Integral mugatua

[1] Kalkulatu hurrengo integralak, adierazitako ordezkapenak erabiliz.

a) $\displaystyle\int_{3/4}^{4/3} \frac{dz}{z\sqrt{z^2+1}} \qquad \left(z = \frac{1}{t}\right)$

$$\int_{3/4}^{4/3} \frac{dz}{z\sqrt{z^2+1}} = \left\| \begin{array}{l} z=1/t \to dz=-dt/t^2 \\ z=4/3 \to t=3/4 \\ z=3/4 \to t=4/3 \end{array} \right\| = \int_{4/3}^{3/4} \frac{1}{\frac{1}{t}\sqrt{\frac{1}{t^2}+1}} \cdot \frac{-dz}{t^2} = \int_{3/4}^{4/3} \frac{1}{\frac{\sqrt{1+t^2}}{\cancel{t^2}}} \cdot \frac{dt}{\cancel{t^2}} = \int_{3/4}^{4/3} \frac{dt}{\sqrt{1+t^2}} = \left[\ln\left|t+\sqrt{1+t^2}\right|\right]_{3/4}^{4/3} =$$

$$= \ln\left|\frac{4}{3}+\sqrt{1+\frac{4^2}{3^2}}\right| - \ln\left|\frac{3}{4}+\sqrt{1+\frac{3^2}{4^2}}\right| = \ln\left|\frac{4}{3}+\sqrt{\frac{25}{9}}\right| - \ln\left|\frac{3}{4}+\sqrt{\frac{25}{16}}\right| =$$

$$= \ln\left|\frac{4+5}{3}\right| - \ln\left|\frac{3+5}{4}\right| = \ln|3| - \ln|2| = \ln|3/2|$$

b) $\displaystyle\int_{1}^{2} \frac{dx}{(x^2-2x+4)^{3/2}} \qquad (x-1=\sqrt{3}\,tg(z))$

$$\int_{1}^{2} \frac{dx}{(x^2-2x+4)^{3/2}} = \int_{1}^{2} \frac{dx}{((x-1)^2+3)^{3/2}} = \left\| \begin{array}{l} x-1=\sqrt{3}tg(z) \\ dx=\sqrt{3}(sec^2(z))dz \\ x=2 \to z=\pi/6 \\ x=1 \to z=0 \end{array} \right\| = \int_0^{\pi/6} \frac{\sqrt{3}\,sec^2(z)\,dz}{((\sqrt{3}\,tg(z))^2+3)^{3/2}} = \int_0^{\pi/6} \frac{\sqrt{3}\,sec^2(z)\,dz}{(3tg^2(z)+3)^{3/2}} =$$

$$= \int_0^{\pi/6} \frac{\cancel{\sqrt{3}}\,sec^2(z)\,dz}{3\cancel{\sqrt{3}}\,(tg^2(z)+1)^{3/2}} = \left\| \begin{array}{l} 1+tg^2(z) \\ \downarrow \\ 1/cos^2(z) \\ \downarrow \\ sec^2(z) \end{array} \right\| = \frac{1}{3}\int_0^{\pi/6} \frac{sec^2(z)\,dz}{(sec^{\cancel{2}}(z))^{\cancel{3/2}}} = \frac{1}{3}\int_0^{\pi/6} \frac{\cancel{sec^2}(z)}{sec^{\cancel{3}}(z)}\,dz = \frac{1}{3}\int_0^{\pi/6} \frac{dz}{sec(z)} =$$

$$= \frac{1}{3}\int_0^{\pi/6} cos(z)\,dz = \frac{1}{3}\left[sin(z)\right]_0^{\pi/6} = \frac{1}{3}\left[sin\left(\frac{\pi}{6}\right) - \cancel{sin(0)}^{\,0}\right] = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

[2] Egiaztatu hurrengo emaitzak

a) $\displaystyle\int_{e}^{e^2} \frac{dx}{x\ln^3(x)} = \frac{3}{8}$

$$\int_{e}^{e^2} \frac{dx}{x\ln^3(x)} = \left\| \begin{array}{l} t=\ln(x) \to dt=dx/x \\ x=e^2 \to t=2 \\ x=e \to t=1 \end{array} \right\| = \int_1^2 \frac{dt}{t^3} = \left[\frac{-1}{2t^2}\right]_1^2 = \frac{-1}{2(2)^2} + \frac{1}{2(1)^2} = \frac{-1+4}{8} = \frac{3}{8}$$

b) $\displaystyle\int_0^1 \frac{x^{1/4}}{1+x^{1/2}}\,dx = \frac{3\pi-8}{3}$

$$\int_0^1 \frac{x^{1/4}}{1+x^{1/2}}\,dx = \left\| \begin{array}{l} t=x^{1/4} \to x=t^4 \\ dx=4t^3dt \\ x=1 \to t=1 \\ x=0 \to t=0 \end{array} \right\| = \int_0^1 \frac{t}{1+t^2}\,4t^3dt = 4\int_0^1 \frac{t^4}{1+t^2}\,dt = 4\int_0^1 \left(t^2-1+\frac{1}{t^2+1}\right)dt =$$

$$= 4\left[\frac{t^3}{3} - t + arctg(t)\right]_0^1 = 4\left[\frac{1}{3} - 1 + arctg(1) - arctg(0)\right] = 4\left[\frac{1}{3} - 1 + \frac{\pi}{4} - 0\right] =$$

$$= \frac{4}{3} - 4 + \pi = \frac{3\pi-8}{3}$$

c) $\int_0^{2\pi} (1-\cos(\varphi))^2 d\varphi = 3\pi$

$$\int_0^{2\pi} (1-\cos(\varphi))^2 d\varphi = \int_0^{2\pi} (1-2\cos(\varphi)+\cos^2(\varphi))d\varphi = \Big[\varphi - 2\sin(\varphi)\Big]_0^{2\pi} + \int_0^{2\pi} \frac{1+\cos(2\varphi)}{2} d\varphi =$$

$$= \Big[\varphi - 2\sin(\varphi) + \frac{\varphi}{2}\Big]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \cos(2\varphi)d\varphi = \Big[\varphi - 2\sin(\varphi) + \frac{\varphi}{2} + \frac{\sin(2\varphi)}{2}\Big]_0^{2\pi} =$$

$$= 2\pi - \cancel{2\sin(2\pi)}^{0} + \frac{2\pi}{2} + \cancel{\frac{\sin(4\varphi)}{2}}^{0} = 3\pi$$

