Chapter 1

Firt Order PDES

1.0 Homogenous Transport Equation

Definition 1.0.1 (Partial differential equation of order k)**.** A partial differential equation of order k, u is unknown F is given

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0.$$

$$D^k u(x) = \left(\frac{\partial^k u}{\partial^k x_1}, \ldots \frac{\partial^k u}{\partial^k x_n}\right).$$

Definition 1.0.2 (Multi-Index)**.** A multi-index $\gamma \in \mathbb{N}_0^n$ of length $|\gamma| = \sum_i \gamma_i$ for example $\gamma = (0, 2, 1) \in \mathbb{N}_0^3$ can be used to denote partial derivatives of higher order as such :

$$\partial^\gamma = \prod_i (\frac{\partial}{\partial x_i})^{\gamma_i}.$$

Only sensible cause partial derivatives commuter as otherwise the index would be ambiguous.

Definition 1.0.3 (Multi-Dimensional Chain Rule)**.** Let $f : U \subset X \to Y$ be differentiable at $x_0 \in U$ and $g : V \subset Y \to Z$ be differentiable at $f(x_0) \in f[U] \subset V$. Then $g \circ f$ is differentiable at x_0 and :

$$(g \circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0).$$

Note the definition uses composition as the derivative of a function :

$$f : R^n \to R^m \quad f' = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

As such composition is used to , and in practice is matrix multiplication.

Simple pde is the transport equation :

$$\dot{u} + b * \nabla u = 0.$$

$u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $b \in \mathbb{R}^n$.

1.0.1 Method of Characteristic intuition

The idea is to reduce the PDE to a system of ODEs by assuming a solution exists and then observing how the solution would change on a (characteristic) curve as follows

Let $u(x,t)$ be the solution to in this examlpe the transport equation. We pick a curve by letting (x,t) depend on $s \in \mathbb{R}$:

$$(x,t) = (x(s), t(s)).$$

this leads us to :

$$z(s) = u(x(s), t(s)) \overset{\text{Specific}}{=} u(x_0 + s * b, t_0 + s).$$

How the curve should look is often determined by further analysis of the pde. By taking the derivative of z in respect to s we get an ode :

$$z'(s) = u(x(s), t(s))' = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s}.$$

for our concrete pick :

$$z'(s) = \underbrace{\nabla u(x_0 + s * b)}_{\frac{\partial u}{\partial x}} \underbrace{b}_{\frac{\partial x}{\partial s}} + \dot{u} \overset{\text{pde}}{=} 0.$$

which means u is constant along all parallel straight lines in direction of (b,1) and is completely determined by the values on all these parallel straight lines.

1.1 Inhomogenous Transport Equation

Extend the Simple transport equation to an arbitrary function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is given :

$$\dot{u} + b * \nabla u = f.$$

$b \in \mathbb{R}^n$ and u unknown real function.

When given an initial value $g : \mathbb{R}^n \to \mathbb{R}$:

$$\dot{u} + b * \nabla u = f \qquad u(x,0) = g(x).$$

and once again using a curve to observe the solution on that curve we can solve it :

$$z'(s) = b\nabla u(x_0 + sb, s) + \dot{u}(x_0 + sb, s) \overset{\text{cond.}}{=} f(x_0 + sb, s).$$

as the right hand side is only a function of s and $z(0) = u(x_0, 0) = g(x_0)$ we can integrate to determine z(s) :

$$\begin{aligned} z(t) = z(0) + \int_0^t z'(s)ds &= g(x_0) + \int_0^t f(x_0 + sb, s)ds \\ &\overset{\text{subs.}}{=} g(x - tb) + \int_0^t f(x + (s - t)b, s)ds. \end{aligned}$$

1.2 Scalar Conservation Laws

As the PDE s so far have been (quasi) linear in nature we picked lines as our characteristics, when faced with a non linear pde, our curve cannot be linear and we need to decide how to determine $x(s), t(s)$.

Definition 1.2.1 (Scalar conservation law)**.** For a smooth function $f : \mathbb{R} \to \mathbb{R}$ and $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$\dot{u}(x,t) + \frac{\partial f(u(x,t))}{\partial x} = \dot{u}(x,t) + f'(u(x,t)) * \frac{\partial u(x,t)}{\partial x} = 0.$$

this non-linear first oder PDE is called scalar conservation law

Corollary 1.2.2. *The meaning of conservation law, is that the change of the integral of $u(\cdot,t)$ over $[a,b]$ is equal to the 'flux' of $f(u(x,t))$ through the boundary $\{a,b\}$*

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \dot{u}(x,t)dx = -\int_a^b \frac{\partial f(u(x,t))}{\partial x}dx = f(u(a,t)) - f(u(b,t)).$$

Assuming u exists and considering arbitrary characteristic curve $z(s) = u(x(s),s)$ we can get an idea of the shape of the curve by taking the derivative

$$z'(s) = \frac{\partial u(x(s),s)}{\partial x}\frac{\partial x(s)}{\partial s} + \dot{u}(x(s),s).$$

Comparing to our PDE

$$\dot{u}(x,t) + f'(u(x,t)) * \frac{\partial u(x,t)}{\partial x}.$$

it is immediate that for the choice $x'(s) = f'(u(x(s),s))$:

$$z'(s) = \frac{\partial u(x(s),s)}{\partial x}\frac{\partial x(s)}{\partial s} + \dot{u}(x(s),s) = \frac{\partial u(x(s),s)}{\partial x}f'(u(x(s),s)) + \dot{u}(x(s),s) = 0.$$

Such that z is constant along these curves.

Question ?

1. Does a curve $x(s)$ with the above property exists
2. What is the value of z

Assuming our characteristic curve begins at $(x_0, 0)$ and the initial value problem :

$$z(0) = u(x(0),0) = g(x_0).$$

since z is constant along this curve it follows :

$$z(s) = g(x_0).$$

from this it also follows that $x(s)$ is constant and equal to :

$$x'(s) = f'(u(x(s),s)) = f'(z(s)) = f'(g(x_0)).$$

and the curve is given as :

$$x(s) - x_0 = \int_0^s x'(t)dt = \int_0^s f'(g(x_0))dt = f'(g(x_0)) * s$$

.

it follows : $x(s) = x_0 + s * f'(g(x_0))$

Example 1.2.3 (Burgers equation)**.** For n=1 and $f(u) = \frac{1}{2}u^2$ burgers equation is given as :

$$\dot{u}(x,t) + \underbrace{u(x,t)}_{f'} \frac{\partial u(x,t)}{\partial x} = 0.$$

The corresponding characteristic equation is given by $x(t) = x_0 + g(x_0)t$ and therefore the solution is :

$$u(x + tg(x), t) = g(x).$$

If g is continuously differentiable and monotonic increasing (unique), then there is a unique C^1 solution

1.3 Noncharacterstic Hypersurfaces

Definition 1.3.1 (General first order PDE)**.** Given a real function $F : W \subset \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ with an unknown function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and boundary condition $u(y) = g(y)$ for $y \in \Sigma := \{x \in \Omega | \phi(x) = \phi(x_0)\}$ a general first oder PDE is given by :

$$F(\nabla u(x), u(x), x) = 0.$$

Given a Cauchy problem, we can transform the problem into the following form :

$$u(y) = g(y) \text{ for all } y \in \Omega \cap H \text{ with } H = \{x \in \mathbb{R}^n | x * e_n = x_0 * e_n\}.$$

Where e_n is the nth element of the canonical basis and H the unique hyperplane (one dimension less than its ambient space) through $x_0 \in \Omega$ orthogonal to e_n.

