Integral hirukoitzak

1 Izan bedi hurrengo [D] domeinua:

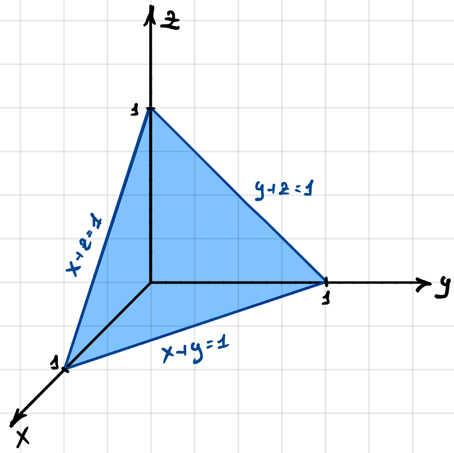
$$D = \{(x,y) \in \mathbb{R}^3 / x \geq 0 \wedge y \geq 0 \wedge z \geq 0 \wedge x+y+z \leq 1)\}$$

Kalkulatu $\iiint_D z\,dx\,dy\,dz$ integral hirukoitzaren balioa.

$x+y+z=1 \rightarrow$ Planoa

$z = 1-x-y$

$z=0:\ x+y=1 \rightarrow y=1-x$



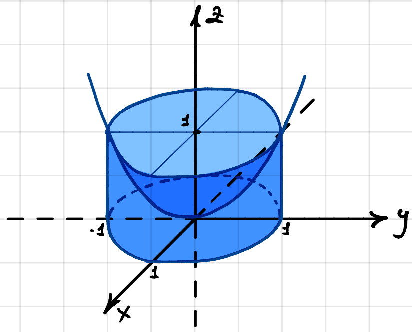
$$\iiint_D z\,dx\,dy\,dz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} z\,dz = \int_0^1 dx \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-x-y} dy =$$

$$= \frac{1}{2}\int_0^1 dx \int_0^{1-x} (1-x-y)^2\,dy = \frac{1}{6}\int_0^1 \left[-(1-x-y)^3\right]_0^{1-x} dx =$$

$$= \frac{1}{6}\int_0^1 (1-x)^3\,dx = \frac{1}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{1}{24}$$

2 Izan bedi XY planoari eta $z = x^2+y^2$ eta $1-x^2-y^2=0$ gainazalen ekuazio Kartesiarrei dagokien [D] domeinu komuna:

a) Kalkulatu [D]. (bolumena)



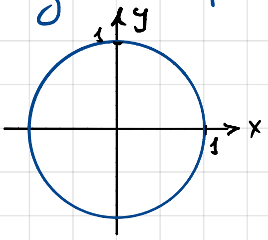
$z = x^2+y^2 \rightarrow$ Paraboloidea

$1-x^2-y^2=0 \rightarrow$ Zilindroa

Ebakidura planoa:

$$\left.\begin{array}{l} x^2+y^2=1 \\ x^2+y^2=z \end{array}\right\} \boxed{z=1}$$

Proiekzioa XOY planoan:



Zirkunferentzia: $C(0,0)$, $R=1$

Koordenatu zilindrikoak:

$$\left.\begin{array}{l} x=\rho\cos(\theta) \\ y=\rho\sin(\theta) \\ z=z \\ J(\rho,\theta,z)=\rho \end{array}\right\} \begin{array}{l} x^2+y^2=2 \rightarrow \rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)=2 \rightarrow \boxed{\rho^2=2} \\ x^2+y^2=1 \rightarrow \rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)=1 \rightarrow \rho^2=1 \rightarrow \boxed{\rho=1} \end{array}$$

$$D=\iiint_D dx\,dy\,dz=\iiint_R \rho\,d\rho\,d\theta\,dz=\int_0^{2\pi}d\theta\int_0^1\rho\,d\rho\int_0^{\rho^2}dz=\int_0^{2\pi}d\theta\int_0^1\rho\left[z\right]_0^{\rho^2}d\rho=\int_0^{2\pi}d\theta\int_0^1\rho^3\,d\rho=\int_0^{2\pi}\left[\frac{\rho^4}{4}\right]_0^1 d\theta=$$

$$=\frac{1}{4}\int_0^{2\pi}d\theta=\frac{1}{4}\left[\theta\right]_0^{2\pi}=\frac{2\pi}{4} \rightarrow \boxed{D=\frac{\pi}{2}u^3}$$

