Let p be a linear combination of u_n and v_n then by (b) we know u_n and v_n are harmonic, since the Laplacian is a linear operator, we conclude p must also be harmonic.

Other direction is missing □

Louisville's Theorem

Liouville's theorem (3.9 in the script) says that if u is bounded and harmonic, then u is constant. In this question we give a geometric proof in $\mathbb{R}^2$ using globular means defined when $B(x,r) \subset \Omega$ through

$$\mathcal{M}[v](x,r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y) dy.$$

Recall from Theorem 3.5 that if u is harmonic then it obeys $u(x) = \mathcal{M}[u](x,r)$.

Exercise (a). Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centered on the two points respectively. Show that the area of the intersection is

$$\text{area} B(a,r) \cap B(b,r) = 2r^2 a\cos(dr^{-1}) - 2d\sqrt{r^2 - d^2}.$$

Solution. W.l.o.g we may assume $a = (0, 0)$ and $b = (2d, 0)$, otherwise we just shift the plane to be centered at a first and rotate (both volume preserving)

In that case we can solve for the points of intersection by

$$x^2 + y^2 = r^2 = (x - 2d)^2 + y^2.$$

Such that

$$\begin{aligned} x^2 - (x - 2d)^2 &= 0 \\ x^2 - (x^2 - 4dx + 4d^2) &= 0 \\ 4dx - 4d^2 &= 0 \\ x &= d. \end{aligned}$$

And y

$$d^2 + y^2 = r^2 \Rightarrow y = \pm\sqrt{r^2 - d^2}.$$

Now either by integrating we get

$$x^2 + y^2 = r^2 \Rightarrow x = \sqrt{r^2 - y^2}.$$

$$2 \cdot \left(\int_{-\sqrt{r^2-d^2}}^{\sqrt{r^2-d^2}} \sqrt{r^2 - y^2} dy - 2d \cdot \sqrt{r^2 - d^2} \right).$$

Or the prettier way

We can get the light-green area by calculating the area of the triangle and subtracting it from the area of the sector and multiplying by 4, this gives the following

Triangle area

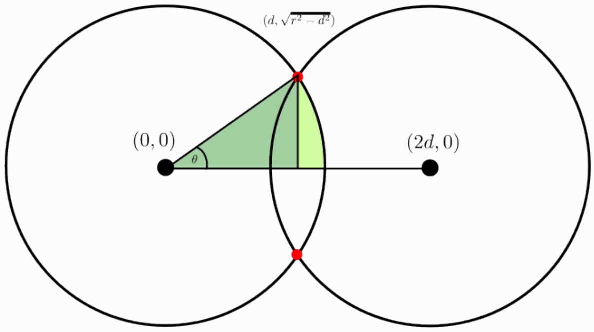
$$\frac{1}{2} \cdot d \cdot \sqrt{r^2 - d^2}.$$

Sector area , since $\theta = \cos^{-1}(\frac{d}{r})$ and the area of the sector is proportional to the angle where $\theta = 2\pi$ gives the full circle we get

$$\frac{r^2\theta}{2}.$$

Putting it together

$$4 \cdot \left(\frac{r^2 \cos^{-1}(\frac{d}{r})}{2} - \frac{d}{2}\sqrt{r^2 - d^2} \right)$$
$$= 2r^2 \cos^{-1}(dr^{-1}) - 2d\sqrt{r^2 - d^2}.$$



□

Exercise (b). Suppose that v is a bounded function on the plane $-C \leq v(x) \leq C$ for all x and some constant C. Show that

$$|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)| \leq \frac{2C}{\omega_2}(\pi - 2a\cos(\frac{d}{r})) - \frac{2d}{r}\sqrt{1 - d^2r^2}.$$

Solution. We get

$$|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)| = \frac{1}{\omega_2 r^2}\left|\left(\int_{B(a,r)} v(y)dy - \int_{B(b,r)} v(y)dy\right)\right|$$

.

leaving the constant

$$\begin{aligned} &= \left|\left(\int_{B(a,r)\setminus B(b,r)} v(y)dy - \int_{B(b,r)\setminus B(a,r)} v(y)dy\right)\right| \\ &\leq \int_{B(a,r)\setminus B(b,r)} |v(y)|dy + \int_{B(b,r)\setminus B(a,r)} |v(y)|dy \\ &\overset{\text{def.}}{\leq} \int_{B(a,r)\setminus B(b,r)} Cdy - \int_{B(b,r)\setminus B(a,r)} Cdy \end{aligned}$$

.

