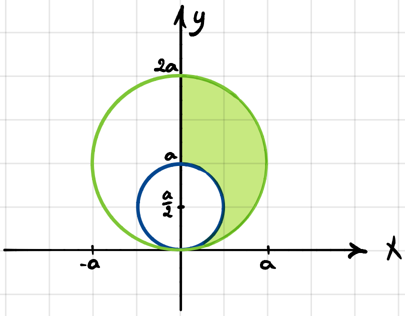
$\boxed{9}$ Aplikatu kalkulu integrala, hurrengo eremu lauen grabitate zentro geometrikoaren koordenatuak lortzeko.

a) [D] lehenengo koadrantean a eta 2a diametroko bi zirkunferentziak mugatzen dute. Zirkunferentziak jatorrian (OX) ardatzaren ukitzaileak dira, x=0.



Urdina → $x^2+\left(y-\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2$

Berdea → $x^2+(y-a)^2=a^2$

$x=\rho\cos(\theta),\ y=\rho\sin(\theta),\ J(\rho,\theta)=\rho$

$x^2+(y-a)^2=a^2 \rightarrow \rho^2\cos^2(\theta)+(\rho\sin(\theta)-a)^2=a^2 \rightarrow \rho^2+\cancel{a^2}-2a\rho\sin(\theta)=\cancel{a^2} \rightarrow$

$\rho^{\cancel{2}}=2a\cancel{\rho}\sin(\theta) \rightarrow \rho=2a\sin(\theta)$ → Goiko muga

$x^2+\left(y-\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2 \rightarrow \rho^2\cos^2(\theta)+\left(\rho\sin(\theta)-\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2 \rightarrow \rho^2+\cancel{\frac{a^2}{4}}-a\rho\sin(\theta)=\cancel{\frac{a^2}{4}} \rightarrow$

$\rho^{\cancel{2}}=a\cancel{\rho}\sin(\theta) \rightarrow \rho=a\sin(\theta)$ → Beheko muga

$$A=\int_0^{\pi/2}d\theta\int_{a\sin(\theta)}^{2a\sin(\theta)}\rho\,d\rho=\int_0^{\pi/2}\left[\frac{\rho^2}{2}\right]_{a\sin(\theta)}^{2a\sin(\theta)}d\theta=\frac{1}{2}\int_0^{\pi/2}\left(4a^2\sin^2(\theta)-a^2\sin^2(\theta)\right)d\theta=\frac{1}{2}\int_0^{\pi/2}3a^2\sin^2(\theta)\,d\theta=$$

$$=\frac{3a^2}{2}\int_0^{\pi/2}\left(\frac{1-\cos(2\theta)}{2}\right)d\theta=\frac{3a^2}{2}\left[\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right]_0^{\pi/2}=\frac{3a^2}{2}\left[\frac{\pi}{4}\right]=\frac{3a^2\pi}{8}\ u^2$$

$$x_c=\frac{1}{A}\iint_D x\,dx\,dy=\frac{8}{3a^2\pi}\iint_R \rho^2\cos(\theta)\,d\rho\,d\theta=\frac{8}{3a^2\pi}\int_0^{\pi/2}\cos(\theta)\,d\theta\int_{a\sin(\theta)}^{2a\sin(\theta)}\rho^2\,d\rho=\frac{8}{3a^2\pi}\int_0^{\pi/2}\cos(\theta)\left[\frac{\rho^3}{3}\right]_{a\sin(\theta)}^{2a\sin(\theta)}d\theta=$$

$$=\frac{8}{9a^2\pi}\int_0^{\pi/2}\cos(\theta)\left[8a^3\sin^3(\theta)-a^3\sin^3(\theta)\right]d\theta=\frac{8}{9a^2\pi}\int_0^{\pi/2}7a^3\cos(\theta)\sin^3(\theta)\,d\theta=\frac{56a^3}{9\cancel{a^2}\pi}\left[\frac{\sin^4(\theta)}{4}\right]_0^{\pi/2}=\frac{56a}{9\pi}\left[\frac{1}{4}\right]=\frac{14a}{9\pi}$$

$$y_c=\frac{1}{A}\iint_D y\,dx\,dy=\frac{8}{3a^2\pi}\iint_R \rho^2\sin(\theta)\,d\rho\,d\theta=\frac{8}{3a^2\pi}\int_0^{\pi/2}\sin(\theta)\,d\theta\int_{a\sin(\theta)}^{2a\sin(\theta)}\rho^2\,d\rho=\frac{8}{3a^2\pi}\int_0^{\pi/2}\sin(\theta)\left[\frac{\rho^3}{3}\right]_{a\sin(\theta)}^{2a\sin(\theta)}d\theta=$$

