Surreal Numbers

janitha

Construction

- Conway's construction of numbers is the following.
- If L, R are two sets of numbers, and if no member of R is $\not\leq$ any member of L, then $\{L|R\}$ is a number. All numbers are constructed in this way.
- Start with $\emptyset$. Then $\{\emptyset|\emptyset\}$ is a number.
- Define $\{\emptyset|\emptyset\} =: 0$.
- Let $x = \{L|R\}$ be a number. Then typical members in L are denoted by x^L, and typical members of R are denoted by x^R.
- $x = \{x^R|x^L\}$.
- x^L is called a left option of x, and x^R is called a right option of x.
- Write $x = \{a, b, c, \ldots|d, e, f, \ldots\}$ when $L = \{a, b, c, \ldots\}$ and $R = \{d, e, f, \ldots\}$.

Construction

- <u>Order</u> : $x \leq y$ iff no $y^R \leq x$ and $y \leq$ no x^L.
- $\leq$ is a total ordering of the surreals. Moreover $x^R \not\leq x$ and $x \not\leq x^L$.
- Three options for the next number; $\{0|0\}, \{0|\emptyset\}$, and $\{\emptyset|0\}$.
- $0 \leq 0$ iff no $0^R \leq 0$ and $0 \leq$ no 0^L. But there's no 0^R or 0^L. So $0 \leq 0$.
- So only two options for the next number; $\{\emptyset|0\}$ and $\{0|\emptyset\}$.
- <u>Negation</u> : $-x := \{-x^R| - x^L\}$.
- $1 := \{0|\emptyset\}$ and $-1 := \{\emptyset|0\}$.
- For $n \in \omega$, $n + 1 := \{0, 1, \ldots, n|\emptyset\}$.
- For ordinals α, $\alpha := \{\beta < \alpha|\emptyset\}$, where β is an ordinal.
- Ordinals correspond to the numbers with no right options.

Construction

- 0 is said to be born on day zero, -1 and 1 are said to be born on day one, and so on. So an ordinal α is born on day α.
- Addition : $x + y := \{x^L + y, x + y^L | x^R + y, x + y^R\}$.
- $0 + x = \{0^L + x, 0 + x^L | 0^R + x, 0 + x^R\} = \{0 + x^L | 0 + x^R\}$ (as $0^L = 0^R = \emptyset$) $= \{x^L | x^R\} = x$. Second equality is by induction hypothesis.
- Induction happens through all the numbers born before x, so they are the set of left options and the set of right options of x.
- $0 + x = x = x + 0$.
- $x + (-x) = \{x^L + (-x), x + (-x^L) | x^R + (-x), x + (-x^R)\}$.

Construction

- Get $x + (-x) \geq 0$ and $x + (-x) \leq 0$, whence $x + (-x) = 0 = (-x) + x$.
- $+$ is commutative and associative.
- $x \leq y$ iff $x + z \leq y + z$.
- Surreal numbers form a totally ordered abelian group under addition.
- $-\omega = \{\emptyset | 0, -1, -2, \ldots\}$.
- $\{0, 1, 2, \ldots, \omega | \emptyset\} = \omega + 1$.
- What's $x = \{0, 1, 2, \ldots | \omega\}$?
- x is a number as every $x^L < \omega$.

Construction

- $x+1 = \{x^L+1, x+1^L | x^R+1, x+1^R\} = \{1, 2, 3, \ldots, x | \omega+1\}$.
- Easy to see that both $x+1 < \omega$ and $\omega < x+1$ give absurdities. So $x+1 = \omega$, or $x = \omega - 1$.
- $\omega - 1 = \{0, 1, 2, \ldots | \omega\}$, $\omega - 2 = \{0, 1, 2, 3, \ldots | \omega, \omega - 1\}$, etc.
- Multiplication :

$$\begin{aligned} xy = \{&x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R\} | \\ &\{x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^R\} \end{aligned}$$

- If $0 \leq x$ and $0 \leq y$, then $0 \leq xy$.
- Surreal numbers form a totally ordered ring.

