

# Chapter 1

## Introduction

Mean Field Particle Systems is about the study of particles which are represented by (stochastic) differential equations. This course in particular is concerned with the behaviour of the system as the size grows to infinity:

**Definition 1.0.1 (Toy Mean Field Particle System).** Let  $N \in \mathbb{N}$  then a Mean Field Particle System of first order is given by :

$$x_1(t), \dots, x_n(t) \in \mathcal{C}^1([0, T]; \mathbb{R}^d) \quad x_i(0) = c_i.$$

Where each particle satisfies

$$dx_i = \frac{1}{N} \sum_{j=1}^N K(x_i, x_j) dt + \sigma dB_i(t).$$

Where  $B_i$  is a Brownian motion; For  $\sigma = 0$  the system is called deterministic.

**Example.** Example choices for  $K$  are :

$$K(x_i, x_j) = \nabla(|x_i - x_j|^2).$$

or :

$$\sigma \gamma = \frac{x_i - x_j}{|x_i - x_j|^d}.$$

Goal is to study what happens at  $N \rightarrow \infty$ , to do so we consider how the measure of a system converges

**Definition 1.0.2 ((Empirical) Measure of a System).** Consider the point measure for every  $x_i : \delta_{x_i(t)}$  , then the measure of the System of order  $N$  is

:

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)}.$$

**Assumption 1.0.1.** For initial data the empirical measure of a system converges  $\mu_N(0) \rightarrow \mu(0)$  where  $\mu$  is absolutely continuous with respect to the Lebesgue Measure

**Corollary.** By Radon Nikodym

$$d\mu = \rho_0 dx \quad \rho_0 \in L^1(\mathbb{R}^d).$$

It can be shown that  $\mu$  solves a PDE, to do so we compute the derivative of  $\mu$  using test functions

$$\forall \phi \in C_0^\infty(\mathbb{R}^d).$$

$$\begin{aligned} \frac{d}{dt} \langle \mu_N(t), \phi \rangle &= \frac{d}{dt} \int_{\mathbb{R}^d} \phi(x) d\mu_N(t)(x) = \frac{d}{dt} \int \frac{1}{N} \sum_{i=1}^N \phi(x) d\delta_{x_i(t)} \\ &= \frac{1}{N} \sum_{i=1}^N \frac{d}{dt} \phi(x_i(t)) \\ &\stackrel{\text{Chain.}}{=} \frac{1}{N} \sum_{i=1}^N \nabla \phi(x_i(t)) \frac{d}{dt} x_i(t) \\ &= \frac{1}{N} \sum_{i=1}^N \nabla_x \phi(x_i(t)) * \underbrace{\frac{1}{N} \sum_{j=1}^N K(x_i(t), x_j(t))}_{\text{Def.}} \\ &= \frac{1}{N} \sum_{i=1}^N \nabla_x \phi(x_i(t)) * \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{R}^d} K(x_i(t), y) d\delta_{x_j(t)}(y) \\ &= \frac{1}{N} \sum_{i=1}^N \nabla_x \phi(x_i(t)) * \int_{\mathbb{R}^d} K(x_i(t), y) d\mu_N(t) \\ &= \int_{\mathbb{R}^d} \nabla \phi(x) \int_{\mathbb{R}^d} K(x, y) d\mu_N(t, y) d\mu_N(t, x) \end{aligned}$$

Where the last line can be rewritten by using Integration by Parts (Divergence Theorem) :

$$\int_{\mathbb{R}^d} \nabla \phi(x) \int_{\mathbb{R}^d} K(x, y) d\mu_N(t, y) d\mu_N(t, x) \stackrel{\text{Part.}}{=} - \langle \nabla * (\mu_N \int_{\mathbb{R}^d} K(\cdot, y) d\mu_N(y)), \phi \rangle$$

This means  $\mu$  satisfies :

$$\partial_t \mu_N + \nabla * (\mu_N * \int K(\cdot, y) d\mu_N(y)) = 0 \xrightarrow{N \rightarrow \infty} \partial_t \mu + \nabla * (\mu * \int K(\cdot, y) d\mu(y)) = 0.$$