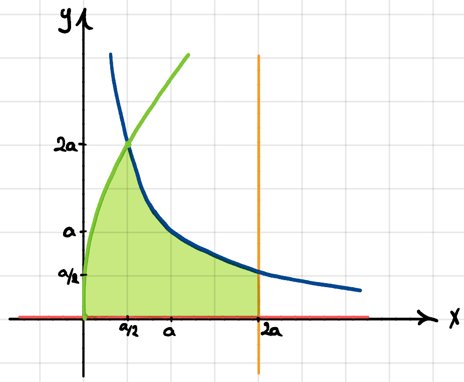
$$=\frac{8}{9a^2\pi}\int_0^{\pi/2}\sin(\theta)\left[8a^3\sin^3(\theta)-a^3\sin^3(\theta)\right]d\theta=\frac{8}{9a^2\pi}\int_0^{\pi/2}7a^3\sin^4(\theta)\,d\theta=\frac{56a^3}{9\cancel{a^2}\pi}\int_0^{\pi/2}\left(\frac{1-\cos(2\theta)}{2}\right)^2d\theta=$$

$$=\frac{56a}{36\pi}\int_0^{\pi/2}\left(1-2\cos(2\theta)+\cos^2(\theta)\right)d\theta=\frac{56a}{36\pi}\int_0^{\pi/2}\left(1-2\cos(2\theta)+\frac{1+\cos(2\theta)}{2}\right)d\theta=$$

$$=\frac{56a}{36\pi}\left[\theta-\sin(2\theta)+\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right]_0^{\pi/2}=\frac{56a}{36\pi}\left[\frac{\pi}{2}+\frac{\pi}{4}\right]=\frac{56a}{36\cancel{\pi}}\cdot\frac{3\cancel{\pi}}{4}=\frac{14a}{12}=\frac{7a}{6}$$

$$\boxed{\text{GRABITATE ZENTROA: }\left(\frac{14a}{9\pi},\frac{7a}{6}\right)}$$

b) [D] eremua $xy=a^2$, $y^2=8ax$, $x=2a$, $y=0$ $[a>0,\ x\geq 0,\ y\geq 0]$ kurbek mugatzen dute



$$xy=a^2\rightarrow y=\frac{a^2}{x}$$

$$y^2=8ax\rightarrow y=\sqrt{8ax}=2\sqrt{2ax}$$

$$A=\int_0^{\frac{a}{2}}dx\int_0^{2\sqrt{2ax}}dy+\int_{\frac{a}{2}}^{2a}dx\int_0^{\frac{a^2}{x}}dy=\int_0^{\frac{a}{2}}2\sqrt{2ax}\,dx+\int_{\frac{a}{2}}^{2a}\frac{a^2}{x}dx=$$

$$=2\sqrt{2a}\left[\frac{2x^{3/2}}{3}\right]_0^{\frac{a}{2}}+a^2\left[\ln|x|\right]_{\frac{a}{2}}^{2a}=\frac{2\sqrt{2a}}{3}\cdot\frac{2\sqrt{a^3}}{2\sqrt{2}}+a^2\left(\ln|2a|-\ln|a/2|\right)=$$

$$=\frac{2\sqrt{a^4}}{3}+a^2\left(\ln|2|+\cancel{\ln|a|}-\cancel{\ln|a|}+\ln|2|\right)=\frac{2a^2}{3}+a^2\,2\ln|2|=\frac{2a^2+6a^2\ln|2|}{3}=\frac{2a^2(1+3\ln|2|)}{3}\,u^2$$

$$x_c=\frac{1}{A}\iint_D x\,dxdy=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\int_0^{\frac{a}{2}}x\,dx\int_0^{2\sqrt{2ax}}dy+\int_{\frac{a}{2}}^{2a}x\,dx\int_0^{\frac{a^2}{x}}dy\right]=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\int_0^{\frac{a}{2}}x2\sqrt{2ax}\,dx+\int_{\frac{a}{2}}^{2a}\cancel{x}\frac{a^2}{\cancel{x}}dx\right]=$$

$$=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[2\sqrt{2a}\int_0^{\frac{a}{2}}x^{3/2}dx+a^2\int_{\frac{a}{2}}^{2a}dx\right]=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[2\sqrt{2a}\left[\frac{2x^{5/2}}{5}\right]_0^{\frac{a}{2}}+a^2[x]_{\frac{a}{2}}^{2a}\right]=$$

$$=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\frac{\cancel{2}\sqrt{2a}}{5}\cdot\frac{\cancel{2}\sqrt{a^5}}{2^{\cancel{2}}\sqrt{2}}+a^2\left(2a-\frac{a}{2}\right)\right]=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\frac{a^3}{5}+\frac{3a^3}{2}\right]=$$

$$=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\frac{17a^3}{10}\right]=\frac{51a^{\cancel{3}}}{20\cancel{a^2}(1+3\ln|2|)}=\frac{51a}{20(1+3\ln|2|)}$$

$$y_c=\frac{1}{A}\iint y\,dxdy=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\int_0^{\frac{a}{2}}dx\int_0^{2\sqrt{2ax}}y\,dy+\int_{\frac{a}{2}}^{2a}dx\int_0^{\frac{a^2}{x}}y\,dy\right]=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\int_0^{\frac{a}{2}}\left[\frac{y^2}{2}\right]_0^{2\sqrt{2ax}}dx+\int_{\frac{a}{2}}^{2a}\left[\frac{y^2}{2}\right]_0^{\frac{a^2}{x}}dx\right]=$$

$$=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\int_0^{\frac{a}{2}}\frac{4(\cancel{2}ax)}{\cancel{2}}dx+\int_{\frac{a}{2}}^{2a}\frac{a^4}{2x^2}dx\right]=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[4a\int_0^{\frac{a}{2}}x\,dx+\frac{a^4}{2}\int_{\frac{a}{2}}^{2a}\frac{1}{x^2}dx\right]=$$