Construction

- Let

$$y = \{0, \frac{1+(x^R - x)y^L}{x^R}, \frac{1+(x^L - x)y^R}{x^L}\} \\ |\{\frac{1+(x^L - x)y^L}{x^L}, \frac{1+(x^R - x)y^R}{x^R}\}$$

- Then $xy = 1$.
- The class **No** of surreal numbers forms an ordered Field.

Simplicity

- The domain of **No** is the class $\textbf{No} = 2^{<\textbf{On}}$.
- So a surreal number is a function of the form $s : \alpha \to \{0, 1\}$, where α is an ordinal.
- α is called the <u>length</u> or the <u>birthday</u> of s.
- x is <u>simpler</u> than y if $x \subseteq y$, ie x is an initial segment of y as a binary sequence. This is denoted by $x \leq_s y$.
- Note that $0 \leq_s 1$ but -1 and 1 are not comparable in this sense.
- $\leq_s$ is a binary tree like partial order on **No**.
- Immediate successors of x are $x^\frown 0$ and $x^\frown 1$.
- Introduce a <u>total order</u> $<$ on $\textbf{No}$: $x^\frown 0 < x < x^\frown 1$.
- In general $x^\frown 0^\frown u < x < x^\frown 1^\frown v$ for all u and v.

Simplicity

- Note that $<$ coincides with the lexicographic order on binary sequences of same length.
- A subclass C of **No** is convex if whenever $x, y \in C$ and $z \in$ **No** satisfies $x < z < y$ we have $z \in C$.
- Every nonempty convex class C contains a simplest element. It's the meet of all elements of C.
- $x \leq_s y$ means x was born before y, equivalently x is in a lower level than y is in the surreal number tree.
- $x < y$ means if you project x and y to a horizontal line passing through the root then x lies to the left of y.
- Given two sets A, B of surreals with $A < B$, the class $\{A|B\} = \{y \in \mathbf{No} : A < y < B\}$ is nonempty and convex. So it has a simplest element x denoted by $A|B$.

Simplicity

- A given surreal can have different representations.
- $0 = \{00\}|\{01\}$.
- $0 = \{000, 00, 001\}|\{010\}$.
- $x = A|B$ is called the canonical representation of x if $\overline{A \cup B = \{y \in \mathbf{No} : y <_s x\}}$. We'll use this notation for Canonical representation of x.
- Canonical representation of 001 is $\{00\}|\{0, 01\}$.
- Note: If $x = A|B$ and $A < y < B$, then $x \leq_s y$. Because y doesn't have room to be in a lower level than x is in the surreal number tree.
- A simple picture shows that $x = A|B$ and $x \leq_s y$ does not imply $A < y < B$.

Simplicity Theorem (when does $x = z$?)

Theorem 1

Suppose for $x = \{x^L\}|\{x^R\}$ we have a number z satisfying the condition $x^L \not\geq z \not\geq x^R$ for all x^L and x^R. Further suppose no option of z satisfies the same condition. Then $x = z$.

Proof.

$x \geq z$ iff no $x^R \leq z$ and $x \leq$ no z^L. But for all z^R and z^L, $x^L \geq z^L, z^R \geq x^R$. Since $x \geq x^L$ we have $x \leq$ no z^L. So $x \geq z$. Unravelling the definition of $x > z$ leads to a contradiction. So $x \leq z$. Hence $x = z$. □

- Why the name "Simplicity Theorem"?
- When we know that x is a number it's the simplest number (earliest created number) lying between x^L and x^R, and if z is the simplest between x^L and x^R, then the simpler numbers z^L and z^R cannot satisfy the same condition.

Dyadic Rationals are contained in **No**

Theorem 2

If x is a rational whose denominator divides 2^n, then $x = \{x - \frac{1}{2^n}\}|\{x + \frac{1}{2^n}\}$.

- Dyadic rationals ($m/2^n$) are exactly the numbers born on finite days.
- Why?

