

1 Linear Algebra

Relevant definitions used throughout Analysis II.

$$\mathbf{A} \in \mathbb{R}^{m \times n}, \quad x, y \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$$

$$\text{Def } \mathbf{Euclidian Norm} \quad \|x\| := \sqrt{\sum_{i=1}^n x_i^2}$$

Used to generalize $|x|$ in many Analysis I definitions

Lem. **Properties of $\|x\|$**

- (i) $\|x\| \geq 0$
- (ii) $\|x\| \iff x = 0$
- (iii) $\|\alpha x\| = |\alpha| \cdot \|x\|$
- (iv) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality)

$$\text{Def } \mathbf{Trace} \quad \text{Tr}(\mathbf{A}) := \sum_{i=0}^{\min(m,n)} \mathbf{A}_{i,i}$$

Used to generalize $\text{Tr}(A)$ in many Analysis I definitions

2 Differential Equations

Def **Differential Equation (DE)**

Equation relating unknown f to derivatives $f^{(i)}$ at same x .

Def **Ordinary Differential Equation (ODE)**

DE s.t. $f : I \rightarrow \mathbb{R}$ is in one variable.

Def **Partial Differential Equation (PDE)**

DE s.t. $f : I^d \rightarrow \mathbb{R}$ is in multiple variables.

Notation $f^{(i)}$ or $y^{(i)}$ instead of $f^{(i)}(x)$ for brevity.

Def **Order** $\text{ord}(F) := \max_{i \geq 0} \{i \mid f^{(i)} \in F, f^{(i)} \neq 0\}$

Remark Any F s.t. $\text{ord}(F) \geq 2$ can be reduced to $\text{ord}(F') = 1$, but using functions of higher dimensions.

Solutions to ODEs

$\forall F : \mathbb{R}^2 \rightarrow \mathbb{R}$ s.t. F is cont. diff. and $x_0, y_0 \in \mathbb{R}$:

$$\exists f : I \rightarrow \mathbb{R}$$

s.t. $\forall x \in I : f'(x) = F(x, f(x))$ and $f(x_0) = y_0$

s.t. I is open and maximal.

Intuition: Solutions always exist (locally!) for nice enough equations.

2.1 Linear Differential Equations

Def **Linear Differential Equation (LDE)**

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

$I \subset \mathbb{R}$ is open, $k \geq 1$, $\forall i < k : a_i : I \rightarrow \mathbb{C}$

Def Homogeneity of LDEs

Homogeneous $\stackrel{\text{def}}{\iff} b = 0$

Inhomogeneous $\stackrel{\text{def}}{\iff} b \neq 0$

Remark $D(y) := y^{(k)} + \dots + a_0y$ is a linear operation:

$$D(z_1f_1 + z_2f_2) = z_1D(f_1) + z_2D(f_2)$$

$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2$ k -times differentiable

Def **Homogeneous Solution Space**

$\mathcal{S}(F) := \{f : I \rightarrow \mathbb{C} \mid f \text{ solves } F, f \text{ is } k\text{-times diff.}\}$

Remark $\mathcal{S}(F)$ is the Nullspace of a lin. map: f to $D(f)$:

$$D(f) = z_1D(f_1) + z_2D(f_2) = 0$$

$$\forall z_1, z_2 \in \mathbb{C}, \quad f_1, f_2 \in \mathcal{S}$$

Solutions for complex homogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous and complex-valued

1. \mathcal{S} is a complex vector space, $\dim(\mathcal{S}) = k$
2. \mathcal{S} is a subspace of $\{f \mid f : I \rightarrow \mathbb{C}\}$
3. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists

Solutions for real homogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous and real-valued

1. \mathcal{S} is a real vector space, $\dim(\mathcal{S}) = k$
2. \mathcal{S} is a subspace of $\{f \mid f : I \rightarrow \mathbb{R}\}$
3. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{R}^k$ a unique sol. exists

Def **Inhomogeneous Solution Space**

$\mathcal{S}_b(F) := \{f + f_0 \mid f \in \mathcal{S}(F), f_0 \text{ is a particular sol.}\}$

Note: This is only a vector space if $b = 0$, where $\mathcal{S}_b = \mathcal{S}$.