d) $\int_1^2 \frac{10x^2}{(x^3+1)^2} dx = \frac{35}{27}$

$$\int_1^2 \frac{10x^2}{(x^3+1)^2} dx = \left\| \begin{matrix} t = x^3+1 \to x = \sqrt[3]{t-1} \\ dx = \frac{dt}{3\sqrt[3]{(t-1)^2}} \\ x=2 \to t=9 \\ x=1 \to t=2 \end{matrix} \right\| = 10\int_2^9 \frac{\cancel{(t-1)^{2/3}}}{(t\cancel{-1+1})^2} \cdot \frac{dt}{3\cancel{(t-1)^{2/3}}} = \frac{10}{3}\int_2^9 \frac{dt}{t^2} = \frac{10}{3}\Big[\frac{-1}{t}\Big]_2^9 = \frac{10}{3}\Big[\frac{-1}{9} + \frac{1}{2}\Big] =$$

$$= \frac{10}{3}\Big[\frac{-2+9}{18}\Big] = \frac{10}{3} \cdot \frac{7}{18} = \frac{70}{54} = \frac{35}{27}$$

e) $\int_0^{\sqrt{3}} x^5\sqrt{x^2+1}\, dx = \frac{848}{105}$

$$\int_0^{\sqrt{3}} x^5\sqrt{x^2+1}\, dx = \int_0^{\sqrt{3}} x^5(1+x^2)^{1/2} dx = \left\| \begin{matrix} t = x^2+1 \to x = \sqrt{t-1} \\ dx = \frac{dt}{2\sqrt{t-1}} \\ x=\sqrt{3} \to t=4 \\ x=0 \to t=0 \end{matrix} \right\| = \int_0^4 (t-1)^{5/2} \cdot t^{1/2} \frac{dt}{2(t-1)^{1/2}} = \frac{1}{2}\int_0^4 (t-1)^2 \cdot t^{1/2} dt =$$

$$= \frac{1}{2}\int_0^4 (t^2-2t+1)t^{1/2} dt = \frac{1}{2}\int_0^4 (t^{5/2}-2t^{3/2}+t^{1/2})dt = \frac{1}{2}\Big[\frac{2}{7}t^{7/2} - \frac{4}{5}t^{5/2} + \frac{2}{3}t^{3/2}\Big]_0^4 =$$

$$= \frac{4^3\sqrt{4}}{7} - \frac{2\cdot 4^2\sqrt{4}}{5} + \frac{4\sqrt{4}}{3} = \frac{128}{7} - \frac{64}{5} + \frac{8}{3} - \frac{1}{7} + \frac{2}{5} - \frac{1}{3} = \frac{127\cdot 15 - 62\cdot 21 + 7\cdot 35}{105} =$$

$$= \frac{1905 - 1302 + 245}{105} = \frac{848}{105}$$

f) $\int_0^2 x\sqrt{4x^2+9}\, dx = \frac{49}{6}$

$$\int_0^2 x\sqrt{4x^2+9}\, dx = \left\| \begin{matrix} t = 4x^2+9 \to x = \frac{(t-9)^{1/2}}{2} \\ dx = \frac{dt}{4(t-9)^{1/2}} \\ x=2 \to t=25 \\ x=0 \to t=9 \end{matrix} \right\| = \int_9^{25} \frac{\cancel{(t-9)^{1/2}}}{2} \cdot t^{1/2} \cdot \frac{dt}{4\cancel{(t-9)^{1/2}}} = \frac{1}{8}\int_9^{25} t^{1/2}dt = \frac{1}{8}\Big[\frac{2}{3}t^{3/2}\Big]_9^{25} =$$

$$\frac{25\cdot 5 - 9\cdot 3}{12} = \frac{125-27}{12} = \frac{98}{12} = \frac{49}{6}$$

$\boxed{3}$ Kalkulatu honako integral inpropio konbergenteak:

a) $$\int_0^{\infty} e^{-ax}\cos(bx)\,dx = \frac{a}{b^2+a^2}$$

$$\int e^{-ax}\cos(bx)\,dx = \left\| \begin{array}{l} u=\cos(bx) \to du = -b\cdot\sin(bx)\,dx \\ dv = e^{-ax}dx \to v = -\frac{e^{-ax}}{a} \end{array} \right\| = -\frac{e^{-ax}}{a}\cos(bx) - \int \frac{e^{-ax}}{a}b\sin(bx)\,dx =$$

$$= \frac{-e^{-ax}}{a}\cos(bx) - \frac{b}{a}\overbrace{\int e^{-ax}\sin(bx)\,dx}^{I} \quad \circledast$$

$$I = \int e^{-ax}\sin(bx)\,dx = \left\| \begin{array}{l} u=\sin(bx) \to du = b\cos(bx)\,dx \\ dv = e^{-ax}dx \to v = -\frac{e^{-ax}}{a} \end{array} \right\| = -\frac{e^{-ax}}{a}\sin(bx) + \int \frac{e^{-ax}}{a}b\cos(bx)\,dx =$$

$$= \frac{-e^{-ax}}{a}\sin(bx) + \frac{b}{a}\int e^{-ax}\cos(bx)\,dx$$

$$\circledast \quad \frac{-e^{-ax}\cos(bx)}{a} + \frac{be^{-ax}\sin(bx)}{a^2} - \frac{b^2}{a^2}\int e^{-ax}\cos(bx)\,dx$$

$$\left(\frac{b^2}{a^2}+1\right)\int e^{-ax}\cos(bx)\,dx = \frac{-e^{-ax}\cos(bx)}{a} + \frac{be^{-ax}\sin(bx)}{a^2}$$

$$\int_0^{\infty} e^{-ax}\cos(bx)\,dx = \frac{a^2}{a^2+b^2}\left[\left(\frac{-e^{-ax}\cos(bx)}{a} + \frac{be^{-ax}\sin(bx)}{a^2}\right]_0^{\infty} =$$

$$= \frac{a^2}{a^2+b^2}\left[\frac{\cancelto{0}{-e^{-a\infty}}\cos(b\infty)}{a} + \frac{b\cancelto{0}{e^{-a\infty}}\sin(b\infty)}{a^2} + \frac{\cancelto{1}{e^{-a\cdot 0}}\cos(b\cdot 0)}{a} - \frac{b\cancelto{1}{e^{-a\cdot 0}}\sin(b\cdot 0)}{a^2}\right] =$$

$$= \frac{a^2}{a^2+b^2}\left[\frac{a\cdot\cancelto{1}{\cos(0)} - b\cancelto{0}{\sin(0)}}{a^2}\right] = \frac{\cancel{a^2}}{a^2+b^2}\cdot\frac{a}{\cancel{a^2}} = \frac{a}{a^2+b^2}$$

b) $$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{\varepsilon\to 0^+}\int_0^{4-\varepsilon}(4-x)^{-1/2}dx = \lim_{\varepsilon\to 0^+}\left[-2(4-x)^{1/2}\right]_0^{4-\varepsilon} = 2\sqrt{4} = 4$$

c) $$\int_1^{\infty}\frac{dx}{x^2\sqrt{1+x^2}} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^2\sqrt{1+x^2}}$$

