A Monadic Logic for Capacity Quantifiers

Janitha Aswedige

University of Hawai'i at Mānoa

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What this talk is about.

- Given a field of study X in mathematics, logicians are interested in languages $\mathcal{L}_X$ corresponding to it.
- $\mathcal{L}_X$ should be able to express important properties and results about X.
- $\mathcal{L}_X$ gives rise to a logic L_X with a satisfaction relation $\vDash_X$ between the structures and sentences of $\mathcal{L}_X$.
- Once there's a logic, we ask what model theoretic theorems can be proved in L_X.

What this talk is about.

- Keisler introduced a logic for probability called <u>*Probability Logic*</u>. His student Hoover proved several model theoretic theorems about it; Model existence theorem, completeness theorem, interpolation theorems etc.
- Probability logic has quantifiers of the form $(P\overline{x} \geq r)$, called <u>*probability quantifiers*</u>, instead of the usual quantifiers $\forall$ and $\exists$.
- The sentence $(P\overline{x} \geq r)\varphi(x)$ is interpreted as "the set $\{\overline{x} : \varphi(\overline{x})\}$ has probability at least r."
- In our work, Hoover and Keisler's logic is generalized to capacities. In particular, we have quantifiers of the form $(\mathscr{I} x \geq r)$ called <u>*capacity quantifiers*</u>, where $(\mathscr{I} x \geq r)\varphi(x)$ is interpreted as "the set $\{x : \varphi(x)\}$ has capacity at least r."

Why Capacities?

- Applications in physics; Newtonian capacities.
- Applications in probability theory; hitting capacities of Brownian motions, theory of random closed sets.
- Applications in computability theory.
- Applications in decision theory; Choquet integral etc.

Capacities.

Definition (Pavings)

Let F be a set. A *paving* $\mathcal{F}$ on F is a set of subsets of F containing the empty set. Call $\mathcal{F}$ *regular* if $\mathcal{F}$ is closed under finite unions and intersections.

- Given a set F and a paving $\mathcal{F}$, we call the pair $\langle F, \mathcal{F} \rangle$ a paved set or a paved space.
- $\{\varnothing\}$ and $\mathcal{P}(F)$ are regular pavings. A topological space $\langle X, \tau \rangle$ is a paved space. A measurable space is a paved space.

Capacities.

Definition

Let $\mathcal{F}$ be a regular paving on a set F. An *$\mathcal{F}$-capacity* on F is an extended real-valued set function I on $\mathcal{P}(F)$ with the following properties.

(1) I is monotone, that is, if $A \subseteq B \subseteq F$ then $I(A) \leq I(B)$.

(2) If $\langle A_n \rangle$ is an increasing sequence of subsets of F, then $I\left(\bigcup_{n=1}^{\infty} A_n\right) = \sup_n I(A_n)$.

(3) If $\langle A_n \rangle$ is a decreasing sequence of elements of $\mathcal{F}$, then $I\left(\bigcap_{n=1}^{\infty} A_n\right) = \inf_n I(A_n)$.

A subset A of F is called *capacitable* if

$$I(A) = \sup\{I(B) : B \in \mathcal{F}_\delta \text{ and } B \subseteq A\}.$$

Recall : $B \in \mathcal{F}_\delta$ iff B can be written as a countable intersection of sets in $\mathcal{F}$.

Capacities

- As measures are for *volumes*, capacities are for *sizes*.
- Measures are defined only on a σ-algebra which is always a proper subset of the power set of a given set.
- Capacities are defined on the entire power set of a given set.

Definition

Let $\langle F, \mathcal{F} \rangle$ be a paved set, and let I be an $\mathcal{F}$-capacity on F. Then I is *strongly subadditive* if and only if

$$I(A \cup B) + I(A \cap B) \leq I(A) + I(B)$$

for all $A, B \in \mathcal{F}$.

Definition

Suppose $\mathcal{F}$ is a regular paving on a set F. A normalized strongly subadditive monotone set function on $\mathcal{F}$ is called an *$\mathcal{F}$-precapacity*.

An Example of a Capacity.

- $\langle \Omega, \mathcal{A}, \mathbb{P} \rangle$ a probability space. Then the outer probability measure

$$\mathbb{P}^*(A) = \inf\{\sum_{n=1}^{\infty} \mathbb{P}(A_n) : A_n \in \mathcal{A} \text{ and } A \subseteq \bigcup_{n=1}^{\infty} E_n\}$$

is a capacity on Ω.

For more about capacities :

(1) Probabilities and Potential - Dellacherie and Meyer.

(2) Dellacherie's chapter in Cabal Seminar 77-79.

Logic Prerequisites.

- Both Hoover-Keisler logic and our logic use ideas from infinitary logic $L_{\omega_1\omega}$.
- In fact our ultimate goal is to prove $L_{\omega_1\omega}$-type results for our logic as Hoover and Keisler did for probability logic.
- We'll look at some very basic ideas.

Infinitary logic $L_{\omega_1\omega}$.

- $L_{\omega_1\omega}$ is similar to the usual first order logic ($L_{\omega\omega}$) except that it allows formulas with countable conjunctions and disjunctions.
- This gives $L_{\omega_1\omega}$ a great deal of expressibility power.

Example

The class of all finite models can be characterized by the $\mathcal{L}_{\omega_1\omega}$-sentence

$$\bigvee_{n<\omega} \exists x_1 \dots x_n \forall y (y = x_1 \vee \dots \vee y = x_n).$$

This cannot be done in first order logic; compactness theorem stands on the way. If a theory has arbitrarily large finite models, then it has an infinite model.

Inifinitary Logic $L_{\omega_1\omega}$.

- However, too much expressibility power causes problems.
- For example, the compactness theorem is not true for $L_{\omega_1\omega}$.

Example

- Let $c_0, c_1, \ldots, c_\omega$ be constant symbols of $\mathcal{L}$.
- Let $\Sigma = \{(\forall x) \bigvee_{n<\omega}(x = c_n), (c_\omega \neq c_0), (c_\omega \neq c_1), \ldots\}$.
- Then Σ is not satisfiable while it is finitely satisfiable.
- Hence $L_{\omega_1\omega}$ is not compact.