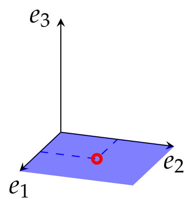

Figure 1.1: Hyperplane

Theorem 1.3.2 (Inverse Function)**.** *A continuously differentiable function $f : \mathbb{R}^n \supset A \to \mathbb{R}^n$ and $f'(a)$ is invertible at a point $a \in A$ (i.e non zero determinant of the Jacobian) then there exist neighborhoods U of a in A and V of $b = f(a)$ such that $f(U) \subset V$ and $f : U \to V$ is bijective.*

Corollary 1.3.3. *Using the Inverse Function Theorem it can be shown that the system of n equations $y_i = f_i(x_1, \dots, x_n)$ where $f = (f_1, \dots, f_n)$ has a unique solution for x in terms of y when $x \in U, y \in V$*

Now if $\nabla\phi(x_0) \neq 0$ we can assume that $\frac{\partial \phi}{\partial x_n}(x_0) \neq 0$ and apply the inverse function theorem to :

$$x \mapsto \Phi(x) = (x_1, \dots, x_{n-1}, \phi(x)).$$

to get a continuously differentiable coordinate transformation (because it is bijective,continuous and differentiable by definition) in a neighbourhood of x_0. This is called "to straighten the boundary at x_0" as $\phi(x) = \phi(x_0)$ if and only if $y * e_n = y_n = \phi(x_0)$.

Using all this we can transform the PDE such that $u = v \circ \Phi$ for a function $v : \Omega' \to \mathbb{R}$:

$$\nabla u(x) = \nabla_y v(\underbrace{\Phi(x)}_{:=y}) * J\Phi(x) = \nabla v(y) * J\Phi(\Phi^{-1}(y)).$$

Note : for v we get a gradient as $v : \mathbb{R}^n \to \mathbb{R}$ and for Φ we get the Jacobian as $\Phi : \mathbb{R}^n \to \mathbb{R}^n$
$J\Phi$ referring to the Jacobian Matrix. THis means that u solves the PDE :

$$F(\nabla u(x), u(x), x) = 0.$$

if and only if v solves the PDE :

$$G(\nabla v(y), v(y), y) := F(\nabla v(y) * J\Phi(\Phi^{-1}(y)), v(y)\Phi^{-1}(y))0.$$

Question? Can we determine anything about u on the hypersurface given the value of u on the hypersurface H. I.e does a solution exist or the value of its derivative. We can compute the partial derivatives in most directions at $x_0 \in H$

$$\frac{\partial u(x_0)}{\partial x_1} = \lim_{h \to 0} \frac{u(x_0 + h * e_1) - u(x_0)}{h} = \lim_{h \to 0} \frac{g(x_0 + h * e_1) - g(x_0)}{h} = \frac{\partial g(x_0)}{\partial x_1}.$$

as $u(x_0) = g(x_0)$ we can substitute g for u in limit. Inserting this into the PDE :

$$F(\nabla u(x_0), u(x_0), x_0) = F((\frac{\partial g(x_0)}{\partial x_1}, \dots, \frac{\partial g(x_0)}{\partial x_{n-1}}), p_n, g(x_0), x_0) = 0.$$

meaning a solution exists depending on F ensuring we can solve the PDE in a neighborhood of x_0.

Definition 1.3.4 (non-characteristic)**.** The Hyperplane $H = \{x_n = 0\}$ is called non-characteristic at x_0 if :

$$\frac{\partial F}{\partial p_n}(p_0, z_0, x_0) \neq 0.$$

Where (p_0, z_0, x_0) solves $F(p, z, x) = 0$.

Can be used as a criterion for invertibility of F in p_n and be used to construct a solution through implicit function theorem (i.e 0 at $(x, p_{0,n})$) and $\frac{\partial F}{\partial p_{0,n}}$ invertierbar

Example 1.3.5. Consider :

$$\frac{\partial u}{\partial x_1} = 0, \qquad u(x_1, 0) = g(x_1).$$

Then $F(p_1, p_2, z, x_1, x_2) = p_1 := \frac{\partial g}{\partial x_1}$ (ka ob das stimmt)

Lemma 1.3.6. *Let $F : W \to \mathbb{R}$ and $g : H \to \mathbb{R}$ be continuously differentiable, $x_0 \in \Omega \cap H$, $z_0 = g(x_0)$ and $p_{0,1} = \frac{\partial g(x_0)}{\partial x_1}, \dots p_{0,n-1} = \frac{\partial g(x_0)}{\partial x_{n-1}}$. If there exists $p_{0,n}$ with $F(p_0, z_0, x_0) = 0$ and H is non-characteristic at x_0 then on an open neighborhood of $x_0 \in \Omega \cap H$ there exists a unique solution q of :*

$$F(q(x), g(x), x) = 0, \quad q_i(x) = \frac{\partial g(x)}{\partial x_i} \quad \textit{and} \quad q(x_0) = p_0.$$

1.4 Method of Characteristics

Generalization of the earlier method of characteristics.

1. Idea : Obtain solution for PDE by observing how a solution u would behave along a curve
2. Method : Plug in arbitrary curve $z(s) = u(x(s))$
3. Determine optimal choice for $x(s)$ such that the PDE reduces to a system of ODE's

Example 1.4.1. Given a PDE :

$$F(\nabla u(x), u(x), x) = 0.$$

Let $z(s) = u(x(s))$ this leads to altered notation :

$$F(p(s), z(s), x(s)).$$

where $p(s) = \nabla u(x(s))$:

$$\frac{dp_i(s)}{ds} = \frac{d}{ds}\frac{\partial u(x(s))}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} x_j'(s).$$

Which is attained by applying the chain rule (take derivative of each component of x for s , which leads to the outer derivative in respect to x)
Taking the derivative of $F(\nabla u(x), u(x), x) = 0$ in respect to fixed x_i gives :

$$\frac{dF}{dx_i} = \sum_{j=1}^{n} \frac{\partial F(\nabla u(x), u(x), x)}{\partial p_j} \frac{\partial^2 u(x)}{\partial x_j \partial x_i} + \frac{\partial F}{\partial u(x)} \frac{\partial u(x)}{\partial x_i} + \frac{\partial F}{\partial x_i}.$$

Goal eliminate dependence on u from all equations, as such we choose the curve as follows :

$$x_j'(s) = \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

Plugging into our first derivative and combining with the derivative of F in respect to x_i

$$\frac{dp_i(s)}{ds} = \frac{d}{ds}\frac{\partial u(x(s))}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} = -\frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) - \frac{\partial F(p(s), z(s), x(s))}{\partial x_i}.$$

Differentiating z :

$$z'(s) = \frac{d}{ds} u(x(s)) \sum_{j=1}^{n} \frac{\partial u(x(s))}{\partial x_j} x_j'(s) = \sum_{j=1}^{h} p_j(s) \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

Obtaining the $2n+1$ system of first order ODEs :

$$\begin{aligned}
x_i'(s) &= \frac{\partial F(p(s), z(s), x(s))}{\partial p_i} \\
p_i'(s) &= \frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \\
z'(s) &= \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s)
\end{aligned}$$

.

