An Ordered Set of Arithmetic Functions Representing the least ϵ-number (Hilbert Levitz)

The set $\mathcal{S}$

- Let $\mathcal{S}$ be the smallest class of functions $\mathbb{N} \to \mathbb{N}$ so that:
  1. $1 \in \mathcal{S}$,
  2. If $f, g \in \mathcal{S}$ and $n \in \mathbb{N}$, then $f+g,\ f \cdot g,\ x^f$, and $n^f \in \mathcal{S}$.
- Levitz : type($\mathcal{S}$)=ϵ_0.

Cantor Normal Form

- Cantor Normal Form : Every ordinal $\alpha > 0$ has the unique form $\alpha = \omega^{\beta_1} k_1 + \cdots + \omega^{\beta_n} k_n$, where $n \geq 1$, $\alpha \geq \beta_1 > \cdots > \beta_n$, and $k_1, \ldots, k_n \in \mathbb{N}^{>0}$.
- Note that we can write

$$\begin{aligned} \alpha &= \underbrace{\omega^{\beta_1} + \cdots + \omega^{\beta^1}}_{k_1-\text{terms}} + \cdots + \underbrace{\omega^{\beta_n} + \cdots + \omega^{\beta_n}}_{k_n-\text{terms}} \\ &= \omega^{\alpha_1} + \cdots + \omega^{\alpha_k}, \end{aligned}$$

  where $\alpha_1 \geq \cdots \geq \alpha_k$ and $k \geq 1$.
- Let $\gamma = \sum_{i=2}^{k} \omega^{\alpha_i}$. Then $\omega^{\alpha_1} + \gamma$ is called the reduced normal form of α.

Reduced Normal Forms

If α, β have reduced normal forms $\omega^{\alpha_1} + \gamma$ and $\omega^{\beta_1} + \delta$ respectively, then $\alpha < \beta$ iff

1. $\omega^{\alpha_1} < \omega^{\beta_1}$ (equivalently $\alpha_1 < \beta_1$), or
2. $\omega^{\alpha_1} = \omega^{\beta_1}$ and $\gamma < \delta$.

Main Ordinals

- Main ordinals : Ordinals of the form ω^{β} ($\beta \geq 0$).
- If $\alpha \neq 1$ is a main ordinal

$$\alpha = \omega^{\omega^{\alpha_1} n_1} \omega^{\omega^{\alpha_2} n_2} \cdots \omega^{\omega^{\alpha_k} n_k}$$

  with $\alpha_1 > \alpha_2 > \cdots > \alpha_k$.
- Write $\beta = \omega^{\alpha_1} k_1 + \cdots + \omega^{\alpha_n} k_n$ with $\beta \geq \alpha_1 > \cdots > \alpha_n$.
- Then $\omega^{\beta} = \omega^{\omega^{\alpha_1} k_1 + \cdots + \omega^{\alpha_n} k_n} = \omega^{\omega^{\alpha_1} k_1} \cdots \omega^{\omega^{\alpha_n} k_n}$.
- $\alpha = \left(\omega^{\omega^{\alpha_1} n_1}\right)\gamma$, where $\gamma = \prod_{i=2}^{k} \omega^{\omega^{\alpha_1} n_i}$ ($\gamma = 1$ if $k = 1$) is called the reduced form of α.

The functions A and B

- Define functions $B : \mathbf{On} \to \mathbf{On}$ and $A : \mathbf{On}^{>0} \to \mathbf{On}$ as follows.

  1. $B(0) = 0$
  2. $A(\alpha + 1) = B(\alpha) + 1 \ (\alpha \geq 0)$
  3. $B(\alpha + 1) = A(\alpha + 1) + 1 \ (\alpha \geq 0)$
  4. $A(\alpha) = \alpha$ if α is a limit
  5. $B(\alpha) = A(\alpha) + 1$ if α is a limit.

The functions A and B

- Observe that we have

$$B(\alpha) = \begin{cases} 0, \text{ if } \alpha = 0, \\ B(\beta) + 2, \text{ if } \alpha = \beta + 1, \\ \alpha + 1, \text{ if } \alpha \text{ is a limit.} \end{cases}$$

B maps finite ordinals to "even numbers", A maps finite nonzero ordinals to "odd numbers". B maps limits to their successors and A does nothing to limits.