- *These are not good traits of a logic.*
- Upward Löwenheim-Skolem-Tarski theorem fails too. Because $(\forall x) \bigvee_{n<\omega}(x = c_n)$ has a countable model but no uncountable model.

Axioms for $L_{\omega_1\omega}$.

- Axioms for $L_{\omega_1\omega}$ are generalizations of the axioms for first order logic.

x, y arbitrary variables, ϕ, ψ arbitrary formulas, and Φ an arbitrary countable set of formulas of $\mathcal{L}_{\omega_1\omega}$.

Definition

The axioms for $L_{\omega_1\omega}$ are the following.

(1) Every instance of a tautology of finitary logic.
(2) $(\neg\varphi) \leftrightarrow (\varphi\neg)$.
(3) $\bigwedge \Phi \to \varphi$ for all $\varphi \in \Phi$.
(4) $\forall x \varphi(x \ldots) \to \varphi(t \ldots)$, where $\varphi(x \ldots)$ is a formula, t is a term which is free for x in $\varphi(x \ldots)$, and $\varphi(t \ldots)$ is obtained by replacing each free occurrence of x by t.
(5) $x = x$.
(6) $x = y \to y = x$.
(7) $(\varphi(x \ldots) \wedge t = x) \to \varphi(t \ldots)$, where $\varphi(x \ldots)$ and $\varphi(t \ldots)$ are as in *(4)*.

Completeness theorem for $L_{\omega_1\omega}$.

- In usual first order logic, we use a technique called *Henkin constructions* to build models. The idea is to throw enough constants to the language and extract a model built out of those constants satisfying various properties.
- A modern technique for this is called the *consistency properties*.

Consistency Properties

- First, augment the language with a countable set C of new constant symbols.
- Put $\mathcal{M} = \mathcal{L} \cup C$.
- Form the logic $M_{\omega_1\omega}$ corresponding to the language $\mathcal{M}_{\omega_1\omega}$.

Consistency Properties

A consistency property for $M_{\omega_1\omega}$ is a set S of countable sets s of $\mathcal{M}_{\omega_1\omega}$-formulas with some special properties.

C1. (Consistency Rule) Either $\varphi \notin s$ or $(\neg\varphi) \notin s$.

C2. ($\neg$-Rule) $(\neg\varphi) \in s$ implies $s \cup \{(\varphi\neg)\} \in S$.

C3. ($\bigwedge$-Rule) $(\bigwedge \Phi) \in s$ implies $s \cup \{\varphi\} \in S$ for all $\varphi \in \Phi$.

C4. ($\forall$-Rule) $(\forall x \varphi(x)) \in s$ implies $s \cup \{\varphi(c)\} \in s$ for all $c \in C$.

C5. ($\bigvee$-Rule) $(\bigvee \Phi) \in s$ implies $s \cup \{\varphi\} \in S$ for some $\varphi \in \Phi$.

C6. ($\exists$-Rule) $(\exists x \varphi(x)) \in s$ implies $s \cup \{\varphi(c)\} \in S$ for some $c \in C$.

C7. (Equality Rules) Let t be a basic term, and let $c, d \in C$.

(α) $(c = d) \in s$ implies $s \cup \{d = c\} \in S$.

(β) $c = t, \varphi(t) \in s$ imply $s \cup \{\varphi(c)\} \in S$.

(γ) $s \cup \{e = t\} \in S$ for some $e \in C$.

Model Existence Theorem.

Theorem (Makkai)

If S is a consistency property and $s_0 \in S$, then s_0 has a countable model.

Proof Theory - Rules of Inference.

Definition

The rules of inference for $L_{\omega_1\omega}$ are the following.

(1) From $\psi, \psi \to \varphi$, infer φ.

(2) From $\psi \to \varphi(x, \ldots)$, infer $\psi \to \forall \varphi(x, \ldots)$, where x does not occur free in ψ.

(3) From for all $\varphi \in \Phi$, $\psi \to \varphi$, infer $\psi \to \bigwedge \Phi$.

Proof Theory - Proofs.

Proofs in $L_{\omega_1\omega}$ are just like proofs in usual first order logic except they can be infinitely long.

Let φ be a sentence in $L_{\omega_1\omega}$.

- φ is a *theorem* of $L_{\omega_1\omega}$ if and only if there is a countable sequence $\varphi_0, \ldots, \varphi_\alpha, \ldots, \varphi_\beta$ such that $\varphi_\beta = \varphi$, and for each $\alpha \leq \beta$, φ_α is either an axiom of $L_{\omega_1\omega}$ or it is inferred from earlier formulas $\varphi_\gamma, \gamma < \alpha$, by an inference rule.
- The sequence $\langle \varphi_\alpha : \alpha \leq \beta \rangle$ is called a *proof* of φ.

Proof Theory - Completeness Theorem.

Carol Karp proved a completeness theorem for $L_{\omega_1\omega}$ with the aid of consistency properties.

Theorem (Karp)

If φ is a sentence of $L_{\omega_1\omega}$, then $\vdash_{L_{\omega_1\omega}} \varphi$ if and only if $\vDash \varphi$.

Proof Theory - Completeness Theorem.

- Idea of the proof : Show that the set of all finite sets s of sentences of $M_{\omega_1\omega}$ such that only finitely many $c \in C$ occur in s and $\nvdash_{M_{\omega_1\omega}} \neg \bigwedge_{\psi \in s} \psi$ is a consistency property.
- How does one pass from a proof in $M_{\omega_1\omega}$ to a proof in $L_{\omega_1\omega}$?
  (a) Only countably many variables occur in a proof.
  (b) But we have more than a countable number of variables.
  (c) Replace *uniformly* each constant in the proof by variables that don't occur in the proof.

Formulas, Set Theoretically.