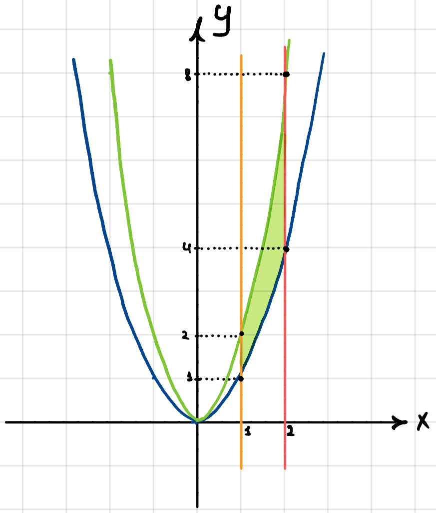
$$=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[4a\left[\frac{x^2}{2}\right]_0^{\frac{a}{2}}+\frac{a^4}{2}\left[\frac{-1}{x}\right]_{\frac{a}{2}}^{2a}\right]=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\cancel{4}a\frac{a^2}{2\cdot\cancel{4}}+\frac{a^4}{2}\left(\frac{-1}{2a}+\frac{2}{a}\right)\right]=$$

$$=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\frac{a^3}{2}+\frac{\cancel{a^4}\,a^3}{2}\cdot\frac{3}{2\cancel{a}}\right]=\left(\frac{3}{2a^2(1+3\ln|2|)}\right)\left[\frac{2a^3+3a^3}{4}\right]=\frac{3\cdot 5a^{\cancel{3}}}{8\cancel{a^2}(1+3\ln|2|)}=\frac{15a}{8(1+3\ln|2|)}$$

$$\boxed{\text{GRABITATE ZENTROA: }\left(\frac{51a}{20(1+3\ln|2|)},\frac{15a}{8(1+3\ln|2|)}\right)}$$

c) $[D]\equiv x^2-y=0,\ 2x^2-y=0,\ x-1=0,\ x-2=0$

$$A=\int_1^2 dx\int_{x^2}^{2x^2}dy=\int_1^2\left[2x^2-x^2\right]dx=\int_1^2 x^2dx=\left[\frac{x^3}{3}\right]_1^2=\frac{8-1}{3}=\frac{7}{3}u^2$$



$$x_c=\frac{1}{A}\iint_D x\,dx\,dy=\int_1^2 x\,dx\int_{x^2}^{2x^2}dy=\frac{3}{7}\int_1^2 x\left[2x^2-x^2\right]dx=\frac{3}{7}\int_1^2 x^3dx=\frac{3}{7}\left[\frac{x^4}{4}\right]_1^2=$$

$$=\frac{3}{7}\left[\frac{16-1}{4}\right]=\frac{45}{28}$$

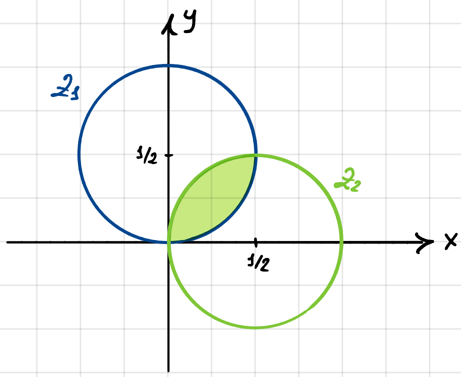
$$y_c=\frac{1}{A}\iint_D y\,dx\,dy=\frac{3}{7}\int_1^2 dx\int_{x^2}^{2x^2}y\,dy=\frac{3}{7}\int_1^2\left[\frac{y^2}{2}\right]_{x^2}^{2x^2}dx=\frac{3}{7}\int_1^2\left(\frac{4x^4-x^4}{2}\right)dx=\frac{3}{7}\cdot\frac{3}{2}\int_1^2 x^4dx=\frac{9}{14}\left[\frac{x^5}{5}\right]_1^2=$$

$$=\frac{9}{14}\left(\frac{32-1}{5}\right)=\frac{279}{70}$$

$$\boxed{\text{GRABITATE ZENTROA: }\left(\frac{45}{28},\frac{279}{70}\right)}$$

d) $[D]$ $\rho=\cos(\theta)$, $\rho=\sin(\theta)$ zirkuluen eremu komuna da

$$\left.\begin{array}{l}x=\rho\cos(\theta)\\ y=\rho\sin(\theta)\\ J(\rho,\theta)=\rho\end{array}\right\}$$



$$\rho=\cos(\theta)\rightarrow\rho^2=\rho\cos(\theta)\rightarrow\rho^2(\cos(\theta)+\sin(\theta))=x$$

$$x^2+y^2=x\rightarrow\left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}\rightarrow Z_2$$

$$\rho=\sin(\theta)\rightarrow\rho^2=\rho\sin(\theta)\rightarrow\rho^2(\cos^2(\theta)+\sin^2(\theta))=y\rightarrow x^2+y^2=y\rightarrow x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}\rightarrow Z_1$$

$$A=\int_0^{\pi/4} d\theta \int_0^{\sin(\theta)} \rho\, d\rho + \int_{\pi/4}^{\pi/2} d\theta \int_0^{\cos(\theta)} \rho\, d\rho = \int_0^{\pi/4} \left[\frac{\rho^2}{2}\right]_0^{\sin(\theta)} d\theta + \int_{\pi/4}^{\pi/2} \left[\frac{\rho^2}{2}\right]_0^{\cos(\theta)} d\theta = \frac{1}{2}\int_0^{\pi/4}\left[\sin^2(\theta)\right] d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}\left[\cos^2(\theta)\right] d\theta =$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(\frac{1-\cos(2\theta)}{2}\right) d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos(2\theta)}{2}\right) d\theta = \frac{1}{2}\left[\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right]_0^{\pi/4} + \frac{1}{2}\left[\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right]_{\pi/4}^{\pi/2} =$$