Corollary 1.4.2. *The above system is closed as in it only depends on these 2n+1 functions and no other information about u*

Formally :

Theorem 1.4.3. *Let F be a real differentiable function on an open subset* $W \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ *and* $u\colon \Omega \to \mathbb{R}$ *a twice differentiable solution on an open subset* $\Omega \subset \mathbb{R}^n$ *of the first order PDE* $F(\nabla u(x), u(x), x) = 0$*. For every solution* $s \mapsto x(s)$ *of the ODE*

$$x_i'(s) = \frac{\partial F(\nabla p(s), z(s), x(s)}{\partial p_i}.$$

the functions $p(s) = \nabla u(x(s))$ *and* $z(s) = u(x(s))$ *solve the ODEs*

$$\begin{aligned}
p_i'(s) &= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \\
z'(s) &= \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s).
\end{aligned}$$

This is idea is extended in the following Theorem where it is shown that a solution to the ODEs locally solves the PDE

Theorem 1.4.4. *Let* $F\colon W \to \mathbb{R}$ *and* $g\colon H \to \mathbb{R}$ *be three times differentiable functions. Suppose we have a point* $(p_0, z_0, x_0) \in W$ *with*

$$F(p_0, z_0, x_0) = 0, \quad z_0 = g(x_0), \quad p_{0,1} = \frac{\partial g(x_0)}{\partial x_1}, \ldots, p_{0,n-1} = \frac{\partial g(x_0)}{\partial x_{n-1}}.$$

Furthermore, assume that H is non-characteristic at x_0 *:*

$$\frac{\partial F(p_0, z_0, x_0)}{\partial p_{0,n}} \neq 0.$$

Then in a neighborhood $\Omega_{x_0} \subset \Omega$ *of* x_0 *there exists a unique solution of the Cauchy problem*

$$F(\nabla u(x), u(x), x) = 0 \text{ for } x \in \Omega_{x_0} \text{ and } u(y) = g(y) \text{ for } y \in \Omega_{x_0} \cap H.$$

Proof. The proof relies on solving the ODE and showing that it solves the PDE, the initial conditions of the PDE can be translated to the initial conditions of the ODEs. By a previous Lemma we know that there exists a solution q on an open neighborhood of x_0 in H of the following :

$$F(q(y), g(y), y) = 0, \quad q_i(y) = \frac{\partial g(y)}{\partial x_i} \text{ for } i = 1, \dots, n-1 \text{ and } q(x_0) = p_0.$$

This relied on H being non-characteristic and then using the implicit function theorem to define the solution. As F is twice and g thrice differentiable we know that solution above is twice differentiable.
By Picard-Lindelöf (Right side Lipschitz) the following initial value problems have for all $y \in H \cap \Omega_{x_0}$ a unique solotuion :

$$\begin{aligned}
x_i'(s) &= \frac{\partial F(\nabla p(s), z(s), x(s)}{\partial p_i} \\
p_i'(s) &= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \\
z'(s) &= \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s).
\end{aligned}$$

with initial conditions :

$$\begin{aligned}
x(0) &= y \\
p(0) &= q(y) \\
z(0) &= g(y)
\end{aligned}$$

.

We get for every y a solution such that we get a family of solutions :

$$(x(y, s), p(y, s), z(y, s)).$$

By the theorem on the dependence of solutions of ODEs on initial values the function :

$$(y, s) \mapsto (x(y, s), p(y, s), z(y, s)).$$

is for some $\epsilon > 0$ on $(\Omega \cap H) \times (-\epsilon, \epsilon)$ continuous and even differentiable. The function $v : (y, s) \mapsto x(y, s)$ which maps initial values to characteristic curves has the Jacobian :

$$\frac{dv}{dy} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \end{pmatrix}.$$

and

$$\frac{dv}{ds} \overset{\text{Def}}{=} \begin{pmatrix} \frac{\partial F(p_0, z_0, x_0)}{\partial p_1} \\ \vdots \\ \frac{\partial F(p_0, z_0, x_0)}{\partial p_{n-1}} \\ \frac{\partial F(p_0, z_0, x_0)}{\partial p_n} \end{pmatrix}.$$

Since H is non characteristic we know that $\frac{\partial F(p_0,z_0,x_0)}{\partial p_n} \neq 0$ and the combined Jacobian is thus invertible.

We can thus use the inverse function theorem which implies the bijection of the inverse and the continuouity such that we get a differentiable homeomorphism ("isomorph"). Thus a function $u : \Omega \to \mathbb{R}$ defined implicit by :

$$u(x(y,s)) = z(y,s).$$

is well defined and satisfies the initial conditions of the PDE.
It remains to show that u solves the PDE. The ODEs imply :

$$\frac{d}{ds}F(p(y,s),z(y,s),x(y,s)) = \frac{\partial F}{\partial p}\frac{\partial p(y,s)}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z(y,s)}{\partial s} + \frac{\partial F}{\partial x}\frac{\partial x(y,s)}{\partial s} = 0.$$

Inserting the system of ODEs and noting that $\frac{\partial F}{\partial p}$ is a row vector and $\frac{\partial p}{\partial s}$ is a column vector (same for x)

$$0 = \sum_{i=1}^{n}\frac{\partial F}{\partial p_i} * (-\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial z}p_i) + \sum_{i=1}^{n}\frac{\partial F}{\partial z}\frac{\partial F}{\partial p_i}p_i + \sum_{i=1}^{n}\frac{\partial F}{\partial x_i}\frac{\partial F}{\partial p_i}.$$

Hence it suffices to show that $p(y,s) = \nabla u(x(y,s))$ for all $(y,s) \in (\Omega \times H) \times (-\epsilon, \epsilon)$ for u to solve the PDE, Noting that :

$$\frac{\partial z(y,s)}{\partial s} = \sum_{i=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial s} \text{ and } \frac{\partial z(y,s)}{\partial s} = \sum_{i=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}.$$

The first follows from the ODE ($z(y,s) = u(x(y,s))$ gives the gradient p), and the second from the initial conditions for $s = 0$. For $s > 0$ Showing that :

$$v(y,s) = \frac{\partial z(y,s)}{\partial s} - \sum_{i=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}.$$

is 0 proves the second. Quite lenghty and mostly transformations while remembering the original PDE.

