

# Analysis II

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TITLE PAGE COMING SOON

*“Multiply it by ai”*

- Özlem Imamoglu, 2025

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Cheat-Sheet based on Lecture notes and Script

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# 1 Introduction

This Cheat-Sheet does not serve as a replacement for solving exercises and getting familiar with the content. There is no guarantee that the content is 100% accurate, so use at your own risk. If you discover any errors, please open an issue or fix the issue yourself and then open a Pull Request here:

<https://github.com/janishutz/eth-summaries>

This Cheat-Sheet was designed with the HS2025 page limit of 10 A4 pages in mind. Thus, the whole Cheat-Sheet can be printed full-sized, if you exclude the title page, contents and this page. You could also print it as two A5 pages per A4 page and also print the [Analysis I summary](#) in the same manner, allowing you to bring both to the exam.

And yes, she did really miss an opportunity there with the quote... But she was also sick, so it's not as unexpected

This summary also uses tips and tricks from this [Exercise Session](#)

## 2 Differential Equations

### 2.1 Introduction

**Ex 2.1.1:**  $f'(x) = f(x)$  has only solution  $f(x) = ae^x$  for any  $a \in \mathbb{R}$ ;  $f' - a = 0$  has only solution  $f(x) = \int_{x_0}^x a(t) dt$

**T 2.1.2:** Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a differential function of two variables. Let  $x_0 \in \mathbb{R}$  and  $y_0 \in \mathbb{R}^2$ . The Ordinary Differential Equation (ODE)  $y' = F(x, y)$  has a unique solution  $f$  defined on a “largest” interval  $I$  that contains  $x_0$  such that  $y_0 = f(x_0)$

### 2.2 Linear Differential Equations

An ODE is considered linear if and only if the  $y$ s are only scaled and not part of powers.

**D 2.2.1:** (Linear differential equation of order  $k$ ) (order = highest derivative)  $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$ , with  $a_i$  and  $b$  functions in  $x$ . If  $b(x) = 0 \ \forall x$ , **homogeneous**, else **inhomogeneous**

**T 2.2.2:** For open  $I \subseteq \mathbb{R}$  and  $k \geq 1$ , for lin. ODE over  $I$  with continuous  $a_i$  we have:

1. Set  $\mathcal{S}$  of  $k \times$  diff. sol.  $f : I \rightarrow \mathbb{C}(\mathbb{R})$  of the eq. is a complex (real) subspace of complex (real)-valued func. over  $I$
2.  $\dim(\mathcal{S}) = k \ \forall x_0 \in I$  and any  $(y_0, \dots, y_{k-1}) \in \mathbb{C}^k$ , exists unique  $f \in \mathcal{S}$  s.t.  $f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1}$ . If  $a_i$  real-valued, same applies, but  $\mathbb{C}$  replaced by  $\mathbb{R}$ .
3. Let  $b$  continuous on  $I$ . Exists solution  $f_0$  to inhom. lin. ODE and  $\mathcal{S}_b$  is set of funct.  $f + f_0$  where  $f \in \mathcal{S}$

The solution space  $\mathcal{S}$  is spanned by  $k$  functions, which thus form a basis of  $\mathcal{S}$ . If inhomogeneous,  $\mathcal{S}$  not vector space.

#### Finding solutions (in general)

- (1) Find basis  $\{f_1, \dots, f_k\}$  for  $\mathcal{S}_0$  for homogeneous equation (set  $b(x) = 0$ ) (i.e. find homogeneous part, solve it)
- (2) If inhomogeneous, find  $f_p$  that solves the equation. The set of solutions is then  $\mathcal{S}_b = \{f_h + f_p \mid f_h \in \mathcal{S}_0\}$ .
- (3) If there are initial conditions, find equations  $\in \mathcal{S}_b$  which fulfill conditions using SLE (as always)

### 2.3 Linear differential equations of first order

**P 2.3.1:** Solution of  $y' + ay = 0$  is of form  $f(x) = ze^{-A(x)}$  with  $A$  anti-derivative of  $a$

#### Inhomogeneous equation

1. Plug all values into  $y_p = \int b(x)e^{A(x)} (A(x) \text{ in the exponent instead of } -A(x) \text{ as in the homogeneous solution})$
2. Solve and the final  $y(x) = y_h + y_p$ . For initial value problem, determine coefficient  $z$

### 2.4 Linear differential equations with constant coefficients

The coefficients  $a_i$  are constant functions of form  $a_i(x) = k$  with  $k$  constant, where  $b(x)$  can be any function.

