Model of perturbed ensemble

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1 Motion of rotlets

Say we have N point-like rotlets on the 2D plane. Each creates a rotating velocity field around it (hence the name) in which all the others move

$$\mathbf{v}_r(\mathbf{r}-\mathbf{r}') = \frac{\boldsymbol{w} \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1}$$

where $\mathbf{r}'$ is the position of the rotlet. $\boldsymbol{w}$ points along a third direction perpendicular to the 2D plane where the motion takes place and the vector product $\times$ works in the usual 3D way.

We can also add a separation-dependent repulsion term

$$\mathbf{v}_f(\mathbf{r}-\mathbf{r}') = f(|\mathbf{r}-\mathbf{r}'|)\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{2}$$

In such a system a few remarkable things happen [cite] . One such computationally observed phenomenon is the apparent attraction between two roughly circular ensembles of rotlets when they are far apart, despite the fact that all pairwise interactions are at least slightly repulsive [cite] Figure!!

distuinguish between the joining that happens for 0-repulsion ensembles and the long-range, smooth attraction

The rest of this document is devoted to explaining the attraction (or lack of repulsion for that matter). For this, we will need a simplified model of the ensembles.

2 Modelling the ensembles

Many body problems especially ones that exhibit some chaotic behaviour are difficult to study precisely. We need a way to make the problem yield to the techniques available to physicists. One way is to model the density of rotors as continuous instead of discrete. For this we need some spatial and temporal averaging. It is only then that we can say that the lone ensembles take circularly symmetric shapes

$$\rho = \rho(|\mathbf{r}|) \tag{3}$$

When in the presence of another ensemble, however, the shape may change. Naturally, this affects the interaction "forces" (they are really just velocities) between the ensembles. This is the key to the problem. Section 2.1 looks into how changing the shape of the ensemble changes the velocity field it creates. Then section 2.2 finds how the presence of another ensemble may affect its shape. Finally, I put this together to find an approximate interaction force between far-apart ensembles.

2.1 Diagonal moment

Say we have an ensemble of rotlets that exhibits the time-averaged density of rotlets $\rho(\mathbf{r})$ and is of finite extent. Say that the ensemble consists of N rotlets

$$N = \int d^2r\, \rho(\mathbf{r}) \tag{4}$$

Place the centre of mass (CoM) of the ensemble at the origin

$$0 = \int d^2r\, \mathbf{r}\rho(\mathbf{r}) \tag{5}$$

Let us calculate the average velocity that the rotation of the rotlets in the ensemble creates at some far away point $\mathbf{L}$. (a figure would help) This is

$$\mathbf{v}_r(\mathbf{L}) = \int d^2r\, \frac{\boldsymbol{w} \times (\mathbf{L}-\mathbf{r})}{|\mathbf{L}-\mathbf{r}|^3}\rho(\mathbf{r}) \tag{6}$$

Assuming that the extent of the ensemble is much less than $L = |\mathbf{L}|$ let us expand the integral to first order in r/L using the relation

$$|\mathbf{L}-\mathbf{r}|^{-3} = L^{-3}\Big(1 + 3\frac{\mathbf{L}\cdot\mathbf{r}}{L^2} + O((r/L)^2)\Big) \tag{7}$$

$$\mathbf{v}_r(\mathbf{L}) \approx L^{-3} \int d^2r \; \boldsymbol{w} \times (\mathbf{L} - \mathbf{r}) \Big(1 + 3 \frac{\mathbf{L} \cdot \mathbf{r}}{L^2} \Big) \rho(\mathbf{r}) \tag{8}$$

$$= \frac{\boldsymbol{w} \times \mathbf{L}}{L^3} \Big(N + 3L^{-2} \int d^2r \; (\mathbf{L} \cdot \mathbf{r}) \, \rho(\mathbf{r}) \Big) - 3L^{-2} \int d^2r \; \boldsymbol{w} \times \mathbf{r} \; (\mathbf{L} \cdot \mathbf{r}) \rho(\mathbf{r}) \tag{9}$$

Here I used equations 4 and 5. The term proportional to N is what we would predict if we simply assumed a point-like ensemble. This gives us the first deviations from this simplistic model of