$$= \frac{1}{2}\left[\cancel{\frac{\pi}{8}}-\frac{1}{4}+\frac{\pi}{4}-\cancel{\frac{\pi}{8}}-\frac{1}{4}\right] = \frac{\pi-2}{8}\, u^2$$

$$x_c = \frac{1}{A}\iint_D x\,dx\,dy = \frac{8}{\pi-2}\iint_R \rho^2\cos(\theta)\,d\rho\,d\theta = \frac{8}{\pi-2}\int_0^{\pi/4}\cos(\theta)\,d\theta\int_0^{\sin(\theta)}\rho^2 d\rho + \frac{8}{\pi-2}\int_{\pi/4}^{\pi/2}\cos(\theta)\int_0^{\cos(\theta)}\rho^2 d\rho =$$

$$= \frac{8}{\pi-2}\int_0^{\pi/4}\cos(\theta)\left[\frac{\rho^3}{3}\right]_0^{\sin(\theta)} d\theta + \frac{8}{\pi-2}\int_{\pi/4}^{\pi/2}\cos(\theta)\left[\frac{\rho^3}{3}\right]_0^{\cos(\theta)} d\theta = \frac{8}{3(\pi-2)}\int_0^{\pi/4}\cos(\theta)\sin^3(\theta)\,d\theta + \frac{8}{3(\pi-2)}\int_{\pi/4}^{\pi/2}\cos^4(\theta)\,d\theta =$$

$$= \frac{8}{3(\pi-2)}\left[\frac{\sin^4(\theta)}{4}\right]_0^{\pi/4} + \frac{8}{3(\pi-2)}\int_{\pi/4}^{\pi/2}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta = \frac{8}{3(\pi-2)}\left[\frac{\left(\sqrt{2}/2\right)^4}{4}\right] +$$

$$+ \frac{8}{3(\pi-2)}\int_{\pi/4}^{\pi/2}\frac{1+2\cos(2\theta)+\cos^2(2\theta)}{4}\,d\theta = \frac{8}{3(\pi-2)}\cdot\frac{1}{16} + \frac{8}{3(\pi-2)}\cdot\frac{1}{4}\int_{\pi/4}^{\pi/2}\left(1+2\cos(2\theta)+\frac{1+\cos(4\theta)}{2}\right)d\theta =$$

$$= \frac{8}{48(\pi-2)} + \frac{8}{12(\pi-2)}\left[\theta+\sin(2\theta)+\frac{\theta}{2}+\frac{\sin(4\theta)}{8}\right]_{\pi/4}^{\pi/2} = \frac{8}{48(\pi-2)} + \frac{8}{12(\pi-2)}\left[\frac{\pi}{2}+\cancel{\frac{\pi}{4}}-\cancel{\frac{\pi}{4}}-1-\frac{\pi}{8}\right] =$$

$$= \frac{8}{48(\pi-2)} + \frac{\cancel{8}}{12(\pi-2)}\left[\frac{3\pi-8}{\cancel{8}}\right] = \frac{2+3\pi-8}{12(\pi-2)} = \frac{3\cancel{(\pi-2)}}{12\cancel{(\pi-2)}} = \frac{1}{4}$$

$$y_c = \frac{1}{A}\iint_D y\,dx\,dy = \frac{8}{\pi-2}\iint_R \rho^2\sin(\theta)\,d\rho\,d\theta = \frac{8}{\pi-2}\int_0^{\pi/4}\sin(\theta)\,d\theta\int_0^{\sin(\theta)}\rho^2 d\rho + \frac{8}{\pi-2}\int_{\pi/4}^{\pi/2}\sin(\theta)\,d\theta\int_0^{\cos(\theta)}\rho^2 d\rho =$$

$$= \frac{8}{\pi-2}\int_0^{\pi/4}\sin(\theta)\left[\frac{\rho^3}{3}\right]_0^{\sin(\theta)} d\theta + \frac{8}{\pi-2}\int_{\pi/4}^{\pi/2}\sin(\theta)\left[\frac{\rho^3}{3}\right]_0^{\cos(\theta)} d\theta = \frac{8}{3(\pi-2)}\int_0^{\pi/4}\sin^4(\theta)\,d\theta + \frac{8}{3(\pi-2)}\int_{\pi/4}^{\pi/2}\sin(\theta)\cos^3(\theta)\,d\theta =$$

$$= \frac{8}{3(\pi-2)}\int_0^{\pi/4}\left(\frac{1-\cos(2\theta)}{2}\right)^2 d\theta + \frac{8}{3(\pi-2)}\left[\frac{-\cos^4(\theta)}{4}\right]_{\pi/4}^{\pi/2} = \frac{8}{3(\pi-2)}\int_0^{\pi/4}\left(\frac{1-2\cos(2\theta)+\cos^2(2\theta)}{4}\right) d\theta +$$

$$+ \frac{8}{3(\pi-2)}\left[\frac{\left(\sqrt{2}/2\right)^4}{4}\right] = \frac{8}{12(\pi-2)}\int_0^{\pi/4}\left(1-2\cos(2\theta)+\frac{1+\cos(4\theta)}{2}\right)d\theta + \frac{8}{3(\pi-2)}\left[\frac{1}{16}\right] =$$

