Modal Invariance

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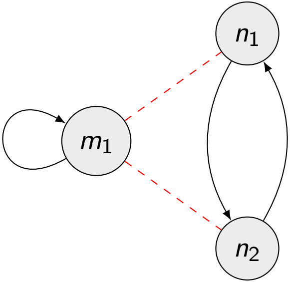
Modal Invariance

- Suppose we have two models $\mathcal{M}$ and $\mathcal{N}$. Further, suppose there is a bisimulation E between $\mathcal{M}$ and $\mathcal{N}$, and suppose $x\,E\,y$, where x and y are two worlds in $\mathcal{M}$ and $\mathcal{N}$ respectively.
- Something cool happens now! If we know that a modal formula φ is true in $\mathcal{M}, x$ then we know it's true in $\mathcal{N}, y$, and vice versa.
- It's cool because it says something about definability of certain properties.
- Here's an example of such a property which fails to be definable in modal logic.

Undefinability of Irreflexivity of R at states



- Consider $\mathcal{M} = (\{m_1\}, R)$ and $\mathcal{N} = (\{n_1, n_2\}, R)$ shown above.
- The red dashed lines depict a bisimulation E.
- Now if φ were a modal formula defining irreflexivity, then since we have bisimulations as shown we would have $\mathcal{M}, m_1 \models \varphi$ because $\mathcal{N}, n_1 \models \varphi$ and $m_1 \ E \ n_1$. But this is clearly not the case because $\mathcal{M}, m_1 \not\models \varphi$.

Proof of the Invariance Lemma

Lemma

For any bisimulation E between models $\mathcal{M}$ and $\mathcal{N}$ and any two worlds x, y with $x\,E\,y$, we have $\mathcal{M}, x \models \varphi$ if and only if $\mathcal{N}, y \models \varphi$ for all modal formulas φ.

<u>Proof:</u>

We'll use induction on formulas. We have $\mathcal{M}, x \models \varphi$ iff $\mathcal{N}, y \models \varphi$ for all atomic φ by the so called "*local harmony*" of x and y as $x\,E\,y$.

Now assume $\varphi \equiv \neg\psi$ for some formula ψ and the result is true for ψ, that is $\mathcal{M}, x \models \psi$ iff $\mathcal{N}, y \models \psi$. Observe the following chain of biconditionals.
$\mathcal{M}, x \models \varphi \iff \mathcal{M}, x \models \neg\psi \iff \mathcal{M}, x \not\models \psi \iff \mathcal{N}, y \not\models \psi \iff \mathcal{N}, y \models \neg\psi \iff \mathcal{N}, y \models \varphi$.

So the result is true for $\neg\psi$ whenever it's true for ψ.

Proof of Invariance Lemma Continued ...

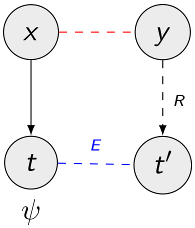
Next assume $\varphi \equiv \psi \vee \rho$ and the result is true for ψ and ρ. Then we have $\mathcal{M}, x \models \psi$ iff $\mathcal{N}, y \models \psi$, and $\mathcal{M}, x \models \rho$ iff $\mathcal{N}, y \models \rho$. So $\mathcal{M}, x \models \varphi \iff \mathcal{M}, x \models \psi \vee \rho \iff \mathcal{M}, x \models \psi$ or $\mathcal{M}, x \models \rho \iff \mathcal{N}, y \models \psi$ or $\mathcal{N}, y \models \rho \iff \mathcal{N}, y \models \psi \vee \rho \iff \mathcal{N}, y \models \varphi$.

So the result is true for $\psi \vee \rho$ whenever it's true for ψ and ρ.

Now assume $\varphi \equiv \Diamond\psi$ and the result is true for ψ. Then $\mathcal{M}, x \models \psi$ iff $\mathcal{N}, y \models \psi$. Suppose $\mathcal{M}, x \models \Diamond\psi$. Then there exists a world t in $\mathcal{M}$ such that $x\, R\; t$ and $\mathcal{M}, t \models \psi$. Now we invoke the *forward* property of the bisimulation E. There exists a world t' in $\mathcal{N}$ so that $y\, R\; t'$ and $t\; E\; t'$.