#### Homogeneous Equation

1. Find **characteristic polynomial** (of form  $\lambda^k + a_{k-1}\lambda^{k-1} + \dots + a_1\lambda + a_0$  for order  $k$  lin. ODE with coefficients  $a_i \in \mathbb{R}$ ).
2. Find the roots of polynomial. The solution space is given by  $\{z_j \cdot x^{v_j-1} e^{\gamma_j x} \mid v_j \in \mathbb{N}, \gamma_j \in \mathbb{R}\}$  where  $v_j$  is the multiplicity of the root  $\gamma_j$ . For  $\gamma_i = \alpha + \beta i \in \mathbb{C}$ , we have  $z_1 \cdot e^{\alpha x} \cos(\beta x), z_2 \cdot e^{\alpha x} \sin(\beta x)$ , representing the two complex conjugated solutions.

#### Inhomogeneous Equation

1. (**Case 1**)  $b(x) = cx^d e^{\alpha x}$ , with special cases  $x^d$  and  $e^{\alpha x}$ :  $f_p = Q(x)e^{\alpha x}$  with  $Q$  a polynomial with  $\deg(Q) \leq j + d$ , where  $j$  is multiplicity of root  $\alpha$  (if  $P(\alpha) \neq 0$ , then  $j = 0$ ) of characteristic polynomial
2. (**Case 2**)  $b(x) = cx^d \cos(\alpha x)$ , or  $b(x) = cx^d \sin(\alpha x)$ :  $f_p = Q_1(x) \cdot \cos(\alpha x) + Q_2(x) \cdot \sin(\alpha x)$ , where  $Q_i(x)$  a polynomial with  $\deg(Q_i) \leq d + j$ , where  $j$  is the multiplicity of root  $\alpha i$  (if  $P(\alpha i) \neq 0$ , then  $j = 0$ ) of characteristic polynomial

#### Other methods

- **Change of variable** Apply substitution method here, substituting for example for  $y' = f(ax + by + c)$   $u = ax + by$  to make the integral simpler. Mostly intuition-based (as is the case with integration by substitution)
- **Separation of variables** For equations of form  $y' = a(y) \cdot b(x)$  (NOTE: Not linear), we transform into  $\frac{y'}{a(y)} = b(x)$  and then integrate by substituting  $y'(x)dx = dy$ , changing the variable of integration. Solution:  $A(y) = B(x) + c$ , with  $A = \int \frac{1}{a}$  and  $B(x) = \int b(x)$ . To get final solution, solve for the above equation for  $y$ .

### 3 Differential Calculus in Vector Space

#### 3.2 Continuity

**D 3.2.1:** (Convergence in  $\mathbb{R}^n$ ) Let  $(x_k)_{k \in \mathbb{N}}$  where  $x_k \in \mathbb{R}^n$  with  $x_k = (x_{k,1}, \dots, x_{k,n})$  and let  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ .  $(x_k)$  converges to  $y$  as  $k \rightarrow +\infty$  if  $\forall \varepsilon > 0 \exists N \geq 1$  s.t.  $\forall n \geq N$  we have  $\|x_k - y\| < \varepsilon$

**L 3.2.2:**  $(x_k)$  converges to  $y$  as  $k \rightarrow +\infty$  iff one of following equiv. statements holds: **(1)**  $\forall 1 \leq i \leq n$ , the sequence  $(x_{k,i})$  with  $x_{k,i} \in \mathbb{R}$  converges to  $y_i$  **(2)**  $(\|x_k - y\|)$  converges to 0 as  $k \rightarrow +\infty$

