

# 1 Linear Algebra

Relevant definitions used throughout Analysis II.

$$\mathbf{A} \in \mathbb{R}^{m \times n}, \quad x, y \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$$

**Def** **Scalar Product**  $x \cdot y := \sum_{i=0}^n (x_i \cdot y_i)$

**Def** **Euclidian Norm**  $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$

**Lem.** **Properties of  $\|x\|$**

- (i)  $\|x\| \geq 0$
- (ii)  $\|x\| \iff x = 0$
- (iii)  $\|\alpha x\| = |\alpha| \cdot \|x\|$
- (iv)  $\|x + y\| \leq \|x\| + \|y\|$  (Triangle Inequality)

**Def** **Definiteness**

Positive Definite  $\iff x^\top \mathbf{A} x > 0 \forall x \in \mathbb{R}_{\neq 0}^n$

Negative Definite  $\iff x^\top \mathbf{A} x < 0 \forall x \in \mathbb{R}_{\neq 0}^n$

If 0 is allowed,  $\mathbf{A}$  is called positive/negative semi-definite.

**Def** **Trace**  $\text{Tr}(\mathbf{A}) := \sum_{i=0}^{\min(m,n)} (\mathbf{A})_{i,i}$

**Lem.** **Inverse of  $\mathbf{A} \in \mathbb{R}^{2 \times 2}$**

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

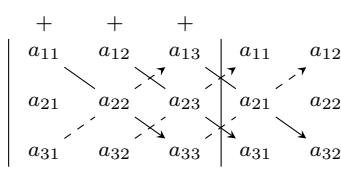
For  $A \in \mathbb{R}^{n \times n}$ : Gauss Algorithm.

**Lem.** **Determinant of  $\mathbf{A} \in \mathbb{R}^{2 \times 2}$**

$$\det(\mathbf{A}) = \det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$$

For  $A \in \mathbb{R}^{n \times n}$ : Cofactor method.

**Lem.** **Determinant of  $A \in \mathbb{R}^{3 \times 3}$  (Sarrus)**



**Lem.** **Properties of Eigenvalues**

$$\text{Tr}(\mathbf{A}) = \sum_{i=0}^n \lambda_i \quad \det(\mathbf{A}) = \prod_{i=0}^n \lambda_i$$

To find  $\lambda_i$  solve  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ .

# 2 Differential Equations

**Def** **Differential Equation (DE)**

Equation relating unknown  $f$  to derivatives  $f^{(i)}$  at same  $x$ .

**Def** **Ordinary Differential Equation (ODE)**

DE s.t.  $f : I \rightarrow \mathbb{R}$  is in one variable.

**Def** **Partial Differential Equation (PDE)**

DE s.t.  $f : I^d \rightarrow \mathbb{R}$  is in multiple variables.

**Notation**  $f^{(i)}$  or  $y^{(i)}$  instead of  $f^{(i)}(x)$  for brevity.

**Def** **Order**  $\text{ord}(F) := \max_{i \geq 0} \{i \mid f^{(i)} \in F, f^{(i)} \neq 0\}$

**Remark** Any  $F$  s.t.  $\text{ord}(F) \geq 2$  can be reduced to  $\text{ord}(F') = 1$ , but using functions of higher dimensions.

## Solutions to ODEs

$\forall F : \mathbb{R}^2 \rightarrow \mathbb{R}$  s.t.  $F$  is cont. diff. and  $x_0, y_0 \in \mathbb{R}$ :

$$\exists f : I \rightarrow \mathbb{R}$$

s.t.  $\forall x \in I : f'(x) = F(x, f(x))$  and  $f(x_0) = y_0$

s.t.  $I$  is open and maximal.

Intuition: Solutions always exist (locally!) for nice enough equations.

## 2.1 Linear Differential Equations

**Def** **Linear Differential Equation (LDE)**

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

$I \subset \mathbb{R}$  is open,  $k \geq 1$ ,  $\forall i < k : a_i : I \rightarrow \mathbb{C}$

**Def** **Homogeneity of LDEs**

**Homogeneous**  $\iff b = 0$

**Inhomogeneous**  $\iff b \neq 0$

**Remark**  $D(y) := y^{(k)} + \dots + a_0y$  is a linear operation:

$$D(z_1f_1 + z_2f_2) = z_1D(f_1) + z_2D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}, f_1, f_2$   $k$ -times differentiable

**Def** **Homogeneous Solution Space**

$\mathcal{S}(F) := \{f : I \rightarrow \mathbb{C} \mid f \text{ solves } F, f \text{ is } k\text{-times diff.}\}$

**Remark**  $\mathcal{S}(F)$  is the Nullspace of a lin. map:  $f$  to  $D(f)$ :

$$D(f) = z_1D(f_1) + z_2D(f_2) = 0$$

$$\forall z_1, z_2 \in \mathbb{C}, f_1, f_2 \in \mathcal{S}$$

## Solutions for complex homogeneous LDEs

$F$  s.t.  $a_0, \dots, a_{k-1}$  continuous and complex-valued

1.  $\mathcal{S}$  is a complex vector space,  $\dim(\mathcal{S}) = k$

2.  $\mathcal{S}$  is a subspace of  $\{f \mid f : I \rightarrow \mathbb{C}\}$

3.  $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$  a unique sol. exists

## Solutions for real homogeneous LDEs

$F$  s.t.  $a_0, \dots, a_{k-1}$  continuous and real-valued

1.  $\mathcal{S}$  is a real vector space,  $\dim(\mathcal{S}) = k$

2.  $\mathcal{S}$  is a subspace of  $\{f \mid f : I \rightarrow \mathbb{R}\}$

3.  $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{R}^k$  a unique sol. exists

**Def** **Inhomogeneous Solution Space**

$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F), f_0 \text{ is a particular sol.}\}$

Note: This is only a vector space if  $b = 0$ , where  $\mathcal{S}_b = \mathcal{S}$ .

## Solutions for real inhomogeneous LDEs

$F$  s.t.  $a_0, \dots, a_{k-1}$  continuous,  $b : I \rightarrow \mathbb{C}$

1.  $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$  a unique sol. exists

2. If  $b, a_i$  are real-valued, a real-valued sol. exists.

**Remark** **Applications of Linearity**

If  $f_1$  solves  $F$  for  $b_1$ , and  $f_2$  for  $b_2$ :  $f_1 + f_2$  solves  $b_1 + b_2$ .