Solutions for real inhomogeneous LDEs

F s.t. a_0, \dots, a_{k-1} continuous, $b : I \rightarrow \mathbb{C}$

1. $\forall x_0 \in I, (y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ a unique sol. exists
2. If b, a_i are real-valued, a real-valued sol. exists.

Remark **Applications of Linearity**

If f_1 solves F for b_1 , and f_2 for b_2 : $f_1 + f_2$ solves $b_1 + b_2$.
Follows from: $D(f_1) + D(f_2) = b_1 + b_2$.

2.2 Linear Solutions: First Order

$I \subset \mathbb{R}$, $a, b : I \rightarrow \mathbb{R}$

Form:

$$y' + ay = b$$

Approach:

1. Hom. Solution f_1 for: $y' + ay = 0$

Note that \mathcal{S} has $\dim(\mathcal{S}) = 1$, so $f_1 \neq 0$ is a Basis for \mathcal{S}

2. Part. Solution f_0 for $y' + ay = b$

Solutions: $f_0 + zf_1$ for $z \in \mathbb{C}$

Explicit Homogeneous Solution

$A(x)$ is a primitive of a , $f(x_0) = y_0$

$$f_1(x) = z \cdot \exp(-A(x))$$

$$f_1(x) = y_0 \cdot \exp(A(x_0) - a(x))$$

Variation of Constants: Treating z as $z(x)$ yields:

Explicit Inhomogeneous Solution

$A(x)$ is a primitive of a

$$f_0(x) = \underbrace{\left(\int b(x) \cdot \exp(A(x)) \right)}_{z(x)} \cdot \exp(-A(x))$$

Method Educated Guess

Usually, y has a similar form to b :

| $b(x)$ | Guess |
|---|--|
| $a \cdot e^{\alpha x}$ | $b \cdot e^{\alpha x}$ |
| $a \cdot \sin(\beta x)$ | $c \sin(\beta x) + d \cos(\beta x)$ |
| $b \cdot \cos(\beta x)$ | $c \sin(\beta x) + d \cos(\beta x)$ |
| $a e^{\alpha x} \cdot \sin(\beta x)$ | $e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$ |
| $b e^{\alpha x} \cdot \cos(\beta x)$ | $e^{\alpha x} (c \sin(\beta x) + d \cos(\beta x))$ |
| $P_n(x) \cdot e^{\alpha x}$ | $R_n(x) \cdot e^{\alpha x}$ |
| $P_n(x) \cdot e^{\alpha x} \sin(\beta x)$ | $e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$ |
| $P_n(x) \cdot e^{\alpha x} \cos(\beta x)$ | $e^{\alpha x} (R_n(x) \sin(\beta x) + S_n(x) \cos(\beta x))$ |

Remark If α, β are roots of $P(X)$ with multiplicity j , multiply guess with a $P_j(x)$.

2.3 Linear Solutions: Constant Coefficients

Form:

$$y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$$

Where $a_0, \dots, a_{k-1} \in \mathbb{C}$ are constants, $b(x)$ is continuous.

2.3.1 Homogeneous Equations

The idea is to find a Basis of \mathcal{S} :