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int x^{-2}(1+x^2)^{-1/2}dx = \left\| \begin{array}{l} p=-1/2 \notin \mathbb{Z}^+ \\ \frac{m+1}{n} = \frac{-1}{2} \notin \mathbb{Z} \\ \frac{m+1}{n}+p = -1 \in \mathbb{Z} \\ 3.\text{mota} \end{array} \middle/ \begin{array}{l} t = x^2 \\ x = t^{1/2} \\ dx = \frac{dt}{2\sqrt{t}} \end{array} \right\| = \int t^{-1}(1+t)^{-1/2}\frac{1}{2}t^{-1/2}dt = \frac{1}{2}\int t^{-3/2}\cdot t^{-1/2}\left(\frac{1+t}{t}\right)^{-1/2}dt =$$

$$= \frac{1}{2}\int t^{-2}\left(\frac{1+t}{t}\right)^{-1/2}dt = \left\| \begin{array}{l} z^2 = \frac{1+t}{t} \to t = \frac{1}{z^2-1} \\ dt = -2z(z^2-1)^{-2}dz \end{array} \right\| = \frac{1}{2}\int \frac{1}{\cancel{(z^2-1)^{-2}}}\cdot\cancel{z^{-1}}(-2)\cancel{z}\cancel{(z^2-1)^{-2}}dz = -\int dz = -z+K =$$

$$= -\left(\frac{1+t}{t}\right)^{1/2} + K = -\frac{\sqrt{1+x^2}}{x} + K$$

$$\lim_{t\to\infty}\left[\frac{-\sqrt{1+x^2}}{x}\right]_1^t = -1+\sqrt{2}$$

$\boxed{4}$ Kalkulatu hurrengo integral inpropioak:

a) $$\int_0^\infty \frac{dy}{(a^2+y^2)^2} = \lim_{t\to\infty}\int_0^t \frac{dy}{(a^2+y^2)^2}$$

$$\int \frac{dy}{(a^2+y^2)^2} = \frac{Ay+B}{a^2+y^2} + \int \frac{Cy+D}{a^2+y^2}\,dy$$

$$\frac{1}{(a^2+y^2)^2} = \frac{A(a^2+y^2)-(Ay+B)2y}{(a^2+y^2)^2} + \frac{Cy+D}{a^2+y^2} = \frac{Aa^2+Ay^2-2Ay^2-2By+Ca^2y+Cy^3+Da^2+Dy^2}{(a^2+y^2)^2}$$

y^3: $0 = Ca^2 \to \boxed{C=0}$

y^2: $0 = -A+D \to A=D \to \boxed{D=1/2a^2}$

y: $0=-2B+Ca^2 \to 2B=0 \to \boxed{B=0}$

y^0: $1 = Aa^2+Da^2 \to 1 = 2Aa^2 \to \boxed{A=1/2a^2}$

$$\frac{y}{2a^2(a^2+y^2)} + \frac{1}{2a^2}\int\frac{dy}{a^2+y^2} = \frac{y}{2a^2(a^2+y^2)} + \frac{1}{2a^3}\operatorname{arctg}\left(\frac{y}{a}\right)+K$$

$$\lim_{t\to\infty}\left[\frac{y}{2a^2(a^2+y^2)}+\frac{1}{2a^3}\operatorname{arctg}\left(\frac{y}{a}\right)\right]_0^t = \frac{1}{2a^3}\cdot\frac{\pi}{2} = \frac{\pi}{4a^3}$$

b) $$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^{1+\varepsilon}\frac{dx}{\sqrt{x(1-x)}}$$

$$\int\frac{dx}{\sqrt{x-x^2}} = \int\frac{dx}{\sqrt{-(x^2-x)}} = \int\frac{dx}{\sqrt{-(x-1/2)^2+1/4}} = \arcsin\left(\frac{x-1/2}{1/2}\right)+K = \arcsin(2x-1)$$

$$\lim_{\varepsilon\to 0^+}\left[\arcsin(2x-1)\right]_\varepsilon^{1+\varepsilon} = \frac{\pi}{2}-\frac{3\pi}{2} = -\pi = \pi$$

c) $$\int_0^\infty \frac{dz}{z^2-3z+2} = \lim_{t\to\infty}\int_0^t\frac{dz}{z^2-3z+2}$$

$$\int\frac{dz}{z^2-3z+2} = \int\frac{dz}{(z-3/2)^2-1/4} = -\ln\left|\frac{1/2+z-3/2}{1/2-z+3/2}\right|+K = \ln\left|\frac{2+z}{z-1}\right|+K = \ln|2+z|-\ln|z-1|+K$$

$$\lim_{t\to\infty}\left[\ln|2+z|-\ln|z-1|\right]_0^t = -\ln(2)$$

5 Frogatu integral hauen dibergentzia

a) $\int_{-2}^{4}\frac{1}{x^2}dx = \lim_{\varepsilon\to 0^+}\left[\int_{-2}^{-\varepsilon}\frac{1}{x^2}dx + \int_{\varepsilon}^{4}\frac{1}{x^2}dx\right] = \lim_{\varepsilon\to 0^+}\left[\left[\frac{-1}{x}\right]_{-2}^{-\varepsilon} + \left[\frac{-1}{x}\right]_{\varepsilon}^{4}\right] = \infty$

b) $\int_{1}^{\infty}\frac{x^2}{4x^2+25}dx = \lim_{t\to\infty}\frac{1}{4}\int_{1}^{t}\frac{x^2}{x^2+\frac{25}{4}}dx = \lim_{t\to\infty}\frac{1}{4}\int_{1}^{t}\frac{x^2+\frac{25}{4}-\frac{25}{4}}{x^2+\frac{25}{4}}dx = \lim_{t\to\infty}\frac{1}{4}\int_{1}^{t}\left(1-\frac{25}{4}\cdot\frac{1}{x^2+\frac{25}{4}}\right)dx$

$= \lim_{t\to\infty}\frac{1}{4}\left[x-\frac{5}{2}\operatorname{arctg}\left(\frac{2x}{5}\right)\right]_{1}^{t} = \infty$

c) $\int_{3}^{6}\frac{\ln(x)}{(x-3)^2}dx = \lim_{\varepsilon\to 0^+}\int_{3+\varepsilon}^{6}\frac{\ln(x)}{(x-3)^2}dx$