Then by drawing a picture and looking at it we know

$$\text{area}(B(a,r) \setminus B(b,r)) + \text{area}(B(a,r) \setminus B(b,r)) = 2\pi r^2 - 2 \cdot \text{area}(B(a,r) \cap B(b,r)).$$

Using (a)

$$|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)| \leq \frac{2C}{\omega_2} \cdot \left(\pi - 2\cos^{-1}(\frac{d}{r}) + \frac{2d}{r^2}\sqrt{r^2 - d^2}\right).$$

□

Exercise (c). Let $u \in \mathcal{C}^2(\mathbb{R}^2)$ be a bounded harmonic function. Complete the proof that u is constant

Solution. By Theorem 3.5 we know that

$$u(x) = \mathcal{M}[u](x,r).$$

Picking any points $a, b \in \mathbb{R}^2$ we check the distance

$$|u(a) - u(b)| = |\mathcal{M}[u](a,r) - \mathcal{M}[u](b,r)| \leq C(r,d).$$

Where $r > d := \|a-b\|$, i.e

$$\begin{aligned}\lim_{h\to 0}\frac{|u(a+h)-u(a)|}{h} &= \lim_{h\to 0}\frac{|\mathcal{M}[u](a+h,r(h))-\mathcal{M}[u](a,r(h))|}{h}\\ &\le \lim_{h\to 0}\frac{1}{h}\cdot C(r(h),h)\\ &= 0.\end{aligned}$$

Where $C(r(h),h)\to 0$ as $h\to 0$, thus u must be constant if it is bounded and harmonic □

Weak Tea

In this question we try to generalise the idea of spherical means to distribution in the way suggested in the lectures. Let $\lambda_\varepsilon : \mathbb{R}\to\mathbb{R}$ be the standard mollifier and define a sphere mollifier

$$\Lambda_{x,r,\varepsilon}(y) = \frac{\lambda_\varepsilon(|y-x|-r)}{n\omega_n|y-x|^{n-1}}.$$

Exercise (a). Describe the support of $\Lambda_{x,r,\varepsilon}$

Solution. By definition it is all y such that $\Lambda_{x,r,\varepsilon}(y)\neq 0$ i.e all y such that $|y-x|-r \in \operatorname{supp}\lambda_\varepsilon := \overline{B(0,\varepsilon)}$, Which gives

$$|y-x|-r\le\varepsilon \Rightarrow |y-x|\le\varepsilon+r.$$

$$y\in B(x,\varepsilon+r).$$

□

Exercise (b). We may try to define the spherical mean of a distribution F as $\lim_{\varepsilon\to 0} F(\Lambda_{x,r,\varepsilon})$. However this does not always exist. Let G be the distribution in Exercise 17(d), submanifold inte- gration on the unit circle. Show that $G(\Lambda_{0,1,\varepsilon}) = \lambda_\varepsilon(1)$ and therefore the limit does not exist.

Solution. First we can check that indeed $\Lambda\in\mathcal{C}_0^\infty$ such that $G(\Lambda_{0,1,\varepsilon})$ is valid

$$\Lambda_{0,1,\varepsilon} = \frac{\lambda_\varepsilon(|y|-1)}{2\omega_2|y|}.$$

We have $C \coloneqq \{x^2 + y^2 = 1\}$ i.e $C \coloneqq \partial B(0,1)$ then

$$\begin{aligned} G(\Lambda_{0,1,\varepsilon}) &= \int_{\{x^2+y^2=1\}} \frac{\lambda_\varepsilon(|y|-1)}{2\omega_2 |y|} d\sigma \\ &= \int_{\partial B(0,1)} \frac{\lambda_\varepsilon(0)}{2\omega_2} d\sigma = \lambda_\varepsilon(0). \end{aligned}$$

If we didn't shift by $r = 1$ it would be 1. For $\varepsilon \to 0$ we have $\lambda_\varepsilon(0) \to \infty$ □

Exercise (c)**.** Show that the limit $\lim_{\varepsilon \to 0} F(\Lambda_{x,r,\varepsilon})$

Solution. Let F be a harmonic distribution i.e $\triangle F = 0$ then F has the weak mean value property by Theorem 3.6 and we know that F vanishes on any test function with $\int \psi d\mu = 0$ where $\psi \in \mathcal{C}_0^\infty((0,r))$
We know that necessarily we have

$$\int_\Omega \Lambda_{x,r,\varepsilon} d\mu = 1.$$

Choose any test function φ such that its integral is 1 then $\tilde{\varphi} = \varphi - \Lambda_{x,r,\varepsilon}$

$$\int_\Omega \tilde{\varphi} = 0.$$

Since distribution are linear operators

$$0 = F(\tilde{\varphi}) = F(\varphi) - F(\Lambda_{x,r,\varepsilon}).$$

Thus the limit exists. □