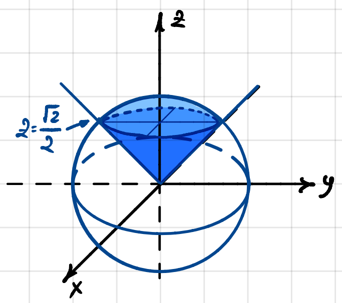
b) Kalkulatu $\iiint_D (x+y+z)\,dx\,dy\,dz$ integral hirukoitzaren balioa.

$$\iiint_D (x+y+z)\,dx\,dy\,dz=\iiint_R \rho(\rho\cos(\theta)+\rho\sin(\theta)+z)\,d\rho\,d\theta\,dz=\int_0^{2\pi}d\theta\int_0^1\rho\,d\rho\int_0^{\rho^2}(\rho\cos(\theta)+\rho\sin(\theta)+z)\,dz=$$

$$=\int_0^{2\pi}d\theta\int_0^1\rho\left[z\rho\cos(\theta)+z\rho\sin(\theta)+\frac{z^2}{2}\right]_0^{\rho^2}d\rho=\int_0^{2\pi}d\theta\int_0^1\left(\rho^4\cos(\theta)+\rho^4\sin(\theta)+\frac{\rho^5}{2}\right)d\rho=\int_0^{2\pi}\left[\frac{\rho^5\cos(\theta)+\rho^5\sin(\theta)}{5}+\frac{\rho^6}{12}\right]_0^1 d\theta=$$

$$=\int_0^{2\pi}\left(\frac{\cos(\theta)+\sin(\theta)}{5}+\frac{1}{12}\right)d\theta=\left[\frac{\sin(\theta)-\cos(\theta)}{5}+\frac{\theta}{12}\right]_0^{2\pi}=\cancel{-\frac{1}{5}}+\frac{2\pi}{12}+\cancel{\frac{1}{5}} \rightarrow \boxed{V=\frac{\pi}{6}u^3}$$

3 Integral hirukoitzaren kontzeptua erabiliz, kalkulatu $x^2+y^2+z^2=1$ eta $z^2=x^2+y^2$ ($z\geq 0$ izanik) gainazalen ekuazio kartesiarrei dagokien eskualde komunaren bolumena.



$x^2+y^2+z^2=1 \rightarrow$ Esfera $C(0,0,0)$, $R=1$

$z^2=x^2+y^2 \rightarrow$ Konoa, erpina $(0,0,0)$

Ebakidura planoa:

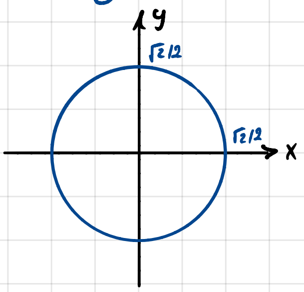
ezin da negatiboa izan $z\geq 0$ delako

$$\left.\begin{array}{l} x^2+y^2=1-z^2 \\ x^2+y^2=z^2 \end{array}\right\} 1-z^2=z^2 \rightarrow 1=2z^2 \rightarrow z=\frac{1}{\pm\sqrt{2}} \rightarrow z=\frac{+\sqrt{2}}{2}$$

