

Chapter 1

Brownian Motion and Martingales

These are merely my thoughts and notes to the stochastic calc script and will only include the entire statement if i had any special thoughts to it.

Definition 1.0.1 (usual conditions). The filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is said to satisfy the usual conditions if :

1. \mathcal{F}_0 contains all Pr-null sets \mathcal{N} ("completeness")
2. $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s > t} \mathcal{F}_s$ for $t \in [0, T)$ ("right-continuity")

Remark. Completeness assures us that any modifications to an adapted stochastic process is again adapted (same null sets).

Right-continuity can be thought of as giving us the ability to slightly peak into the future, consider the following hitting time :

$$\tau_A = \inf\{t \in [0, T] | X_t \in A\}.$$

For some open set $A \subset \mathbb{R}$ then for any $s \in [0, T]$ the event :

$$\{\tau_A = s\} = \{X_s \in \overline{A}\}.$$

Meaning that at time s we do not know if X_s is already in A or just right on the boundary of entering, such that we need the ability to peak slightly into the future.

Proposition 1.0.1. Let $(B_t)_{t \in [0, T]}$ be a Brownian motion. The completed

natural filtration $(\mathcal{F}_t)_{t \in [0, T]}$ of a Brownian motion $(B_t)_{t \in [0, T]}$ is defined by

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^B, \mathcal{N}).$$

is right-continuous

Proof. Idea is to show $\mathcal{F}_{t+} \subseteq \mathcal{F}_t$ by taking any continuous and bounded $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and showing that for $d \in \mathbb{N}$, $0 \leq t_1 < t_2 < \dots < t_d$

$$\mathbb{E}[f(B_{t_1}, \dots, B_{t_d}) \mid \mathcal{F}_{t+}] \text{ is } \mathcal{F}_t\text{-measurable.}$$

We know that by the properties of Brownian motions any increment $B_t - B_s$ is independent of \mathcal{F}_s^B , take $k \in \{1, \dots, d-1\}$ such that $t_k \leq t \leq t_{k+1}$. For $n \in \mathbb{N}$ large :

$$t + \frac{1}{n} < t_{k+1}.$$

and :

$$\lim_{n \rightarrow \infty} t + \frac{1}{n} = t.$$

Idea is to first show that $\mathbb{E}[f(B_{t_1}, \dots, B_{t_d}) \mid \mathcal{F}_{t+\frac{1}{n}}]$ converges against a \mathcal{F}_t measurable limit and converges against $\mathbb{E}[\cdot \mid \mathcal{F}_{t+}]$ which concludes the proof.

$$\begin{aligned} & \mathbb{E}[f(B_{t_1}, \dots, B_{t_d}) \mid \mathcal{F}_{t+\frac{1}{n}}] \\ &= \mathbb{E}[f(B_{t_1}, \dots, B_{t_k}, \underbrace{B_{t+\frac{1}{n}} + (B_{t_{k+1}} - B_{t+\frac{1}{n}}))}_{=0}, \dots, \underbrace{B_{t+\frac{1}{n}} + (B_{t_d} - B_{t+\frac{1}{n}}))}_{=0}) \mid \mathcal{F}_{t+\frac{1}{n}}] \\ &= \int_{\mathbb{R}^{d-k}} f(B_{t_1}, \dots, B_{t_k}, B_{t+\frac{1}{n}} + x_1, \dots, B_{t+\frac{1}{n}} + x_{d-k}) \rho_n(x) dx. \end{aligned}$$

Question? why do we ignore the first t_k elements

Convergence of integral is shown by DCT against :

$$\int_{\mathbb{R}^{d-k}} f(B_{t_1}, \dots, B_{t_k}, B_t + x_1, \dots, B_t + x_{d-k}) \rho(x) dx.$$

Which is clearly \mathcal{F}_t mb.

Convergence of left hand side is shown by backward martingale theorem.

Plugging in $\mathbb{1}_A$

□

Chapter 2

Ito-Integration

Lemma 2.0.1 (Ito's isometry). For $f \in \mathcal{H}_0^2$ (f is of form $f(\omega, s) = \sum_{i=0}^{n-1} a_i(\omega) \mathbb{1}_{(t_i]}$, a_i is any random variable, its just a discrete case)

Proof. Idea is similar to law of large numbers , i.e split the quadratic and non quadratic terms are 0 as $\mathbb{E}[(B_{t_{i+1}} - B_{t_i})] = 0$ is 0 and for a brownian motion the increments are Normal with mean $t_{i+1} - t_i$ \square

Remark. The idea is to define a space of simple functions that lies dense in the space of functions (Stochastic processes) this allows us to prove properties of simple functions and transfer that to more complex one , think Stochastic 1