The functions A and B

The functions A and B have the following properties. If $\alpha < \beta$,

1. $A(\alpha) < A(\beta)$
2. $B(\alpha) < B(\beta)$
3. $A(\alpha) < B(\beta)$
4. $B(\alpha) < A(\beta)$
5. $\text{Range}(A) \cap \text{Range}(B) = \emptyset$
6. $A(\alpha) < B(\alpha)$.

$\mathcal{S}'$-Normal Form

- $\alpha = \omega^{\omega^{\alpha_1} n_1} \cdots \omega^{\omega^{\alpha_k} n_k}$, where $\alpha_1 \geq \cdots \geq \alpha_k$ and $n_i = 1$ if α_i is of the form $B(x)$ for some x. This is called the $\mathcal{S}'$-normal form of α.
- Reduced $\mathcal{S}'$-normal form is $\omega^{\omega^{\alpha_1} n_1}\gamma$, where $\gamma = \omega^{\omega^{\alpha_2} n_2} \cdots \omega^{\omega^{\alpha_k} n_k}$ ($\gamma = 1$ if $k = 1$).
- Example :

$$\begin{aligned}\omega^{\omega^2 2 + \omega 3} &= (\omega^{\omega^2})(\omega^{\omega^2})(\omega^{\omega 3}) \ (\mathcal{S}' - \text{normal form}) \\ &= (\omega^{\omega^2})\gamma \ (\text{reduced } \mathcal{S}' - \text{normal form})\end{aligned}$$

$$\omega^{\omega^3 2 + \omega^2 3} = (\omega^{\omega^3 2})\underbrace{(\omega^{\omega^2})(\omega^{\omega^2})(\omega^{\omega^2})}_{\gamma}$$

Main Ordinals

Lemma 1

Say $\alpha = (\omega^{\omega^{\alpha_1} n_1})\gamma$ *and* $\beta = (\omega^{\omega^{\beta_1} m_1})\delta$ *are reduced* $\mathcal{S}'$*-normal forms. Then* $\alpha < \beta$ *if and only if*

1. $\alpha_1 < \beta_1$,
2. $\alpha_1 = \beta_1$ *and* $n_1 < m_1$, *or*
3. $\alpha_1 = \beta_1$ *and* $n_1 = m_1$ *and* $\gamma < \delta$.

Primes

- $\mathcal{S}$ doesn't have an additive identity 0.
- Put $\mathcal{S}' = \mathcal{S} \cup \{0\}$.
- $f \in \mathcal{S}'$ is called
  1. An additive prime if $f \neq g + h$ for all nonzero $g, h \in \mathcal{S}'$,
  2. A multiplicative prime if $f \neq 0$ and $f \neq gh$ for all $g, h \neq 1$ in $\mathcal{S}'$, and
  3. An exponential prime if $f \neq 0$ and $f \neq g^h$ for all $g, h \neq 1$ in $\mathcal{S}'$.
- The only functions that are prime in all three senses are x and 1.

Primes

- $\prec$ is a well-ordering on $\mathcal{S}'$.
- So every nonzero $f \in \mathcal{S}'$ has a representation $f = p_1 + \cdots + p_k$ $(k \geq 1)$, where $p_1 \succeq p_2 \succeq \cdots \succeq p_k$ are additive primes. This is called the additive normal form of f.
- $f = p_1 + q, \ q = \sum_{i=2}^{k} p_i$ $(q = 0$ if $k = 1)$ is called the reduced additive normal form of f.
- Each additive prime has a representation $p = q_1 q_2 \cdots q_k$ $(k \geq 1)$ with each q_i a multiplicative prime.
- In such representations q_i is also an additive prime.

Primes

- Suppose $q \in \mathcal{S}'$ and $q \neq 1$ and q is both an additive and multiplicative prime.

Write $q = p_1 + \cdots p_k$ where p_i's are additive primes with $p_1 \succeq \cdots \succeq p_k$. We can write $p_i = u_1^{(i)} \cdots u_{k_i}^{(i)}$ for each i where $u_j^{(i)}$'s are multiplicative primes. Since q is a multiplicative prime p_i's cannot have common factors other than 1. Since q is an additive prime $k = 1$. So $q = u_1 \cdots u_m$ for some m where u_i's are multiplicative primes with no common factors.

