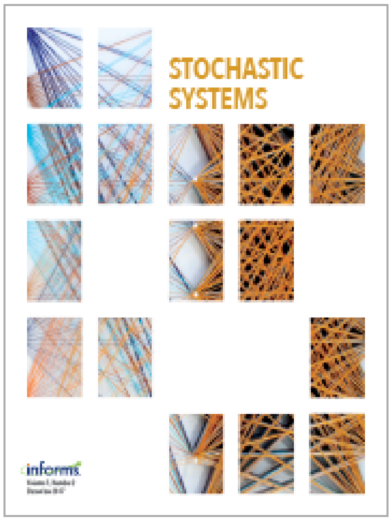

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Optimal Liquidity-Based Trading Tactics

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Abstract. We consider an agent who needs to buy (or sell) a relatively small amount of assets over some fixed short time interval. We work at the highest frequency meaning that we wish to find the optimal tactic to execute our quantity using limit orders, market orders, and cancellations. To solve the agent's control problem, we build an order book model and optimize an expected utility function based on our price impact. We derive the equations satisfied by the optimal strategy and solve them numerically. Moreover, we show that our optimal tactic enables us to outperform significantly naive execution strategies.

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Keywords: market microstructure • limit order book • high frequency trading • queuing model • Markov jump processes • ergodic properties • adverse selection • execution probabilities • market impact • optimal trading strategies • optimal tactics • stochastic control

1. Introduction

Most electronic exchanges use an order book mechanism. In such markets, buyers and sellers send their orders to a continuous-time double auction system. These orders are then matched according to price and time priority. Each submitted order has a specific price and size, and the order book is the collection of all submitted and unmatched limit orders. This is illustrated in Figure 1, which shows a classical representation of an order book at a given time.

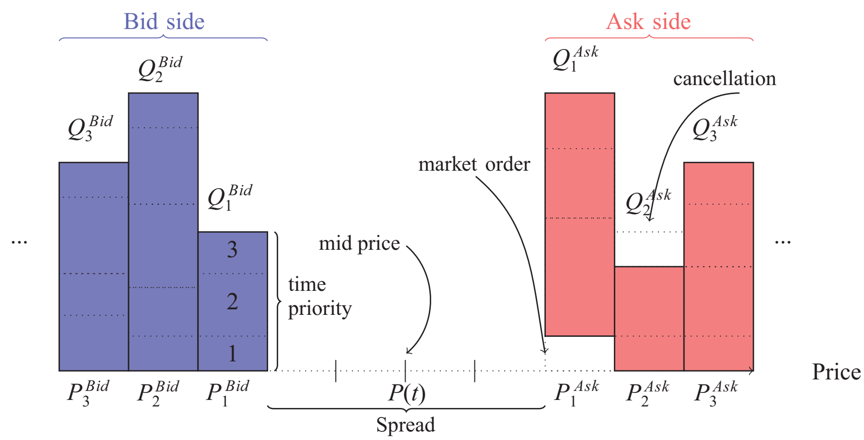
In this limit order book setting, we consider the following issue: an agent has to buy or sell a given quantity of asset before a fixed time horizon. During the execution process, the agent can take four elementary decisions:

• Insert limit orders in the order book, hoping to avoid crossing the spread. We will assume that the agent does not place limit orders above the best limits; however, he can insert them within the spread.
• Stay in the order book with an already existing limit order to keep his tactical placement.
• Cancel existing limit orders.
• Send market orders to get immediate execution.

Note that this is the microstructural version of the classical Almgren-Chriss optimal scheduling problem for the liquidation of a large quantity of asset over a time interval $[0, T]$ (Bertsimas and Lo 1998, Grinold and Kahn 2000, Almgren and Chriss 2001, Cartea et al. 2015, Guéant 2016 for various extensions). In the setting of Almgren and Chriss (2001), $[0, T]$ is split in sub time windows (typically a few minutes per window) and one derives the number of shares to be executed in each window. In our case, we want to specify how to act optimally within each window. Indeed, our buyer or seller reacts to every order book move and handles reasonably small quantities during short periods of time.

In order to solve this problem, we of course need to model the order book dynamic. There are essentially two order book modelling approaches in the literature. First, "equilibrium models," based on interactions between rational agents who take optimal decisions (Parlour 1998, Foucault 1999, Roşu 2009). Second, "statistical models" where the order book is seen as a suitable random process (Smith et al. 2003; Cont et al. 2010; Abergel and Jedidi 2013, 2015; Cont and De Larrard 2013; Lachapelle et al. 2016; Lakner et al. 2016; Bayer et al. 2017). Statistical models focus on reproducing many salient features of real markets rather than individual agents' behaviours and interactions between them. In this paper, we use a statistical model. In such models, the arrival and cancellation

Figure 1. Order Book Representation at a Given Time



Notes. Here P_i^{Ask} (resp. P_i^{Bid}) with $i \geq 1$ are the sellers (resp. buyers) limit prices and they are increasingly (resp. decreasingly) ordered. For a given price P_i^{Ask} (resp. P_i^{Bid}), the limit Q_i^{Ask} (resp. Q_i^{Bid}) is the available selling (resp. buying) quantity.

flows often follow independent Poisson processes. The Poisson assumption allows for the derivation of simple, and often closed-form, formulas, for example for the probabilities of various order book events (Cont et al. 2010, Abergel and Jedidi 2013, Lachapelle et al. 2016, Toke 2017).

However, as clearly shown in Huang et al. (2015), this assumption is not realistic and it is necessary to take into account accurately the local state-dependent behaviour of the order book. So in Huang et al. (2015) and Huang and Rosenbaum (2017), the authors introduce the Queue-Reactive order book model where order flows follow a Markov jump process. They also provide ergodicity conditions and model parameters calibration methodology.

1.1. Order Book Model

Here we refine the Queue-Reactive model to make it compatible with a stochastic control framework enabling us to solve important practical issues. To do so, we only consider the best bid and ask limits to work with a reasonably small state space. Furthermore, in order to get a truly good fit to real order book dynamics, we focus on the so-called regeneration process, which models the order book state right after the total depletion of a limit. In our setting, when a limit is totally depleted, the order book is regenerated in a new state whose regeneration law depends on the order book state just before the depletion. In general, order book models consider several bid and ask limits and use a regeneration process independent from the order book state (Cont et al. 2010, Abergel and Jedidi 2015, Huang et al. 2015). Here, we model the order book by a three-dimensional Markov jump process (Q_t^1, Q_t^2, S_t) where Q_t^1 is the available quantity at the best bid, Q_t^2 is the available quantity at the best ask, and S_t is the spread. In addition, we provide a dynamic for the mid price P_t, which depends on the order book state.

1.2. The Agent's Control Problem

Let us now introduce the agent's control problem. We formulate it for a buy order of size q^a (it can be changed to a sell order in an obvious way). From time zero to the final time T, we assume that, at every decision time, the buyer can do nothing or use one of the three following actions: insert a fraction of the remaining quantity to buy (if not already inserted) at the top of the bid queue or within the spread (decision l), cancel the already inserted limit orders (decision c), or send a market order (decision m) for a fraction of the remaining quantity. If the agent does not obtain the total execution of q^a at time T, he cancels the remaining quantity in the order book and sends a market order. Thus, the trader's strategy is modelled by the sequence $\mu = (\tau_i, \upsilon_i)_{i\geq 0}$ of random variables where $(\tau_i)_{i\geq 0}$ is an increasing sequence of stopping times that represents the optimal decision times and $\upsilon_i \in \mathbb{E} = \mathbb{T} \times \mathbb{N} \times \mathbb{P}$ refers to the optimal decision. Here, the set $\mathbb{T} = \{l, c, m\}$ is that of the type of the order, $\mathbb{N}$ that of the order size and $\mathbb{P}$ that of the order posting price.[1] The agent aims at determining the optimal sequence of decisions to reduce its price impact PI_∞^μ, which is defined by

$$PI_\infty^\mu = -\lim_{t\to\infty} \mathbb{E}[q^a P_t^\mu - P^{Exec,\mu}], \tag{1}$$

with P_t^μ the mid price, $P^{Exec,\mu}$ the acquisition price of the quantity q^a, t the current observation time, and μ the agent's control.

Let I_t^μ be the agent's inventory, that is the remaining quantity he has to buy at time t. We denote by $P_t^{Exec,\mu}$ the acquisition price of the quantity $q^a - I_t^\mu$ and write T_{Exec}^μ for the time where q^a is totally executed. To take into account the waiting cost, the sensitivity to the price impact and to work in a slightly more general setting, we consider the following optimisation problem:

$$\sup_\mu \mathbb{E}\Bigg[\underbrace{f\Big(\lim_{t\to\infty}\mathbb{E}\big[q^a P_t^\mu - P^{Exec,\mu}_{T^\mu_{Exec}} | \mathcal{F}_{T^\mu_{Exec}}\big]\Big)}_{\text{final constraint}} - \gamma \underbrace{\int_0^{T^\mu_{Exec}} I_s^\mu \, ds}_{\text{running cost}}\Bigg],$$

where $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function and γ is a nonnegative constant representing the waiting cost. Here we use a conditional expectation to account for the fact that agents collect information along their own trading. Note that in general, in optimal execution problems, one minimizes the acquisition (resp. liquidation) price of the quantity to buy (resp. sell), which corresponds here to $P^{Exec,\mu}_{T^\mu_{Exec}}$. However, because of the price relaxation typically following the end of a metaorder, this acquisition (resp. liquidation) price is not an appropriate benchmark for agents such as brokers who need to sell (resp. buy) back at least a portion of the executed shares in the future. In this work, we place ourselves in a setting where we can define a notion of postrelaxation average price $P_\infty^\mu = \lim_{t\to\infty}\mathbb{E}[P_t^\mu]$ and use it as a benchmark. Our own trading has an influence on P_∞^μ that we will be able to compute. Note that we send t to infinity to derive the new stationary value of the price after our execution. Thus, infinity should be seen as the time scale where the resilience of the market takes place. This notion of resilience after order execution has been widely studied in the literature (see e.g., Gatheral et al. 2011, Farmer et al. 2013). In our case, its order of magnitude corresponds to a few hours. We stress the fact that the reduction of the price impact is crucial for brokers. As a matter of fact, they interact thousands of times per day with the order book to execute hundreds of metaorders, which are all subjected to the impact of previous orders.

1.3. Positioning of the Paper

This paper is obviously not the first work where a stochastic control framework involving limit orders, market orders, and cancellations is used to solve a high frequency trading problem. For example, in Guéant et al. (2012), Laruelle et al. (2013), and Lehalle and Mounjid (2017), the authors consider the problem of optimal posting of a limit order while market making issues are addressed in Avellaneda and Stoikov (2008), Baradel et al. (2019), Cartea et al. (2015), Guéant et al. (2013), and Guo et al. (2017). The problem of optimal execution using limit and market orders is also investigated in Bulthuis et al. (2016), Cartea et al. (2017), Cartea and Jaimungal (2015, 2016), Cont and Kukanov (2017), Guéant (2015), and Guéant and Lehalle (2015). However, the interactions between market participants decisions, liquidity, and behaviour of the order book are not really taken into account. In our work, the decision of the agent depends on its current position in the queue and the available liquidity in the order book. Some of the few papers considering an approach close to ours are Guilbaud and Pham (2013), Jacquier and Liu (2018), and Lehalle and Mounjid (2017). Beyond a slightly more general setting, compared with these papers, our main contribution is to optimize our trading tactic not only with respect to its local profit and loss but also to the endogenous price impact it generates. This is of primary importance for any market participant with intense trading activity.

1.4. Results

In this paper, we propose an order book model in reduced dimension, with state-dependent regeneration and nonconstant spread. Within this framework, we provide a closed-form formula for the endogenous price impact. This allows us to compute the sequence of orders solving the optimal execution problem for an agent who wants to minimize its impact. Moreover, we prove the ergodicity of our order book process under mild assumptions on the intensity functions.

The paper is organized as follows. In Section 2, we introduce our order book model, prove its ergodicity, and provide the formula for the price impact. In Section 3, we formulate the agent's control problem. Our main theorems including the equations satisfied by the value function and the numerical methodology to solve them are provided in Section 4. Finally, numerical experiments are given in Section 5. The proofs are relegated to appendices.

2. Order Book Modelling

In this section, we first confirm on data that agents' behaviours depend on order book liquidity (see Huang et al. 2015, Lehalle and Mounjid 2017, Lehalle and Neuman 2019 for closely related results). Then we describe our order book dynamic.

2.1. Preliminary: Empirical Evidences

One objective of our work is that we wish to carefully model the interactions between market participants and liquidity. We first show on real data that market participants act differently when facing different liquidity conditions.

2.1.1. Database Presentation. Data used here are from Bund futures on Eurex exchange Frankfurt. We focus on this product because it is a good example of a very liquid and large tick asset. The database records during one week, from 1 to September 5, 2014, the state of the order book (i.e., available quantities and prices at best limits) event by event with microsecond accuracy. For each day, our data cover the time period from 8 a.m. to 10 p.m. Frankfurt time. Each event has a type, a side (i.e., bid/ask), a price, and a size. We consider three types of events: insertion of limit orders, cancellation of existing limit orders, and market orders. The database accounts for 3,407,574 events.

Let t be the time when an event happens in the order book. We define the imbalance Imb_t and the mid price move δ seconds after the event time t, $\Delta P_\delta^{mid}(t)$, by

$$\begin{cases} \mathrm{Imb}_t = \epsilon_t \dfrac{Q_t^1 - Q_t^2}{Q_t^1 + Q_t^2}, \\ \Delta P_\delta^{mid}(t) = \epsilon_t \dfrac{P_{\delta+t} - P_t}{s_t}, \end{cases}$$

where Q_t^1 (resp. Q_t^2) is the available quantity at the best bid (resp. ask), P_t is the mid price, ϵ_t is the event sign (i.e., $\epsilon_t = 1$ when it is a buy order and -1 otherwise), and s_t is the spread (i.e., $s_t = P_t^{Ask} - P_t^{Bid}$ with P_t^{Ask} the best ask price and P_t^{Bid} the best bid price).

We want to confirm that agents' decisions depend on the order book liquidity. A simple way to do so is to summarize the state of the order book liquidity through the imbalance. Figure 2(a) shows the average imbalance value for each event type. We give the interpretation of Figure 2(a) in the case of a buy limit/limit spread/[2] cancellation/market order, because the event sign is taken into account in the expression of Imb_t. We see that essentially, market participants insert limit orders when imbalance is negative (execution highly probable), cancel orders or send a limit spread order when imbalance is positive (less chance to be executed), and use market orders when imbalance is highly positive (rushing for liquidity when it is scarce).

Figure 2(b) shows the distribution of imbalance just before a liquidity provision event (i.e., insertion of limit order) and a liquidity consumption event (i.e., cancellation of limit order or market order). We see that agents are highly active at extreme imbalance values.[3] Indeed, in these cases, they identify a profit opportunity to catch or on the contrary an adverse selection effect to avoid (e.g., buying just before a price decrease). This is related to the predictive power of the imbalance. As can be seen in Figure 2(c), $\Delta P_\delta^{mid}(t)$ after two minutes (i.e., $\delta = 2$ min) is highly correlated to the imbalance. This means that market participants use the imbalance as a signal to anticipate next price moves.[4] Hence, our empirical results clearly confirm that agents' decisions depend on the order book liquidity.