In practical applications (Theoretical Physics , Biology) systems that are considered are often of second order

**Definition 1.0.3 (Toy Second Order System).** Given  $N \in \mathbb{N}$  a Second Order System is given by

$$(x_i(t), v_i(t)), \dots, (x_N(t), v_N(t)) \in \mathbb{R}^{2d}.$$

Such that :

$$\begin{aligned} \frac{d}{dt} x_i(t) &= v_i(t) \\ \frac{d}{dt} v_i(t) &= \frac{1}{N} \sum_{j=1}^N F(\underbrace{x_i(t), v_i(t)}_{\text{Position and Velocity of itself}}; x_j(t), v_j(t)) + \sigma \frac{dB_t}{dt} \end{aligned}$$

**Example (Gravitational Force).** An example of F could be :

$$F(x, v, y, u) = \frac{x - y}{|x - y|^d}.$$

**Definition 1.0.4 (Second Order Measure).** The Measure of a second order System is given by :

$$\mu_N(x, v) = \frac{1}{N} \sum_{i=1}^N \delta_{(x_i(t), v_i(t))}.$$

**Exercise 1.** Show what PDE  $\mu$  solves for  $\sigma = 0$ , *Hint* : Calculate  $\frac{d}{dt} \langle \mu_N, \phi \rangle$  for some test function  $\phi \in C_0^\infty(\mathbb{R}^{2d})$

## Chapter 2

# Deterministic Mean Field Particle Systems

### 2.1 Goal

**Definition 2.1.1** (Deterministic Mean Field Particle System). For  $N \in \mathbb{N}$  a deterministic mean field particle system is given by  $N$  particles :

$$x_1(t), \dots, x_N(t) \in \mathcal{C}^1([0, T]; \mathbb{R}^d) \quad x_i(0) = c_i.$$

With initial points :

$$x_i(0) = x_{i,0} \in \mathbb{R}^d.$$

And the relation :

$$\frac{d}{dt}x_i = \frac{1}{N} \sum_{j=1}^N K(x_i, x_j).$$

The system is then given by :

$$X_N = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix} \in \mathbb{R}^{dN}.$$

The goal is to solve the above  $d * N$  dimensional system under assumptions on  $K$ .

### 2.2 ODE Theory

**Definition 2.2.1** (Initial Value Problem (standard)). For  $\forall T > 0$  the standard

ode system is given by :

$$\begin{aligned} x' &= f(t, x) \\ x|_{t=0} &= x_0 \in \mathbb{R}^n. \end{aligned}$$

with  $t \in [0, T]$ ,  $x(t) \in \mathbb{R}^n$

**Assumption 2.2.1.** Condition for global existence :  $f$  is continuous in  $(t, x) \in [0, T] \times \mathbb{R}^n$

Condition for uniqueness :  $f$  is Lipschitz continuous in  $x$

**Theorem 2.2.1 (Picard Iteration).** Whenever Assumption 2.2.1 holds the standard IVP has a unique solution  $x \in \mathcal{C}^1([0, T]; \mathbb{R}^n)$

**Proof.** Rewriting the IVP using integration :

$$x(t) - x(0) = \int_0^t f(s, x(s)) ds \quad \forall t \in [0, T].$$

We construct the following sequence by :

$$\begin{aligned} x_1(t) &= x_0 + \int_0^t f(s, x_0) ds \\ x_2(t) &= x_0 + \int_0^t f(s, x(s)) ds & \vdots \\ x_m(t) &= x_0 + \int_0^t f(s, x_{m-1}(s)) ds. \end{aligned}$$

**Step 1:** First part of the proof consists of proving the above sequence is converging

**Step 2:** The second part is then showing the limit is a solution to the IVP

Under Assumption 2.2.1 we know that  $f$  is continuous such that  $(x_n)_{n \in \mathbb{N}} \subset \mathcal{C}^1([0, T]; \mathbb{R}^n)$ , As  $\mathcal{C}^1$  is complete any sequence that is cauchy must also converge against a limit in the space.

