Integral bikoitzak:

$\boxed{1}$ Ebatzi hurrengo integral bikoitzak:

a) $$\int_0^2 dx \int_x^{\sqrt{3}x} \frac{x\,dy}{x^2+y^2} = \int_0^2 x\left[\frac{1}{x}\operatorname{arctg}\left(\frac{y}{x}\right)\right]_x^{\sqrt{3}x} dx = \int_0^2 \left(\operatorname{arctg}(\sqrt{3}) - \operatorname{arctg}(1)\right)dx =$$

$$= \int_0^2 \left(\frac{\pi}{3} - \frac{\pi}{4}\right)dx = \left[\frac{\pi}{3}x - \frac{\pi}{4}x\right]_0^2 = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{4\pi - 3\pi}{6} = \frac{\pi}{6}$$

b) $$\int_1^3 dy \int_2^5 \frac{dx}{(x+2y)^2} = \int_1^3 \left[-\frac{1}{x+2y}\right]_2^5 dy = \int_1^3 \left(-\frac{1}{5+2y} + \frac{1}{2+2y}\right)dy = \frac{1}{2}\int_1^3 \left(\frac{1}{1+y} - \frac{1}{5/2 - y}\right)dy =$$

$$= \frac{1}{2}\left[\ln|1+y| - \ln\left|\frac{5}{2}+y\right|\right]_1^3 = \frac{1}{2}\left(\ln|4| - \ln\left|\frac{11}{2}\right| - \ln|2| + \ln\left|\frac{7}{2}\right|\right) =$$

$$= \frac{1}{2}\left(\ln\left|\frac{4}{11/2}\right| + \ln\left|\frac{7/2}{2}\right|\right) = \frac{1}{2}\left(\ln\left|\frac{8}{11}\right| + \ln\left|\frac{7}{4}\right|\right) = \frac{1}{2}\ln\left|\frac{\overset{2}{\cancel{8}}\cdot 7}{11\cdot\cancel{4}}\right| = \frac{1}{2}\ln\left|\frac{14}{11}\right|$$

c) $$\int_1^3 dx \int_{x^3}^{x} (x-y)\,dy = \int_1^3 \left[xy - \frac{y^2}{2}\right]_{x^3}^{x} dx = \int_1^3 \left(x^2 - \frac{x^2}{2} - x^4 + \frac{x^6}{2}\right)dx = \left[\frac{x^3}{3} - \frac{x^3}{6} - \frac{x^5}{5} + \frac{x^7}{14}\right]_1^3 =$$

$$= \frac{3^3}{3} - \frac{3^3}{6} - \frac{3^5}{5} + \frac{3^7}{14} - \frac{1}{3} + \frac{1}{6} + \frac{1}{5} - \frac{1}{14} = \frac{26}{3} - \frac{26}{6} - \frac{242}{5} + \frac{2186}{14} = \frac{26\cdot 70 - 26\cdot 35 - 242\cdot 42 + 2186\cdot 15}{210} =$$

$$= \frac{1820 - 910 - 10164 + 32790}{210} = \frac{23536}{210} = \frac{11768}{105}$$

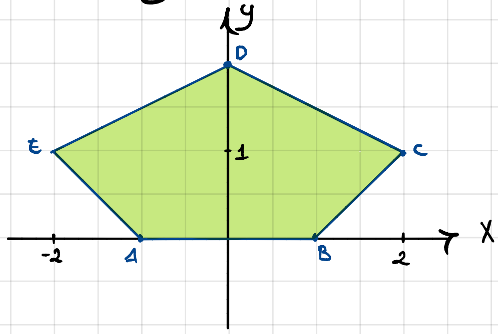
d) $$\int_0^{\pi/2} d\theta \int_{a\cos(\theta)}^{a(1+\cos(\theta))} \rho\,d\rho = \int_0^{\pi/2} \left[\frac{\rho^2}{2}\right]_{a\cos(\theta)}^{a(1+\cos(\theta))} d\theta = \frac{1}{2}\int_0^{\pi/2} \left[a^2(1+\cos(\theta))^2 - a^2\cos^2(\theta)\right]d\theta =$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \left[1 + 2\cos(\theta) + \cos^2(\theta) - \cos^2(\theta)\right]d\theta = \frac{a^2}{2}\int_0^{\pi/2} (1+2\cos(\theta))\,d\theta = \frac{a^2}{2}\left[\theta + 2\sin(\theta)\right]_0^{\pi/2} =$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + 2\right] = \frac{a^2\pi}{4} + a^2 = \frac{a^2(\pi+4)}{4}$$

$\boxed{2}$ Hurrengo [D] eremuetarako, jarri $I=\iint_D f(x,y)\,dx\,dy$ integralaren integrazio limiteak bi ordena posibleetan:

a) $A(-1,0)$, $B(1,0)$, $C(2,1)$, $D(0,2)$ eta $E(-2,1)$ erpinezko poligonoa da:



$\overline{AB}$: $y-0=m(x+1) \rightarrow y=0$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{0-0}{1+1}=0$$