**D 3.2.3:** (Continuity) Let  $X \subseteq \mathbb{R}^n$  and  $f : X \rightarrow \mathbb{R}^m$ . **(1)** Let  $x_0 \in X$ .  $f$  continuous in  $\mathbb{R}^n$  if  $\forall \varepsilon > 0 \exists \delta > 0$  s.t. if  $x \in X$  satisfies  $\|x - x_0\| < \delta$ , then  $\|f(x) - f(x_0)\| < \varepsilon$  **(2)**  $f$  continuous on  $X$  if continuous at  $x_0 \forall x_0 \in X$  **P 3.2.4:** Let  $X$  and  $f$  as prev. Let  $x_0 \in X$ .  $f$  continuous at  $x_0$  iff  $\forall (x_k)_{k \geq 1}$  in  $X$  s.t.  $x_k \rightarrow x_0$  as  $k \rightarrow +\infty$ ,  $(f(x_k))_{k \geq 1}$  in  $\mathbb{R}^m$  converges to  $f(x_0)$

**D 3.2.5:** (Limit) Let  $X$ ,  $f$  and  $x_0$  as prev. and  $y \in \mathbb{R}^m$ .  $f$  has limit  $y$  as  $x \rightarrow x_0$  with  $x \neq x_0$  if  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.  $\forall x \neq x_0 \in X, \|x - x_0\| < \delta$  we have  $\|f(x) - y\| < \varepsilon$ . We write  $\lim_{x \rightarrow x_0, x \neq x_0} f(x) = y$  **R 3.2.6:** Also possible without ass. that  $x_0 \in X$

**P 3.2.7:** Let  $X$ ,  $f$ ,  $x_0$  and  $y$  as prev. We have  $\lim_{x \rightarrow x_0, x \neq x_0} f(x) = y$  iff  $\forall (x_k)$  in  $X$  s.t.  $x_k \rightarrow x_0$  as  $k \rightarrow +\infty$  and  $x_k \neq x_0$   $(f(x_k))$  in  $\mathbb{R}^m$  converges to  $y$  **P 3.2.9:** Let  $X \subseteq \mathbb{R}^n$ ,  $Y \subseteq \mathbb{R}^m$ ,  $p \in \mathbb{N}$  and let  $f : X \rightarrow Y$  and  $g : Y \rightarrow \mathbb{R}^p$  be cont. Then  $g \circ f$  is continuous

**Remark:** To find the limits, we have two tricks (for  $\lim_{(x,y) \rightarrow (a,b)}$ ):

- (Substitution)** Substitute  $y = x + (b - a)$ , then limit is  $\lim_{x \rightarrow (a-b)}$
- (Polar coordinates)** Substitute  $x = r \cos(\varphi)$  and  $y = r \sin(\varphi)$  and the limit is  $\lim_{r \rightarrow 0}$

**Ex 3.2.10:** **(1)**  $f_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{m_1}$  and  $f_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{m_2}$  continuous  $\Rightarrow f = (f_1, f_2) : \mathbb{R}^n \rightarrow \mathbb{R}^{m_1+m_2}$  is continuous (Cartesian product) **(2)** Any linear map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous. In particular, the identity map is continuous **(3)** If  $f_1, \dots, f_n$  continuous, then  $f(x_1, \dots, x_n) = f_1(x_1) \cdot \dots \cdot f_n(x_n)$  is continuous **(4)** Polynomials in  $x_1, \dots, x_n$  are continuous **(5)**  $f_1 f_2$  is continuous if  $f_1$  and  $f_2$  are continuous and if  $f_2(x) \neq 0 \forall x \in X$ , then  $f_1 \div f_2$  is continuous. (see Theorem 2.1.8 in Analysis I)