Def **Characteristic Polynomial** $P(X) = \prod_{i=1}^k (X - \alpha_i)$

Remark The unique roots $\alpha_1, \dots, \alpha_l$ form a Basis:

$$\text{span}(\mathcal{S}) = \{x^j e^{\alpha_i x} \mid i \leq l, 0 \leq j \leq v_i\}$$

v_1, \dots, v_k are the Multiplicities of $\alpha_1, \dots, \alpha_k$

Remark If $\alpha_j = \beta + \gamma i \in \mathbb{C}$ is a root, $\bar{\alpha}_j = \beta - \gamma i$ is too. To get a real-valued solution, apply:

$$e^{\alpha_j x} = e^{\beta x} (\cos(\gamma x) + i \sin(\gamma x))$$

Explicit Homogeneous Solution

Using $\alpha_1, \dots, \alpha_k$ from $P(X)$ s.t. $\alpha_i \neq \alpha_j$, $z_i \in \mathbb{C}$ arbitrary

$$f(x) = \prod_{i=1}^k z_i \cdot e^{\alpha_i x} \quad \text{with} \quad f^{(j)(x)} = \prod_{i=1}^k z_i \cdot \alpha_i^j e^{\alpha_i x}$$

Multiple roots: same scheme, using the basis vectors of \mathcal{S}

Solutions exist $\forall Z = (z_1, \dots, z_k)$ since that system's $\det(M_Z) \neq 0$.

2.3.2 Inhomogeneous Equations

Method **Undetermined Coefficients**: An educated guess.

1. $b(x) = cx^d \cdot e^{\alpha x} \implies f_p(x) = Q(x)e^{\alpha x}$
 $\deg(Q) \leq d + v_\alpha$, where v_α is α 's multiplicity in $P(X)$

2. $b(x) = cx^d \cdot \cos(\alpha x)$
 $b(x) = cx^d \cdot \sin(\alpha x)$
 $\deg(Q_{i,2}) \leq d + v_\alpha$, where v_α is α 's multiplicity in $P(X)$

Remark **Applying Linearity**

If $b(x) = \sum_{i=1}^n b_i(x)$, A solution for $b(x)$ is $f(x) = \sum_{i=1}^n f_i(x)$
Sometimes called *Superposition Principle* in this context

2.4 Other Methods

Method **Change of Variable**

If $f(x)$ is replaced by $h(y) = f(g(y))$, then h is a sol. too.
Changes like $h(t) = f(e^t)$ may lead to useful properties.

Separation of Variables

Form:

$$y' = a(y) \cdot b(x)$$

Solve using:

$$\int \frac{1}{a(y)} dy = \int b(x) dx + c$$

Usually $\int 1/a(y) dy$ can be solved directly for $\ln|a(y)| + c$.

2.5 Method Overview

| Method | Use case |
|---------------------------|---------------------------------|
| Variation of constants | LDE with $\text{ord}(F) = 1$ |
| Characteristic Polynomial | Hom. LDE w/ const. coeff. |
| Undetermined Coefficients | Inhom. LDE w/ const. coeff. |
| Separation of Variables | ODE s.t. $y' = a(y) \cdot b(x)$ |
| Change of Variables | e.g. $y' = f(ax + by + c)$ |

3 Continuous functions in \mathbb{R}^n

Treating functions $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}/\mathbb{C}/\mathbb{R}^m$, $m, n \geq 1$

Notation $f(x)$ for $f : I \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ means:
 $x = (x_1, \dots, x_n)$, $f(x) = f(f_1(x), \dots, f_m(x))$

3.1 Multivariate functions

Def Linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

In other words: $f(x) = \mathbf{A}x$, $\mathbf{A} \in \mathbb{C}^{m \times n}$

Linear Maps are continuous

Def Affine Linear map $f(x) \mapsto \mathbf{A}x + c$

Def Quadratic form $Q : \mathbb{R}^n \rightarrow \mathbb{R}$

In other words: $Q(x) = \sum_{i=0}^n \sum_{j=0}^m (a_{i,j} x_i x_j)$

Def Monomials $M(x) : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto \alpha x_1^{d_1} \cdots x_n^{d_n}$

For example: $f(x, y, z) = 16x^2yz^5$

Def $\deg(M) := e = \sum_{i=1}^n d_i$

For example: $\deg(16x^2yz^5) = 8$

Def Polynomials $P(x) := \sum_{i=0}^n M_i(x)$

For example: $P(x, y, z) = x^3 + 25x^2y^6z + xy$

Polynomials are continuous.