2.2. Order Book Framework

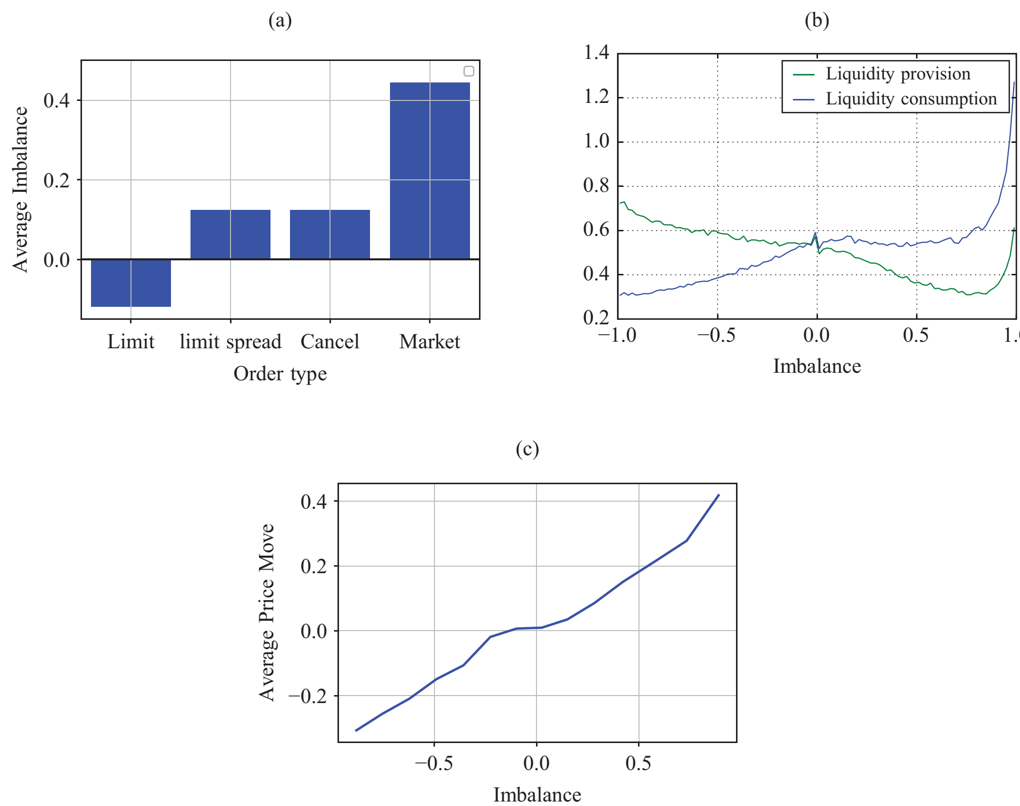
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space with $\mathcal{F}_0$ the trivial σ-algebra. The order book state is modelled by the Markov process $U_t = \left(Q_t^1, Q_t^2, S_t\right) \in \mathbb{U}$ where Q_t^1 (resp. Q_t^2) is the best bid (resp. ask) quantity and S_t is the spread. We denote by P_t^1 (resp. P_t^2) the best bid (resp. ask) price, P_t the mid price, and b_0 the tick size. The mid price dynamic will be described in Section 2.4, and in this section we focus on the dynamic of U_t. For simplification, we take the state space $\mathbb{U} = (\mathbb{N}^*)^2 \times b_0\mathbb{N}^*$.[5]

The Markov process U is characterized by its the infinitesimal generator $\mathcal{Q}$. For $u = (q^1, q^2, s) \in \mathbb{U}$, $u' = (q'^1, q'^2, s') \in \mathbb{U}$, $n \in \mathbb{N}^*$, $k \in \left\{1, \dots, \frac{s}{b_0} - 1\right\}$, $e_1 = (1, 0)$, $e_2 = (0, 1)$, and $i \in \{1, 2\}$, we consider the following form for $\mathcal{Q}$:

$$\begin{array}{ll} \text{(Insertion of orders)} & \\ \text{(Consumption of orders)} & \\ \text{(Insertion within the spread)} & \\ \text{(Otherwise)} & \end{array} \begin{cases} \mathcal{Q}_{(q,s),(q+ne_i,s)} = \lambda^{i,+}(u,n) + R(u,(q+ne_i,s)), & \\ \mathcal{Q}_{(q,s),(q-ne_i,s)} = \lambda^{i,-}(u,n,\mathbf{0}) + R(u,(q-ne_i,s)), & \text{if } q^i > n, \\ \mathcal{Q}_{(q,s),(q+(n-q^i)e_i,s-kb_0)} = \lambda^{i,k}(u,n) + R(u,(q+(n-q^i)e_i,s-kb_0)), & \text{if } s > b_0, \\ \mathcal{Q}_{(q,s),(q',s')} = R(u,(q''s')), & \text{if } s' \neq s, \end{cases} \quad (2)$$

with $\mathcal{Q}_{u,u} = -\sum_{u' \neq u} \mathcal{Q}_{u,u'}$, $q = (q^1, q^2)$, $q' = (q'^1, q'^2)$, and

Figure 2. Some Statistics About the Imbalance



Notes. (a) Average imbalance before limit/limit spread/cancel/market order. (b) Imbalance density before liquidity provision/consumption event. (c) Average price move after two minutes against imbalance.

- $\lambda^{1,+}(u,n)$ (resp. $\lambda^{2,+}(u,n)$) represents the arrival rate of limit orders of size n at the best bid (resp. ask) when the order book state is u.
- $\lambda^{1,-}(u,n,\mathbf{0})$ (resp. $\lambda^{2,-}(u,n,\mathbf{0})$) is the arrival rate of liquidity consumption orders of size n that do not deplete the best bid (resp. ask), when the order book state is u.
- $\lambda^{1,k}(u,n)$ (resp. $\lambda^{2,k}(u,n)$) represents the arrival rate of buying (resp. selling) limit orders of size n within the spread at the price P^1+kb_0 (resp. P^2-kb_0).
- $\lambda^{1,-}(u,n,u')$ (resp. $\lambda^{2,-}(u,n,u')$) represents the arrival rate of liquidity consumption orders of size n that deplete the best bid (resp. ask) and lead to a new state u' when the order book state is u.
- $R(u,u')$ satisfies $R(u,u')=\sum_{j=1}^{2}\sum_{m\geq q^j}\lambda^{j,-}(u,m,u')$ and represents the order book regeneration component.

The arrival rates and regeneration component are chosen so that the process U is irreducible.

2.2.1. Order Book Regeneration. In our framework, when one limit is totally depleted, the order book is regenerated in a new state whose law depends on the order book state just before the depletion and the depleted side (i.e., best bid/ask). The regeneration of the process U is described through the quantity $R(u,u')$ where $u'\in\mathbb{U}$ is the order book state after the depletion and u is the order book state before the depletion. A simple choice is to consider the case where the spread increases by one tick when the best bid or ask is depleted and to draw new best bid and ask quantities from a fixed stationary distribution (Cont and De Larrard 2013).

2.2.2. Symmetry Relations. Additionally, for every $u=(q^1,q^2,s)\in\mathbb{U}, u'=(q'^1,q'^2,s')\in\mathbb{U}$, $n\in\mathbb{N}^*$ and $k\leq\frac{s}{b_0}-1$, we assume the following bid-ask symmetry relations:

$$\begin{cases}\lambda^{1,+}(u,n) & = & \lambda^{2,+}(u^{sym},n),\\ \lambda^{1,k}(u,n) & = & \lambda^{2,k}(u^{sym},n),\\ \lambda^{1,-}(u,n,u') & = & \lambda^{2,-}(u^{sym},n,u'^{sym}),\end{cases}\tag{3}$$

with $u^{sym} = (q^2, q^1, s)$. This classical bid-ask symmetry relation ensures no statistical arbitrage and allows us to aggregate bid and ask side data in the calibration of the model parameters.

2.3. Ergodicity

We now give a theoretical result on the ergodicity of the process $U_t = (Q_t^1, Q_t^2, S_t)$ under the two general assumptions. A definition of the notion of ergodicity is provided in Appendix B.

For any $i \in \{1,2\}$, we denote by $\lambda_Q^{i,+}(u,n)$ (resp. $\lambda_Q^{i,-}(u,n)$) and $\lambda_S^{+}(u,n)$ (resp. $\lambda_S^{-}(u,n)$) the arrival rate of events that increase (resp. decrease) by n respectively the size of the limit Q^i and the spread S starting from a state u. For sake of completeness, we give an explicit expression for $\lambda_Q^{i,\pm}$ and $\lambda_S^{\pm}$

$$\lambda_Q^{i,\pm}(u,n) = \sum_{(q''s')\in\mathbb{U}} \mathcal{Q}_{u,(q''s')}\mathbf{1}_{q'^i = q^i \pm n}, \qquad \lambda_S^{\pm}(u,n) = \sum_{(q''s')\in\mathbb{U}} \mathcal{Q}_{u,(q''s')}\mathbf{1}_{s'=s\pm nb_0}, \qquad \forall u = (q^1,q^2,s)\in\mathbb{U}, \;\; \forall n \geq 1.$$

Assumption 1 (Negative Individual Drift). *There exist three positive constants* C_{bound}, $z_0 > 1$, *and* $\delta > 0$ *such that for any* $u = (q^1, q^2, s) \in \mathbb{U}$

$$\sum_{n\geq 0}(z_0^n - 1)\left(\lambda_Q^{1,+}(u,n) - \lambda_Q^{1,-}(u,n)\frac{1}{z_0^n}\right) \leq -\delta, \quad \text{when } q^1 \geq C_{bound},$$
$$\sum_{k\geq 0}(z_0^{b_0 k} - 1)\left(\lambda_S^{+}(u,k) - \lambda_S^{-}(u,k)\frac{1}{z_0^{b_0 k}}\right) \leq -\delta, \quad \text{when } s \geq C_{bound}.$$

Assumption 1 ensures that the queue size and spread value tend to decrease when they become too large. Using Equation (3), we also have

$$\sum_{n\geq 0}(z_0^n - 1)\left(\lambda_Q^{2,+}(u,n) - \lambda_Q^{2,-}(u,n)\frac{1}{z_0^n}\right) \leq -\delta, \quad \forall q^2 \geq C_{bound}.$$

Assumption 2 (Local Bound on the Incoming Flow). *There exists* $z_1 > 1$ *such that for any* $B \geq 0$ *we have*

$$\sum_{n\geq 0} z_1^n \lambda_Q^{1,+}(u,n) \leq H^B, \quad \text{when } q^1 \leq B,$$
$$\sum_{k\geq 0} z_1^{b_0 k} \lambda_S^{+}(u,k) \leq H^B, \quad \text{when } s \leq B,$$

with $u = (q^1, q^2, s) \in \mathbb{U}$ *and* H^B *a positive constant.*

Assumption 2 ensures no explosion in the system: the order arrival speed stays bounded within any bounded set of $\mathbb{U}$. Using the symmetry relation, we have $\sum_{n\geq 0} z_1^n \lambda_Q^{2,+}(u,n) \leq H^B$ when $q^2 \leq B$. For example, Assumption 2 is satisfied in the particular case when both the size of the best limits and the spread cannot get beyond a maximum threshold. Assumptions 1 and 2 are close to those used in Huang and Rosenbaum (2017) and slightly more general. We have the following result.

Theorem 1 (Ergodicity). *Under Assumptions 1 and 2, the process U_t is ergodic (i.e., converges toward a unique invariant distribution). Additionally, we have the following speed of convergence:*

$$\|P_u^t(.) - \pi\|_{TV} \leq B(u)\rho^t,$$

with $\|\cdot\|_{TV}$ the total variation norm, $P_u^t(.)$ the Markov kernel of the process U_t starting from the point $u \in \mathbb{U}$, π the invariant distribution, $\rho < 1$, and $B(u)$ a constant depending on the initial state u, see Appendix B.

Remark 1. To prove the ergodicity, we do not require the intensities to be uniformly bounded.

This theorem is the basis for the asymptotic study of the order book dynamic in Section 2.1, because it ensures the convergence of the order book state toward an invariant probability distribution. Thus, the stylized facts observed on market data can be explained by a law of large numbers type phenomenon for this invariant distribution. The proof of this result is given in Appendix B and is quite inspired from Huang et al. (2015) and Huang and Rosenbaum (2017).

2.4. Computation of $\lim_{t\to\infty}\mathbb{E}_u[P_t - P_0]$

In this section, we show how to compute

$$D(u) = \lim_{t\to\infty}\mathbb{E}_u[P_t - P_0], \tag{4}$$

with $u \in \mathbb{U}$. The computation of $D(u)$ is interesting for at least two reasons: first, it allows us to predict the long-term average mid price move for any initial order book state u. Second, the quantity $D(u)$ is useful for the computation of the price impact PI^{μ}_{∞} defined in (1), see Section 4.1 for a detailed connection between the computation of PI^{μ}_{∞} and Equation (4). To compute numerically $D(u)$, we need to bound the domain $\mathbb{U}$. This is why we replace Assumption 2 by slightly less general assumptions.

Assumption 3 (Insertion Bound). *There exists a positive quantity Q^{max} such that for any $u = (q^1, q^2, s) \in \mathbb{U}$, $u' \in \mathbb{U}$, $n \in \mathbb{N}^*$ and $k \le \frac{s}{b_0} - 1$,*

$$\begin{cases} \lambda^{1,+}(u,n) = 0, & \text{when} \quad q^1 + n > Q^{max}, \\ \lambda^{1,k}(u,n) = 0, & \text{when} \quad q^1 + n > Q^{max}, \\ \lambda^{1,-}(u,n,u') = 0, & \text{when} \quad q^1 > Q^{max}. \end{cases}$$

This assumption is not restrictive because quantities at the best limits remain bounded. Using the symmetry relation, we have as well, for any $u = (q^1, q^2, s) \in \mathbb{U}$, $u' \in \mathbb{U}$, $n \ge 0$ and $k \le \frac{s}{b_0} - 1$,

$$\begin{cases} \lambda^{2,+(k)}(u,n) = 0, & \text{when} \quad q^2 + n > Q^{max}, \\ \lambda^{2,-}(u,n,u') = 0, & \text{when} \quad q^2 > Q^{max}. \end{cases}$$

Assumption 4 (Regeneration Bound 2). *There exists a positive constant $\tilde{Q}^{max}$ such that for all $(u,u') \in \mathbb{U}^2$ and $i \in \{1,2\}$, we have*

$$\lambda^{i,-}(u,n,u') = 0, \quad \text{when} \quad \|u'\|_{\infty} > \tilde{Q}^{\max},$$

with $\|x\|_{\infty} = \sup_{j \le N} |x_j|$ for any vector $x \in \mathbb{R}^N$.

Under the symmetry relation, Assumption 4 needs to be satisfied for only one $i \in \{1,2\}$.

2.4.1. Price Dynamic. The mid price after the n-th order book event P_n satisfies $P_n = P_0 + \sum_{i=1}^{n} \Delta P_i$ with $\Delta P_i = P_i - P_{i-1}$. The price jump ΔP_i is a deterministic function of the order book state before the jump and the order book event causing the jump. For example, we can consider the simple case where the mid price decreases (resp. increases) by one tick when the best bid (resp. ask) is depleted.