**(6)** If both  $f$  and  $g$  have limits, then  $\lim_{x \rightarrow x_0} (f(x) + g(x)) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$  and analogous for  $\times$  **(7)** If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  continuous, then  $g(x) = f(x, y_0)$  for  $y_0 \in \mathbb{R}$  is continuous. The converse is not true

**D 3.2.11:** **(1)**  $X \subseteq \mathbb{R}^n$  is **bounded** if the set of  $\|x\|$  for  $x \in X$  is bounded in  $\mathbb{R}$  **(2)**  $X \subseteq \mathbb{R}^n$  is **closed** if  $\forall (x_k)$  in  $X$  that converge in  $\mathbb{R}^n$  to some vector  $y \in \mathbb{R}^n$ , we have  $y \in X$  **(3)**  $X \subseteq \mathbb{R}^n$  is **compact** if it is bounded and closed

**Ex 3.2.12:** **(1)**  $\emptyset$  and  $\mathbb{R}^n$  are closed. **(2)** The open disc  $D = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$  for  $r > 0$  and  $x_0 \in \mathbb{R}^n$  is bounded and not closed. **(3)** The closed disc  $\Delta = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$  is bounded and closed. In particular, a closed interval is a closed set. An interval is compact if it is bounded **(4)** If  $X_1 \subseteq \mathbb{R}^n$  and  $X_2 \subseteq \mathbb{R}^m$  are bounded (also closed or compact), then so is  $X_1 \times X_2 \subseteq \mathbb{R}^{n+m}$

**P 3.2.13:** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a continuous map. For any closed  $Y \subseteq \mathbb{R}^m$ , the set  $f^{-1}(Y) = \{x \in \mathbb{R}^n : f(x) \in Y\} \subseteq \mathbb{R}^n$  is closed

**Ex 3.2.14:** The zero set  $Z = \{x \in \mathbb{R}^n : f(x) = 0\}$  is closed in  $\mathbb{R}^n$  because  $\{0\} \subseteq \mathbb{R}$  is closed. More generally: for any  $r \geq 0$ ,  $\{x \in \mathbb{R}^n : |f(x)| \leq r\}$  is  $f^{-1}([-r, r])$  and is closed, since  $[-r, r]$  is closed. Furthermore:  $\{x \in \mathbb{R}^3 : \|x - x_0\| = r\}$  is closed

**T 3.2.15:** Let  $(X \neq \emptyset) \subseteq \mathbb{R}^n$  compact and  $f : X \rightarrow \mathbb{R}$  continuous. Then  $f$  bounded, has max and min, i.e.  $\exists x_+, x_- \in X$  s.t.  $f(x_+) = \sup_{x \in X} f(x)$  and  $f(x_-) = \inf_{x \in X} f(x)$

#### 3.3 Partial derivatives

**D 3.3.1:**  $X \subseteq \mathbb{R}^n$  **open** if for any  $x = (x_1, \dots, x_n) \in X \exists \delta > 0$  s.t.  $\{y = (y_1, \dots, y_n) \in \mathbb{R}^n : |x_i - y_i| < \delta \forall i\}$  is contained in  $X$ . (= changing a coordinate of  $x$  by  $< \delta \rightarrow x' \in X$ ) **P 3.3.2:**  $X \subseteq \mathbb{R}^n$  open  $\Leftrightarrow$  **complement**  $Y = \{x \in \mathbb{R}^n : x \notin X\}$  is closed

**C 3.3.3:** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  cont. and  $Y \subseteq \mathbb{R}^m$  open, then  $f^{-1}(Y)$  is open in  $\mathbb{R}^n$  **Ex 3.3.4:** **(1)**  $\emptyset$  and  $\mathbb{R}^n$  are both open and closed. **(2)** Open ball  $D = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$  is open in  $\mathbb{R}^n$  ( $x_0$  the center and  $r$  radius) **(3)**  $I_1 \times \dots \times I_n$  is open in  $\mathbb{R}^n$  for  $I_i$  open **(4)**  $X \subseteq \mathbb{R}^n$  open  $\Leftrightarrow \forall x \in X \exists \delta > 0$  s.t. open ball of center  $x$  and radius  $\delta$  is contained in  $X$