$$\begin{aligned}
 |x_2 - x_1| &= \left| \int_0^t f(s, x_1(s)) ds - \int_0^t f(s, x_0(s)) ds \right| = \left| \int_0^t f(s, x_{m-1}(s)) - f(s, x_{n-1}(s)) ds \right| \\
 &\leq \int_0^t |f(s, x_1(s)) - f(s, x_0(s))| ds \\
 &\stackrel{\text{Lip.}}{\leq} L \int_0^t |x_1(s) - x_0(s)| ds \\
 &= L \int_0^t \left| \int_0^{s_0} f(s, x_0) ds \right| ds_0 \\
 &\leq L * \int_0^t \int_0^{s_0} |f(s, x_0)| ds ds_0 \\
 &\leq L \underbrace{M}_{=\max_{s \in [0, T]} |f(s, x_0)|} \frac{t^2}{2}.
 \end{aligned}$$

By repeatedly using the Lipschitz continuity of  $f$  the following induction assumption is motivated :

$$|x_m(t) - x_{m-1}(t)| \leq ML^{m-1} \frac{t^m}{m!}. \quad (\text{IA})$$

(IS) :  $m \rightarrow m + 1$

$$\begin{aligned}
 |x_{m+1}(t) - x_m(t)| &\stackrel{\text{Lip.}}{\leq} L \int_0^t |x_m(s) - x_{m-1}(s)| ds \\
 &\stackrel{\text{IA.}}{\leq} L \int_0^t \frac{ML^{m-1}s^m}{m!} ds = ML^m \frac{t^{m+1}}{(m+1)!}.
 \end{aligned}$$

For any  $n, m \in \mathbb{N}$  and assuming without loss of generality that  $n > m$  such that  $n = m + p$  for  $p \in \mathbb{N}$  :

$$\begin{aligned}
 |x_n(t) - x_m(t)| &= |x_{m+p}(t) - x_m(t)| \leq \sum_{k=m+1}^{m+p} |x_k(t) - x_{k-1}(t)| \stackrel{\text{Ind.}}{\leq} M \sum_{k=m+1}^{m+p} \frac{L^{k-1}T^k}{k!} \\
 &= \frac{M}{L} \sum_{k=m+1}^{m+p} \frac{(LT)^k}{k!} = \frac{M}{L} \frac{(LT)^{m+1}}{(m+1)!} \sum_{k=0}^{p-1} \frac{(LT)^k}{k!} \\
 &\leq \frac{M}{L} \frac{(LT)^{m+1}}{(m+1)!} e^{LT} \xrightarrow{m \rightarrow \infty} 0 \text{ uniformly in } t \in [0, T].
 \end{aligned}$$

This shows that  $(x_m)_{m \in \mathbb{N}}$  is Cauchy and has a limit  $x \in \mathcal{C}([0, T]; \mathbb{R}^n)$  with :

$$\max_{t \in [0, T]} |x_m(t) - x(t)| \rightarrow 0.$$

It remains to show that  $x(t)$  is a solution to the IVP i.e :

$$x(t) = \lim_{m \rightarrow \infty} x_0 + \int_0^t f(s, x_{m-1}(s)) ds \leftrightarrow x_0 + \int_0^t f(s, x(s)) ds.$$

Which can be shown by :

$$\begin{aligned} \left| \lim_{m \rightarrow \infty} \int_0^t f(s, x_{m-1}(s)) - f(s, x(s)) ds \right| &\leq \lim_{m \rightarrow \infty} \int_0^t |f(s, x_{m-1}(s)) - f(s, x(s))| ds \\ &\leq \lim_{m \rightarrow \infty} L \int_0^t |x_{m-1}(s) - x(s)| ds \\ &\leq \lim_{m \rightarrow \infty} Lt * \max_{s \in [0, t]} |x_{m-1}(s) - x(s)| \\ &\leq \lim_{m \rightarrow \infty} Lt * \max_{s \in [0, T]} |x_{m-1}(s) - x(s)| \\ &= 0. \end{aligned}$$