Assumption 5 (Mid Price Bound). *The process U_t, see Section 2.2, is irreducible, and there exists a state $u \in \mathbb{U}$ such that $D(u) < \infty$.*

Remark 2. Because U_t is irreducible, Assumption 5 implies that $D(u) < \infty$ for all $u \in \mathbb{U}$.

Computation methodology of the price impact: A mid price move can be caused by only one of the following events:

- **event 1:** depletion of the best bid or sell limit order within the spread.
- **event 2:** depletion of the best ask or buy limit order within the spread.

We aggregate events that decrease the price and those that increase it because their impact in practice is quite similar. This simplifies the notations otherwise we need to keep track of the four types of events. Let t_1 (resp. t_2) be the first time when an event of type 1 (resp. 2) happens. At t_1 (resp. t_2), the mid price moves on average by $\alpha_i^- = \mathbb{E}_{U_i}[\Delta P_{t_1}]$ (resp. $\alpha_i^+ = \mathbb{E}_{U_i}[\Delta P_{t_2}]$) and the order book is regenerated according to a distribution $d^1_{i,\cdot}$ (resp. $d^2_{i,\cdot}$). Here we refer to the state U_i through the index i. We consider the following notations:

- $q^-_{ii'} = \mathbb{P}_{U_i}[\{t_1 < t_2\} \cap \{U_{t_1^-} = U_{i'}\}]$ (resp. $q^+_{ii'} = \mathbb{P}_{U_i}[\{t_2 \le t_1\} \cap \{U_{t_2^-} = U_{i'}\}]$) is the probability that $t_1 < t_2$ (resp. $t_1 \ge t_2$) and the exit state is $U_{i'}$.
- $d^1_{i,k}$ (resp. $d^2_{i,k}$) are transition probabilities from the state U_i to U_k when $t_1 < t_2$ (resp. $t_1 \ge t_2$).
- $q_i = \sum_{i'} (q^+_{ii'} \alpha^+_{i'} + q^-_{ii'} \alpha^-_{i'})$ and $p_{i,k} = \sum_{i'} (q^+_{ii'} d^2_{i',k} + q^-_{ii'} d^1_{i',k})$ represent respectively the average mid price move after the first regeneration and the probability to reach the state U_k starting from the initial point U_i right after the first regeneration.
- $U_i^{sym} = (q^2, q^1, s)$ is the symmetric state of $U_i = (q^1, q^2, s)$ and i^{sym} is the index of the symmetric state U_i^{sym}, see (3).
- D is a vector satisfying $D_i = \lim_{t \to \infty} \mathbb{E}_{U_i}[P_t - P_0]$ for every state $U_i = (q^1, q^2, s) \in \mathcal{D}$. We write $\mathcal{D}$ for the set $\mathcal{D} = \{(q^1, q^2, s);\ q^1 \ge q^2\}$.[6]
- A is a matrix defined as $A_{i,k} = \frac{p_{i,k} - p_{i,k^{sym}}}{1 - (p_{i,i} - p_{i,i^{sym}})}$ when $i \ne k$ and $A_{i,i} = 0$ for any $(U_i, U_k) \in \mathcal{D}^2$. Note that the matrix $I - A$ is invertible under Assumptions 3, 4, and 5.

We have the following result.

Proposition 1 (Price Impact). *Under Assumptions 3, 4, and 5, the vector D satisfies*

$$D = (I - A)^{-1}F.$$

The vector F is defined by $F_i = \frac{q_i}{1-(p_{i,i}-p_{i,i^{sym}})}$ *for any* $U_i \in \mathcal{D}$.[7]

The proof of this result is given in Appendix C.1. A numerical computation of the vector D is given in Section 2.5, Figure 3.

Remark 3. To compute D, we need to estimate the regeneration distributions $d^1_{\cdot,\cdot}$, $d^2_{\cdot,\cdot}$, $\alpha^{\pm}_{\cdot}$ and $q^{\pm}_{\cdot\cdot}$. The quantities $d^1_{\cdot,\cdot}$, $d^2_{\cdot,\cdot}$ and $\alpha^{\pm}_{\cdot}$ can be estimated from the order book empirical distribution after a price change. Then, we only need to estimate $q^{\pm}_{\cdot\cdot}$. The estimation methodology of $q^{\pm}_{\cdot\cdot}$ is detailed in Lemma 3 of Appendix C.

2.5. Numerical Application

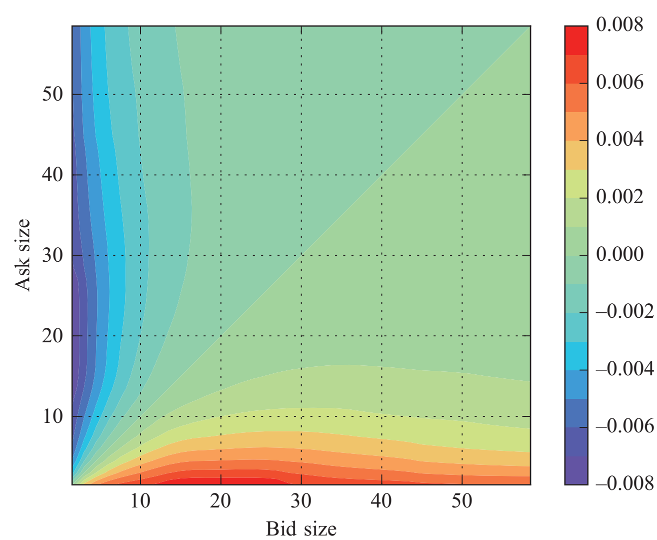
In this section, we compute numerically $\lim_{t\to\infty} \mathbb{E}_u[P_t - P_0]$ using Proposition 1. We also assess the model by comparing the theoretical and empirical distributions of (Q^1, Q^2) at long-term and short-term horizons.

2.5.1. Approximation of $\lim_{t\to\infty} \mathbb{E}_{U_0}[P_t - P_0]$. Figure 3 shows the quantity $\lim_{t\to\infty} \mathbb{E}_{U_0}[P_t - P_0]$, which is defined in Section 4.1 and computed using Proposition 1, for different values of the initial state $U_0 = (Q^1, Q^2, b_0)$ where the spread here is equal to one tick. Figure 3 highlights the predictive power of the imbalance: when the imbalance is positive the price increases on average and conversely. We also note that the bid-ask symmetry relation is respected.

2.5.2. Model Approximation at Short-Time Horizon. Figure 4, (a) and (b) show respectively the empirical and theoretical distributions of Q^1 after 20 events. We choose 20 events because it is the order of magnitude of the duration of our control. The estimation of the theoretical distribution is based on a Monte-Carlo simulation of the order book. We can see that both distributions are close and consequently that our model is consistent with the empirical order book dynamic at least during the control duration.

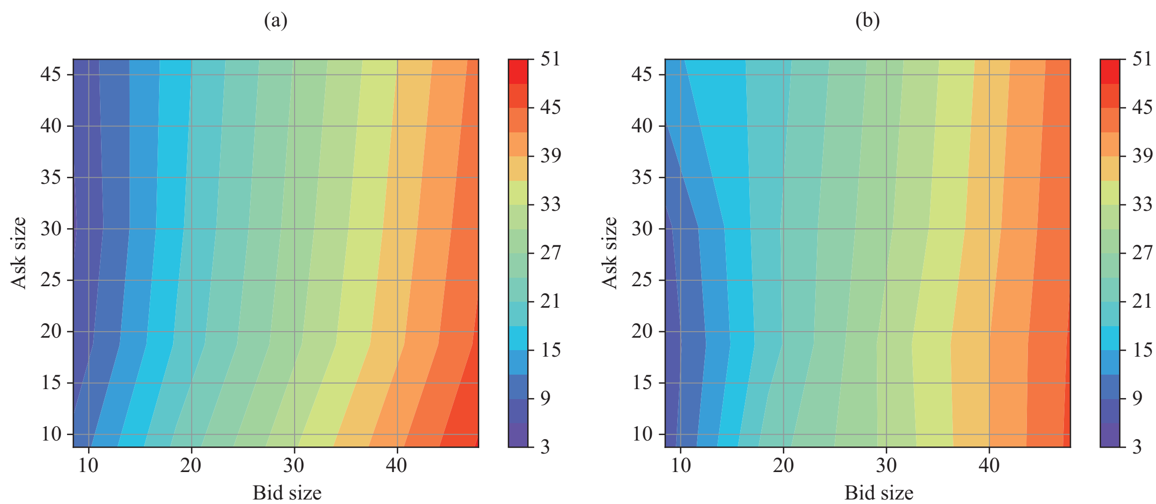
2.5.3. Model Approximation at a Long-Term Horizon. Figure 5, (a) and (b) display respectively the empirical and theoretical distribution of Q^1 and Q^2. These two distributions are close. This is consistent with Huang et al. (2015).

Figure 3. Price Impact $\lim_{t\to\infty} \mathbb{E}_{U_0}[P_t - P_0]$



Notes. The quantities Q^1 and Q^2 are divided by the average event size and the tick $b_0 = 0.01$.

Figure 4. (a) Empirical Distribution of Q^1 After 20 Events and (b) Theoretical Distribution of Q^1



Notes. The quantities Q^1 and Q^2 are divided by the average event size. (a) Empirical Q^1 distribution after 20 events. (b) Theoretical Q^1 distribution after 20 events.

3. Optimal Tactic Control Problem

We express the control problem for a buy order of size q^a. It can be changed to a sell order in an obvious way.

3.1. Order Book Dynamic

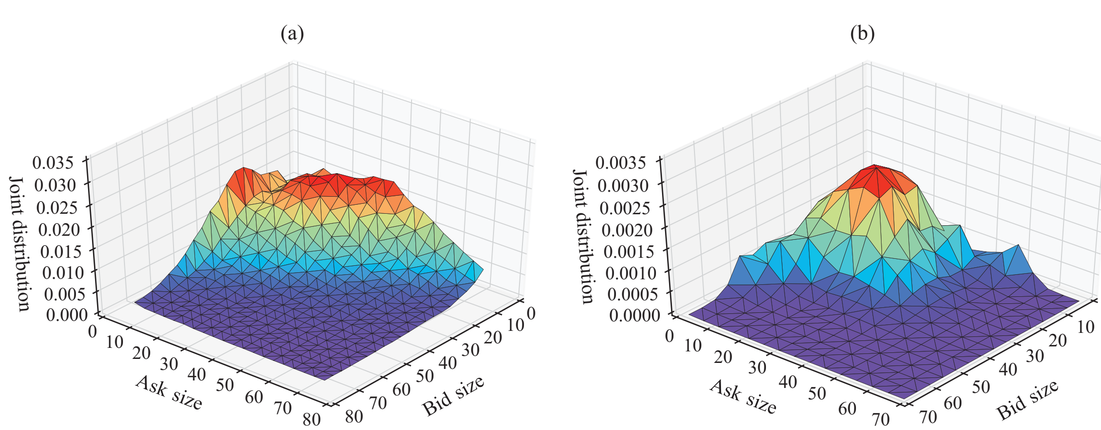
The agent state is modelled by the process

$$\overline{U}_t^\mu = \left(Q_t^{Bef,\mu}, Q_t^{a,\mu}, Q_t^{Aft,\mu}, Q_t^{2,\mu}, I_t^\mu, S_t^\mu, P_t^\mu, P_t^{Exec,\mu}\right),$$

where $Q_t^{a,\mu}$ is the size of the agent's limit order inserted at the best bid, $Q_t^{Bef,\mu}$ is the quantity inserted before $Q_t^{a,\mu}$, $Q_t^{Aft,\mu}$ represents orders inserted after $Q_t^{a,\mu}$ (see Figure 6), $P_t^{Exec,\mu}$ is the acquisition price of $q^a - I_t^\mu$, I_t^μ is the agent's inventory, and μ is the control of the agent. We recall that $Q_t^{2,\mu}$ is the best ask limit, P_t^μ is the mid price, and S_t^μ is the spread. Then $Q_t^{1,\mu} = Q_t^{Bef,\mu} + Q_t^{a,\mu} + Q_t^{Aft,\mu}$ is the total volume at the best bid. It is split into three quantities to take into account the order placement. We add minor changes to the order book dynamic:

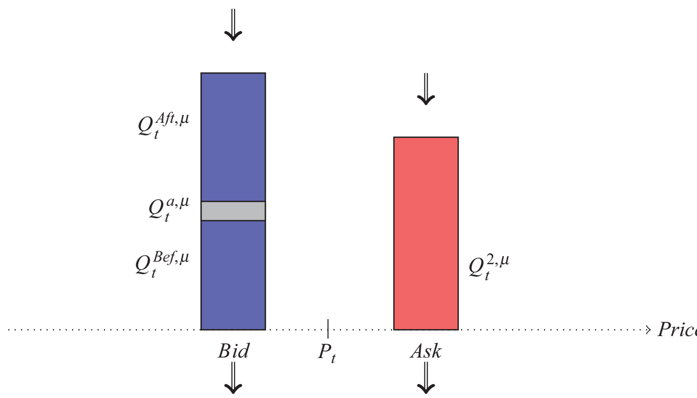
- **Arrival rates:** for the best bid, we differentiate market orders consumption rate $\lambda_m^{1,-}$ from limit orders cancellation rate $\lambda_c^{1,-}$. Cancellation orders consume $Q_t^{Aft,\mu}$ first and market orders $Q_t^{Bef,\mu}$ first.[8]

Figure 5. Empirical Distribution of Q^1 on x-axis and Q^2 y-axis in (a) and Theoretical One in (b)



Notes. (a) Empirical distribution of (Q^1, Q^2). (b) Stationary distribution of (Q^1, Q^2).

Figure 6. Diagram Representing the Position of the Agent Order



• **Regeneration:** the regeneration of the process $\overline{U}_t^\mu$ is deduced from that of $U_t^\mu = (Q_t^{1,\mu}, Q_t^{2,\mu}, S_t^\mu)$ and P^μ, which are described in Section 2.2. Here we explain the regeneration of the three variables $Q_t^{Bef,\mu}$, $Q_t^{a,\mu}$, and $Q_t^{Aft,\mu}$ since the one of $Q_t^{2,\mu}$, S_t^μ and P_t^μ is detailed in Section 2. After a regeneration $Q_t^{a,\mu} = 0$ when the best bid is totally depleted and remains unchanged otherwise. Furthermore, the quantity $Q_t^{Aft,\mu} + Q_t^{Bef,\mu}$ is equal to the regenerated best bid,[9] and the position of $Q_t^{a,\mu}$ is drawn from a distribution ζ_u^i depending on the order book state just before the regeneration u and the depleted side i (i.e., best ask in our case). A natural choice is to set $Q_t^{Aft,\mu} = 0$ and $Q_t^{Bef,\mu}$ equal to the new best bid when the best bid is depleted or the price moves, and keep the quantities $Q_t^{Bef,\mu}, Q_t^{a,\mu}$ and $Q_t^{Aft,\mu}$ unchanged when the best ask is depleted with no price move.

The symmetry relation (3) satisfied by $(Q_t^{1,\mu}, Q_t^{2,\mu}, S_t^\mu)$ is unchanged.