**D 3.3.5:** (Partial derivative) Let  $X \subseteq \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$  and  $1 \leq i \leq n$ . Then  $f$  has partial derivative on  $X$  with respect to the  $i$ -th variable (or coordinate), if  $\forall x_0 = (x_{0,1}, \dots, x_{0,n}) \in X$ ,  $g(t) = f(x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n})$  on set  $I = \{t \in \mathbb{R} : (x_{0,1}, \dots, x_{0,i-1}, t, x_{0,i+1}, \dots, x_{0,n}) \in X\}$  is differentiable at  $t = x_{0,i}$ . The derivative  $g'(x_{0,i})$  at  $x_{0,i}$  is denoted:  $\frac{\partial f}{\partial x_i}(x_0)$ ,  $\partial_{x_i} f(x_0)$  or  $\partial_i f(x_0)$

**P 3.3.7:** Let  $X \subseteq \mathbb{R}^n$  open,  $f, g : X \rightarrow \mathbb{R}^m$  and  $1 \leq i \leq n$ . Then: **(1)** If  $f$  &  $g$  have  $\partial_i$  on  $X$ , then so does  $f + g$  and  $\partial_{x_i}(f + g) = \partial_{x_i}(f) + \partial_{x_i}(g)$  **(2)** If  $m = 1$  (i.e.  $\mathbb{R}^1$ ) and  $f$  &  $g$  have  $\partial_i$  on  $X$ , then so does  $fg$  and  $\partial_{x_i}(fg) = \partial_{x_i}(f)g + f\partial_{x_i}(g)$  and if  $g(x) \neq 0 \forall x \in X$ , then if  $f \div g$  has  $\partial_i$  on  $X$ , then so does  $\partial_{x_i}(f \div g) = (\partial_{x_i}(f)g - f\partial_{x_i}(g)) \div g^2$

**D 3.3.8:** (Jacobi Matrix  $J$ ) Element  $J_{ij} = \partial_{x_j} f_i(x)$  for function  $f : X \rightarrow \mathbb{R}^m$  with  $X \subseteq \mathbb{R}^n$  open.  $x_j$  is the  $j$ -th variable,  $f_i$  is the  $i$ -th component of the equation (i.e. in the vector of the function).  $J$  has  $m$  rows and  $n$  columns.

**D 3.3.10:** (Gradient, Divergence) for  $f : X \rightarrow \mathbb{R}$  with  $X \subseteq \mathbb{R}^n$  open, the **gradient** is given by  $\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{pmatrix}$  and the trace of the Jacobi Matrix,  $\text{div}(f)(x_0) = \text{Tr}(J_f(x_0)) = \sum_{i=1}^n \partial_{x_i} f_i(x_0)$  is called the **divergence** of  $f$  at  $x_0$ .

### 3.4 The differential

**D 3.4.2:** (*Differentiable function*) We have function  $f : X \rightarrow \mathbb{R}^m$ , linear map  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $x_0 \in X$ .  $f$  is differentiable at  $x_0$  with differential  $u$  if  $\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{1}{\|x - x_0\|} (f(x) - f(x_0) - u(x - x_0)) = 0$  where the limit is in  $\mathbb{R}^m$ . We denote  $df(x_0) = u$ . If  $f$  is

differentiable at every  $x_0 \in X$ , then  $f$  is differentiable on  $X$ . **P 3.4.4:** Let  $f : X \rightarrow \mathbb{R}^m$  be differentiable on  $X$

- $f$  is continuous on  $X$
- $f$  admits partial derivatives on  $X$  with respect to each variable
- Assume  $m = 1$ , let  $x_0 \in X$  and let  $u(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n$  be diff. of  $f$  at  $x_0$ . Then  $\partial_{x_i}f(x_0) = a_i$  for  $1 \leq i \leq n$