Hahn Fields

- Start with an ordered abelian group Γ.
- The Hahn field $\mathbb{R}((\Gamma))$ with coefficients in $\mathbb{R}$ and monomials in Γ is the following.
- Domain of $\mathbb{R}((\Gamma))$ is the set of functions $f : \Gamma \to \mathbb{R}$ with support $S(f)$ a <u>reverse well-ordered subset</u> of Γ.
- For $f \in \mathbb{R}((\Gamma))$ which is not identically zero, $S(f)$ has a maximum element $\mathfrak{m}$. We say f is positive if $f(\mathfrak{m}) > 0$.
- Fix $\mathfrak{m} \in \Gamma$. <u>Truncation</u> of f at $\mathfrak{m}$ is $f|\mathfrak{m} : \Gamma \to \mathbb{R}$; $f|\mathfrak{m}$ coincides with f for arguments $> \mathfrak{m}$ and equals 0 on arguments $\leq \mathfrak{m}$.
- <u>Addition</u> : $(f+g)(\mathfrak{m}) = f(\mathfrak{m}) + g(\mathfrak{m})$.
- <u>Multiplication</u> : $(fg)(\mathfrak{m}) = \sum_{\mathfrak{n}+\mathfrak{o}=\mathfrak{m}} f(\mathfrak{n})g(\mathfrak{o})$.

Hahn Fields

- $\mathbb{R}((\Gamma))$ is an ordered field.
- If Γ is divisible, $\mathbb{R}((\Gamma))$ is a real closed field.[1]
- $x, y \in \mathbf{No}$ are in the same archimedean class if there exists $k \in \mathbb{N}$ so that $|x| \leq k|y|$ and $|y| \leq k|x|$.
- A positive $x \in \mathbf{No}$ is called a <u>monomial</u> if x is the simplest positive element in its archimedean class.
- A <u>term</u> is a nonzero real r multiplied by a monomial.
- The class of all terms is denoted by $\mathbb{R}\mathfrak{M}$, where $\mathfrak{M} \subseteq \mathbf{No}$ is the class of all monomials.

[1] Γ is divisible if for all $n \in \mathbb{Z}^+$ and for all $h \in \Gamma$ there exists $g \in \Gamma$ so that $ng = h$. Equivalently $n\Gamma = \Gamma$ for all $n \in \mathbb{Z}^+$

Hahn Field $\mathbb{R}((\mathfrak{M}))$ and $\sum$-map

- We can identify **No** with $\mathbb{R}((\mathfrak{M}))$ as follows.
- Given $f \in \mathbb{R}((\mathfrak{M}))$, write $f_{\mathfrak{m}}$ for the real number $f(\mathfrak{m})$.
- $f_{\mathfrak{m}}\mathfrak{m} = f(\mathfrak{m})\mathfrak{m} \in \mathbb{R}\mathfrak{M}$.
- Define the map $\sum : \mathbb{R}((\mathfrak{M})) \to \mathbf{No}$ by induction on the order type of the support.
- If $S(f) = \emptyset$, define $\sum f := 0 \in \mathbf{No}$.
- If $S(f)$ has a smallest monomial $\mathfrak{n}$, define $\sum f := \sum f|\mathfrak{n} + f_{\mathfrak{n}}\mathfrak{n}$.
- If $S(f) \neq \emptyset$ and $S(f)$ has no smallest monomial, define

$$\sum f := \{\sum f|\mathfrak{m} + q'\mathfrak{m}\}|\{\sum f|\mathfrak{m} + q''\mathfrak{m}\},$$

where $\mathfrak{m}$ varies in $S(f)$ and q', q'' vary among rationals such that $q' < f_{\mathfrak{m}} < q''$.

Hahn Field $\mathbb{R}((\mathfrak{M}))$ and $\sum$-map

Theorem 3

For every $f \in \mathbb{R}((\mathfrak{M}))$,

$$\sum f = \{\sum f|\mathfrak{m} + q'\mathfrak{m}\}|\{\sum f|\mathfrak{m} + q''\mathfrak{m}\},$$

where $\mathfrak{m}$ varies in $S(f)$ and q', q'' varies among rationals with $q' < f_{\mathfrak{m}} < q''$.

$\mathbb{R}((\mathfrak{M})) \cong \mathbf{No}$ as ordered Fields

- Write $\sum_{\mathfrak{m}\in\mathfrak{M}} f_{\mathfrak{m}}\mathfrak{m}$ for $\sum f$.
- Think of $\sum f$ as a decreasing formal infinite sum of terms $f_{\mathfrak{m}}\mathfrak{m}$ with reverse well-ordered support.

