Inference Rules for Probability Logic by Marija Boricic

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From A to $A[a, b]$

- Start with a sentence A.
- Then $A[a, b]$ means "*The probability of A being true lies in the interval* $[a, b]$".
- **NKprob** - Probabilized natural deduction system.

Plan of the paper

- Develop the syntax i.e. get the best probability bounds for sentences that involve logical connectives.
- Prove the obtained logic is sound and complete.

The system **NKprob**

- I is a finite subset of $[0, 1]$ containing 0 and 1.
- I is closed under $+$ as follows.
- $a + b := \min(1, a + b)$.
- $a + b - 1 := \max(0, a + b - 1)$.
- Example : $I = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ is such a set. If I_1, I_2 are two such sets then so is $I_1 \cap I_2$.

The system **NKprob**

Definition

For each propositional formula A and each $a, b \in I$, the object $A[a, b]$ is called the *probabilized formula*.

- Write $A\emptyset$ when $b < a$.
- $A[a, a]$ for $a = b$.
- Note that $A[a, b] = A[c, c]$ for some $c \in [a, b]$.
- Every I and formula A generate a finite list of probabilized formulas $A[a, b]$ for $a, b \in I$.

The system **NKprob**

- Combining probabilized formulas is not allowed.
- For example we cannot write things like $A[a, b] \wedge B[c, d]$.
- But we can infer, for instance, $(A \vee B)[?, ?]$ from $A[a, b]$ and $B[c, d]$.

Axioms for **NKprob**

1. For each propositional formula A provable in classical logic, $A[1,1]$ is an axiom of **NKprob**.
2. A list of rules of inference.

Conjunctions

$$\frac{A[a,b] \qquad B[c,d]}{(A \wedge B)[a+c-1, \min(b,d)]}\,(\mathrm{I}\wedge)$$

$$\frac{A[a,b] \qquad (A \wedge B)[c,d]}{B[c, 1+d-a]}\,(\mathrm{E}\wedge)$$

Disjunctions

$$\frac{A[a,b] \qquad B[c,d]}{(A \vee B)[\max(a,c), b+d]}\,(\mathrm{I}\vee)$$

$$\frac{A[a,b] \qquad (A \vee B)[c,d]}{B[c-b,d]}\,(\mathrm{E}\vee)$$

Implication

$$\frac{A[a,b] \qquad B[c,d]}{(A \to B)[\max(1-b,c), 1-a+d]}\,(\mathrm{I}\!\to)$$

$$\frac{A[a,b] \qquad (A \to B)[c,d]}{B[a+c-1,d]}\,(E_1 \to)$$

$$\frac{B[a,b] \qquad (A \to B)[c,d]}{A[1-d, 1-c+b]}\,(E_2 \to)$$

Negation

$$\frac{A[a, b]}{(\neg A)[1-b, 1-a]}\ (\mathsf{I}\neg)$$

$$\frac{(\neg A)[a, b]}{A[1-b, 1-a]}\ (\mathsf{E}\neg)$$

Let's take a break!

- We know $A \to B \equiv \neg A \vee B$.
- So we must have $(A \to B)[?, ?] = (\neg A \vee B)[?, ?]$ once we know $A[a, b]$ and $B[c, d]$.
- This is true indeed. By (I$\neg$) we get $(\neg A)[1 - b, 1 - a]$ from $A[a, b]$.
- By (I$\vee$) we get

$$(\neg A \vee B)[\max(1 - b, c), 1 - a + d].$$

- And by (I$\to$) we have

$$(A \to B)[\max(1 - b, c), 1 - a + d].$$

- It turns out this result can be generalized to all logically equivalent formulas in classical logic.

Additivity Rule

$$\frac{A[a, b] \qquad B[c, d] \qquad (A \wedge B)[e, f]}{(A \vee B)[a + c - f, b + d - e]}\ (\mathrm{ADD})$$

Monotonicity Rules

$$\frac{A[a, b] \qquad A[c, d]}{A[\max(a, c), \min(b, d)]}\ (\mathrm{M}\!\downarrow)$$

$$\frac{A[a, b]}{A[c, d]}\ (\mathrm{M}\!\uparrow)$$

- For M$\uparrow$ we suppose $[a, b] \subseteq [c, d]$.
- What M$\downarrow$ does is taking the intersection of $[a, b]$ and $[c, d]$.