$\overline{BC}$: $y-0=m(x-1) \rightarrow y=x-1 \rightarrow x=y+1$

$$m=\frac{1-0}{2-1}=1$$

$\overline{CD}$: $y-1=m(x-2) \rightarrow y=2-x/2=\frac{4-x}{2} \rightarrow x=4-2y$

$$m=\frac{2-1}{0-2}=\frac{-1}{2}$$

$\overline{DE}$: $y-2=m(x-0) \rightarrow y=\frac{4+x}{2} \rightarrow x=2y-4$

$$m=\frac{1-2}{-2-0}=\frac{1}{2}$$

$\overline{EA}$: $y-1=m(x+2) \rightarrow y=-x-1 \rightarrow x=-y-1$

$$m=\frac{0-1}{-1+2}=-1$$

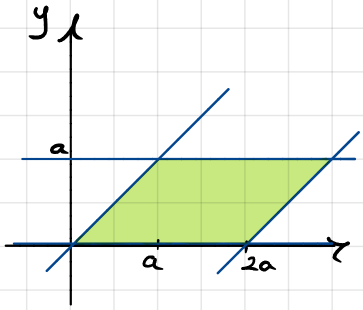
x lehenengo integrazio aldagaitzat hartuz:

$$I=\int_0^1 dy\int_{-y-1}^{y+1} f(x,y)\,dx+\int_1^2 dy\int_{2y-4}^{4-2y} f(x,y)\,dx$$

y lehenengo integrazio aldagaitzat hartuz:

$$I=\int_{-2}^{-1} dx\int_{-x-1}^{\frac{4+x}{2}} f(x,y)\,dy+\int_{-1}^{0} dx\int_{0}^{\frac{4+x}{2}} f(x,y)\,dy+\int_{0}^{1} dx\int_{0}^{\frac{4-x}{2}} f(x,y)\,dy+\int_{1}^{2} dx\int_{y-1}^{\frac{4-x}{2}} f(x,y)\,dy$$

b) [D]-ko puntuek $y \leq x \leq y+2a$, $0 \leq y \leq a$ desberdintzak betetzen dituzte



$y = x$

$x = y+2a \rightarrow y = x-2a$

$y = 0$

$y = a$

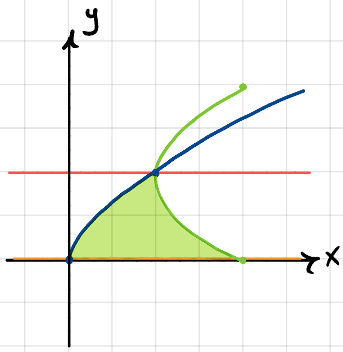
x lehenengo integrazio aldagaitzat hartuta:

$$I = \int_0^a dy \int_y^{y-2a} f(x,y)\,dx$$

y lehenengo integrazio aldagaitzat hartuta:

$$I = \int_0^a dx \int_0^x f(x,y)\,dy + \int_a^{2a} dx \int_0^a f(x,y)\,dy + \int_{2a}^{3a} dx \int_{x-2a}^{a} f(x,y)\,dy$$

c) [D] eremua $y=0$; $y=1$; $x=y^{3/2}$; $x=2-\sqrt{2y-y^2}$ kurbek mugatuta dago.



- $x = y^{3/2} \rightarrow y = x^{2/3}$
- $x = 2-\sqrt{2y-y^2} \rightarrow 2y-y^2 = (x-2)^2 \rightarrow 1-(y-1)^2 = (x-2)^2 \rightarrow$

$\rightarrow y-1 = \sqrt{1-(x-2)^2} \rightarrow y = 1 \pm \sqrt{1-(x-2)^2}$

x lehenengo integrazio aldagaitzat hartuz

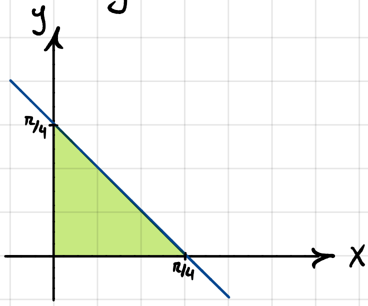
$$I = \int_0^1 dy \int_{y^{3/2}}^{2-\sqrt{2y-y^2}} f(x,y)\,dx$$

y lehenengo integrazio aldagaitzat hartuz

$$I = \int_0^1 dx \int_0^{x^{2/3}} f(x,y)\,dy + \int_1^2 dx \int_0^{1-\sqrt{1-(x-2)^2}} f(x,y)\,dy$$

$\boxed{3}$ Kalkulatu hurrengo integral bikoitzak adierazitako [D] eremuen gainean:

a) $\iint_D [\cos(2x)+\sin(y)]\,dx\,dy$



$[D] \equiv x=0,\ y=0,\ 4x+4y-\pi=0$

$$V=\int_0^{\pi/4} dx \int_0^{\frac{\pi}{4}-x} [\cos(2x)+\sin(y)]\,dy = \int_0^{\pi/4} \Big[y\cos(2x)-\cos(y)\Big]_0^{\frac{\pi}{4}-x} dx =$$

$$=\int_0^{\pi/4} \Big[\underbrace{\frac{\pi}{4}\cos(2x)}_{A}-\underbrace{x\cos(2x)}_{B}-\underbrace{\cos\left(\frac{\pi}{4}-x\right)}_{C}+1\Big]dx$$

$$A=\frac{\pi}{4}\int_0^{\pi/4}\cos(2x)\,dx=\frac{\pi}{4}\left[\frac{\sin(2x)}{2}\right]_0^{\pi/4}=\frac{\pi}{8}$$

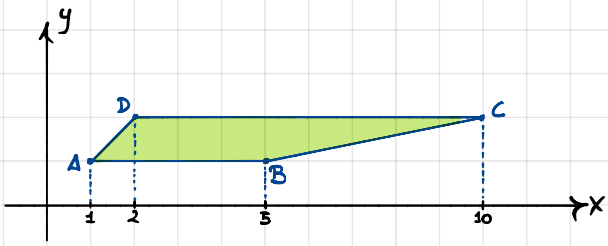
$$B=\int_0^{\pi/4} x\cos(2x)\,dx = \left\| \begin{matrix} u=x \rightarrow du=dx \\ dv=\cos(2x)dx \rightarrow v=\frac{\sin(2x)}{2} \end{matrix} \right\| = \left[\frac{x\sin(2x)}{2}\right]_0^{\pi/4} - \int_0^{\pi/4}\frac{\sin(2x)}{2}dx = \left[\frac{x\sin(2x)}{2}+\frac{\cos(2x)}{4}\right]_0^{\pi/4} =$$