Def $\deg(P) := d \geq \max\{\deg(M_i) \mid M_i \text{ in } P\}$

For example: $\deg(x^3 + 25x^2y^6z + xy) = 9$

Visualisations for some function types:

Def Graph $G_f := \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$

Only for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Visually, this is a surface in \mathbb{R}^3

Def Vector Plots for $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

Points in $(x, y) \in \mathbb{R}^2$ are displayed as vectors $f(x, y)$

3.2 Sequences in \mathbb{R}^n

Def Sequences in \mathbb{R}^n

$(x_k)_{k \geq 1}$ s.t. $x_k \in \mathbb{R}^n$ where $x_k = (x_{k,1}, \dots, x_{k,n})$

Def Convergence in \mathbb{R}^n

$$\lim_{k \rightarrow \infty} (x_k) = y \iff \forall \epsilon > 0, \exists N \geq 1 : \forall k \geq N : \|x_k - y\| < \epsilon$$

Using this definition preserves many familiar results:

Lem. Equivalent conditions to Convergence

- (i) $\forall i \text{ s.t. } 1 \leq i \leq n : \lim_{k \rightarrow \infty} (x_{k,i}) = y_i$
- (ii) $\lim_{k \rightarrow \infty} \|x_k - y\| = 0$

Def Limits at points

$$\lim_{x \neq x_0 \rightarrow x_0} (f(x)) = y \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 : \forall x \neq x_0 \in X : \|x - x_0\| < \delta \implies \|f(x) - y\| < \epsilon$$

$X \subset \mathbb{R}^n$, $f : X \rightarrow \mathbb{R}^m$, $x_0 \in X$, $y \in \mathbb{R}^m$

The sequence test for Continuity works for point-limits too.

3.3 Continuity in \mathbb{R}^n

Def Continuity in \mathbb{R}^n

f continuous at $x_0 \in X \stackrel{\text{def}}{\iff} \forall \epsilon > 0, \exists \delta > 0 :$

$$\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \epsilon$$

f continuous $\stackrel{\text{def}}{\iff} \forall x \in X : f$ continuous at x

$X \subset \mathbb{R}^n$, $f : X \rightarrow \mathbb{R}^m$

Lem. Continuity using Sequences

f continuous at x_0 if and only if:

$$\forall (x_k)_{k \geq 1} : \lim_{k \rightarrow \infty} (x_k) = x_0 \implies \lim_{k \rightarrow \infty} (f(x_k)) = f(x_0)$$

$X \subset \mathbb{R}^n$, $f : X \rightarrow \mathbb{R}^m$

Lem. Continuity of Compositions

$f : X \rightarrow Y$, $g : Y \rightarrow \mathbb{R}^p$ continuous $\implies g \circ f$ continuous

$X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $p \geq 1$

Lem. Continuity using Coordinate Functions

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ continuous $\iff \forall i \leq m : f_i$ continuous

3.4 Subsets of \mathbb{R}^n

Def Bounded

$X \subset \mathbb{R}^n$ bounded $\stackrel{\text{def}}{\iff} \left\{ \|x\| \mid x \in X \right\} \subset \mathbb{R}$ bounded.

Example: The open disc $D = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$ is bounded.

Def Closed

$X \subset \mathbb{R}^n$ closed $\stackrel{\text{def}}{\iff} \forall (x_k)_{k \geq 1} \in X : \lim_{k \rightarrow \infty} (x_k) \in X$

Example: \emptyset, \mathbb{R}^n are closed.

Def Compact if closed and bounded.

Example: The closed Disc $\Lambda = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ is compact.

Def Open

$X \subset \mathbb{R}^n$ open $\stackrel{\text{def}}{\iff} \forall x \in X, \exists \delta > 0 :$

$$\{y \in \mathbb{R}^n \mid |x_i - y_i| < \delta, \forall i \leq n\} \subset X$$

In other words: Changing any coord. x_i by δ keeps x' in X

Example: \emptyset, \mathbb{R}^n are open (and closed)

Lem. The Cartesian Product preserves bounded/closed.