**P 3.4.6:** Let  $f, g : X \rightarrow \mathbb{R}^m$  with  $X \subseteq \mathbb{R}^n$  open

- The function  $f + g$  is differentiable with differential  $d(f + g) = df + dg$ . If  $m = 1$ , then  $fg$  is differentiable
- If  $m = 1$  and if  $g(x) \neq 0 \forall x \in X$ , then  $f \div g$  is differentiable

**P 3.4.7:** If  $f$  as above has all partial derivatives on  $X$  and if they are all continuous on  $X$ , then  $f$  is differentiable on  $X$ . The differential is the Jacobi Matrix of  $f$  at  $x_0$ . This implies that most elementary functions are differentiable.

**P 3.4.8:** (*Chain Rule*) For  $X \subseteq \mathbb{R}^n$  and  $Y \subseteq \mathbb{R}^m$  both open and  $f : X \rightarrow Y$  and  $g : Y \rightarrow \mathbb{R}^p$  are both differentiable. Then  $g \circ f$  is differentiable on  $X$  and for any  $x \in X$ , its differential is given by  $d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$ . The Jacobi matrix is  $J_{g \circ f}(x_0) = J_g(f(x_0))J_f(x_0)$  (RHS is a matrix product)

**D 3.4.11:** (*Tangent space*) The graph of the affine linear approximation  $g(x) = f(x_0) + u(x - x_0)$ , or the set

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = f(x_0) + u(x - x_0)\}$$

**D 3.4.13:** (*Directional derivative*)  $f$  has a directional derivative  $w \in \mathbb{R}^m$  in the direction of  $v \in \mathbb{R}^n$ , if the function  $g$  defined on the set  $I = \{t \in \mathbb{R} : x_0 + tv \in X\}$  by  $g(t) = f(x_0 + tv)$  has a derivative at  $t = 0$  and is equal to  $w$ . **R 3.4.14:** Because  $X$  is open, the set  $I$  contains an open interval  $]-\delta, \delta[$  for some  $\delta > 0$ . **P 3.4.15:** Let  $f$  as previously be differentiable. Then for any  $x \in X$  and non-zero  $v \in \mathbb{R}^n$ ,  $f$  has a directional derivative at  $x_0$  in the direction of  $v$ , given by  $df(x_0)(v)$ . **R 3.4.16:** The values of the above directional derivative are linear with respect to the vector  $v$ . Suppose we know the dir. der.  $w_1$  and  $w_2$  in directions  $v_1$  and  $v_2$ , then the directional derivative in direction  $v_1 + v_2$  is  $w_1 + w_2$

### 3.5 Higher derivatives

**D 3.5.1:**  $f$  is in class  $C^1$  if  $f$  is differentiable and all its partial derivatives are continuous.  $f$  is of class  $C^k$  if it is differentiable and each of its partial derivatives are in  $C^{k-1}$ . If  $f \in C^k(X; \mathbb{R}^m)$  for all  $k \geq 1$ , then  $f \in C^\infty(X; \mathbb{R}^m)$

**P 3.5.4:** (*Mixed derivatives commute*)  $\partial_{x,y}f = \partial_{y,x}$ , as well as  $\partial_{x,y,z} = \partial_{x,z,y} = \dots$ , etc (all mixed derivatives commute) Since we have symmetry, we can use the notation  $\partial_{x_1^{m_1}, \dots, x_n^{m_n}} f = \frac{\partial^k}{\partial x^m} f = D^m f = \partial^m f$ , where  $m = (m_1, \dots, m_n)$  and  $m_1 + \dots + m_n = k$ . There are  $\binom{n+k-1}{k}$  possible values for  $m$  and e.g.  $(1, 1, 2)$  corresponds to the derivative  $\frac{\partial^4 f}{\partial x \partial y \partial^2 z}$

**R 3.5.6:** Due to linearity of the partial derivative  $\partial_x^m (af_1 + bf_2) = a\partial_x^m f_1 + b\partial_x^m f_2$