Follows from:  $D(f_1) + D(f_2) = b_1 + b_2$ .

### 3 Solutions to Differential Equations

#### 3.1 Linear Solutions: First Order

**Form:**  $y' + ay = b \quad I \subset \mathbb{R}, \quad a, b : I \rightarrow \mathbb{R}$

**Approach:**

1. Hom. Solution  $f_1$  for:  $y' + ay = 0$

Note that  $\mathcal{S}$  has  $\dim(\mathcal{S}) = 1$ , so  $f_1 \neq 0$  is a Basis for  $\mathcal{S}$

2. Part. Solution  $f_0$  for  $y' + ay = b$

**Solutions:**  $f_0 + zf_1 \quad \text{for } z \in \mathbb{C}$

##### Explicit Homogeneous Solution

$A(x)$  is a primitive of  $a$ ,  $f(x_0) = y_0$

$$f_1(x) = z \cdot \exp(-A(x))$$

$$f_1(x) = y_0 \cdot \exp(A(x_0) - a(x))$$

**Method Variation of Constants:** Treating  $z$  as  $z(x)$  yields:

##### Explicit Inhomogeneous Solution

$A(x)$  is a primitive of  $a$

$$f_0(x) = \underbrace{\left( \int b(x) \cdot \exp(A(x)) \right)}_{z(x)} \cdot \exp(-A(x))$$

##### Method Educated Guess

Usually,  $y$  has a similar form to  $b$ :

$b(x)$	Guess
$a \cdot e^{\alpha x}$	$b \cdot e^{\alpha x}$
$a \cdot \sin(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$b \cdot \cos(\beta x)$	$c \sin(\beta x) + d \cos(\beta x)$
$a e^{\alpha x} \cdot \sin(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$b e^{\alpha x} \cdot \cos(\beta x)$	$e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$
$P_n(x) \cdot e^{\alpha x}$	$R_n(x) \cdot e^{\alpha x}$
$P_n(x) \cdot e^{\alpha x} \sin(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$
$P_n(x) \cdot e^{\alpha x} \cos(\beta x)$	$e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$

**Remark** If  $\alpha, \beta$  are roots of  $P(X)$  with multiplicity  $j$ , multiply guess with a  $P_j(x)$ .

#### 3.2 Linear Solutions: Constant Coefficients

**Form:**  $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$

Where  $a_0, \dots, a_{k-1} \in \mathbb{C}$  are constants,  $b(x)$  is continuous.

##### 3.2.1 Homogeneous Equations

The idea is to find a Basis of  $\mathcal{S}$ :

**Def Characteristic Polynomial**  $P(X) = \prod_{i=1}^k (X - \alpha_i)$

**Remark** The unique roots  $\alpha_1, \dots, \alpha_l$  form a Basis:

$$\text{span}(\mathcal{S}) = \{x^j e^{\alpha_i x} \mid i \leq l, \quad 0 \leq j \leq v_i\}$$

$v_1, \dots, v_k$  are the Multiplicities of  $\alpha_1, \dots, \alpha_k$

**Remark** If  $\alpha_j = \beta + \gamma i \in \mathbb{C}$  is a root,  $\bar{\alpha}_j = \beta - \gamma i$  is too.

If only solutions in  $\mathbb{R}$  are considered, this can be rewritten:

$$c_1 e^{\alpha_j x} + c_2 e^{\alpha_i x} = e^{\beta x} (c_1 \cos(\gamma x) + c_2 \sin(\gamma x))$$

##### Explicit Homogeneous Solution

Using  $\alpha_1, \dots, \alpha_k$  from  $P(X)$  s.t.  $\alpha_i \neq \alpha_j$ ,  $z_i \in \mathbb{C}$  arbitrary

$$f(x) = \prod_{i=1}^k z_i \cdot e^{\alpha_i x} \quad \text{with} \quad f^{(j)(x)} = \prod_{i=1}^k z_i \cdot \alpha_i^j e^{\alpha_i x}$$

Multiple roots: same scheme, using the basis vectors of  $\mathcal{S}$

Solutions exist  $\forall Z = (z_1, \dots, z_k)$  since that system's  $\det(M_Z) \neq 0$ .

##### 3.2.2 Inhomogeneous Equations

**Method Undetermined Coefficients:** An educated guess.

1.  $b(x) = cx^d \cdot e^{\alpha x} \implies f_p(x) = Q(x)e^{\alpha x}$   
 $\deg(Q) \leq d + v_\alpha$ , where  $v_\alpha$  is  $\alpha$ 's multiplicity in  $P(X)$

2.  $b(x) = cx^d \cdot \cos(\alpha x)$   
 $b(x) = cx^d \cdot \sin(\alpha x)$   
 $\deg(Q_{1,2}) \leq d + v_\alpha$ , where  $v_\alpha$  is  $\alpha$ 's multiplicity in  $P(X)$

**Remark Applying Linearity**

If  $b(x) = \sum_{i=1}^n b_i(x)$ , A solution for  $b(x)$  is  $f(x) = \sum_{i=1}^n f_i(x)$ .  
Sometimes called *Superposition Principle* in this context.

#### 3.3 Other Methods

**Method Change of Variable**

If  $f(x)$  is replaced by  $h(y) = f(g(y))$ , then  $h$  is a sol. too.  
Changes like  $h(t) = f(e^t)$  may lead to, i.e. ODEs in constant coeffs

**Example:**  $2xy' - y = 0$

Using substitution:  $x = e^t, \quad h(t) = y(e^t), \quad h'(t) = e^t \cdot y'(e^t)$

$$1. \quad 2x \cdot y'(x) = 2 \cdot h'(t)$$

$$2. \quad -y(x) = -h(t)$$

So:  $2xy' - y \stackrel{\text{sub}}{=} 2h'(t) - h(t) = 0$

Yields:  $h(t) = \alpha \cdot e^{\frac{t}{2}}$   $\stackrel{\text{resub}}{\implies} y(x) = \alpha \cdot e^{\frac{\ln(x)}{2}} = \alpha \cdot \sqrt{x}$

##### Separation of Variables

**Form:**  $y' = a(y) \cdot b(x)$

$$\int \frac{1}{a(y)} dy = \int b(x) dx + c$$

Usually  $\int 1/a(y) dy$  can be solved directly for  $\ln|a(y)| + c$ .