Two cases now - (1) $u_i = x$ for some i; (2) $u_i \neq x$ for each i. If $u_i = x$ for some i then $u_j = x$ for all $j \leq m$ because otherwise q would have at least two distinct factors. So q has the form x^m. But now m must be an additive prime because otherwise x^m would split into two smaller factors x^{m_1} and x^{m_2} with $m_1 + m_2 = m$. Therefore $q = x^h$ for some additive prime h.

Primes

If $u_i \neq x$ for each i then u_i must be a constant ≥ 2 ($q = f^m \implies q$ not a multiplicative prime). Say $u_i = n$ ($n \geq 2$). So $q = n^m$. Now if m were not a multiplicative prime we'd have $m = m_1 m_2$, and $q = n^{m_1 m_2} = \underbrace{n^{m_1} \cdots n^{m_1}}_{m_2-\text{terms}}$.

Primes

- If $q \in \mathcal{S}'$ and $q \neq 1$ and q is both an additive and multiplicative prime, then q is of one of the following forms.
  1. $x^{h(x)}$ with $h(x)$ an additive prime
  2. $n^{h(x)}$ with $n \geq 2$ and $h(x) \neq 1$ a multiplicative prime.
- So each additive prime $f \neq 1$ can be written as

$$f = u_1^{f_1} u_2^{f_2} \cdots u_k^{f_k} \ (k \geq 1),$$

  where each u_i is of the form 1 or 2.
- We can ensure that each u_i has a distinct exponent. Multiply together any two such factors - group the exponents.

Frame Title

- Summary : $f \neq 1$ is an additive prime. Then f can be written as $f = u_1^{f_1} u_2^{f_2} \cdots u_k^{f_k}$ $(k \geq 1)$, where
  1. Each f_i is an additive prime
  2. $2 \preceq u_i$ for all i
  3. $u_i \in \mathbb{N} \cup \{x\}$
  4. $u_i, u_j \in \mathbb{N}$ $(i \neq j) \implies f_i \neq f_j$
  5. $u_i \in \mathbb{N} \implies f_i \neq 1$
  6. $u_1^{f_1} \succeq u_2^{f_2} \succeq \cdots \succeq u_k^{f_k}$

Reduced Multiplicative Form

- Reduced multiplicative form of f is $f = u_1^{f_1} y$, where $y = u_2^{f_2} \cdots u_k^{f_k}$ ($y = 1$ if $k = 1$).

Results

Lemma 2

If $f \succeq x$, then $\lim f(x) = +\infty$.

Proof.

Suppose $f \succeq x$. Then $\exists n_0 \in \mathbb{N}\ \forall x \geq n_0\ x \leq f(x)$. Therefore since $\lim x = +\infty$ we have $\lim f(x) = +\infty$. $\square$

Multiplicative normal form - Results

Theorem 3

Let $g \neq 1$ be an additive prime.

1. *For every additive prime $f \neq 1$, if f, g have reduced multiplicative normal forms $f = u_1^{f_1} y$ and $g = v_1^{g_1} z$, then $f \prec g$ if and only if one of the following holds.*
   a. $f_1 \prec g_1$
   b. $f_1 = g_1$ *and* $u_1 \prec v_1$
   c. $f_1 = g_1$ *and* $u_1 = v_1$ *and* $y \prec z$
2. *For any additive prime f, if $f \prec g$ then $f \cdot x \preceq g$.*
3. *The multiplicative normal form representation for g is unique.*

Proof

By transfinite induction on g over the well ordering $\prec$. The inductive hypothesis is

Items $1, 2, 3$ hold for all additive primes $\psi \neq 1$ with $\psi \prec g$.

Sufficiency of conditions in 1 :

Case 1 : $f_1 \prec g_1$. Let $f = u_1^{f_1} \cdots u_k^{f_k}$ be in normal form with $u_1^{f_1} \succeq \cdots \succeq u_k^{f_k}$. Thus $f \preceq u_1^{kf_1}$. By IH for 2 we get $f_1 x \preceq g_1$. Since $u_1 \preceq x$ get

$$\frac{\log f}{\log v_1^{g_1}} \preceq \frac{\log u_1^{kf_1}}{\log v_1^{g_1}} \preceq \frac{kf_1 \log u_1}{g_1 \log v_1} \preceq \frac{kf_1 \log u_1}{f_1 x \log v_1} \preceq \frac{k \log u_1}{x \log v_1} \preceq \frac{k \log x}{x \log v_1}$$

$$\preceq \frac{k \log x}{x}\Big(\frac{1}{\log v_1}\Big)$$

Proof

Since $\lim_{x\to\infty} \dfrac{k \log x}{x}\left(\dfrac{1}{\log v_1}\right) = 0$ we have $\log f \preceq \log v_1^{g_1}$. So $f \prec v_1^{g_1} \preceq v_1^{g_1} z = g$.