$\int\frac{\ln(x)}{(x-3)^2}dx = \left/\!\!\left/\begin{matrix}u=\ln(x)\to du=\frac{1}{x}dx\\ dv=(x-3)^{-2}dx\to v=-(x-3)^{-1}\end{matrix}\right/\!\!\right/ = \frac{-\ln(x)}{x-3} - \int\frac{-dx}{x(x-3)} = \left/\!\!\left/\begin{matrix}\frac{1}{x(x-3)}=\frac{A}{x}+\frac{B}{x-3}=\frac{Ax-3A+Bx}{x(x-3)}\\ x:\ 0=A+B\to\boxed{B=1/3}\\ x^0:\ 1=-3A\to\boxed{A=-1/3}\end{matrix}\right/\!\!\right/ =$

$= \frac{-\ln(x)}{(x-3)} + \left[\int\frac{-1/3}{x}dx + \int\frac{1/3}{(x-3)}dx\right] = \frac{-\ln(x)}{(x-3)} - \frac{\ln(x)}{3} + \frac{\ln(x-3)}{3} + K$

$\lim_{\varepsilon\to 0^+}\left[\frac{\ln\left|\frac{x-3}{x}\right|}{3} - \frac{\ln|x|}{(x-3)}\right]_{3+\varepsilon}^{6} = \frac{\cancel{\ln\left|\frac{6-3}{3}\right|}^{0}}{3} - \frac{\ln|6|}{(6-3)} + \frac{\cancel{\ln|3|}^{\infty}}{\cancel{\varepsilon}} = \infty$

6 Kalkulatu $y=\ln\frac{e^x-1}{e^x+1}$ kurba arkuaren luzera, $x=2$ eta $x=4$ artean

$y=\ln\frac{e^x-1}{e^x+1} = \ln(e^x-1)-\ln(e^x+1)$ $\quad // L=\int\sqrt{1+(y')^2}dx //$

$y' = \frac{e^x}{e^x-1} - \frac{e^x}{e^x+1} = \frac{e^x(e^x+1)-e^x(e^x-1)}{(e^x-1)(e^x+1)} = \frac{\cancel{e^{2x}}+e^x-\cancel{e^{2x}}+e^x}{e^{2x}+\cancel{e^x}-\cancel{e^x}-1} = \frac{2e^x}{e^{2x}-1} \to (y')^2 = \frac{4e^{2x}}{(e^{2x}-1)^2}$

$(y')^2+1 = \frac{4e^{2x}}{(e^{2x}-1)^2}+1 = \frac{4e^{2x}+e^{4x}-2e^{2x}+1}{(e^{2x}-1)^2} = \frac{e^{4x}+2e^{2x}+1}{(e^{2x}-1)^2} = \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}$

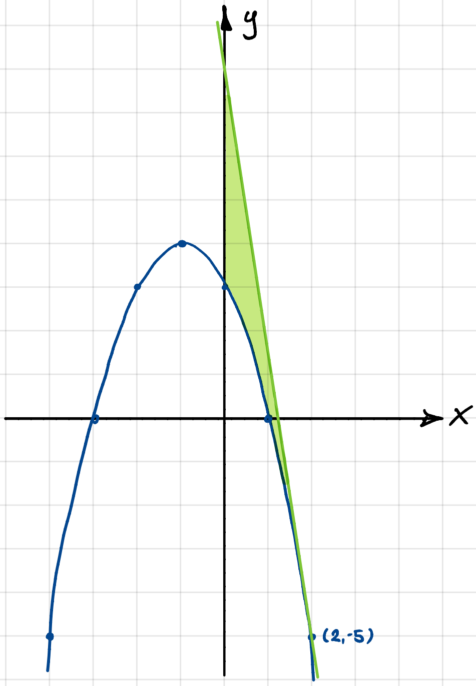
$L=\int_2^4\sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}dx = \int_2^4\frac{e^{2x}+1}{e^{2x}-1}dx = \left/\!\!\left/\begin{matrix}t=e^{2x}\to x=\frac{1}{2}\ln(t)\\ dx=\frac{dt}{2t}\\ x=4\to t=e^8\\ x=2\to t=e^4\end{matrix}\right/\!\!\right/ = \int_{e^4}^{e^8}\frac{(t+1)}{(t-1)}\cdot\frac{dt}{2t} = \frac{1}{2}\int_{e^4}^{e^8}\frac{t+1}{t(t-1)}dt$

$\frac{t+1}{t(t-1)} = \frac{A}{t}+\frac{B}{t-1} = \frac{A(t-1)+Bt}{t(t-1)}$ $\quad t=0\to\boxed{A=-1}$ $\quad t=1\to\boxed{B=2}$

$L=\frac{1}{2}\int_{e^4}^{e^8}\left(\frac{-1}{t}+\frac{2}{t-1}\right)dt = \frac{1}{2}\left[2\ln|t-1|-\ln|t|\right]_{e^4}^{e^8} = \frac{1}{2}\left[2\ln|e^8-1| - \cancel{\ln|e^8|}^{8} - 2\ln|e^4-1| + \cancel{\ln|e^4|}^{4}\right] =$

$= \frac{1}{2}\left[2\ln\left|\frac{e^8-1}{e^4-1}\right|-8+4\right] = \frac{1}{2}\left[2\ln\left|\frac{(e^4+1)\cancel{(e^4-1)}}{\cancel{e^4-1}}\right|-4\right] = \ln|e^4+1|-2$

$\boxed{7}$ Kalkulatu $y=-x^2-2x+3$ parabola, bere ukitzailea $P(2,-5)$ puntuan eta ordenatu ardatzak mugatutako azalera.



PARABOLA:

- $y'=-2x-2=0 \rightarrow x=-1,\ y=4 \rightarrow$ Erpina $(-1,4)$
- $\boxed{y=0} \rightarrow -x^2-2x+3=0$

$$x=\frac{2\pm\sqrt{2^2-4(-1)\cdot 3}}{2(-1)}=\frac{2\pm\sqrt{16}}{-2} \begin{cases}\boxed{x=-3}\\ \boxed{x=1}\end{cases}$$

UKITZAILEA $(2,-5)$:

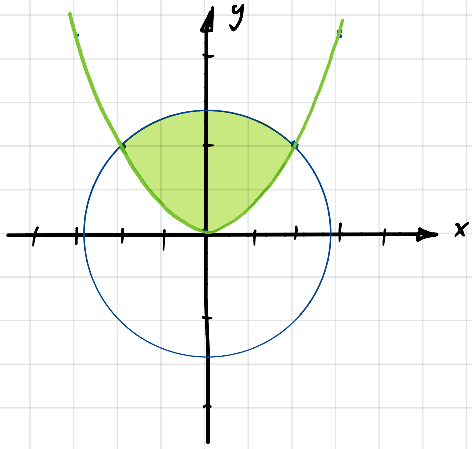
- $m=y'(2)=-6$
- $y+5=m(x-2) \rightarrow y+5=-6(x-2) \rightarrow y=-6x+7$

$$A=\int_0^2\left[(-6x+7)-(-x^2-2x+3)\right]dx=\left[-3x^2+7x+\frac{x^3}{3}+x^2-3x\right]_0^2=-\cancel{12}+\cancel{14}+\frac{8}{3}+\cancel{4}-\cancel{6}=\frac{8}{3}\,u^2$$

$\boxed{8}$ Kalkulatu $x^2+y^2=8$, eta $x^2=2y$ kurbek barnean mugatzen duten azalera.