Koordenatu zilindrikoak

$$\left.\begin{array}{l} x=\rho\cos(\theta) \\ y=\rho\sin(\theta) \\ z=z \\ J(\rho,\theta,z)=\rho \end{array}\right\} \begin{array}{l} \rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)+z^2=1 \rightarrow \rho^2+z^2=1 \rightarrow \boxed{z=+\sqrt{1-\rho^2}} \rightarrow \text{Goi muga} \\ \rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)=z^2 \rightarrow \rho^2=z^2 \rightarrow \boxed{z=\rho} \rightarrow \text{Behe muga} \end{array}$$

Proiekzioa XOY planoan:



$\theta=[0,2\pi]$

$\rho=[0,\sqrt{2}/2]$

$z=[\rho,\sqrt{1-\rho^2}]$

$$V=\iiint_D dx\,dy\,dz=\iiint_R \rho\,d\rho\,d\theta\,dz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}/2}\rho\,d\rho\int_\rho^{\sqrt{1-\rho^2}}dz=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}/2}\rho\left(\sqrt{1-\rho^2}-\rho\right)d\rho=\int_0^{2\pi}\left[\frac{-(1-\rho^2)^{3/2}}{3}-\frac{\rho^3}{3}\right]_0^{\sqrt{2}/2}d\theta=$$

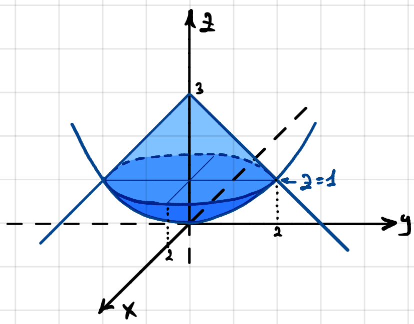
$$=\frac{-1}{3}\left(\frac{1}{2\sqrt{2}}+\frac{\sqrt{2}}{4}-1\right)[\theta]_0^{2\pi}=\frac{-1}{3}\left(\frac{\sqrt{2}+\sqrt{2}}{4}-1\right)(2\pi)=\frac{-2\pi}{3}\left(\frac{2\sqrt{2}}{4}-1\right)=\frac{-2\pi}{3}\left(\frac{\sqrt{2}}{2}-1\right)\rightarrow \boxed{V=\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)u^3}$$

$\boxed{4}$ Kalkulatu hurrengo [D] domeinuaren bolumena:

$D=\{(x,y)\in\mathbb{R}^3 / x^2+y^2\leq(z-3)^2 \wedge x^2+y^2\leq 4z \wedge z\leq 3)\}$

$x^2+y^2\leq(z-3)^2 \rightarrow$ Konoa, erpina $(0,0,3)$

$x^2+y^2\leq 4z \rightarrow$ Paraboloidea, erpina $(0,0,0)$



Koordenatu zilindrikoak:

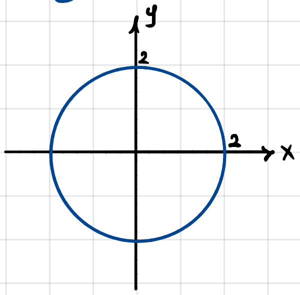
z≤3 denez, ρ≥0 izateko ⊖ hartzen da

$$\left.\begin{array}{l} x=\rho\cos(\theta) \\ y=\rho\sin(\theta) \\ z=z \\ J(\rho,\theta,z)=\rho \end{array}\right\} \begin{array}{l} \rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)=(z-3)^2 \rightarrow \rho^2=(z-3)^2 \rightarrow \rho=-(z-3) \rightarrow \rho=3-z \rightarrow \\ \rightarrow \boxed{z=3-\rho} \\ \rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)=4z \rightarrow \rho^2=4z \rightarrow \boxed{z=\rho^2/4} \end{array}$$