Theorem 4

$\sum : \mathbb{R}((\mathfrak{M})) \to \mathbf{No}$ *is an isomorphism of ordered Fields.*

- Identify $f \in \mathbb{R}((\mathfrak{M}))$ with $\sum f = \sum_{\mathfrak{m}} f_{\mathfrak{m}}\mathfrak{m} \in \mathbf{No}$.
- Write $\mathbf{No} = \mathbb{R}((\mathfrak{M}))$.

Conway's ω-map

- Fix $x \in \mathbf{No}$.
- Define $\omega^x := \{0, k\omega^{x'}\}|\{\frac{1}{2^k}\omega^{x''}\}$, where k ranges in $\mathbb{N}$, x' ranges in surreals x with $x' <_s x$ and $x' < x$, and x'' ranges in surreals x with $x'' <_s x$ and $x < x''$.
- $\omega^0 = \{0\}|\emptyset = 1$, $\omega^1 = \{0, 1, 2, \ldots\}|\emptyset = \omega$, and so on.

Lemma 5

If $x \leq_s y$, then $\omega^x \leq_s \omega^y$.

Theorem 6

The ω-map is an isomorphism from $(\mathbf{No}, +, <)$ to $(\mathfrak{M}, \cdot, <)$. And ω^x is the simplest positive representative of its archimedean class, $\omega^0 = 1$, and $\omega^{x+y} = \omega^x \cdot \omega^y$.

Conway's ω-map

- We have $\mathbf{No} = \mathbb{R}((\mathfrak{M}))$ and $\mathfrak{M} = \omega^{\mathbf{No}}$.
- So $\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}}))$.
- So every surreal number has the unique form

  $$x = \sum_{y \in \mathbf{No}} a_y \omega^y,$$

  where $a_y \in \mathbb{R}$ and $a_y \neq 0$ iff $\omega^y \in S(x)$. This is called the <u>normal form</u> of x.
- The normal form of x coincides with the Cantor normal form of x when $x \in \mathbf{On}$.

On the subgroup $\mathfrak{M}$

- Recall that a multiplicative subgroup $\mathfrak{M} \subseteq K^{>0}$ of an ordered field K is a set of monomials if for each nonzero $x \in K$, there's one and only one $\mathfrak{m} \in \mathfrak{M}$ so that $x \asymp \mathfrak{m}$.
- WTS: $\mathfrak{M}$ of monomials in our construction is a multiplicative subgroup of **No**.
- This follows via the ω-map.
- We have $\omega^{x+y} = \omega^x \cdot \omega^y$, $(\mathbf{No}, +)$ is a group, and $(\mathbf{No}, +) \cong^{\omega} (\mathfrak{M}, \cdot)$.
- So $(\mathfrak{M}, \cdot)$ is a multiplicative group.

Extending notions about Hahn fields to **No**

Let $x \in \mathbf{No}$. Write $x = \sum_{\mathfrak{m}} x_{\mathfrak{m}} \mathfrak{m}$.

- The support of x is $S(x) = \{\mathfrak{m} \in \mathfrak{M} : x_{\mathfrak{m}} \neq 0\}$.
- Terms of x are the numbers in $\{x_{\mathfrak{m}}\mathfrak{m} : x_{\mathfrak{m}} \neq 0\} \subseteq \mathbb{R}\mathfrak{M}$.
- The coefficient of $\mathfrak{m}$ in x is $x_{\mathfrak{m}}$.
- Leading monomial of x = maximal monomial in $S(x)$.
- Leading term of x = Leading monomial multiplied by its coefficient.
- Let $\mathfrak{n} \in \mathfrak{M}$. Truncation of x at $\mathfrak{n}$ is $x|\mathfrak{n} := \sum_{\mathfrak{m} > \mathfrak{n}} x_{\mathfrak{m}} \mathfrak{m}$.
- Write $y \trianglelefteq x$ if $y \in \mathbf{No}$ is a truncation of x, and $y \triangleleft x$ if moreover $x \neq y$.

$\trianglelefteq$ is a partial order with a tree-like structure

Proposition 7

$\trianglelefteq$ is a partial order with a tree-like structure.

$x \trianglelefteq y$ implies $x \leq_s y$

Theorem 8

If $x \trianglelefteq y$, then $x \leq_s y$.