**Ex 3.5.8:** (*Laplace operator*)  $f \in C^2(X)$ ,  $\nabla f \in C_1(X; \mathbb{R}^n)$ , so  $\text{div}(\nabla f) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{\partial f}{\partial x_i} \right) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$  (called **Laplacian**,  $\Delta f$ )

**D 3.5.9:** (*Hessian*)  $f : X \rightarrow \mathbb{R}$  in  $C^2$ . For  $x \in X$ , the **Hessian matrix** of  $f$  at  $x$  is the symmetric square matrix

$$\text{Hess}_f(x) = (\partial_{x_i, x_j} f)_{1 \leq i, j \leq n} = H_f(x) \quad (i\text{-th row, } j\text{-th column})$$

### 3.6 Change of variable

The idea is to substitute variables for others that make the equation easier to solve. A common example is to switch to polar coordinates from cartesian coordinates, as already demonstrated with continuity checks

### 3.7 Taylor polynomials

**D 3.7.1:** (*Taylor polynomials*) Let  $f : X \rightarrow \mathbb{R}$  with  $f \in C^k(X, \mathbb{R})$  and  $y \in X$ . The Taylor-Polynomial of order  $k$  of  $f$  at  $y$  is:

$$T_k f(y; x - y) = \sum_{|i| \leq k} \frac{\partial_i f(y)(x - y)^i}{i!}$$

where  $i$  is a *multi-index*, so:

- $i = (i_1, \dots, i_n)$  (each  $i_j \geq 0$ )
- $|i| = i_1 + \dots + i_n$
- $\partial_i = \partial_1^{i_1} \dots \partial_n^{i_n}$
- $(x - y)^i = (x_1 - y_1)^{i_1} \dots (x_n - y_n)^{i_n}$
- $i! = i_1! \dots i_n!$

The concept this formula uses is that we iterate through all possible partial derivatives of  $f$  and assigns each a multi-index  $i$ . To denote that we want to take the partial derivative  $\partial_{112}$ , we use  $i = (2, 1, 0)$ , since we take the derivative of the first variable twice, of the second variable once and never of the third variable. So the expression is thus now:

$$\frac{\partial_{112} f(y)(x_1 - y_1)^2 (x_2 - y_2)^1 (x_3 - y_3)^0}{2!1!0!} = \frac{\partial_{112} f(y)(x_1 - y_1)^2 (x_2 - y_2)}{2}$$

### 3.8 Critical points

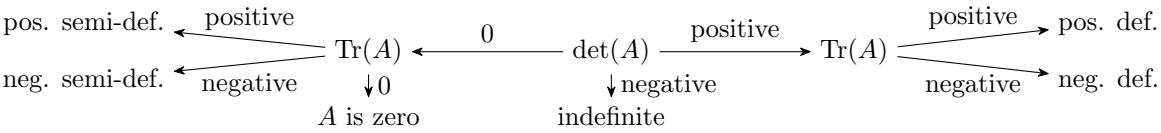
**D 3.8.2:** (*Critical Point*) For  $f : X \rightarrow \mathbb{R}^n$  differentiable,  $x_0 \in X$  is called a **critical point** of  $f$  if  $\nabla f(x_0) = 0$  **R 3.8.3:** As in 1 dimensional case, check edges of the interval for the critical point.

To determine the kind of critical point, we need to determine if  $H_f(x_0)$  is definite:

- positive definite  $\Rightarrow x_0$  local max
- negative definite  $\Rightarrow x_0$  local min
- indefinite  $\Rightarrow x_0$  point of inflection

**D 3.8.6:** (*Non-degenerate critical point*) If  $\det(H_f(x_0)) \neq 0$  (if  $H_f(x_0)$  is semi-definite, then  $\det(H_f(x_0)) = 0$ , thus degenerate) To figure out if a matrix is definite, we can compute the eigenvalues.  $A$  is positive (negative) definite, if and only if all eigenvalues are greater (lower) than 0.  $A$  is indefinite if and only if it has both positive and negative eigenvalues.  $A$  is positive (negative) semi-definite if and only if all eigenvalues are greater (lower) or equal to 0. (Compute Eigenvalues using  $\det(A - \lambda I) = 0$ )