(i) the angular velocity around the origin

$$w(\mathbf{L}) = \frac{|\mathbf{L} \times \mathbf{v}_r(\mathbf{L})|}{L^2} \approx \frac{w}{L^3} \Big(N + 3L^{-1} \int d^2r \, \rho(\mathbf{r}) \, r \cos\theta \Big); \tag{10}$$

(ii) the velocity along $\mathbf{L}$

$$v_{r\|}(\mathbf{L}) = \frac{\mathbf{v}_r(\mathbf{L})}{L} \approx 3 \frac{w}{L^4} \int d^2r \, \rho(\mathbf{r}) \, r^2 \cos\theta \, \sin\theta; \tag{11}$$

where θ is the angle between $\mathbf{L}$ and $\mathbf{r}$ (counting from $\mathbf{L}$ to $\mathbf{r}$). (11) can imply a velocity towards or away from the CoM, depending on the distribution, taking the maximal values when the rotlets lie as much as possible along a $\pm 45°$ diagonal. Hence I define the *diagonal moment*

$$D(\hat{\mathbf{L}}) \equiv \frac{3}{N} \int d^2r \, \rho(\mathbf{r}) \, r^2 \cos\theta \, \sin\theta \tag{12}$$

$$v_{r\|}(\mathbf{L}) \approx \frac{wN}{L^4} D(\hat{\mathbf{L}}) \tag{13}$$

The discrete version of (12) is

$$D = \frac{3}{N} \sum_i r_i^2 \cos\theta_i \sin\theta_i = \frac{3}{N} \sum_i x_i y_i \tag{14}$$

where $\hat{\mathbf{x}}$ points along $\hat{\mathbf{L}}$.

Now consider the effect that the repulsion has.

$$\mathbf{v}_f(\mathbf{L}) = \int d^2r f(|\mathbf{L} - \mathbf{r}|) \frac{\mathbf{L} - \mathbf{r}}{|\mathbf{L} - \mathbf{r}|} \rho(\mathbf{r}) \tag{15}$$

The component along $\mathbf{L}$ will itself have to compete with $v_{r\|}(\mathbf{L})$, so it is unnecessary to expand it in powers of r/L.

$$v_{f\|}(\mathbf{L}) \approx N f(L) \tag{16}$$

We arrive at an approximate expression for the average radial velocity that an ensemble creates at a faraway point

$$v_{\|}(\mathbf{L}) \approx N f(L) + \frac{wN}{L^4} D(\hat{\mathbf{L}}) \tag{17}$$

aaa

$$D_{\text{true}}/D_{\text{predicted}} = 12 \pm 1$$

2.2 Perturbed ensemble

The goal of this section is to obtain the diagonal moment that one ensemble may induce in another. First I approximate how the motion of the rotlets inside an ensemble may be affected. Then I see how that changes the density distribution $\rho(\mathbf{r})$ from which I obtain the diagonal moment.

2.2.1 Adjusted motion

Consider an ensemble of rotlets which is ordered or close enough to being ordered for all the constituents to be rotating roughly as a rigid body[1]. Then we know that each rotlet follows the velocity field

$$\begin{cases} \dot{x} = & -\omega y + \alpha x \\ \dot{y} = & +\omega x + \alpha y \end{cases} \tag{18}$$

where ω is the angular velocity of the ensemble (calculable as J/I) around its centre and α sets the rate of expansion of the ensemble.[2]

Now we introduce a perturbation from a far away rotlet, which creates a velocity field

$$\mathbf{v}(\mathbf{r}) = \frac{w}{r^2}\hat{\mathbf{e}}_\theta + f(r)\hat{\mathbf{e}}_r \tag{19}$$

where the origin $\mathbf{r} = 0$ is the position of the rotlet. (illustration needed) We want to find how this perturbation changes the shape of the ensemble.