Lem. Continuous functions preserve closed/open

\forall closed/open $Y :$

$$f^{-1}(Y) = \{x \in \mathbb{R}^n \mid f(x) \in Y\} \text{ is closed/open.}$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous, $Y \subset \mathbb{R}^m$

Lem. The complement of open sets is closed

$X \subset \mathbb{R}^n$ is open $\iff \underbrace{\{x \in \mathbb{R}^n \mid x \notin X\}}_{\text{Complement}}$ is closed

Min-Max Theorem

For compact, non-empty $X \subset \mathbb{R}^n$, continuous $f : X \rightarrow \mathbb{R}$:

$$\exists x_1, x_2 \in X : f(x_1) = \sup_{x \in X} f(x), f(x_2) = \inf_{x \in X} f(x)$$

4 Differential Calculus in \mathbb{R}^n

4.1 Partial Derivatives

Partial Derivative

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$, $1 \leq i \leq n$, $x_0 \in X$

$$\frac{\partial f}{\partial x_i}(x_0) := g'(x_{0,i})$$

for $g : \{t \in \mathbb{R} \mid (x_{0,1}, \dots, t, \dots, x_{0,n}) \in X\} \rightarrow \mathbb{R}^n$

$$g(t) := \underbrace{f(x_{0,1}, \dots, x_{0,t-1}, t, x_{0,t+1}, \dots, x_{0,n})}_{\text{Freeze all } x_{0,k} \text{ except one } x_{0,i} \rightarrow t}$$

Notation $\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$

Lem. Properties of Partial Derivatives

Assuming $\partial_{x_i} f$ and $\partial_{x_i} g$ exist :

- (i) $\partial_{x_i}(f + g) = \partial_{x_i} f + \partial_{x_i} g$
- (ii) $\partial_{x_i}(fg) = \partial_{x_i}(f)g + \partial_{x_i}(g)f$ if $m = 1$
- (iii) $\partial_{x_i}\left(\frac{f}{g}\right) = \frac{\partial_{x_i}(f)g - \partial_{x_i}(g)f}{g^2}$ if $g(x) \neq 0 \forall x \in X$

$X \subset \mathbb{R}^n$ open, $f, g : X \rightarrow \mathbb{R}^n$, $1 \leq i \leq n$

The Jacobian

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^n$ with partial derivatives existing

$$\mathbf{J}_f(x) := \begin{bmatrix} \partial_{x_1} f_1(x) & \partial_{x_2} f_1(x) & \cdots & \partial_{x_n} f_1(x) \\ \partial_{x_1} f_2(x) & \partial_{x_2} f_2(x) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_m(x) & \partial_{x_2} f_m(x) & \cdots & \partial_{x_n} f_m(x) \end{bmatrix}$$

Think of f as a vector of f_i , then \mathbf{J}_f is that vector stretched for all x_j

Def Gradient $\nabla f(x_0) := \begin{bmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{bmatrix} = \mathbf{J}_f(x)^{\top}$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}$

Def Divergence $\operatorname{div}(f)(x_0) := \operatorname{Tr}(\mathbf{J}_f(x_0))$

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^n$, \mathbf{J}_f exists

4.2 The Differential

Partial derivatives don't provide a good approx. of f , unlike in the 1-dimensional case. The *differential* is a linear map which replicates this purpose in \mathbb{R}^n .

Differentiability in \mathbb{R}^n

$X \subset \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^n$, $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear map

$$df(x_0) := u$$

If f is differentiable at $x_0 \in X$ with u s.t.

$$\lim_{x \neq x_0 \rightarrow x_0} \frac{1}{\|x - x_0\|} \left(f(x) - f(x_0) - u(x - x_0) \right) = 0$$