#### 3.4 Method Overview

Method	Use case
Variation of constants	LDE with $\text{ord}(F) = 1$
Characteristic Polynomial	Hom. LDE w/ const. coeff.
Undetermined Coefficients	Inhom. LDE w/ const. coeff.
Separation of Variables	ODE s.t. $y' = a(y) \cdot b(x)$
Change of Variables	e.g. $y' = f(ax + by + c)$

## 4 Continuous functions in $\mathbb{R}^n$

Treating functions  $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}/\mathbb{C}/\mathbb{R}^m$ ,  $m, n \geq 1$

**Notation**  $f(x)$  for  $f : I \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  means:  
 $x = (x_1, \dots, x_n)$ ,  $f(x) = f(f_1(x), \dots, f_m(x))$

### 4.1 Multivariate functions

**Def Linear map**  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

In other words:  $f(x) = \mathbf{A}x$ ,  $\mathbf{A} \in \mathbb{C}^{m \times n}$

Linear Maps are continuous

**Def Affine Linear map**  $f(x) \mapsto \mathbf{A}x + c$

**Def Quadratic form**  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$

In other words:  $Q(x) = \sum_{i=0}^n \sum_{j=0}^m (a_{i,j} x_i x_j)$

**Def Monomials**  $M(x) : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto \alpha x_1^{d_1} \cdots x_n^{d_n}$

For example:  $f(x, y, z) = 16x^2yz^5$

**Def**  $\deg(M) := e = \sum_{i=1}^n d_i$

For example:  $\deg(16x^2yz^5) = 8$

**Def Polynomials**  $P(x) := \sum_{i=0}^n M_i(x)$

For example:  $P(x, y, z) = x^3 + 25x^2y^6z + xy$

Polynomials are continuous.

**Def**  $\deg(P) := d \geq \max\{\deg(M_i) \mid M_i \text{ in } P\}$

For example:  $\deg(x^3 + 25x^2y^6z + xy) = 9$

Visualisations for some function types:

**Def Graph**  $G_f := \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$

Only for  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Visually, this is a surface in  $\mathbb{R}^3$

**Def Vector Plots** for  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Points in  $(x, y) \in \mathbb{R}^2$  are displayed as vectors  $f(x, y)$

## 4.2 Sequences in $\mathbb{R}^n$

**Def Sequences in  $\mathbb{R}^n$**

$(x_k)_{k \geq 1}$  s.t.  $x_k \in \mathbb{R}^n$  where  $x_k = (x_{k,1}, \dots, x_{k,n})$

**Def Convergence in  $\mathbb{R}^n$**

$$\lim_{k \rightarrow \infty} (x_k) = y \iff \forall \epsilon > 0, \exists N \geq 1 : \forall k \geq N : \|x_k - y\| < \epsilon$$

Using this definition preserves many familiar results:

**Lem. Equivalent conditions to Convergence**

- (i)  $\forall i \text{ s.t. } 1 \leq i \leq n : \lim_{k \rightarrow \infty} (x_{k,i}) = y_i$
- (ii)  $\lim_{k \rightarrow \infty} \|x_k - y\| = 0$

**Def Limits at points**

$$\lim_{x \neq x_0 \rightarrow x_0} (f(x)) = y \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 : \forall x \neq x_0 \in X : \|x - x_0\| < \delta \implies \|f(x) - y\| < \epsilon$$

$X \subset \mathbb{R}^n$ ,  $f : X \rightarrow \mathbb{R}^m$ ,  $x_0 \in X$ ,  $y \in \mathbb{R}^m$

The sequence test for Continuity works for point-limits too.

## 4.3 Continuity in $\mathbb{R}^n$

**Def Continuity in  $\mathbb{R}^n$**

$f$  continuous at  $x_0 \in X \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$

$$\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$$

$f$  continuous  $\stackrel{\text{def}}{\iff} \forall x \in X : f$  continuous at  $x$

$X \subset \mathbb{R}^n$ ,  $f : X \rightarrow \mathbb{R}^m$

**Lem. Continuity using Sequences**

$f$  continuous at  $x_0$  if and only if:

$$\forall (x_k)_{k \geq 1} : \lim_{k \rightarrow \infty} (x_k) = x_0 \implies \lim_{k \rightarrow \infty} (f(x_k)) = f(x_0)$$

$X \subset \mathbb{R}^n$ ,  $f : X \rightarrow \mathbb{R}^m$

**Lem. Continuity of Compositions**

$f : X \rightarrow Y$ ,  $g : Y \rightarrow \mathbb{R}^p$  continuous  $\implies g \circ f$  continuous

$X \subset \mathbb{R}^n$ ,  $Y \subset \mathbb{R}^m$ ,  $p \geq 1$

**Lem. Continuity using Coordinate Functions**

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  continuous  $\iff \forall i \leq m : f_i$  continuous

## 4.4 Subsets of $\mathbb{R}^n$

**Def Bounded**

$X \subset \mathbb{R}^n$  bounded  $\stackrel{\text{def}}{\iff} \left\{ \|x\| \mid x \in X \right\} \subset \mathbb{R}$  bounded.

Example: The open disc  $D = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$  is bounded.

**Def Closed**

$X \subset \mathbb{R}^n$  closed  $\stackrel{\text{def}}{\iff} \forall (x_k)_{k \geq 1} \in X : \lim_{k \rightarrow \infty} (x_k) \in X$

Example:  $\emptyset, \mathbb{R}^n$  are closed.

**Def Compact** if closed and bounded.

Example: The closed Disc  $\Lambda = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$  is compact.