The CoM of the ensemble then moves with the approximate velocity

$$\mathbf{v}(\mathbf{L}) = \frac{w}{L^2}\hat{\mathbf{e}}_y + f(L)\hat{\mathbf{e}}_x \tag{20}$$

where $\mathbf{L}$ is the instantaneous position of the CoM. This defines the basis $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y\} = \{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}\theta\}$ at $\mathbf{L}$. The true velocity of the CoM is not important here. It only matters how the velocity perturbation varies across the ensemble. In the frame where the CoM is stationary at the origin ($x = y = 0$), the perturbation of the velocity field up to linear order in distance from the centre of the ensemble is

$$\mathbf{v}(x,y) = \big(-2\frac{v}{L}\,\hat{\mathbf{e}}_y + f'(L)\,\hat{\mathbf{e}}_x\big)x + \big(-v\,\hat{\mathbf{e}}_x + f(L)\,\hat{\mathbf{e}}_y\big)\frac{y}{L} + \text{higher order terms} \tag{21}$$

where $v = w/L^2$.

Now we just add the perturbation of the velocity field to (18) to obtain our new equation of motion[3]

$$\dot{\mathbf{r}} = M\mathbf{r} \tag{22}$$

with

$$M = \begin{pmatrix} \alpha + f' & -\omega - v/L \\ +\omega - 2v/L & \alpha + f/L \end{pmatrix} \tag{23}$$

The resulting motion is along an exponentially growing ellipse (figure maybe) . To see this consider motion according to a matrix of the type in (18)

$$\bar{M} = \begin{pmatrix} \bar{\alpha} & -\bar{\omega} \\ \bar{\omega} & \bar{\alpha} \end{pmatrix} \tag{24}$$

Now do a stretch Q along the x-axis by a factor q followed by a rotation R by an angle ϕ and insist that we arrive at matrix M

$$M = RQ\bar{M}Q^{-1}R^{-1} = \begin{pmatrix} \bar{\alpha} + \bar{\omega}(q - q^{-1})\cos\phi\sin\phi & -\bar{\omega}(q\cos^2\phi + q^{-1}\sin^2\phi) \\ \bar{\omega}(q^{-1}\cos^2\phi + q\sin^2\phi) & \bar{\alpha} - \bar{\omega}(q - q^{-1})\cos\phi\sin\phi \end{pmatrix} \tag{25}$$

[1] If we lift this requirement ω would gain some radial dependence.

[2] I treat ω and α as constants despite the fact that they do change over time. You only need to assume they change very little over one rotation ($\dot{\omega} \ll \omega^2$ and $\dot{\alpha} \ll \alpha^2$) and the time it takes for the perturbation from the other ensemble to take effect.

[3] You may object to this saying that this will surely change the density distribution to something which is not radially symmetric anymore, at which point (18) becomes invalid. But the deviation from radial symmetry for sufficiently small (far away) perturbations will be proportional to the size of the perturbation(?) Hence the way it changes the self-motion of the ensemble must also be ... first order. hmm... Anyway we stick with this for now, it must be somewhat close to the truth, maybe off by some multiplication factors.

According to Wolfram Mathematica, this requires

$$\bar{\alpha} = \alpha + \frac{f' + f/L}{2} \tag{26}$$

$$\bar{\omega} = \left[\omega - \frac{v}{2L} - \sqrt{\left(\frac{f' - f/L}{2}\right)^2 + \left(\frac{3v}{2L}\right)^2}\,\right] q \tag{27}$$

$$q = \text{big expression} \tag{28}$$

$$\phi = \text{enormous expression that would be foolish to try and write here} \tag{29}$$

We can and should approximate these expressions keeping in mind that we are in the regime where

- repulsion is much weaker than the torque - $\omega \gg \alpha$; $v \gg f \,\&\, f'$;
- the perturbing droplet is far away - $\omega \gg v/L$; for the exponential force usually $f' > f/L$.

This achieves the approximations [4]

$$q \approx 1 + \frac{3v}{2L\omega} \tag{30}$$

$$\tan\phi \approx \phi \approx -\frac{f' - f/L}{6v/L} \tag{31}$$

Now we see why the repulsion is instrumental - without it there would only be elongation along x and hence no diagonal moment.