3.2. Trader's Controls

At every decision time, the trader can do nothing or take one of the following three decisions:

• l: He can insert a fraction of I^μ at the top of the bid queue or within the spread if not already inserted.

• c: He can cancel his already existing limit order $Q^{a,\mu}$. By acting this way, the trader can wait for a better order book state. This control will essentially be used to avoid adverse selection, that is, obtaining a transaction just before a price decrease.

• m: He can send a market order to get an immediate execution of a fraction of I^μ.

Every decision of the agent is also characterized by a price level p and an order size q. The price level is equal to $p \geq 0$ when the order is inserted at the limit price $P^1 + p$. We consider the set of controls $\mathcal{C}$ where the strategy of the trader is modelled by the sequence $\mu = (\tau_i, \upsilon_i)_{i\geq 0}$ of random variables with $(\tau_i)_{i\geq 0}$ an increasing sequence of stopping times (with respect to the filtration $\mathcal{F}_t^\mu = \sigma(\overline{U}_s^\mu, s \leq t)$ generated by $\overline{U}^\mu$), which represents the optimal decision times and such that $\tau_n \rightarrow_{n\rightarrow\infty} +\infty\, a.s.$. The quantity $\upsilon_i \in \mathbb{E}$ is an $\mathcal{F}_{\tau_i^-}^\mu$-measurable random variable that refers to the optimal decision. We define the set $\mathbb{E}$ such that $\mathbb{E} = \mathbb{T} \times \mathbb{N} \times \mathbb{P}$ with $\mathbb{T} = \{l, c, m\}$ and $\mathbb{P} = b_0\mathbb{N}$. For any $i \in \mathbb{N}$, we assume that $\tau_i \in \Delta\mathbb{N}$ with $\Delta > 0$ a constant. This means that the agent's decisions are taken at fixed frequency Δ^{-1}. Moreover, we suppose that $p_i \leq S_{\tau_i^-}^\mu$ and $q_i \leq I_{\tau_i^-}^\mu - Q_{\tau_i^-}^{a,\mu}$ for any admissible $\upsilon_i = (o_i, p_i, q_i) \in \mathbb{E}$. Finally, if the agent has no order inserted in the order book and does nothing at the beginning, the initial control is c.[10] We focus on the set of strategies $\mathcal{C}^a \subset \mathcal{C}$ that satisfies the condition.

Assumption 6 (Admissible Strategies). *There exist fixed constants $\bar{P}$ and $\bar{I}$ such that the agent liquidates its remaining quantity when $P^\mu \geq \bar{P}$ or $I^\mu \geq \bar{I}$.*

Assumption 6 ensures the boundedness of P^μ, the execution price $P^{Exec,\mu}$, the inventory I^μ, and $Q_t^{a,\mu}$. Assumptions 3 and 4 guarantee the boundedness of $(Q_t^{Bef,\mu}, Q_t^{Aft,\mu}, Q_t^{2,\mu}, S_t^\mu)$. When these assumptions are combined, the process $\overline{U}_t^\mu$ is bounded.

3.3. Optimal Control Problem

We fix a finite horizon time $T < \infty$ and we want to compute

$$V_T(0, \bar{u}) = \sup_{\mu\in\mathcal{C}^a} \mathbb{E}\Bigg[\underbrace{f\Big(\lim_{s\rightarrow\infty} \mathbb{E}[\Delta P_s^\mu | \mathcal{F}_{T_{Exec}^\mu}]\Big)}_{\text{final constraint}} - \gamma \underbrace{\int_0^{T_{Exec}^\mu} I_s^\mu \, ds}_{\text{running cost}}\Bigg], \tag{5}$$

where

- $\bar{u} = (q^{bef}, q, q^{aft}, q^2, i, s, p, p^{exec})$ is the initial agent state.
- $T^{\mu}_{Exec} = \inf\{t \geq 0, s.t \quad I^{\mu}_t = 0\} \wedge T$ represents the final execution time.
- $\Delta P^{\mu}_t = (q^a P^{\mu}_t - P^{Exec,\mu}_{T^{\mu}_{Exec}})$ represents the price impact and q^a is the order size.[11]
- γ is a nonnegative constant representing the waiting cost or the risk aversion of the agent and $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function.

In this work, we solve the agent's control problem and provide a numerical methodology to approximate the value function when decisions are taken at fixed frequency Δ^{-1}. The obtained scheme can actually be seen as a finite difference scheme coming from the approximation of the value function in the control problem where decisions can be taken at any time. We refer to Mounjid (2019) for details on this case.

4. Theoretical results

In this section, we compute $\lim_{t\to\infty} \mathbb{E}[\Delta P^{\mu}_t | \mathcal{F}_{T^{\mu}_{Exec}}]$, discuss the existence and uniqueness of the solution of our control problem, and give equations satisfied by the value function.

4.1. Price Impact Computation

In this section, Assumptions 3, 4, and 5 are in force. Let $U^{\mu}_{T^{\mu}_{Exec}}$ be the order book state at the end of the execution. We split ΔP^{μ}_t into two quantities

$$\Delta P^{\mu}_t = \left(q^a P^{\mu}_t - P^{Exec,\mu}_{T^{\mu}_{Exec}}\right) = \left(q^a P^{\mu}_t - q^a P^{\mu}_{T^{\mu}_{Exec}}\right) + \left(q^a P^{\mu}_{T^{\mu}_{Exec}} - P^{Exec,\mu}_{T^{\mu}_{Exec}}\right)$$
$$= q^a \Delta P'^{,\mu}_t + \left(q^a P^{\mu}_{T^{\mu}_{Exec}} - P^{Exec,\mu}_{T^{\mu}_{Exec}}\right),$$

where

- $P^{\mu}_{T^{\mu}_{Exec}}$ is the mid price at the end of the execution (that is, $P^{\mu}_{T^{\mu}_{Exec}}$ and $P^{Exec,\mu}_{T^{\mu}_{Exec}}$ are known at the execution).
- $\Delta P'^{,\mu}_t = P^{\mu}_t - P^{\mu}_{T^{\mu}_{Exec}}$ is the long-term mid price move after the execution.

Thus, we only need to compute $\Delta P'^{\mu}_t$. Because we place ourselves after the execution, we have $Q^{a,\mu}_{T^{\mu}_{Exec}} = 0$ and $Q^{Aft,\mu}_{T^{\mu}_{Exec}} = 0$, which means that $Q^{1,\mu}_{T^{\mu}_{Exec}} = Q^{Bef,\mu}_{T^{\mu}_{Exec}}$. Because the price jumps depend only on the dynamics of the order book state U^{μ} and U^{μ} is Markov,[12] we have $\lim_{t\to\infty} \mathbb{E}[\Delta P'^{\mu}_t | \mathcal{F}_{T^{\mu}_{Exec}}] = \lim_{t\to\infty} \mathbb{E}_{U^{\mu}_{T^{\mu}_{Exec}}}[\Delta P'^{\mu}_t]$. Proposition 1 provides an explicit formula for the computation of $\lim_{t\to\infty} \mathbb{E}_{U^{\mu}_{T^{\mu}_{Exec}}}[\Delta P'^{\mu}_t]$.

4.2. Existence and Uniqueness of the Optimal Strategy

In the rest of the article, Assumptions 1, 3, 4, and 5 are in force. In this section, we discuss existence and uniqueness of the optimal strategy. First, for a finite horizon time T, we define the reward function

$$J_T(t, \bar{u}) = \mathbb{E}\left[f\left(\lim_{s\to\infty} \mathbb{E}\left[\Delta P^{\mu}_s | \mathcal{F}_{T^{\mu}_{Exec}}\right]\right) - \gamma \int_t^{T^{t,\mu}_{Exec}} I^{\mu}_s \, ds | \overline{\mathrm{U}}^{\mu}_t = \bar{u}\right],$$

and the value function

$$V_T(t, \bar{u}) = \sup_{\mu \in \mathcal{C}^a} J_T(t, \bar{u}),$$

with $0 \leq t \leq T$, $\bar{u} \in \bar{\mathbb{U}} = \mathbb{N}^5 \times b_0 \mathbb{N}^* \times \left(\frac{b_0}{2}\mathbb{Z}\right)^2$ and $T^{t,\mu}_{Exec} = \inf\{s \geq t, s.t \quad I^{\mu}_s = 0\} \wedge T$.[13] Note that because decisions are taken at fixed frequency Δ^{-1}, the optimal strategy exists because we have a finite number of available strategies. However, there is a priori no uniqueness of the optimal strategy.

4.2.1. Regularization of the Problem.

To force the uniqueness of the optimal strategy, we propose a practical approach. First, we define an order relation between trader's decisions $c < l < m$. The intuition behind is that m is the least risky decision because we get direct execution, l is riskier than m but less risky than c because there is no delay of the execution. Then, we order the sizes (resp. prices) increasingly (resp. decreasingly) because a large size accelerates the liquidation of the inventory and a small posting price reduces instantaneously the impact. Finally, we consider the following order relation between $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$:

$$x \leq y \text{ iff } (x_1 < y_1) \text{ or } (x_1 = y_1 \text{ and } (x_2, \ldots, x_n) \leq (y_2, \ldots, y_n)).$$

Hence, we can choose the least risky decision among the optimal ones in the above sense.

4.3. Dynamic Programming Equation

In this section, we provide the system of equations satisfied by the value function V_T of the optimal control problem. The constant $\bar{P}$ is defined in Section 3. We have the following result.

Theorem 2. *Let $\bar{u} = (q^{bef}, q^a, q^{aft}, q^2, i, s, p, p^{exec})$ be an initial state and $t \in [0,T]$. Then $V(t,\bar{u})$ satisfies*

- *When $i > 0$ and $p < \bar{P}$:*
  - *At the decision time $t = k\Delta < T$:*

$$V(k\Delta,\bar{u}) = \sup_{e \in \mathbb{E}(i,\bar{u})} \{V^e((k\Delta)_+,\bar{u})\}, \tag{6}$$

where $V^e(t,\bar{u}) = \mathbb{E}[V(t,\bar{u}^e)]$ and $\mathbb{E}(i,\bar{u})$ is the set of admissible actions. The variable $\bar{u}^e$ is the new order book state when the decision $e \in \mathbb{E}(i,\bar{u})$ is taken.[14]

  - *At $t \neq k\Delta < T$:*

$$0 = -\gamma i + \mathcal{A}V(t,\bar{u}), \tag{7}$$

where $\mathcal{A} = \partial_t + \bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}$ is the infinitesimal generator of the process $\overline{\mathbf{U}}_t^{\mu}$.

- *When i = 0 or $p \geq \bar{P}$ (execution time condition):*

$$V(t,\bar{u}) = \tilde{g}(\bar{u}), \qquad \forall t < T, \tag{8}$$

with $\tilde{g}(\bar{u}) = f(\lim_{t\to\infty} \mathbb{E}_u[\Delta P_t])$.

- *The terminal condition is:*

$$V(T,\bar{u}) = g(\bar{u}), \tag{9}$$

with $g(\bar{u}) = f(\mathbb{E}_u[\lim_{t\to\infty} \mathbb{E}_{\bar{u}^{m^t}}[\Delta P_t]])$ and the decision m^t represents the liquidation of the remaining inventory.

The proof of this result is given in Appendix D.

Remark 4. At every decision time, as long as the order is not fully executed, the agent compares the value function given by each control and takes the highest one, see Equation (6). When, the order is executed, the agent gain is $\tilde{g}(u)$ with u the order book state at the end of the execution. If the order is not executed before T, the agent sends a market order to obtain immediate execution and earns $g(u)$.

Remark 5. Without the controls c and l, Equations (6), (7), and (9) are equivalent, in dimension 1, to the classical problem of finite horizon Bermudan options.

4.3.1. An Explicit Solution. In general, we use the numerical scheme introduced in Section 4.4 to approximate the value function V. However, it is possible to exhibit a solution for the equations of Theorem 2 when Condition (11) is met. For this, we construct our solution backward and step by step within each interval $[k\Delta,(k+1)\Delta \wedge T)$ with $k \leq k_1 = \lfloor \frac{T}{\Delta} \rfloor$ an integer.

- *Step 1 - Initialization:* We take $k = k_1$ and place ourselves in $[k\Delta,(k+1)\Delta \wedge T)$. Let $\mathbf{V}(t)$ and $\mathbf{g}$ be two vectors such that $\mathbf{V}(t)_i = V(t,\bar{u}_i)$ and $\mathbf{g}_i = g(\bar{u}_i)$ for every state $\bar{u}_i$[7]. Note that we can reformulate equations of Theorem 2 to obtain that $\mathbf{V}$ satisfies

$$\begin{cases} \mathbf{V}(T) = \mathbf{g}, \\ 0 = -\gamma \mathbf{I} + \tilde{\mathbf{g}} + \tilde{\mathbf{A}}V, \qquad \forall t \in (k\Delta,(k+1)\Delta \wedge T), \end{cases} \tag{10}$$

where

- $\mathbf{I}$ is a vector that encodes the inventory of all the states.
- $\tilde{\mathbf{g}}$ is a vector that incorporates the execution time constraint associated to (8).
- the operator $\tilde{\mathcal{Q}}$ is obtained by removing all the transitions to states where the inventory is zero or the price exceeds $\bar{P}$.
- the vectorial operator $\tilde{\mathbf{A}}$ is defined such that $\tilde{\mathbf{A}} = \partial_t + \tilde{\mathbf{Q}}$ and $\tilde{\mathbf{Q}}$ verifies $(\tilde{\mathbf{Q}}\mathbf{V})_i = \tilde{\mathcal{Q}}V(t,\bar{u}_i)$ for every $t \in [0,T]$ and state $\bar{u}_i$.

