Appendix

Proof. (Lemma 1) For better clarity, we carry out the computations in dimension 1 but all the arguments are valid in higher dimension and we will clarify delicate points throughout the proof.

Differentiating both sides of the optimality condition (6) and rearranging yields

$$u'(x) = \int c'(x,y)\gamma_\varepsilon(x,y)\beta(y)dy. \tag{13}$$

Notice that $\gamma_\varepsilon'(x,y) = \frac{u'(x)-c'(x,y)}{\varepsilon}\gamma_\varepsilon(x,y)$. Thus by immediate recurrence (differentiating both sides of the equality again) we get that

$$u^{(n)}(x) = \int g_n(x,y)\gamma_\varepsilon(x,y)\beta(y)dy, \tag{14}$$

where $g_{n+1}(x,y) = g_n'(x,y) + \frac{u'(x)-c'(x,y)}{\varepsilon}g_n(x,y)$ and $g_1(x,y) = c'(x,y)$

To extend this first lemma to the d-dimensional case, we need to consider the sequence of indexes $\sigma = (\sigma_1, \sigma_2, \dots) \in \{1, \dots, d\}^{\mathbb{N}}$ which corresponds to the axis along which we successively differentiate. Using the same reasoning as above, it is straightforward to check that

$$\frac{\partial^k u}{\partial x_{\sigma_1} \dots \partial x_{\sigma_k}} = \int g_{\sigma,k}\gamma_\varepsilon$$

where $g_{\sigma,1} = \frac{\partial c}{\partial x_{\sigma_1}}$ and $g_{\sigma,k+1} = \frac{\partial g_{\sigma,k+1}}{\partial x_{\sigma_{k+1}}} + \frac{1}{\varepsilon}\left(\frac{\partial u}{\partial x_{\sigma_{k+1}}} - \frac{\partial c}{\partial x_{\sigma_{k+1}}}\right) g_{\sigma,k+1}$

□

Proof. (Lemma 2) The proof is made by recurrence on the following property :
P_n : *For all* $j = 0, \dots, k$, *for all* $k = 0, \dots, n-2$, $\left\| g_{n-k}^{(j)} \right\|_\infty$ *is bounded by a polynomial in* $\frac{1}{\varepsilon}$ *of order* $n-k+j-1$.
Let us initialize the recurrence with $n = 2$

$$\begin{aligned} g_2 &= g_1' + \frac{u'-c'}{\varepsilon}g_1 && (15)\\ \|g_2\|_\infty &\leqslant \|g_1'\|_\infty + \frac{\|u'\|_\infty + \|c'\|_\infty}{\varepsilon}\|g_1\|_\infty && (16) \end{aligned}$$

Recall that $\|u'\|_\infty = \|g_1\|_\infty = \|c'\|_\infty$. Let $C = \max_k \left\| c^{(k)} \right\|_\infty$, we get that $\|g_2\|_\infty \leqslant C + \frac{C+C}{\varepsilon}C$ which is of the required form.

Now assume that P_n is true for some $n \geqslant 2$. This means we have bounds on $g_{n-k}^{(i)}$, for $k = 0, \dots, n-2$ and $i = 0, \dots, k$. To prove the property at rank $n+1$ we want bounds on $g_{n+1-k}^{(i)}$, for $k = 0, \dots, n-1$ and $i = 0, \dots, k$. The only new quantity that we need to bound are $g_{n+1-k}^{(k)}, k = 0, \dots, n-1$. Let us start by bounding $g_2^{(n-1)}$ which corresponds to $k = n-1$ and we will do a backward recurrence on k. By applying Leibniz formula for the successive derivatives of a product of functions, we get

