What is the MVE

The Mckean-Vlasov Equation is in a sense the limiting equation of a Stochastic Many Particle System

$$\text{(SDEN)}\begin{cases} dX_i^N(t) = b(X_i^N(t), \mu_N(t))dt + \sigma(X_i^N(t), \mu_N(t))dW_t^i \\ X_i^N(0) = X_{i,0}^N \end{cases}.$$

Now as $N \to \infty$ we get

$$\text{(MVE)}\begin{cases} dY(t) = b(Y(t), \mu(t))dt + \sigma(Y(t), \mu(t))dW_t \\ Y(0) = \xi \in L^2 \\ \mu \sim \mathcal{L}(Y) \end{cases}.$$

What is the MVE

Using an SDE approach we get a Solution to the MVE as long as b and σ are Lipschitz. The Mean-Field-Limit can then be formulated by considering an intermediate Empirical Measure (of Y_i), then

$$\mathbb{E}[d_t^2(\mu_N, \mu)] \leq 2\mathbb{E}[d_t^2(\mu_N, \mu_N^Y)] + 2\mathbb{E}[d_t^2(\mu, \mu_N^Y)].$$

PDE Approach

The PDE setting makes the following observations, that if

$$b(Y(t), u) = \int F(Y(t) - y)u(y)dy = \int F(y)u(Y(t) - y)dy.$$

And

$$\sigma = \sqrt{2}.$$

Then

$$(\text{MVE*}) \begin{cases} dY(t) = \Big[F \star \mu(Y(t))\Big]dt + \sqrt{2}dW_t \\ Y(0) = \xi \in L^2 \\ \mu(t) \sim \mathcal{L}(Y(t)) \end{cases}.$$

Has a solution if $F \star \mu$ is bounded Lipschitz. This allows the possibility of singularities.

PDE Approach

We check for $\phi \in \mathcal{C}_0^\infty$, by Itôs formula we see

$$\begin{aligned}\phi(Y(t),t)-\phi(Y(0),0) &= \int_0^t \partial_t\phi + \nabla\phi * (F \star \mu(s))(Y(s))ds \\ &+ \int_0^t \Delta\phi ds + \int_0^t \nabla\phi\sqrt{2}dW_s.\end{aligned}$$

Then by taking the expectation we see that, μ satisfies the parabolic pde

$$\begin{cases} \partial_t\mu - \Delta\mu + \nabla * [(F \star \mu) * \mu] = 0 \\ \mu(0) = \mu_0 \end{cases}.$$

If μ has density u then the PDE gives us additional regularity such that we can indeed consider "worse" F.

Goal

We notice that if we get a solution $d\mu = u$ to the above PDE, then the SDE

$$(\text{MVE*})\begin{cases} dY(t) = (F \star u)(Y(t))dt + \sqrt{2}dW_t \\ Y(0) = \xi \in L^2 \end{cases}.$$

has a solution Y if $F \star u$ is bounded and Lipschitz, in turn the Law of $\mathcal{L}(Y) = \overline{\mu}$ solves

$$\begin{cases} \partial_t \overline{\mu} - \Delta \overline{\mu} + \nabla * [(F \star \mu) * \overline{\mu}] = 0 \\ \overline{\mu}(0) = \mu_0 \end{cases}.$$

or with densities

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} + \nabla * [(F \star u) * \overline{u}] = 0 \\ \overline{u}(0) = u_0 \end{cases}.$$

This implies that if $u = \overline{u} = \mathcal{L}(Y)$, then we solve the MVE* (knowing the law is enough ?)

Technique

We seek to solve the non-local parabolic PDE

$$\text{FIN}\begin{cases} \partial_t \mu - \Delta\mu + \nabla * [(F \star \mu) * \mu] = 0 \\ \mu(0) = \mu_0 \end{cases}.$$

1. Solve simple Heat-Equation by Heat-kernel/Fundamental-Solution Representation
2. We break up the above PDE into a couple intermediate ones :

   $$\text{LDE} \to \text{PDE}(v) \to \text{FIN}.$$

   Roughly that means, a fixpoint of PDE(v) is a solution to FIN, we get that PDE(v) is well defined by the previous LDE

Steps

$$\text{(LDE)} \begin{cases} \partial_t u - \Delta u + \underbrace{\nabla \cdot (b(x,t) * u)}_{\approx f} = 0 \\ u(0) = u_0 \end{cases} .$$

Has a solution by Fundamental-Solution Representation

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^d} K(x-y,t) u_0(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \nabla K(x-y,t-\tau) * (b(y,\tau) u(y,\tau)) dy d\tau \\ &= I + II. \end{aligned}$$

We need $b \in L^q((0,T); L^\infty)$, $u_0 \in L^1$. Tools are,

1. Fix point iteration (acting on u), by contraction it is unique
2. $I \leq \|u_0\|_{L^1}$, (integral of K is $= 1$)
3. $II \leq \|b\| * C$

Steps

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$$(\text{PDE})_\epsilon \begin{cases} u_t^\epsilon - \Delta u^\epsilon + \nabla * (\tilde{j}_\epsilon \star (F \star v(1_{|x|\leq \frac{1}{\epsilon}}\ u^\epsilon)) = 0 \\ u^\epsilon\big|_{t=0} = j_\epsilon \star (1_{|x|\leq \frac{1}{\epsilon}} u_0) \end{cases}.$$

Has a solution by the LDE solution