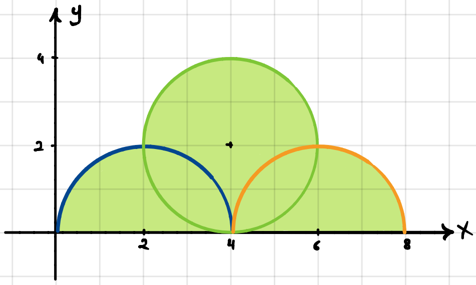
$$= \frac{8}{12(\pi-2)}\left[\theta-\sin(2\theta)+\frac{\theta}{2}+\frac{\sin(4\theta)}{2}\right]_0^{\pi/4} + \frac{2}{12(\pi-2)} = \frac{8}{12(\pi-2)}\left[\frac{\pi}{4}-1+\frac{\pi}{8}\right] + \frac{2}{12(\pi-2)} =$$

$$= \frac{\cancel{8}}{12(\pi-2)}\left[\frac{3\pi-8}{\cancel{8}}\right] + \frac{2}{12(\pi-2)} = \frac{3\cancel{(\pi-2)}}{12\cancel{(\pi-2)}} = \frac{1}{4}$$

$$\boxed{\text{GRABITATE ZENTROA: } \left(\frac{1}{4}, \frac{1}{4}\right)}$$

10 Alderantzizkatu honako integral hauen integrazio ordena:

$$I=\int_0^2 dx\int_0^{\sqrt{4x-x^2}} f(x,y)\,dy+\int_2^6 dx\int_0^{2+\sqrt{4-(x-4)^2}} f(x,y)\,dy+\int_6^8 dx\int_0^{\sqrt{4-(x-6)^2}} f(x,y)\,dy$$



$$y=\sqrt{4x-x^2}\rightarrow x^2-4x+y^2=0\rightarrow (x-2)^2+y^2=4$$

$$C(2,0)\ ,\ R=2$$

$$x-2=-\sqrt{4-y^2}\rightarrow x=2-\sqrt{4-y^2}$$

$$y=2+\sqrt{4-(x-4)^2}\rightarrow (y-2)^2=4-(x-4)^2\rightarrow (x-4)^2+(y-2)^2=4 \quad C(4,2)\ ,\ R=2$$

$$x-4=\sqrt{4-(y-2)^2}\rightarrow x=4\pm\sqrt{4-(y-2)^2}$$

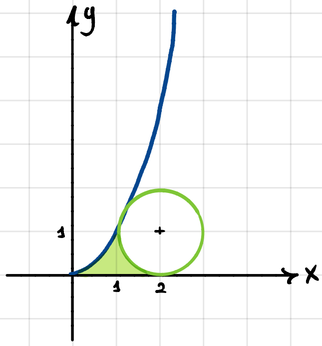
$$y=\sqrt{4-(x-6)^2}\rightarrow (x-6)^2+y^2=4 \quad C(6,0)\ ,\ R=2$$

$$x-6=\sqrt{4-y^2}\rightarrow x=6+\sqrt{4-y^2}$$

$$I=\int_0^2 dy\int_{2-\sqrt{4-y^2}}^{6+\sqrt{4-y^2}} f(x,y)\,dx+\int_2^4 dy\int_{4-\sqrt{4-(y-2)^2}}^{4+\sqrt{4-(y-2)^2}} f(x,y)\,dx$$

11 Alderantzizkatu I-ren integrazio ordena:

$$I=\int_0^1 dx\int_0^{x^2} f(x,y)\,dy+\int_1^2 dx\int_0^{1-\sqrt{4x-x^2-3}} f(x,y)\,dy$$



$$y=x^2\rightarrow x=+\sqrt{y}$$

$$y=1-\sqrt{4x-x^2-3}\rightarrow (y-1)^2=4x-x^2-3\rightarrow (x-2)^2+(y-1)^2=1$$

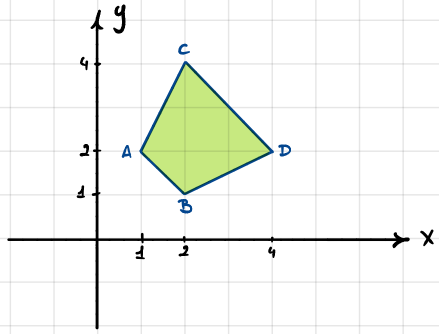
$$x-2=-\sqrt{1-(y-1)^2}\rightarrow x=2-\sqrt{1-(y-1)^2}$$

$$I=\int_0^1 dy\int_{\sqrt{y}}^{2-\sqrt{1-(y-1)^2}} f(x,y)\,dx$$

12 Izan bedi [D] honako erpin hauek dauzkan laukizuzena:

$A(1,2)$, $B(2,1)$, $C(2,4)$, $D(4,2)$.

Egin aldaketa egokiak $\iint_D \frac{x+y}{x^2}\,dx\,dy$ integrala ebaluatzeko.