Ebakidura planoa:

$$\left.\begin{array}{l} x^2+y^2=(z-3)^2 \\ x^2+y^2=4z \end{array}\right\} (z-3)^2=4z \rightarrow z^2-6z+9=4z \rightarrow z^2-10z+9=0 \rightarrow z=\frac{10\pm\sqrt{100-36}}{2} \begin{cases} \cancel{z=9} \rightarrow z\leq 3 \\ z=1 \end{cases}$$

Proiekzioa XOY planoan:



$$z=\left[\frac{\rho^2}{4},\ 3-\rho\right]$$

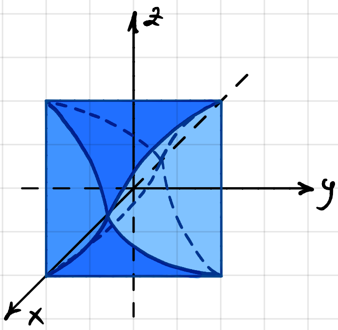
$$\rho=[0,2]$$

$$\theta=[0,2\pi]$$

$$V=\iiint_D dx\,dy\,dz=\iiint_R \rho\,d\rho\,d\theta\,dz=\int_0^{2\pi}d\theta\int_0^2\rho\,d\rho\int_{\rho^2/4}^{\rho-3}dz=\int_0^{2\pi}d\theta\int_0^2\rho\left[3-\rho-\frac{\rho^2}{4}\right]d\rho=\int_0^{2\pi}d\theta\int_0^2\left(3\rho-\rho^2-\frac{\rho^3}{4}\right)d\rho=$$

$$=\left[\frac{3\rho^2}{2}-\frac{\rho^3}{3}-\frac{\rho^4}{16}\right]_0^2\cdot[\theta]_0^{2\pi}=2\pi\left(6-\frac{8}{3}-1\right)=2\pi\left(\frac{15-8}{3}\right)=2\pi\left(\frac{7}{3}\right)\rightarrow \boxed{V=\frac{14\pi}{3}u^3}$$

$\boxed{5}$ Integral hirukoitza erabiliz, kalkulatu $4x^2+y^2=4$, $4x^2+z^2=4$ zilindroek mugatzen duten [V] domeinuaren bolumena



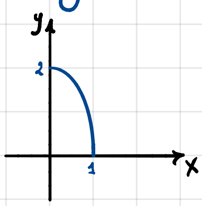
$4x^2+y^2=4 \rightarrow$ Zilindroa, z ardatzean $a=1$, $b=2$

$$x^2+\frac{y^2}{4}=1 \rightarrow \frac{y^2}{4}=1-x^2 \rightarrow y^2=4(1-x^2)\rightarrow y=2\sqrt{1-x^2}$$

$4x^2+z^2=4 \rightarrow$ Zilindroa, Y ardatzean $a=1$, $b=2$

$$x^2+\frac{z^2}{4}=1 \rightarrow \frac{z^2}{4}=1-x^2 \rightarrow z^2=4(1-x^2)\rightarrow z=2\sqrt{1-x^2}$$

Proiekzioa XOY planoan (1.go oktantea)



$$z=[0,\ 2\sqrt{1-x^2}]$$

$$y=[0,\ 2\sqrt{1-x^2}]$$

$$x=[0,\ 1]$$

8 oktante daude

$$V=8\int_0^1 dx\int_0^{2\sqrt{1-x^2}}dy\int_0^{2\sqrt{1-x^2}}dz=8\int_0^1 4(1-x^2)\,dx=32\left[x-\frac{x^3}{3}\right]_0^1=32\left[\frac{2}{3}\right]\rightarrow \boxed{V=\frac{64}{3}u^3}$$