We also give the following explicit expression for $\tilde{\mathbf{g}}$ and $\tilde{\mathcal{Q}}$:

$$\begin{cases} \tilde{\mathbf{g}}_j = \mathbf{1}_{i_j \neq 0, p_j < \bar{P}} + \sum_{q_j^{bef} + q_j^a \leq n < q_j^1} \lambda_m^{1,-}(u_j,n,\mathbf{0})\tilde{g}(\bar{u}_j) + \mathbf{1}_{i_j \neq 0, p_j < \bar{P}} \sum_{u'' n \geq q_j^1} \lambda_m^{1,-}(u_j,n,u')\tilde{g}(\bar{u}_j'), \\ \tilde{\mathcal{Q}}(u_j,u') = \bar{\mathcal{Q}}(u_j,u')\mathbf{1}_{i \neq 0, p < \bar{P}, i' = 0}, \end{cases}$$

for any $\bar{u}_j = (q_j^{bef}, q_j^a, q_j^{aft}, q_j^2, i_j, s_j, p_j, p_j^{exec}) \in \bar{\mathbb{U}}$ and $\bar{u}' = (q'^{bef}, q'^a, q'^{aft}, q'^2, i'' s'' p'' p'^{exec}) \in \overline{\mathbb{U}}$ with $q_j^1 = q_j^{bef} + q_j^a + q_j^{aft}$, $q'^1 = q'^{bef} + q'^a + q'^{aft}$, $u_j = (q_j^1, q_j^2, s_j)$, $u' = (q'^1, q'^2, s')$, and $n \in \mathbb{N}^*$. Let $Im(\tilde{\mathbf{Q}})$ and $Ker(\tilde{\mathbf{Q}})$ be respectively the image and the kernel of $\tilde{\mathbf{Q}}$. We consider the following assumption:

$$-\gamma \mathbf{I} + \tilde{\mathbf{g}} \in Im(\tilde{\mathbf{Q}}) + Ker(\tilde{\mathbf{Q}}), \tag{11}$$

which means that

$$-\gamma \mathbf{I} + \tilde{\mathbf{g}} = \tilde{\mathbf{g}}_{Im(\tilde{\mathbf{Q}})} + \tilde{\mathbf{g}}_{Ker(\tilde{\mathbf{Q}})},$$

with $\tilde{\mathbf{g}}_{Im(\tilde{\mathbf{Q}})} \in Im(\tilde{\mathbf{Q}})$ and $\tilde{\mathbf{g}}_{Ker(\tilde{\mathbf{Q}})} \in Ker(\tilde{\mathbf{Q}})$. Because $\tilde{\mathbf{g}}_{Im(\tilde{\mathbf{Q}})} \in Im(\tilde{\mathbf{Q}})$, there exists $\tilde{z}$ such that $\tilde{\mathbf{Q}}\tilde{z} = \tilde{\mathbf{g}}_{Im(\tilde{\mathbf{Q}})}$. Then, we can check that the following variable is solution of (10):

$$V_t^0 = e^{(T-t)\tilde{\mathbf{Q}}}\mathbf{g} + (T-t)\tilde{\mathbf{g}}_{Ker(\tilde{\mathcal{Q}})} - \tilde{z}, \quad \forall t \in (k\Delta, T]. \tag{12}$$

Indeed, we have

$$\begin{aligned}\tilde{\mathbf{A}}V_t^0 = \partial_t V_t^0 + \tilde{\mathbf{Q}}V_t^0 &= -\tilde{\mathbf{Q}}e^{(T-t)\tilde{\mathbf{Q}}}\mathbf{g} - \tilde{\mathbf{g}}_{Ker(\tilde{\mathbf{Q}})} \\ &\quad + \tilde{\mathbf{Q}}e^{(T-t)\tilde{\mathbf{Q}}}\mathbf{g} + (T-t)\underbrace{\tilde{\mathbf{Q}}\tilde{\mathbf{g}}_{Ker(\tilde{\mathbf{Q}})}}_{=0} - \tilde{\mathbf{Q}}\tilde{z} \\ &= -[\tilde{\mathbf{g}}_{Ker(\tilde{\mathbf{Q}})} + \tilde{\mathbf{g}}_{Im(\tilde{\mathbf{Q}})}] = \gamma \mathbf{I} - \tilde{\mathbf{g}},\end{aligned}$$

which ensures that V_t^0 is solution of (10).

• *Step 2 - Iteration:* At time $k\Delta$, the agent can take a decision. So he compares expressions of Equation (6) and takes the maximum. After that, he reiterates Step 1 with new initial values and $k = k_1 - 1$.

Remark 6 (Regularity of the Solution of Theorem 2). The explicit solution given by Equation (12) has a time derivative $\partial_t V$, which is continuous in each subinterval $I_k = (k\Delta, (k+1)\Delta)$ with $k \in \mathbb{N}$. Thus, V is Lipschitz in time and space within each subinterval I_k,[15] and $\partial_t V$ is continuous within each I_k.

4.4. Numerical Resolution of the Optimal Execution Problem

To solve numerically the preceding optimal control problem, we consider a discrete framework. We show here how this discrete framework can be used to approximate the value function of the control problem. Furthermore, an error estimate is provided.

4.4.1. Discrete-Time Markov Chain Approximation. Let δ be a positive constant that divides Δ (i.e., $\Delta = m\delta$ with $m \in \mathbb{N}$). We define the operator $P^{*,\delta,\mu}$ such that

$$P_{u,u'}^{*,\delta,\mu} = \mathbb{P}_u[\overline{\mathbb{U}}_\delta^\mu = u'], \qquad \forall\, (u,u') \in \bar{\mathbb{U}}^2, \tag{13}$$

with $\overline{\mathbb{U}}_t^\mu$ the process defined in Section 3. Given the infinitesimal generator $\tilde{\mathcal{Q}}^\mu$ of $\overline{\mathbb{U}}_t^\mu$, the transition matrix $P^{*,\delta,\mu}$ can be easily computed since $P^{*,\delta,\mu} = e^{\delta\tilde{\mathcal{Q}}^\mu}$.[16] After that, we introduce a Markov chain $\overline{\mathbb{U}}_n^{\delta,\mu}$ with a transition matrix denoted by $P^{\delta,\mu}$. We will specify later on how to choose $P^{\delta,\mu}$. Finally, for every $k \geq 0$, we define the piecewise constant process $\tilde{\mathbb{U}}^{\delta,\mu}$ associated to $\overline{\mathbb{U}}_n^{\delta,\mu}$ such that

$$\tilde{\mathbb{U}}_t^{\delta,\mu} = \overline{\mathbb{U}}_k^{\delta,\mu}, \qquad \forall t \in [k\delta, (k+1)\delta).$$

We denote by $\tilde{V}^\delta(t, \bar{\mathbb{U}})$ the value function of the control problem 3 where the process $\bar{\mathbb{U}}^\mu$ is replaced by $\tilde{\mathbb{U}}^{\delta,\mu}$.

4.4.2. Solving Numerically the Optimal Control Problem in the Discrete Framework. We denote by $V^\delta(n, \bar{u})$ the value function associated to the discrete control problem (i.e., the state process $\overline{\mathbb{U}}_n^{\delta,\mu}$), with n the period and $\bar{u}$ the order book state. The dynamic programming principle reads

$$V^\delta(i, \bar{u}) = \sup_{e \in \mathbb{E}(i,\bar{u})} \mathbb{E}\Big[V^\delta\Big((i+1), \bar{\mathbb{U}}_{i+1}^{\delta,\mu}\Big) - \gamma I\,\delta | \overline{\mathbb{U}}_i^{\delta,\mu} = \bar{u},\ \mu_i = e\Big], \tag{14}$$

with the terminal constraint $V^\delta(n_f, \bar{u}) = g(\bar{u})$, I the agent's inventory, μ the control of the agent, and n_f the final period. Equation (14) provides a numerical scheme to compute $V^\delta(0, \bar{u})$. At the final time T, we can compute

$V^\delta(n_f, \bar{u})$ for each reachable state. Using the backward Equation (14), we can compute $V^\delta(i,u)$ knowing $V^\delta(i+1,\bar{u})$ to get the initial value $V^\delta(0,\bar{u})$. Thus, one can estimate $\tilde{V}^\delta(n\delta,\bar{u})$ because $\tilde{V}^\delta(n\delta,\bar{u}) = V^\delta(n,\bar{u})$. The numerical results of simulations are presented in Section 5. To compute efficiently the value function, computations can be carried on in parallel. Let us consider the following assumption.

Assumption 7. *There exists a constant D such that*

$$\sup_{\mu \in \mathcal{C}^a} \|P^{\delta,\mu} - P^{*,\delta,\mu}\|_F \le D\Delta^2,$$

with $\|.\|_F$ *the Frobenius norm.*

In fact, the finite difference scheme associated to the equations of Theorem 2 is the application of the discrete-time approximation with a Markov chain $\overline{U}_n^{\delta,\mu}$ whose transition matrix $P^{\delta,\mu} = I + \mathcal{Q}^\mu\delta$. This matrix $P^{\delta,\mu}$ satisfies Assumption 7. We have the following error estimate result.

Theorem 3. *Under Assumption 7,* $\tilde{V}^\delta(t,\bar{u})$ *converges toward* $V(t,\bar{u})$ *for every* $(t,\bar{u}) \in [0,T] \times \bar{\mathbb{U}}$. *Additionally, we have the following error estimate:*

$$|\tilde{V}^\delta(t,\bar{u}) - V(t,\bar{u})| \le R(T-t)\delta, \tag{15}$$

with $R > 0$ *a constant that depends on the model parameters.*

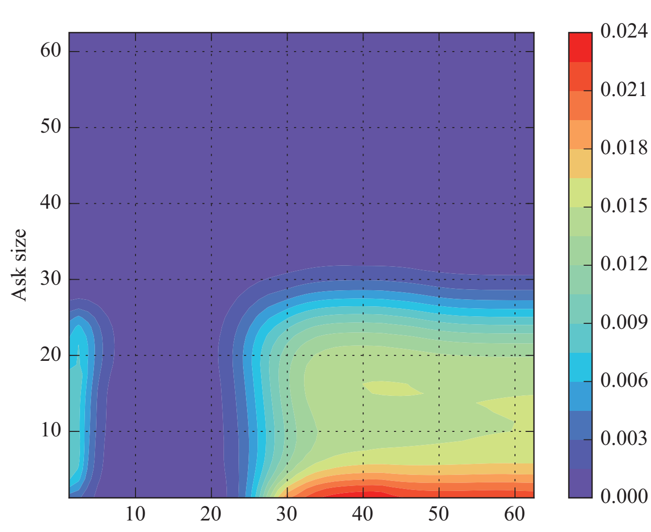
The proof of this result is given in Appendix E.

5. Numerical Experiments

In this section, we show the relevance of the optimal strategy. To do so, we compare the optimal gain given by our strategy and the one given by the standard strategy "join the bid": stay in the order book at the best bid until the final time. Here, we write Q^1 (resp. Q^2) for the best bid (resp. ask) limit.

Figure 7 shows for an order of size one the difference between the average gain (i.e., the initial value function) of the optimal strategy and the one of the strategy "join the bid" for different values of the initial Q^1 and Q^2. The gain of the optimal strategy is obviously always higher than that of the "join the bid." However, because of the priority value, that is the advantage of a limit order compared with another limit order standing at the rear of the same queue, it is more useful to be active (i.e., cancel the order or send a market order) when imbalance is highly positive than when it is negative. Finally, note that the optimal strategy reaches the maximum value of 2.4 ticks (the tick b_0 being equal to 0.01).

Figure 7. Difference Between the Gain of the Optimal Strategy and the One of the Strategy "Join the Bid"



Note. The initial parameters are fixed as follows: the time frequency is equal to $\Delta = 10$ seconds; the final time $T = 100$ seconds; arrival and consumption rates are estimated on data (see Appendix A); the new bid (resp. ask) is set to 5 and the new ask (resp. bid) to 3 after the total depletion of the bid (resp. ask) limit; the quantity $q^a = 1$; the waiting cost $c = 0$; the price increases (resp. decreases) by $b_0 = 0.01$ when the ask limit (resp. bid limit) is totally consumed; and the function f is equal to the identity.

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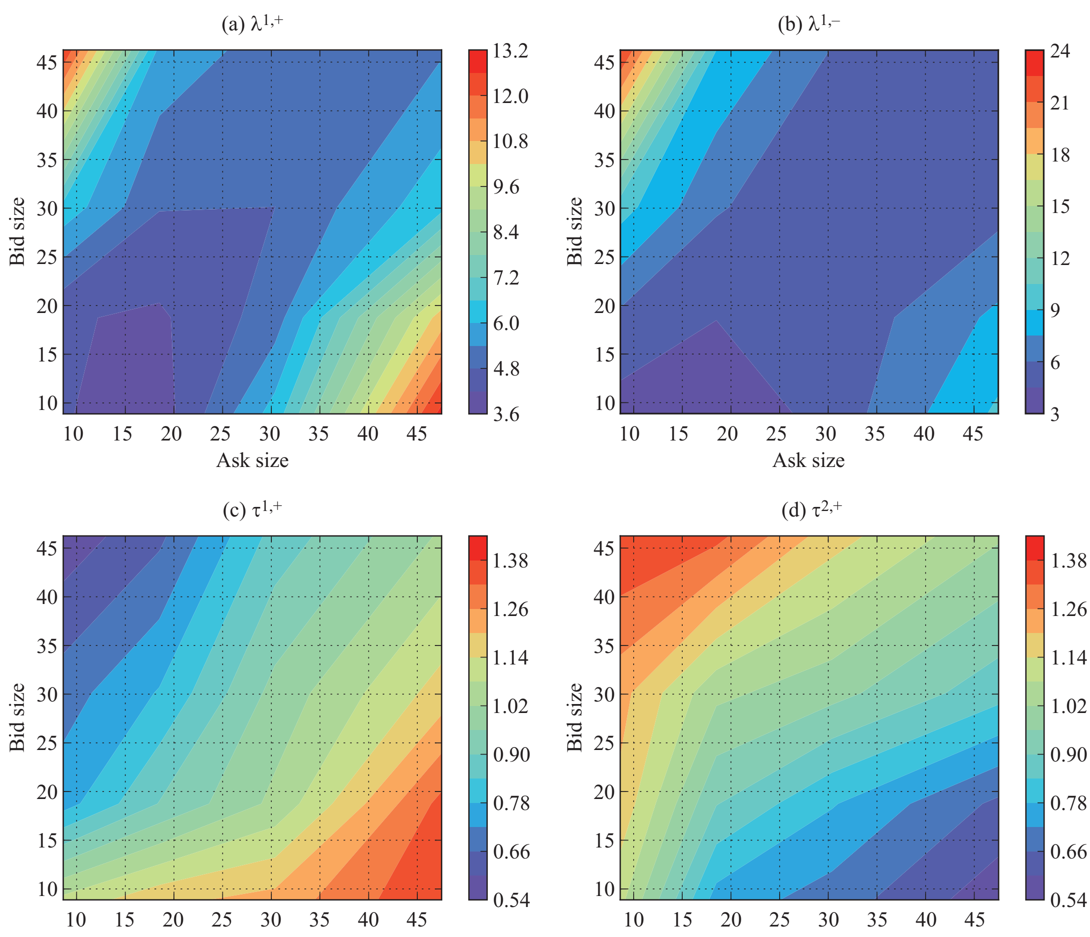
Appendix A. Model Parameters Estimation
The estimation methodology of the arrival and cancellation rates of limit orders is similar to that in Huang et al. (2015). The regeneration distribution of the order book is estimated from the empirical distribution of order book states after a depletion. In what follows, we provide the calibration results of our order book model using the database described in Section 2.1. Here, we write $Q_t = (Q_t^1, Q_t^2)$ with Q_t^1 (resp. Q_t^2) the best bid (resp. ask) quantity and consider that intensities and regeneration distributions depend only on Q_t.

Intensities estimation. For every $Q = (Q^1, Q^2)$, we write $\tau^{1,+}(Q) = \lambda^{1,+}/\lambda^{1,-}$ and $\tau^{2,+}(Q) = \lambda^{1,+}/\lambda^{1,-}$ respectively for the bid and ask side growth ratios. Given the bid-ask symmetry relation, we can aggregate data and focus on the bid side only. Figure A.1, (a), (b), (c), and (d) show respectively $\lambda^{1,+}$, $\lambda^{1,-}$, $\tau^{1,+}$ and $\tau^{2,+}$ for different values of Q. As expected, we can see that participants insert more limit orders when the imbalance is negative (see Figure A.1(a) when $Q^2 \gg Q^1$) while they cancel more when the imbalance is positive (see Figure A.1(b) when $Q^1 \gg Q^2$). Finally, Figure A.1(c) (resp. Figure A.1(d)) shows that $\tau^{1,+}$ (resp. $\tau^{2,+}$) is high when imbalance is negative (resp. positive) and becomes low when imbalance is positive (resp. negative), which means that the bid limit (resp. ask limit) tends to increase (resp. decrease) when $Q^1 \ll Q^2$ and tends to decrease (resp. increase) when $Q^1 \gg Q^2$.