It remains to show that the solution is unique, for that assume  $x, \hat{x} \in \mathcal{C}([0, T]; \mathbb{R}^n)$  are both solutions to the IVP. Meaning that :

$$\begin{aligned} x(t) &= x_0 + \int_0^t f(s, x(s)) ds \\ \hat{x}(t) &= x_0 + \int_0^t f(s, \hat{x}(s)) ds. \end{aligned}$$

Then :

$$\begin{aligned} |x - \hat{x}| &\leq \int_0^t |f(s, x(s)) - f(s, \hat{x}(s))| ds \leq L * \int_0^t |x(s) - \hat{x}(s)| ds \\ &= L \int_0^t \underbrace{e^{-\alpha s} |x(s) - \hat{x}(s)|}_{=\rho(s)} e^{\alpha s} ds \\ &\leq L \max_{t \in [0, T]} \rho(t) * \frac{1}{\alpha} (e^{\alpha t} - 1) \\ &\leq L \max_{t \in [0, T]} \rho(t) * \frac{1}{\alpha} * e^{\alpha t}. \end{aligned}$$

By rearranging with the initial term :

$$\begin{aligned} \rho(t) = e^{-\alpha t} |x(t) - \hat{x}(t)| &\leq \frac{L}{\alpha} \max_{t \in [0, T]} \rho(t) \\ \max_{t \in [0, T]} \rho(t) &\leq \frac{L}{\alpha} \max_{t \in [0, T]} t \rho(t). \end{aligned}$$

by choosing  $\alpha = 2L$  :

$$\max_{t \in [0, T]} e^{-2Lt} |x(t) - \hat{x}(t)| = 0.$$

And the solutions must be equal for  $\forall t \in [0, T]$ . □

The reason this proof deviates from the standard Picard-Lindelöf theorem, is that for our systems we require Global existence, doing so by requiring  $f$  to be globally Lipschitz continuous.

**Theorem 2.2.2.** The solution  $x(t, t_0, x_0) \in \mathcal{C}$  is continuously dependent on  $(t_0, x_0)$

**Theorem 2.2.3 (Gronwalls inequality).** For  $\alpha, \beta, \phi \in \mathcal{C}([a, b]; \mathbb{R})$   $\beta \geq 0$  and

$$0 \leq \phi(t) \leq \alpha(t) + \int_a^t \beta(s)\phi(s)ds, \quad \forall t \in [a, b].$$

then :

$$\phi(t) \leq \alpha(t) + \int_a^t \beta(s) \exp\left(\int_s^t \beta(\tau)d\tau\right) \alpha(s)ds.$$

**Proof.** Denote  $\psi(t) = \int_a^t \beta(s)\phi(s)ds$  then

$$\begin{aligned} \psi'(t) &= \beta(t)\phi(t) \leq \beta(t)\alpha(t) + \beta(t)\psi(t) \\ &= \beta(t) * (\alpha(t) + \psi(t)) \end{aligned}$$

Recall  $\dot{x} + a(t)x + b(t) = 0$

$$\begin{aligned} (\dot{\psi}(t) - \beta(t)\psi(t))e^{-\int_a^t \beta(s)ds} &\leq \beta(t)\alpha(t) * e^{-\int_a^t \beta(s)ds} \\ (e^{-\int_a^t \beta(s)ds}\psi(t))' &\leq \beta(t)\alpha(t) * e^{-\int_a^t \beta(s)ds}. \end{aligned}$$

Integrating gives :

$$(e^{-\int_a^t \beta(s)ds}\psi(t)) \leq \int_a^t \beta(s)\alpha(s)e^{-\int_a^s \beta(r)dr}ds.$$

□

**Definition 2.2.2 (Regularity).** A function  $K : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$  is called regular if :

1.  $K \in \mathcal{C}^1(\mathbb{R}^{2d}; \mathbb{R}^d)$  (gives local lipschitz )
2. And  $\exists L > 0$  s.t. :

$$\sup_y |\nabla_x K(x, y)| + \sup_x |\nabla_y K(x, y)| \leq L.$$



**Remark.** We further assume  $K$  has the following properties :

$$\begin{aligned} K(x, y) &= -K(y, x) & (\text{antisymmetric}) \\ K(x, x) &= 0. \end{aligned}$$