$$=\frac{\pi}{8}-\frac{1}{4}=\frac{\pi-2}{8}$$

$$C=\int_0^{\pi/4}\cos\left(\frac{\pi}{4}-x\right)dx=\left[\sin\left(\frac{\pi}{4}-x\right)\right]_0^{\pi/4}=-\sin\left(\frac{\pi}{4}\right)=\frac{-\sqrt{2}}{2}$$

$$V=\frac{\pi}{8}-\frac{\pi-2}{8}+\frac{\sqrt{2}}{2}+\frac{\pi}{4}=\frac{\cancel{\pi}-\cancel{\pi}+2+4\sqrt{2}+2\pi}{8} \rightarrow \boxed{V=\frac{\pi+2\sqrt{2}+1}{4}\,u^3}$$

b) $\iint_D \sqrt{xy-y^2}\,dx\,dy$



$[D] \equiv A(1,1),\ B(5,1),\ C(10,2),\ D(2,2)$

$\overline{AB} \equiv y=1,\ \overline{DC} \equiv y=2,\ \overline{AD} \equiv y=x$

$\overline{BC} \equiv y-1=m(x-5) \rightarrow y=x/5 \rightarrow x=5y \qquad m_{BC}=\frac{2-1}{10-5}=\frac{1}{5}$

$$V=\int_1^2 dy\int_y^{5y}\sqrt{xy-y^2}\,dx = \left\| \begin{matrix} t=xy-y^2 \rightarrow dt=y\,dx \\ dx=dt/y \\ x=5y \rightarrow t=4y^2 \\ x=y \rightarrow t=0 \end{matrix} \right\| = \int_1^2 dy \int_0^{4y^2}\sqrt{t}\,\frac{dt}{y} = \int_1^2 \frac{1}{y}\left[\frac{2t^{3/2}}{3}\right]_0^{4y^2} dy = \frac{2}{3}\int_1^2 \frac{1}{y}\left[t^{3/2}\right]_0^{4y^2} dy =$$

$$=\frac{2}{3}\int_1^2 \frac{\sqrt{(4y^2)^3}}{y}dy = \frac{2}{3}\int_1^2 \frac{4y^2(2\cancel{y})}{\cancel{y}}dy = \frac{16}{3}\int_1^2 y^2\,dy = \frac{16}{3}\left[\frac{y^3}{3}\right]_1^2 = \frac{16}{9}[8-1] = \frac{16\cdot 7}{9}$$

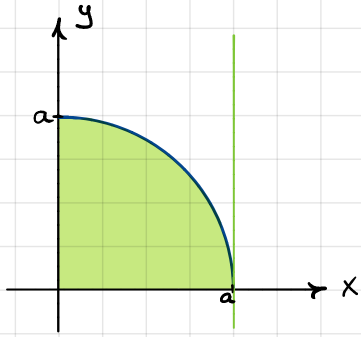
$$\boxed{V=\frac{112}{9}\,u^3}$$

c) $\iint_D \sqrt{a^2+x^2}\,dx\,dy$

$[D] \equiv y^2 - x^2 = a^2,\ x=0,\ x=a,\ y=0 \quad (y \geq 0)$

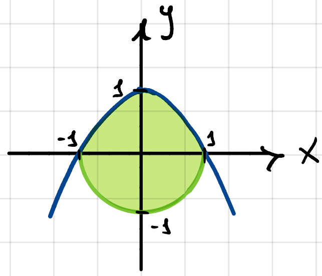
$$V = \int_0^a dx \int_0^{\sqrt{a^2+x^2}} \sqrt{a^2+x^2}\,dy = \int_0^a \sqrt{a^2+x^2}\,[y]_0^{\sqrt{a^2+x^2}}\,dx = \int_0^a (\cancel{\sqrt{a^2+x^2}})^{\cancel{2}}\,dx =$$

$$= \left[a^2x + \frac{x^3}{3}\right]_0^a = a^3 + \frac{a^3}{3} = \frac{3a^3+a^3}{3} \rightarrow \boxed{V = \frac{4a^3}{3}\,u^3}$$



$\boxed{4}$ Alderantzizkatu integrazio ordena hurrengo integraletan:

a) $I = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy$

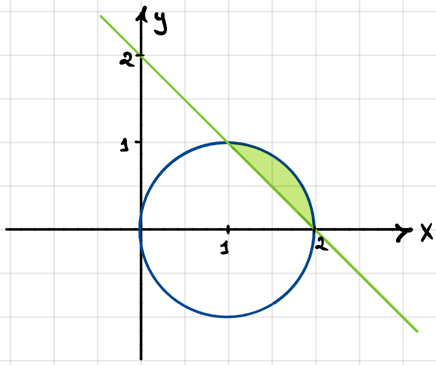


$y = 1 - x^2 \rightarrow x = \sqrt{1-y^2}$

$y = -\sqrt{1-x^2} \rightarrow 1 - x^2 = y^2 \rightarrow x = -\sqrt{1-y^2}$

$$I = \int_{-1}^{0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx + \int_{0}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

b) $I = \int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx$



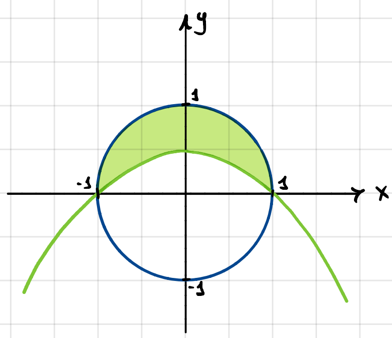
$x = 1 + \sqrt{1-y^2} \rightarrow (x-1)^2 = 1 - y^2 \rightarrow y = \sqrt{1-(x-1)^2} \rightarrow$