Given the two equalities and note $\nabla u(x(y,s)) = (\frac{\partial u}{\partial x_1}, \ldots)$

$$\begin{aligned}\frac{\partial u}{\partial x_j} &= \frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j} = (\sum_{k=1}^{n} p_k\frac{\partial x_k}{\partial s})\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}(\sum_{k=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i})\frac{\partial y_i}{\partial x_j} \\ &= \sum_{k=1}^{n} p_k(\frac{\partial x_k}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\frac{\partial x_k}{\partial y_i}\frac{\partial y_i}{\partial x_j}) \\ &= \sum_{k=1}^{n} p_k\frac{\partial x_k}{\partial x_j} = p_j.\end{aligned}$$

Notice in the last its just the derivative of $\frac{\partial x(y,s)}{\partial x_j}$, and $= p_j$ because the derivatives are 0 except for $k = j \implies 1$

The uniqueness follows from Lindelöf as the ODE solutions are unique □

1.5 Weak Solutions

Generally Weak Solutions refer to solutions that are almost a solution but might deviate from some requirements, in our context of pde's a weak solution refers to one that has restricted differentiability on our domain.

Definition 1.5.1 (Rankine Hugnoit Condition)**.** Let the PDE problem be given by conserved integrals :

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)).$$

then a function u with discontinuities along the graph $\{(x,t)|x = y(t)\}$ for $y \in C^1$, splitting the interval $[a,b]$ into $[a,b] = [a,y(t)] \cup [y(t),b]$ we can calculate the derivative :

$$\begin{aligned}
\frac{d}{dt}\int_a^b u(x,t)dx &= \frac{d}{dt}\int_a^{y(t)} u(x,t)dx + \frac{d}{dt}\int_{y(t)}^b u(x,t)dx \\
&= \dot{y}(t)\lim_{x\uparrow y(t)} u(x,t) + \int_a^{y(t)} \dot{u}(x,t)dx - \dot{y}(t)\lim_{x\downarrow y(t)} u(x,t) + \int_{y(t)}^b \dot{u}(x,t)
\end{aligned}$$

.

Letting $u^l(y(t),t) = \lim_{x\uparrow y(t)} u(x,t)$ and u^r analog, and assume that on both sides of the graph of y the function u is a solution of the conservation law :

$$\begin{aligned}
\frac{d}{dt}\int_a^b u(x,t)dx &= \dot{(y)}(u^l - u^r) - \int_a^{y(t)} \frac{d}{dx}f(u(x,t))dx - \int_{y(t)}^b \frac{d}{dx}f(u(x,t))dx \\
&= \dot{(y)}(u^l - u^r) + \underbrace{f(u(a,t)) - f(u(b,t))}_{=\frac{d}{dt}\int_a^b u(x,t)dx} + f(u^r(y(t),t)) - f(u^l(y(t),t)).
\end{aligned}$$

the $\frac{d}{dx}f$ comes from the scalar conversation pdf :

$$\dot{u} + \frac{df(u(x,t))}{du} * \frac{\partial u(x,t)}{\partial x} = 0.$$

Solving for $\dot{y}(t)$ gives the Rankine-Hugonoit condition:

$$\dot{y}(t) = \frac{f(u^r(y,t)) - f(u^l(y,t))}{u^r(y,t) - u^l(y,t)}.$$

Corollary 1.5.2. *The condition says that the conservation law still holds for a piecewise solution u^r, u^l and y if the condition is true.*

Basically if the "derivative" of y matches the rate of change at the discontinuity.

Example 1.5.3. Burgers equation :

$$\dot{u} + u\frac{\partial u}{\partial x} = 0.$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}^n$ with $u(x,0) = g(x)$ and :

$$g(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 1-x, & \text{if } 0 \leq x < 1 \\ 0, & \text{if } 1 \leq x \end{cases}.$$

We know the solutions to the characteristic problem is $x(t) = x_0 + g(x_0)t$, such that we get crossing characteristics for $t = 1$

$$x + tg(x) = \begin{cases} x+t, & \text{if } x \leq 0 \\ x+t(1-x), & \text{if } 0 \leq x < 1 \\ x, & \text{if } 1 \leq x \end{cases}.$$

For $t < 1$ it is a homeomorphism from $\mathbb{R} \to \mathbb{R}$ with inverse :

$$x = \begin{cases} x-t, & \text{if } x \leq 0 \\ \frac{x-1}{t-1}, & \text{if } 0 \leq x < 1 \\ 0, & \text{if } 1 \leq x \end{cases}.$$

such that the solution at $0 < t < 1$ is :

$$u(x,t) = \begin{cases} 1, & \text{if } x < t \\ \frac{x-1}{t-1}, & \text{if } t < x < 1 \\ 0, & \text{if } 1 \leq x \end{cases}.$$

For $t = 1$ we need a solution that is 1 on $(-\infty, y(t))$ and 0 on $(y(t), \infty)$

We get that :

$$\dot{u} + u\frac{\partial u}{\partial x} = 0 \leftrightarrow \dot{u} + f'(u)\frac{\partial u}{\partial x} = 0.$$

With $f'(u) = u \implies f(u) = \frac{1}{2}u^2$:

$$\dot{y} = \frac{1}{2}\frac{u_r^2 - u_l^2}{u_r - u_l} = \frac{1}{2}u_r - u_l = \frac{1}{2}.$$

We can determine y now by considering that it starts at $(x,t) = (1,1)$

$$y(t) = 1 + \frac{t-1}{2}.$$

Condition for uniqueness of weak solutions for scalar conversation law

Definition 1.5.4 (Lax Entropy condition)**.** A discontinuity of a weak solution along a C^1 path $y(t)$ satisfies the Lax entropy condition, if along the path the following inequality is fulfilled :

$$f'(u^l(y,t)) > \dot{y}(t) > f'(u^r(y,t)).$$

A weak solution with discontinuities along C^1 paths is called an admissible solution if along the path both the Rankine-Hugonoit condition and the Lax Entropy condition are satisfied

In the case of scalar Conservation laws a crossing of characteristics only occurs if $f'(g(x_1)) > f'(g(x_2))$ for $x_1 < x_2$, basically if one curve catches up to the other i guess.

Theorem 1.5.5. *Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be convex and u and v be two admissible solutions of L*

$$\dot{u} + f'(u)\frac{\partial u}{\partial x} = 0.$$

in $L^1(\mathbb{R})$. Then $t \mapsto \|u(\cdot, t) - v(\cdot, t)\|_{L^1(\mathbb{R})}$ is monotonically decreasing.

Chapter 2

General Concepts

2.1 Classification of Second order PDEs

2.1.1 General Problem

For PDEs of order above one no general methods exist to solve them and methods for solving differ quite a bit from each other, thus PDEs are classified by methods that solve them and once a new method to solve is found all pdes that can be solved by it are classified under it.

A general second order linear PDE has the following form :

Definition 2.1.1 (Second Order Linear PDE)**.** A general second order linear PDE is given by :

$$Lu(x) = \sum_{i,j=1}^{n} a_{i,j}(x)\partial_i\partial_j u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u(x) = 0.$$

i.e, second order terms , first order terms and 0th order terms.