Derivations in **NKprob**

$A[a, b]$ is derived from a set of hypotheses Γ in **NKprob** if there is a finite sequence of probabilized formulas ending with $A[a, b]$ such that each formula is either

- an axiom,
- from Γ, or
- obtained by a rule of inference in **NKprob** applied to some previous formulas from the sequence.

We denote this by $\Gamma \vdash A[a, b]$. Note that the inference rules can be expressed as derivation rules.

Rules for Inconsistency

Let $a, b \in I = \{c_1 \ldots c_n\}$. Recall that $A\emptyset$ stands for $A[a, b]$ with $b < a$. Then the rule (I$\emptyset$) is the following.

- From $\Gamma \cup \{A[c_1, c_1]\} \vdash A\emptyset, \ldots \Gamma \cup \{A[c_n, c_n]\} \vdash A\emptyset$ we can deduce $\Gamma \vdash A\emptyset$.

The rule (E$\emptyset$) is

- From $\Gamma \vdash A\emptyset$ we can deduce $\Gamma \vdash B[a, b]$.

I'd like to stop here and discuss these two rules. 😄

Equivalent formulas have equal probabilities of being true

Lemma

If $A \leftrightarrow B$ is provable in classical logic and $A[a, b]$ is provable in **NKprob**, *then $B[a, b]$ is provable in* **NKprob**.

Proof.

- If $A \leftrightarrow B$ is provable in classical logic then both $(A \to B)[1, 1]$ and $(B \to A)[1, 1]$ are among the axioms of **NKprob**.
- From $A[a, b]$ and $(A \to B)[1, 1]$ we obtain $B[a + 1 - 1, 1] \equiv B[a, 1]$ by an application of $(E_1 \to)$.
- From $A[a, b]$ and $(B \to A)[1, 1]$ we get $B[1 - 1, 1 - 1, b] \equiv B[0, b]$ by applying $(E_2 \to)$.
- Applying $(M \downarrow)$ to $B[a, 1]$ and $B[0, b]$ we get $B[a, b]$.

$\square$

Equivalent formulas have equal probabilities of being true

Corollary

If $A \leftrightarrow B$ is provable in classical logic and $A[a, a]$ is provable in **NKprob**, *then $B[a, a]$ is provable in* **NKprob**.

NKprob Theories

- An **NKprob** theory is a set of formulas derivable from a set of hypotheses $\{A_1[a_1, b_1] \ldots A_n[a_n, b_n]\}$ in **NKprob**.
- This is denoted by **NKprob**$\{A_1[a_1, b_1] \ldots A_n[a_n, b_n]\}$.
- The theory **NKprob**$\{A_1[a_1, b_1] \ldots A_n[a_n, b_n]\}$ is inconsistent if there is a formula A so that both $A[a, b]$ and $A[c, d]$ are in the theory with $[a, b] \cap [c, d] = \emptyset$.

Consistent theories to Maximal consistent theories

Lemma

Each consistent **NKprob***-theory can be extended to a maximal consistent theory.*

Proof.

Let $\mathcal{T}$ be a consistent theory. Let $A_0, A_1, \ldots$ be the list of all propositional formulas. Since I is finite we can order the elements of I and write $I = \{c_1, \ldots, c_m\}$. So we get the list of all probabilized formulas $A_0[c_1, c_1] \ldots A_n[c_m, c_m] \ldots$. Now build a sequence $\langle \mathcal{T}_n \rangle$ recursively as follows.

- Put $\mathcal{T}_0 = \mathcal{T}$
- Put $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{A_n[c_1, c_1]\}$ if $\mathcal{T}_n \cup \{A_n[c_1, c_1]\}$ is consistent.
- If NOT, put $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{A_n[c_2, c_2]\}$ if $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{A_n[c_2, c_2]\}$ is consistent.

  $\vdots$
- If NOT, put $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{A_n[c_m, c_m]\}$.

Now let $\mathcal{T}' = \bigcup_{n \in \omega} \mathcal{T}_n$.