**Theorem 2.2.4.** For regular  $K$  the MPS has a solution for all  $T > 0$

$$\begin{cases} \frac{d}{dt}x_i &= \frac{1}{N} \sum_{j=1}^N K(x_i, x_j), 1 \leq i \leq N \\ x_i(0) &= x_{i,0} \in \mathbb{R}^d \end{cases}.$$

has a unique solution by Picard-Iteration :

$$X_N(t) = (x_1(t), x_2(t), \dots, x_N(t)) \in \mathcal{C}^1([0, T]; \mathbb{R}^{dN}).$$

**Definition 2.2.3 (Empirical Measure of a System).** Consider the point measure for every  $x_i : \delta_{x_i(t)}$ , then the measure of the System of order  $N$  is given by

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)}.$$

As shown in the introduction  $\mu_N$  is a (weak-) solution to the following PDE

$$\partial_t \mu_N + \nabla * (\mu_N * \int K(\cdot, y) d\mu_N(y)) = 0.$$

**Intuition.** Now for  $N \rightarrow \infty$  if we have  $\mu_N \xrightarrow{\text{in some sense}} \mu$  then  $\mu$  is a (weak) solution to

$$\partial_t \mu + \nabla * (\mu * \int K(\cdot, y) d\mu(y)) = 0.$$

with

$$\mu_0 \leftarrow \mu_N(0).$$

## 2.3 Weak Solutions and Distributions

Distributions are a more general class of functions and can be seen as the dual space of the space of test functions

**Definition 2.3.1 (Multi-Index).** A multi-index  $\gamma \in \mathbb{N}_0^n$  of length  $|\gamma| = \sum_i \gamma_i$  for example  $\gamma = (0, 2, 1) \in \mathbb{N}_0^3$  can be used to denote partial derivatives of

higher order as such :

$$\partial^\gamma = \prod_i \left( \frac{\partial}{\partial x_i} \right)^{\gamma_i}.$$

**Remark.** Only sensible cause partial derivatives commute as otherwise the index would be ambiguous.

**Definition 2.3.2 (Test Functions).** For  $\Omega \subset \mathbb{R}^d$  the space of test functions  $\mathcal{D}(\Omega) \subset \mathcal{C}_0^\infty(\Omega)$ . We say a sequence of test functions  $(\phi_m)_{m \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\Omega)$  converges against some limit  $\phi \in \mathcal{C}_0^\infty(\Omega)$  iff.

1.  $\exists$  a compact set  $K \subset \Omega$  s.t.  $\text{supp } \phi_m \subset K$  for all  $m \in \mathbb{N}$
2.  $\forall$  multi-indexes  $\alpha \in \mathbb{N}_0^n$  :

$$\sup_K |\partial^\alpha \phi_m - \partial^\alpha \phi| \xrightarrow{m \rightarrow \infty} 0.$$

**Remark.**  $\mathcal{D}(\Omega)$  is a linear space

**Definition 2.3.3 (Distribution).** The space of distributions  $\mathcal{D}(\Omega)'$  is the dual space of  $\mathcal{D}(\Omega)$  i.e.  $\mathcal{D}(\Omega)'$  contains all the continuous linear functionals  $T$

$$T : \mathcal{D}(\Omega) \rightarrow \mathbb{K}.$$

**Remark.** Continuity refers to the notion that for a sequence  $(\phi_m)_{m \in \mathbb{N}} \subset \mathcal{D}(\Omega)$  with limit  $\phi$  then :

$$\phi_m \rightarrow \phi \Rightarrow T(\phi_m) \rightarrow T(\phi).$$

linearity :

$$T(\alpha\phi_1 + \beta\phi_2) = \alpha T(\phi_1) + \beta T(\phi_2).$$

We sometimes write  $\langle T, \phi \rangle$  instead of  $T(\phi)$

**Definition 2.3.4 (Convergence).** For a sequence of distributions  $(T_m)_{m \in \mathbb{N}} \subset \mathcal{D}(\Omega)'$  we say it converges against a limit  $T \in \mathcal{D}(\Omega)'$  iff

$$\langle T_m, \phi \rangle \rightarrow \langle T, \phi \rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