Where $a_{i,j}$ is a matrix of coefficients and pde's can be classified by the shape they take. The matrix $a_{i,j}$ is symmetric and diagonalizable as the partial derivatives are symetric **Schwarz's Theorem** ($a_{i,j} \equiv \frac{1}{2}(a_{i,j} + a_{j,i})$)

Elliptic PDEs

Definition 2.1.2 (Elliptic PDEs)**.** If the matrix $a_{i,j}$ is the unity matrix and $b = c = 0$ then they are called elliptic pdes

Example 2.1.3 (Laplace Equation)**.** Laplace Equation is given by :

$$\triangle u := \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Solutions are called harmonic functions. Important tool : a priori estimates i.e lower order derivatives can be estimated in terms of second order derivatives.

Major example whose investigation played a role in the development of elliptic theory is :

Example 2.1.4 (Minimal surface equation)**.** Note $\nabla * u = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) * \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

$$\nabla * \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}.$$

The graphs of such solutions describe minimal surfaces. The area of such hypersurfaces in $\mathbb{R}^{n+1}$ does not change with infinitesimal variation.

Example : Soap bubbles are one example.

Boundary value problem is called Plateaus problem, first proof of existence received Field's Medal (Jesse Douglas)

2.1.2 Parabolic PDEs

Parabolic PDEs are linear PDEs where the matrix $a_{i,j}$ is considered as a symmetric bilinear form which is only semi-definite and belong to the boundary of the class of elliptic PDEs. semi-definite (all eigenvalues are non-negative / non positive).

Example 2.1.5 (Heat equation)**.** The heat equation is given by :

$$\dot{u} - \triangle u = 0.$$

And describes diffusion processes, named after the prominent example of temperature.
Many stochastic processes have this property.

These are processes which level inhomogeneities of some quantity by some flow along the negative gradient of the quantity.
Interpretation : the rate $\dot{u}$ at which the material at a point will heat up (or cool down) is proportional to how much hotter (or cooler) the surrounding is.

Example 2.1.6 (Ricci Flow)**.**

$$\dot{g}_{i,j} = -2R_{i,j}.$$

This PDE describes a diffusion-like process on Riemannian manifolds, it levels the inhomogeneities of the metric (g).

Definition 2.1.7. Riemannian Manifold A Manifold is a locally euclidean space but not globally, common example are maps of an atlas, i.e we can locally embed the maps into $\mathbb{R}^n$ but globally thats impossible, a Riemanniann manifold is a n dimensional manifold with a function g that assigns every point $p \in M$ a scalar product.

2.1.3 Hyperbolic PDE

Hyperbolic PDEs are the second most important class of linear PDEs. The matrix $a_{i,j}$ has one eigenvalue of opposite sign than all other eigenvalues. An example is :

Example 2.1.8 (Wave equation)**.**

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = 0.$$

The wave equation describes the behavior of waves with constant finite speed. The investigation of these PDEs depend on understanding all trajectories which propagate by given speed.

2.2 Existence of Solutions

There exists PDEs with smooth coefficients without solutions, an example to this is :

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = f(x, y).$$

Sucht that :

$f(-x, y) = f(x, y)$

there exists a sequence of positive numbers $\rho_n \downarrow 0$,such that f vanishes on a neighbourhood of the circles bo

The idea to proof there exists no solution in a neighbourhood of $(0, 0) \in \mathbb{R}^2$ is to :

1. If the function $u(x, y)$ is a solution then due to (i) $-u(-x, y)$ is also a solution. Such that $u \equiv frac12(u(x, y) - u(-x, y))$ and assume $u(-x, y) = -u(x, y)$
2. We claim that every solution u vanishes on the circles boundary $\partial B(0, \rho_n)$.

This leads to a contradiction by the Divergence Theorem :

$$\int_{B(0,\rho_n)} f dxdy = \int_{B(0,\rho_n)} (\frac{\partial u}{\partial x} + ix\frac{\partial u}{\partial y})dxdy = \int_{B(0,\rho_n)\nabla *} \begin{pmatrix} u \\ ixu \end{pmatrix} dxdy$$

$$\overset{\text{Div Th.}}{=} \int_{\partial B(0,\rho_n)} \begin{pmatrix} u \\ ixu \end{pmatrix} * N(x, y)d\sigma(x, y) = 0$$

.

2.3 Regularity of Solutions

Regularity of a differential equation refers to the local properties of the corresponding functions. The most general functions we consider are distributions, which have the lowest regularity.

Distributions contain measurable functions with the next highest regularity. The highest regularity are smooth functions and analytic functions.

2.4 Boundary Value Problems

In general partial differential equations have an infinite dimensional space of solutions. Similar to how solutions in the ODE case can be uniquely determined (by fixing the values of the derivatives), in PDEs solutions are functions on higher dimensional domains $\Omega \subset \mathbb{R}^n$ such that a natural condition is the specification of the values of the solution and some of its derivatives on the boundary of the domain.

2.5 Divergence Theorem

The divergence theorem is a generalization of the fundamental theorem of calculus to higher dimensions. It states that the surface integral of a vector field over a closed surface, which is called the "flux" through the surface, is equal to the volume integral of the divergence over the region inside the surface. The Idea behind it can be classified into two definitions

Definition 2.5.1. A continuously differentiable homeomorphism $\Phi : \mathbb{R}^k \supset U \to A \subset \mathbb{R}^n$ is called a $k-$dimensional parameterization of A. It is called regular if the Jacobian ϕ' has full rank k at every point of U.

Definition 2.5.2. Let $A \subset \mathbb{R}^n$ be a subset with a regular parameterization Φ and f a continuous function on A. We define :

$$\int_A f d\sigma := \int_U f \circ \Phi \sqrt{det((\Phi')^T \Phi')} d\mu_{\mathbb{R}^k}.$$

Φ' is the Jacobian.

Some subsets cannot be regularly parameterised, usually this is because the cannot be covered by a single parameterisation an example of this is a sphere, as a sphere is compact there cannot exists a homeomorphism between the sphere and any open set $U \subset \mathbb{R}^k$ this can be solved by using more than one parameterisation, thus the following definition

Definition 2.5.3 (Submanifold)**.** A subster $A \subset \mathbb{R}^n$ is called a $k-$dimensional submanifold if there exists subsets A_i such that each A_i has a regular $k-$dimensional parameterization and $A = \cup A_i$

Issue : subsets can overlap ,which leads to double counts when integrating over the parameterisations. An answer to this are partitions of unity (not practically useful)

Definition 2.5.4. Let $\Omega \subset \mathbb{R}^n$ be covered by a countable family $U_{i \in \mathbb{N}}$ of open subsets. A smooth partition of unity is a countable family $(h_i)_{i \in \mathbb{N}}$ of smooth functions $h_i : \Omega \to [0, 1]$ such that :

1. Each $x \in \Omega$ has a neighbourhood on which all but finite many h_i vanish identically
2. For all $x \in \Omega$ we have $\sum_{i=1}^{\infty} h_i(x) = 1$
3. Each h_i vanishes outside of U_i

For every family of open subsets of $\mathbb{R}^n$ there exists a smooth partition of unity.