$\rightarrow y = \sqrt{\cancel{1} - x^2 + 2x - \cancel{1}} \rightarrow \quad y = \sqrt{2x - x^2}$

$x = 2 - y \rightarrow y = 2 - x$

$$I = \int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy$$

c) $I = \int_0^{1/2} dy \int_{\sqrt{1-2y}}^{\sqrt{1-y^2}} f(x,y)\,dx + \int_{1/2}^{1} dy \int_0^{\sqrt{1-y^2}} f(x,y)\,dx$

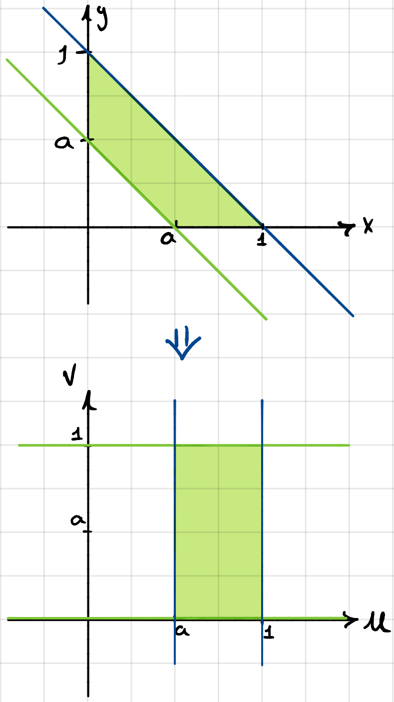


$x = \sqrt{1-y^2} \rightarrow \quad y = \sqrt{1-x^2}$

$x = \sqrt{1-2y} \rightarrow \quad y = \dfrac{1-x^2}{2}$

$$I = \int_{-1}^{1} dx \int_{\frac{1-x^2}{2}}^{\sqrt{1-x^2}} f(x,y)\,dy$$

5 Ordezkapen egokiak erabiliz, kalkulatu hurrengo integralek adierazitako eremuen gainean.



a) $I=\iint_D \frac{y\,dx\,dy}{x+y}$

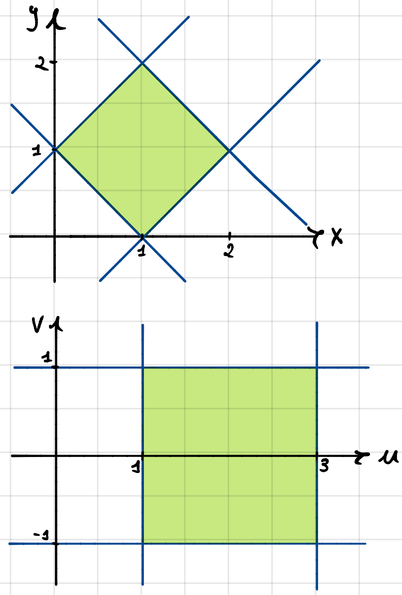
$[D] \equiv x=0,\ y=0,\ x+y=a,\ x+y=1\ \ (0<a<1)$

$$\left.\begin{aligned} x+y&=u \\ y&=uv \end{aligned}\right\} \rightarrow u=x+uv \rightarrow x=u-uv \rightarrow x=u(1-v)$$

$$J(u,v)=\frac{D(x,y)}{D(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}=$$

$$=u(1-v)+uv=u-\cancel{uv}+\cancel{uv}=u$$

$$I=\iint_D \frac{y}{x+y}\,dx\,dy=\int_0^1 v\,dv\int_a^1 u\,du=\int_0^1 v\left[\frac{u^2}{2}\right]_a^1 dv=$$



b) $I=\iint_D (x+y)^3(x-y)^2\,dx\,dy$

$[D] \equiv x+y=1,\ x+y=3,\ x-y=1,\ x-y=-1$

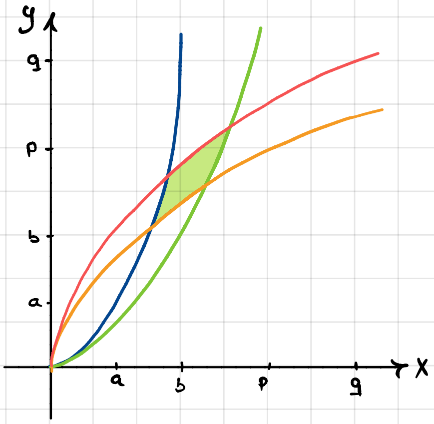
$$\left.\begin{aligned} u&=x+y \\ v&=x-y \end{aligned}\right| \begin{aligned} &\rightarrow x=u-y \rightarrow x=u-\frac{u-v}{2} \rightarrow x=\frac{u+v}{2} \\ &\rightarrow v=u-y-y \rightarrow y=\frac{u-v}{2} \end{aligned}$$

$$J(u,v)=\frac{D(x,y)}{D(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}=-1/2$$

$$I=\iint u^3\cdot v^2\,|J(u,v)|\,du\,dv=\frac{1}{2}\int_{-1}^{1} v^2\,dv\int_1^3 u^3\,du=\frac{1}{2}\int_{-1}^{1} v^2\left[\frac{u^4}{4}\right]_1^3 dv=\frac{3^4-1}{8}\int_{-1}^{1} v^2\,dv=\frac{80}{8}\left[\frac{v^3}{3}\right]_{-1}^{1}=$$