**Def Open**

$X \subset \mathbb{R}^n$  open  $\stackrel{\text{def}}{\iff} \forall x \in X, \exists \delta > 0 :$

$$\{y \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \forall i \leq n\} \subset X$$

In other words: Changing any coord.  $x_i$  by  $\delta$  keeps  $x'$  in  $X$

Example:  $\emptyset, \mathbb{R}^n$  are open (and closed)

**Lem.** The Cartesian Product preserves bounded/closed.

**Lem.** Continuous functions preserve closed/open

$\forall$  closed/open  $Y :$

$$f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \text{ is closed/open.}$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous,  $Y \subset \mathbb{R}^m$

Note:  $X$  open/closed, does not imply  $f(X)$  open/closed

**Lem.** The complement of open sets is closed

$X \subset \mathbb{R}^n$  is open  $\iff \underbrace{\{x \in \mathbb{R}^n \mid x \notin X\}}_{\text{Complement}}$  is closed

### Min-Max Theorem

For compact, non-empty  $X \subset \mathbb{R}^n$ , continuous  $f : X \rightarrow \mathbb{R}$ :

$$\exists x_1, x_2 \in X : f(x_1) = \sup_{x \in X} f(x), f(x_2) = \inf_{x \in X} f(x)$$

# 5 Differential Calculus in $\mathbb{R}^n$

## 5.1 Partial Derivatives

### Partial Derivative

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}$ ,  $1 \leq i \leq n$ ,  $x_0 \in X$

$$\frac{\partial f}{\partial x_i}(x_0) := g'(x_{0,i})$$

for  $g : \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, \dots, x_{0,n}) \in X\} \rightarrow \mathbb{R}^n$

$$g(t) := \underbrace{f(x_{0,1}, \dots, x_{0,t-1}, t, x_{0,t+1}, \dots, x_{0,n})}_{\text{Freeze all } x_{0,k} \text{ except one } x_{0,i} \rightarrow t}$$

Notation  $\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$

### Lem. Properties of Partial Derivatives

Assuming  $\partial_{x_i} f$  and  $\partial_{x_i} g$  exist :

$$(i) \quad \partial_{x_i}(f + g) = \partial_{x_i} f + \partial_{x_i} g$$

$$(ii) \quad \partial_{x_i}(fg) = \partial_{x_i}(f)g + \partial_{x_i}(g)f \quad \text{if } m = 1$$

$$(iii) \quad \partial_{x_i}\left(\frac{f}{g}\right) = \frac{\partial_{x_i}(f)g - \partial_{x_i}(g)f}{g^2} \quad \text{if } g(x) \neq 0 \forall x \in X$$

$X \subset \mathbb{R}^n$  open,  $f, g : X \rightarrow \mathbb{R}^n$ ,  $1 \leq i \leq n$

### The Jacobian

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^n$  with partial derivatives existing

$$\mathbf{J}_f(x) := \begin{bmatrix} \partial_{x_1}f_1(x) & \partial_{x_2}f_1(x) & \cdots & \partial_{x_n}f_1(x) \\ \partial_{x_1}f_2(x) & \partial_{x_2}f_2(x) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1}f_m(x) & \partial_{x_2}f_m(x) & \cdots & \partial_{x_n}f_m(x) \end{bmatrix}$$

Think of  $f$  as a vector of  $f_i$ , then  $\mathbf{J}_f$  is that vector stretched for all  $x_j$

$$\text{Def Gradient } \nabla f(x_0) := \begin{bmatrix} \partial_{x_1}f(x_0) \\ \vdots \\ \partial_{x_n}f(x_0) \end{bmatrix} = \mathbf{J}_f(x)^\top$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}$ , i.e. must map to 1 dimension

**Remark**  $\nabla f$  points in the direction of greatest increase.

This generalizes that in  $\mathbb{R}$ ,  $\text{sgn}(f)$  shows if  $f$  increases/decreases

**Def Divergence**  $\text{div}(f)(x_0) := \text{Tr}(\mathbf{J}_f(x_0))$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^n$ ,  $\mathbf{J}_f$  exists

## 5.2 The Differential

Partial derivatives don't provide a good approx. of  $f$ , unlike in the 1-dimensional case. The *differential* is a linear map which replicates this purpose in  $\mathbb{R}^n$ .

### Differentiability in $\mathbb{R}^n$ & the Differential

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^n$ ,  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear map

$$df(x_0) := u$$

If  $f$  is differentiable at  $x_0 \in X$  with  $u$  s.t.

$$\lim_{x \neq x_0 \rightarrow x_0} \frac{1}{\|x - x_0\|} \left( f(x) - f(x_0) - u(x - x_0) \right) = 0$$

Applying this directly can be useful for strange functions like  $|xy|$ .

### Lem. Properties of Differentiable Functions

(i) Continuous on  $X$

(ii)  $\forall i \leq m, j \leq n : \partial_{x_j} f_i$  exists

(iii)  $m = 1 : \partial_{x_i} f(x_0) = a_i$

for:  $u(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$  differentiable on  $X$

### Lem. Preservation of Differentiability

(i)  $f + g$  is differentiable:  $d(f + g) = df + dg$

(ii)  $fg$  is differentiable, if  $m = 1$

(iii)  $\frac{f}{g}$  is differentiable, if  $m = 1$ ,  $g(x) \neq 0 \forall x \in X$

$X \subset \mathbb{R}^n$  open,  $f, g : X \rightarrow \mathbb{R}^m$  differentiable on  $X$

### Lem. Cont. Partial Derivatives imply Differentiability

if all  $\partial_{x_j} f_i$  exist and are continuous:

$f$  differentiable on  $X$ ,  $df(x_0) = \mathbf{J}_f(x_0)$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$

**Lem. Chain Rule**  $g \circ f$  is differentiable on  $X$

$$\begin{aligned} d(f \circ g)(x_0) &= df(g(x_0)) \circ dg(x_0) \\ J_{f \circ g}(x_0) &= J_f(g(x_0)) \cdot J_g(x_0) \end{aligned}$$

$X \subset \mathbb{R}^n$  open,  $Y \subset \mathbb{R}^m$  open,  $f : X \rightarrow Y, g : Y \rightarrow \mathbb{R}^p$ ,  $f, g$  diff.-able  
 $f \circ g$  meaning  $f(g(x))$ .

**Example:** Let  $\nabla f(-\frac{\sqrt{3}}{2}, \frac{3}{2}, 7) = (6, 2, 0)$ .

We want  $\partial_r f(\sqrt{3}, \frac{2}{3}\pi, 7)$  in cyl. coords, i.e. the 1st entry of  $J_{f \circ g}$  with:

$$g(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z)$$

Applying the Chain Rule:

$$J_{f \circ g} = J_f(g(x_0)) \cdot J_g(x_0) = J_f(g(\sqrt{3}, \frac{3}{2}\pi, 7)) \cdot J_g$$

Where  $J_g$  is the Jacobian for cylindrical coordinates.