$\overline{AB}$: $y-2=-1(x-1) \rightarrow y-2=1-x \rightarrow x+y=3$

$\overline{AC}$: $y-2=2(x-1) \rightarrow y-2=2x-2 \rightarrow y=2x \rightarrow y/x=2$

$\overline{BD}$: $y-1=\frac{1}{2}(x-2) \rightarrow y=\frac{x}{2} \rightarrow y/x=1/2$

$\overline{CD}$: $y-4=-1(x-2) \rightarrow x+y=6$

$u=x+y \quad , \quad v=y/x$

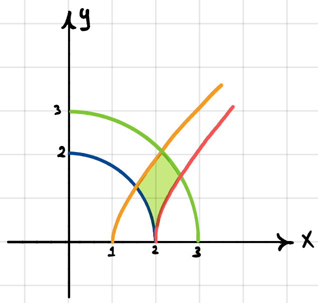
$$J(x,y)=\frac{D(u,v)}{D(x,y)}=\begin{vmatrix} 1 & 1 \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix}=\frac{1}{x}+\frac{y}{x^2}=\frac{x+y}{x^2}$$

$$J(u,v)=\frac{1}{J(x,y)}=\frac{x^2}{x+y}$$

$$\iint_D \frac{x+y}{x^2}\,dx\,dy=\iint_R \frac{x+y}{x^2}\cdot\frac{x^2}{x+y}\,du\,dv=\int_3^6 du\int_{1/2}^{2} dv=[u]_3^6\,[v]_{1/2}^{2}=3\cdot\frac{3}{2}=\frac{9}{2}u^2$$

13 Kalkulatu $I=\iint_D x\cdot y\,dx\,dy$ integralaren balioa, [D] eremua honako kurba hauek lehenengo koadrantean mugatutako esparrua izanik:

$x^2+y^2=4$, $x^2+y^2=9$, $x^2-y^2=1$, $x^2-y^2=4$



$u=x^2+y^2 \quad , \quad v=x^2-y^2$

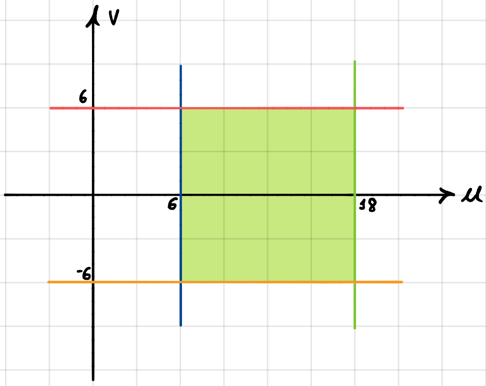
$$J(x,y)=\frac{D(u,v)}{D(x,y)}=\begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix}=-4xy-4xy=-8xy$$

$$J(u,v)=\frac{1}{J(x,y)}=\frac{-1}{8xy}$$

$$I=\iint_D xy\,dx\,dy=\iint_R xy\frac{1}{8xy}\,du\,dv=\frac{1}{8}\int_1^4 dv\int_4^9 du=\frac{1}{8}[u]_4^9\,[v]_1^4=\frac{5\cdot 3}{8}=\frac{15}{8}u^2$$

14 Kalkulatu XY planoan honako desberdintza hauek mugatzen duten eremuaren azalera: $x+y\geq 6,\ x+y\leq 18,\ x-y\geq -6,\ x-y\leq 6$

$u=x+y\ ,\ v=x-y$



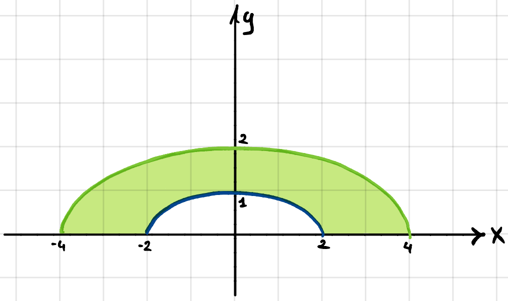
$$J(x,y)=\frac{D(u,v)}{D(x,y)}=\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}=-1-1=-2$$

$$J(u,v)=\frac{1}{J(x,y)}=\frac{-1}{2}$$

$$A=\iint_R \frac{1}{2}\,du\,dv=\frac{1}{2}\int_{-6}^{6}dv\int_{6}^{18}du=\frac{1}{2}[u]_6^{18}[v]_{-6}^{6}=\frac{12\cdot 12}{2}=\frac{144}{2}=72\ u^2$$

15 Honako kurba hauek mugatzen duten [D] eremu laua kontsideratzen da. Grabitate zentroa kalkulatu.

$x^2+4y^2-4=0\ ,\ x^2+4y^2-16=0\ ,\ y\geq 0$



$x^2+4y^2=4 \rightarrow \frac{x^2}{4}+y^2=1 \qquad C(0,0) \quad a=2,\ b=1$

$x^2+4y^2=16 \rightarrow \frac{x^2}{16}+\frac{y^2}{4}=1 \qquad C(0,0) \quad a=4\ \ b=2$

$$\left.\begin{aligned} x&=4\rho\cos(\theta) \\ y&=2\rho\sin(\theta) \\ J(\rho,\theta)&=\overset{a\cdot b\cdot\rho}{8\rho} \end{aligned}\right\} \begin{aligned} &\frac{16\rho^2\cos^2(\theta)}{4}+4\rho^2\sin^2(\theta)=1 \rightarrow 4\rho^2(\cos^2(\theta)+\sin^2(\theta))=1 \rightarrow \rho^2=\frac{1}{4} \rightarrow \rho=\frac{1}{2} \\ &\frac{16\rho^2\cos^2(\theta)}{16}+\frac{4\rho^2\sin^2(\theta)}{4}=1 \rightarrow \rho^2(\cos^2(\theta)+\sin^2(\theta))=1 \rightarrow \rho=1 \end{aligned}$$

$$A=\int_0^{\pi}d\theta\int_{1/2}^{1}8\rho\,d\theta \rightarrow 8\int_0^{\pi}\left[\frac{\rho^2}{2}\right]_{1/2}^{1}d\theta=4\int_0^{\pi}\left(1-\frac{1}{4}\right)d\theta=4\int_0^{\pi}\frac{3}{4}\,d\theta=3[\theta]_0^{\pi}=3\pi\ u^2$$