2.2.2 Adjusted density

As a sort of steady-state, I assume that the shape of the density isolines in the ensemble does not evolve in time. This is true if the isolines are the elliptical paths traced by

$$\dot{\mathbf{r}} = \mathfrak{M}\mathbf{r} \;;\; \mathfrak{M} = RQ\bar{\mathfrak{M}}Q^{-1}R^{-1} \;;\; \bar{\mathfrak{M}} = \begin{pmatrix} 0 & -\bar{\omega} \\ \bar{\omega} & 0 \end{pmatrix} \tag{32}$$

The only difference compared to (22) is that the repulsive expansion is omitted. To see this consider a radially symmetric density distribution $\rho(\mathbf{r}) = \bar{\rho}(r)$ moving circularly and then transforming it under Q and R. Then the density at all points is simply scaled by q, and the density isolines are elliptical. Adding the expansion does not change this fact. So we may write the density as

$$\rho(\mathbf{r}) = \bar{\rho}(|RQ\mathbf{r}|) \tag{33}$$

which is the form of a stretched radially symmetric density distribution.

2.2.3 Diagonal moment

This is what we are really after. As a reminder the definition of the diagonal moment is (12):

$$D = \frac{3}{N}\int d^2r\, \rho(\mathbf{r})\, r^2 \cos\theta\, \sin\theta$$

Conveniently, θ has the usual definition of angle in the x-y plane. Using the density parameterization (33)

$$D = \frac{3}{N}\int d^2r\, \bar{\rho}(|RQ\mathbf{r}|)\, r^2 \cos\theta\, \sin\theta = \frac{3}{N}\iint dx\, dy\, \bar{\rho}(|RQ\mathbf{r}|)\, xy \tag{34}$$

As you may expect, this integral is simpler in the unstretched coordinates where $\bar{\rho}$ is radially symmetric. So let us transform the coordinates under

$$\mathbf{r} \to \mathbf{r}' = Q^{-1}R^{-1}\mathbf{r} \;;\; Q^{-1}R^{-1} = \begin{pmatrix} q^{-1}\cos\phi & q^{-1}\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tag{35}$$

meaning that

$$d^2r = q\, d^2r' \;;\; \bar{\rho}(|RQ\mathbf{r}|) = \bar{\rho}(r') \;;\; xy = (q^2x'^2 - y'^2)\cos\phi\sin\phi + qx'y'(\cos^2\phi - \sin^2\phi)$$

[4] I'm really not sure about how correct these are, but they make sense.

$$D = \frac{3}{N} q \cos\phi \sin\phi \int d^2r' \; (q^2 x'^2 - y'^2)\, \bar{\rho}(r') \tag{36}$$

since for the now radially symmetric distribution $\int d^2r' \; x'y' \, \bar{\rho}(r') = 0$.

We can characterize this using the "moment of inertia"

$$\bar{I} \equiv \int d^2r' \, \bar{\rho}(r')\, r'^2 \text{ and } I = \int d^2r \, \rho(\mathbf{r})\, r^2 = \frac{q^2+1}{2} \bar{I} \tag{37}$$

because $r^2 = x^2 + y^2 = q^2 x'^2 + y'^2$ and $\int d^2r' \, \bar{\rho}(r')\, r'^2 = 2 \int d^2r' \, \bar{\rho}(r')\, x'^2 = \int d^2r' \, \bar{\rho}(r')\, y'^2$. Then

$$D = \frac{3\bar{I}}{2N} (q^2 - 1) \cos\phi \sin\phi \tag{38}$$

$$= \frac{3I}{N} \frac{q^2-1}{q^2+1} \cos\phi \sin\phi \tag{39}$$

With the approximations (30),(31) this becomes

$$D \approx -\frac{3I}{2N\omega} (f' - f/L) \tag{40}$$

from which we can obtain a radial force between two interacting ensembles!

2.2.4 New force law

Remember that the previous section considers the perturbation to be from just a single rotlet. Now let it be a whole ensemble of N_2 particles. Let the second ensemble be small enough and far enough away that we can assume its influence to be the same as that of a point-like ensemble at its CoM. Considering the finite size or the perturbed shape of it would lead to second order effects, which we are not doing here. With the assumptions already made in 2.1 it suffices to assert that both ensembles are comparable in size i.e.

$$\frac{I_1}{N_1} \sim \frac{I_2}{N_2} \tag{41}$$

using I/N as a measure for how spatially spread out an ensemble is.