For  $2 \times 2$  matrices, we can use the following scheme:



## 4 Integral Calculus in Vector Space

### 4.1 Line integrals

**D 4.1.1:** Let  $I = [a, b]$  be a closed and bounded interval in  $\mathbb{R}$ .  $f : I \rightarrow \mathbb{R}$  with  $f(t) = (f_1(t), \dots, f_n(t))$  continuous (also  $f_i$  cont.).

(1) Then 
$$\int_a^b f(t) \, dt = \left( \int_a^b f_1(t), \dots, \int_a^b f_n(t) \right)$$

(2) **Parametrized Curve** in  $\mathbb{R}^n$  is a continuous map  $\gamma : I \rightarrow \mathbb{R}^n$ , piecewise in  $C^1$ , i.e. for  $k \geq 1$ , we have partition  $a = t_0 < t_1 < \dots < t_k = b$ , such that if  $f$  is restricted to interval  $]t_{j-1}, t_j]$ , restriction is  $C^1$ .  $\gamma$  is a *path* between  $\gamma(a)$  and  $\gamma(b)$

(3) **Line integral**  $X \subseteq \mathbb{R}^n$  is the image of  $\gamma$  as above and  $f : X \rightarrow \mathbb{R}^n$  cont.

Integral  $\int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt \in \mathbb{R}$  is line integral of  $f$  along  $\gamma$ , denoted  $\int_\gamma f(s) \, ds$  or  $\int_\gamma f(s) \, d\vec{s}$  or  $\int_\gamma \omega$ ,

with  $\omega = f_1(x) \, dx_1 + \dots + f_n(x) \, dx_n$

We usually call  $f : X \rightarrow \mathbb{R}^n$  a **vector field**, which maps each point  $x \in X$  to a vector in  $\mathbb{R}^n$ , displayed as originating from  $x$

**D 4.1.4:** (*Oriented reparametrization*) of  $\gamma$  is parametrized curve  $\sigma : [c, d] \rightarrow \mathbb{R}^n$  s.t.  $\sigma = \gamma \circ \varphi$ , with  $\varphi : [c, d] \rightarrow I$  cont. map, differentiable on  $]a, b[$  and for which  $\varphi(a) = c$  and  $\varphi(b) = d$ . Conversely,  $\gamma = \sigma \circ \varphi^{-1}$

**P 4.1.5:** For  $f : X \rightarrow \mathbb{R}^n$  with  $X$  containing the image of  $\gamma$  and equivalently  $\sigma$ , we have  $\int_\gamma f(s) \cdot d\vec{s} = \int_\sigma f(s) \cdot d\vec{s}$

**D 4.1.8:** (*Conservative Vector Field*) If for any  $x_1, x_2 \in X$  the line integral  $\int_\gamma f(s) \, d\vec{s}$  is independent choice of  $\gamma$  in  $X$

**R 4.1.9:**  $f$  conservative iff  $\int_\gamma f(s) \, d\vec{s} = 0$  for a *closed* ( $\gamma(a) = \gamma(b)$ ) parametrized curve

**T 4.1.10:** Let  $X$  be open set,  $f$  conservative vector field. Then  $\exists C^1$  function  $g$  s.t.  $f = \nabla g$ . If any two points of  $X$  can be joined by a parametrized curve, then  $g$  is unique up to a constant: if  $\nabla g_1 = f$ , then  $g - g_1$  is constant on  $X$

**R 4.1.11:** If  $f$  vec. field on  $X$ , then  $g$  is called a **potential** for  $f$

### 4.2 Riemann integral in Vector Space

### 4.3 Improper integrals

### 4.4 Change of Variable Formula

### 4.5 The Green Formula