Case 2 : $f_1 = g_1$ and $u_1 \prec v_1$. Since $u_1, v_1 \in \{2, 3, \ldots, x\}$ we have $u_1 = n$ for some natural number $n \geq 2$. Because if $u_1 = x$ then no room for v_1 in $\{2, 3, \ldots, x\}$ as $u_1 \prec v_1$. By IH-2 $u_1^{f_1} \prec g$. Let $i \geq 2$. If $f_1 \prec f_i$ we'd have by IH-1 that $u_i^{f_i} \prec u_1^{f_1}$ which is impossible as $f = u_1^{f_1} y$ is the multiplicative normal form of f. Therefore $f_1 \succeq f_i$. Now claim that $f_i \prec f_1$. If $f_1 = f_i$, since $u_1^{f_1} \succeq u_i^{f_i}$, we'd have $n^{f_1} \succeq u_i^{f_1}$, so $u_i = m$ for some natural number $m \succeq 2$. But this contradicts the fact that in multiplicative normal forms the factors with numerical bases have distinct exponents. Therefore $f_i \prec f_1$. Since by definition of multiplicative normal form $f_1 \neq 1$, by IH-2 $f_i x \preceq f_1$.

Proof

$$\frac{\log \prod_{i=1}^{k} u_i^{f_i}}{\log v_1^{g_1}} = \frac{\sum_{i=1}^{k} f_i \log u_i}{g_i \log v_1} = \frac{\sum_{i=1}^{k} f_i \log u_i}{f_1 \log v_1} = \sum_{i=1}^{k} \frac{f_i \log u_i}{f_1 \log v_1}.$$

$\dfrac{f_1 \log u_1}{f_1 \log v_1} = \dfrac{\log u_1}{\log v_1}$. If $v_1 = x$, since the numerator is a constant, $\dfrac{f_1 \log u_1}{f_1 \log v_1} \to 0$ as $x \to \infty$. If v_1 is a constant, then since $u_1 \prec v_1$ (hypothesis of case 2), $\dfrac{f_1 \log u_1}{f_1 \log v_1}$ tends to a limit (constant) less than 1. For $i \geq 2$, since $f_i x \preceq f_1$ we have

$$\frac{f_i \log u_i}{f_1 \log v_1} \preceq \frac{f_i \log u_i}{f_i x \log v_1} = \frac{\log u_i}{x}\left(\frac{1}{\log v_1}\right).$$

Proof

Since $v_1 \preceq x$ and $u_i \preceq x$, $\frac{\log u_i}{x}\left(\frac{1}{\log v_1}\right) \to 0$ as $x \to \infty$. Hence $f \prec v_1^{g_1} \preceq g$.

Case 3 : $f_1 = g_1,\ u_1 = v_1$, and $y \prec z$. Trivial.

Necessity follows from sufficiency. Why? if conditions a,b,c fail and $f \neq g$, then one of a,b,c holds with roles of f, g reversed giving $g \prec f$. This completes the proof of 1.

Now want to show that 3 holds for g. Suppose $g = \prod v_i^{g_i} = \prod w_i^{h_i}$ are two representations of g in multiplicative normal form. Then have reduced forms $g = v_1^{g_1} y = w_1^{h_1} z$. By 1 we have $g_1 = h_1$, because otherwise we'd have $g \prec g$. Now by IH, y and z have the same normal form. This proves 3.

Proof

Finally want to show that 2 holds for g. Use the fact that 1 and 3 hold for g. Let f be an additive prime such that $f \prec g$. If $f = 1$, then done since $x \preceq g$. So suppose $f \neq 1$. Consider the unique normal form representations for f and g. Let w be the product of the common factors. Then $f = f'w$ and $g = g'w$, where $f' = 1$, or $f' \neq 1$ and f', g' have normal forms with no common factors. We'll show $f'x \preceq g'$. If $f' = 1$, done since $x \preceq g'$. If $f' \neq 1$, let $s_1^{t_1} h$ and $r_1^{q_1} l$ be the reduced normal forms for f', g' respectively. From $f' \prec g'$ and 1 one of the following holds.