$\boxed{6}$ Aldagai aldaketa egokia erabiliz, kalkulatu hurrengo zilindro parabolikoek mugatzen duten [C] gorputzaren bolumena:

$$y-z^2=0,\ 2y-z^2=0,\ z-x^2=0,\ 2z-x^2=0,\ x-y^2=0,\ 2x-y^2=0$$

$$u=\frac{z^2}{y}\rightarrow u=[1,2]\quad,\quad v=\frac{x^2}{z}\rightarrow v=[1,2]\quad,\quad w=\frac{y^2}{x}\rightarrow w=[1,2]$$

$$J(x,y,z)=\frac{D(u,v,w)}{D(x,y,z)}=\begin{vmatrix}0 & -z^2/y^2 & 2z/y\\ 2x/z & 0 & -x^2/z^2\\ -y^2/x^2 & 2y/x & 0\end{vmatrix}=-1+8=7$$

$$J(u,v,w)=\frac{1}{J(x,y,z)}=\frac{1}{7}$$

$$V=\iiint_D dx\,dy\,dz=\iiint_R \frac{1}{7}\,du\,dv\,dw=\frac{1}{7}\int_1^2 du\int_1^2 dv\int_1^2 dw=\frac{1}{7}[u]_1^2[v]_1^2[w]_1^2\rightarrow \boxed{V=\frac{1}{7}u^3}$$

$\boxed{7}$ Determinatu lehen oktantean dagoen dentsitate konstanteko gorputz baten masa zentroa, baldin gorputza hurrengo gainazalek mugatzen badute.

$$x^2+y^2=z\ ,\ x^2+y^2=2z\ ,\ xy=1,\ xy=4,\ y=x\ ,\ y=3x$$

$$u=\frac{x^2+y^2}{z}\qquad z=\frac{x^2+y^2}{u}\xrightarrow{①②} z=\frac{(v/w)+vw}{u}=\frac{v+vw^2}{uw}\rightarrow z=\frac{v(1+w^2)}{uw}$$

$$v=xy\qquad x=\frac{v}{y}\xrightarrow{①} x=\frac{v}{xw}\rightarrow x=\frac{v^{1/2}}{w^{1/2}}\ ②$$

$$w=\frac{y}{x}\qquad y=xw\ ①\xrightarrow{②} y=v^{1/2}w^{1/2}$$

$u=[1,2]$

$v=[1,4]$

$w=[1,3]$

$$J(x,y,z)=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{2x}{z} & \frac{2y}{z} & \boxed{\frac{-(x^2+y^2)}{z^2}} \\ y & x & 0 \\ \frac{-y}{x^2} & \frac{1}{x} & 0 \end{vmatrix} = \frac{-(x^2+y^2)}{z^2}\left(\frac{y}{x}+\frac{\cancel{x}y}{x^{\cancel{2}}}\right)=$$

$$=\frac{-2y(x^2+y^2)}{xz^2}=-2\frac{u}{z}w=\frac{-2u^2w^2}{v(1+w^2)}$$

$$J(u,v,w)=\frac{-v(1+w^2)}{2u^2w^2}$$

Dentsitatea Konstantea denez, $M=V$

$$V=\iiint_V \frac{v(1+w^2)}{2u^2w^2}\,du\,dv\,dw=\frac{1}{2}\int_1^2\frac{1}{u^2}\,du\int_1^4 v\,dv\int_1^3\frac{1+w^2}{w^2}\,dw=\frac{1}{2}\left[\frac{-1}{u}\right]_1^2\left[\frac{v^2}{2}\right]_1^4\left[w-\frac{1}{w}\right]_1^3=\frac{1}{\cancel{2}}\cdot\frac{1}{\cancel{2}}\cdot\frac{15}{\cancel{2}}\cdot\frac{\cancel{8}}{3}=5u^3$$