As we saw, $\mathcal{L}_{\omega_1\omega}$ is badly behaved. Having an uncountable number of formulas is a nuisance. However, there's hope. We restrict to nicely behaved fragments of $L_{\omega_1\omega}$ with nicely behaved countable sets of formulas.

- We can build the formulas of a language in a set theoretic fashion, in the sense that every formula can be treated as a set.
- Atomic formulas are finite sequences of symbols.
- $\neg\varphi$ is $\langle\neg, \varphi\rangle$, $\bigwedge\varphi$ is $\langle\bigwedge, \varphi\rangle$, $\forall v_\alpha\varphi$ is $\langle\forall, v_\alpha, \varphi\rangle$, and so on.
- So the set of all formulas of $\mathcal{L}_{\omega_1\omega}$ can be treated as a set.
- Given any set $\mathscr{A}$, we can form $L_{\mathscr{A}} = L_{\omega_1\omega} \cap \mathscr{A}$.
- $L_{\mathscr{A}}$ is called a *fragment* of $L_{\omega_1\omega}$ if $L_{\mathscr{A}}$ has some nice closure properties; closed under $\neg$, countable disjunctions and conjunctions, introducing quantifiers etc.

Admissible Fragments.

- *Admissible sets* are a special kind of sets. They are transitive models of Kripke-Platek set theory.
- The set of all hereditarily finite sets and the set of hereditarily countable sets are some examples of admissible sets.
- But why do we care about admissible sets?
- Most of the properties that we lose once we step in to the world of infinitary logic can be regained when we restrict to admissible sets. For example, we gain Barwise Compactness.
- The set of formulas of $L_{\omega_1\omega}$ is uncountable. But when we restrict to a countable admissible set, we obtain a nicely-behaved countable set of formulas called an *admissible fragment* of $L_{\omega_1\omega}$, that is a set of the form $L_{\omega_1\omega} \cap \mathbb{A}$, where $\mathbb{A}$ is an admissible set.

Probability Logic as Motivation.

We are now ready to discuss some key ideas from the Hoover-Keisler logic.

Probability Logic

The language of probability logic is built in the following manner.

- Start with an admissible set $\mathbb{A} \subseteq \mathbf{HC}$ with $\omega \in \mathbb{A}$.
- Let $\mathcal{L}$ be a countable set of constant and finitary relation symbols.
- There are no function symbols in $\mathcal{L}$.
- Let the logical symbols be:
  (1) A countable list of variables $v_0, v_1, \ldots$
  (2) The connectives $\neg$ and $\bigwedge$.
  (3) The quantifiers $(P\overline{x} \geq r)$, where $r \in \mathbb{A} \cap [0, 1]$.
  (4) The equality symbol $=$.
- This way, the language $\mathcal{L}_{\mathbb{A}P}$ is formed.

$\mathcal{L}_{\mathbb{A}P}$-formulas

The set of formulas of $\mathcal{L}_{\mathbb{A}P}$ is the least set X such that:

1. Every atomic formula of first order logic is in X.
2. If $\varphi \in X$, then so is $\neg\varphi$.
3. If $\Phi \subseteq X$, then $\bigwedge \Phi \in X$.
4. If $\varphi(\overline{x}) \in X$, then $(P\overline{x} \geq r)\varphi(\overline{x}) \in X$, where $r \in \mathbb{A} \cap [0,1]$.

We make the formulas set theoretically as before. So $X \subseteq \mathbb{A}$. Since $\mathbb{A}$ is countable, Φ above is automatically countable as well. So we apply $\bigwedge$ to only at most countable sets of formulas.

Probability Models

Definition

A probability structure for $\mathcal{L}$ is a structure

$$\mathfrak{M} = \langle M, R_i^{\mathfrak{M}}, c_j^{\mathfrak{M}}, \mu \rangle_{i \in I,\ j \in J},$$

where μ is a countably additive probability measure on M such that each singleton is measurable, each $R_i^{\mathfrak{M}}$ is $\mu^{(n_i)}$-measurable, and each $c_j^{\mathfrak{M}} \in M$.

The logic $L_{\mathbb{A}P}$

It remains to define the satisfaction relation. Everything else other than the following is defined as usual.

Definition

Let $\mathfrak{M}$ be a probability structure. Let $\varphi(\overline{x}, \overline{y})$ be a formula of $\mathcal{L}_{\mathbb{A}P}$. Then

$$\mathfrak{M} \models (P\overline{y} \geq r)\varphi(\overline{x}, \overline{y})[\overline{a}] \text{ iff } \{\overline{b} \in M^n : \mathfrak{M} \models \varphi[\overline{a}, \overline{b}]\}$$

is $\mu^{(n)}$-measurable and has measure at least r.

In the context of our work, we do not worry about higher dimensional sets, or measures, as our logic is monadic. But this is the type of model we use in our logic. See Kiesler's chapter [Kei85] in Model Theoretic Logics for further details.

Capacity Quantifiers.

Analogous to probability quantifiers, we define a notion called *capacity quantifiers*.

- $(\mathscr{I} x \geq r)\varphi(x)$ is to be interpreted as "the set $\{x : \varphi(x)\}$ has capacity at least r."
- Our language $\mathcal{L}_{\mathbb{A}\mathscr{I}}$ is the same as $\mathcal{L}_{\mathbb{A}P}$ with the only exceptions of having capacity quantifiers instead of probability quantifiers and <u>unary</u> relation symbols instead of relation symbols of arbitrary arities.
- So our logic is monadic; we only consider unary relations in our logic. The reason for this is we don't have a natural way to extend a given capacity to higher dimensions.

Definable Sets.

Definition

We say a relation $R \subseteq A$ is *parametrically definable* if there is some $\mathcal{L}_{\mathbb{A},\mathscr{I}}$-formula $\varphi(x, y_1, \ldots, y_k)$ and $b_1, \ldots, b_k \in A$ so that $R = \{a : \mathfrak{A} \vDash \varphi(a, b_1, \ldots, b_k)\}$. Here the parameters are the elements $b_1, \ldots, b_k$. The relation R is said to be *definable* if we do not need any parameters.