1. $t_1 \prec q_1$
2. $t_1 = q_1$ and $s_1 \prec r_1$
3. $t_1 = q_1$ and $s_1 = r_1$ and $h \prec l$.

Proof

But III is impossible because otherwise f', g' would have a common factor $s_1^{t_1}$. So I or II holds. In either case $s_1^{t_1}(hx)$ is a reduced normal form for $f'x$. So by 1

$$f'x = s_1^{t_1}(hx) \prec r_1^{q_1} l = g'.$$

This completes the proof. ■

Additive normal form - Results

Theorem 4

If f and g have reduced additive normal forms $f = f_1 + v$ and $f = g_1 + w$, then $f \prec g$ if and only if one of the following holds.

- (a) $f_1 \prec g_1$
- (b) $f_1 = g_1$ *and* $v \prec w$.

Additive normal forms are unique

Corollary 5

Additive normal form representation for nonzero members of $\mathcal{S}'$ is unique.

The Principal Mapping

- Exhibit an order preserving mapping G from $\mathcal{S}'$ into the initial segment of **On** determined by ϵ_0.
- Skolem : A subset of $\mathcal{S}'$ has order type ϵ_0.
- So type$(\mathcal{S}') = \epsilon_0$.
- Define $P : \mathbf{On} \to \mathbf{On}$ as follows.

$$P(\alpha) = \begin{cases} \beta \text{ if } \alpha = \omega^\beta \text{ for some } \beta \in \mathbf{On}, \\ 0 \text{ otherwise.} \end{cases}$$

- The function P sort of ignores non-main-ordinals.

The Principal Mapping

Define $G : \mathcal{S}' \to \mathbf{On}$ by recursion as follows.

1. $G(0) = 0$.
2. $G(1) = 1$.
3. $G(x^g) = \omega^{\omega^{B(P(G(g)))}}$ if g is an additive prime.
4. $G((n+1)^g) = \omega^{\omega^{A(P(G(g)))}n}$ if $g \neq 1$ is an additive prime and $n \geq 1$.
5. If $f \neq 1$ is an additive prime with multiplicative normal form $f = u_1^{f_1} \cdots u_k^{f_k}$, then $G(f) = G(u_1^{f_1}) \cdots G(u_k^{f_k})$.
6. If f has additive normal form $f = p_1 + \cdots + p_k$ then $G(f) = G(p_1) + \cdots + G(p_k)$.

- The mapping G is well-defined as the normal forms are unique.

Examples

$$G(x) = \omega^{\omega^{B(P(G(1)))}} = \omega^{\omega^{B(P(1))}} = \omega^{\omega^{B(0)}} = \omega^{\omega^0} = \omega^1 = \omega.$$

$$G(x^n) = \underbrace{G(x) \cdots G(x)}_{n-\text{terms}} = \omega^n.$$

$$\begin{aligned} G(2^x) &= G((1+1)^x) \\ &= \omega^{\omega^{A(P(G(x)))}} \\ &= \omega^{\omega^{A(P(\omega))}} \ (\because G(x) = \omega) \\ &= \omega^{\omega^{A(1)}} \ (\because P(\omega) = 1) \\ &= \omega^{\omega} \ (\because A(1) = 1). \end{aligned}$$

Examples

$G(3^x) = \omega^{\omega 2}$.

$$
\begin{aligned}
G(x^x) &= \omega^{\omega^{B(P(G(x)))}} \\
&= \omega^{\omega^{B(P(\omega))}} \ (\because G(x) = \omega) \\
&= \omega^{\omega^{B(1)}} \ (\because P(\omega) = 1) \\
&= \omega^{\omega^2} \ (\because B(1) = 2).
\end{aligned}
$$

$$
\begin{aligned}
G(2^{x^2}) &= G((1+1)^{x^2}) \\
&= \omega^{\omega^{A(P(G(x^2)))}} \\
&= \omega^{\omega^{A(P(\omega^2))}} \ (\because G(x^2) = 2) \\
&= \omega^{\omega^{A(2)}} \ (\because P(\omega^2) = 2) = \omega^{\omega^3}.
\end{aligned}
$$

The Principal Mapping

Theorem 6

If $f \prec g$, then $G(f) < G(g)$.

$\star$ $G(f) < \epsilon_0$ by transfinite induction on f. Important : Initial segment of **On** determined by ϵ_0 is closed under ordinal addition, multiplication and exponentiation.