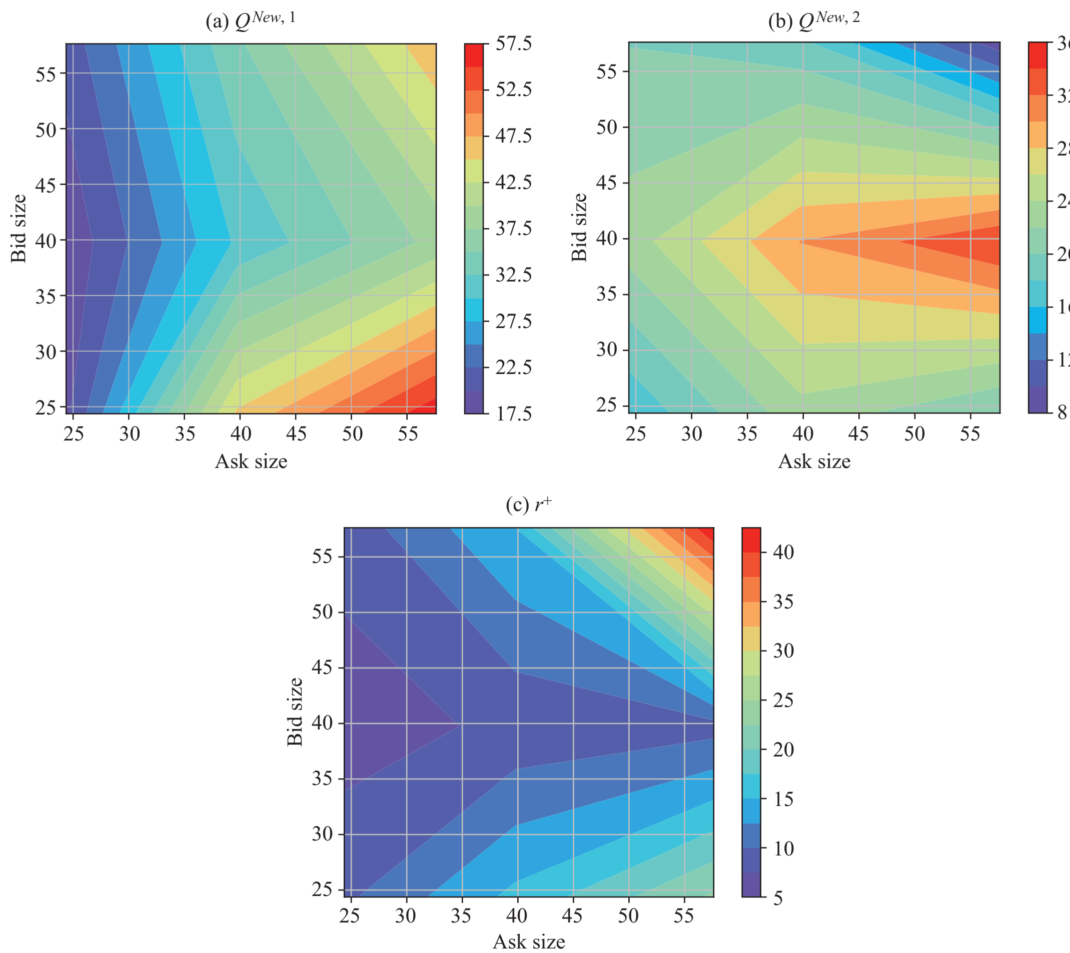
Quantities after depletion. When one limit is depleted, we write $Q^{New,1}$ (resp. $Q^{New,2}$) for the new best bid (resp. ask). Figure A.2, (a), (b), and (c) show respectively $Q^{New,1}$, $Q^{New,2}$ and the ratio $r^+(Q_1, Q_2) = \frac{Q^{New,1}}{Q^{New,2}}$ for different values of Q^1 and Q^2 before the mid price move. Because we aggregate data, the bid queue is always the depleted queue and the ask limit is the nonconsumed limit. Figure A.2, (a) and (b) show that $Q^{New,1}$ depends mainly on Q^2 while $Q^{New,2}$ depends on both

Figure A.1 (a) $\lambda^{1,+}$, (b) $\lambda^{1,-}$, (c) $\tau^{1,+}$, and (d) $\tau^{2,+}$ for Different Values of (Q^1, Q^2)



Note. Q^1 and Q^2 are divided by the average event size.

Figure A.2 (a) $Q^{New,1}$, (b) $Q^{New,2}$, and (c) r^+ for Different Values of Q^1 and Q^2



Note. Q^1 and Q^2 are divided by the average event size.

Q^1 and Q^2. However, the interesting point is that r^+ reach its maxima in two cases, see Figure A.2(c). The first case, when the bid is low and the ask is high, can be explained by a mean reversion effect while the second one, when both queues are initially high, is due to the arrival of a large order consuming market liquidity.

Appendix B. Ergodicity of the Process (U_t)

B.1. Outline of the Proof

Let Z_t be a Markov process defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and valued in $(W, \mathcal{W})$ and $P_t(x, A)$ the transition probability of Z_t.

Definition B.1 (Ergodicity). The process Z_t is ergodic if there exists an invariant probability measure π which satisfies

$$\lim_{t\to\infty} \|P_t(x,.) - \pi(.)\|_{TV} = 0, \qquad \forall x \in W,$$

where $\|\mu - \mu'\|_{TV} = \sup_{A\in\mathcal{F}} |\mu(A) - \mu'(A)|$.

To prove that U_t is ergodic, we design a Lyapunov function $V : \mathbb{U} \to (0, \infty)$, on which the following negative drift condition is satisfied for some $c > 0$ and $d > 0$:

$$\mathcal{Q}V(u) \le -cV(u) + d,$$

with $\mathcal{Q}$ the infinitesimal generator of U_t. Then, using theorem 6.1 in Meyn and Tweedie (1993), the Markov process U_t is nonexplosive and V-uniformly ergodic. Furthermore, by theorem 4.2 in Meyn and Tweedie (1993), it is Harris positive recurrent.

B.2. Proof

Let $u = (q^1, q^2, s) \in \mathbb{U}$ and $z = \min(z_0, z_1)$, we define

$$V(u) = \sum_{i \in \{1,2\}} z^{q^i - C_{bound}} + z^{s - C_{bound}}.$$

To simplify notations, we do not write the dependence of $\lambda_Q^{i,\pm}(n)$ and $\lambda_S^{\pm}(n)$ on u. For any $t \geq 0$, we have

$$\mathcal{Q}V(u) = \sum_{u' \neq u} \mathcal{Q}_{u,u'}\big[V(u') - V(u)\big].$$

By rearranging the terms, we get

$$\begin{aligned}\mathcal{Q}V(u) &= \sum_{i=1}^{2}\sum_{1\leq n}\Big[\lambda_Q^{i,+}(n)(z^{q^i+n-C_{bound}} - z^{q^i - C_{bound}}) + \lambda_Q^{i,-}(n)(z^{q^i-n-C_{bound}} - z^{q^i-C_{bound}})\Big] \\ &\quad + \sum_{1\leq n}\Big[\lambda_S^{+}(n)(z^{s+nb_0-C_{bound}} - z^{s-C_{bound}}) + \lambda_S^{-}(n)(z^{s-nb_0-C_{bound}} - z^{s-C_{bound}})\Big] \\ &= \sum_{i=1}^{2} z^{q^i - C_{bound}} \sum_{1\leq n}(z^n - 1)\bigg[\lambda_Q^{i,+}(n) - \lambda_Q^{i,-}(n)\frac{1}{z^n}\bigg] \\ &\quad + z^{s-C_{bound}}\sum_{1\leq n}(z^{nb_0} - 1)\bigg[\lambda_S^{+}(n) - \lambda_S^{-}(n)\frac{1}{z^{nb_0}}\bigg].\end{aligned} \tag{B.1}$$

Step (*i*). When $q^i \leq C^{bound}$, the quantity $z^{q^i - C_{bound}}$ is bounded and the intensities are bounded by Assumption 2. Hence, there exist $c^1 > 0$ and $d^1 > 0$ such that

$$\begin{aligned} z^{q^i - C_{bound}}\sum_{1\leq n}(z^n - 1)\bigg[\lambda_Q^{i,+}(n) - \lambda_Q^{i,-}(n)\frac{1}{z^n}\bigg] &\leq z^{q^i - C_{bound}}\bigg(\sum_{1\leq n} z^n \lambda_Q^{i,+}(n)\bigg) \\ &\leq -c^1 z^{q^i - C_{bound}} + d^1. \end{aligned} \tag{B.2}$$

Similarly, when $s \leq C^{bound}$ we have

$$z^{s-C_{bound}}\sum_{1\leq n}(z^{nb_0} - 1)\bigg[\lambda_S^{+}(n) - \lambda_S^{-}(n)\frac{1}{z^{nb_0}}\bigg] \leq -c^1 z^{s-C_{bound}} + d^1. \tag{B.3}$$

Step (*ii*). Using Assumption 1, we deduce that

$$\begin{aligned} z^{q^i - C_{bound}}\sum_{1\leq n}(z^n - 1)\Big[\lambda_Q^{i,+}(n) - \lambda_Q^{i,-}(n)\frac{1}{z^n}\Big] &\leq -2\delta z^{q^i - C_{bound}}, \quad \text{when } q^i > C^{bound} \\ z^{s-C_{bound}}\sum_{1\leq n}(z^{nb_0} - 1)\Big[\lambda_S^{+}(n) - \lambda_S^{-}(n)\frac{1}{z^{nb_0}}\Big] &\leq -2\delta z^{s-C_{bound}}, \quad \text{when } s > C^{bound}. \end{aligned} \tag{B.4}$$

Step (*iii*). By combining inequalities (B.2), (B.3), and (B.4), we have

$$\mathcal{Q}V(u) \leq -cV(u) + d,$$

with $c = \min(c^1, 2\delta)$ and $d = d^1$. This completes the proof.

Appendix C. Proof of the Computation of $\lim_{t\to\infty} \mathbb{E}_u[P_t - P_0]$

C.1. Proof of Proposition 1

For simplification, we fix the spread equal to one tick. Thus, t_1 (resp. t_2) is the first depletion time of the best bid (resp. ask). Under Assumptions 3 and 4, the number of order book states N is finite. For any $(i,i') \in \mathbb{N}_U^2$ with $\mathbb{N}_U = \{1, \ldots, N\}$ and $(s,r) \in \mathbb{R}_+ \times \{1,2\}$, we denote by $\mu_{ii'}^r(s)$ the density of the random variable $t_{ii'}^r$, which is the first depletion time of the limit r when the initial state is U_i and the state before the depletion is $U_{i'}$.[17] We write $\mu_{ii'}^-$ (resp. $\mu_{ii'}^+$) for the density of the random variable $\tilde{t}_{ii'}^-$ (resp. $\tilde{t}_{ii'}^+$) the first time when the bid (resp. ask) limit is consumed before the ask (resp. bid) one and where the initial state is U_i and the state before the depletion is $U_{i'}$.[18] To show Proposition 1, we use the two following lemmas that we prove after the proof of Proposition 1.

Lemma C.1. *There exist $m_1 > 0$ and $m_2 > 0$ such that*

$$\mu_{ii'}^r(s) \leq m_1 e^{-m_2 s}, \quad \mu_{ii'}^{\pm}(s) \leq m_1 e^{-m_2 s}, \quad \forall s \in \mathbb{R}_+,\ \forall r \in \{1,2\},\ \forall (i,i') \in \mathbb{N}_U^2.$$

Let $i \in \mathbb{N}_U$, $s \geq 0$ and $D_s^i = \mathbb{E}_{U_i}[P_s - P_0]$. We denote by $\mu_{ii'}^{-t}$ (resp. $\mu_{ii'}^{+t}$) the density of $\tilde{t}_{ii'}^{-t}$ (resp. $\tilde{t}_{ii'}^{+t}$) the first time smaller than t when the bid (resp. ask) limit is depleted before the ask (resp. bid) one and when the initial state is U_i and the state before the depletion is $U_{i'}$. Under Assumption 5, we have $D_\infty^i = \lim_{t\to\infty} \mathbb{E}_{U_i}[P_t - P_0] < \infty$ at least for one $i \in \mathbb{N}_U$. Because U_t is irreducible it means that $D_\infty^i < \infty$, $\forall i \in \mathbb{N}_U$. We have the following result.

Lemma C.2. *We have*

$$\lim_{t\to\infty}\int_0^t \mu_{ii'}^{\pm t}(s)(D_t^k - D_{t-s}^k)\,ds = 0, \quad \forall\,(i,i',k)\in\mathbb{N}_U^3, r\in\{1,2\}.$$

The proof of Lemma C.2 is given at the end of this section. Let us now prove Proposition 1.