Capacity Structures.

Definition

A *weak capacity structure* for $\mathcal{L}$ is a quadruple $\mathfrak{A} = \langle A, R_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, T \rangle_{i \in I, j \in J}$, where $R_i^{\mathfrak{A}} \subseteq A$ for each $i \in I$, $c_j^{\mathfrak{A}} \in A$ for each $j \in J$, where I and J are indexing sets, and T is a $\mathcal{K}$-precapacity, where $\mathcal{K}$ is the regular paving on A generated by formulas of $L_{\mathbb{A}\mathscr{I}}$.

Definition

A *strong capacity structure* for $\mathcal{L}$ is the same as a weak capacity structure except that T is a normalized strongly subadditive $\mathcal{K}$-capacity on $\mathcal{P}(A)$.

Satisfaction.

Definition

Suppose $\langle \mathfrak{A}, T \rangle$ is a capacity structure. The satisfaction relation $\langle \mathfrak{A}, T \rangle \models \varphi[\overline{a}]$, where $\varphi(x) \in L_{\mathbb{A}\mathscr{I}}$ and $a \in A$ is defined recursively as usual with the following modification.

$\langle \mathfrak{A}, T \rangle \models (\mathscr{I} y \geq r)\varphi(y, \overline{x})[\overline{a}]$ if and only if $T(\{b \in A : \mathfrak{A} \models \varphi[b, \overline{a}]\}) \geq r$.

And $\langle \mathfrak{A}, T \rangle$ is a model of a sentence φ if $\langle \mathfrak{A}, T \rangle \models \varphi$.

Note that we don't have to worry about measurability issues, as we did in $L_{\mathbb{A}P}$, because a capacity applies to the entire power set.

Axioms.

The logic $L_{\mathbb{A}\mathscr{I}}$ has two types axioms; *weak axioms* and *strong axioms*. Let $\varphi, \psi \in L_{\mathbb{A}\mathscr{I}}$, let $\Phi \subseteq L_{\mathbb{A}\mathscr{I}}$ be countable, and let $r, s, t \in \mathbb{A} \cap [0,1]$.

The weak axioms are the following.

$W0$. All axioms of $L_{\mathbb{A}}$ without quantifiers.

$W1$. (*Normalized nonnegativity*) $(\mathscr{I} x \leq 1)(x = x)$ and $(\mathscr{I} x \geq 0)(x \neq x)$.

$W2$. (*Monotonicity of the quantifier*) $(\mathscr{I} x \geq r)\varphi \to (\mathscr{I} x \geq s)\varphi$ whenever $r \geq s$.

$W3$. (*Monotonicity of the capacity*)

(a) $\big((\mathscr{I} x \geq 1)(\varphi(x) \to \psi(x)) \wedge (\mathscr{I} x \leq r)\psi(x)\big) \to (\mathscr{I} x \leq r)\varphi(x)$.

(b) $\big((\mathscr{I} x \geq 1)(\varphi(x) \to \psi(x)) \wedge (\mathscr{I} x \geq r)\varphi(x)\big) \to (\mathscr{I} x \geq r)\psi(x)$.

Axioms.

$W4$. (*Strong subadditivity*)

$(\mathscr{I}x \leq r)\varphi \wedge (\mathscr{I}x \leq s)\psi \wedge (\mathscr{I}x \leq t)(\varphi \wedge \psi) \rightarrow (\mathscr{I}x \leq r+s-t)(\varphi \vee \psi)$,

whenever $t \leq \min\{r, s\}$.

$W5$. (*Archimedean property*)

$$(\mathscr{I}x < r)\varphi \leftrightarrow \bigvee_{n=1}^{\infty}(\mathscr{I}x \leq r - \frac{1}{n})\varphi.$$

The following together with the weak axioms are the strong axioms.

$S1$. $\bigwedge_{\Psi \subseteq \Phi}(\mathscr{I}x \leq r) \bigwedge \Psi \rightarrow (\mathscr{I}x \leq r) \bigvee \Phi$, where Ψ ranges over the finite subsets of Φ.

$S2$. $\bigvee_{\Psi \subseteq \Phi}(\mathscr{I}x \geq r) \bigwedge \Psi \rightarrow (\mathscr{I}x \geq r) \bigwedge \Phi$, where Ψ ranges over the finite subsets of Φ.

$S1$ and $S2$.

We claim that $S1$ and $S2$ correspond to continuity from below and continuity from above respectively. The following lemma justifies our claim.

Lemma

Let $\langle \mathfrak{A}, T \rangle$ be a weak capacity structure. Suppose $\langle \mathfrak{A}, T \rangle \vDash (S1 \wedge S2)$. Further, suppose each A_n below is definable. Then:

(1) *If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$, and $\bigcup_{i=1}^{\infty} A_i$ is also definable, then $T\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sup_i T(A_i)$.*

(2) *If $A \supseteq A_1 \supseteq A_2 \supseteq \cdots$, then $T\left(\bigcap_{i=1}^{\infty} A_i\right) \geq \inf_i T(A_i)$.*

Proof of Lemma.

Proof.

We will prove (1), and (2) will follow by a similar argument. Suppose $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$. Let $i \in \mathbb{N}^+$. Then for some appropriate formula $\varphi_i(x, \underline{a}_i)$ we have $A_i = \{x : \varphi_i(x, \underline{a}_i)\}$. Let

$$\Phi = \{\varphi_i(x, \underline{a}_i) : i \in \mathbb{N}^+\}.$$

Now for any finite subset Ψ of Φ, we have

$$\langle \mathfrak{A}, T \rangle \models (\mathscr{I} x \leq \sup_i T(A_i)) \bigwedge \Psi.$$

Therefore, by $S1$, it follows that

$$\langle \mathfrak{A}, T \rangle \models (\mathscr{I} x \leq \sup_i T(A_i)) \bigvee \Phi.$$

Hence $T(\bigcup_{i=1}^{\infty} A_i) \leq \sup_i T(A_i)$. $\square$

Expressive Power.