Theorem 2.5.5 (Divergence Theorem). *Let $\Omega \subset \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)-$ dimensional sub-manifold of $\mathbb{R}^n$. Let $F : \overline{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously to $\partial\Omega$. Then we have :*

$$\int_\Omega \nabla * F d\mu = \int_{\partial\Omega} F * N d\sigma.$$

where N is the outward pointing normal. (last component is positive)

2.6 Distributions

Main trick is to use integration by parts to "transfer" the integration from one function to the other, see :

$$F_f(\Phi) = \int_\Omega f\Phi d\mu.$$

$$F_{f'}(\Phi) = \int_\Omega f'\Phi d\mu = -\int_\Omega f\Phi' d\mu.$$

Where the boundary terms vanish as Φ is a test function, that vanish on outside of a compact set ?.

In general distributions are a way to define solutions for partial differential equations that may not posses a regular solution that is continuously differentiable. Thus distributions are a special case of functions that act as weak solutions for linear differential equations (solutions in the sense of distributions)

2.6.1 Test Functions

Test functions are infinitely differentiable functions that vanish outside of their compact support. We say for an open set $\Omega \subset \mathbb{R}^n$ the set of test functions $\mathcal{D}(\Omega)$ are such functions with the following notion of convergence : We say test functions converge $f_n \to f$ if there is a compact subset $K \subset \Omega$ such that $\forall n \in \mathbb{N} : supp f_n \in K$ and $\partial t^\alpha f_n \to \partial^\alpha f$ in the supremum norm on K for every multi-index α.

Mollifier

Mollifier or approximate identities is a subset of test functions $(\lambda_\epsilon)_{\epsilon>0}$ with $supp\lambda_\epsilon = \overline{B(0,\epsilon)}$ and $\int \lambda_\epsilon d\mu = 1$, the standard molifier is defined as :

$$\lambda(x) := \begin{cases} C\exp(\frac{1}{|x|^2-1}), & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}.$$

Then the standard mollifier is given by :

$$\lambda_\epsilon(x) = \epsilon^{-n}\lambda(\frac{x}{\epsilon}).$$

They have the property that for any continuous function f on Ω and suppose $0 \in \Omega$ then :

$$\int_{\Omega} f \lambda_\epsilon d\mu \approx \int_{B(0,\epsilon)} f(0)\lambda_\epsilon d\mu = f(0).$$

This is in fact an equality as $\epsilon \downarrow 0$, the proof can be summarized as choosing a compact subset of Ω then taking an ϵ ball around any point x such that the Ball $B(x,\epsilon) \subset \Omega$:

$$|f_\epsilon(x) - f(x)| = |\int_{\Omega} \lambda_\epsilon(x-y)(f(y) - f(x))d^n y| \leq \sup_{y \in B(x,\epsilon)} |f(y) - f(x)|.$$

when $\epsilon \downarrow 0$ the sup goes to 0 uniformly.

Usecase : Mollifiers are used to prove that properties valid for smooth functions are also valid in nonsmooth situations and in our case to give notion to product of distributions.

2.6.2 Formal Definition

Definition 2.6.1. For any function $f \in L^1_{loc}(\Omega)$ a distribution is given by :

$$F_f : \mathcal{D}(\Omega) \to \mathbb{R}, \qquad \Phi \mapsto \int_{\Omega} f \Phi d\mu.$$

We define the space of distributions as

Definition 2.6.2. On an open subset $\Omega \subset \mathbb{R}^n$ the space of distributions $\mathcal{D}'(\Omega)$ is defined as the vector space of all linear maps $F : \mathcal{D}(\Omega) \to \mathbb{R}$ which are continuous with respect to the seminorms

$$\|\cdot\|_{K,\alpha} : C_0^\infty(\Omega) \to \mathbb{R} \qquad \Phi \mapsto \|\Phi\|_{K,\alpha} := \sup_{x \in K} |\partial^\alpha \Phi(x)|.$$

meaning for each compact $K \subset \Omega$ there exist finite many multiindicies α_i and constants $C_i > 0$ such that the following holds for all testfunctions $\Phi \in \mathcal{D}(\Omega)$:

$$|F(\Phi)| \leq C_1 \|\Phi\|_{K,\alpha_1} + \ldots + C_M \|\Phi\|_{K,\alpha_M}.$$

The space of distributions $\mathcal{D}'$ can be regarded as the dual space of $\mathcal{D}$

Corollary 2.6.3. *We get the following convergence property :*

If $\Phi_n \to \Phi$ in $\mathcal{D}(\Omega)$ then the values $F(\Phi_n) \to F(\Phi)$ and we say a sequence of distributions F_n converges to F if $F_n(\Phi) \to F(\Phi)$ for all test functions Φ

Definition 2.6.4. The delta distortion is a special case of distortion such that it does not correspond to an element of $L^1_{loc}(\mathbb{R}^n)$:

$$\delta : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R} \qquad \Phi \mapsto \Phi(0).$$

It is the limit of the sequence of distributions corresponding to the mollifier λ_ϵ

2.6.3 Opereations on Distributions

The goal is to define as many operations on distributions as possible. The first operation we define is convolution, by first defining it on $C_0^\infty(\mathbb{R}^n)$

$$(g \star f)(x) := \int_{\mathbb{R}^n} g(x-y)f(y)d^n y = \int_{\mathbb{R}^n} g(z)f(x-z)d^n z.$$

by using integration by parts we get :

$$\partial^\alpha (g \star f) = (\partial^\alpha g) \star f = g \star (\partial^\alpha f).$$

convolution is well behaved in respect to integration by using volume preserving transformation $z = y - x, y = y$, and preservers symmetry of functions, also proven by coordinate transformation $y = Oz + b$

Chapter 3

Laplace Equation

The Laplace equation is given by :

$$\triangle u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

With the corresponding inhomogeneous PDE (Poissons equation) :

$$-\triangle u = f.$$

Note :

$$\triangle u = \nabla \cdot \nabla u.$$

Definition 3.0.1 (Harmonic Functions)**.** A function is called harmonic if it solves the Laplace equation

They describe the potential of an electric field in vacuum (with some distribution of charges f)

3.1 Fundamental Solution

The Laplace equation is invariant with respect to all rotations and translations of the Euclidean space $\mathbb{R}^n$ This means that any harmonic function must be invariant to translations and rotations and only depends on the length of the vector :

$$u(x) = v(r) = v(\sqrt{x * x}).$$

Taking the derivative

$$\nabla_x u(x) = v'(\sqrt{x * x})\nabla r = v'(\sqrt{x * x})\frac{2x}{2r}.$$

Such that the Laplace Equation simplifies to an ODE :

$$\triangle u(x) = \nabla * \nabla u = v(r)'' \frac{x^2}{r^2} + v'(r)\frac{n}{r} - v'(r)\frac{x^2}{r^2 r} = v''(r) + \frac{n-1}{r}v'(r) = 0.$$