Proof of Proposition 1. Let $\Delta P_t^0 = P_t - P_0$ and $q_{ii'}^{\pm t} = \mathbb{P}[\tilde{t}_{ii'}^{\pm} \le t]$ for $t \ge 0$. We can write

$$\begin{aligned}
\mathbb{E}_{U_i}[\Delta P_t^0] &= \mathbb{E}_{U_i}[\Delta P_t^0 \mathbf{1}_{t_2\le t_1<t}] + \mathbb{E}_{U_i}[\Delta P_t^0 \mathbf{1}_{t_1<t_2<t}]\\
&= \mathbb{E}_{U_i}[\mathbb{E}[\Delta P_t^0|\mathcal{F}_{t_2}]\mathbf{1}_{t_2\le t_1<t}] + \mathbb{E}_{U_i}[\mathbb{E}[\Delta P_t^0|\mathcal{F}_{t_1}]\mathbf{1}_{t_1<t_2<t}]\\
&= \sum_{i'} q_{ii'}^{+t}\left[\alpha_{i'}^+ + \int_0^t \mu_{ii'}^{+t}(s)\left(\sum_{k=1}^N d_{i'\prime k}^2 D_{t-s}^k\right)ds\right] + \sum_{i'} q_{ii'}^{-t}\left[\alpha_i^- + \int_0^t \mu_{ii'}^{-t}(s)\left(\sum_{k=1}^N d_{i'\prime k}^1 D_{t-s}^k\right)ds\right]\\
&= \overbrace{\sum_{i'} q_{ii'}^{+t}\left(\alpha_{i'}^+ + \sum_{k=1}^N d_{i'\prime k}^2\int_0^t \mu_{ii'}^{+t}(s) D_t^k\,ds\right) + \sum_{i'} q_{ii'}^{-t}\left(\alpha_i^- + \sum_{k=1}^N d_{i'\prime k}^1\int_0^t \mu_{ii'}^{-t}(s) D_t^k\,ds\right)}^{(1)}\\
&\quad + \overbrace{\sum_{i'} q_{ii'}^{+t}\left(\alpha_{i'}^+ + \sum_{k=1}^N d_{i'\prime k}^2\int_0^t \mu_{ii'}^{+t}(s)\Delta D_{t,s}^k\,ds\right) + \sum_{i'} q_{ii'}^{-t}\left(\alpha_i^- + \sum_{k=1}^N d_{i'\prime k}^1\int_0^t \mu_{ii'}^{-t}(s)\Delta D_{t,s}^k\,ds\right)}^{(2)},
\end{aligned} \tag{C.1}$$

with $\Delta D_{t,s}^k = D_{t-s}^k - D_t^k$. Using Lemma C.2, the quantity (2) goes to 0 when t tends to infinity. Hence by sending t to infinity in (C.1), we find

$$D_\infty^i = \sum_{i'} q_{ii'}^+\alpha_{i'}^+ + q_{ii'}^-\alpha_{i'}^- + \sum_{k=1}^N\sum_{i'}(q_{ii'}^+ d_{i'\prime k}^2 + q_{ii'}^- d_{i'\prime k}^1)D_\infty^k = q_i + \sum_{k=1}^N p_{i,k}D_\infty^k. \tag{C.2}$$

Additionally, using the symmetry relation, we have $D_\infty^i = -D_\infty^{i^{sym}}$. We write $\mathcal{D}$ for the set $\mathcal{D} = \{(q^1,q^2,s);\, p=0,\, q^1 \ge q^2\}$. Consequently, Equation (C.2) reads

$$D_\infty^i(1-(p_{i,i}-p_{i,i^{sym}})) = q_i + \sum_{k\in\mathcal{D}}(p_{i,k}-p_{i,k^{sym}})D_\infty^k.$$

Given that $0\le p_{i,i}<1$ (the price moves with a nonzero probability when one limit is totally consumed), we have $(1-(p_{i,i}-p_{i,i^{sym}}))>0$. This proves the result of Proposition 1. □

Proof of Lemma C.1. Let $(i,i')\in\mathbb{N}_U^2$ and $r\in\{1,2\}$. Under Assumptions 3 and 4, the intensities are bounded by a constant λ_∞. Let N_∞ be the Poisson process that admits λ_∞ as an intensity. We have

$$\mathbb{P}_{U_i}[t_*\le t_1\le t^*, U_{t_1^-}=U_i'] \le \mathbb{P}[N_t - N_{t_*}\ne 0,\ \forall t\in[t_*,t^*]] = \int_{t_*}^{t^*}\lambda_\infty e^{-\lambda_\infty s}\,ds, \quad \forall (t_*,t^*)\in\mathbb{R}_+^2.$$

Consequently, we get

$$\mu_{ii'}^r(\underline{t}) = \lim_{t^*\to t_*}\frac{\mathbb{P}_{U_i}[t_*\le t_1\le t^*, U_{t_1^-}=U_i']}{t^*-\underline{t}} \le \lim_{t^*\to t_*}\frac{\int_{t_*}^{t^*}\lambda_\infty e^{-\lambda_\infty s}\,ds}{t^*-t_*} = \lambda_\infty e^{-\lambda_\infty t_*}.$$

By following the same methodology, we also have

$$\mu_{ii'}^{\pm} \le \lambda_\infty e^{-\lambda_\infty t_*}.$$

This completes the proof. □

Proof of Lemma C.2. Using Lemma C.1, we have

$$\mathbb{P}[\tilde{t}_{ii'}^{\pm t}\le s] = \mathbb{P}[\tilde{t}_{ii'}^{\pm}\le s|\tilde{t}_{ii'}^{\pm}\le t] = \frac{\mathbb{P}[\tilde{t}_{ii'}^{\pm}\le s]}{\mathbb{P}[\tilde{t}_{ii'}^{\pm}\le t]} \le \frac{m_1 e^{-m_2 s}}{q_{ii'}^{\pm t}}, \quad \forall s\le t,$$

with $q_{ii'}^{\pm t} = \mathbb{P}[\tilde{t}_{ii'}^{\pm}\le t]$. Under Assumptions 3 and 4, the intensities are bounded and the price jumps are also bounded because the state space is finite. Hence, there exists $m_3>0$ such that

$$D_t^k \le m_3 t, \quad \forall t\ge 0.$$

Because $\lim_{t\to\infty} D^k_t = D^k_\infty < \infty$ is finite by Assumption 5, we have $\epsilon_t = \sup_{s\le\frac{t}{2}}|D^k_t - D^k_{t-s}| \to_{t\to\infty} 0$. Thus, we deduce

$$\lim t\to\infty \left|\int_0^t \mu^{\pm,t}_{ii'}(s)(D^k_t - D^k_{t-s})\,ds\right| \le \lim_{t\to\infty} \epsilon_t \int_0^{\frac{t}{2}} \mu^{\pm t}_{ii'}(s)\,ds + \int_{\frac{t}{2}}^t \mu^{\pm t}_{ii'}(s)|D^k_t - D^k_{t-s}|\,ds$$

$$\le \lim_{t\to\infty} \epsilon_t + \frac{2m_3}{q^{\pm t_0}_{ii'}} t \int_{\frac{t}{2}}^t m_1 e^{-m_2 s}\,ds$$

$$\le \lim_{t\to\infty} \epsilon_t + \frac{2m_1 m_2 m_3}{q^{\pm t_0}_{ii'}} t(e^{-m_2 t} - e^{-m_2\frac{t}{2}}) = 0,$$

with t_0 a fixed positive number such that $q^{\pm t_0}_{ii'} > 0$. This completes the proof. □

C.2. Computation Methodology of $q^{\pm}_{..}$

For simplification, we fix the added/cancelled quantity $n = 1$ and the spread constant. To take into account nonunitary jumps, we can simply fill the zero values of the matrix $\tilde{Q}^*$ with the right probabilities, see Equation (C.3).

Let $R = [R^-, R^+]$ be the matrix such that $R^-_{ii'} = q^-_{ii'}$ and $R^+_{ii'} = q^+_{ii'}$ for any $(i,i') \in \mathbb{N}^2_U$ with $\mathbb{N}_U = \{1,\ldots,N\}$, see Section 2.4. To compute R, we first fix the price $P = 0$ because there is no price move before the total depletion of a limit and model the order book state only by $u = (q^1, q^2)$ with q^1 (resp. q^2) the best bid (resp. ask) quantity. Then, we introduce the absorbing states $U_{0,q'}$ (resp. $U_{q''0}$) with $q' \ge 1$ associated to the cases $u' = (0,q')$ (resp. $u' = (q''0)$) where the best bid (resp. ask) is consumed before the ask (resp. bid) and the ask (resp. bid) value is $Q^2 = q'$ (resp. $Q^1 = q'$). We want to compute the probabilities to visit $U_{0,q'}$ and $U_{q''0}$ starting from U'_i. To do this, we consider the infinitesimal generator Q^* of the Markov process (Q^1, Q^2),[19]

$$Q^* = \begin{bmatrix} 0_{2\bar{Q}^{max}} & 0 \\ [\,Q^- \quad Q^+\,] & \tilde{Q}^* \end{bmatrix},$$

where $\bar{Q}^{max} = \max(Q^{max}, \tilde{Q}^{max})$, see Assumptions 3 and 4, $0_{2\bar{Q}^{max}}$ is the zero square matrix of size $2\bar{Q}^{max}$, Q^- encodes transitions to the absorbing states $U_{0,q'}$ and Q^+ encodes transitions to the absorbing states $U_{q''0}$ with $1 \le q' \le \bar{Q}^{max}$, and $\tilde{Q}^*$ is similar to the infinitesimal generator of the process U_t without regeneration. The matrix $\tilde{Q}^*$ has the following form:

$$\tilde{Q}^* = \begin{bmatrix} \tilde{Q}^{*,(1)}_1 & \tilde{Q}^{*,(1)}_0 & 0 & 0 & \ldots \\ \tilde{Q}^{*,(2)}_2 & \tilde{Q}^{*,(2)}_1 & \tilde{Q}^{*,(2)}_0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & 0 & 0 & \tilde{Q}^{*,(Q^{max})}_2 & \tilde{Q}^{*,(Q^{max})}_1 \end{bmatrix}, \tag{C.3}$$

where $\tilde{Q}^{*,(l)}_0$ encodes transitions from level $Q^1 = l$ to level $Q^1 = l+1$, matrix $\tilde{Q}^{*,(l)}_2$ encodes transition from level $Q^1 = l$ to $Q^1 = l-1$, and matrix $\tilde{Q}^{*,(l)}_1$ encodes transitions within the level $Q^1 = l$. $\bar{Q}^{max}$ is the maximum quantity available on each limit. Within each submatrix $\tilde{Q}^{*,(l)}_i$ with $i \in \{0,1,2\}$, Q^1 is equal to l and Q^2 vary from 1 to $\bar{Q}^{max}$. The submatrices $\tilde{Q}^{*,(l)}_i$, for $i = 0, 1$, can be written

$$\tilde{Q}^{*,(l)}_0 = \begin{pmatrix} \lambda^{1,+}(l,1) & & \\ & \ddots & \\ & & \lambda^{1,+}(l,Q^{max}) \end{pmatrix} \text{ and } \tilde{Q}^{*,(l)}_2 = \begin{pmatrix} \lambda^{1,-}(l,1) & & \\ & \ddots & \\ & & \lambda^{1,-}(l,Q^{max}) \end{pmatrix}.$$

Let $\lambda^*(l,l') = \sum_{i=1}^2 \lambda^{i,+}(l,l') + \lambda^{i,-}(l,l')$ for every $l,l' \in \{1,\ldots,\bar{Q}^{max}\}$. For $l \le \bar{Q}^{max}$, we have

$$\tilde{Q}^{*,(l)}_1 = \begin{pmatrix} -\lambda^*(l,1) & \lambda^{2,+}(l,1) & 0 & 0 & \ldots \\ \lambda^{2,-}(l,2) & -\lambda^*(l,2) & \lambda^{2,+}(l,2) & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & 0 & 0 & \lambda^{2,-}(l,Q^{max}) & -\lambda^*(l,Q^{max}) \end{pmatrix}.$$

Finally, we define the matrix Q^- such that $Q^-_{ii} = \lambda^{1,-}(1,i)$ for $1 \le i \le \bar{Q}^{max}$ and 0 otherwise, and the matrix Q^+ such that $Q^+_{i\bar{Q}^{max}+1,i+1} = \lambda^{2,-}(i,1)$ for $0 \le i \le \bar{Q}^{max} - 1$ and 0 otherwise.

Lemma C.3 (Computation of $q^{\pm}_{ii'}$). *Let $R = [R^-, R^+]$ be the matrix such that $R^-_{ii'} = q^-_{ii'}$ and $R^+_{ii'} = q^+_{ii'}$ for any $i \in \mathbb{N}_U = \{1,\ldots,N\}$ and $i' \in \mathbb{N}^d_U$,[20] see Section 2.4. Then, R is the minimal nonnegative solution to the system of linear equations*

$$\tilde{Q}^* R = -z^1,$$

where $\tilde{Q}^*$ is defined in (C.3) and $z^1 = [Q^-, Q^+]$ with Q^- and Q^+ defined in (C.4).

The proof of this lemma is given.

Example 1. The case where the best bid dynamics are independent from the ones of the best ask is a classical setting that was studied in Huang et al. (2015). In such situation, the matrix $\tilde{Q}^*$ is diagonalisable, see Appendix C.3. Additionally, when the intensities are constant, we have a closed-form formula for the diagonalisation of $\tilde{Q}^*$, see Appendix C.3.

Proof of Lemma C.3. Using theorem 3.3.1 in Norris (1998), the matrices $(q^-_{ii'})_{(i,i')\in\mathbb{N}_U\times\mathbb{N}^d_U}$ and $(q^+_{ii'})_{(i,i')\in\mathbb{N}_U\times\mathbb{N}^d_U}$ are the minimal nonnegative solution to the system of linear equations

$$\begin{cases} q^-_{U_{0,q},U_{0,q}} = 1, \quad q^+_{U_{0,q},U_{i'}} = 0, \quad q^-_{U_{q,0},U_{i'}} = 0, \quad q^+_{U_{q,0},U_{q,0}} = 1, \\ \sum_j Q^*_{i,j} q^{\pm}_{j,i'} = 0 \qquad \forall i \in [2\bar{Q}^{max}+1, (\bar{Q}^{max})^2 + 2\bar{Q}^{max}], \end{cases}$$

for all $q \in \{1, \bar{Q}^{max}\}$ and every absorbing state $U_{i'}$. In the above equations, we use a slight abuse of notation and do not differentiate the state $U_{i'}$ from the index i'. The equation above reads

$$\tilde{Q}^* R = -z^1, \tag{C.4}$$

with $z^1 = [Q^-, Q^+]$ and $R = [R^-, R^+]$ the matrix such that $R^-_{ii'} = q^-_{ii'}$ and $R^+_{ii'} = q^+_{ii'}$. When queues are independent $\tilde{Q}^*$ is diagonalisable, see next subsection. In the simple case of constant intensities, $\tilde{Q}^*$ diagonalisation is explicitly computable. □

C.3. Diagonalisation of $\tilde{Q}^*$

C.3.1.Symmetrization of $\tilde{Q}^*$ Under the Assumption of Independent Queues.

The idea is to find a matrix P such that $P^{-1}\tilde{Q}^*P$ is symmetric with $P = LH$. First, we consider the block-diagonal matrix $H = diag\{H_1, H_2, \ldots H_{Q^{max}}\}$ where every H_i is a square matrix of size Q^{max} such that

$$\begin{cases} H_1 = I, \\ H_{i+1} = H_i \sqrt{\tilde{Q}_2^{*,(i)} \left(\tilde{Q}_0^{*,(i-1)}\right)^{-1}}, \qquad \forall i \geq 1. \end{cases}$$

Here $\sqrt{\cdot}$ refers to the square root of a matrix. The existence of such a matrix in this case is trivial because $\tilde{Q}_2^{*,(i)}$ and $\tilde{Q}_0^{*,(i-1)}$ are diagonal with strictly positive coefficients.

Next, we consider the block-diagonal matrix $L = diag\{L_1, L_1, \ldots L_1\}$ where L_1 is a diagonal matrix with diagonal coefficients $L_1(1,1) = 1$ and $L_1(i+1,i+1) = L_1(i,i)\sqrt{\frac{\tilde{Q}_1^{*,(0)}(i+1,i)}{\tilde{Q}_1^{*,(0)}(i,i+1)}}$ for all $i \geq 1$. Given that queues are independent, we have $\tilde{Q}_1^{*,(0)} = \tilde{Q}_i^{*,(0)}$ for all $i \geq 1$. Finally, we note that $P^{-1}\tilde{Q}^*P$, with $P = LH$, is symmetric.

C.3.2. Diagonalisation of the Symmetric Matrix $P^{-1}\tilde{Q}^*P$: Constant Coefficients.