Proof (Cont.)

- Claim : If $\mathcal{T}_n$ is consistent, then $\mathcal{T}_{n+1}$ is consistent. $\mathcal{T}_0 = \mathcal{T}$ is consistent. Suppose $\mathcal{T}_1 = \mathcal{T} \cup \{A_1[c_i, c_i]\}$ is not consistent for any $i \leq m-1$. Then $\mathcal{T} \cup \{A_1[c_m, c_m]\}$ must be consistent. If not, there's A so that

$$\mathcal{T} \cup \{A_1[c_m, c_m]\} \vdash A[a, b] \wedge A[c, d]$$

with $[a, b] \cap [c, d] = \emptyset$. So $\mathcal{T} \cup \{A_1[c_m, c_m]\} \vdash A\emptyset$ by $(M\downarrow)$. So for all $1 \leq i \leq m$ we have

$$\mathcal{T} \cup \{A_1[c_i, c_i]\} \vdash A\emptyset,$$

and by $(E\emptyset)$

$$\mathcal{T} \cup \{A_1[c_i, c_i]\} \vdash A_1\emptyset$$

for all $1 \leq i \leq m$. But then by $(I\emptyset)$

$$\mathcal{T} \vdash A_1\emptyset.$$

Proof (Cont.)

Now by applying $(E\emptyset)$ one more time we get

$$\mathcal{T} \vdash A_1[a,b] \wedge A_1[c,d]$$

for some choice of $a,b,c,d \in I$ with $[a,b] \cap [c,d] = \emptyset$, contradiction.

- Claim : $\mathcal{T}'$ is maximally consistent. Let $\mathcal{T}^*$ be a proper extension of $\mathcal{T}'$. Then $\mathcal{T}^*$ has some $A_k[a,b]$ which is not in $\mathcal{T}'$. But $A_k[a,b] \equiv A_k[c,c]$ for some $c \in [a,b]$. So $A_k[c,c] \notin \mathcal{T}'$. But $A_k[d,d] \in \mathcal{T}_{k+1}$ for some $d \in I$. So $c \neq d$. Since $\mathcal{T}^*$ extends $\mathcal{T}_{k+1}$ we have

$$\mathcal{T}^* \vdash A_k[c,c] \wedge A_k[d,d].$$

But $[c,c] \cap [d,d] = \emptyset$. Therefore $\mathcal{T}^*$ is inconsistent. Hence $\mathcal{T}'$ is maximally consistent. $\square$

NKprob-models

- **For** $=$ the set of all propositional formulas.
- $I \subseteq [0,1]$, finite, contains 0 and 1, and closed under addition.
- A map $p : \mathbf{For} \to I$ is an **NKprob**-model if
  1. $p(\top) = 1$ and $p(\bot) = 0$.
  2. If $p(A \wedge B) = 0$, then $p(A \vee B) = p(A) + p(B)$.
  3. If $A \leftrightarrow B$ in classical logic, then $p(A) = p(B)$.
- Satifiability in a model : $\vDash_p A[a,b]$ iff $a \leq p(A) \leq b$.
- The inference rules can be justified now.

Towards Soundness of **NKprob**

1. $p(A) + p(B) = p(A \vee B) + p(A \wedge B)$. (Additivity Rule)

Proof.

$p(A) = p(A \wedge B) + p(A \wedge \neg B)$ as $(A \wedge B) \wedge (A \wedge \neg B) \equiv \bot$.
Similarly, $p(A \vee B) = p((A \wedge \neg B) \vee B) = p(A \wedge \neg B) + p(B)$. So
$p(A) + p(B) = p(A \vee B) + p(A \wedge B)$. $\square$

2. $p(\neg A) = 1 - p(A)$.

Proof.

By $p(\bot) = 0$, $p(A \wedge \neg A) = 0$. So
$1 = p(\top) = p(A \vee \neg A) = p(A) + p(\neg A)$. So
$p(\neg A) = 1 - p(A)$. $\square$

Towards Soundness of **NKprob**

3. If $A \to B$ in classical logic, then $p(A) \leq p(B)$.

Proof.