If T satisfies $T(A_1 \cup A_2) = \max\{T(A_1), T(A_2)\}$ in its paving, then T is called a *maxitive* capacity. We claim that being maxitive can be expressed in $\mathcal{L}_{\mathbb{A}\mathscr{I}}$ as follows.

$$\mathrm{MAX}: \Big((\mathscr{I}x \leq r)\varphi(x) \wedge (\mathscr{I}x \leq s)\psi(x) \Big) \to (\mathscr{I}x \leq \max\{r, s\})\big(\varphi(x) \vee \psi(x)\big).$$

The following proposition justifies our claim.

Proposition

Let $\langle \mathfrak{A}, T \rangle$ be a weak capacity model and suppose $\langle \mathfrak{A}, T \rangle \models \mathrm{MAX}$. Then $T(A_1 \cup A_2) = \max\{T(A_1), T(A_2)\}$ for all definable subsets A_1 and A_2 of A.

Proof of Proposition.

Proof.

Let $A_1 = \{x : \varphi_1(x, \underline{a}_1)\}$ and $A_2 = \{x : \varphi_2(x, \underline{a}_2)\}$ for appropriate formulas φ_1 and φ_2. Then

$$\langle \mathfrak{A}, T \rangle \vDash (\mathscr{I} x \leq T(A_1))\varphi_1(x, \underline{a}_1)$$

and

$$\langle \mathfrak{A}, T \rangle \vDash (\mathscr{I} x \leq T(A_2))\varphi_2(x, \underline{a}_2).$$

Therefore, by $W2$,
$\langle \mathfrak{A}, T \rangle \vDash (\mathscr{I} x \leq \max\{T(A_1), T(A_2)\})(\varphi_1(x, \underline{a}_1) \vee \varphi_2(x, \underline{a}_2))$, that is,

$$T(A_1 \cup A_2) = T\big(\{x : \varphi_1(x, \underline{a}_1) \vee \varphi_2(x, \underline{a}_2)\}\big) \leq \max\{T(A_1), T(A_2)\}.$$

But by $W3$, we have $T(A_1 \cup A_2) \geq \max\{T(A_1), T(A_2)\}$, whence $T(A_1 \cup A_2) = \max\{T(A_1), T(A_2)\}$. □

$L_{\mathbb{A}\mathscr{I}}$ is not compact.

$L_{\mathbb{A}\mathscr{I}}$ is not compact just like $L_{\mathbb{A}P}$ isn't.

Example

Let R be a unary relation symbol and let Φ be the set of sentences

$$\{(\mathscr{I}x > 0)R(x)\} \cup \{(\mathscr{I}x \leq \frac{1}{n})R(x) : n \in \mathbb{N}^+\}.$$

Then any finite subset of Φ has a capacity model but Φ does not, owing to the axiom $W5$ (Archimedean property).

Formal Negation.

Let φ be a formula of $\mathcal{L}_{\mathbb{A}\mathscr{I}}$.

Definition

$$\begin{aligned}
\varphi\neg &\equiv \neg\varphi \text{ if } \varphi \text{ is atomic.}\\
(\neg\varphi)\neg &\equiv \varphi.\\
\Big(\bigwedge_n \varphi_n\Big)\neg &\equiv \bigvee_n \neg\varphi_n.\\
\Big(\bigvee_n \varphi_n\Big)\neg &\equiv \bigwedge_n \neg\varphi_n.\\
\big((\mathscr{I}x \geq r)\varphi\big)\neg &\equiv \big(\mathscr{I}x > 1-r\big)\neg\varphi.\\
\big((\mathscr{I}x > r)\varphi\big)\neg &\equiv \big(\mathscr{I}x \geq 1-r\big)\neg\varphi.
\end{aligned}$$

Consistency Properties for $L_{\mathbb{A}\mathscr{I}}$.

- Start with a countable set of constant symbols $\mathcal{C} = \{c_n : n \in \mathbb{N}\}$.
- Put $\mathcal{L}(C) = \mathcal{L} \cup \mathcal{C}$.
- Let $L_{\mathbb{A}\mathscr{I}}(C)$ be the set of $\mathcal{L}(C)_{\omega_1 \mathscr{I}}$-formulas consisting of all formulas derived from formulas in $L_{\mathbb{A}\mathscr{I}}$ by substituting finitely many $c \in C$ for free variables.

Consistency Properties for $L_{\mathbb{A}\mathscr{I}}$.

A *consistency property* for $L_{\mathbb{A}\mathscr{I}}$ is a set S of sets s, where each s is a set of sentences of $L_{\mathbb{A}\mathscr{I}}(C)$ satisfying each of the following properties.

$C1$. (Consistency Rule) Either $\varphi \notin s$ or $(\neg\varphi) \notin s$.

$C2$. ($\neg$-Rule) $(\neg\varphi) \in s$ implies $s \cup \{(\varphi\neg)\} \in S$.

$C3$. ($\bigwedge$-Rule) $(\bigwedge \Phi) \in s$ implies $s \cup \{\varphi\} \in S$ for all $\varphi \in \Phi$.

$C4$. ($\bigvee$-Rule) $(\bigvee \Phi) \in s$ implies $s \cup \{\varphi\} \in S$ for some $\varphi \in \Phi$.

$C5$. (Equality Rules) Let $b, c, d \in C$.

(α) $(c = d) \in s$ implies $s \cup \{d = c\} \in S$.

(β) $c = b, \varphi(b) \in s$ imply $s \cup \{\varphi(c)\} \in S$.

(γ) $s \cup \{e = b\} \in S$ for some $e \in C$.

Consistency Properties for $L_{\mathbb{A}\mathscr{I}}$.

$C6$. (($\mathscr{I}x > 0$)-Rule) $(\mathscr{I}x > 0)\varphi(x) \in s$ implies $s \cup \{\varphi(c)\} \in S$ for some $c \in C$.