$x^2+y^2=8 \rightarrow$ Zirkunferentzia $C(0,0)$ $R=\sqrt{8}$

$x^2=2y \rightarrow$ Parabola



EBAKI PUNTUAK:

$2y+y^2=8 \rightarrow y^2+2y-8=0$

$$y=\frac{-2\pm\sqrt{2^2-4\cdot 1(-8)}}{2}=\frac{-2\pm\sqrt{36}}{2} \begin{cases}\boxed{y=2}\rightarrow\boxed{x=\pm 2}\\ \cancel{y=-4}\end{cases}$$

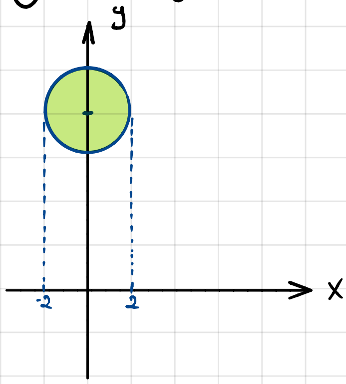
$$A=2\int_0^2\left(\sqrt{8-x^2}-\frac{x^2}{2}\right)dx=\underbrace{2\int_0^2\sqrt{8-x^2}\,dx}_{I}-\left[\frac{x^3}{3}\right]_0^2=I-\frac{8}{3}$$

$$I=2\int_0^2\sqrt{8-x^2}\,dx=\left\|\begin{matrix}x=\sqrt{8}\sin(t)\rightarrow t=\arcsin(x/\sqrt{8})\\ dx=\sqrt{8}\cos(t)dt\\ x=2\rightarrow t=\pi/4\\ x=0\rightarrow t=0\end{matrix}\right\|=2\int_0^{\pi/4}\sqrt{8-8\sin(t)}\,\sqrt{8}\cos(t)dt=16\int_0^{\pi/4}\sqrt{1-\sin^2(t)}\cdot\cos(t)dt=$$

$$=16\int_0^{\pi/4}\cancel{\sqrt{\cos^2(t)}}\cdot\cos(t)dt=16\int_0^{\pi/4}\cos^2(t)dt=16\int_0^{\pi/4}\frac{1+\cos(2t)}{2}dt=16\left[\frac{t}{2}+\frac{\sin(2t)}{4}\right]_0^{\pi/4}=16\left[\frac{\pi}{8}+\frac{\sin(\pi/2)}{4}\right]=$$

$$A=2\pi+4-\frac{8}{3}=2\pi+\frac{12-8}{3}\longrightarrow\boxed{A=2\pi+\frac{4}{3}}$$

9 Kalkulatu $x^2+(y-8)^2=4$ zirkunferentziak abzisa ardatzaren inguruan biratzean sortutako gorputzaren bolumena



$x^2+(y-8)^2=4 \to$ zirkunferentzia

$y=8\pm\sqrt{4-x^2}$ $\quad C(0,8), R=2$

$$V=\pi\int_a^b [f(x)]^2\,dx$$

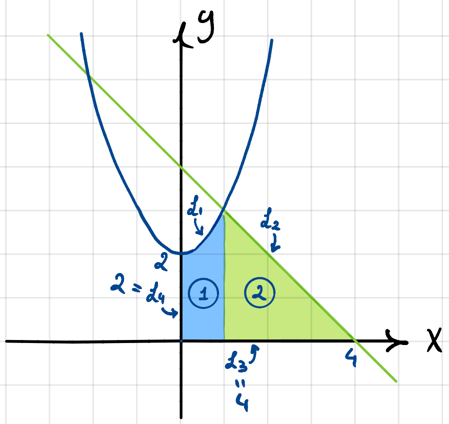
$$V=2\pi\int_0^2 \left[\left(8+\sqrt{4-x^2}\right)^2-\left(8-\sqrt{4-x^2}\right)^2\right]dx = 2\pi\int_0^2 32\sqrt{4-x^2}\,dx = 64\pi\int_0^2\sqrt{4-x^2}\,dx = \left\| \begin{array}{l} x=2\sin(t)\to t=\arcsin(x/2)\\ dx=2\cos(t)dt \\ x=2\to t=\pi/2\\ x=0\to t=0\end{array}\right\| =$$

$$=64\pi\int_0^{\pi/2}\sqrt{4-(2\sin(t))^2}\,2\cos(t)dt = 128\pi\int_0^{\pi/2}\sqrt{4\overset{\cos^2(t)}{\cancel{(1-\sin^2(t))}}}\,\cos(t)dt = 128\pi\int_0^{\pi/2}2\cos^2(t)dt = 128\pi\int_0^{\pi/2}(\cos(2t)+1)\,dt=$$

$$=128\pi\left[\frac{\sin(2t)}{2}+t\right]_0^{\pi/2} = 128\pi\left[\frac{\overset{0}{\cancel{\sin(\pi)}}}{2}+\frac{\pi}{2}-\frac{\overset{0}{\cancel{\sin(0)}}}{2}-0\right] = 128\pi\frac{\pi}{2} = 64\pi^2\ u^3$$

10 Izan bedi 1. koadrantean honako kurba hauek definitutako eskualdea:

$$\begin{cases} x^2+2 \geq y & \to \text{Parabola} \\ x+y\leq 4 & \to y\leq 4-x \to \text{Zuzena} \end{cases}$$



Kalkulatu:

a) Eskualdearen azalera eta perimetroa:

Azalera:

$$A = A_1+A_2 = \int_0^1 (x^2+2)dx + \int_1^4 (4-x)dx = \left[\frac{x^3}{3}+2x\right]_0^1 + \left[4x-\frac{x^2}{2}\right]_1^4 =$$

$$= \frac{1}{3}+2+16-\frac{16}{2}-4+\frac{1}{2} = \frac{2+36+3}{6} \longrightarrow \boxed{A=\frac{41}{6}\ u^2}$$

Perimetroa:

- Funtzioen deribatuak:

$y_1 = x^2+2 \to y_1' = 2x \to (y')^2 = 4x^2$

$y_2 = 4-x \to y_2' = -1 \to (y_2')^2 = 1$

• Arku luzerak: $(\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4)$

$$\mathcal{L}_1=\int_0^1\sqrt{1+(y_1')^2}\,dx=\int_0^1\sqrt{1+4x^2}\,dx=\int_0^1\frac{1+4x^2}{\sqrt{1+4x^2}}\,dx$$

$$\int\frac{1+4x^2}{\sqrt{1+4x^2}}\,dx=(Ax+B)\sqrt{1+4x^2}+M\int\frac{dx}{\sqrt{1+4x^2}}=I$$

$$\frac{1+4x^2}{\sqrt{1+4x^2}}=A\sqrt{1+4x^2}+\frac{(Ax+B)4x}{\sqrt{1+4x^2}}+\frac{M}{\sqrt{1+4x^2}}=\frac{A+4Ax^2+4Ax^2+4Bx+M}{\sqrt{1+4x^2}}$$

$$x^2:\ 4=4A+4A\ \rightarrow\ \boxed{A=1/2}$$

$$x:\ 0=4B\ \rightarrow\ \boxed{B=0}$$

$$x^0:\ 1=A+M\ \rightarrow\ \boxed{M=1/2}$$

$$\mathcal{L}_1=\frac{\sqrt{1+4x^2}}{2}+\frac{1}{2}\int_0^1\frac{dx}{\sqrt{1+4x^2}}=\frac{\sqrt{1+4x^2}}{2}+\frac{1}{4}\int_0^1\frac{dx}{\sqrt{x^2+1/4}}=\left[\frac{x}{2}\sqrt{1+4x^2}+\frac{\ln\left|x+\sqrt{x^2+1/4}\right|}{4}\right]_0^1=$$

$$=\frac{1}{2}\sqrt{1+4}+\frac{\ln\left|1+\sqrt{1+1/4}\right|}{4}-\frac{\ln\left|\sqrt{1/4}\right|}{4}=\frac{\sqrt{5}}{2}+\frac{1}{4}\left(\ln\left|\frac{2+\sqrt{5}}{2}\right|-\ln\left|\frac{1}{2}\right|\right)=$$

$$=\frac{\sqrt{5}}{2}+\frac{1}{4}\left(\ln|2+\sqrt{5}|+\cancel{\ln\left|\frac{1}{2}\right|}-\cancel{\ln\left|\frac{1}{2}\right|}\right)=\frac{\sqrt{5}}{2}+\frac{\ln|2+\sqrt{5}|}{4}\ u$$

$$\mathcal{L}_2=\int_1^4\sqrt{1+1}\,dx=\left[\sqrt{2}x\right]_1^4=4\sqrt{2}-\sqrt{2}=3\sqrt{2}\ u$$

$$\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4=\frac{\sqrt{5}}{2}+\frac{\ln|2+\sqrt{5}|}{4}+3\sqrt{2}+4+2$$

$$\boxed{\mathcal{L}=6+3\sqrt{2}+\frac{\sqrt{5}}{2}+\frac{\ln|2+\sqrt{5}|}{4}\ u}$$

b) Eskualdeak OX ardatzaren inguruan biratzean sortutako gorputz bolumena

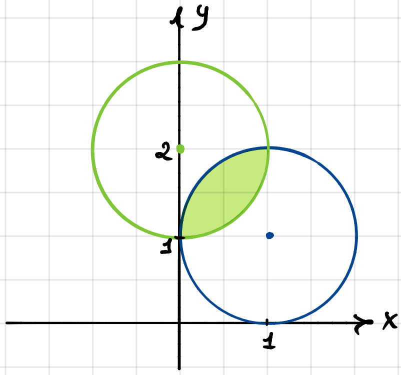
$$V=\pi\int_0^1(x^2+2)^2dx+\pi\int_1^4(4-x)^2dx=\pi\int_0^1(x^4+4x^2+4)dx+\pi\int_1^4(x^2-8x+16)dx=$$

$$=\pi\left[\frac{x^5}{5}+\frac{4x^3}{3}+4x\right]_0^1+\pi\left[\frac{x^3}{3}-4x^2+16x\right]_1^4=\pi\left[\frac{1}{5}+\frac{4}{3}+4+\frac{4^3}{3}-\cancel{4^3}+\cancel{16\cdot4}-\frac{1}{3}+4-16\right]=$$

$$=\pi\left[\frac{3+4^3\cdot5-7\cdot15}{15}\right]=\pi\left[\frac{3+320-105}{15}\right]\rightarrow\boxed{V=\pi\frac{218}{15}\ u^3}$$

11 Kalkulatu hurrengo kurbek bornatzen duten eskualdearen azalera:

$$\begin{cases} x^2+y^2-2x-2y+1\le 0 \to (x-1)^2+(y-1)^2\le 1 \to \text{Zirkunferentzia } R=1 \ C(1,1) \\ x^2+y^2-4y+3\le 0 \to x^2+(y-2)^2\le 1 \to \text{Zirkunferentzia } R=1 \ C(0,2) \end{cases}$$



$$(x-1)^2+(y-1)^2=1 \to y-1=\sqrt{1-(x-1)^2} \to y=1+\sqrt{1-(x-1)^2}$$

Zirkunferentziaren beheko aldea

$$x^2+(y-2)^2=1 \to y-2=-\sqrt{1-x^2} \to y=2-\sqrt{1-x^2}$$

$$A=\int_0^1\Big(1+\underbrace{\sqrt{1-(x-1)^2}}_{I_1}\Big)dx-\int_0^1\Big(2-\underbrace{\sqrt{1-x^2}}_{I_2}\Big)dx=\big[x-2x\big]_0^1+I_1+I_2$$

$$I_1=\int_0^1\sqrt{1-(x-1)^2}\,dx=\left\|\begin{matrix} x-1=\sin(t)\to t=\arcsin(x-1) \\ dx=\cos(t)dt \\ x=1\to t=0 \quad , x=0\to t=-\pi/2 \end{matrix}\right\|=$$

$$=\int_{-\pi/2}^{0}\sqrt{\cancel{1-\sin^2(t)}}^{\cos^2(t)}\cos(t)\,dt=\int_{-\pi/2}^{0}\cos^2(t)\,dt=\int_{-\pi/2}^{0}\frac{1+\cos(2t)}{2}dt=\left[\frac{x}{2}+\frac{\sin(2t)}{4}\right]_{-\pi/2}^{0}=$$

$$=\frac{\cancel{\sin(0)}^{0}}{4}+\frac{\pi}{4}-\frac{\cancel{\sin(-\pi)}^{0}}{4}=\frac{\pi}{4}$$

$$I_2=\int_0^1\sqrt{1-x^2}\,dx=\left\|\begin{matrix} x=\sin(t) \\ dx=\cos(t) \\ x=1\to t=\pi/2 \\ x=0\to t=0 \end{matrix}\right\|=\int_0^{\pi/2}\sqrt{1-\sin^2(t)}\cos(t)\,dt=\int_0^{\pi/2}\cos^2(t)\,dt=\int_0^{\pi/2}\frac{1+\cos(2t)}{2}dt=\left[\frac{x}{2}+\frac{\sin(2t)}{4}\right]_0^{\pi/2}=$$