$p(\neg A) + p(B) = p(\neg A \vee B) + p(\neg A \wedge B)$. So $p(B) = p(\neg A \vee B) + p(\neg A \wedge B) - p(\neg A)$ Since $\neg A \vee B \equiv A \to B$ and $A \to B$ in classical logic, $p(\neg A \vee B) = 1$. So $p(B) = 1 + p(\neg A \wedge B) - 1 + p(A) \geq p(A)$. $\square$

4. $p(A) + p(B) - 1 \leq p(A \wedge B) \leq \min(p(A), p(B))$. (This is the rule $I\wedge$)

Proof.

Since $p(A \vee B) \leq 1$, $p(A \wedge B) = p(A) + p(B) - p(A \vee B) \geq p(A) + p(B) - 1$. $p(A) = p((A \wedge B) \vee (A \wedge \neg B)) = p(A \wedge B) + p(A \wedge \neg B)$. Therefore $p(A \wedge B) = p(A) - p(A \wedge \neg B) \leq p(A)$. Changing roles of A and B, $p(A \wedge B) \leq p(B)$. So $p(A \wedge B) \leq \min(p(A), p(B))$. $\square$

Towards Soundness of **NKprob**

- We can justify the rest of the rules similarly.
- If we can justify the rules $I\emptyset$ and $E\emptyset$ then **NKprob** is sound.
- So let's do that. ????

Towards Completeness . . . Canonical Model

- $\mathrm{Cn}(\mathbf{NKprob}(\sigma_1 \ldots \sigma_n))$ is the set of all $\mathbf{NKprob}(\sigma_1 \ldots \sigma_n)$-provable formulas.
- $\mathrm{ConExt}(\mathrm{Cn}(\mathbf{NKprob}(\sigma_1 \ldots \sigma_n)))$ is the class of all maximally consistent extensions of $\mathrm{Cn}(\mathbf{NKprob}(\sigma_1 \ldots \sigma_n))$.
- *Canonical Model* : Let $X \in \mathrm{ConExt}(\mathrm{Cn}(\mathbf{NKprob}(\sigma_1 \ldots \sigma_n)))$.
- Define $\vDash_{p^X} A[a, b]$ iff

$$a \leq \max\{c : A[c, 1] \in X\}$$

  and

$$b \geq \min\{c : A[0, c] \in X\}.$$

- p^X is called a canonical model.

Towards Completeness ... Canonical Model

- A canonical model is a model by virtue of the results we have in hand.
- We also have $\models_{p^X} A[a, b]$ iff $A[a, b] \in X$. This is due to the fact that X is maximally consistent/deductively closed.
- So if we have a consistent theory then we can extend it to a maximal consistent theory.
- Then we take a *world* X and the canonical model p^X.
- And by the second bullet point we have completeness of **NKprob**.
- So we'll prove the claims in the first two bullet points.

Canonical model is a model

Lemma

Let p^X be a canonical model. Then p^X is a model.

Proof.

Note because we are in the realm of X we have the additivity rule for free. So if $p^X(A \wedge B) = 0$, then $p^X(A \vee B) = p^X(A) + p^X(B)$. Also unravelling the definition of $\vDash_{p^X} A[a, b]$ proves $p^X(\top) = 1$ and $p^X(\bot) = 0$. Finally if $A \leftrightarrow B$ in classical logic, in X we have "the probability of A being true is the same as the probability of B being true". So $p^X(A) = p^X(B)$. □

$\vDash_{p^X} A[a, b]$ iff $A[a, b] \in X$

Lemma

$\vDash_{p^X} A[a, b]$ *iff* $A[a, b] \in X$.

Proof.

X is deductively closed. So we have the following. By $M\uparrow$, $A[a, 1] \in X$ and $A[0, b] \in X$. By $M\downarrow$, $A[a, b] \in X$. Conversely suppose $A[a, b] \in X$. Then by $M\uparrow$, $A[a, 1] \in X$. So

$$a \leq \max\{c : A[c, 1] \in X\}.$$

By $M\uparrow$ again, $A[0, b] \in X$. So

$$b \geq \min\{c : A[0, c] \in X\}.$$

Hence $\vDash_{p^X} A[a, b]$. $\square$