$C7$. (Axiom-Rule) If $\varphi(x) \in L_{\mathbb{A}\mathscr{I}}(C)$ is an axiom, then each of the following holds.

(α) $s \cup \{(\mathscr{I}x \geq 1)\varphi(x)\} \in S$.

(β) For any constant c of $\mathcal{L}(C)$, $s \cup \{\varphi(c)\} \in S$.

$C6$ and $C7$ are the only two new rules.

Goal: Prove that if $s_0 \in S$ and S is a consistency property, then s_0 has a capacity model. This is our main theorem.

Model Existence Theorem for $L_{\mathbb{A}\mathscr{I}}$.

Theorem (Model Existence Theorem for $L_{\mathbb{A}\mathscr{I}}$)

Suppose S is a consistency property for $L_{\mathbb{A}\mathscr{I}}$. Then any $s_0 \in S$ has a weak capacity model.

Proof of the model existence theorem.

- Let $\mathcal{S} = \{\varphi_n : n \in \mathbb{N}\}$ be an enumeration of the sentences of $L_{\mathbb{A}\mathscr{I}}(C)$.
- Let $\mathcal{D} = \{d_n : n \in \mathbb{N}\}$ be an enumeration of the constants of $L_{\mathbb{A}\mathscr{I}}(C)$.

We want to construct a sequence $s_0 \subseteq s_1 \subseteq \cdots$ of sets in S with some properties. We do this by induction. Suppose s_n is given.

(1) Get the first constant symbol c in $\mathcal{D}$ such that $s_n \cup \{c = d_n\} \in S$. This is possible by $C5$. Put $s_n' = s_n \cup \{c = d_n\}$.

(2) If $s_n' \cup \{\varphi_n\} \notin S$, let $s_{n+1} = s_n'$. If $s_n' \cup \{\varphi_n\} \in S$, let $s_n'' = s_n' \cup \{\varphi_n\}$.

Proof of the model existence theorem.

(3) (i) If φ_n does not begin with $\bigvee$, let $s_{n+1} = s''_n$.

(ii) If φ_n is of the form $(\mathscr{I} x > 0)\psi$, get the first $d \in \mathcal{D}$ such that $s''_n \cup \{\psi(d)\} \in S$, and put $s_{n+1} = s''_n \cup \{\psi(d)\}$. This is possible by $C6$.

(iii) If φ_n is $\bigvee \Phi$, use $C4$ to find the least $\psi \in \Phi$ in the enumeration $\mathcal{S}$ such that $s''_n \cup \{\psi\} \in S$, and put $s_{n+1} = s''_n \cup \{\psi\}$.

- Let $s_\omega = \bigcup_{n \in \mathbb{N}} s_n$.
- Define an equivalence relation $\approx$ on $\mathcal{D}$ by declaring $c \approx d$ if and only if $(c = d) \in s_\omega$.
- Put $M = \{[c] : c \in \mathcal{D}\}$, where $[c]$ is the $\approx$-equivalence class of $c \in \mathcal{D}$.
- Let $\mathfrak{M} = \langle M, R_i^{\mathfrak{M}}, c^{\mathfrak{M}} \rangle_{i \in I, c \in \mathcal{D}}$.

Now, we need to interpret the relation symbols and the constant symbols.

Proof of the model existence theorem.

- For each relation symbol R_i of L, $R_i^{\mathfrak{M}}([c])$ if and only if $R_i([c]) \in s_\omega$. This interpretation is well-defined by $C5$.
- Interpret the constant symbols as $c^{\mathfrak{M}} = [c]$.

It remains to construct a precapacity on the family $\mathcal{K}$ of all the definable subsets of M with parameters from M.

- Define, for any $\varphi(x, \underline{d}) \in L_{\mathbb{A}\mathscr{I}}$,

$$T\big(\{c^{\mathfrak{M}} : \varphi(c, \underline{d})\}\big) = \inf\{r : (\mathscr{I}x \le r)\varphi(x, \underline{d}) \in s_\omega\}.$$

The infimum is attained by $W5$ (Archimedean Property). T is well defined by $W2$ (Monotonicity of the quantifier) which says

$$(\mathscr{I}x \ge r)\varphi \to (\mathscr{I}x \ge s)\varphi$$

whenever $r \ge s$.

Proof of the model existence theorem.

It remains to verify that T is a $\mathcal{K}$-precapacity. Let

$$X = \{c^{\mathfrak{M}} : \varphi(c, \underline{d})\} \text{ and } Y = \{c^{\mathfrak{M}} : \psi(c, \underline{e})\}$$

for appropriate formulas φ and ψ. Let's show $T(X) \leq T(Y)$ if $X \subseteq Y$.

- Suppose $X \subseteq Y$.
- Then by Monotonicity axiom $W3$, we have

$$\{r : (\mathscr{I} x \leq r)\psi(c, \underline{d}) \in s_\omega\} \subseteq \{r : (\mathscr{I} x \leq r)\varphi(c, \underline{e}) \in s_\omega\}.$$

- Therefore, $T(X) \leq T(Y)$ as desired.

Proof of the model existence theorem.

Let's prove that T is strongly subadditive, that is

$$T(X \cup Y) + T(X \cap Y) \leq T(X) + T(Y).$$

Let $n \in \mathbb{N}^+$ be arbitrary. Observe:

- There exists r_1 such that $r_1 < T(X) + \frac{1}{n}$ with $(\mathscr{I} x \leq r_1)\varphi(c, \underline{d}) \in s_\omega$.
- There exists r_2 such that $r_2 < T(Y) + \frac{1}{n}$ with $(\mathscr{I} x \leq r_2)\psi(c, \underline{e}) \in s_\omega$.
- $X \cap Y$ is definable by the conjunction of φ and ψ.
- $\therefore$ There exists r_3 such that $r_3 < T(X \cap Y) + \frac{1}{n}$ with

$$(\mathscr{I} x \leq r_3)(\varphi(c, \underline{d}) \wedge \psi(c, \underline{e})) \in s_\omega.$$

- By the strong subadditivity axiom $W4$, it follows that

$$\begin{aligned} T(X \cup Y) &\leq T(X) + \frac{1}{n} + T(Y) + \frac{1}{n} - \left(T(X \cap Y) + \frac{1}{n}\right) \\ &= T(X) + T(Y) - T(X \cap Y) + \frac{1}{n}. \end{aligned}$$

Proof of the model existence theorem.