$$x_c=\frac{1}{A}\iint_D x\,dx\,dy=\frac{1}{3\pi}\iint_R 32\rho^2\cos(\theta)\,d\rho\,d\theta=\frac{32}{3\pi}\int_0^{\pi}\cos(\theta)\,d\theta\int_{1/2}^{1}\rho^2\,d\rho=\frac{32}{3\pi}\int_0^{\pi}\cos(\theta)\left[\frac{\rho^3}{3}\right]_{1/2}^{1}d\theta=\frac{32}{9\pi}\int_0^{\pi}\cos(\theta)\left(1-\frac{1}{8}\right)d\theta=$$

$$=\frac{32}{9\pi}\int_0^{\pi}\frac{7}{8}\cos(\theta)\,d\theta=\frac{28}{9\pi}[\sin(\theta)]_0^{\pi}=0$$

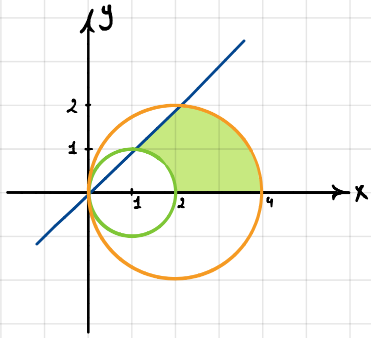
$$y_c = \frac{1}{A}\iint_D y\,dx\,dy = \frac{1}{3\pi}\iint_R 16\rho^2\sin(\theta)\,d\rho\,d\theta = \frac{16}{3\pi}\int_0^{\pi}\sin(\theta)\,d\theta\int_{1/2}^{1}\rho^2\,d\rho = \frac{16}{3\pi}\int_0^{\pi}\sin(\theta)\left[\frac{\rho^3}{3}\right]_{1/2}^{1}d\theta =$$

$$= \frac{16}{9\pi}\int_0^{\pi}\sin(\theta)\left(1-\frac{1}{8}\right)d\theta = \frac{14}{9\pi}\int_0^{\pi}\sin(\theta)\,d\theta = \frac{14}{9\pi}\left[-\cos(\theta)\right]_0^{\pi} = \frac{14}{9\pi}(1+1) = \frac{28}{9\pi}$$

$$\boxed{\text{GRABITATE ZENTROA: } \left(0, \frac{28}{9\pi}\right)}$$

16 Integral bikoitzaren integrazio ordena alderantzizkatu. Kalkulatu integrazio eremuaren grabitate zentroaren ordenatua (y_c).

$$I = \int_1^2 dx\int_{\sqrt{1-(x-1)^2}}^{x} f(x,y)\,dy + \int_2^4 dx\int_0^{\sqrt{4-(x-2)^2}} f(x,y)\,dy$$



$y = x$

$y = \sqrt{1-(x-1)^2} \rightarrow (x-1)^2 + y^2 = 1 \rightarrow x = 1 + \sqrt{1-y^2}$

$y = \sqrt{4-(x-2)^2} \rightarrow (x-2)^2 + y^2 = 4 \rightarrow x = 2 + \sqrt{4-y^2}$

$$I = \int_0^1 dy\int_{1+\sqrt{1-y^2}}^{2+\sqrt{4-y^2}} f(x,y)\,dx + \int_1^2 dy\int_y^{2+\sqrt{4-y^2}} f(x,y)\,dx$$

Koordenatu polar orokortuak:

$x = \rho\cos(\theta)$
$y = \rho\sin(\theta)$
$J(\rho,\theta) = \rho$

- $\rho^2\cos^2(\theta) - 2\rho\cos(\theta) + \cancel{1} + \rho^2\sin^2(\theta) = \cancel{1} \rightarrow \rho^{\cancel{2}} = 2\cancel{\rho}\cos(\theta) \rightarrow \rho = 2\cos(\theta)$
- $\rho^2\cos^2(\theta) - 4\rho\cos(\theta) + \cancel{4} + \rho^2\sin^2(\theta) = \cancel{4} \rightarrow \rho^{\cancel{2}} = 4\cancel{\rho}\cos(\theta) \rightarrow \rho = 4\cos(\theta)$
- $\cancel{\rho}\cos(\theta) = \cancel{\rho}\sin(\theta) \rightarrow \theta = \pi/4$

$$A = \int_0^{\pi/4} d\theta\int_{2\cos(\theta)}^{4\cos(\theta)}\rho\,d\rho = \int_0^{\pi/4}\left[\frac{\rho^2}{2}\right]_{2\cos(\theta)}^{4\cos(\theta)}d\theta = \frac{1}{2}\int_0^{\pi/4}\left(16\cos^2(\theta) - 4\cos(\theta)\right)d\theta = \frac{1}{2}\int_0^{\pi/4}12\cos^2(\theta)\,d\theta = 6\int_0^{\pi/4}\frac{1+\cos(2\theta)}{2}\,d\theta =$$

$$= 6\left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{\pi/4} = 6\left[\frac{\pi}{8} + \frac{1}{4}\right] = \frac{6(\pi+2)}{8} = \frac{3(\pi+2)}{4}\,u^2$$

$$y_c=\frac{1}{A}\iint_D y\,dx\,dy=\frac{4}{3(\pi+2)}\iint_R \rho^2\sin(\theta)\,d\theta=\frac{4}{3(\pi+2)}\int_0^{\pi/4}\sin(\theta)\,d\theta\int_{2\cos(\theta)}^{4\cos(\theta)}\rho^2\,d\rho=\frac{4}{3(\pi+2)}\int_0^{\pi/4}\sin(\theta)\left[\frac{\rho^3}{3}\right]_{2\cos(\theta)}^{4\cos(\theta)}d\theta=$$