**Example.** Every locally integrable function  $f \in L_{\text{loc}}^1(\Omega) := \{f \mid \forall K \subset$

$\Omega, \int_K f(x)dx < \infty\}$  defines a Distribution by :

$$T_f(\phi) = \langle T_f, \phi \rangle = \int_{\Omega} f(x)\phi(x)dx. \quad \forall \phi \in \mathcal{D}(\Omega).$$

i.e.  $L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$

**Example.** Probability densities are distributions in the same sense as for  $L^1_{\text{loc}}$  functions

**Example.** (Probability - ) Measures  $\mu \in \mathcal{M}(\Omega)$  define a distributions , by :

$$\langle T_{\mu}, \phi \rangle = \int_{\mathbb{R}^d} \phi(x)d\mu(x) < \infty \quad \forall \phi \in \mathcal{D}(\Omega).$$

A prominent example is the  $\delta$  distribution defined by :

$$\langle \delta, \phi \rangle = \int_{\mathbb{R}^d} \phi(x)d\delta = \phi(0).$$

Remember for a measurable set  $E$

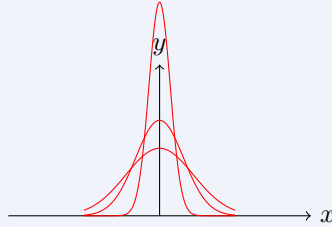
$$\delta_x(E) = \begin{cases} 1, & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}.$$

Coming back to our empirical measure from 2.2.3, we can see the corresponding distribution is defined by :

$$\langle \mu_n, \phi \rangle = \frac{1}{N} \sum_{i=1}^N \phi(x_i).$$

Some examples in approximation of  $\delta$  distribution

**Example (Heat Kernel).** The heat kernel  $f_t(x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$  approximates the  $\delta$  distribution



**Proof.**

$$\begin{aligned}\lim_{t \rightarrow 0+} \langle f_t, \phi \rangle &= \lim_{t \rightarrow 0+} \int_{\mathbb{R}} f_t(x) \phi(x) dx = \lim_{t \rightarrow 0+} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \phi(x) dx \\ &= \lim_{t \rightarrow 0+} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \phi(2ty) dy \\ &\stackrel{*}{=} \phi(0) = \langle \delta, \phi \rangle.\end{aligned}$$

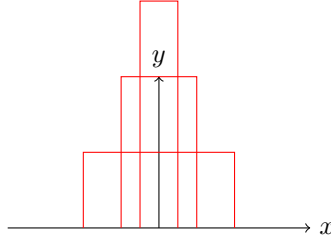
with the transformation  $\frac{x}{\sqrt{2t}} = y$

□

**Exercise 2.** Show the change of integration and limit is valid in \*

Further examples include :

$$Q_n(x) = \begin{cases} \frac{n}{2}, & \text{if } |x| \leq \frac{1}{n} \\ 0 & \text{if } |x| > \frac{1}{n} \end{cases}.$$



And the dirichlet kernel

$$D_n(x) = \frac{\sin(n + \frac{1}{2})x}{\sin(\frac{x}{2})} = 1 + 2 \sum_{k=1}^n \cos(kx) \rightarrow 2\pi\delta.$$

To define the notion of a distribution solving a PDE we need to first define the way we take the derivatives of distributions

**Definition 2.3.5** (Weak derivative of Distributions).  $\forall T \in \mathcal{D}(\Omega)'$  .  $\partial_i T$  is given by

$$\langle \partial_i T, \phi \rangle := -\langle T, \partial_i \phi \rangle \quad \forall \phi \in \mathcal{D}(\Omega).$$

We first show this for all distributions that are defined by a  $f \in L^1_{\text{loc}}$ , every other distribution  $T$  also has to satisfy this property.

**Exercise 3.** Proof the above equality for distributions  $f \in L^1_{\text{loc}}$  and show for arbitrary distribution  $T$  that :

$$-\langle T, \partial_i \phi \rangle.$$

is continuous and linear.