- Since $n \in \mathbb{N}^+$ was arbitrary we obtain

$$T(X \cup Y) + T(X \cap Y) \leq T(X) + T(Y),$$

  which proves that T is strongly subadditive on $\mathcal{K}$.
- Hence $\langle \mathfrak{M}, T \rangle = \langle M, r_i^{\mathfrak{M}}, c^{\mathfrak{M}}, T \rangle_{i \in I, c \in \mathcal{D}}$ is a weak capacity structure.

We're done if we verify that $\langle \mathfrak{M}, T \rangle \vDash \varphi$ whenever $\varphi \in s_\omega$. We do this by an induction on complexity argument.

Proof of the model existence theorem.

- Suppose $\varphi \in s_\omega$ is atomic. Then by the definition of $R_i^{\mathfrak{M}}$ and $c_i^{\mathfrak{M}}$, we see that $\langle \mathfrak{M}, T \rangle \vDash \varphi$.
- By $C1$, it follows that $\langle \mathfrak{M}, T \rangle \vDash \neg\varphi$ whenever $\neg\varphi \in s_\omega$.
- The properties $C3$ and $C4$ handle conjunctions and disjunctions respectively.
- For instance, if $\varphi = \bigwedge \Phi$ and $\langle \mathfrak{M}, T \rangle \vDash \phi$ whenever $\phi \in s_\omega$ for each $\phi \in \Phi$, then by $C3$ we have $\langle \mathfrak{M}, T \rangle \vDash \varphi$.
- It remains to verify for φ of the form $(\mathscr{I} x \le r)\psi(x)$.
- Suppose $\inf\{t : (\mathscr{I} x \le t)\psi(x) \in s_\omega\} \le r$. Then $T(\{c^{\mathfrak{M}} : \mathfrak{M} \vDash \psi[c]\}) \le r$ by the induction hypothesis, whence $\langle \mathfrak{M}, T \rangle \vDash (\mathscr{I} x \le r)\psi(x)$.

This completes the induction argument. Thus $\langle \mathfrak{M}, T \rangle$ is a weak capacity model of s_ω. Hence $\langle \mathfrak{M}, T \rangle$ is a weak capacity model of s_0. This completes the proof. $\dashv$

Nonstandard capacity theory.

Note that the model existence theorem we proved is “weak” in the sense that we only obtained a *weak* capacity structure. But we conjecture that we can pass to a *strong* capacity structure using nonstandard analysis. We describe it now.

Definition

Let F be a set with a regular paving $\mathcal{F}$. Let $T : \mathcal{F} \to [0,1]$ be a precapacity on $\mathcal{F}$. Define $L(T) : \mathcal{P}(F) \to [0,1]$ by

$$L(T)(E) = \inf_{\substack{E \subseteq D \\ D \in \mathcal{F}_\sigma}} \sup_{\substack{X \subseteq D \\ X \in \mathcal{F}}} T(X).$$

Nonstandard capacity theory.

Let F be an internal set and let $\mathcal{F}$ be an internal (standardly) regular paving on F. And let $T : \mathcal{F} \to {}^*[0,1]$ be internal precapacity. Let ${}^\circ T(A)$ be the standard part of $T(A)$ for each $A \in \mathcal{F}$. Thus we get a mapping ${}^\circ T : \mathcal{F} \to [0,1]$. It is easy to see that ${}^\circ T$ is an $\mathcal{F}$-precapacity on F. We have the following theorem by David Ross.

Theorem ([Ros90])

The mapping $L({}^\circ T)$ induced by T is a strongly subadditive $\mathcal{F}$-capacity on F.

Denote $L({}^\circ T)$ by $\widehat{T}$. Consider the superstructure $\mathbb{V} = \mathbb{V}(M, \mathbb{R})$ with $\mathbb{V}_0$ containing the transitive closure of M, and $\mathbb{R}$ as urelements, and we will assume it is $\aleph_1$-saturated.

Conjecture.

Conjecture (Strong Model Existence Theorem)

Let $\langle \mathfrak{M}, T \rangle$ be a weak model for $L_{\mathbb{A}\mathscr{I}}$. Then $\langle {}^*\mathfrak{M}, \widehat{T} \rangle$ is a strong model. Moreover, for each $\varphi(x) \in L_{\mathbb{A}\mathscr{I}}$ and $a \in M$, we have

$$\langle \mathfrak{M}, T \rangle \vDash \varphi(a) \text{ if and only if } \langle {}^*\mathfrak{M}, \widehat{T} \rangle \vDash \varphi(a).$$

In our attempt at a proof we face the following issues.

- In the nonstandard universe the quantifier $(\mathscr{I} x \leq r)$ takes the values $r \in {}^*\mathbb{R}$.
- For formulas of the kind $\varphi \equiv \bigvee_{n \in \mathbb{N}} \varphi_n$, the $*$-transform yields formulas of the kind ${}^*\varphi \equiv \bigvee_{n \in {}^*\mathbb{N}} {}^*\varphi_n$.

We think we can fix the second issue by a saturation argument by proving that ${}^*\varphi$ is equivalent to a finite disjunction of ${}^*\psi_n$'s. But for now, the strong model existence theorem will remain as a conjecture.

Proof Theory.

We prove a "weak" completeness theorem for $L_{\mathbb{A}\mathscr{I}}$.

- *Weak* $L_{\mathbb{A}\mathscr{I}}$ is the capacity logic with only the weak axioms.