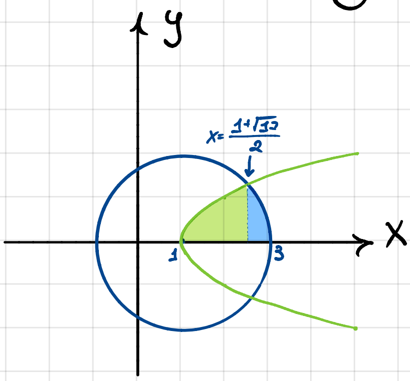
$$=\frac{\pi}{4}+\frac{\cancel{\sin(\pi)}^{0}}{4}-\frac{\cancel{\sin(0)}^{0}}{4}=\frac{\pi}{4}$$

$$A=-1+\frac{\pi}{4}+\frac{\pi}{4}\to \boxed{A=\frac{\pi}{2}-1}$$

12 Honako kurba hauek definitutako eskualdearentzako:

$$\begin{cases} (x-1)^2+y^2\le 4 \to \text{Zirkunferentzia } C(1,0),\ R=2 \\ y^2\le x-1 \to \text{Parabola erpina } (1,0) \end{cases}$$

Kalkulatu eskualdeak OX ardatzaren inguruan sortutako gorputzaren bolumena:



$$(x-1)^2+y^2=4 \to y^2=4-(x-1)^2 \to y=\sqrt{4-(x-1)^2}$$

$$y^2=x-1 \to y=\sqrt{x-1}$$

Ebaki puntuak:

$$4-(x-1)^2 = x-1 \rightarrow x^2-x-4=0$$

$$x=\frac{1\pm\sqrt{1-4(-4)}}{2}=\frac{1\pm\sqrt{17}}{2}=\frac{1+\sqrt{17}}{2}$$

$$V=\pi\left[\int_1^{\frac{1+\sqrt{17}}{2}}\left(\sqrt{x-1}\right)^2dx+\int_{\frac{1+\sqrt{17}}{2}}^{3}\left(\sqrt{4-(x-1)^2}\right)^2dx\right]=\pi\int_1^{\frac{1+\sqrt{17}}{2}}(x-1)dx+\pi\int_{\frac{1+\sqrt{17}}{2}}^{3}\left(4-(x-1)^2\right)dx=$$

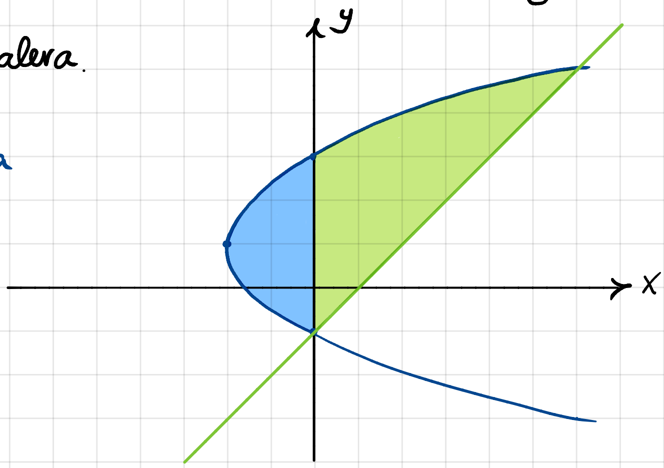
$$=\pi\left[\frac{x^2}{2}-x\right]_1^{\frac{1+\sqrt{17}}{2}}+\pi\left[4x-\frac{(x-1)^3}{3}\right]_{\frac{1+\sqrt{17}}{2}}^{3}=\pi\left[\frac{\left(\frac{1+\sqrt{17}}{2}\right)^2}{2}-\frac{1+\sqrt{17}}{2}-\frac{1}{2}+1\right]+\pi\left[12-\frac{8}{3}-2(1+\sqrt{17})+\frac{\left(\frac{1+\sqrt{17}}{2}-1\right)^3}{3}\right]=$$

$$=\pi\left[\frac{1+17+2\sqrt{17}-\cancel{4}-4\sqrt{17}-\cancel{4}+\cancel{8}}{8}+\frac{240-64-48\sqrt{17}+20\sqrt{17}-52}{24}\right]=$$

$$=\pi\left[\frac{18-2\sqrt{17}}{8}+\frac{124-28\sqrt{17}}{24}\right]=\pi\left[\frac{54-6\sqrt{17}+124-28\sqrt{17}}{24}\right]=\pi\left[\frac{178-34\sqrt{17}}{24}\right]$$

$$V=\pi\left[\frac{89-17\sqrt{17}}{12}\right]u^3$$

13 Kalkulatu $(y-1)^2-2(2+x)=0$ kurbak eta $A(0,-1)$ eta $B(6,5)$ puntuak lotzen dituen zuzenak barnean mugatzen duten azalera.



$$(y-1)^2=2(2+x)\rightarrow x=\frac{(y-1)^2}{2}-2 \rightarrow \text{Parabola}$$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{5+1}{6}=1$$

$$y-y_1=m(x-x_1)\rightarrow y=x-1\rightarrow \text{Zuzena}$$

Ebakidura puntuak:

$$\left.\begin{array}{l} x=\frac{(y-1)^2}{2}-2 \\ \\ x=y+1 \end{array}\right\} \quad y+3=\frac{(y-1)^2}{2}\rightarrow 2y+6=y^2-2y+1\rightarrow y^2-4y-5=0$$

$$y=\frac{4\pm\sqrt{4^2-4(-5)}}{2}=\frac{4\pm\sqrt{36}}{2} \begin{array}{l} y=5,\ x=6 \\ y=-1,\ x=0 \end{array}$$

$$A=\int_{-1}^{5}\left[(y+1)-\left(\frac{(y-1)^2}{2}-2\right)\right]dy=\left[\frac{(y+1)^2}{2}-\frac{(y-1)^3}{6}+2y\right]_{-1}^{5}=\frac{6^2}{2}-\frac{4^3}{6}+10-\frac{2^3}{6}+2=$$

$$=18-\frac{64}{6}-\frac{8}{6}+12=30-\frac{72}{6}=30-12\rightarrow \boxed{A=18u^2}$$