In the simple case of constant coefficients, the matrix P defined in Appendix C.3 satisfies

$$P^{-1}\tilde{Q}^*P = \begin{bmatrix} A(a,b) & V & 0 & 0 & 0 \\ V & A(a,b) & V & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & V & A(a,b) \end{bmatrix} \quad \text{and} \quad A(a,b) = \begin{pmatrix} a & b & 0 & 0 & \ldots \\ b & a & b & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & 0 & 0 & b & a \end{pmatrix},$$

where $V = \beta I$ with $\beta > 0$, I the identity matrix, and a and b are some fixed constants. In such framework, the eigenvalues of $P^{-1}\tilde{Q}^*P$ are

$$\lambda^{k,j}_{a,b,\beta} = a + 2b\cos\left(\frac{k\pi}{n+1}\right) + 2\beta\cos\left(\frac{j\pi}{n+1}\right), \qquad \forall 1 \leq k,j \leq n,$$

with $n = \bar{Q}^{\max}$ and the associated eigenspace is generated by the eigenvector $X^{k,j} = \left(v^j_1 X^k, v^j_2 X^k \ldots, v^j_{Q^{max}} X^k\right)$, where v^j satisfies

$$v^j_r = \sin\left(r\frac{j\pi}{n+1}\right), \qquad \forall 1 \leq r,\ j \leq n,$$

and X^k is a vector such that

$$X^k_l = \sin\left(l\frac{k\pi}{n+1}\right), \qquad \forall 1 \leq k,l \leq n.$$

Appendix D. Proof of Theorem 2

The main idea of the proof is to show existence and uniqueness of the solution within each interval $I_k = (k\Delta, (k+1)\Delta \wedge T]$ with $k \leq k_1 = \lfloor \frac{T}{\Delta} \rfloor$ an integer using a backward induction. First, we reformulate the equations of Theorem 2 as in Section 4.3

to show that V satisfies (10) in the interval I_{k_1}. The existence of a solution for (10) on I_{k_1} uses a fixed point argument and the uniqueness comes from an application of the Cauchy-Lipschitz theorem.

Step 1—existence: Let $t_1 = k_1\Delta$, $t_2 = T$, L^0 be the space of functions that are measurable with respect to the Lebesgue measure, $L^1 = \{f \in L^0; \|f\|^1 = \int_{t_1}^{t_2} \|f(u)\| du < \infty\}$, and $c > 0$ be a constant whose value will be fixed later on. We consider the norm

$$\|f\|_c^1 = \int_{t_1}^{t_2} e^{-cu} \|f(u)\| du, \qquad \forall f \in L^1.$$

Clearly the norms $\|.\|^1$ and $\|.\|_c^1$ are equivalent in L^1. We define the map U as follows:

$$\mathrm{U}(f)(s) = \int_s^{t_2} (\gamma \mathbf{I} - \tilde{\mathbf{g}} - \tilde{\mathbf{Q}} f(u)) du, \quad \forall t_1 \le s \le t_2,$$

with $\mathbf{I}$, $\tilde{\mathbf{g}}$, and $\tilde{\mathbf{Q}}$ introduced in (10). Let us first prove that $\mathrm{U}(f) \in L^1$. The definition of U gives

$$\|\mathrm{U}(f)\|^1 = \int_{t_1}^{t_2}\int_s^{t_2} (\gamma \mathbf{I} - \tilde{\mathbf{g}} - \tilde{\mathbf{Q}} f(u)) du \le T^2(\gamma\|\mathbf{I}\|_\infty - \|\tilde{\mathbf{g}}\|_\infty) + \int_{t_1}^{t_2}\int_s^{t_2} \|\tilde{\mathbf{Q}} f(u)\| du, \tag{D.1}$$

with $\|h\|_\infty = \sup_{\bar{u} \in \bar{\mathbb{U}}} |h(\bar{u})|$ for any function h. Because the intensity functions are bounded under Assumptions 3 and 4 and the operator $\tilde{\mathbf{Q}}$ consists in a finite linear combination of the intensity functions, there exists $K \ge 0$ such that $\|\tilde{\mathbf{Q}} f(u)\| \le K\|f(u)\|$. Using (D.1), we deduce that

$$\|\mathrm{U}(f)\|^1 \le T^2(\gamma\|\mathbf{I}\|_\infty - \|\tilde{\mathbf{g}}\|_\infty) + T^2 K \|f\|^1 < \infty.$$

Second, we show that U is a contracting map for an appropriate choice of c. Because L^1 is a Hilbert space, this contracting property is enough to prove the existence. For any $(f^1, f^2) \in (L^1)^2$ such that $f^1(t_2) = f^2(t_2)$, we have

$$\|(\mathrm{U}(f^1) - \mathrm{U}(f^2))(s)\| = \|\int_{t_1}^{t_2} \tilde{\mathbf{Q}}(f^1 - f^2)(u) du\| \le K\int_{t_1}^{t_2} \|(f^1 - f^2)(u)\| du.$$

Integrating the above inequality between t_1 and t_2 shows $\|\mathrm{U}(f^1) - \mathrm{U}(f^2)\|_c^1 \le \frac{K}{c}\|f^1 - f^2)\|_c^1$. Hence, U is a contracting map for a large enough c.

Step 2—uniqueness: Let us check that the assumptions of the Cauchy-Lipschitz theorem are satisfied. Note that the variable $\bar{u}$ is valued on the Banach space $G = ([0, \bar{Q}^{\max}])^6 \times [0, \bar{P}] \times [0, \bar{P}^{Exec}]$ with $\bar{Q}^{\max} = \max(Q^{\max}, \tilde{Q}^{\max})$, $\bar{P}$ the maximum mid price value, and $\bar{P}^{Exec}$ the upper bound for the execution price. The value function V is also valued on a Banach subspace of $\mathbb{R}$ because it is bounded, see (5) and Proposition 1. Moreover, the operator $\tilde{\mathbf{Q}}$ introduced in (10) is Lipschitz mainly because it is a finite linear combination of the intensity functions, which are Lipschitz. The intensity functions are Lipschitz because they are defined on a finite space. The variable $\tilde{\mathbf{g}}$ is Lipschitz as well because f and the function D, see (4), are both Lipschitz. Here again, the function D is Lipschitz because it is valued on a finite space. We can then apply the Cauchy-Lipschitz theorem to prove the uniqueness of a solution for (10) on I_{k_1}.

We repeat the same methodology several times to define V step by step uniquely on each I_k, which completes the proof.

Appendix E. Proof of Theorem 3

Let us show using a backward induction the following inequality:

$$|\tilde{V}^\delta(n\delta, \bar{u}) - V(n\delta, \bar{u})| \le R'(n_f - n)\delta^2, \qquad \forall n \le n_f, \ \forall \bar{u} \in \mathbb{U}, \tag{E.1}$$

with n_f the final period and $R' > 0$ a constant. Because δ divides Δ and V is Lipschitz on each subinterval $I_k = [k\Delta, (k+1)\Delta)$ with k a nonnegative integer k satisfying $(k+1)\Delta < T$, see Remark 6, we have $|V(k\delta, \bar{u}) - V(t, \bar{u})| \le R^1\delta$ for all $t \in [k\delta, (k+1)\delta)$ with $R^1 > 0$ and therefore we only need to prove (E.1).

1. *Initialisation part.* When $n = n_f$, we have $\tilde{V}^\delta(n\delta, \bar{u}) = V(n\delta, \bar{u}) = g$, which gives (E.1).
2. *Induction step.* Let us write $t^n = n\delta$ for any $n \in \mathbb{N}$. Note that V satisfies the following dynamic programming principle:

$$V(t^n, \bar{u}) = \sup_{e \in \mathbb{E}(n, \bar{u})} \mathbb{E}\left[V\left(t^{n+1}, \bar{\mathrm{U}}^\mu_{t^{n+1}}\right) - \gamma\int_{t^n}^{t^{n+1} \wedge T^\mu_{Exec}} I^\mu_s \, ds | \bar{\mathrm{U}}^\mu_{t^n} = \bar{u}, \mu_i = e\right]. \tag{E.2}$$

Using (14) and (E.2), we have

$$|\tilde{V}^\delta(t^n, \bar{u}) - V(t^n, \bar{u})| \le \sup_{e \in \mathbb{E}(n, \bar{u})} \left\{\mathbb{E}\left[\gamma\left|\int_{t^n}^{t^{n+1} \wedge T^\mu_{Exec}} I^\mu_s \ ds - \tilde{I}^\mu_{t^n}\delta\right|\right] + \mathbb{E}\left[\left|V\left(t^{n+1}, \bar{\mathrm{U}}^\mu_{t^{n+1}}\right) - \tilde{V}^\delta\left(t^{n+1}, \tilde{\mathrm{U}}^{\mu,\delta}_{t^{n+1}}\right)\right|\right]\right\} = (1) + (2).$$

- First, we have

$$(1) \le \gamma\mathbb{E}\left[\mathbf{1}_{T^\mu_{Exec} \le t^{n+1}}\left|\int_{t^n}^{T^\mu_{Exec}} I^\mu_s \ ds - \tilde{I}^\mu_{t^n}\delta\right|\right] + \gamma\mathbb{E}\left[\mathbf{1}_{T^\mu_{Exec} > t^n}\left|\int_{t^n}^{t^{n+1}} I^\mu_s ds - \tilde{I}^\mu_{t^n}\delta\right|\right] = (a) + (b).$$

For Part (a), we get

$$(a) \leq \gamma \bar{Q}^{\max} \mathbb{E}\Big[\mathbf{1}_{T^{\mu}_{Exec} \leq t^{n+1}}((T^{\mu}_{Exec} - t^n) + \delta)\Big] \leq \gamma \bar{Q}^{\max} \delta \mathbb{E}\Big[2\mathbf{1}_{T^{\mu}_{Exec} \leq t^{n+1}}\Big] \leq \gamma \bar{Q}^{\max} \delta^2 2H,$$

because $I^{\mu} \leq \bar{Q}^{\max}$ with $\bar{Q}^{\max}$ the maximum available inventory and $\mathbb{E}[\mathbf{1}_{T^{\mu}_{Exec} \leq t^{n+1}}] \leq \delta H$. The constant H is an upper bound for the intensity functions that exists under Assumptions 3 and 4. For Part (b), we use

$$(b) \leq \gamma \mathbb{E}\Big[\mathbf{1}_{T^{\mu}_{Exec} > t^{n+1}} \int_{t^n}^{t^{n+1}} |\tilde{I}^{\mu}_s - I^{\mu}_s| ds\Big] \leq \gamma \delta \mathbb{E}\Big[\sup_{t^n \leq s \leq t^{n+1}} |\tilde{I}^{\mu}_s - I^{\mu}_s|\Big].$$

Using standard computations, we have $\mathbb{E}[\sup_{t^n \leq s \leq t^{n+1}} |\tilde{I}^{\mu}_s - I^{\mu}_s|] \leq C\delta$ with $C > 0$ a constant. This gives

$$(b) \leq \gamma C \delta^2.$$

By combining the previous inequalities, we deduce that $(1) \leq C_1 \delta^2$ with C_1 a positive constant.

• Second, using (13), Assumption 7, and the induction assumption, we have

$$\begin{aligned}
(2) = \mathbb{E}[|\tilde{V}^{\delta}(t^{n+1}, \tilde{U}^{\mu,\delta}_{t^{n+1}}) - V(t^{n+1}, \bar{U}^{\mu}_{t^{n+1}})|] &= \sum_{u'} |P^{\delta,\mu}_{u,u'} \tilde{V}^{\delta}(t^{n+1}, \bar{u}') - P^{*,\delta,\mu}_{u,u'} V(t^{n+1}, \bar{u}')| \\
&\leq \sum_{u'} P^{*,\delta,\mu}_{u,u'} |(\tilde{V}^{\delta}(t^{n+1}, \bar{u}') - V(t^{n+1}, \bar{u}')| + \sum_{u'} |P^{*,\delta,\mu}_{u,u'} - P^{\delta,\mu}_{u,u'}| \tilde{V}^{\delta}(t^{n+1}, \bar{u}) \\
&\leq \sum_{u'} [P^{*,\delta,\mu}_{u,u'} |(\tilde{V}^{\Delta}(t^{n+1}, \bar{u}') - V(t^{n+1}, \bar{u}')|] + D\|g\|_{\infty} \delta^2 \\
&\leq R'(n_f - (n+1))\delta^2 + D\| g\|_{\infty} \delta^2.
\end{aligned}$$

By combining above inequalities and taking $R' = \max(C_1, D\| g\|_{\infty})$, we conclude

$$|\tilde{V}^{\delta}(t^n, \bar{u}) - V(t^n, \bar{u})| \leq R'(n_f - (n+1))\delta^2 + R'\delta^2 \leq R'(n_f - n)\delta,$$

which completes the proof of (E.1).

Endnotes

[1] See Section 3 for a detailed description of the control μ and the set $\mathbb{P}$.

[2] Limit order inserted inside the spread.

[3] The high rate of liquidity provision for very positive imbalance can be surprising at first sight. However, it may be due to orders inserted within the spread creating a new best limit.

[4] Quoting Sasha Stoikov: “Imbalance is the worst kept secret of high frequency trading” (Stoïkov 2014).

[5] However, from a theoretical viewpoint, we can work on the following state space $\mathbb{U} = (\mathbb{R}^*_+)^3$ and recover similar results.

[6] Note that $D_i = -D_{i^{sym}}$ under the symmetry relation (3), thus it is enough to know the value of D_i for any $U_i \in \mathcal{D}$.

[7] Recall that we refer to the state U_i through the index i.

[8] This modelling is conservative because we delay the order execution as long as possible. It corresponds to the worst case scenario for the trader.

[9] The best bid $Q^{1,\mu}_t$ after a depletion is defined in Section 2.2.

[10] To track only the first limits, we assume that the agent follows the best bid.

[11] We will see that $\lim_{t\to\infty} \mathbb{E}[\Delta P^{\mu}_t | \mathcal{F}_{T^{\mu}_{Exec}}]$ is well-defined and an explicit computation of this quantity is given in Section 4.1.

[12] U^{μ} does not depend on μ after the execution because the agent leaves the market.

[13] The results remain valid when $\overline{\mathbb{U}} = \mathbb{R}^5_+ \times \mathbb{R}^*_+ \times (\mathbb{R})^2$.

[14] We keep in mind that a control may lead to several states because of the regeneration. Moreover, the set $\mathbb{E}(i, \bar{u})$ depend on the current state, see Section 3.

[15] One can show that V is globally Lipschitz in space because V is bounded and $\mathbb{U}$ is finite.

[16] Because $\delta < \Delta$, no decision is taken during the time interval $(0, \delta]$. Moreover, using that the transition probabilities of $\bar{U}^{\mu}$ do not depend on the time, the process U^{μ} is a stationary Markov-chain on $[0, \delta]$ and therefore the expression of $P^{*,\delta,\mu}$ can be found in Norris (1998)[section 3.1].

[17] The variable $t^r_{ii'}$ admits a density function because it can be seen as the first jump time of a Markov jump process U^r and the interarrival times of Markov jump processes are always exponentially distributed. The process U^r_t used here is the order book state at time t after the last depletion of the limit r. It is Markov because U_t is Markov.

[18] The variables $\tilde{t}^-_{ii'}$ and $\tilde{t}^+_{ii'}$ admit densities for the same reason as $t^r_{ii'}$.

[19] The price P is fixed equal to 0.

[20] The set $\mathbb{N}_U^d$ contains the indexes of the possible order book states before a depletion. These state are of the form $U_{0,q'}$ or $U_{0,q'}$ with $q' \in \{1, \ldots, \bar{Q}^{max}\}$. Thus, we have $\#\mathbb{N}_U^d = 2\bar{Q}^{max}$.

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