$$=10\,\frac{2}{3}=\frac{20}{3}$$

c) $I=\iint_D xy\,dx\,dy$

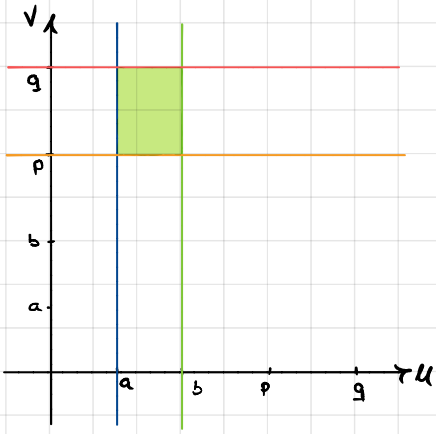


[D] $y=ax^3,\ y=bx^3,\ y^2=px,\ y^2=qx \quad [0<a<b,\ 0<p<q]$

$$\left.\begin{array}{l} \frac{y}{x^3}=u \\ \\ \frac{y^2}{x}=v \end{array}\right\} \begin{array}{l} \to y=ux^3 \to y=u\frac{v^{3/5}}{u^{6/5}} \to y=\frac{v^{3/5}}{u^{1/5}} \\ \\ \to v=\frac{u^2x^{\cancel{6}5}}{\cancel{x}} \to x=\frac{v^{1/5}}{u^{2/5}} \end{array}$$

$$J(x,y)=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}=\begin{vmatrix} \frac{-3y}{x^4} & \frac{1}{x^3} \\ \frac{-y^2}{x^2} & \frac{2y}{x} \end{vmatrix}=$$

$$=\frac{-6y^2}{x^5}+\frac{y^2}{x^5}=\frac{-5y^2}{x^5}$$

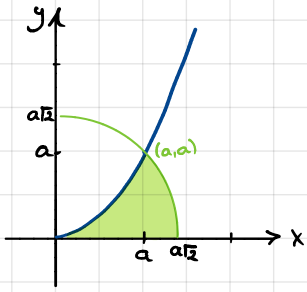


$$J(u,v)=\frac{1}{J(x,y)}=\frac{x^5}{-5y^2}=\frac{-(v/u^2)}{5(v^{6/5}/u^{2/5})}=\frac{-vu^{2/5}}{5v^{6/5}u^2}=\frac{-1}{5v^{1/5}u^{8/5}}$$

$$I=\iint_R \frac{v^{4/5}}{u^{3/5}}\cdot\frac{1}{5v^{1/5}u^{8/5}}\,du\,dv=\frac{1}{5}\int_p^q v^{3/5}dv\int_a^b\frac{1}{u^{11/5}}du=\frac{1}{6}\left[\frac{-1}{u^{6/5}}\right]_a^b\int_p^q v^{3/5}dv=\frac{1}{6}\left(\frac{1}{a^{6/5}}-\frac{1}{b^{6/5}}\right)\frac{5}{8}\left[v^{8/5}\right]_p^q=$$

$$=\frac{5}{48}\left(a^{-6/5}-b^{-6/5}\right)\left(q^{8/5}-p^{8/5}\right)$$

d) $I=\iint_D \frac{x\,dx\,dy}{x^2+y^2}$ $\quad [D]\equiv x^2=ay,\ x^2+y^2=2a^2,\ y=0 \quad [x>0,\ a>0]$



$$\left.\begin{array}{l} x=\rho\cos(\theta) \\ y=\rho\sin(\theta) \end{array}\right| \begin{array}{l} \rho^{\cancel{2}}\cos^2(\theta)=a\cancel{\rho}\sin(\theta)\to\rho=\frac{a\,tg(\theta)}{\cos(\theta)} \\ \rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)=2a^2\to\rho^2=2a^2\to\rho=a\sqrt{2} \end{array}$$

$$J(\rho,\theta)=\rho$$

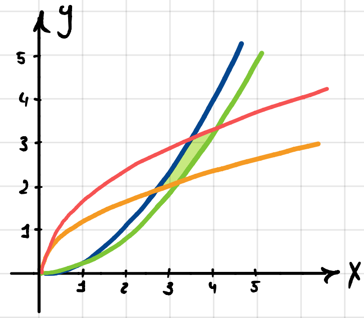
$$I=\iint_R\frac{\cancel{\rho}\cos(\theta)}{\cancel{\rho^2}}\cdot\cancel{\rho}\,d\rho\,d\theta=\int_0^{\pi/4}\cos(\theta)d\theta\int_{\frac{a\,tg(\theta)}{\cos(\theta)}}^{a\sqrt{2}}d\rho=\int_0^{\pi/4}\cos(\theta)\left[a\sqrt{2}-\frac{a\,tg(\theta)}{\cos(\theta)}\right]d\theta=$$

$$=a\int_0^{\pi/4}\left(\sqrt{2}\cos(\theta)-tg(\theta)\right)d\theta=a\left[\sqrt{2}\sin(\theta)+\ln|\cos(\theta)|\right]_0^{\pi/4}=a\left[1+\ln\left|\frac{\sqrt{2}}{2}\right|\right]=$$

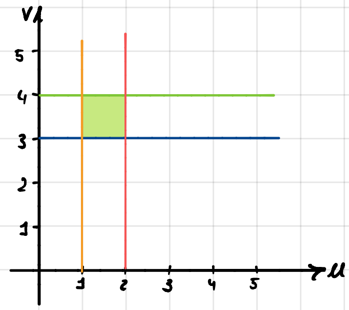
$$=a\left[1+\ln\left|\frac{1}{\sqrt{2}}\right|\right]=a\left[1+\ln|1|-\ln|\sqrt{2}|\right]=a\left[1-\frac{1}{2}\ln|2|\right]=\frac{a(2-\ln|2|)}{2}$$