The following Proposition does just that and tells us that this sequence exists

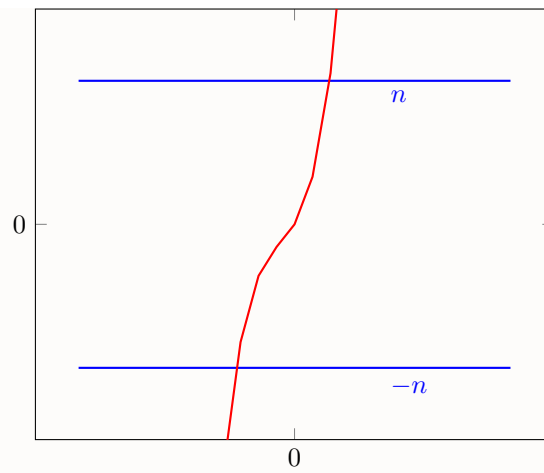
Proposition 2.0.1. For every $f \in \mathcal{H}^2$ there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{H}_0^2$ such that

$$\|f_n - f\|_{\mathcal{H}^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. In essence the proof first shows that we can define us f_n as

$$f_n = \max(-n, \min(f, n)).$$

This is just this



Then as f_n is bounded by f clearly we can use *DCT* to get the convergence, (swap integral and limit)

Second step

□

Chapter 3

Problem Sheet 4

3.1 4.1

Question 1. Let $(B_t)_{t \in [0, T]}$ be a brownian motion. Without using Ito formula show that

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t).$$

Solution. Without using Ito formula we only know how to evaluate Simple Functions $f \in \mathcal{H}_0^2$ this is done by

$$\int_0^t f dB_s = I(f \mathbb{1}_{[0, t]}) = \sum_{i=0}^{n-1} f_i (B_{t_{i+1}} - B_{t_i}).$$

In our case we have $f(\cdot, s) = B_s \in \mathcal{H}^2$ we approximate by

$$f_n(s) = \sum_{i=1}^{n-1} \mathbb{1}_{(t_i, t_{i+1}]}(s) B_{t_i}.$$

Where we decompose the interval $[0, T]$ into $t_i = \frac{i}{n}T$ First show that

$f_n \rightarrow f$, consider

$$\begin{aligned}
 \|f_n - f\|_{\mathcal{H}^2}^2 &= \mathbb{E} \left[\int_0^T \sum_{i=0}^{n-1} (\mathbb{1}_{(t_i, t_{i+1}]}(t) B_{t_i} - B_t)^2 dt \right] \\
 &\stackrel{\text{Fub}}{=} \int_0^T \sum_{i=0}^{n-1} \mathbb{E} [(\mathbb{1}_{(t_i, t_{i+1}]}(t) B_{t_i} - B_t)^2] dt \\
 &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E} [(B_{t_i} - B_t)^2] dt \\
 &\stackrel{\text{Def.}}{=} \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} |t - t_i| dt \\
 &= \sum_{i=0}^{n-1} \frac{1}{2} (t_{i+1} - t_i)^2 \\
 &= \sum_{i=0}^{n-1} \frac{T^2}{2n^2} \\
 &= \frac{T}{2n} \xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}$$

Such that we get

$$I(f \mathbb{1}_{[0,t]}) = \lim_{n \rightarrow \infty} I(f_n \mathbb{1}_{[0,t]}) = \lim_{n \rightarrow \infty} \sum_{t_i \leq t}.$$

□

Chapter 4

Uebung

Definition 4.0.1. For $X = (X_t)_{t \in [0, T]}$ continuous local martingale then, The process $\langle X \rangle = (\langle X \rangle_t)_{t \in [0, T]}$ is given by

$$\langle X \rangle_t = \lim_{n \rightarrow \infty} \sum_{J \in \Pi_n} (\Delta_{J \cap [0, t]} X)^2.$$

The limit is needed such that the non decreasing property is satisfied , i.e while the summands are always non negative , the increments might be different in size such that the process is not non-decreasing , note that the difference between t and $t + \varepsilon$ is that the intersection $J \cap [0, t]$ is taken over a bigger intervall

Dont think this is true (he said it) look below , the granularity does not change if we fix an n , the limit is just needed for the uniqueness tbh.

Remark. The Process is well defined as it is independent of the sequence of partitions Π_n

1. $\langle X \rangle_0 = 0$, and non decreasing
2. And $(X_t^2 - \langle X \rangle_{t \in [0, T]})$ is again a continuous local martingale

Why is it unique, as it is non decreasing and 0 at time 0, we get a unique process by (ii) as the only way for it to be a martingale is to be the same at every time t as the Martingale property allows us to trace back at any time t to time 0

$$\langle X \rangle_{t+1} - \langle X \rangle_t = \sum_{J \in \Pi_n} (\Delta_{J \cap [0, t+1]} X)^2 - (\Delta_{J \cap [0, t]} X)^2 = \sum_{J \in \Pi_n} (\Delta_{J \cap [t, t+1]} X)^2 > 0.$$