Definition

- The set of *theorems* of weak $L_{\mathbb{A}\mathscr{I}}$ is the least set of formulas of weak $L_{\mathbb{A}\mathscr{I}}$ containing all the weak axioms so that it is closed under the rules of inference.
- $\vdash_{L_{\mathbb{A}\mathscr{I}}} \varphi$ denotes that φ is a theorem of weak $L_{\mathbb{A}\mathscr{I}}$.
- A formula φ of weak $L_{\mathbb{A}\mathscr{I}}$ is *valid*, denoted $\vDash_{L_{\mathbb{A}\mathscr{I}}} \varphi$, if and only if φ is satisfied in every weak model by every interpretation of the free variables of φ.

Rules of Inference.

Let φ, ψ be formulas of weak $L_{\mathbb{A}\mathscr{I}}$ and let Ψ be a set of formulas of weak $L_{\mathbb{A}\mathscr{I}}$.

R1. *Modus Ponens.* $\varphi, \varphi \to \psi \vdash \psi$.

R2. *Conjunction.* $\{\varphi \to \psi : \psi \in \Psi\} \vdash \varphi \to \bigwedge \Psi$, where Ψ is at most countable.

R3. *Generalization.* $\varphi \to \psi(x) \vdash \varphi \to (\mathscr{I} x \geq 1)\psi(x)$, provided x is not free in φ.

Weak Soundness Theorem.

We close the weak axioms under rules of inference and get the theorems of weak $L_{\mathbb{A}\mathscr{I}}$.

Theorem (Weak Soundness Theorem)

The weak axioms of $L_{\mathbb{A}\mathscr{I}}$ are sound and the rules of inference preserve validity.

Proof.

Obviously, any weak capacity model satisfies each of the weak axioms. Let's show $R3$ preserves validity. Let $\langle\mathfrak{A}, T\rangle$ be an arbitrary weak model for $L_{\mathbb{A}\mathscr{I}}$, and suppose that

$$\langle\mathfrak{A}, T\rangle \models \varphi \rightarrow \psi(x),$$

$\langle\mathfrak{A}, T\rangle \models \varphi$, and x is not free in φ. Then by R1 we have $\langle\mathfrak{A}, T\rangle \models \psi(x)$ and

$$T\Big(\{a \in A : \mathfrak{A} \models \psi(x)[a]\}\Big) = T(A) = 1 \geq 1.$$

Hence $\langle\mathfrak{A}, T\rangle \models (\mathscr{I}x \geq 1)\psi(x)$. □

Weak Completeness Theorem.

- $L_{\mathbb{A}\mathscr{I}}(C)$ is the set of $\mathcal{L}(C)_{\omega_1\mathscr{I}}$-formulas consisting of all formulas derived from formulas in $L_{\mathbb{A}\mathscr{I}}$ by substituting only finitely many $c \in C$ for free variables.
- A formula of $L_{\mathbb{A}\mathscr{I}}(C)$ is of the form $\varphi(x_1, \ldots, x_n, c_1, \ldots, c_m)$, where $\varphi(x_1, \ldots, x_n, y_1, \cdots, y_m)$ is a formula of $L_{\mathbb{A}\mathscr{I}}$, where we have replaced each free occurrence of y_i by $c_i \in C$ for $1 \leq i \leq m$.
- If $z_1, \ldots, z_m$ are variables in $L_{\mathbb{A}\mathscr{I}}$ not occurring in $\varphi(x_1, \ldots, x_n, c_1, \ldots, c_m)$, then $\varphi(x_1, \ldots, x_n, z_1, \ldots, z_m)$ is in $L_{\mathbb{A}\mathscr{I}}$.
- So, if φ is a sentence of $L_{\mathbb{A}\mathscr{I}}$ and $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \varphi$, then $\vdash_{L_{\mathbb{A}\mathscr{I}}} \varphi$.

Weak Completeness Theorem.

Lemma

Let S be the set of all finite sets s of sentences in $L_{\mathbb{A}\mathscr{I}}(C)$ such that $\nvdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg \bigwedge s$. Then S is a consistency property.

- Let's prove $C6$; if $(\mathscr{I} x > 0)\varphi(x) \in s$ then $s \cup \{\varphi(c)\} \in S$ for some $c \in C$.
- Suppose $(\mathscr{I} x > 0)\varphi(x) \in s \in S$ but for all $c \in C$ we have $s \cup \{\varphi(c)\} \notin S$.
- Let $c \in C$ be such that c does not occur in s. This is possible since s is a finite set of sentences in $L_{\mathbb{A}\mathscr{I}}(C)$.
- Then $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg \bigwedge(s \cup \{\varphi(c)\})$.
- $\therefore \vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg(\bigwedge s \wedge \varphi(c))$.
- $\therefore \vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \bigwedge s \rightarrow \neg\varphi(c)$.
- Let y be a variable not occurring in s.

Proof of Lemma.

- Replacing c by y in the proof gives $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \bigwedge s \rightarrow \neg\varphi(y)$.
- $\therefore \vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \bigwedge s \rightarrow (\mathscr{I} y \geq 1)\neg\varphi(y)$, or rather $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \bigwedge s \rightarrow \neg(\mathscr{I} y > 0)\varphi(y)$.
- Hence $\vdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg \bigwedge s$, which is a contradiction. This proves $C6$.

Weak Completenss Theorem.

- Observe that $\nvdash_{L_{\mathbb{A}\mathscr{I}}(C)} \varphi$ if and only if $\nvdash_{L_{\mathbb{A}\mathscr{I}}(C)} \neg(\neg\varphi)$.
- So by the weak model existence theorem for $L_{\mathbb{A}\mathscr{I}}$, since S of Lemma is a consistency property, we get a weak model $\langle\mathfrak{M}, T\rangle \models \neg\varphi$.
- So $\not\models \varphi$.
- By the Weak Soundness Theorem, we obtain ...

Theorem (Weak Completeness Theorem for $L_{\mathbb{A}\mathscr{I}}$)

If φ is a sentence of weak $L_{\mathbb{A}\mathscr{I}}$, then $\vdash_{L_{\mathbb{A}\mathscr{I}}} \varphi$ if and only if $\models \varphi$.

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