$\boxed{6}$ Erabili aldagai aldaketa egokiak hurrengo kurbek mugatzen dituzten domeinuen azalerak kalkulatzeko

a) $x^2-3y=0,\ x^2-4y=0,\ x-y^2=0,\ 2x-y^2=0\quad (x\geqslant 0,\ y\geqslant 0)$



$$\left.\begin{aligned} u&=\frac{y^2}{x} \\ v&=\frac{x^2}{y}\end{aligned}\right\}\qquad \begin{aligned} &x=\frac{y^2}{u}\ \rightarrow\ x=\frac{(vu^2)^{2/3}}{u}\ \rightarrow\ x=(vu)^{1/3} \\ &v=\frac{y^4/u^2}{y}=\frac{y^3}{u^2}\ \rightarrow\ y=(vu^2)^{1/3}\end{aligned}$$

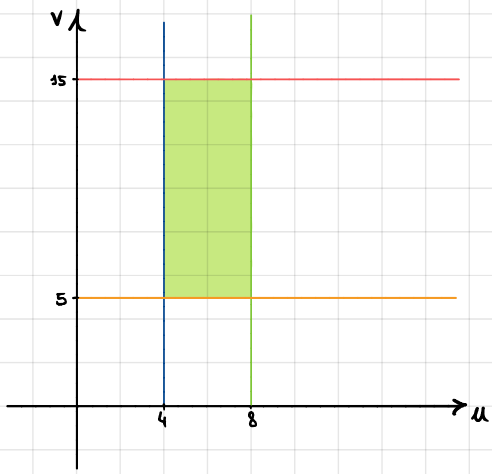


$$J(x,y)=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\dfrac{-y^2}{x^2} & \dfrac{2y}{x}\\[2mm] \dfrac{2x}{y} & \dfrac{-x^2}{y^2}\end{vmatrix}=1-4=-3$$

$$J(u,v)=\frac{1}{J(x,y)}=\frac{-1}{3}$$

$$A=\frac{1}{3}\int_3^4 dv\int_1^2 du=\frac{1}{3}[u]_1^2\,[v]_3^4\rightarrow\ \boxed{A=\frac{1}{3}u^2}$$

b) $xy=4,\ xy=8,\ xy^3=5,\ xy^3=15\quad (x\geqslant 0,\ y\geqslant 0)$



$$\left.\begin{aligned} u&=xy \\ v&=xy^3\end{aligned}\right\}\qquad \begin{aligned} &y=\frac{u}{x}\ \rightarrow\ y=\frac{uv^{1/2}}{u^{3/2}}\ \rightarrow\ y=\frac{v^{1/2}}{u^{1/2}} \\ &v=x\frac{u^3}{x^3}=\frac{u^3}{x^2}\ \rightarrow\ x=\frac{u^{3/2}}{v^{1/2}}\end{aligned}$$

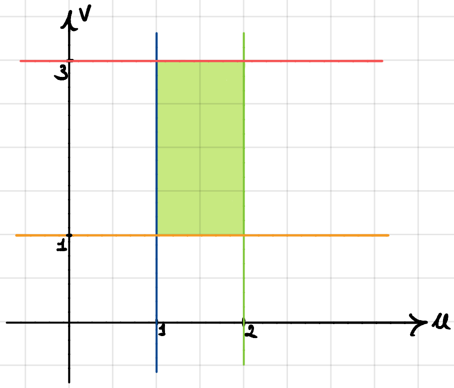
$$J(x,y)=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix} y & x\\ y^3 & 3xy^2\end{vmatrix}=3xy^3-xy^3=2xy^3$$

$$J(u,v)=\frac{1}{J(x,y)}=\frac{1}{2xy^3}=\frac{\cancel{u^{3/2}}\,v^{1/2}}{2\cancel{u^{3/2}}\,v^{3/2}}=\frac{1}{2v}$$

$$A=\int_5^{15}\frac{1}{2v}\,dv\int_4^8 du=\frac{1}{2}[u]_4^8\,[\ln|v|]_5^{15}=\frac{1}{2}4(\ln(15)-\ln(5))=2\ln\left(\frac{15}{5}\right)\rightarrow\ \boxed{A=2\ln(3)\,u^2}$$

c) $xy=1$, $xy=2$, $y=x$, $y=3x$

$$\left.\begin{array}{l} u=xy \\ v=\frac{y}{x}\end{array}\right\} \quad \begin{array}{l} x=\frac{u}{y} \rightarrow x=\frac{u}{(uv)^{1/2}} \rightarrow x=\frac{u^{1/2}}{v^{1/2}} \\ v=\frac{y}{u/y} \rightarrow v=\frac{y^2}{u} \rightarrow y=(uv)^{1/2} \end{array}$$



$$J(x,y)=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix} = \begin{vmatrix} y & x \\ \frac{-y}{x^2} & \frac{1}{x}\end{vmatrix} = \frac{y}{x}+\frac{y}{x}=\frac{2y}{x}$$

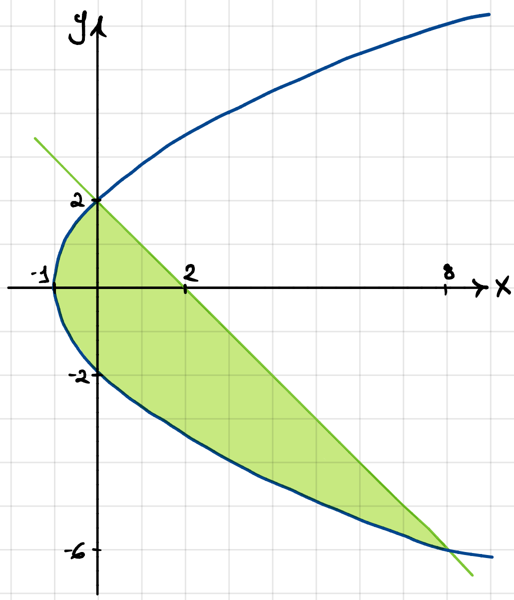
$$J(u,v)=\frac{1}{J(x,y)}=\frac{x}{2y}=\frac{\cancel{u^{1/2}}/v^{1/2}}{2(\cancel{u}v)^{1/2}}=\frac{1}{2v}$$

$$I=\frac{1}{2}\int_1^3 \frac{1}{v}dv\int_1^2 du=\frac{1}{2}[u]_1^2[\ln(v)]_1^3=\frac{1}{2}[\ln(3)-\cancel{\ln(1)}] \rightarrow \boxed{A=\frac{\ln(3)}{2}u^2}$$