### Def Tangent Space

$$T_f(x_0) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = f(x_0) + u(x - x_0)\}$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$  diff.-able,  $x_0 \in X$ ,  $u = df(x_0)$

**Example:** Find  $T_f$  for  $x_0 = (1, 1, 1)$  for  $S = \{(x, y, e^{x-y}) \mid (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ .

$$u = df(x_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ e^{x-y} & -e^{x-y} \end{bmatrix}(x_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

So we can define  $T_f(x_0)$  as  $g(x, y) = \underbrace{f(x_0)}_{(1,1,1)} + J_f(x_0) \cdot \begin{bmatrix} x - x_0, 1 \\ y - y_0, 2 \end{bmatrix}$ .

### Def Directional Derivative

$$D_v f(x_0) = \lim_{t \neq 0 \rightarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$ ,  $v \neq 0 \in \mathbb{R}^n$ ,  $x_0 \in X$

### Lem. Directional Derivatives for Diff.-able Functions

$$D_v f(x_0) = df(x_0)(v) = \mathbf{J}_f(x_0) \cdot v$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$  diff.-able,  $v \neq 0 \in \mathbb{R}^n$ ,  $x_0 \in X$

**Remark**  $D_v f$  is linear w.r.t  $v$ , so:  $D_{v_1+v_2} f = D_{v_1} f + D_{v_2} f$

**Remark**  $D_v f(x_0) = \nabla f(x_0) \cdot v = \|\nabla f(x_0)\| \cos(\theta)$

In the case  $f : X \rightarrow \mathbb{R}$ , where  $\theta$  is the angle between  $v$  and  $\nabla f(x_0)$

## 5.3 Higher Derivatives

### Def Differentiability Classes

- $f \in C^1(X; \mathbb{R}^m) \iff f \text{ diff.-able on } X, \text{ all } \partial_{x_j} f_i \text{ exist}$
- $f \in C^k(X; \mathbb{R}^m) \iff f \text{ diff.-able on } X, \text{ all } \partial_{x_j} f_i \in C^{k-1}$
- $f \in C^\infty(X; \mathbb{R}^m) \iff f \in C^k(X; \mathbb{R}^m) \forall k \geq 1$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$

**Lem.** Polynomials, Trig. functions and exp are in  $C^\infty$

**Lem.** Operations preserve Differentiability Classes

- (i)  $f + g \in C^k$
- (ii)  $fg \in C^k \text{ if } m = 1$
- (iii)  $\frac{f}{g} \in C^k \text{ if } m = 1, g(x) \neq 0 \forall x \in X$   
 $f, g \in C^k$

**Lem.** Composition preserves Differentiability Classes

$$g \circ f \in C^k$$

$f \in C^k, f(X) \subset Y, Y \subset \mathbb{R}^m$  open,  $g : Y \rightarrow \mathbb{R}^p, g \in C^k$

### Partial Derivatives commute in $C^k$

$k \geq 2, X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m, f \in C^k$

$$\forall x, y : \partial_{x,y} f = \partial_{y,x} f$$

This generalizes for  $\partial_{x_1, \dots, x_n} f$  if  $k \geq n$ .

**Remark** Linearity of Partial Derivatives

$$\partial_x^m (af_1 + bf_2) = a\partial_x^m f_1 + b\partial_x^m f_2$$

Assuming both  $\partial_x f_{1,2}$  exist.

### Def Laplace Operator

$$\Delta f := \operatorname{div}\left(\nabla f(x)\right) = \sum_{i=0}^n \frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_i} \right) = \sum_{i=0}^n \frac{\partial^2 f}{\partial x_i^2}$$

### The Hessian

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}, f \in C^2, x_0 \in X$

$$\mathbf{H}_f(x) := \begin{bmatrix} \partial_{1,1}f(x_0) & \partial_{2,1}f(x_0) & \cdots & \partial_{n,1}f(x_0) \\ \partial_{1,2}f(x_0) & \partial_{2,2}f(x_0) & \cdots & \partial_{n,2}f(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1,n}f(x_0) & \partial_{2,n}f(x_0) & \cdots & \partial_{n,n}f(x_0) \end{bmatrix}$$

Where  $(\mathbf{H}_f(x))_{i,j} = \partial_{x_i, x_j} f(x)$

Note that  $f : X \rightarrow \mathbb{R}$ , i.e.  $\mathbf{H}_f$  only exists for 1-dimensionally valued  $f$

**Notation**  $\mathbf{H}_f(x) = \operatorname{Hess}_f(x) = \nabla^2 f(x)$

**Remark**  $\mathbf{H}_f(x_0)$  is symmetric:  $(\mathbf{H}_f(x_0))_{i,j} = (\mathbf{H}_f(x_0))_{j,i}$

### Def Polar Coordinates

$$g(r, \theta) = (r \cos(\theta), r \sin(\theta))$$

$$\mathbf{J}_g(r, \theta) = \begin{bmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{bmatrix}$$

$$\partial_x f = \cos(\theta) \partial_r f - \frac{1}{r} \sin(\theta) \partial_\theta f$$

$$\partial_y f = \sin(\theta) \partial_r f + \frac{1}{r} \cos(\theta) \partial_\theta f$$

$(r, \theta) \in (0, +\infty) \times \mathbb{R}, \det(\mathbf{J}_g) = r$

## 5.4 Taylor Polynomials

**Def**  $|m| := \sum_{i=1}^n m_i$

**Def**  $y^m := y_1^{m_1} \cdots y_n^{m_n}$

**Def**  $m! := m_1! \cdots m_n!$

for  $m = (m_1, \dots, m_n), y = (y_1, \dots, y_n)$

### Taylor Polynomials

$k \geq 1, f : X \rightarrow \mathbb{R}, f \in C^k, x_0 \in X$

$$T_k f(y; x_0) := \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m$$

### Lem. Taylor Approximation

$$\lim_{x \neq x_0 \rightarrow x_0} \frac{E_k f(x; x_0)}{\|x - x_0\|^k} = 0$$

Where  $f(x) = T_k f(x - x_0; x_0) + E_k f(x; x_0)$   
 $k \geq 1, X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}, f \in C^k, x_0 \in X$

**Remark** Taylor polynomials of degree 1, 2:

$$T_1 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y$$

$$T_2 f(y; x_0) = f(x_0) + \nabla f(x_0) \cdot y + \frac{1}{2} (y^\top \cdot \mathbf{H}_f(x_0) \cdot y)$$

**Method** Calculating  $T_k f(y; x_0)$  also yields  $\mathbf{H}_f$  for  $k \geq 2$ .

$$T_2 f((x_0, y_0); (x, y)) = \dots + ax^2 + by^2 + cxy$$

$$\implies \mathbf{H}_f(x_0, y_0) = \begin{bmatrix} 2a & c \\ c & 2b \end{bmatrix}$$