$$\begin{aligned} g_2 &= g_1' + \frac{u'-c'}{\varepsilon}g_1 && (17)\\ g_2^{(n-1)} &= g_1^{(n)} + \sum_{p=0}^{n-1}\binom{n-1}{p}\frac{u^{(p+1)} - c^{(p+1)}}{\varepsilon}g_1^{(n-1-p)} && (18)\\ \left\| g_2^{(n-1)} \right\|_\infty &\leqslant \left\| g_1^{(n)} \right\|_\infty + \sum_{p=0}^{n-1}\binom{n-1}{p}\frac{\left\| u^{(p+1)} \right\|_\infty + \left\| c^{(p+1)} \right\|_\infty}{\varepsilon}\left\| g_1^{(n-1-p)} \right\|_\infty && (19)\\ &\leqslant C + \sum_{p=0}^{n-1}\binom{n-1}{p}\frac{\|g_{p+1}\|_\infty + C}{\varepsilon}C && (20) \end{aligned}$$

Thanks to P_n we have that $\|g_p\|_\infty \leqslant \sum_{i=0}^{p} a_{i,p}\frac{1}{\varepsilon^i}, p = 1, \dots, n$ so the highest order term in ε in the above inequality is $\frac{1}{\varepsilon^n}$. Thus we get $\left\|g_2^{(n-1)}\right\|_\infty \leqslant \sum_{i=0}^{n+1} a_{i,2,n-1}\frac{1}{\varepsilon^i}$ which is of the expected order

Now assume $g_{n+1-j}^{(j)}$ are bounded with the appropriate polynomials for $j < k \leqslant n-1$. Let us bound $g_{n+1-k}^{(k)}$

$$\left\|g_{n+1-k}^{(k)}\right\|_\infty \leqslant \left\|g_{n-k}^{(k+1)}\right\|_\infty + \sum_{p=0}^{k}\binom{k}{p}\frac{\left\|u^{(p+1)}\right\|_\infty + \left\|c^{(p+1)}\right\|_\infty}{\varepsilon}\left\|g_{n-k}^{(k-p)}\right\|_\infty \tag{21}$$

$$\leqslant \left\|g_{n-k}^{(k+1)}\right\|_\infty + \sum_{p=0}^{k}\binom{k}{p}\frac{\|g_{p+1}\|_\infty + C}{\varepsilon}\left\|g_{n-k}^{(k-p)}\right\|_\infty \tag{22}$$

The first term $\left\|g_{n-k}^{(k+1)}\right\|_\infty$ is bounded with a polynomial of order $\frac{1}{\varepsilon^{n+1}}$ by recurrence assumption. Regarding the terms in the sum, they also have all been bounded and

$$\|g_{p+1}\|_\infty \left\|g_{n-k}^{(k-p)}\right\|_\infty \leqslant \left(\sum_{i=0}^{p} a_{i,p+1}\frac{1}{\varepsilon^i}\right)\left(\sum_{i=0}^{n-p} a_{i,n-k,k-p}\frac{1}{\varepsilon^i}\right) \leqslant \sum_{i=0}^{n}\tilde{a}_i\frac{1}{\varepsilon^i}$$

So $\left\|g_{n+1-k}^{(k)}\right\|_\infty \leqslant \sum_{i=0}^{n+1} a_{i,n+1-k,k}\frac{1}{\varepsilon^i}$

To extend the result in $\mathbb{R}^d$, the recurrence is made on the the following property

$$\left\|g_{\sigma,n-k}^{(j)}\right\|_\infty \leqslant \sum_{i=0}^{n-k+|j|-1} a_{i,n-k,j,\sigma}\frac{1}{\varepsilon^i} \qquad \forall j \mid |j| = 0, \dots, k \quad \forall k = 0, \dots, n-2 \quad \forall \sigma \in \{1, \dots, d\}^{\mathbb{N}} \tag{23}$$

where j is a multi-index since we are dealing with multi-variate functions, and $g_{\sigma,n-k}$ is defined at the end of the previous proof. The computations can be carried out in the same way as above, using the multivariate version of Leibniz formula in (18) since we are now dealing with multi-indexes. □