7 Ebaluatu adierazitako kurbek mugatzen dituzten eremu lauen azalerak:

a) $x+y-2=0$, $y^2-4x-4=0$

$$\left.\begin{array}{l} x=2-y \\ x=\frac{y^2}{4}-1\end{array}\right\} \quad 2-y=\frac{y^2}{4}-1 \rightarrow 3-y=\frac{y^2}{4} \rightarrow 12-4y=y^2$$



$$y^2+4y-12=0 \rightarrow y=\frac{-4\pm\sqrt{4^2-4(-12)}}{2} < \begin{array}{l} y=2 \\ y=-6\end{array}$$

$$A=\int_{-6}^{2}dy\int_{\frac{y^2}{4}-1}^{2-y}dx=\int_{-6}^{2}\left[2-y-\frac{y^2}{4}+1\right]dy=$$

$$=\int_{-6}^{2}\left[3-y-\frac{y^2}{4}\right]dy=\left[3y-\frac{y^2}{2}-\frac{y^3}{12}\right]_{-6}^{2}=$$

$$=6-2-\frac{2}{3}+18+\cancel{18}-\cancel{18}=22-\frac{2}{3}=\frac{66-2}{3}$$

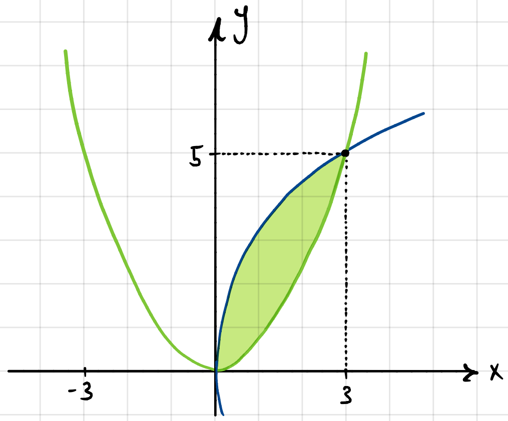
$$\boxed{A=\frac{64}{3}u^2}$$

b) $3y^2-25x=0\ ,\ 5x^2-9y=0$

$x=\frac{3y^2}{25}\qquad x=\frac{3\sqrt{y}}{\sqrt{5}}$

$$A=\int_0^5 dy\int_{\frac{3y^2}{25}}^{\frac{3\sqrt{y}}{\sqrt5}}dx=\int_0^5\left[\frac{3\sqrt{y}}{\sqrt5}-\frac{3y^2}{25}\right]dy=$$

$$=\frac{\cancel{3}}{\sqrt5}\left[\frac{2y^{3/2}}{\cancel{3}}\right]_0^5-\frac{\cancel{3}}{25}\left[\frac{y^3}{\cancel{3}}\right]_0^5=\frac{2\cdot5^{3/2}}{\cancel{5^{1/2}}}-\frac{5^{\cancel{3}}}{\cancel{25}}=10-5=5$$

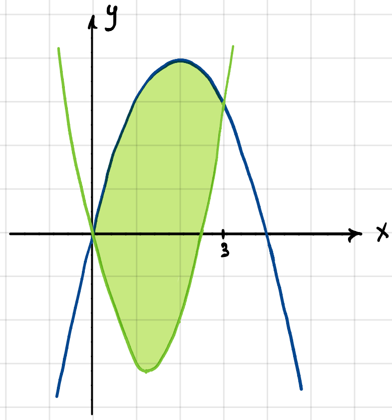


c) $y=x(4-x)\ ,\quad y=x(2x-5)$

$y=4x-x^2\qquad y=2x^2-5x$

$$A=\int_0^3 dx\int_{2x^2-5x}^{4x-x^2}dy=\int_0^3(4x-x^2-2x^2+5x)dx=\int_0^3(9x-3x^2)dx=$$

$$=\left[\frac{9x^2}{2}-x^3\right]_0^3=\frac{9\cdot3^2}{2}-3^3=\frac{81-27\cdot2}{2}\to\boxed{A=\frac{27}{2}u^2}$$



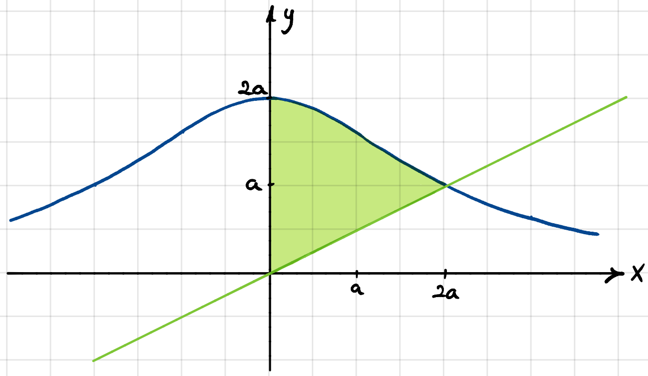
d) $y=\frac{8a^3}{x^2+4a^2}\ ,\ x=2y\ ,\ x=0\quad(a>0)$

$$A=\int_0^{2a}dx\int_{\frac{x}{2}}^{\frac{8a^3}{x^2+4a^2}}dy=\int_0^{2a}\left[\frac{8a^3}{x^2+4a^2}-\frac{x}{2}\right]dx=$$

$$=8a^3\left[\frac{1}{2a}\operatorname{arctg}\left(\frac{x}{2a}\right)\right]_0^{2a}-\frac{1}{2}\left[\frac{x^2}{2}\right]_0^{2a}=$$