$\boxed{14}$ Kalkulatu $\int_0^{3/2} \frac{1}{x^2-4}dx$

a) Trapezioen metodoa erabiliz
b) Simpson-en metodoa erabiliz

$\}$ Ez dira sartzen

c) Integralaren balio zehatza

$$\int_0^{3/2} \frac{1}{x^2-4}dx = -\int_0^{3/2}\frac{1}{4-x^2}dx = -\left[\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|\right]_0^{3/2} = -\frac{1}{4}\ln\left|\frac{2+3/2}{2-3/2}\right| + \cancelto{0}{\frac{1}{4}\ln\left|\frac{2}{2}\right|} =$$

$$= -\frac{1}{4}\ln\left|\frac{7/\cancel{2}}{1/\cancel{2}}\right| = -\frac{1}{4}\ln|7|$$

$\boxed{15}$

$\boxed{16}$ $\}$ Trapezioen eta Simpsonen metodoak $\rightarrow$ Ez dira sartzen

$\boxed{17}$

$\boxed{18}$ Kalkulatu honako integral inpropio konbergenteak:

a) $$\int_0^{\infty} \frac{5}{5+x^2}dx = \lim_{t\to\infty}\int_0^t \frac{5}{5+x^2}dx$$

$$\int \frac{5}{5+x^2}dx = \sqrt{5}\,\mathrm{arctg}\left(\frac{x}{\sqrt{5}}\right) + K$$

$$\lim_{t\to\infty}\left[\sqrt{5}\,\mathrm{arctg}\left(\frac{x}{\sqrt{5}}\right)\right]_0^t = \frac{\sqrt{5}\pi}{2}$$

b) $$\int_0^5 \frac{1}{\sqrt[3]{1-x}}dx = \lim_{\varepsilon\to 0^+}\left[\int_0^{1-\varepsilon}\frac{1}{\sqrt[3]{1-x}}dx + \int_{1+\varepsilon}^{5}\frac{1}{\sqrt[3]{1-x}}dx\right]$$

$$\int \frac{1}{\sqrt[3]{1-x}}dx = \frac{-3\sqrt[3]{(1-x)^2}}{2} + K$$

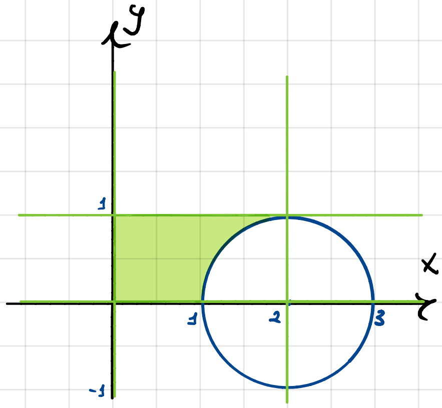
$$\lim_{\varepsilon\to 0^+}\left[\frac{-3\sqrt[3]{(1-x)^2}}{2}\right]_0^{1-\varepsilon} + \left[\frac{-3\sqrt[3]{(1-x)^2}}{2}\right]_{1+\varepsilon}^{5} = \frac{-3}{2} - \frac{6\sqrt[3]{2}}{2} = \frac{-3}{2}(1-2\sqrt[3]{2})$$

19 Izan bedi honako eran definitutako [D] eskualdea:

$$D=\{(x,y)\in\mathbb{R}^2 / (x-2)^2+y^2\geq 1;\ y\leq 1;\ x\geq 0;\ x\leq 2;\ y\geq 0\}$$

Integral mugatua erabiliz, kalkulatu:

a) [D] domeinuaren azalera



$(x-2)^2+y^2=1 \rightarrow$ Zirkunferentzia $C(2,0)$ $R=1$

$$y=\sqrt{1-(x-2)^2}$$

$$A=\int_0^2 dx-\int_1^2\sqrt{1-(x-2)^2}\,dx=[x]_0^2-\int_1^2\sqrt{1-(x-2)^2}\,dx$$

$$\int_0^2\sqrt{1-(x-2)^2}\,dx=\left\|\begin{array}{l}x-2=\sin(t)\\ dx=\cos(t)\\ x=2\rightarrow t=0\\ x=1\rightarrow t=-\pi/2\end{array}\right\|=\int_{-\pi/2}^0\sqrt{1-\sin^2(t)}\cos(t)dt=\int_{-\pi/2}^0\cos^2(t)dt=\int_{-\pi/2}^0\frac{1+\cos(2t)}{2}dt=\left[\frac{t}{2}+\frac{\sin(2t)}{4}\right]_{-\pi/2}^0=$$

$$A=2-\frac{\pi}{4}\rightarrow \boxed{A=\frac{8-\pi}{4}u^2}$$

b) [D] eskualdeak abzisa ardatzaren inguruan biratzean sortutako gorputzaren bolumena.

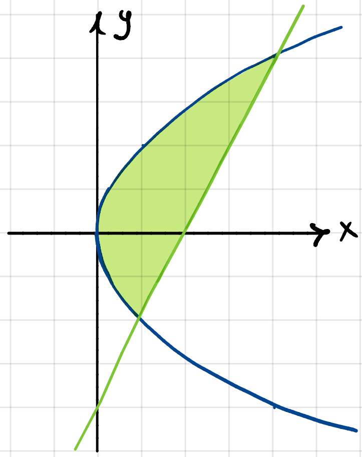
$$V=\pi\int_0^2(1)^2dx-\pi\int_1^2\left(\sqrt{1-(x-2)^2}\right)^2dx=\pi\int_0^2dx-\pi\int_1^2(1-(x-2)^2)dx=$$

$$=\pi[x]_0^2-\pi\left[x-\frac{(x-2)^3}{3}\right]_1^2=\cancel{2\pi}-\cancel{2\pi}+\pi+\frac{\pi}{3}\rightarrow\boxed{V=\frac{4\pi}{3}u^3}$$

20 Izan bedi $y^2\leq 4x$; $y\geq 2x-4$ kurbek bornatzen duten [D] eskualdea:

a) [D] eskualdearen azalera (oharra: erabili y integrazio aldagaitzat)

$y^2=4x\rightarrow x=y^2/4\rightarrow$ Parabola



$y=2x-4\rightarrow x=\frac{y+4}{2}\rightarrow$ Zuzena

$$A=\int_{-2}^4\left(\frac{y+4}{2}-\frac{y^2}{4}\right)dx=\left[\frac{y^2}{4}+2y-\frac{y^3}{12}\right]_{-2}^4=$$

$$=4+8-\frac{4^3}{12}-\frac{(-2)^2}{4}+4+\frac{(-2)^3}{12}=16-\frac{16}{3}-1-\frac{2}{3}=$$

$$=15-\frac{18}{3}\rightarrow\boxed{V=9u^3}$$