**Method** Taylor Polynomials can be found by combination.

$$\text{Example: } f(x, y) = \underbrace{e^{y^4}}_1 + \underbrace{\sin(xy)}_2 + \underbrace{2xy^2}_3 - \underbrace{\ln(x^2 + 1)}_4, k = 3$$

$$1. e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \implies e^{y^4} \approx 1 + y^4 + \frac{y^8}{2} + \frac{y^{12}}{6}$$

Since  $k = 3$ , discarding all terms with  $\deg > 3$  yields:  $e^{y^3} \approx 1$

$$2. \sin(x) \approx x - \frac{x^3}{6} \implies \sin(xy) \approx xy$$

$$3. 2xy^2 \approx 2xy^2 \quad (\text{Since it's already a polynomial, } \deg = 3)$$

$$4. \ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \implies \ln(x^2 + 1) \approx x^2$$

$$\text{Thus: } f(x) \approx 1 + xy + 2xy^2 - x^2 = T_3 f((0, 0); (x, y))$$

## 5.5 Critical Points

### Lem. Local Maxima & Minima

$$\left. \begin{array}{l} f(y) \leq f(x_0) \forall y \text{ close} \\ f(y) \geq f(x_0) \forall y \text{ close} \end{array} \right\} \quad \frac{\partial f}{\partial x_i}(x_0) = 0 \quad \forall i \leq n$$

In other words:  $df(x_0) = \nabla f(x_0) = 0$

$f : X \rightarrow \mathbb{R}$ ,  $X \subset \mathbb{R}^n$  open,  $f$  diff.-able

### Def Critical Point

$$x_0 \in X \text{ is critical} \iff \nabla f(x_0) = 0$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}$  diff.-able

### Remark Existence of Maxima/Minima

Don't have to exist if  $X$  is open, only if  $X$  is compact.

However, for compact sets, the lemma above no longer applies.

### Method Critical points on Compact Sets

Decompose  $X = X' \cup B$ , s.t.  $X'$  (Interior) is open,  $B$  is a boundary.

1. Find critical points in  $X'$ : via  $\nabla f$ , check state using  $H_f$
2. Check if any  $x \in B$  is a maximum/minimum  
For this: try to parametrize (sections of)  $B$ , check corners.

### Def Non-degenerate Critical Point

$$x_0 \in X \text{ non-deg.} \iff \det(H_f(x_0)) \neq 0$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}$ ,  $f \in C^2$ ,  $x_0 \in X$  is critical

### Lem. Definiteness of the Hessian

$H_f(x_0)$  positive definite  $\implies x_0$  is a local min.

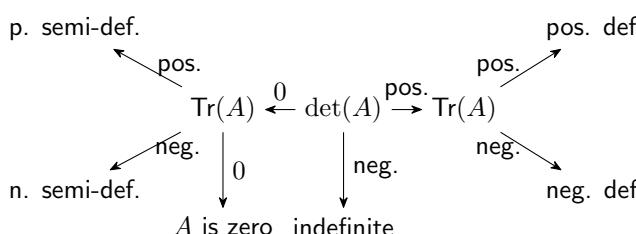
$H_f(x_0)$  negative definite  $\implies x_0$  is a local max.

$H_f(x_0)$  indefinite  $\implies x_0$  is a saddle point.

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}$ ,  $f \in C^2$ ,  $x_0 \in X$  non-deg. critical

### Method Determining Definiteness for $2 \times 2$ Matrices

p. semi-def.



## 6 Integral Calculus in $\mathbb{R}^n$

### 6.1 Line Integrals

#### Integrals for $f : I \rightarrow \mathbb{R}^n$

$I = [a, b]$  closed & bounded,  $f : I \rightarrow \mathbb{R}^n$  cont.

$$\int_a^b f(t) dt = \left( \int_a^b f_1(t) dt, \dots, \int_a^b f_n(t) dt \right)$$

#### Def Piecewise Continuity

$\exists k \geq 1$ , and a Partition  $a = t_0 < \dots < t_k = b$

s.t.  $f_j : [t_{j-1}, t_j] \rightarrow \mathbb{R}^n$  has  $f_j \in C^1$  for all  $j \leq k$

For  $f : I \rightarrow \mathbb{R}^n$

#### Def Parametrized Curve $\gamma : [a, b] \rightarrow \mathbb{R}^n$ pw.-cont.

Also called Path from  $\gamma(a)$  to  $\gamma(b)$

#### Line Integral

$\gamma : [a, b] \rightarrow \mathbb{R}^n$  is path,  $X \subset \mathbb{R}^n$  s.t.  $\gamma([a, b]) \subset X$

$f : X \rightarrow \mathbb{R}^n$  continuous

$$\int_{\gamma} f(s) \cdot ds := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

#### Def Continuous integrals are linear

$$\int_a^b (f(t) + g(t)) dt = \int_a^b f(t) dt + \int_a^b g(t) dt$$

$f, g : I \rightarrow \mathbb{R}^n$  continuous

Remark  $f : X \rightarrow \mathbb{R}^n$  is called a Vector Field.

#### Def Oriented Reparametrization

For  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  (param. curve),  $\phi : [c, d] \rightarrow [a, b]$  continuous

$$\sigma : [c, d] \rightarrow \mathbb{R}^n \text{ s.t. } \sigma = \gamma \circ \phi$$

diff.-able on  $(c, d)$ , strictly increasing and  $\phi(c) = a, \phi(d) = b$

### Lem. Oriented Reparametrizations preserve Integrals

$$\int_{\gamma} f(s) \cdot ds = \int_{\sigma} f(s) \cdot ds$$

$\gamma : [a, b] \rightarrow \mathbb{R}^n$  param. curve,  $\sigma$  oriented reparam.,  $\gamma([a, b]) \subset X$ ,  $f : X \rightarrow \mathbb{R}^n$  cont.