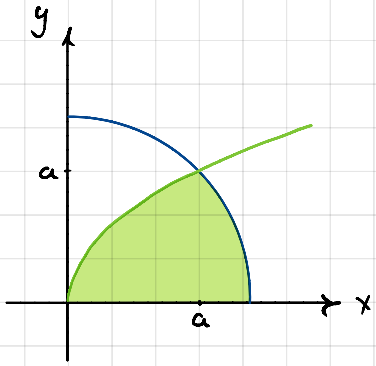
$$=4a^2\left[\operatorname{arctg}(1)-\cancel{\operatorname{arctg}(0)}^{\,0}\right]-\frac{(2a)^2}{4}=$$

$$=\cancel{4}a^2\frac{\pi}{\cancel{4}}-\frac{\cancel{4}a^2}{\cancel{4}}=\pi a^2-a^2\to\boxed{A=a^2(\pi-1)u^2}$$



e) $x^2+y^2-2a^2=0$, $y^2-ax=0$, $y=0$, 1. koadrantea

$$\left.\begin{array}{l} x=\rho\cos(\theta) \\ y=\rho\sin(\theta) \end{array}\right| \begin{array}{l} \rho^2(\underbrace{\cos^2(\theta)+\sin^2(\theta)}_{1})=2a^2 \to \rho=a\sqrt{2} \\ \rho^{\cancel{2}}\sin^2(\theta)=a\cancel{\rho}\cos(\theta) \to \rho=\dfrac{a\cos(\theta)}{\sin^2(\theta)} \end{array}$$



$$J(\rho,\theta)=\rho$$

$$A=\int_0^{\pi/4} d\theta \int_{\frac{a\cos(\theta)}{\sin^2(\theta)}}^{a\sqrt{2}} \rho\, d\rho = \int_0^{\pi/4}\left[\frac{\rho^2}{2}\right]_{\frac{a\cos(\theta)}{\sin(\theta)}}^{a\sqrt{2}} = \frac{1}{2}\int_0^{\pi/4}\left[2a^2-\frac{a^2\cos^2(\theta)}{\sin^4(\theta)}\right]d\theta$$

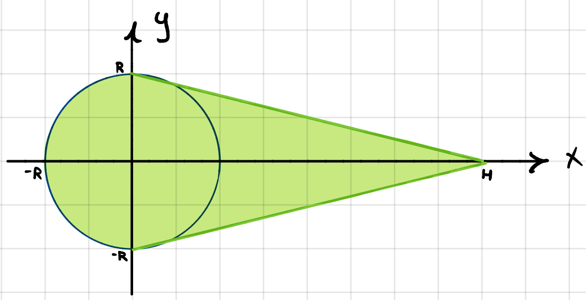
$$I=\int\frac{\cos^2(\theta)}{\sin^4(\theta)}d\theta = \left/\!\!\left/ \begin{array}{l} t=\frac{\cos(\theta)}{\sin(\theta)} \\ dt=\frac{-dx}{\sin^2(\theta)} \end{array}\right/\!\!\right/ = -\int t^2 dt = -\frac{t^3}{3}+K = \frac{-\cos^3(\theta)}{3\sin^3(\theta)}+K$$

$$A=a^2[\theta]_0^{\pi/4}-\frac{a^2}{2}\left[\frac{-\cos^3(\theta)}{3\sin^3(\theta)}\right]_0^{\pi/4} = a^2\left[\frac{\pi}{4}\right]-\frac{a^2}{2}\left[\frac{-1}{3}\right]=\frac{a^2\pi}{4}+\frac{a^2}{6} \to \boxed{A=\frac{a^2(3\pi+2)}{12}\,u^2}$$

$\boxed{8}$ r erradioko erdizirkulu bati triangelu isoszele bat lotzen zaio, halako moldez non triangeluaren alde desberdina erdizirkuluaren diametroarekin bat baitator. Determinatu triangeluaren h altuera, halako moldez non figuraren grabitate zentroa erdizirkuluaren diametroaren erdiko puntuan baitago.

$x_c=0$, $y_c=0$

Zirkunferentzia: $x^2+y^2=R^2 \to y=\sqrt{R^2-x^2}$



Triangelua: $y-R=\frac{-R}{H}(x-0) \to y=\frac{R}{H}(H-x)$

$$x_c=\frac{1}{A}\iint_D x\,dx\,dy=0 \to \iint_D x\,dx\,dy=0$$

$$\iint_D x\,dx\,dy=\int_{-R}^{0}x\,dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}dy+\int_0^H x\,dx\int_{\frac{R}{H}(x-H)}^{\frac{R}{H}(H-x)}dy=\int_{-R}^{0}2x\sqrt{R^2-x^2}\,dx+\int_0^H 2x\frac{R}{H}(H-x)\,dx=\left[\frac{-2(R^2-x^2)^{3/2}}{3}\right]_{-R}^{0}+\left[Rx^2-\frac{2Rx^3}{3H}\right]_0^H=$$

$$=\frac{-2R^3}{3}+RH^2-\frac{2RH^{\cancel{3}\,2}}{3\cancel{H}}=\frac{-2R^3}{3}+\frac{3RH^2-2RH^2}{3}=\frac{RH^2-2R^3}{3}$$

$$\frac{RH^2-2R^3}{3}=0 \to \frac{\cancel{R}H^2}{\cancel{3}}=\frac{2R^{\cancel{3}\,2}}{\cancel{3}} \to H^2=2R^2 \to \boxed{H=\sqrt{2}R}$$