So the only change to the diagonal moment is gaining a factor of N_2

$$D_1 \approx -\frac{3 I_1 N_2}{2 N_1 \omega_1} (f' - f/L) \tag{42}$$

By symmetry the first ensemble must perturb the second in an analogous way $(1 \leftrightarrow 2)$

$$D_2 \approx -\frac{3 I_2 N_1}{2 N_2 \omega_2} (f' - f/L) \tag{43}$$

Finally, using the long-distance assumption one last time, we plug this result into (17) as if the perturbed velocity field of one ensemble acts everywhere the same in the other. Again, assuming otherwise would dip into higher order contributions.

$$v_{\parallel 1} = N_2 f(L) - \frac{3 w I_1 N_2}{2 \omega_1 L^4} (f' - f/L) \tag{44}$$

and $(1 \leftrightarrow 2)$.

Newton's 3rd law should hold here, since all the basic pairwise interactions are symmetric. This means

$$N_1 v_{\parallel 1} = N_2 v_{\parallel 2} \tag{45}$$

which leads to

$$\frac{I_1}{\omega_1} = \frac{I_2}{\omega_2} \tag{46}$$

which is questionable

Finally, we get the rate of growth of the separation between the centres of the ensembles

$$\dot{L} = (N_1 + N_2) f(L) - \frac{3w}{2L^4} \Big[\frac{I_1 N_2}{\omega_1} + \frac{I_2 N_1}{\omega_2} \Big] (f' - f/L) \tag{47}$$

3 Predictions to be tested

aaa

$$\frac{D_{\text{true}}}{D_{\text{predicted}}} = 12 \pm 1$$

$$I = \sum_i r_i^2 \ ; \ \ J = \sum_i \mathbf{r}_i \times \dot{\mathbf{r}}_i \ ; \ \ \omega = J/I$$

$$\mathbf{v}(\mathbf{r}) = \frac{1}{r^2}\hat{\mathbf{e}}_\theta + f(r)\hat{\mathbf{e}}_r \tag{48}$$

$$\mathbf{v}(\mathbf{r}) = \frac{1}{r^2}\hat{\mathbf{e}}_\theta \tag{49}$$

$$f(r) = F \exp\left(-r/l\right)$$

$$\dot{\mathbf{r}}_i = \sum_j \frac{\hat{\boldsymbol{\theta}}_{ij}}{r_{ij}^2} + \sum_j f(r_{ij})\hat{\mathbf{r}}_{ij} \tag{50}$$

$$J^2 I = \text{const.}$$

$$\delta M = \begin{pmatrix} f/2 & -3/2L^3 \\ -3/2L^3 & -f/2 \end{pmatrix} \tag{51}$$

$$D_{\text{predicted}} = \frac{3If}{2N\omega}$$

$$D_{\text{empirical}} = \frac{9If}{10\omega}$$

$$\rho(t_2, r) = \gamma^2 \rho(t_1, \gamma r)$$

$$
\begin{aligned}
\mathbf{r}_i &= \sum_j \frac{\hat{\mathbf{e}}_z \times (\mathbf{r}_i - \mathbf{r}_j)}{|(\mathbf{r}_i - \mathbf{r}_j)|^3} \\
&= \sum_j \frac{-(y_i - y_j)\hat{\mathbf{e}}_x + (x_i - x_j)\hat{\mathbf{e}}_y}{|(\mathbf{r}_i - \mathbf{r}_j)|^3}
\end{aligned}
$$