We can solve this ODE by :

$$v''(r) = -\frac{n-1}{r}v'(r) \quad \implies \quad \frac{v''}{v'} = \frac{1-n}{r}.$$

Using the ln trick :

$$\frac{v''}{v'} = \frac{1-n}{r} \implies \frac{d}{dr}\ln(v'(r)) = \frac{d}{dr}\frac{1-n}{r}.$$

Integrating :

$$\ln(v'(r)) = (1-n)\ln(r) + C \implies v(r) = \begin{cases} C'\ln(r) + C'', & \text{if } n = 2 \\ \frac{C'}{r^{n-2}} + C'' & \text{if } n \geq 3 \end{cases}.$$

Calculating is just $\exp((1-n)\ln(r)) = \exp(\ln(r))^{1-n}$

Meaning we get a two dimensional solution space (C', C'')

Definition 3.1.1 (Fundamental Solutions)**.** Let $\Phi(x)$ be the following solutions of the Laplace equation :

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x|, & \text{if } n = 2 \\ \frac{1}{n(n-2)\omega_n|x|^{n-2}} & \text{if } n \geq 3 \end{cases}.$$

Where ω_n denotes the volume of the unit ball $B(0,1)$ in Euclidean space $\mathbb{R}^n$.

Choosing $C'' = 0$ shows this solution lies in the space of symmetric solutions. Where as C' is chosen such that the following holds :

Theorem 3.1.2. *For $f \in C_0^2(\mathbb{R}^n)$ a solution of Poisson's equations $-\triangle u = f$ is given by*

$$u(x) = \Phi \star f = \int_{\mathbb{R}^n} \Phi(y) f(x-y) d^n y.$$

The distribution corresponding to the fundamental solution obeys $-\triangle F_\Phi = \delta$

Proof . Differentiating u twice , noting that integral and Differentiating can be switched as A , f has compact support it vanishes on the boundary and it converges as it can be decomposed into a finite sum of bounded integrals.

Next step is splitting the integral at the singularity of Φ in terms of $\epsilon-$balls around it. The two integrals are shown to both converge to 0 for $\epsilon \to 0$ by upper bounding them (first take max norm as a constant, then limit the integral of Φ which includes an epsilon term)

Distribution step works by using the gradient operation :

$$(\triangle F_\Phi)(\phi) = F_\Phi(\triangle\phi).$$

This is the same as setting $\phi(y) = f(0-y)$ such that $-\triangle F_\Phi(\phi) = \phi(0)$ which is the definition of the delta distribution □

3.2 Mean Value Property

Mean value property shows that any harmonic function u is equal to its mean on any ball given the ball is part of the domain by :

$$u(x) = M(u, x, r).$$

for any r , this allows us to prove the Maximum principle which says that any harmonic function if it takes a maximum does so on its boundary which gives an easy way to show that solutions to the dirichtlet problem are unique by $max(-v) = -min(v) = 0$ where $v = u_1 - u_2$ for two solutions.

Goal is to prove that for any harmonic functions on an open domain $\Omega \subset \mathbb{R}^n$ the following holds :

Definition 3.2.1 (Mean Value Property)**.** If u is a harmonic function on an open domain $\Omega \subset \mathbb{R}^n$ then the value $u(x)$ at the center of any ball $B(x,r)$ with compact closure in Ω is equal to the mean of u on the boundary of the ball. And the opposite if this holds for all balls with compact closure then u is harmonic.

Definition 3.2.2. Given a function u the spherical mean is given by :

$$S[u](x,r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y)d\sigma(y) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(x+rz)d\sigma(z).$$

Where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$

The mean of u on the Ball $B(x,r)$ is the mean over $r' \in [0,r]$ of the spherical means of u on $\partial B(x,r')$ (basically taking the ball and unfolding it into $|[0,r]|$ many lines and integrating over it and taking the mean this is similar to integrating two dimensional shapes , remember video , https://www.youtube.com/watch?v=jNpKKDekS6k).

$$\int_{B(x,r)} u d\mu = \int_0^r (\int_{\partial B(x,s)} u d\sigma) ds.$$

Corollary 3.2.3. *Spherical mean and means have several nice properties, the normalisation constant in $S[u]$ ensures that :*

$$S[c] = c.$$

for any constant function c , and the linearity of the integral gives

$$S[au + bv] = aS[u] + bS[v].$$

And :

$$u \le v \implies S[u] \le S[v].$$

Lemma 3.2.4. *If u is a continuous function then* $\lim_{r \downarrow 0} S[u](x,r) = u(x)$

Proof. Just using the constant properties and continuous nature :

$$|S[u] - u(x)| = |S[u] - S[u(x)]| = |S[u - u(x)]| \le S[|u - u(x)|] < S[\epsilon] = \epsilon.$$

Which proves the statement. □

We get the following property using the divergence theorem :

$$\begin{aligned}\frac{\partial}{\partial r}S(r) &= \frac{1}{n\omega_n}\int_{\partial B(0,1)}\frac{d}{dr}u(x+rz)d\sigma(z) = \frac{1}{n\omega_n}\int_{\partial B(0,1)}\nabla u(x+rz)*zd\sigma(z) \\ &= \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)}\nabla u(y)*Nd\sigma(y) = \frac{1}{n\omega_n r^{n-1}}\int_{B(x,r)}\triangle u d\mu.\end{aligned}$$

Chapter 4

Task Sheets

4.1 Sheet 2

Exercise 1. Inhomogeneous Transport Equation (a)
Find a solution $u : \mathbb{R} \to \mathbb{R}$:

$$\frac{du}{dt} = f(t).$$

Solution 1.
Integrating gives :

$$u(t) = u(0) + \int_0^t f(t)dt.$$

Exercise 2. Inhomogeneous Transport Equation (b)
Show that for every c the solution to the PDE with initial value $u(0) = c$ is unique

Solution 2.
Let $u, v \in C(\mathbb{R}; \mathbb{R})$ be two solutions to the IVP then $u - v$ is a solution to the homogeneous PDE

$$\frac{dh}{dt} = 0.$$

Meaning $u - v$ is a constant function and since both solve the IVP $u(0) = v(0) = c$ and the functions are equal.

Exercise 3. Inhomogeneous Transport Equation (c)
Show that the integral term itself solves the Inhomogeneous transport equation. What initial value problem does it solve

Solution 3.
Taking both derivatives of the Integral part :

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{\partial}{\partial t}\int_0^t f(x + (s-t)b, s)ds \\ &= f(x + (s-t)b, s)|_{s=t} + \int_0^t \frac{\partial f(x + (s-t)b, s)}{\partial t} \\ &= f(x,t) + \int_0^t -b\nabla f(x + (s-t)b, s)\end{aligned}$$

.