*Hint :* Integration by parts; Why does it vanish on the boundary ?

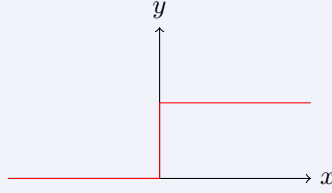
**Example.** For the  $\delta$  distribution :

$$\langle \delta^{(k)}, \phi \rangle = (-1)^k \phi^{(k)}(0).$$

**Example.** Heaviside The Heaviside step function is defined by :

$$H = \begin{cases} 1, & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \in L^1_{\text{loc}}.$$

$$\begin{aligned} \langle H', \phi \rangle &:= -\langle H, \phi' \rangle = -\int_{-\infty}^{\infty} H(x) \phi'(x) dx \\ &= -\int_0^{\infty} \phi'(x) dx = \phi(0) = \langle \delta, \phi \rangle. \end{aligned}$$



Using all the above we can rewrite our many particle system (MPS) by using the empiric measure and distributions

$$\begin{cases} \frac{d}{dt} x_i &= \langle K(x_i, \cdot), \mu_N \rangle = \int K(x_i, y) d\mu_N(y) \\ x_i(0) &= x_{i,0} \end{cases}.$$

**Definition 2.3.6 (Weak Solution of MFE).** We say  $\mu$  is a weak solution of the Mean-Field-Equation (MFE) iff for  $\forall t \in [0, T]$  ,  $\mu_t \in \mathcal{M}(\mathbb{R}^d)$  satisfies for all test functions  $\forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$  the following equation

$$\langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle = \int_0^t \langle \mu_s K \mu_s, \phi \rangle ds.$$

**Remark.** If  $\mu_0 = \mu_{N,0}$  then  $\mu_{N,t}$  is a weak solution of the MFE

**Theorem 2.3.1.** Let the empirical measure 2.2.3 be denoted by  $\mu_N$  then

for "good" (regular ?)  $K(x, y)$  we have

$$\frac{d}{dt} \langle \mu_N, \phi \rangle = \langle \mu_N, \underbrace{\int K(x, y) d\mu_N(y)}_{=K\mu_N} \rangle = -\langle \nabla * \nabla(\mu_N K \mu_N), \phi \rangle.$$

Such that  $\mu_N$  is a weak solution of the MPDE

$$\partial_t \mu_N + \nabla * \nabla(\mu_N K \mu_N) = 0.$$

**Remark.** Note when we talk about weak solution, it means the PDE is solved in the sense of distributions.

**Exercise 4.** Show  $\mu_N K \mu_N$  as defined above is a distribution for regular/ good  $K(x, y)$

**Proof.**

□

**Definition 2.3.7** (characteristic problem for MFE). The corresponding characteristic is given by :

$$\begin{cases} \frac{d}{dt} x(t, x_0, \mu_0) &= \int_{\mathbb{R}^d} K(x(t, x_0, \mu_0), y) d\mu_t(y) \\ x(0, x_0, \mu_0) &= x_0 \in \mathbb{R}^d \\ \mu_t &= x(t, *, \mu_0) \# \mu_0 \end{cases}.$$

## Chapter 3

# Stochastic Mean Field Particle Systems

5-6 weeks

3.1 Basics of probability

3.2 Bad K

3.3 Convergence

## Chapter 4

# New Results

### 4.1 Relative entropy Method

Goal to prove "strong" convergence in  $L^1$



## Chapter 5

# Appendix

**Theorem 5.0.1** (Divergence Theorem ). Let  $\Omega \subset \mathbb{R}^n$  be bounded and open with  $\partial\Omega$  being a  $(n-1)$ - dimensional sub-manifold of  $\mathbb{R}^n$ . Let  $F : \overline{\Omega} \rightarrow \mathbb{R}^n$  be continuous and differentiable on  $\Omega$  such that  $\nabla F$  continuously to  $\partial\Omega$ . Then we have :

$$\int_{\Omega} \nabla * F d\mu = \int_{\partial\Omega} F * N d\sigma.$$

where  $N$  is the outward pointing normal. (last component is positive)