$$
\begin{aligned}
J &= \sum_i \mathbf{r}_i \times \dot{\mathbf{r}}_i \\
&= \sum_{i,j} \frac{\mathbf{r}_i \times [-(y_i - y_j)\hat{\mathbf{e}}_x + (x_i - x_j)\hat{\mathbf{e}}_y]}{|(\mathbf{r}_i - \mathbf{r}_j)|^3} \\
&= \sum_{i,j} \frac{x_i(x_i - x_j) + y_i(y_i - y_j)}{|(\mathbf{r}_i - \mathbf{r}_j)|^3} \\
&= \frac{1}{2} \sum_{i,j} \frac{x_i(x_i - x_j) + y_i(y_i - y_j)}{|(\mathbf{r}_i - \mathbf{r}_j)|^3} + \frac{1}{2} \sum_{i,j} \frac{x_j(x_j - x_i) + y_j(y_j - y_i)}{|(\mathbf{r}_j - \mathbf{r}_i)|^3} \\
&= \frac{1}{2} \sum_{i,j} \frac{x_i^2 - 2x_i x_j + x_j^2 + y_j^2 - 2y_j y_i + y_i^2}{|(\mathbf{r}_i - \mathbf{r}_j)|^3} \\
&= \frac{1}{2} \sum_{i,j} \frac{(x_i - x_j)^2 + (y_i - y_j)^2}{|(\mathbf{r}_i - \mathbf{r}_j)|^3} \\
&= \frac{1}{2} \sum_{i,j} \frac{1}{|(\mathbf{r}_i - \mathbf{r}_j)|}
\end{aligned}
$$

Expansion of an ordered ensemble

My definition of the equations of motion:

$$\mathbf{v}_i = \sum_i \left(\frac{\hat{\mathbf{z}} \times \mathbf{r}_{ji}}{r_{ji}^3} + f(r_{ij})\mathbf{r}_{ji} \right)$$

where $\mathbf{r}_{ji} = \mathbf{r}_i - \mathbf{r}_j$ lie in the x-y plane and we have chosen to study $f(r) = f_0 \exp(-r/r_0)/r$.

Further I will be considering motion in the *co-rotating frame* in which the velocities are

$$\mathbf{u}_i = \mathbf{v}_i - \boldsymbol{\omega} \times \mathbf{r}_i$$

where

$$\boldsymbol{\omega} = \mathbf{J}/I; \quad \mathbf{J} = \sum_i \mathbf{v}_i \times \mathbf{r}_i = \sum_{i \neq j} \frac{\hat{\mathbf{z}}}{2r_{ij}}; \quad I = \sum_i r_i^2$$

and the centre of mass is placed at the origin.

Now the main assumption of my simplified model of the expansion of an **ordered** ensemble of rotlets:

The shape of the ensemble remains contant up to rescaling. In other words it expands visually just as you would zoom into an image. This is expressed by

$$\mathbf{u}_i \approx \alpha(R)\mathbf{r}_i$$

where R is a characteristic scale of the ensemble, which I here will define as the mean radius of the convex hull.[5] With this assumption the various characteristic scales, say the rms distance from the centre $I^{1/2}$ or the mean nearest-neighbour (n-n) separation $\bar{a}$, are all proportional to each other throughout the expansion.

This immediately leads to

$$I \propto R^2; \quad J \propto R^{-1}; \quad \implies IJ^2 = const.$$

which I tested last September.[6]

To actually work out the time dependence of the expansion, we need a further assumption. *The rotlets making up the convex hull are pushed outwards with velocity $\gamma a f(a)$ where a is the characteristic (n-n) separation between the rotlets of the convex hull and the next inner convex hull.* γ is a proportionality coefficient. I find this to be the shakiest assumption here, because the effective number of involved rotlets from the inner hull may change, and so would their relative importances and thus local effective a. I'll denote the proportionality constant $\beta = a/R$. With this

$$\dot{R} = \gamma a f(a) = \gamma f_0 \exp(-a/r_0) = \gamma f_0 \exp(-\beta R/r_0)$$

[5] Deviations from this assumption could make a nice parameter for measuring the amount of chaotic motion in an ensemble.

[6] I have yet to add the scaling with respect to N - for J it may be nontrivial.

Integrating with time (and yes, $R = 0, t = 0$ is well-defined, but may not align with your simulations' definition of $t = 0$.)

$$\int_0^{R(t)} dR' \exp(\beta R'/r_0) = \gamma f_0 \int_0^t dt'$$

$$\exp(\beta R(t)/r_0) - 1 = \frac{\beta\gamma f_0}{r_0} t$$

I don't know how to linearise this to fit nicely on a plot. The nicest plot that I can think of to test this model would be $\ln(\dot{R})$ *vs.* R.

$$\ln(\dot{R}) = \ln(\gamma f_0) - \beta R/r_0$$