Derivative is a result of chain rule and fundamental theorem of calculus (derivative of upper bound gives back original function at that value) And :

$$\begin{aligned}\nabla u := \frac{\partial u(x,t)}{\partial x} &= \frac{\partial}{\partial x}\int_0^t f(x + (s-t)b, s)ds \\ &= \int_0^t \nabla f(x + (s-t)b, s)ds.\end{aligned}$$

Plugging into the PDE :

$$\frac{\partial u(x,t)}{\partial t} + b * \nabla u = \underbrace{f(x,t) + \int_0^t -b\nabla f(x+(s-t)b,s)}_{=\frac{\partial u}{\partial t}} + \underbrace{b * \int_0^t \nabla f(x+(s-t)b,s)ds}_{=b*\nabla u} = f(x,t).$$

Set $u(x,t) = \int_0^t f(x+(s-t)b,s)ds$ then :

$$u(x,0) = \int_0^0 f ds = 0.$$

The Integral solves the initial value problem prob also not wrong

$$\frac{\partial u_s}{\partial t} + b * \nabla u_s = 0$$
$$u_s(x,\tau = 0) = f(x,s).$$

Where $\tau = t - s$ is a time translation to get an initial value problem (see ODE), alternatively the Cauchy Problem:

$$\frac{\partial u_s}{\partial t} + b * \nabla u_s = 0$$
$$u_s(x,\tau = 0) = f(x,s).$$

Exercise 4. Inhomogeneous Transport Equation (d)

Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique)

Solution 4.

Given two solutions u_1, u_2 to the Inhomogeneous transport equation the difference between them : $u_1 - u_2$ is a solution to the homogeneous transport equation with initial value 0, which by assumption is unique (compare ODE)

Exercise 5. Method of characteristics for an Inhomogeneous PDE

Use the method of characteristics to solve the following Inhomogeneous PDE :

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

for $(x,y) \in \mathbb{R}^{>0} \times \mathbb{R}$ with initial condition $u(1,y) = y$.

Solution 5.

Let $z(s) = \ln(u(x(s), y(s)))$ then :

$$z'(s) = \underbrace{\frac{1}{u}}_{\text{outer}} * \underbrace{\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)}_{\text{inner}}.$$

This means :

$$x'(s) = x(s)$$
$$y'(s) = y(s).$$

Such that :

$$x(s) = x_0 * e^s$$
$$y(s) = y_0 * e^s.$$

Which means :

$$z'(s) = 2.$$

Now given the initial condition $u(1, y) = y$ we want :

$$z(0) = \ln(u(1, y)) = \ln(y_0) \implies x_0 = 1.$$

Integrating z' :

$$z(t) = z(0) + \int_0^t z'(s)ds = \ln(y_0) + \int_0^t 2ds = \ln(y_0) * 2t.$$

Such that :

$$u(x(s), y(s)) = \exp(z(s)) = y_0 * e^{2s}.$$

Given any point $(\hat{x}, \hat{y}) \in \mathbb{R}^{>0} \times \mathbb{R}$ we can determine the value $u(\hat{x}, \hat{y})$ by determining the characteristic it lies on and the value of s and checking $z(s)$, using our above :

$$x(s) = \hat{x} \implies s = \ln(\hat{x}).$$

and :

$$y(\ln(x)) = y_0 * \hat{x} \implies y_0 = \frac{\hat{y}}{\hat{x}}.$$

such that for any $(\hat{x}, \hat{y})$ the value of u is given by

$$u(x(s), y(s)) = y_0 * e^{2s} = \frac{\hat{y}}{\hat{x}} * (\hat{x})^2 = \hat{y}\hat{x}.$$

Disregarding the characteristics :

$$u(x, y) = xy.$$

Which indeed solves the PDE.

The solution is unique as the characteristics for different (x, y) cannot cross as every point in the domain belongs to exactly one characteristic as seen by the above relations

Exercise 6. Duhamel's Principle (1)

Consider an Inhomogeneous PDE on $\mathbb{R}^n \times \mathbb{R}$ of the following form :

$$\frac{\partial u}{\partial t} - Lu = f(x,t), \quad u(x,0) = 0.$$

where L is a linear differential operator on $\mathbb{R}^n$ with constant coefficients.
Show that if u_s is a solution to the following homogeneous equation :

$$\frac{\partial u_s}{\partial t} - Lu_s = 0, \quad u_s(x,s) = f(x,s).$$

Then :

$$u(x,t) = \int_0^t u_s(x,t)ds.$$

solves the Inhomogeneous problem

Solution 6.

Let the homogeneous problem be given as :

$$\frac{\partial u_s}{\partial t} - Lu_s = 0, \quad u_s(x,s) = f(x,s).$$

and define :

$$u(x,t) = \int_0^t u_s(x,t)ds.$$

Such that :

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\int_0^t u_s(x,t)ds \\
&\overset{\text{Leib.}}{=} u_s(x,t)|_{s=t} + \int_0^t \frac{\partial u_s}{\partial t}ds \\
&= \underbrace{f(x,t)}_{=u_t(x,t)} + \int_0^t \frac{\partial u_s}{\partial t}ds.
\end{aligned}$$

And :

$$Lu_s = L\int_0^t u_s(x,t)ds \overset{\text{Descr.}}{=} \int_0^t Lu_s(x,t)ds.$$

Together :

$$\frac{\partial u_s}{\partial t} + Lu_s = f(x,t) + \underbrace{\int_0^t \frac{\partial u_s}{\partial t}ds + Lu_s x, t}_{=0} = f(x,t).$$

Which shows $u(x,t) = \int_0^t u_s(x,t)ds$ is a solution to the Inhomogeneous PDE

Exercise 7. Duhamel's Principle (2)

Use the method to solve the Inhomogeneous Transport equation

Solution 7.

Let the homogeneous pde be given as :

$$\frac{\partial u_s}{\partial t} + b\nabla u_s = 0, \quad u_s(x,s) = f(x,s).$$

With initial value at s, then the following time translation :

$$v_s(x,\tau) = u_s(x,\tau+s) \implies v(x,0) = f(x,s).$$

leads to a initial value problem at 0.
Solving by characteristics, let $z(\tau) = v(x(\tau),\tau)$:

$$z' = \frac{\partial v}{\partial x}x' + \frac{\partial v}{\partial t}.$$

Choosing $x'(\tau) = b \implies x(\tau) = b * \tau + x_0$:

$$z' = 0.$$

Meaning z is constant i.e $v_s(x,\tau)$ is constant along the characteristics with value :

$$z(0) = v_s(x_0,0) = f(x_0,s).$$

Now for any $(x,\tau) \in \mathbb{R}^n \times \mathbb{R}$ we get $v_s(x,\tau)$ by determining the characteristics it lies on (i.e determine x_0):

$$x = x_0 + \tau * b \implies x_0 = x - \tau * b.$$

meaning :

$$v_s(x,\tau) = f(x - \tau * b) \implies u_s(x,t) = f(x-(t-s)b,s).$$

Then :

$$u(x,t) = \int_0^t f(x+(s-t)b,s)ds.$$

solves the Inhomogeneous transport equation with initial value 0 (4 .c)