Remark Line Integrals of the form  $\int_{\gamma} \nabla f(s) \cdot ds$  have:

$$\int_{\gamma} \nabla f(s) \cdot ds = \int_a^b \sum_{i=1}^n \frac{\partial g}{\partial x_i}(\gamma(t)) \gamma'_i(t) dt = f(\gamma(b)) - f(\gamma(a))$$

Follows from the Chain rule for  $h(t) = g(\gamma(t))$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}$ ,  $f \in C^1$ ,  $\gamma : [a, b] \rightarrow X$  param. curve

#### Def Conservative Vector Field

$f : X \rightarrow \mathbb{R}$  conservative  $\overset{\text{def}}{\iff} \forall \gamma_1, \gamma_2$  s.t. start & end points match:

$$\int_{\gamma_1} f(s) \cdot ds = \int_{\gamma_2} f(s) \cdot ds$$

No matter which path, if start & end match, the integral matches

#### Remark Closed Curves in Conservative Vector Fields

$$\forall \gamma : [a, a] \rightarrow \mathbb{R} : \int_{\gamma} f(s) \cdot ds = 0$$

This is actually equivalent to  $f$  being conservative.

#### The Potential exists in Conservative Vector Fields

$X \subset \mathbb{R}^n$  open,  $f$  conservative

$$\exists g \in C^1 : f = \nabla g$$

If  $x_1, x_2 \in X$  are joined by a  $\gamma$ ,  $g$  is unique up to  $C \in \mathbb{R}$

$$\nabla g_1 = f \implies g - g_1 \text{ is constant on } X$$

#### Def Path-Connected Set

$\forall x_1, x_2 \in X : \exists \gamma : [a, b] \rightarrow X$  s.t.  $\gamma(a) = x_1, \gamma(b) = x_2$

## Lem. Property of Conservative Vector Fields

Easy way to e.g. disprove  $f$  being conservative:

$$\forall 1 \leq i \neq j \leq n : \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

Equivalently:

$$\mathbf{J}_f(x) = \mathbf{J}_f(x)^\top$$

$X \subset \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^n$ ,  $f \in C^1$ ,  $f$  conserv.

Only this way: This being true (alone) does not imply  $f$  is conservative!

## Def Star Shaped Set

$\exists x_0 \in X : \forall x \in X$  Line seg.  $x_0 \rightarrow x$  is in  $X$

## Def Convex Set

$\forall x_1, x_2 \in X : \text{Line seg. } x_1 \rightarrow x_2 \text{ is in } X$

Convex implies star shaped.

## Th. Some Star Shaped Sets are conservative

In open star-shaped sets  $X \subset \mathbb{R}^n$ :  $f \in C^1$

$$\forall 1 \leq i \neq j \leq n : \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \implies f \text{ conservative}$$

$$\text{Def } \text{curl}(f) := \begin{bmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{bmatrix} \quad f : X \rightarrow \mathbb{R}^3, \quad f \in C^1$$

$$\text{Remark } \text{curl}(f) = 0 \iff \forall 1 \leq i \neq j \leq 3 : \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

$$\text{Remark } \text{curl}(\nabla f) = 0 \text{ if } f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is in } C^2.$$

i.e. if  $f$  has a potential.

## Method Finding the Potential

We want  $g$  s.t.  $\nabla g = f$  for some conservative  $f$

1. Find  $\int f_i dx_i$  for all  $i \leq n$

2. Define  $g$  as the union of terms in  $\int f_i dx_i$

3.  $g$  now has  $\partial x_i g_i = f_i$  thus  $\nabla g = f$

Note that the union step *only works* if  $f$  is conservative.

## 6.2 The Riemann Integral in $\mathbb{R}^n$

For  $f : X \rightarrow \mathbb{R}$  ( $X \subset \mathbb{R}^n$  bounded & closed),  $\int_X f(x) dx$  fulfills:

### 1. Composability

$$\int_X f(x) dx = \int_a^b f(x) dx \quad n = 1, X = [a, b]$$

### 2. Linearity

$$\int_X (af_1(x) + bf_2(x)) dx = a \int_X f_1(x) dx + b \int_X f_2(x) dx$$

$f, g$  cont. on  $X$ ,  $a, b \in \mathbb{R}$

### 3. Positivity

$$f \leq g \implies \int_X f(x) dx \leq \int_X g(x) dx$$

### 4. Upper Bound

$$\left| \int_X f(x) dx \right| \leq \int_X |f(x)| dx$$

### 5. Triangle Inequality

$$\left| \int_X (f(x) + g(x)) dx \right| \leq \int_X |f(x)| dx + \int_X |g(x)| dx$$

### 6. Volume

$\int_X f(x) dx$  is the volume of  $\{(x, y) \in X \times \mathbb{R} \mid 0 \leq y \leq f(x)\}$   
So the intuitive idea of  $\int_a^b f(x) dx$  being the area carries over.

### 7. Domain Additivity

$$\int_{X_1 \cup X_2} f(x) dx + \int_{X_1 \cap X_2} f(x) dx = \int_{X_1} f(x) dx + \int_{X_2} f(x) dx$$

If  $X_1, X_2$  are compact,  $f$  is cont. on  $X_1 \cup X_2$

## Fubini's Theorem: Multiple Integrals

$f : X \rightarrow \mathbb{R}$ ,  $n = n_1 + n_2$ ,  $n_1, n_2 \geq 1$

$$X_{x_1} := \{x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X\} \subset \mathbb{R}^{n_2}$$

$$X_1 := \{x_1 \in \mathbb{R}^{n_1} \mid X_{x_1} \neq \emptyset\} \subset \mathbb{R}^{n_1}$$

If  $g(x_1) := \int_{X_{x_1}} f((x_1, x_2)) dx_2$  is continuous on  $X_1$ :

$$\int_X f(x) dx = \int_{X_1} \left( \int_{X_{x_1}} f((x_1, x_2)) dx_2 \right) dx_1$$

The role of  $x_1, x_2$  can be swapped, if  $f$  is continuous.