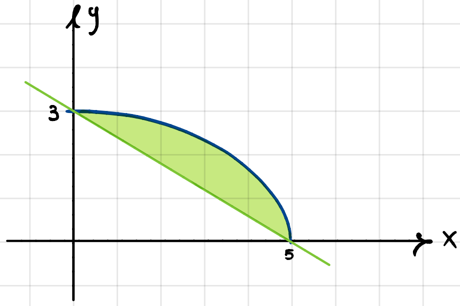
$$=\frac{4}{9(\pi+2)}\int_0^{\pi/4}\sin(\theta)\left(64\cos^3(\theta)-8\cos^3(\theta)\right)d\theta=\frac{224}{9(\pi+2)}\int_0^{\pi/4}\sin(\theta)\cos^3(\theta)\,d\theta=\frac{224}{9(\pi+2)}\left[\frac{-\cos^4(\theta)}{4}\right]_0^{\pi/4}=$$

$$=\frac{56}{9(\pi+2)}\left[-\frac{2^{4/2}}{2^4}+1\right]=\frac{56}{9(\pi+2)}\left[\frac{3}{4}\right]\to\boxed{y_c=\frac{14}{3(\pi+2)}}$$

$\boxed{17}$ Kalkulatu lehenengo koadrantean $\frac{x^2}{25}+\frac{y^2}{9}=1$ elipseak eta bere sokak $\frac{x}{5}+\frac{y}{3}=1$, mugatzen duten irudiaren grabitate zentro geometrikoaren ordenatua.

$$\frac{x^2}{25}+\frac{y^2}{9}=1\to\frac{y^2}{9}=1-\frac{x^2}{25}\to y=\sqrt{9-\frac{9x^2}{25}}=3\sqrt{1-\frac{x^2}{25}}$$

$$\frac{x}{5}+\frac{y}{3}=1\to\frac{y}{3}=1-\frac{x}{5}\to y=3-\frac{3x}{5}$$



$$A=\int_0^5 dx\int_{3-\frac{3x}{5}}^{3\sqrt{1-\frac{x^2}{25}}}dy=\int_0^5\left[3\sqrt{1-\frac{x^2}{25}}-3+\frac{3x}{5}\right]dx=\frac{3}{5}\int_0^5\left(\sqrt{25-x^2}-5+x\right)dx=$$

$$=\frac{3}{5}I_1+\frac{3}{5}\left[\frac{x^2}{2}-5x\right]_0^5=\frac{3}{5}I_1+\frac{3}{5}\left[\frac{25}{2}-25\right]=\frac{3}{5}I_1+\frac{3}{5}\left[\frac{-25}{2}\right]=\frac{3}{5}I_1-\frac{15}{2}$$

$$I_1=\int_0^5\sqrt{25-x^2}\,dx=\left/\!\!\left/\begin{array}{l}x=5\sin(t)\\dx=5\cos(t)\\x=0\to t=0\\x=5\to t=\pi/2\end{array}\right/\!\!\right/=\int_0^{\pi/2}\sqrt{25-25\sin^2(t)}\,5\cos(t)\,dt=25\int_0^{\pi/2}\sqrt{1-\sin^2(t)}\cos(t)\,dt=25\int_0^{\pi/2}\cos^2(t)\,dt=$$

$$=25\int_0^{\pi/2}\frac{1+\cos(2t)}{2}dt=25\left[\frac{t}{2}+\frac{\sin(2t)}{4}\right]_0^{\pi/2}=25\left[\frac{\pi}{4}\right]=\frac{25\pi}{4}$$

$$A=\frac{3}{5}\left[\frac{25\pi}{4}\right]-\frac{15}{2}=\frac{15\pi}{4}-\frac{15}{2}=\frac{15\pi-30}{4}=\frac{15(\pi-2)}{4}u^2$$

$$y_c=\frac{1}{A}\iint_D y\,dx\,dy=\frac{4}{15(\pi-2)}\int_0^5 dx\int_{3-\frac{3x}{5}}^{3\sqrt{1-\frac{x^2}{25}}}y\,dy=\frac{4}{15(\pi-2)}\int_0^5\left[\frac{y^2}{2}\right]_{3-\frac{3x}{5}}^{3\sqrt{1-\frac{x^2}{25}}}dx=\frac{2}{15(\pi-2)}\int_0^5\left[9\left(1-\frac{x^2}{25}\right)-9-\frac{9x^2}{25}+\frac{18x}{5}\right]dx=$$

$$=\frac{6}{25\cdot5(\pi-2)}\int_0^5\left(\cancel{25}-x^2-\cancel{25}-x^2+10x\right)dx=\frac{6}{5^3(\pi-2)}\int_0^5\left(10x-2x^2\right)dx=\frac{6}{5^3(\pi-2)}\left[5x^2-\frac{2x^3}{3}\right]_0^5=$$

$$=\frac{6}{5^3(\pi-2)}\left[5^3-\frac{2\cdot5^3}{3}\right]=\frac{6}{\cancel{5^3}(\pi-2)}\left[\frac{\cancel{5^3}(3-2)}{3}\right]\to\boxed{y_c=\frac{2}{\pi-2}}$$