Proof of Invariance Lemma Continued . . .

Here's a picture.



So $\mathcal{N}, t' \models \psi$ by the inductive hypothesis. Thus $y\ R\ t'$ and $\mathcal{N}, t' \models \psi$. Therefore $\mathcal{N}, y \models \Diamond\psi$.

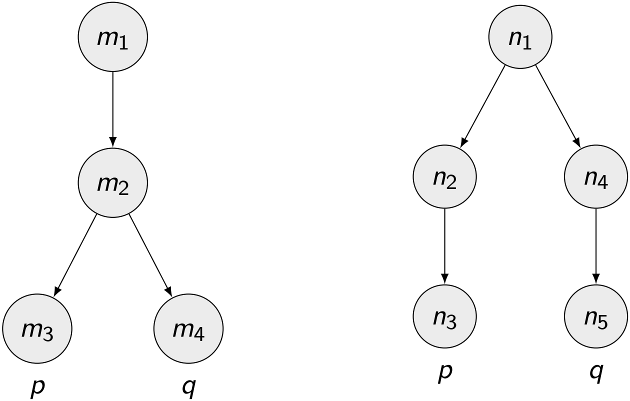
Proof of Invariance Lemma Continued ...

Conversely suppose $\mathcal{N}, y \models \Diamond\psi$. Then there exists a world u in $\mathcal{N}$ such that $y\ R\ u$ and $\mathcal{N}, u \models \psi$. Now invoke the *backward* property of the bisimulation E. There exists a world v in $\mathcal{M}$ so that $v\ E\ u$ and $x\ R\ v$. So by the inductive hypothesis $\mathcal{M}, v \models \psi$. Thus $x\ R\ v$ and $\mathcal{M}, v \models \psi$. Therefore $\mathcal{M}, x \models \Diamond\psi$. Hence, whenever the result is true for ψ it's also true for $\Diamond\psi$.
Finally, we'll show the result is true for $\Box\psi$ whenever it's true for ψ. We just make the following observation.

$$\Box\psi \text{ is logically equivalent to } \neg\Diamond\neg\psi,$$

and use the proof for $\neg\psi$ and $\Diamond\psi$. Here's how. Since the result is true for ψ it's true for $\neg\psi$; so it's true for $\Diamond\neg\psi$; so it's true for $\neg\Diamond\neg\psi$; hence it's true for $\Box\psi$. This completes the proof. 😄

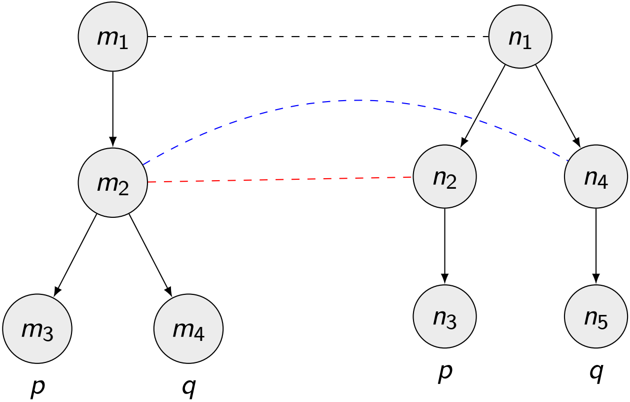
Example



- There's no bisimulation E such that $m_1 \; E \; m_2$.

Example Continued...

- Suppose $m_1 \; E \; m_2$. Then by the forward property of E we'd have $m_2 \; E \; n_2$ (red dashed line) or $m_2 \; E \; n_4$ (blue dashed line).



- Note that $\mathcal{M}, m_2 \models \Diamond p$ and $\mathcal{M}, m_2 \models \Diamond q$. But $\mathcal{N}, n_2 \not\models \Diamond q$ and $\mathcal{N}, n_4 \not\models \Diamond p$. So both the blue dashed line and red dashed line contradict the invariance lemma! ☺

Example Continued...

Before reading the invariance lemma I had to go one extra level and use *local harmony* to get a contradiction. Maybe we can nail it right at m_1 and n_1. I couldn't.