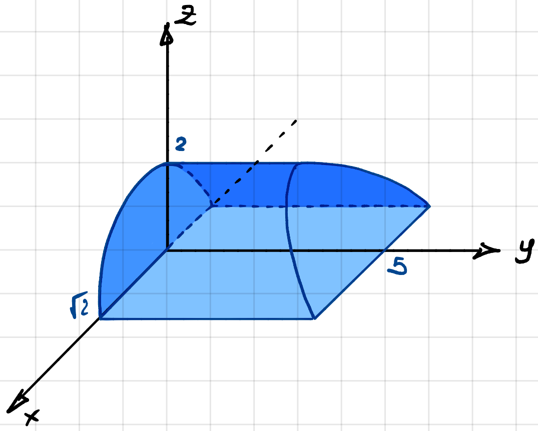
$$X_m=\frac{1}{V}\iiint_V x\,dx\,dy\,dz=\frac{1}{5}\iiint_V \frac{v^{3/2}}{w^{1/2}}\cdot\frac{v(1+w^2)}{2u^2w^2}\,du\,dv\,dw=\frac{1}{10}\int_1^2\frac{1}{u^2}\,du\int_1^4 v^{3/2}\,dv\int_1^3\frac{1+w^2}{w^{5/2}}\,dw=$$

$$=\frac{1}{10}\left[\frac{-1}{u}\right]_1^2\left[\frac{2v^{5/2}}{5}\right]_1^4\int_1^3\left(\frac{1}{w^{5/2}}+\frac{1}{w^{1/2}}\right)dw=\frac{1}{10}\cdot\frac{1}{2}\cdot\frac{62}{5}\left[\frac{-2}{3w^{3/2}}+2w^{1/2}\right]_1^3=\frac{62}{100}\cdot$$

$$y_m=\frac{1}{V}\iiint_V y\,dx\,dy\,dz=\frac{1}{5}\iiint$$

$\boxed{8}$ $x^2 + z - 2 = 0 \longrightarrow$ ZILINDRO PARABOLIKOA

$$\left.\begin{aligned} y + z &= 5 \\ x &= 0 \\ y &= 0 \end{aligned}\right\} \text{PLANOAK}$$

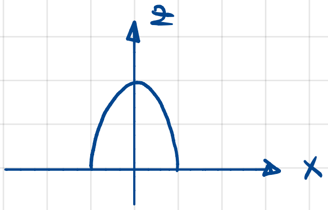


Erpina:

$z = 2 - x^2 \rightarrow z' = -2x$

$-2x = 0 \longrightarrow x = 0 \rightarrow z = 2$

PROIEKZIOA XOZ PLANOAN



$y = [0, 5 - z]$
$z = [0, 2 - x^2]$
$x = [0, \sqrt{2}]$

$$V = 2\int_0^{\sqrt{2}} dx \int_0^{2-x^2} dz \int_0^{5-z} dy = 2\int_0^{\sqrt{2}} dx \int_0^{2-x^2} (5-z)\,dz = 2\int_0^{\sqrt{2}} \left(5(2-x^2) - \frac{(2-x^2)^2}{2}\right) dx =$$

$$= 10\int_0^{\sqrt{2}} (2-x^2)\,dx - \int_0^{\sqrt{2}} (4 - 4x^2 + x^4)\,dx = 10\left[2x - \frac{x^3}{3}\right]_0^{\sqrt{2}} - \left[4x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right]_0^{\sqrt{2}} =$$

$$= 10\left[2\sqrt{2} - \frac{2\sqrt{2}}{3}\right] - \left[4\sqrt{2} - \frac{8\sqrt{2}}{3} + \frac{4\sqrt{2}}{5}\right] = 10\left[\frac{2\sqrt{2}(3-1)}{3}\right] - \left[\frac{4\sqrt{2}(15-10+3)}{15}\right] =$$

$$= \frac{40\sqrt{2}}{3} - \frac{32\sqrt{2}}{15} = 8\sqrt{2}\left(\frac{5}{3} - \frac{4}{15}\right) = 8\sqrt{2}\left(\frac{25-4}{15}\right) = 8\sqrt{2}\left(\frac{21}{15}\right) = 8\sqrt{2}\,\frac{7}{5} = \frac{56\sqrt{2}}{5}\,u^3$$