Proof.

Suppose $x \trianglelefteq y$. By Theorem 1, $x = A|B$, where

$$A = \{x|\mathfrak{n} + q'\mathfrak{n}\},\ B = \{x|\mathfrak{n} + q''\mathfrak{n}\},$$

with $\mathfrak{n}$ varying in $S(x)$ and q', q'' varying in $\mathbb{Q}$ with $q' < x_{\mathfrak{n}} < q''$. Similarly $y = A'|B'$, where

$$A' = \{y|\mathfrak{n} + q'\mathfrak{n}\},\ B' = \{y|\mathfrak{n} + q''\mathfrak{n}\}.$$

- Since $x \trianglelefteq y$ we have $S(x) \subseteq S(y)$.
- For every $\mathfrak{n} \in S(x)$ we have $x|\mathfrak{n} = y|\mathfrak{n}$ and $x_{\mathfrak{n}} = y_{\mathfrak{n}}$.
- Therefore $A \subseteq A'$ and $B \subseteq B'$.
- Hence $x \leq_s y$.

$\square$

Extending the notion of infinite sum to **No**

Summability :

- Let I be a set and $(x_i : i \in I)$ be an indexed family of surreals.
- $(x_i : i \in I)$ is summable if $\bigcup_i S(x_i)$ is reverse well-ordered and if for every $\mathfrak{m} \in \bigcup_i S(x_i)$ there are only finitely many $i \in I$ so that $\mathfrak{m} \in S(x_i)$.

Another Representation

- Let $x = \sum_{m \in \mathfrak{M}} x_m m$.
- The support of x, $S(x) = \{m \in \mathfrak{M} : x_m \neq 0\}$ is reverse well-ordered.
- So there's some $\alpha \in \mathbf{On}$ so that $S(x) = \{m_i : i < \alpha\}$ and (m_i) is a decreasing sequence.
- So $x = \sum_{i<\alpha} m_i r_i$, with (m_i) a decreasing sequence in $\mathfrak{M}$ and $0 \neq r_i \in \mathbb{R}$ for each $i < \alpha$.
- Moreover $x > 0$ iff $r_0 > 0$.
- Now truncations make more sense: $g \in \mathbb{R}((\mathfrak{M}))$ is a truncation of $f = \sum_{i<\alpha} m_i r_i \in \mathbb{R}((\mathfrak{M}))$ iff there's some $\beta \in \mathbf{On}$ so that $g = \sum_{i<\beta} m_i r_i$ and $\beta < \alpha$.

Ressayre Form

- Let $x = \sum_{i<\alpha} m_i r_i$.
- x is purely infinite if $m_i > 1$ for all $i < \alpha$. This makes sense as (m_i) is s decreasing sequence.
- $\mathbf{No}^{\uparrow}$ = Non unitary ring of purely infinite surreals.
- x has the unique form $x^{\uparrow} + x^{\circ} + x^{\downarrow}$, where $x^{\uparrow} \in \mathbf{No}^{\uparrow}, x^{\circ} \in \mathbb{R}$, and $x^{\downarrow} \prec 1$.
- $\mathbf{No} = \mathbf{No}^{\uparrow} + \mathbb{R} + o(1)$.
- Gonshor defined a group isomorphism $\exp : (\mathbf{No}, +, <) \to (\mathbf{No}^{>0}, \cdot, <)$ extending the real exponential function and satisfying $\exp(x) \geq 1 + x$ for all $x \in \mathbf{No}$ and $\exp(x) = \sum_{n\in\mathbb{N}} \frac{x^n}{n!}$ for $x \prec 1$. Moreover $\exp(\mathbf{No}^{\uparrow}) = \mathfrak{M}$. So $\mathbf{No} = \mathbb{R}((e^{\mathbf{No}^{\uparrow}}))$.

Ressayre Form

- Every surreal can be written uniquely in the form $\sum_{i<\alpha} e^{\gamma_i} a_i$, where $\alpha \in \mathbf{On}$, (γ_i) is a decreasing sequence in $\mathbf{No}^{\uparrow}$, and $0 \neq a_i \in \mathbb{R}$ for each i.
- This representation is called the Ressayre Form.