## Def Parametrized $m$ -Set in $\mathbb{R}^n$

$f : [a_1, b_1] \times \cdots \times [a_m, b_m] \rightarrow \mathbb{R}$   
s.t.  $f \in C^1$  on  $(a_1, b_1) \times \cdots \times (a_m, b_m)$   
A param. 1-set in  $\mathbb{R}^n$  is just a param. curve

## Def Negligible Subset

$B \subset \mathbb{R}^n$  s.t.  $\exists k \geq 0$  param.  $m_i$ -sets:  $f_i : X_i \rightarrow \mathbb{R}^n$  s.t.

$$B \subset f_1(X_1) \cup \cdots \cup f_k(X_k)$$

$1 \leq i \leq k, \quad m_i < n$

**Remark** For an affine subspace  $H \subset \mathbb{R}^n$  with  $\dim(H) < n$ , any  $X \subset \mathbb{R}^n$  contained in  $H$  is Negligible

**Remark** The image of a param. curve  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  is negligible.  
 $\gamma$  is a 1-set in  $\mathbb{R}^n$

## Lem. Integral of Negligible Sets

For continuous  $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}$ :

$$X \text{ negligible} \implies \int_X f(x) dx = 0$$

## 6.3 Improper Integrals

$I \subset \mathbb{R}$  bounded,  $J = [a, +\infty)$  for  $a \in \mathbb{R}$ ,  $f$  cont. on  $X = J \times I$

$$\lim_{x \rightarrow \infty} \int_{[a, x] \times I} f(x, y) dxdy = \underbrace{\int_a^\infty \left( \int_I f(x, y) dy \right) dx}_{\text{Order of Integration may change}}$$

If this Limit is equal for both orders of Integration:

## Def Improper Integral in $\mathbb{R}^2$

$$\int_{J \times I} f(x, y) dxdy := \lim_{x \rightarrow \infty} \int_{[a, x] \times I} f(x, y) dxdy$$

## Def Integral over $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} f(x, y) dxdy := \lim_{R \rightarrow \infty} \int_{[-R, R]^2} f(x, y) dxdy$$

**Remark** if  $|f| \leq g$ , and an impr. Integr. exists on  $g$ , it exists on  $f$ .

## 6.4 Change of Variable

This is to provide an Analogue of the Change of Variable in  $\mathbb{R}$

$$\int f(g(x))g'(x) dx = \int f(y) dy$$

### Prerequisites

$\bar{X}, \bar{Y} \subset \mathbb{R}^n$  compact,  $\varphi : \bar{X} \rightarrow \bar{Y}$  cont.

We have:  $\bar{X} = X \cup B$ ,  $\bar{Y} = Y \cup C$  s.t.

1.  $X, Y$  are open
2.  $B, C$  are negligible
3.  $\varphi$  on  $X$  is a  $C^1$  map  $\varphi : X \rightarrow Y$

### Change of Variable in $\mathbb{R}^n$

$\bar{X}, \bar{Y}$  as above,  $f$  cont. on  $\bar{Y}$  arbitrary

$$\int_{\bar{X}} f(\varphi(x)) \cdot |\det(\mathbf{J}_\varphi(x))| dx = \int_{\bar{Y}} f(y) dy$$

**Remark** Translations:  $\varphi(x) = x + x_0$  have  $\mathbf{J}_\varphi(x) = \mathbf{I}_n$

so the volume is preserved:  $\int_{\bar{X}} f(x + x_0) dx = \int_{x_0 + \bar{X}} f(x) dx$

**Remark** Linear maps:  $\varphi(x) = \mathbf{Ax}$  have  $\mathbf{J}_\varphi(x) = \mathbf{A}$

The change of variable is:  $\int_{\bar{X}} f(\varphi(x)) dx = \frac{1}{|\det(\mathbf{A})|} \int_{\bar{Y}} f(y) dy$

### Remark Common Changes

#### 1. Polar Coordinates

$$\varphi(r, \theta) = (r \cos(\theta), r \sin(\theta)) \quad \theta \in [0, 2\pi]$$

$$dxdy = r dr d\theta$$

#### 2. Cylindrical Coordinates

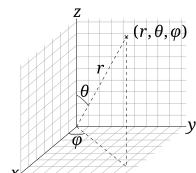
$$\varphi(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z) \quad \theta \in [0, \pi], \phi \in [0, 2\pi]$$

$$dxdydz = r dr d\theta dz$$

#### 3. Spherical Coordinates

$$\varphi(r, \theta, \phi) = (r \sin(\phi) \cos(\theta), r \sin(\phi) \sin(\theta), r \cos(\phi))$$

$$dxdydz = r^2 \sin(\phi) dr d\theta d\phi$$



## 6.5 Green's Theorem

An analogue of the Fundamental Theorem of Calculus in  $\mathbb{R}^2$ .

**Def Simple Closed Parametrized Curve**

$\gamma : [a, b] \rightarrow \mathbb{R}^2$  closed param. curve s.t.

1.  $\gamma(t) \neq \gamma(s)$  unless  $s = t$ , or  $\{s, t\} = \{a, b\}$
2.  $\gamma'(t) \neq 0 \quad \forall a < t < b$

Example:  $\varphi(t) = (x_0 + r \cos(t), y_0 + r \sin(t))$

(A circle, traversed once, i.e. for  $0 \leq t \leq 2\pi$ )

### Green's Theorem

$X \subset \mathbb{R}^2$  compact with Boundary  $\partial X = \bigcup_{1 \leq i \leq n} \gamma_i$  as above

Assume:  $\gamma_i : [a_i, b_i] \rightarrow \mathbb{R}^2$  s.t.  $X$  is always left of  $\gamma'_i(t)$  at  $\gamma_i(t)$

$$\int_X \left( \underbrace{\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}}_{\text{curl}(f)} \right) dxdy = \sum_{i=1}^k \int_{\gamma_i} f \cdot ds$$

For a  $C^1$  Vector field  $f = (f_1, f_2)$  containing  $X$

So, a sum of line integrals can be written as the Integral of the curl.  
This is very useful for computing complex line integrals.

### Lem. Volume using Green

$$\text{Vol}(X) = \sum_{i=1}^k \int_{\gamma_i} x \cdot ds = \sum_{i=1}^k \int_{a_i}^{b_i} \gamma_{i,1}(t) \cdot \gamma'_{i,2}(t) dt$$

Same assumptions as above.

**Remark** The Gauss-Ostrogradski Formula exists for  $\mathbb{R}^3$ .

**Example:** Finding an Area using Green:

$$\text{Area}(X) = \int_X 1 dxdy$$

To use Green, we want  $f$  s.t.  $(\partial_x f_y - \partial_y f_x) = 1$ . Common Choices:

1.  $f(x, y) = (0, x)$
2.  $f(x, y) = (-y, 0)$

We can then use Green, assuming we have a  $\gamma$  for the Boundary of  $X$ :

$$\text{Area}(X) = \int_X 1 dxdy = \int_X \left( \partial_x f_y - \partial_y f_x \right) dxdy = \int_{\gamma} f(s) ds$$